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Scattering boundary rigidity in the presence of a magnetic field

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It has been shown in [10] that, under some additional assumptions, two simple domains with the same scattering data are equivalent. We show that the simplicity of a region can be read from the metric in the boundary and the scattering data. This lets us extend the results in [10] to regions with the same scattering data, where only one is known a priori to be simple. We will then use this results to resolve a local version of a question by Robert Bryant. That is, we show that a surface of constant curvature cannot be modified in a small region while keeping all the curves of some fixed constant geodesic curvatures closed.

1. Introduction

A magnetic field on a Riemannian manifold can be represented by a closed two-form Ω, or equivalently by the $(1,1)$ tensor $Y : TM \to TM$ defined by $\Omega(\xi, \nu) = \langle Y(\xi), \nu\rangle$ for all $x \in M$ and $\xi, \nu \in T_xM$. The trajectory of a charged particle in such a magnetic field is then modeled by the equation

$$\nabla_{\gamma'}\gamma' = Y(\gamma'),$$

we will call such curves *magnetic geodesics*. In contrast to regular (or *straight*) geodesics, magnetic geodesics are not reversible, and cannot be rescaled, i.e., the trajectory depends on the energy $|\gamma'|^2$.

Magnetic geodesics and the magnetic flow where first considered by Arnold [2] and Anosov and Sinai [1]. The existence of closed magnetic geodesics, and the magnetic flow in general, has been widely studied since then. Some of the approaches to this subject are, the Morse–Novikov theory for variational functionals (e.g., [20, 21, 26]), Aubry–Mather's theory (e.g., [6]), the theory of dynamical systems (e.g., [15, 19, 24]) and using methods from symplectic geometry (e.g., [3, 13, 14]).

In the special case of surfaces, where the two-form has the form $\Omega = k(x)dA$, a magnetic geodesic of energy c has geodesic curvature $k_g = k(x)/\sqrt{c}$.

This relates magnetic geodesics with the problem of prescribing geodesic curvature, in particular with the study of curves of constant geodesic curvature. This relation was used by Arnold in [4], and later by many others (see, e.g., [17, 25]), to study the existence of closed curves with prescribed geodesic curvature.

It is clear that on surfaces of constant curvature the curves of large constant geodesic curvature are circles, therefore closed. The study of these curves goes back to Darboux, who in 1894 claimed (in a footnote in his book [11]) that the converse is true, that is, if all curves of constant (sufficiently large) geodesic curvature are closed, then the surface has to be of constant Gauss curvature.

The proof of this result depends strongly on the fact that curves of geodesic curvature are closed for all large curvature, or equivalently low energy. This raises the following question, brought to my attention by R. Bryant.

Question 1.1. *Are surfaces of constant Gauss curvature the only surfaces for which all curves of a fixed constant nonzero geodesic curvature are closed?*

For the case where the constant is 0, this question corresponds to existence of surfaces all of whose geodesics are closed. The first examples of such surfaces where given by Zoll [27] who, in 1903, constructed a surface of revolution with this property.

Using the relation between geodesic curvature and magnetic geodesics we can approach this question by studying magnetic geodesics on a surface, in the presence of a constant magnetic field. A first step in this direction is to determine whether a surface of constant curvature can be locally changed keeping all the curves of a fixed constant geodesic curvature closed. We show in Section 5 that the metric cannot be changed in a small region without loosing this property. In fact, more generally, we show that a Riemannian surface with a magnetic flow whose orbits are closed cannot be changed locally without "opening" some of its orbits.

The easier way of changing the metric in a small region without opening the orbits is to require that all magnetic geodesics that enter the region leave it at the same place and in the same direction as before, to join the outside part of the orbit. This is the *magnetic scattering data* of the region; for each point and inward direction on the boundary, it associates the exit point and direction of the corresponding unit speed magnetic geodesic.

With this problem in mind, we can ask the following boundary rigidity question for magnetic geodesics. In a Riemannian manifold with boundary, in the presence of a magnetic field, is the metric determined by the metric on the boundary and the magnetic scattering data?

In general this is not true, even for the geodesic case. For example, a round sphere with a small disk removed has the same scattering data as a round $\mathbb{R}P^2$ with a disk of the same size removed. One of the usual conditions to obtain boundary rigidity is to assume that the region is *simple.* In our setting *simple* means a compact region that is magnetically convex, and where the magnetic exponential map has no conjugate points (see Section 4).

For simple domains, scattering rigidity for geodesics is equivalent to distance boundary rigidity (see [9]) and it has been widely studied. It is known to hold for simple subdomains of $\mathbb{R}^n$ [16] or [8] , an open round hemisphere [18], hyperbolic space [5, 9] and some spaces of negative curvature [7, 22] among others. For a discussion on the subject, see [9].

Recently Dairbekov, Paternain, Stefanov and Uhlmann [10] proved magnetic boundary rigidity between two simple manifolds in several classes of metrics, including simple conformal metrics, simple analytic metrics and all two-dimensional simple metrics.

To be able to apply this results to local perturbations of existing metrics we would like to be able to compare a simple domain with any other (not necessarily simple) domain with the same boundary behavior. For this we prove the following theorem. Thus, proving magnetic rigidity for the simple domains considered in [10].

Theorem 1.1. *Magnetic simplicity can be read from the metric on the boundary and the scattering data.*

To change the metric in a small region without opening the orbits, it is not necessary to preserve the scattering data. It could be the case, in principle, that orbits exit the region in a different place, but after some time, came back to the region and leave it in the proper place to close up again (see figure 1). In Section 5 we look at this case in two dimensions, and we show the following theorem.

Theorem 1.2. *Let M and $\widehat{M}$ be compact surfaces with magnetic fields, all of whose magnetic geodesics are closed. Let $R \subset \widehat{M}$ be a strictly magnetically convex region, such that every magnetic geodesic passes through R at most once.*

If the metric and magnetic fields of M and $\widehat{M}$ agree outside R, then they have the same scattering data.

Applying these two theorems for a constant magnetic field on a surface of constant curvature, we conclude that:

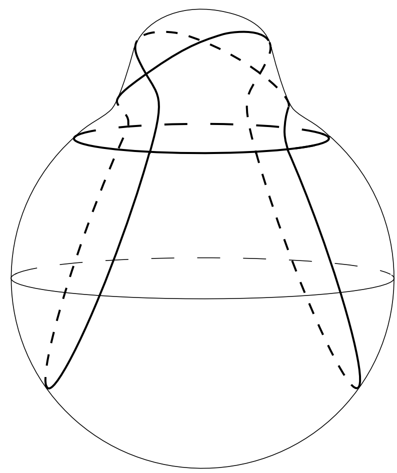

Figure 1: A closed orbit going through R twice.

Corollary 1.1. *Consider a surface of constant curvature and a fixed k such that all the circles of geodesic curvature k are simple. Then the surface can not be perturbed in a small enough region while keeping all the curves of geodesic curvature k closed.*

2. Magnetic Jacobi fields and conjugate points

Let (M, g) be a compact Riemannian manifold with boundary, with a magnetic field given by the closed two-form Ω on M. Denote by ω_0 the canonical symplectic form on TM, that is, the pull back of the canonical symplectic form of T^*M by the Riemmanian metric. The geodesic flow can be described as the Hamiltonian flow of H w.r.t. ω_0, where $H : TM \to \mathbb{R}$ is defined as

$$H(v) = \frac{1}{2}|v|_g^2, \quad v \in TM.$$

In a similar way, the magnetic flow $\psi^t : TM \to TM$ can be described as the Hamiltonian flow of H with respect to the modified symplectic form $\omega = \omega_0 + \pi^*\Omega$. This flow has orbits $t \to (\gamma(t), \gamma'(t))$, where γ is a magnetic geodesic, i.e., $\nabla_{\gamma'}\gamma' = Y(\gamma')$. Note that when $\Omega = 0$ we recover the geodesic flow, whose orbits are geodesics.

It follows from the above definitions that the magnetic geodesics have constant speed. In fact,

$$\frac{d}{dt}\langle \gamma'(t), \gamma'(t)\rangle = 2\langle Y(\gamma'), \gamma'\rangle = 2\Omega(\gamma', \gamma') = 0.$$

Moreover, the trajectories of the magnetic geodesics depend on the energy level. Unlike geodesics, a rescaling of a magnetic geodesic is no longer a magnetic geodesic. We will restrict our attention to a single energy level, or equivalently to unit speed magnetic geodesics. Therefore, from now on, we will only consider the magnetic flow $\psi^t : SM \to SM$.

The choice of energy level is not a restriction, since we can study other energy levels by considering the form $\tilde{\Omega} = \lambda\Omega$, for any $\lambda \in \mathbb{R}$.

For $x \in M$ we define the *magnetic exponential map* at x to be the partial map $\exp^\mu_x : T_xM \to M$ given by

$$\exp^\mu_x(t\xi) = \pi \circ \psi^t(\xi), \quad t \geq 0, \quad \xi \in S_xM.$$

This map takes a vector $t\xi \in T_xM$ to the point in M that corresponds to following the magnetic geodesic with initial direction ξ, a time t. This function is C^∞ on $T_xM \setminus \{0\}$ but in general only C^1 at 0. The lack of smoothness at the origin can be explained by the fact that magnetic geodesics are not reversible. When we pass through the origin we change from γ_ξ to $\gamma_{-\xi}$, that in general only agree up to first order. For a proof see Appendix A in [10].

We will say that a point $p \in M$ is *conjugate* to x along a magnetic geodesic γ if $p = \gamma(t_0) = \exp^\mu_x(t_0\xi)$ and $v = t_0\xi$ is a critical point of $\exp^\mu_x$. The *multiplicity* of the conjugate point p is then the dimension of the kernel of $d_v \exp^\mu_x$.

In what follows, and throughout this paper, if V is a vector field along a geodesic $\gamma(t)$, V' will denote the covariant derivative $\nabla_{\gamma'}V$.

We want to give an alternative characterizations of conjugate points. For this consider a variation of γ through magnetic geodesics. That is

$$f(t,s) = \gamma_s(t),$$

where $\gamma_s(t)$ is a magnetic geodesic for each $s \in (-\epsilon, \epsilon)$ and $t \in [0,T]$. Therefore $\frac{D}{\partial t}\frac{\partial f}{\partial t} = Y(\frac{\partial f}{\partial t})$. Using this and the definition of the curvature tensor we can write

$$\begin{aligned}\frac{D}{\partial s}\left(Y\left(\frac{\partial f}{\partial t}\right)\right) &= \frac{D}{\partial s}\frac{D}{\partial t}\frac{\partial f}{\partial t} = \frac{D}{\partial t}\frac{D}{\partial s}\frac{\partial f}{\partial t} - R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial t}\\ &= \frac{D}{\partial t}\frac{D}{\partial t}\frac{\partial f}{\partial s} + R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t}.\end{aligned}$$

If we call the variational field $J(t) = \frac{\partial f}{\partial s}(t,0)$ we get for $s = 0$

$$\nabla_J(Y(\gamma')) = J'' + R(\gamma', J)\gamma'.$$

Note also that

$$\nabla_J(Y(\gamma')) = Y(\nabla_J\gamma') + (\nabla_J Y)(\gamma')$$

and

$$\nabla_J\gamma'(t) = \frac{D}{\partial s}\frac{\partial f}{\partial t}(t,0) = \frac{D}{\partial t}\frac{\partial f}{\partial s}(t,0) = J'(t), \tag{2.1}$$

so we can rewrite the above equation as

$$J'' + R(\gamma', J)\gamma' - Y(J') - (\nabla_J Y)(\gamma') = 0.$$

Since magnetic geodesics cannot be rescaled, we have $|\gamma_s'| = 1$ for all the magnetic geodesics in the variation. This equation together with Equation (2.1) gives, for any such variational field J,

$$\langle J', \gamma'\rangle = \langle \nabla_J\gamma', \gamma'\rangle = J\langle\gamma', \gamma'\rangle = 0.$$

This equations characterize the variational field of variations through magnetic geodesics, in a way analogous to the characterization of Jacobi fields. We will use this equation as a definition as follows.

Given a magnetic geodesic $\gamma : [0, T] \to M$, let $\mathcal{A}$ and $\mathcal{C}$ be the operators on smooth vector fields Z along γ defined by

$$\begin{aligned}\mathcal{A}(Z) &= Z'' + R(\gamma', Z)\gamma' - Y(Z') - (\nabla_Z Y)(\gamma'),\\ \mathcal{C}(Z) &= R(\gamma', Z)\gamma' - Y(Z') - (\nabla_Z Y)(\gamma').\end{aligned}$$

A vector field J along γ is said to be a *magnetic Jacobi field* if it satisfies the equations

$$\mathcal{A}(J) = 0 \tag{2.2}$$

and

$$\langle J', \gamma'\rangle = 0. \tag{2.3}$$

Note that from Equation (2.2) we can see that

$$\begin{aligned}\frac{d}{dt}\langle J', \gamma'\rangle &= \langle J'', \gamma'\rangle + \langle J', Y(\gamma')\rangle\\ &= \langle -R(\gamma', J)\gamma' + Y(J') + (\nabla_J Y)(\gamma'), \gamma'\rangle - \langle Y(J'), \gamma'\rangle = 0,\end{aligned}$$

where we used that $\langle(\nabla_J Y)(\gamma'), \gamma'\rangle = 0$ because Y is skew-symmetric. Therefore, it is enough to check condition (2.3) at a point.

Note: In [10] magnetic Jacobi fields were defined without condition (2.3), although where used only for vector fields that satisfied this condition. We will add it to the definition since it helps preserve the relation between (magnetic) Jacobi fields and variations through (magnetic) geodesics.

A magnetic Jacobi field along a magnetic geodesic γ is uniquely determined by its initial conditions $J(0)$ and $J'(0)$. To see this, consider the orthonormal basis defined by extending an orthonormal basis $e_1, \dots, e_n$ at $\gamma(0)$ by requiring that

$$e_i' = Y(e_i) \tag{2.4}$$

along γ. This extension gives an orthonormal basis at each point since

$$\frac{d}{dt}\langle e_i, e_j\rangle = \langle Y(e_i), e_j\rangle + \langle e_i, Y(e_j)\rangle = 0.$$

Using this basis,

$$z = \sum_{i=1}^{n} f_i e_i$$

and we can write Equation (2.2) as the system

$$f_j'' + \sum_{i=1}^{n} f_i' y_{ij} + \sum_{i=1}^{n} f_i a_{ij} = 0,$$

where $y_{ij} = \langle Y(e_i), e_j\rangle$ and

$$a_{ij} = \langle \nabla_{\gamma'} Y(e_i) + R(\gamma', e_i)\gamma' - Y(Y(e_i)) - (\nabla_{e_i} Y)(\gamma'), e_j\rangle.$$

This is a linear second-order system, and therefore it has a unique solution for each set of initial conditions.

Magnetic Jacobi fields correspond exactly to variational field of variations through magnetic geodesics. In the case of magnetic Jacobi fields J along γ_ξ that vanish at 0 can be seen by considering

$$f(t,s) = \gamma_s(t) = \exp_x^\mu(t\xi(s)), \tag{2.5}$$

where $\xi : (-\epsilon, \epsilon) \to S_xM$ is a curve with $\xi(0) = \xi = \gamma_\xi$ and $\xi'(0) = J'(0)$. This is clearly a variation through magnetic geodesics, and therefore its variational field $\frac{\partial f}{\partial s}(t,0)$ satisfies (2.2). The variational field $\frac{\partial f}{\partial s}(t,0)$ and the magnetic Jacobi field $J(t)$ are then solutions of (2.2) with the same initial conditions, therefore they must agree.

For magnetic Jacobi fields J that do not vanish at 0, we can use the variation

$$f(t,s) = \gamma_s(t) = \exp^{\mu}_{\tau(s)}(t\xi(s)),$$

where $\tau(s)$ is any curve with $\tau'(0) = J(0)$ and $\xi(s)$ is a vector field along τ with $\xi(0) = \gamma'(0)$ and $\xi'(0) = J'(0)$.

It is easy to see from the definition and the equation $\gamma'' = Y(\gamma')$ that γ' is always a magnetic Jacobi field. Unlike the case of straight geodesics, this is the only magnetic Jacobi field parallel to γ'. Another difference from the straight geodesic case is that magnetic Jacobi fields that are perpendicular to γ' at $t = 0$ do not stay perpendicular for all t. For this reason we will sometimes consider instead the orthogonal projection $J^{\perp} = J - f\gamma'$ where $f = \langle J, \gamma' \rangle$. The component $f\gamma'$ of J parallel to γ' is uniquely determined by $J^{\perp}$ and $J(0)$, since

$$f' = \langle J', \gamma' \rangle + \langle J, \gamma'' \rangle = \langle J, Y(\gamma') \rangle = \langle J^{\perp}, Y(\gamma') \rangle.$$

We will need one more property of magnetic Jacobi fields that vanish at 0. Let $\gamma : [0,T] \to M$ be a magnetic geodesic with $\gamma(0) = x$. Let $v \in T_{\gamma'} S_x M$, or equivalently under the usual identification, $v \in T_x M$ perpendicular to γ'.

Let $f(t,s)$ be a variation through magnetic geodesics of the form (2.5) where $t \in [0,T]$ and $\xi : (-\epsilon, \epsilon) \to S_x M$ with $\xi(0) = \gamma'(0)$ and $\xi'(0) = v$. The variational field J_v of this variation is

$$\begin{aligned} J_v(t) &= \frac{\partial f}{\partial s}(t,0) = \frac{\partial}{\partial s}[\pi \circ \psi^t(\xi(s))]\Big|_{s=0} = d_{\xi(0)}[\pi \circ \psi^t](\xi'(0)) \\ &= d_{\psi^t(\xi)}\pi \circ d_{\xi}\psi^t(v) = d_{\gamma'(t)}\pi \circ d_{\gamma'(0)}\psi^t(v) \end{aligned} \tag{2.6}$$

and its derivative is given by

$$\begin{aligned} J_v'(t) &= \frac{D}{\partial t}\frac{\partial f}{\partial s}(t,0) = \frac{D}{\partial s}\frac{\partial}{\partial t}[\pi \circ \psi^t(\xi(s))]\Big|_{s=0} \\ &= \frac{D}{\partial s}[\psi^t(\xi(s))]\Big|_{s=0} = d_{\xi(0)}\psi^t(\xi'(0)) = d_{\gamma'(0)}\psi^t(v). \end{aligned}$$

This equations are independent of the variation f.

Since the magnetic flow is a Hamiltonian flow with respect to the symplectic form $\omega = \omega_0 + \pi^*\Omega$, this form is invariant under the magnetic flow ψ^t [23, p. 10]. Therefore for any two magnetic Jacobi fields J_v and J_w as above, we have that

$$\omega(d_{\gamma'}\psi^t(v), d_{\gamma'}\psi^t(w)) = \omega_0(d_{\gamma'}\psi^t(v), d_{\gamma'}\psi^t(w)) + \pi^*\Omega(d_{\gamma'}\psi^t(v), d_{\gamma'}\psi^t(w))$$
$$= \langle J_v(t), J'_w(t)\rangle - \langle J'_v(t), J_w(t)\rangle + \Omega(J_v(t), J_w(t))$$

is independent of t. Using also that $J_v(0) = 0$, we get

$$\langle J_v, J'_w\rangle - \langle J'_v, J_w\rangle + \langle Y(J_v), J_w\rangle = 0 \tag{2.7}$$

for any two such Jacobi fields.

We will now relate the concepts of magnetic Jacobi fields and conjugate points.

Proposition 2.1. *Let $\gamma_\xi : [0,T] \to M$ be the magnetic geodesic with $\gamma(0) = x$ and $\gamma'(0) = \xi$. The point $p = \gamma(t_0)$ is conjugate to x along γ if and only if there exist a magnetic Jacobi field J along γ, not identically zero, with $J(0) = 0$ and $J(t_0)$ parallel to γ'.*

Moreover, the multiplicity of p as a conjugate point is equal to the number of linearly independent such Jacobi fields.

Consider a variation through magnetic geodesics as in (2.5), with $\xi'(0) = v$ perpendicular to γ'. Then

$$J_v(t) = \frac{\partial f}{\partial s}(t,0) = d_{t\xi}\exp^\mu_x(tv)$$

is a nontrivial magnetic Jacobi field. If there is a vector v for which $J_v(t_0)$ is parallel to γ', then $d_{t_0\xi}\exp^\mu_x(t_0 v)$ and $d_{t_0\xi}\exp^\mu_x(\xi) = \gamma'$ will be parallel, and $t_0\xi$ is a critical point of $\exp^\mu_x$. Conversely, if $t_0\xi$ is a critical point there must be a vector v such that $d_{t_0\xi}\exp^\mu_x(v) = 0$. Let $v^\perp = v - \langle v,\xi\rangle\xi$, this is not 0 since $d_{t_0\xi}\exp^\mu_x(\xi) = \gamma' \neq 0$, and

$$J_{v^\perp}(t_0) = d_{t_0\xi}\exp^\mu_x(t_0 v) - d_{t_0\xi}\exp^\mu_x(t_0\langle v,\xi\rangle\xi) = -t_0\langle v,\xi\rangle\gamma'(t_0).$$

To prove the second statement, note that Jacobi fields J_{v_i} as above are linearly independent iff the vectors v_i are. Since all v_i are perpendicular to $\gamma'(0)$, the number of linearly independent vectors will be the dimension of the kernel of $d_{t_0\xi}\exp^\mu_x$, that is the multiplicity of the conjugate point.

3. The index form

Let Λ denote the $\mathbb{R}$-vector space of piecewise smooth vector fields Z along γ. Define the quadratic form $\mathrm{Ind} : \Lambda \to \mathbb{R}$ by

$$\mathrm{Ind}_\gamma(Z) = \int_0^T \{|Z'|^2 - \langle \mathcal{C}(Z), Z\rangle - \langle Y(\gamma'), Z\rangle^2\} dt.$$

Note that

$$\begin{aligned}\mathrm{Ind}_\gamma(Z) = -&\int_0^T \{\langle \mathcal{A}(Z), Z\rangle + \langle Y(\gamma'), Z\rangle^2\} dt \\ &+ \langle Z, Z'\rangle|_0^T + \sum \langle Z, Z'^- - Z'^+\rangle|_{t_i},\end{aligned}$$

where $Z'^{\pm}$ stands for the left and right derivatives of Z at the points t_i where the derivative is discontinuous.

The $\mathrm{Ind}_\gamma(Z)$ generalizes the index form of a geodesic in a Riemannian manifold. It is easy to see that when $\Omega = 0$ these are of the same form. We will see throughout this section that, when restricted to orthogonal vector fields, we retain some of the relations between (magnetic) Jacobi fields, index form and conjugate points.

Let Λ_0 denote the $\mathbb{R}$-vector space of piecewise smooth vector fields Z along γ such that $Z(0) = Z(T) = 0$, $\Lambda^\perp$ the subspace of piecewise smooth vector fields that stay orthogonal to γ', and $\Lambda_0^\perp = \Lambda_0 \cap \Lambda^\perp$.

For any magnetic Jacobi field J along a magnetic geodesic γ, let $f = \langle J, \gamma'\rangle$ and $J^\perp = J - f\gamma'$ the component of J orthogonal to γ, using that $\gamma'' = Y(\gamma')$ we have

$$\begin{aligned}\mathcal{A}(J) = J^{\perp\prime\prime} + f''\gamma' + 2f'Y(\gamma') + f\gamma''' + R(\gamma', J^\perp + f\gamma')\gamma' - Y(J^{\perp\prime} \\ + f'\gamma' + f\gamma'') - (\nabla_{Z+f\gamma'}Y)(\gamma')\end{aligned}$$

so

$$0 = A(J^\perp) + fA(\gamma') + f''\gamma' + f'Y(\gamma'). \tag{3.1}$$

On the other hand, any magnetic Jacobi field satisfies $\langle J', \gamma'\rangle = 0$. Using this together with $f = \langle J, \gamma'\rangle$ and $\gamma'' = Y(\gamma')$ we see that $f' = \langle J, Y(\gamma')\rangle = \langle J^\perp, Y(\gamma')\rangle$. Since γ' is a Jacobi field, $A(\gamma') = 0$, and we have from (3.1)

$$\langle A(J^\perp), Z\rangle = -f'\langle Y(\gamma'), J^\perp\rangle = -\langle Y(\gamma'), J^\perp\rangle^2,$$

therefore if J is a magnetic Jacobi field and its orthogonal component $J^\perp$ is in Λ_0

$$\mathrm{Ind}_\gamma(Z) = -\int_0^T \{\langle \mathcal{A}(J^\perp), J^\perp\rangle + \langle Y(\gamma'), J^\perp\rangle^2\}dt = 0. \tag{3.2}$$

Also, if there are no conjugate points along γ, we can restrict our attention to a neighborhood of γ for which the magnetic exponential is a diffeomorphism. In this case we can adapt the proof given in [10] for the case of simple domains to prove the following version of the Index Lemma. We include the proof below.

Lemma 3.1. *Let γ be a magnetic geodesic without conjugate points to $\gamma(0)$. Let J be a magnetic Jacobi field, and $J^\perp$ the component orthogonal to γ'. Let $Z \in \Lambda^\perp$ be a piecewise differentiable vector field along γ, perpendicular to γ'. Suppose that*

$$J^\perp(0) = Z(0) = 0 \quad \text{and} \quad J^\perp(T) = Z(T). \tag{3.3}$$

Then

$$\mathrm{Ind}_\gamma(J^\perp) \le \mathrm{Ind}_\gamma(Z)$$

and equality occurs if and only if $Z = J^\perp$.

Note that in the case where the vector field Z satisfies $Z(0) = Z(T) = 0$, $J = \gamma'$ is a Jacobi field that satisfies the above hypothesis, giving the following corollary.

Corollary 3.1. *If γ has no conjugate points and $Z \in \Lambda_0^\perp$, then* $\mathrm{Ind}_\gamma(Z) \geq 0$*, with equality if and only if $Z = 0$.*

In other words, if γ has no conjugate points, the quadratic form $\mathrm{Ind}_\gamma : \Lambda_0^\perp \to \mathbb{R}$ is positive definite.

Proof of Lemma 3.1. Given a vector $v \in T_{\gamma(0)}M$ orthogonal to γ', we can define a magnetic Jacobi field J_v along γ as in (2.6) that has $J_v(0) = 0$ and $J_v'(0) = v$. Since there are no conjugate points to $\gamma(0)$ along γ, these Jacobi fields are never 0 nor parallel to γ'. There exist a basis $\{v_1, \dots, v_n\}$ of $T_{\gamma(0)}M$, with $v_1 = \gamma'$ such that $J_1(t) = \gamma'(t)$ and $J_i(t) = J_{v_i}(t)$, $i = 2, \dots, n$ form a basis for $T_{\gamma(t)}M$ for all $t \in (0, T]$.

If Z is a vector field with $Z(0)=0$, we can write, for $t\in(0,T]$

$$Z(t)=\sum_{i=1}^{n} f_i(t)J_i(t),$$

where $f_1,\ldots,f_n$ are smooth functions. This function can be smoothly extended to $t=0$, as we now show. For $i\geq 2$, we can write $J_i(t)=tA_i(t)$ where A_i are smooth vector fields with $A_i(0)=J_i'(0)$. Then each $A_i(t)$ is parallel to $J_i(t)$ and $\{\gamma'(t),A_2(t),\ldots,A_n(t)\}$ is a basis for all $t\in[0,T]$, so

$$Z(t)=g_1\gamma'+\sum_{i=2}^{n} g_i(t)A_i(t).$$

It follows that for $t\in(0,T]$, $g_1=f_1$ and $g_i(t)=tf_i(t)$ for $i\geq 2$, and since $Z(0)=0$, $g_i(0)=0$ and f_i extends smoothly to $t=0$.

Using this representation we can write

$$\begin{aligned}\operatorname{Ind}_\gamma(Z)=&-\sum_{i,j}\int_0^T\langle\mathcal{A}(f_iJ_i),f_jJ_j\rangle dt\\&-\int_0^T\langle Y(\gamma'),Z\rangle^2dt+\langle Z,Z'\rangle|_0^T+\sum\langle Z,Z'^{-}-Z'^{+}\rangle|_{t_k},\end{aligned}$$

where $Z'^{\pm}$ stands for the left and right derivatives of Z at the points t_k where the derivative is discontinuous.

And

$$\begin{aligned}\mathcal{A}(f_iJ_i)&=f_i''J_i+2f_i'J_i'+f_iJ_i''+f_iR(\gamma',J_i)\gamma'\\&\quad-f_i'Y(J_i)-f_iY(J_i')-f_i(\nabla_{J_i}Y)(\gamma')\\&=f_i\mathcal{A}(J_i)+f_i''J_i+2f_i'J_i'-f_i'Y(J_i),\end{aligned}\tag{3.4}$$

where the first term is 0 since J_i is a magnetic Jacobi field.

We know, moreover, from (2.7) that

$$\langle J_i,J_j'\rangle-\langle J_i',J_j\rangle+\langle Y(J_i),J_j\rangle=0,$$

so we can write

$$\begin{aligned}\langle\mathcal{A}(f_iJ_i),J_j\rangle&=\langle f_i''J_i,J_j\rangle+2\langle f_i'J_i',J_j\rangle-\langle f_i'Y(J_i),J_j\rangle\\&=f_i''\langle J_i,J_j\rangle+f_i'\langle J_i',J_j\rangle+f_i'\langle J_i,J_j'\rangle\\&=\frac{d}{dt}(f_i'\langle J_i,J_j\rangle)\end{aligned}$$

and

$$\int_0^T \langle \mathcal{A}(f_iJ_i), f_jJ_j\rangle dt = \int_0^T f_j\frac{d}{dt}(f_i'\langle J_i, J_j\rangle)dt$$
$$= \langle f_i'J_i, f_jJ_j\rangle|_0^T - \int_0^T \langle f_i'J_i, f_j'J_j\rangle dt.$$

Using this, and that $Z(0) = 0$ we can write the index form as

$$\begin{aligned}
\mathrm{Ind}_\gamma(Z) &= \int_0^T \left\|\sum_1^n f_i'J_i\right\|^2 dt - \left\langle \sum_1^n f_i'J_i, Z\right\rangle\Bigg|_0^T \\
&\quad - \sum\left\langle \sum_1^n (f_i'^- - f_i'^+)J_i, Z\right\rangle\Bigg|_{t_k} \\
&\quad - \int_0^T \langle Y(\gamma'), Z\rangle^2 dt + \langle Z, Z'\rangle\Big|_0^T + \sum\langle Z, Z'^- - Z'^+\rangle\Big|_{t_k} \\
&= \int_0^T \left\|\sum_1^n f_i'J_i\right\|^2 - \langle Y(\gamma'), Z\rangle^2 dt + \left\langle Z(T), \sum_1^n f_i(T)J_i'(T)\right\rangle \\
&\quad + \sum\left\langle Z, \sum_1^n f_i(J_i'^- - J_i'^+)\right\rangle\Bigg|_{t_k},
\end{aligned}$$

where the last term is 0 because J_i is differentiable.

Let $W = \sum_2^n f_i'J_i$, and remember that $J_1 = \gamma'$, then we get

$$\left\|\sum_1^n f_i'J_i\right\|^2 = \langle f_1'\gamma' + W, f_1'\gamma' + W\rangle = f_1'^2 + 2f_1'\langle\gamma', W\rangle + \langle W, W\rangle.$$

Since Z is orthogonal to γ', $\langle Z, \gamma'\rangle = 0$ and by differentiating $\langle Z', \gamma'\rangle = -\langle Z, Y(\gamma')\rangle$. Using also that for magnetic Jacobi fields $\langle J', \gamma'\rangle = 0$ we have

$$-\langle Z, Y(\gamma')\rangle = \left\langle \sum_1^n f_i'J_i + f_iJ_i', \gamma'\right\rangle = f_1' + \langle W, \gamma'\rangle$$

and

$$\langle Z, Y(\gamma')\rangle^2 = f_1'^2 + 2f_1'\langle W, \gamma'\rangle + \langle W, \gamma'\rangle^2,$$

so

(3.5)
$$\begin{aligned}
\operatorname{Ind}_\gamma(Z) &= \int_0^T \left\| \sum_1^n f_i' J_i \right\|^2 - \langle Y(\gamma'), Z\rangle^2 dt + \left\langle Z(T), \sum_1^n f_i(T) J_i'(T) \right\rangle \\
&= \int_0^T \langle W, W\rangle - \langle W, \gamma'\rangle^2 dt + \left\langle Z(T), \sum_1^n f_i(T) J_i'(T) \right\rangle \\
&= \int_0^T \|W^\perp\| dt + \left\langle Z(T), \sum_1^n f_i(T) J_i'(T) \right\rangle,
\end{aligned}$$

where $W^\perp$ is the component of W orthogonal to γ'.

Let J be the magnetic Jacobi field $J = \sum_2^n f_i(T) J_i$. The fact that it is a magnetic Jacobi field is easy to see since $\mathcal{A}$ is linear over $\mathbb{R}$, it is also clear that $J(0) = 0$ and $J^\perp(T) = Z(T)$. Since

$$J^\perp = \sum_2^n f_i(T) J_i - \sum_2^n f_i(T) \langle J_i, \gamma'\rangle \gamma',$$

the corresponding functions $f_i(T)$ are constant for $i \geq 2$, so $W = 0$ and we can see from Equation (3.5) that

$$\operatorname{Ind}_\gamma(J^\perp) = \left\langle J(T), \sum_1^n f_i(T) J_i'(T) \right\rangle$$

that gives

$$\operatorname{Ind}_\gamma(Z) - \operatorname{Ind}_\gamma(J^\perp) = \int_0^T ||W^\perp|| dt \geq 0$$

with equality iff $W^\perp$ vanishes everywhere. That is

$$W^\perp = \sum_2^n f_i' J_i - \langle W, \gamma'\rangle \gamma' = 0$$

and therefore f_i constant for $i \geq 2$. So $Z = f_1\gamma' + \sum_2^n f_i J_i = f_1 \gamma' + J$, and since Z is orthogonal to γ' this implies that $Z = J^\perp$. $\square$

In what follows, we want to use the above lemma for more general vector fields that do not vanish at 0 but vanish at T. Since magnetic flows are not reversible, we cannot simply reverse time. We will consider instead the associated magnetic flow $(M, g, -\Omega)$. This magnetic flow has the same magnetic geodesics, but with opposite orientation.

Lemma 3.2. *Let (M, g, Ω) be a magnetic field. Then $(M, g, -\Omega)$ is also a magnetic field.*

(1) *Magnetic geodesics in both magnetic fields agree, but with opposite orientation.*

(2) *Jacobi fields agree in both magnetic fields.*

(3) *Index form is independent of its orientation.*

If $\gamma : [0, T] \to M$ is a magnetic geodesic in (M, g, Ω), denote by γ_- the geodesic with opposite orientation, that is $\gamma_-(t) = \gamma(-t)$, for $t \in [-T, 0]$. Then $\gamma_-''(t) = \gamma''(-t) = Y(\gamma'(-t)) = -Y(\gamma_-'(t))$, so γ_- is a magnetic geodesic in $(M, g, -\Omega)$. Part 2 follows from 1 and the fact that magnetic Jacobi fields are variational fields of variations through magnetic geodesics. Alternatively, we can check that for $J_-(t) = J(-t)$:

$$\begin{aligned}\mathcal{A}_-(J_-(t)) &= J_-''(t) + R(\gamma_-'(t), J_-(t))\gamma_-'(t) + Y(J_-'(t)) + (\nabla_{J_-}Y)(\gamma_-'(t))\\ &= J''(-t) + R(\gamma'(-t), J(-t))\gamma'(-t) - Y(J'(-t))\\ &\quad - (\nabla_J Y)(\gamma'(-t))\\ &= \mathcal{A}(J(-t))\end{aligned}$$

and

$$\langle J_-'(t), \gamma_-'(t)\rangle = \langle -J'(-t), -\gamma'(-t)\rangle = \langle J'(-t), \gamma'(-t)\rangle.$$

From the above computation follows also that $\mathcal{C}_-(Z_-(t)) = \mathcal{C}(Z(t))$. So

$$\begin{aligned}\mathrm{Ind}_{\gamma_-}(Z_-) &= \int_{-T}^{0} \{|Z_-'(t)|^2 - \langle \mathcal{C}_-(Z_-(t)), Z_-(t)\rangle - \langle Y_-(\gamma_-'(t)), Z_-(t)\rangle^2\}dt\\ &= \int_{-T}^{0} \{|Z'(-t)|^2 - \langle \mathcal{C}(Z(-t)), Z(-t)\rangle - \langle Y(\gamma'(-t)), Z(-t)\rangle^2\}dt\\ &= \int_{0}^{T} \{|Z'(t)|^2 - \langle \mathcal{C}(Z(t)), Z(t)\rangle - \langle Y(\gamma'(t)), Z(t)\rangle^2\}dt\\ &= \mathrm{Ind}_\gamma(Z).\end{aligned}$$

Corollary 3.2. *Lemma 3.1 holds when we replace Equation* (3.3) *with $Z(0) = J^\perp(0)$ and $Z(T) = J^\perp(T) = 0$.*

When $Z(T) = J^\perp(T) = 0$, we can consider Z_- and $J_-^\perp$, this will satisfy the hypothesis of Lemma 3.1, so we have

$$\mathrm{Ind}_\gamma(J^\perp) = \mathrm{Ind}_{\gamma_-}(J_-^\perp) \le \mathrm{Ind}_{\gamma_-}(Z_-) = \mathrm{Ind}_\gamma(Z).$$

Lemma 3.3. *If $\gamma(t_0)$ is conjugate to $\gamma(0)$ along γ, for some $t_0 < T$, then there is a vector field $Z \in \Lambda_0^\perp$ with $\mathrm{Ind}_\gamma(\mathrm{Z}) < 0$.*

Let J a Jacobi field along γ with $J(0) = J(t_0) = 0$, and $\tilde{J}$ be $J^\perp$ for $t \in [0, t_0]$ and 0 for $t \in [t_0, T]$. Then $\mathrm{Ind}_\gamma(\tilde{J}) = 0$. We can use the Index Lemma and Corollary 3.2 to show that cutting the corner at t_0 by replacing $\tilde{J}|_{[t_0-\epsilon, t_0+\epsilon]}$ by a Jacobi field with the same endpoints decreases the value of the index form. So this new vector field has $\mathrm{Ind}_\gamma(Z) < 0$.

Lemma 3.4. *If $\gamma(T)$ is the first conjugate point to $\gamma(0)$ along γ and $Z \in \Lambda_0^\perp$, then $\mathrm{Ind}_\gamma(\mathrm{Z}) \geq 0$, with equality if and only if $Z = 0$ or Z is the perpendicular component of a Jacobi field.*

This is an extension of Lemma 3.1, and it is proved by a similar argument. We will follow the proof of Lemma 3.1, using the same notation. Suppose $\gamma(T)$ is a first conjugate point and has multiplicity k, then we can find a basis $\{v_0, \ldots, v_{n-1}\}$ of $T_{\gamma(0)}M$, with $v_0 = \gamma'$ such that $J_{v_i}(T)$ are parallel to $\gamma'(T)$ for $i = 1, \ldots, k$ and are not parallel to $\gamma'(T)$ for $i = k+1, \ldots, n-1$. Then $J_0(t) = \gamma'(t)$ and $J_i(t) = J_{v_i}(t)$, $i = 1, \ldots, n-1$ form a basis for $T_{\gamma(t)}M$ for all $t \in (0, T)$.

If Z is a vector field in $\Lambda_0^\perp$, we can write, for $t \in (0, T)$

$$Z(t) = \sum_{i=0}^{n-1} f_i(t) J_i(t),$$

where $f_0, \ldots, f_{n-1}$ are smooth functions. We can extend these functions to $t = 0$ as before. To extend f_i to $t = T$ we can write $J_i = (t - T)A_i + J_i(T)$, for $i = 1, \ldots, k$. Then A_i are smooth vector fields with $A_i(T) = J_i'(T)$ that is orthogonal to γ' since $\langle J', \gamma' \rangle = 0$ for all Jacobi fields. Then $\{\gamma', A_1, \ldots, A_k, J_{k+1}, \ldots, J_{n-1}\}$ are a basis for all $t \in (0, T]$, and

$$Z(t) = g_0 \gamma' + \sum_{i=1}^{k} g_i(t) A_i(t) + \sum_{i=k+1}^{n-1} g_i(t) J_i(t).$$

It follows that for $t \in (0, T)$, $g_i = f_i$ for $i > k$, $g_i(t) = (t - T) f_i(t)$ for $0 < i \leq k$, and $g_0 = f_0 + \sum_{i=1}^{k} f_i(t) \langle J_i(T), \gamma' \rangle$. Since $Z(T) = 0$, $g_i(T) = 0$ and f_i extends smoothly to $t = T$.

Following the proof of Lemma 3.1, we get from (3.5)

$$\begin{aligned}\mathrm{Ind}_\gamma(Z) &= \int_0^T \|W^\perp\| dt + \left\langle Z(T), \sum_1^n f_i(T) J_i'(T) \right\rangle \\ &= \int_0^T \|W^\perp\| dt \geq 0\end{aligned}$$

with equality iff $W^\perp$ vanishes everywhere. That is when f_i constant for $i > 0$. So $Z = f_1\gamma' + J$ for some Jacobi field J, and since Z is orthogonal to γ' this implies that $Z = J^\perp$.

Corollary 3.3. $\mathrm{Ind}_\gamma(Z)$ *restricted to* $\Lambda_0^\perp$ *is positive definite if and only if* γ *has no conjugate points.*

When γ has no conjugate points, it follows directly from Corollary 3.1 that Ind_γ is positive definite. In the case that the endpoints are conjugate to each other Ind_γ has nontrivial kernel, as can be seen from Equation (3.2). If γ has conjugate points, we saw from Lemma 3.3 that there is a vector field in $\Lambda_0^\perp$ with $\mathrm{Ind}_\gamma < 0$, therefore it is not positive definite.

We will be interested in the dependence of the index form on its parameters. For this consider a continuous (possibly constant) family of vectors $\xi(s) \in S_xM$ and the correspondent family of magnetic geodesics $\gamma_s(t) = \exp_x^\mu(t\xi(s))$. Let T_s, the length of each geodesic, be continuous on s. Let Λ_s denote the vector space of piecewise smooth vector fields Z_s along γ_s, perpendicular to γ_s' and such that $Z(0) = Z(T_s) = 0$.

Let $\{v_1, \ldots, v_n\}$ be an orthonormal basis with $v_1 = \gamma_0'(0)$, and extend it to a continuous family $\{v_1(s), \ldots, v_n(s)\}$ of orthonormal basis for each s with $v_1(s) = \xi(s)$. This can be done by defining

$$v_i(s) = \rho_s(v_i),$$

where ρ_s is a rotation of S^n with $\rho_s(\xi(0)) = \xi(s)$. We extend this for all t by requiring that

$$\nabla_{\gamma_s'} e_i = Y(e_i) \tag{3.6}$$

along each magnetic geodesic. As in (2.4) this gives an orthonormal basis for each point.

Using this basis, we can extend any vector field $Z = \sum_2^n a_i(t)e_i(t)$ in Λ_0 to a vector field over the family of geodesics by

$$Z(s,t) = \sum_2^n a_i\left(t\frac{T_0}{T_s}\right) e_i(s,t)$$

that belongs to Λ_s when restricted to each γ_s. We will denote the set of such vector fields by $\Lambda_{[0,1]}$.

Since everything depends continuously on s, so does

$$\mathrm{Ind}_{\gamma_s}(Z) = \int_0^{T_s} \{|Z'|^2 - \langle \mathcal{C}(Z), Z\rangle - \langle Y(\gamma_s'), Z\rangle^2\} dt.$$

We will be mostly interested on whether the index form is positive definite. For such a family of curves, the fact that the index form is positive definite (and therefore the nonexistence of conjugate points) depends continuously on s in the following sense. If the index form is positive definite for some s_0, and has a negative value for some s_1 there must be some $s \in (s_0, s_1)$ where it has nontrivial kernel. Moreover, the first such s will be when γ_s has conjugate endpoints and no conjugate points in the interior.

4. Simple metrics and boundary data

Consider a manifold M_1 such that $M \subset \mathrm{int}(M_1)$, extend g and Ω smoothly. We say that M is *magnetic convex* at $x \in \partial M$ if there is a neighborhood U of x in M_1 such that all unit speed magnetic geodesics in U, passing through x and tangent to ∂M at x, lie in $U \setminus \mathrm{int}(M)$. If M is not magnetic convex at x, then there is a magnetic geodesic tangent to ∂M that goes into $\mathrm{int}(M)$ at arbitrarily small time, therefore for all small enough time. It is not hard to see that this will depend neither on the choice of M_1 nor on the way we extend g and Ω to M_1.

Let II stand for the second fundamental form of ∂M and $\nu(x)$ for the inward pointing normal. Then if M is magnetic convex

$$\mathrm{II}(x,\xi) \geq \langle Y_x(\xi), \nu(x)\rangle$$

for all $(x,\xi) \in TM$ [10, Lemma A.6].

We say that ∂M is *strictly magnetic convex* if

$$\mathrm{II}(x,\xi) > \langle Y_x(\xi), \nu(x)\rangle$$

for all $(x,\xi) \in TM$.

This condition implies that the tangent magnetic geodesics do not intersect M except for x, as shown in [10, Lemma A.6].

We say that M is *simple*(w.r.t.(g,Ω)) if ∂M is strictly magnetic convex and the magnetic exponential map $\exp_x^\mu : (\exp_x^\mu)^{-1}(M) \to M$ is a diffeomorphism for every $x \in M$.

For $(x,\xi) \in SM$, let $\gamma_\xi : [l^-(x,\xi), l(x,\xi)] \to M$ be the magnetic geodesic such that $\gamma_\xi(0) = x$, $\gamma_\xi'(0) = \xi$ and $\gamma_\xi(l^-(x,\xi)), \gamma_\xi(l(x,\xi)) \in \partial M$. Where l^- and l can take the values $\pm\infty$ if the magnetic geodesic γ_ξ stays in the interior of M for all time in the corresponding direction.

Let $\partial_+ SM$ and $\partial_- SM$ denote the bundles of inward and outward unit vectors over ∂M:

$$\partial_+ SM = \{(x,\xi) \in SM : x \in \partial M, \langle \xi, \nu(x)\rangle \geq 0\},$$
$$\partial_- SM = \{(x,\xi) \in SM : x \in \partial M, \langle \xi, \nu(x)\rangle \leq 0\},$$

where ν is the inward unit normal to ∂M. Note that $\partial(SM) = \partial_+ SM \cup \partial_- SM$ and $\partial_+ SM \cap \partial_- SM = S(\partial M)$.

In the case that M is simple, is clear that the functions $l^-(x,\xi)$ and $l(x,\xi)$ are continuous and, on using the implicit function theorem, they are easily seen to be smooth near a point (x,ξ) such that the magnetic geodesic $\gamma_\xi(t)$ meets ∂M transversely at $t = l^-(x,\xi)$ and $l(x,\xi)$, respectively. By the definition of strict magnetic convexity, $\gamma_\xi(t)$ meets ∂M transversely for all $(x,\xi) \in SM \setminus S(\partial M)$. In fact, these functions are smooth in $\partial_- SM$ and $\partial_+ SM$, respectively, as was shown by Dairbekov, Paternain, Stefanov and Uhlmann in the following lemma.

Lemma 4.1 [10, Lemma 2.3]**.** *For a simple magnetic system, the function $\mathbb{L} : \partial(SM) \to \mathbb{R}$, defined by*

$$\mathbb{L}(x,\xi) := \begin{cases} l(x,\xi) & \text{if}(x,\xi) \in \partial_+ SM, \\ l^-(x,\xi) & \text{if}(x,\xi) \in \partial_- SM \end{cases}$$

is smooth. In particular, $l : \partial_+ SM \to \mathbb{R}$ is smooth. The ratio

$$\frac{\mathbb{L}(x,\xi)}{\langle \nu(x), \xi\rangle}$$

is uniformly bounded on $\partial(SM) \setminus S(\partial M)$.

This lemma was proved as stated, for simple magnetic systems, but the proof is a local argument using only the strong magnetic convexity of the region.

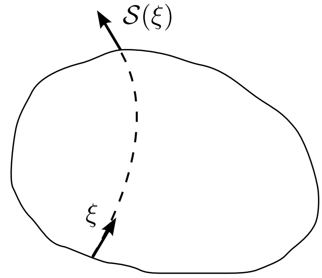

Figure 2: The scattering relation.

The *scattering relation* $\mathcal{S} : \partial_+SM \to \partial_-SM$ of a magnetic system (M, g, Ω) is defined as follows:

$$\mathcal{S}(x, \xi) = (\gamma_\xi(l(x, \xi)), \gamma'_\xi(l(x, \xi)))$$

when the value $l(x, \xi)$ is finite, otherwise it is not defined (see figure 2).

The *restricted scattering relation* $\mathtt{s} : \partial_+SM \to \partial M$ is defined to be the postcomposition of the scattering relation with the natural projection of ∂_-SM to ∂M, i.e.,

$$\mathtt{s}(x, \xi) = \gamma_\xi(l(x, \xi))$$

when properly defined.

We are interested only in simple domains, and domains that have the same scattering data as a simple domain, so we will assume that l^- and l are finite and smooth on $\partial(SM)$. Moreover, it follows from the smoothness of l and their definitions that both $\mathcal{S}$ and $\mathtt{s}$ are smooth everywhere on ∂_+SM.

Let $\widehat{M}$ be a compact simple domain with respect to $\widehat{\Omega}$ in the interior of a manifold $(\widehat{M_1}, \hat{g})$. Let M be a compact domain in the interior of a manifold (M_1, g) with Ω. Related in such a way that $\hat{g} = g$ and $\widehat{\Omega} = \Omega$ on $\widehat{M_1} \setminus \widehat{M} = M_1 \setminus M$, and the (restricted) scattering relations $\widehat{\mathcal{S}}, \mathcal{S}$ agree on $\partial M (= \partial\widehat{M})$. To be able to compare the magnetic flows in M and $\widehat{M}$ we would like to say that M is also simple, without having to impose it as a condition. The purpose of this section is to prove the following theorem.

Theorem 4.1. *Given M and $\widehat{M}$ as above, then M is also simple.*

The fact that this is true in the geodesic case, when $\Omega = 0$, follows from the equivalence between the scattering and boundary distance data for simple domains, and being able to read simplicity from the boundary distance function [18]. To prove that the magnetic exponential is a diffeomorphism we need to show that it has no conjugate points. For this we need the following lemma.

Lemma 4.2. *If there are conjugate points in M, then there is a pair of points in ∂M conjugate to each other.*

Suppose there is a point in the interior conjugate to $x \in \partial M$ along a geodesic γ_ξ. Let $\tau : [0,1] \to S_xM$ be a curve joining $S_x\partial M$ to ξ, and consider the family of magnetic geodesics $\gamma_s = \exp^\mu_x(t\tau(s))$. These geodesics exit M at time $l(x, \gamma'_s)$, that by the simplicity of $\widehat{M}$ is a continuous function of s. Close enough to x the magnetic exponential is a diffeomorphism, and by Lemma 4.1 there is a $C > 0$ such that $l(x, \eta) \leq C\langle \nu(x), \eta\rangle$ for all $\eta \in S_xM$. This implies that for s small enough, the magnetic geodesic from x to $\mathtt{s}(\tau(s))$ is short, and stays inside a neighborhood where the magnetic exponential is a diffeomorphism. Therefore, it has no conjugate points, and the index form is positive definite close to x. On the other hand, there is a perpendicular vector field along γ_ξ for which the index form is negative. Then Ind_{γ_s} is positive definite for $s = 0$ and not for $s = 1$. Let s_0 be the smallest s for which Ind_{γ_s} has nontrivial kernel. Then, by the results on the previous section, $\mathtt{s}(\tau(s_0))$ is conjugate to x along the magnetic geodesic γ_s that joins them.

If there are points conjugate to each other along a magnetic geodesic γ_ξ, and both lie in the interior of M, there must be a point conjugate to $\gamma_\xi(0)$ along this magnetic geodesic. Therefore reducing the problem to the case above. This can be proved by a similar argument using the family of geodesics $\gamma_\xi|_{[0,sT]}$.

Proof of Theorem 4.1. It is easy to see from the definition that the domain M has to be strictly magnetic convex, since the metrics and magnetic flows agree outside M.

To prove that the magnetic exponential map is a diffeomorphism form $(\exp^\mu_x)^{-1}(M)$ to M we need to show that it has no conjugate points, i.e., there are no points in M that are conjugate to each other along a magnetic geodesic. For this purpose assume such points exist, then by Lemma 4.2 there are points $x, y \in \partial M$ conjugate to each other along a magnetic geodesic γ_ξ, where $\gamma_\xi(0) = x$ and $\gamma_\xi(t_0) = y$ for some $t_0 > 0$.

Let J be a magnetic Jacobi field along γ_ξ that vanishes at 0, and $f(s,t)$ a variation through magnetic geodesics with $f(0,t) = \gamma_\xi$ and J as a variational field. We can use $f(s,t) = \gamma_s(t) = \exp^\mu_x(t\xi(s))$ where $\xi : (-\epsilon, \epsilon) \to S_xM$ is a curve with $\xi(0) = \xi$, $\xi'(0) = J'(0)$. f is well defined in M for $(s,t) \in (-\epsilon, \epsilon) \times [0, T_s]$ where $T_s = l(x, \xi(s))$. Consider $c(s) = f(s, T_s) \in \partial M$ the curve of the exit points in ∂M. Then

$$\begin{aligned}\frac{dc}{ds}(0) &= \frac{df}{ds}(0,T_s) + \frac{df}{dt}\frac{dT_s}{ds}(0,T_s)\\ &= J(T_s) + \frac{dT_s}{ds}(0,T_s)\gamma'_\xi.\end{aligned}$$

If $\gamma_\xi(l(x,\xi))$ is conjugate to $\gamma_\xi(0)$ along γ_ξ, there is a Jacobi field J that is 0 at $t=0$ and parallel to γ_ξ at T_s, then $dc/ds(0)$ is parallel to γ'_ξ. On the other hand, if $dc/ds(0)$ is parallel to γ'_ξ for any Jacobi field with $J(0)=0$ then $J(T_s)$ is parallel to γ'_ξ. Therefore $\gamma_\xi(T_s)$ is conjugate to $\gamma_\xi(0)$ along γ_ξ.

Note that we can write $c(s) = \mathtt{s}(\xi(s))$, that depends only on the scattering data, so the scattering relation detects conjugate points in the boundary. Since there are no conjugate points in the boundary of $\widehat{M}$, there can be none in M. Therefore the magnetic exponential is a local diffeomorphism.

We will now see that $\exp^\mu_x$ is a global diffeomorphism from $(\exp^\mu_x)^{-1}(M)$ to M. To see that it is surjective let $x \in \partial M$, and y any point in M. Let $c : [0,1] \to M$ be a path from x to y, and consider the set $A \subset [0,1]$ of points such that $c(s)$ is in the image of $\exp^\mu_x$. This set is open, since $\exp^\mu_x$ is a local diffeomorphism. To see that it is closed, choose a sequence $s_n \in A$ converging to s_0. Then $c(s_n) = \exp^\mu_x(t(s_n)\xi(s_n))$, and there is a subsequence such that $t(s_n)$ and $\xi(s_n)$ converge to t_0 and ξ_0 respectively. If $t_0\xi_0 \notin (\exp^\mu_x)^{-1}(M)$, there must be a first $t_1 < t_0$ such that $\exp^\mu_x(t_1\xi_0) \in \partial M$. Then $\exp^\mu_x(t\xi_0)$ must be tangent to ∂M and inside M for $t < t_1$, which contradicts the magnetic convexity of M. Then, A is both open and closed, therefore $A = [0,1]$ and y is in the image of $\exp^\mu_x$.

To see that $\exp^\mu_x$ is injective for $x \in \partial M$, note that it is a covering map. The point x has only one preimage, since by the simplicity of $\widehat{M}$ there are no magnetic geodesics form x to x. Therefore $\exp^\mu_x$ is a covering map of degree 1.

To prove this for $x \notin \partial M$, we need to see that there are no trapped magnetic geodesics, that is, that there are no magnetic geodesics that stay inside M for an infinite time. Note that, since any magnetic geodesic that enters the region at ξ has to exit at $\mathtt{s}(\xi)$, it is enough to see that all geodesics enter the region at a finite time. Let γ be a magnetic geodesic. We know that we can reach the point $\gamma(0)$ from the boundary, so there is a variation through magnetic geodesics $\gamma_s(t)$ with $\gamma_0 = \gamma$, $\gamma_s(0) = \gamma(0)$ for all $s \in [0,1]$, and $\gamma_1(t_1) \in \partial M$ for some $t_1 < 0$. If γ_{s_0} intersects ∂M, by the magnetic convexity of M it has to be a transverse intersection, therefore intersecting ∂M is an open condition on $[0,1]$. It is also a closed condition, by continuity of the geodesic flow and compactness of ∂M. Therefore, since γ_1 intersects ∂M, so does γ_s for all s, and γ is not trapped.

Now we see that $\exp_x^\mu$ is a global diffeomorphism from $(\exp_x^\mu)^{-1}(M)$ to M for $x \notin \partial M$. Since $\exp_x^\mu$ is injective for $x \in \partial M$, and all geodesics come from some point x in ∂M, magnetic geodesics in M have no self-intersections. In particular, any $x \in M$ has only one preimage under $\exp_x^\mu$. We can then follow the same argument as for $x \in \partial M$ to show that $\exp_x^\mu$ is a global diffeomorphism from $(\exp_x^\mu)^{-1}(M)$ to M, for all $x \in M$. □

5. Rigidity for surfaces

Consider a magnetic field on a surface $\widehat{M}$ all of whose orbits are closed, and consider a magnetically simple region R on it. We want to prove that there is no way of changing the metric and magnetic field in this region in such a way that all orbits are still closed.

In the previous section we saw that such a region is magnetically rigid, therefore it cannot be changed on the region preserving the scattering data. Here we will look at the general behavior of such a magnetic flow to ensure that there are no other metrics with all its orbits closed. We want to rule out the case where a magnetic geodesic that passes through the region, after coming out at a different spot and following the corresponding orbit, goes back into the region and exits at the exit point and direction of the original first magnetic geodesic, therefore forming a closed orbit out of two (or more) segments of the original orbits, like in figure 1.

To show this, assume that we have such a magnetic field. Assume, moreover, that the region R is such that every magnetic geodesic passes through R at most once. This condition restricts both the size of R and the flow, since in a flow where the orbits have many self-intersections such a region might not exist. On the other hand, if all orbits are simple we can see by compactness that there are small regions with this property.

Proof of Theorem 1.2. Consider the unit tangent bundle SM, and the magnetic geodesic vector field G i.e., the vector field that generates the magnetic flow on the unit tangent bundle. For the sake of simplicity of the exposition we will assume first that SM is oriented. Then SM is a compact orientable three-dimensional manifold, and G is a smooth vector field that foliates SM by circles. By a theorem of Epstein [12], this foliation is C^∞ diffeomorphic to a Seifert fibration. In particular, any orbit has a neighborhood diffeomorphic to a standard fibered torus.

Note that to each orbit on SM we can uniquely associate a magnetic geodesic, by projecting the orbit back to M. We will use this correspondence freely. As a Seifert fibration, the base B or space of orbits of SM is a two-dimensional orbifold.

Let SR be the subset of SM that corresponds to the region R, and $S\partial R$ the subset of SM corresponding to vectors tangent to the boundary of R. The orbit of a point in $S\partial R$ corresponds to a magnetic geodesic that is tangent to R, and since R is strictly magnetically convex, it is tangent only at one point. This magnetic geodesic corresponds exactly to one in $\widehat{M}$, and therefore stays away from R thereafter. This means that each orbit contains at most one point of $S\partial R$, so the set of orbits passing through it forms a one-dimensional submanifold on B, we will denote it by R_0.

Let $m : B \to \mathbb{N}$ be a function that counts the number of times the orbit pases through SR in a common period. For regular orbits this is the number of times it passes through SR. If the orbit is singular it has a neighborhood diffeomorphic to an (a, b) torus, that is a torus obtained by gluing two faces of a cylinder with a rotation by an angle of $2\pi b/a$. In this case the common period is a times the period of the singular orbit. Therefore, m will be a times the number of times the orbit passes through SR. Since the other orbits in the neighborhood will be completed when the singular orbit is traveled a times, m will be, in general, continuous at such points. In fact, if a magnetic geodesic intersects ∂R transversally (or not at all), we can chose a neighborhood small enough that all intersections are transverse, and therefore m will be constant. The only discontinuities occur when a magnetic geodesic is tangent to ∂R, that is exactly at the orbits in R_0.

We will now look at these discontinuities. If a magnetic geodesic γ corresponds to an orbit b in R_0, it is tangent to R at a point $\gamma(0)$. It agrees with a magnetic geodesic in $\widehat{M}$, so it never reaches R again and $m(b) = 1$. By the magnetic convexity of M, there is a δ small enough that each magnetic geodesic in the ball $B_\delta(\gamma(0))$ goes through R at most once. Moreover, since the magnetic geodesic is compact, we can find an ε neighborhood $N_\varepsilon(\gamma)$ such that it only intersects R close to $\gamma(0)$, i.e., $N_\varepsilon(\gamma) \cap R = B_\delta(\gamma(0)) \cap R$.

If the orbit b corresponding to γ is regular, orbits in a small enough neighborhood will correspond to nearby magnetic geodesics, completely contained in $N_\varepsilon(\gamma)$. These magnetic geodesics will then intersect R only inside $B_\delta(\gamma(0))$, and therefore at most once. Thus, these orbits will have m equal to 0 or 1.

We have that B is a two-dimensional orbifold, and R_0 is a continuous curve on it. The function m is constant on each connected component of $B \setminus R_0$, and takes values 0 or 1. On R_0, the function $m = 1$, except maybe at isolated singular orbits. Since on regular orbits $m = 1$, we can say that magnetic geodesics go through R at most once, except maybe at a finite number of singular ones. Any singular magnetic geodesics that is tangent must go through R only once. If a singular magnetic geodesics cuts ∂R transversely, we know that $m = 1$. But the corresponding orbit pases through SR exactly m/a times, so $a = m = 1$ and the geodesic is not singular.

In the case where SM is nonorientable, consider instead its orientable double cover $\widetilde{SM}$, and the associated vector field $\widetilde{G}$. Then $\widetilde{SM}$ is a compact orientable three-dimensional manifold, and $\widetilde{G}$ is a smooth vector field that foliates $\widetilde{SM}$ by circles. We can follow the same arguments with a few modifications.

The correspondence between orbit on $\widetilde{SM}$ and magnetic geodesics is not a one–one correspondence, a magnetic geodesic lifts either to an orbit that covers it twice, or two disjoint orbits. Nonetheless, will use this correspondence freely, keeping in mind this possible duplicity.

Let $\widetilde{SR}$ be the subset of $\widetilde{SM}$ that corresponds to the region R, and $\widetilde{S\partial R}$ the subset of $\widetilde{SM}$ corresponding to vectors tangent to the boundary of R. Let $\widetilde{B}$ be space of orbits of $\widetilde{SM}$ and $\widetilde{R}_0$ the set of orbits passing through $\widetilde{S\partial R}$. The counting function $m : \widetilde{B} \to \mathbb{N}$ can then take value 2, since an orbit that covers a magnetic geodesic twice will pass through $\widetilde{SR}$ twice. In fact, when $m(b) \neq 0$, it will be 1 if the magnetic geodesic corresponds to two disjoint orbits, and 2 when it corresponds to an orbit that covers it twice.

Since on regular orbits $m = 2$ only on orbits that cover a magnetic geodesic twice, we can say that magnetic geodesics go through R at most once, except maybe for a finite number of singular ones. Any singular one that is tangent must go through R only once, by assumption. If a singular orbit cuts ∂R transversely, we know that m is at most 2. But the orbit passes through $\widetilde{SR}$ m/a times, so if it is singular $a = m = 2$ and the geodesic goes through R only once.

Every magnetic geodesic goes through R at most once, and outside R they agree with the magnetic geodesics from $\widehat{M}$. For the magnetic geodesics to close, they have to exit R in the same place and direction, therefore preserving the scattering data. □

If the region R is simple, we can use this result together with Theorem 4.1 to get rigidity. For surfaces of constant curvature is easy to see that any circular disk that is strictly smaller than one of the orbit circles is a simple domain. Corollary 1.1 can be stated in a more precise way as the following theorem.

Theorem 5.1. *Let M be a surface of constant curvature K, and $k > 0$ big enough that all circles of curvature k are simple. Let r_k be the radius of a circle of curvature k, and $0 < r < r_k$ such that $r + r_k$ is smaller than the injectivity radius of M. Let R be a compact region contained in the interior of a disk of radius r. Then the region R cannot be perturbed while keeping all the circles of curvature k closed.*

Consider M with the constant magnetic field that has circles of curvature k as magnetic geodesics. If R is contained in a disk D of radius r we can consider any perturbation $\widetilde{R}$ of R as a perturbation $\widetilde{D}$ of D. Since $r < r_k$ the disk D is simple. Also, since $r + r_k$ is smaller than the injectivity radius of M, any circle of curvature k will go through D at most once. We can then use Theorem 1.2 to show that D and $\widetilde{D}$ have the same scattering data. Since D is simple, and they have the same scattering data, by Theorem 4.1 $\widetilde{D}$ is also simple. But in [10, Theorem 7.1] Dairbekov, Paternain, Stefanov and Uhlmann proved that 2 two-dimensional simple magnetic systems with the same scattering data are gauge equivalent.

If M is not compact, consider instead of M a compact quotient that contains all the magnetic geodesics that pass through D. This can be achieved since all this magnetic geodesics are inside a disk of radius $4r$, where r is the radius of a circle of curvature k.

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