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$W^{2,2}$-conformal immersions of a closed Riemann surface into $\mathbb{R}^n$

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We study sequences $f_k : \Sigma_k \to \mathbb{R}^n$ of conformally immersed, compact Riemann surfaces with fixed genus and Willmore energy $\mathcal{W}(f) \leq \Lambda$. Assume that Σ_k converges to Σ in moduli space, i.e., $\phi_k^*(\Sigma_k) \to \Sigma$ as complex structures for diffeomorphisms ϕ_k. Then we construct a branched conformal immersion $f : \Sigma \to \mathbb{R}^n$ and Möbius transformations σ_k, such that for a subsequence $\sigma_k \circ f_k \circ \phi_k \to f$ weakly in $W^{2,2}_{\mathrm{loc}}$ away from finitely many points. For $\Lambda < 8\pi$ the map f is unbranched. If the Σ_k diverge in moduli space, then we show $\liminf_{k\to\infty} \mathcal{W}(f_k) \geq \min(8\pi, \omega_p^n)$. Our work generalizes results in [12] to arbitrary codimension.

1. Introduction

Let Σ be a closed oriented surface of genus $p \in \mathbb{N}_0$. For an immersion $f : \Sigma \to \mathbb{R}^n$ the Willmore functional is defined by

$$\mathcal{W}(f) = \frac{1}{4} \int_\Sigma |H|^2 \, d\mu_g,$$

where H is the mean curvature vector and g is the induced metric on Σ. The infimum among closed immersed surfaces of genus p is denoted by β_p^n. We have $\beta_0^n = 4\pi$, which is attained only by round spheres [21]. For $p \geq 1$ we have the inequalities $4\pi < \beta_p^n < 8\pi$ [8, 18]. In this paper we study compactness properties of sequences $f_k : \Sigma \to \mathbb{R}^n$ with $\mathcal{W}(f_k) \leq \Lambda$. By the Gauß equations and Gauß–Bonnet, the second fundamental form is then equivalently bounded by

$$\int_\Sigma |A_{f_k}|^2 \, d\mu_{g_k} \leq 4\Lambda + 8\pi(p-1).$$

In [14] Langer proved a compactness theorem for surfaces with $\|A\|_{L^q} \leq \Lambda$ for $q > 2$, using that the surfaces are represented as C^1-bounded graphs over discs of radius $r(n, q, \Lambda) > 0$. Clearly, the relevant Sobolev embedding fails for $q = 2$. For surfaces with $\|A\|_{L^2}$ small in a ball, L. Simon proved an approximate graphical decomposition, see [18], and showed the existence of Willmore minimizers for any $p \geq 1$, assuming for $p \geq 2$ that

$$\beta_p^n < \min \left\{ 4\pi + \sum_i (\beta_{p_i}^n - 4\pi) : \sum_i p_i = p,\, 1 \leq p_i < p \right\} = \omega_p^n.$$

This inequality was confirmed later in [2]. As $\lim_{p\to\infty} \beta_p^n = 8\pi$ by [10], we have $\omega_p^n > 8\pi$ for large p. Recently, using the annulus version of the approximate graphical decomposition lemma, a compactness theorem was proved in [12] for surfaces in $\mathbb{R}^3$ under the assumptions

$$\liminf_{k\to\infty} \mathcal{W}(f_k) < \begin{cases} 8\pi, & \text{if } p = 1, \\ \min(8\pi, \omega_p^3), & \text{if } p \geq 2. \end{cases}$$

Moreover, it was shown that these conditions are optimal. For $n = 4$ the result was proved under the additional assumption $\liminf_{k\to\infty} \mathcal{W}(f_k) < \beta_p^4 + \frac{8\pi}{3}$. In [13] these compactness theorems were applied to prove the existence of a Willmore minimizer with prescribed conformal type.

Here we develop a new approach to compactness, generalizing the results of [12] to any codimension. As main tools we use a convergence theorem of Hurwitz type for conformal immersions, which is due to Hélein [6], and the estimates for the conformal factor by Müller and Šverák [15]. The paper is organized as follows. In Section 2 we introduce the notion of $W^{2,2}$ conformal immersions, and recall the main estimate from [15] as well as the monotonicity formula from [18]. In Section 3 we adapt the analysis of [15] to show that isolated singularities of conformal immersions with square integrable second fundamental form and finite area are branchpoints, in a suitable weak sense. The compactness theorem for conformal immersions is presented in Section 4. We first deal with the case of a fixed Riemann surface in Proposition 4.1, and extend the result to sequences of Riemann surfaces converging in moduli space in Theorem 4.1. Finally in Section 5, we study surfaces whose conformal type degenerates and show that the lower bound from [12] extends to higher codimension. Along the lines, we state a version of Theorem 5.1.1 in [6] with optimal constants.

2. $W^{2,2}$ conformal immersions

Definition 2.1. Let Σ be a Riemann surface. A map $f \in W^{2,2}_{\text{loc}}(\Sigma, \mathbb{R}^n)$ is called a conformal immersion, if in any local conformal coordinates

$\varphi : U \to \Sigma$ with $U \subset \mathbb{C}$ the metric $g_{ij} = \langle \partial_i f, \partial_j f \rangle$ is given by

$$g_{ij} = \mathrm{e}^{2u} \delta_{ij}, \qquad \text{where } u \in L^\infty_{\mathrm{loc}}(U).$$

The set of all conformal immersions $f \in W^{2,2}_{\mathrm{loc}}(\Sigma, \mathbb{R}^n)$ is denoted by $W^{2,2}_{\mathrm{conf,loc}}(\Sigma, \mathbb{R}^n)$, and by $W^{2,2}_{\mathrm{conf}}(\Sigma, \mathbb{R}^n)$ if Σ is compact.

It is easy to see that for $f \in W^{2,2}_{\mathrm{conf}}(\Sigma, \mathbb{R}^n)$ one has in local conformal coordinates on U

$$u = \frac{1}{2} \log \left(\frac{1}{2} |Df|^2 \right) \in W^{1,2}_{\mathrm{loc}}(U).$$

The induced measure μ_g, the second fundamental form A and the mean curvature vector H are given by the standard coordinate formulae. We define K_{g} by the Gauß equation

$$K_{\mathrm{g}} = \frac{1}{2}(|H|^2 - |A|^2_{\mathrm{g}}) = \mathrm{e}^{-4u}(\langle A_{11}, A_{22} \rangle - |A_{12}|^2).$$

In a local parameter, we will now verify the weak Liouville equation

$$\int_U \langle Du, D\varphi \rangle = \int_U K_{\mathrm{g}} \mathrm{e}^{2u} \varphi, \qquad \text{for all } \varphi \in C^\infty_0(U).$$

In particular, this shows that K_{g} is intrinsic. We start by computing

$$\langle \partial^2_{ij} f, \partial_k f \rangle + \langle \partial^2_{ki} f, \partial_j f \rangle = 2\, \mathrm{e}^{2u} \partial_i u \, \delta_{jk},$$

which implies after permutation of the indices that

$$\langle \partial^2_{ij} f, \partial_k f \rangle = \mathrm{e}^{2u} (\partial_i u \, \delta_{jk} + \partial_j u \, \delta_{ik} - \partial_k u \, \delta_{ij}).$$

Expanding explicitly yields

$$\begin{aligned}
\partial^2_{11} f &= A_{11} + \partial_1 u \, \partial_1 f - \partial_2 u \, \partial_2 f, \\
\partial^2_{22} f &= A_{22} - \partial_1 u \, \partial_1 f + \partial_2 u \, \partial_2 f, \\
\partial^2_{12} f &= A_{12} + \partial_2 u \, \partial_1 f + \partial_1 u \, \partial_2 f,
\end{aligned}$$

and we obtain

$$\langle A_{11}, A_{22} \rangle - |A_{12}|^2 = \langle \partial^2_{11} f, \partial^2_{22} f \rangle - |\partial^2_{12} f|^2 + 2\mathrm{e}^{2u} |Du|^2.$$

For any $u \in W^{1,2} \cap L^\infty(U)$, $f \in W^{2,2}_{\mathrm{loc}}(U, \mathbb{R}^n)$ and $\varphi \in C^\infty_0(U)$, we have the formula

$$\begin{aligned}\int_U & \Big(\langle \partial^2_{11} f, \partial^2_{22} f\rangle - |\partial^2_{12} f|^2\Big)\, \mathrm{e}^{-2u}\varphi \\ &= \int_U \Big(\langle \partial_1 f, \partial^2_{12} f\rangle \partial_2\big(\mathrm{e}^{-2u}\varphi\big) - \langle \partial_1 f, \partial^2_{22} f\rangle \partial_1\big(\mathrm{e}^{-2u}\varphi\big)\Big).\end{aligned}$$

This follows by approximation from the case when f is smooth. Now for f conformal

$$\langle \partial_1 f, \partial^2_{12} f\rangle = \mathrm{e}^{2u}\partial_2 u \quad \text{ and } \quad \langle \partial_1 f, \partial^2_{22} f\rangle = -\mathrm{e}^{2u}\partial_1 u,$$

which yields

$$\int_U \Big(\langle \partial^2_{11} f, \partial^2_{22} f\rangle - |\partial^2_{12} f|^2\Big)\, \mathrm{e}^{-2u}\varphi = \int_U \langle Du, D\varphi\rangle - 2\int_U |Du|^2\varphi,$$

and the Liouville equation follows.

Remark 2.1. More generally if $g = \mathrm{e}^{2u} g_0$ where g_0 is any smooth conformal metric, then

$$-\Delta_{g_0} u = K_{\mathrm{g}} \mathrm{e}^{2u} - K_{g_0} \quad \text{ weakly.}$$

Testing with a constant function, we infer for closed Σ the Gauß–Bonnet formula

$$\int_\Sigma K_{\mathrm{g}}\, d\mu_g = 2\pi\chi(\Sigma).$$

$W^{2,2}$ conformal immersions f can be approximated by smooth immersions in the $W^{2,2}$ norm. In fact, a standard mollification f_ε will be immersed for small $\varepsilon > 0$, by an argument of [17].

2.1. Gauß map and compensated compactness

By assumption the right-hand side $K_{\mathrm{g}}\,\mathrm{e}^{2u}$ of the Liouville equation belongs to L^1. In [15] Müller and Šverák discovered that the term can be written as a sum of Jacobi determinants, and that improved estimates can be obtained from the Wente lemma [20] or from [4]. The following result is Corollary 3.5.7

of [15]. Recall that in their notation, ω denotes twice the standard Kähler form and $W_0^{1,2}(\mathbb{C})$ is the space of functions $v \in L^2_{\mathrm{loc}}(\mathbb{C})$ with $Dv \in L^2(\mathbb{C})$.

Theorem 2.1. *Let $\varphi \in W_0^{1,2}(\mathbb{C}, \mathbb{C}P^n)$ satisfy*

$$\int_{\mathbb{C}} \varphi^* \omega = 0 \quad \text{and} \quad \int_{\mathbb{C}} J\varphi \leq \gamma < 2\pi.$$

Then there is a unique function $v \in W_0^{1,2}(\mathbb{C})$ solving the equation $-\Delta v = *\varphi^*\omega$ in $\mathbb{C}$ with boundary condition $\lim_{z\to\infty} v(z) = 0$. Moreover

$$\|v\|_{L^\infty(\mathbb{C})} + \|Dv\|_{L^2(\mathbb{C})} \leq C(\gamma) \int_{\mathbb{C}} |D\varphi|^2.$$

For $f \in W^{2,2}_{\mathrm{conf}}(D, \mathbb{R}^n)$ let $G \in W^{1,2}(D, \mathbb{C}P^{n-1})$ be the associated Gauß map. Here we embed the Grassmannian $G(2,n)$ of oriented 2-planes into $\mathbb{C}P^{n-1}$ by sending an orthonormal basis $e_{1,2}$ to $[(e_1 + \mathrm{i} e_2)/\sqrt{2}]$. Then

$$K_{\mathrm{g}}\, \mathrm{e}^{2u} = *G^*\omega \quad \text{and} \quad \int_D |DG|^2 = \frac{1}{2}\int_D |A|^2\, d\mu_g.$$

Corollary 2.1. *For $f \in W^{2,2}_{\mathrm{conf}}(D, \mathbb{R}^n)$ with induced metric $g_{ij} = \mathrm{e}^{2u}\delta_{ij}$, assume*

$$\int_D |A|^2\, d\mu_g \leq \gamma < \gamma_n = \begin{cases} 8\pi, & \text{if } n = 3, \\ 4\pi, & \text{if } n \geq 4. \end{cases}$$

Then there exists a function $v : \mathbb{C} \to \mathbb{R}$ solving the equation

$$-\Delta v = K_{\mathrm{g}}\, \mathrm{e}^{2u} \quad \text{in } D,$$

and satisfying the estimates

$$\|v\|_{L^\infty(\mathbb{C})} + \|Dv\|_{L^2(\mathbb{C})} \leq C(\gamma) \int_D |A|^2\, d\mu_g.$$

Proof. We follow [15]. Define the map $\varphi : \mathbb{C} \to \mathbb{C}P^{n-1}$ by

$$\varphi(z) = \begin{cases} G(z), & \text{if } z \in D, \\ G(\frac{1}{\bar{z}}), & \text{if } z \in \mathbb{C}\backslash\overline{D}. \end{cases}$$

Then $\varphi \in W_0^{1,2}(\mathbb{C}, \mathbb{C}P^{n-1})$ and $\int_{\mathbb{C}} \varphi^*\omega = 0$. For $n \geq 4$, we have

$$\int_{\mathbb{C}} J\varphi = 2\int_D JG \leq \frac{1}{2}\int_D |A|^2\, d\mu_g \leq \frac{\gamma}{2} < 2\pi.$$

Thus the result follows from Theorem 2.1. The same is true for $n = 3$, since then

$$\int_{\mathbb{C}} J\varphi = \frac{1}{2}\int_D |K_{\mathrm{g}}|\, d\mu_g \leq \frac{1}{4}\int_D |A|^2\, d\mu_g \leq \frac{\gamma}{4} < 2\pi.$$

□

The function $K_{\mathrm{g}}\,\mathrm{e}^{2u}$ belongs actually to the Hardy space $\mathcal{H}^1$; see [4]. This implies that v has second derivatives in L^1, and in particular that v is continuous [15]. As $u-v$ is harmonic, it follows that u is also continuous, but this will not be used here. The following is an immediate consequence of Corollary 2.1.

Corollary 2.2. *Let $f \in W^{2,2}_{\mathrm{conf}}(D, \mathbb{R}^n)$ with induced metric $g_{ij} = \mathrm{e}^{2u}\delta_{ij}$. If*

$$\int_D |A|^2\, d\mu_g \leq \gamma < \gamma_n,$$

then we have the estimate

$$\|u\|_{L^\infty(D_{\frac{1}{2}})} + \|Du\|_{L^2(D_{\frac{1}{2}})} \leq C(\gamma)\left(\int_D |A|^2\, d\mu_g + \|u\|_{L^1(D)}\right).$$

2.2. Simon's monotonicity formula

We briefly review the monotonicity identity from [18] for proper $W^{2,2}$ conformal immersions $f : \Sigma \to \mathbb{R}^n$. For more details we refer to [11]. Since f is locally Lipschitz, the measure $\mu = f(\mu_g)$ is an integral varifold with multiplicity function $\theta^2(\mu, x) = \# f^{-1}\{x\}$ and approximate tangent space $T_x\mu = Df(p) \cdot T_p\Sigma$ a.e. when $x = f(p)$. The immersion f satisfies

$$\int_\Sigma \mathrm{div}_g X\, d\mu_g = -\int_\Sigma \langle X, H\rangle\, d\mu_g, \qquad \text{for any } X \in W_0^{1,1}(\Sigma, \mathbb{R}^n).$$

For the varifold μ this implies the first variation formula

$$\int \mathrm{div}_\mu \phi\, d\mu = -\int \langle \phi, H_\mu\rangle\, d\mu, \qquad \text{for } \phi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n),$$

where the weak mean curvature is given by

$$H_\mu(x) = \begin{cases} \dfrac{1}{\theta^2(\mu,x)} \displaystyle\sum_{p \in f^{-1}\{x\}} H(p), & \text{if } \theta^2(\mu,x) > 0, \\ 0, & \text{else.} \end{cases}$$

From the definition we have trivially the inequality

$$\mathcal{W}(\mu, V) := \frac{1}{4} \int_V |H_\mu|^2 \, d\mu \leq \frac{1}{4} \int_{f^{-1}(V)} |H|^2 \, d\mu_g.$$

Observing that $H_\mu(x)$ is μ a.e. perpendicular to $T_x\mu$, the proof of the monotonicity identity in [18] extends to show that for $B_\sigma(x_0) \subset B_\varrho(x_0)$ one has

$$g_{x_0}(\varrho) - g_{x_0}(\sigma) = \frac{1}{16\pi} \int_{B_\varrho(x_0) \setminus B_\sigma(x_0)} \left| H_\mu + 4 \frac{(x - x_0)^\perp}{|x - x_0|^2} \right|^2 d\mu,$$

where

$$g_{x_0}(r) = \frac{\mu(B_r(x_0))}{\pi r^2} + \frac{1}{4\pi} \mathcal{W}(\mu, B_r(x_0)) + \frac{1}{2\pi r^2} \int_{B_r(x_0)} \langle x - x_0, H_\mu \rangle \, d\mu.$$

Applications include the existence and upper semicontinuity of $\theta^2(\mu,x)$ and, for closed surfaces, the Li–Yau inequality; see [9],

$$\theta^2(\mu, x) \leq \frac{1}{4\pi} \mathcal{W}(f).$$

Another consequence is the diameter bound from [18]. If Σ is compact and connected, then for $f \in W^{2,2}_{\mathrm{conf}}(\Sigma, \mathbb{R}^n)$ one obtains

$$\left(\frac{\mu_g(\Sigma)}{\mathcal{W}(f)} \right)^{\frac{1}{2}} \leq \operatorname{diam} f(\Sigma) \leq C(\mu_g(\Sigma)\, \mathcal{W}(f))^{\frac{1}{2}}. \tag{2.1}$$

3. Classification of isolated singularities

In [15] Müller and Šverák studied the behavior at infinity of complete, conformally parameterized surfaces with square integrable second fundamental form. Here we adapt their analysis to the case of finite isolated singularities.

Theorem 3.1. *Suppose that $f \in W^{2,2}_{\mathrm{conf,loc}}(D\backslash\{0\}, \mathbb{R}^n)$ satisfies*

$$\int_{D\backslash\{0\}} |A|^2 \, d\mu_g < \infty \quad \textit{and} \quad \mu_g(D\backslash\{0\}) < \infty,$$

where $g_{ij} = \mathrm{e}^{2u}\delta_{ij}$ is the induced metric. Then $f \in W^{2,2}_{\mathrm{loc}}(D, \mathbb{R}^n)$ and we have

$$\begin{aligned} u(z) &= m \log|z| + \omega(z), \quad \textit{where } m \in \mathbb{N}_0,\ \omega \in C^0 \cap W^{1,2}(D), \\ -\Delta u &= -2m\pi\delta_0 + K_{\mathrm{g}}\,\mathrm{e}^{2u}, \quad \textit{in } D. \end{aligned}$$

The multiplicity of the immersion at $f(0)$ is given by

$$\theta^2(f(\mu_g \llcorner D_\sigma(0)), f(0)) = m + 1 \quad \textit{for any small } \sigma > 0.$$

Moreover, if $m = 0$ then f is a conformal immersion on D.

Proof. We may assume $\int_D |A|^2 \, d\mu_g < 4\pi$; hence the Gauß map $G : D \to G(n,2)$ has energy

$$\int_D |DG|^2 = \frac{1}{2}\int_D |A|^2 \, d\mu_g < 2\pi.$$

Extending by $G(z) := G(1/\bar{z})$ for $|z| > 1$ yields $G \in W^{1,2}_0(\mathbb{R}^2, G(n,2))$, where

$$\int_{\mathbb{R}^2} G^*\omega = 0 \quad \text{and} \quad \int_{\mathbb{R}^2} JG \le \int_D |DG|^2 < 2\pi.$$

Thus there is a function $v \in C^0 \cap W^{1,2}_0(\mathbb{R}^2)$ such that

$$\begin{gathered} -\Delta v = K_{\mathrm{g}}\,\mathrm{e}^{2u} \quad \text{and} \quad \lim_{z\to\infty} v(z) = 0, \\ \|v\|_{C^0(\mathbb{R}^2)} + \|Dv\|_{L^2(\mathbb{R}^2)} \le C \int_D |A|^2 \, d\mu_g. \end{gathered}$$

Now consider the harmonic function $h : D\backslash\{0\} \to \mathbb{R}$, $h(z) = u(z) - v(z) - \alpha \log|z|$, where

$$\alpha = \frac{1}{2\pi}\int_{\partial D_r(0)} \frac{\partial(u-v)}{\partial r}\, ds \in \mathbb{R}, \quad \text{for } r \in (0,1).$$

We claim that h has a removable singularity at the origin. Let $h = \operatorname{Re}\phi$ where $\phi : D\backslash\{0\} \to \mathbb{C}$ is holomorphic, and compute for $m = \min\{k \in \mathbb{Z} : k \geq \alpha\}$

$$|z^m\,\mathrm{e}^{\phi(z)}| = |z|^m\,\mathrm{e}^{h(z)} \leq \mathrm{e}^{u(z)-v(z)} \leq C\,\mathrm{e}^{u(z)} \in L^2(D).$$

Thus $z^m\,\mathrm{e}^{\phi(z)} = z^k g(z)$ for $k \in \mathbb{N}_0$ and $g : D \to \mathbb{C}\backslash\{0\}$ holomorphic, which yields $h(z) = (k-m)\log|z| + \log|g(z)|$. But the choice of α in the definition of h implies $k = m$, thereby proving our claim. Moreover from $|z|^\alpha = \mathrm{e}^{u(z)-v(z)-h(z)} \in L^2(D)$, we conclude that

$$u(z) = \alpha\log|z| + \omega(z), \qquad \text{where } \alpha > -1,\, \omega \in C^0 \cap W^{1,2}(D).$$

Next we perform a blow-up to show that $\alpha = m$. For any $z_0 \in \mathbb{C}\backslash\{0\}$ and $0 < \lambda < 1/|z_0|$ we let

$$f_\lambda : D_{\frac{1}{\lambda}}(0) \to \mathbb{R}^n,\ f_\lambda(z) = \frac{1}{\lambda^{\alpha+1}}(f(\lambda z) - f(\lambda z_0)).$$

The f_λ have induced metric $(g_\lambda)_{ij} = \mathrm{e}^{2u_\lambda}\delta_{ij}$, where

$$u_\lambda(z) = u(\lambda z) - \alpha\log\lambda = \alpha\log|z| + \omega(\lambda z).$$

Putting $\omega_0 = \omega(0)$, we have

$$u_\lambda(z) \to \alpha\log|z| + \omega_0 \quad \text{ in } C^0_{\mathrm{loc}} \cap W^{1,2}_{\mathrm{loc}}(\mathbb{C}\backslash\{0\}).$$

Furthermore, the Gauß map of f_λ is given by $G_\lambda(z) = G(\lambda z)$, in particular $DG_\lambda \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{C}\backslash\{0\})$. Using the formula

$$|D^2 f_\lambda|^2 = 2\,\mathrm{e}^{2u_\lambda}(|DG_\lambda|^2 + 2|Du_\lambda|^2),$$

we obtain by Vitali's theorem

$$|D^2 f_\lambda|(z) \to \frac{2\,\mathrm{e}^{\omega_0}\alpha}{|z|^{1-\alpha}} \quad \text{ in } L^2_{\mathrm{loc}}(\mathbb{C}\backslash\{0\}).$$

As $f_\lambda(z_0) = 0$, we can find a sequence $\lambda_k \searrow 0$ such that the f_{λ_k} converge in $C^0_{\mathrm{loc}}(\mathbb{C}\backslash\{0\})$ and weakly in $W^{2,2}_{\mathrm{loc}}(\mathbb{C}\backslash\{0\})$ to a limit map $f_0 : \mathbb{C}\backslash\{0\} \to \mathbb{R}^n$ satisfying $f_0(z_0) = 0$. After passing to a further subsequence, we can also assume that $G_{\lambda_k} \to L$ in $W^{1,2}_{\mathrm{loc}}(\mathbb{C}\backslash\{0\})$, where $L \in G(n,2)$ is a constant. It is then easy to see that f_0 maps into the plane L. Further we have

$$\langle \partial_i f_0(z), \partial_j f_0(z)\rangle = \mathrm{e}^{2\omega_0}|z|^{2\alpha}\delta_{ij}.$$

Using that f_0 is locally in $W^{2,2} \cap W^{1,\infty}$ we verify the identity

$$\langle \Delta f_0, \partial_j f_0\rangle = \partial_i \langle \partial_i f_0, \partial_j f_0\rangle - \frac{1}{2}\partial_j \langle \partial_i f_0, \partial_i f_0\rangle.$$

Since f_0 is conformal, maps into L and has rank two almost everywhere we see that f_0 is harmonic on $\mathbb{C}\backslash\{0\}$. Identifying $L \cong \mathbb{C}$ by choosing an orthonormal frame $e_{1,2}$, the conformality relations are

$$4\frac{\partial f_0}{\partial z}\left(\overline{\frac{\partial f_0}{\partial \overline{z}}}\right) = \left|\frac{\partial f_0}{\partial x}\right|^2 - \left|\frac{\partial f_0}{\partial y}\right|^2 - 2i\left\langle \frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial y}\right\rangle = 0.$$

Since the two factors on the left are holomorphic, the identity principle implies that f_0 is holomorphic on $\mathbb{C}\backslash\{0\}$, after replacing e_1, e_2 by $e_1, -e_2$ if necessary. Now $|f_0'(z)| = \mathrm{e}^{\omega_0}|z|^\alpha$ and thus for some $\beta \in [0, 2\pi)$

$$f_0'(z) = \mathrm{e}^{\omega_0 + \mathrm{i}\beta} z^\alpha \qquad \text{on } \mathbb{C}\backslash[0,\infty).$$

As f_0' is single-valued, we must have $\alpha = m \in \mathbb{N}_0$ and

$$f_0(z) = \frac{\mathrm{e}^{\omega_0+\mathrm{i}\beta}}{m+1}\big(z^{m+1} - z_0^{m+1}\big).$$

In particular, we have the desired expansion $u(z) = m\log|z| + v(z) + h(z)$, and u satisfies the stated differential equation. Furthermore,

$$|D^2 f|^2 = 2\,\mathrm{e}^{2u}\big(|DG|^2 + 2|Du|^2\big) \in L^1(D),$$

thus $f \in W^{2,2}(D,\mathbb{R}^n)$. Assuming without loss of generality $f(0) = 0$, we claim that

$$\lim_{z\to 0}\frac{|f(z)|}{|z|^{m+1}} = \frac{\mathrm{e}^{\omega_0}}{m+1}.$$

Since $|Df(z)| = |z|^m\,\mathrm{e}^{\omega(z)}$ with ω bounded, we have $|f(z)| \leq C|z|^{m+1}$. Now let $z_k \to 0$ be a given sequence. We can assume that $\zeta_k := \frac{z_k}{|z_k|} \to \zeta$ with

$|\zeta| = 1$, and compute

$$\begin{aligned}\left|\frac{|f(z_k)|}{|z_k|^{m+1}} - \frac{e^{\omega_0}}{m+1}\right| &= \Bigg|\left|f_{\lambda_k}(\zeta_k) + \frac{1}{\lambda_k^{m+1}} f(\lambda_k z_0)\right| \\ &\qquad - \left|\frac{e^{\omega_0 + i\beta}}{m+1}(\zeta^{m+1} - z_0^{m+1}) + \frac{e^{\omega_0+i\beta}}{m+1} z_0^{m+1}\right|\Bigg| \\ &\le \left|f_{\lambda_k}(\zeta_k) - \frac{e^{\omega_0+i\beta}}{m+1}(\zeta^{m+1} - z_0^{m+1})\right| + C|z_0|^{m+1}.\end{aligned}$$

Letting $k \to \infty$ we obtain, for a constant $C < \infty$ depending only on m and ω,

$$\liminf_{k\to\infty} \left|\frac{|f(z_k)|}{|z_k|^{m+1}} - \frac{e^{\omega_0}}{m+1}\right| \le C|z_0|^{m+1}.$$

This proves our claim since $z_0 \in \mathbb{C}\backslash\{0\}$ was arbitrary. Now

$$\lim_{\varrho\searrow 0} \frac{\mu_g\big(D_\varrho(0)\big)}{\pi r(\varrho)^2} = m+1, \qquad \text{where } r(\varrho) = \frac{e^{\omega_0}}{m+1}\varrho^{m+1}.$$

Choose $\sigma \in (0,1)$ such that $f(z) \neq 0$ for $z \in \overline{D_\sigma(0)}\backslash\{0\}$, and let $\varrho_{1,2} > 0$ be such that

$$\frac{1}{\gamma} r(\varrho_1) = r = \gamma r(\varrho_2), \qquad \text{where } \gamma \in (0,1).$$

Then for $r > 0$ sufficiently small we have the inclusions

$$D_{\varrho_1}(0) \subset \big(f^{-1}(B_r(0)) \cap D_\sigma(0)\big) \subset D_{\varrho_2}(0).$$

It follows that

$$\gamma^2 \frac{\mu_g(D_{\varrho_1}(0))}{\pi r(\varrho_1)^2} \le \frac{f(\mu_g \llcorner D_\sigma(0))(B_r(0))}{\pi r^2} \le \frac{1}{\gamma^2}\frac{\mu_g(D_{\varrho_2}(0))}{\pi r(\varrho_2)^2}.$$

Letting $r \searrow 0$, $\gamma \nearrow 1$ proves that $\theta^2(f(\mu_g \llcorner D_\sigma(0)), 0) = m+1$. $\square$

A map $f : \Sigma \to \mathbb{R}^n$ is called a branched conformal immersion (with locally square integrable second fundamental form), if $f \in W^{2,2}_{\mathrm{conf,loc}}(\Sigma\backslash\mathcal{S}, \mathbb{R}^n)$ for some discrete set $\mathcal{S} \subset \Sigma$ and

$$\int_\Omega |A|^2 \, d\mu_g < \infty \quad \text{and} \quad \mu_g(\Omega) < \infty, \qquad \text{for all } \Omega \subset\subset \Sigma.$$

The number $m(p)$ as in Theorem 3.1 is the branching order, and $m(p)+1$ is the multiplicity at $p \in \Sigma$. The map f is unbranched at p if and only if $m(p)=0$. For a closed Riemann surface Σ and a branched conformal immersion $f : \Sigma \to \mathbb{R}^n$, consider now

$$\hat{f} = I_{x_0} \circ f : \Sigma \backslash f^{-1}\{x_0\} \to \mathbb{R}^n, \qquad \text{where } I_{x_0}(x) = x_0 + \frac{x - x_0}{|x - x_0|^2}.$$

Then $\hat{g} = \mathrm{e}^{2v} g$ where $v = -\log|f - x_0|^2$. The weak Liouville equation says that

$$\int_\Sigma \varphi K_{\hat{g}}\, d\mu_{\hat{g}} - \int_\Sigma \varphi K_{\mathrm{g}}\, d\mu_g = -\int_\Sigma \langle D \log|f - x_0|^2, D\varphi\rangle_g\, d\mu_g,$$
$$\text{for all } \varphi \in C_0^\infty(\Sigma \backslash f^{-1}\{x_0\}).$$

A simple computation shows $|\hat{A}^\circ|^2 d\mu_{\hat{g}} = |A^\circ|^2\, d\mu_g$; hence by the Gauß equation

$$\frac{1}{4}\int_\Sigma \varphi|\hat{H}|^2\, d\mu_{\hat{g}} - \frac{1}{4}\int_\Sigma \varphi|H|^2\, d\mu_g = -\int_\Sigma \langle D\log|f - x_0|^2, D\varphi\rangle_g\, d\mu_g.$$

At a point $p \in f^{-1}\{x_0\}$ of order $m \in \mathbb{N}_0$, choose conformal coordinates on the unit disc D and introduce the rescaled maps

$$f_\lambda : D \to \mathbb{R}^n,\ f_\lambda(z) = \frac{1}{\lambda^{m+1}}(f(\lambda z) - x_0).$$

From the proof of Theorem 3.1, we have that $f_{\lambda_k} \to f_0$ weakly in $W^{2,2}$ away from the origin for a subsequence $\lambda_k \searrow 0$, where f_0 is given by

$$f_0(z) = \frac{\mathrm{e}^{\omega_0}}{m+1} L z^{m+1}, \qquad \text{for some } L \in \mathbb{O}(2,n),\ \omega_0 \in \mathbb{R}.$$

Fix a smooth function $\varphi : \mathbb{R}^n \to [0,1]$, such that $|\varphi(z)| = 1$ for $|z| \geq 1$ and $\varphi = 0$ in a neighborhood of the origin. Using as cut-off function $\varphi_\lambda(z) = \varphi(\frac{z}{\lambda})$, we obtain

$$\begin{aligned}\int_D \langle D\log|f - x_0|^2, D\varphi_\lambda\rangle_g\, d\mu_g &= \int_D \frac{2\langle f(z) - x_0, \partial_i f(z)\rangle}{|f(z) - x_0|^2}\frac{1}{\lambda}\partial_i\varphi\left(\frac{z}{\lambda}\right)\, dz \\ &= \int_D \frac{2\langle f_\lambda(\zeta), \partial_i f_\lambda(\zeta)\rangle}{|f_\lambda(\zeta)|^2}\partial_i\varphi(\zeta)\, d\zeta.\end{aligned}$$

Now for $\lambda_k \searrow 0$ the right-hand side converges to

$$\int_D \frac{2\langle f_0, \partial_i f_0\rangle}{|f_0|^2}\partial_i\varphi = 2(m+1)\int_D \frac{\partial_r\varphi}{r} = 4\pi(m+1).$$

Adding the contribution of the finitely many preimages we conclude

$$\mathcal{W}(\hat{f}) = \mathcal{W}(f) - 4\pi \sum_{p\in f^{-1}\{x_0\}} (m(p)+1). \tag{3.1}$$

4. Weak compactness of conformal immersions

Proposition 4.1. *Let Σ be a closed Riemann surface and $f_k \in W^{2,2}_{\mathrm{conf}}$ $(\Sigma, \mathbb{R}^n)$ be a sequence of conformal immersions satisfying*

$$\mathcal{W}(f_k) \leq \Lambda < \infty.$$

Then for a subsequence there exist Möbius transformations σ_k and a finite set $\mathcal{S} \subset \Sigma$, such that

$$\sigma_k \circ f_k \to f, \quad \text{weakly in } W^{2,2}_{\mathrm{loc}}(\Sigma\backslash\mathcal{S}, \mathbb{R}^n),$$

where $f : \Sigma \to \mathbb{R}^n$ is a branched conformal immersion with square integrable second fundamental form. Moreover, if $\Lambda < 8\pi$ then f is unbranched and topologically embedded.

We will use the following standard estimate.

Lemma 4.1. *Let Σ be a two-dimensional, closed manifold with smooth Riemannian metric g_0, and suppose that $u \in W^{1,2}(\Sigma)$ is a weak solution of the equation*

$$-\Delta_{g_0} u = F, \quad \text{where } F \in L^1(\Sigma).$$

Then for any Riemannian ball $B_r(p)$ and $q \in [1,2)$ we have

$$\|Du\|_{L^q(B_r^{g_0}(p))} \leq C\, r^{\frac{2}{q}-1}\|F\|_{L^1(\Sigma)}, \quad \text{where } C = C(\Sigma, g_0, q) < \infty.$$

Proof. We may assume that $\|F\|_{L^1(\Sigma)} = 1$ and $\int_\Sigma u\, d\mu_{g_0} = 0$. The function u is given by

$$u(x) = \int_\Sigma G(x,y)F(y)\, d\mu_{g_0}(y),$$

where $G(x,y)$ is the Riemannian Green function; see Theorem 4.13 in [1]. In particular, $G(x,y) = G(y,x)$, and we have the estimate

$$|D_x G(x,y)| \leq \frac{C}{d(x,y)}, \quad \text{where } C = C(\Sigma, g) < \infty.$$

By Jensen's inequality we get

$$\begin{aligned}
\int_{B_r(p)} |Du|^q \, d\mu_{g_0} &\leq \int_{B_r(p)} \Big(\int_\Sigma |D_x G(x,y)||F(y)|\, d\mu_{g_0}(y) \Big)^q \, d\mu_{g_0}(x) \\
&\leq \int_\Sigma |F(y)| \int_{B_r(p)} |D_x G(x,y)|^q \, d\mu_{g_0}(x)\, d\mu_{g_0}(y) \\
&\leq C \int_\Sigma |F(y)| \int_{B_r(p)} \frac{1}{d(x,y)^q}\, d\mu_{g_0}(x)\, d\mu_{g_0}(y).
\end{aligned}$$

Now if $d(p,y) < 2r$ we can estimate

$$\int_{B_r(p)} \frac{1}{d(x,y)^q}\, d\mu_{g_0}(x) \leq \int_{B_{3r}(y)} \frac{1}{d(x,y)^q}\, d\mu_{g_0}(x) \leq Cr^{2-q}.$$

In the other case $d(p,y) \geq 2r$, we have $d(x,y) \geq r$ on $B_r(p)$, which implies

$$\int_{B_r(p)} \frac{1}{d(x,y)^q}\, d\mu_{g_0}(x) \leq \frac{C}{r^q}\, \mu_g(B_r(p)) \leq Cr^{2-q}.$$

The statement of the lemma follows. $\square$

Proof of Proposition 4.1: We may assume $\mu_{g_k} \llcorner |A_k|^2 \to \alpha$ as Radon measures, and put

$$\mathcal{S} = \{p \in \Sigma : \alpha(\{p\}) \geq \gamma_n\}.$$

Choose a smooth, conformal background metric g_0 and write $g_k = \mathrm{e}^{2u_k} g_0$. Then

$$\int_\Sigma |K_{g_k}\, \mathrm{e}^{2u_k}|\, d\mu_{g_0} = \int_\Sigma |K_{g_k}|\, d\mu_{g_k} \leq \frac{1}{2} \int_\Sigma |A_k|^2 \, d\mu_{g_k} \leq C(\Lambda).$$

From the equation $-\Delta_{g_0} u_k = K_{g_k}\, e^{2u_k} - K_{g_0}$, we thus obtain using Lemma 4.1 for arbitrary $q \in (1,2)$ the bound

$$\int_\Sigma |Du_k|^q \, d\mu_{g_0} \leq C = C(\Lambda, \Sigma, g_0, q).$$

By dilating the f_k appropriately we can arrange that

$$\int_\Sigma u_k \, d\mu_{g_0} = 0,$$

and then get by the Poincaré inequality; see Theorem 2.34 in [1],

$$\|u_k\|_{W^{1,q}(\Sigma)} \leq C.$$

In particular, we can assume that $u_k \to u$ weakly in $W^{1,q}(\Sigma)$. For any $p \notin \mathcal{S}$, we choose conformal coordinates on a neighborhood $U_\delta(p) \cong D_\delta(0)$, where $U_\delta(p) \subset\subset \Sigma \backslash \mathcal{S}$. Putting $(g_k)_{ij} = e^{2v_k} \delta_{ij}$ we have $(g_0)_{ij} = e^{2(v_k - u_k)} \delta_{ij}$ and hence, for a constant depending on $U_\delta(p)$,

$$\|v_k\|_{W^{1,q}(U_\delta(p))} \leq C.$$

Passing to a smaller $\delta > 0$ if necessary, we obtain from Corollary 2.2 the estimate

$$\|v_k\|_{L^\infty(U_\delta(p))} + \|Dv_k\|_{L^2(U_\delta(p))} \leq C.$$

Hence we can assume that v_k converges to v on $U_\delta(p)$ weakly in $W^{1,2}$ and pointwise almost everywhere. But now $|Df_k| = e^{v_k}$ and $\Delta f_k = e^{2v_k} H_k$, where by assumption

$$\int_{U_\delta(p)} |H_k|^2 e^{2v_k} \, dx\, dy \leq \Lambda.$$

Translating the f_k such that $f_k(p) = 0$ for some fixed $p \in \Sigma \backslash \mathcal{S}$, we finally obtain

$$\|f_k\|_{W^{2,2}(\Omega)} \leq C, \qquad \text{for any } \Omega \subset\subset \Sigma \backslash \mathcal{S}.$$

In particular, the f_k converge weakly in $W^{2,2}_{\mathrm{loc}}(\Sigma \backslash \mathcal{S})$ to some $f \in W^{2,2}_{\mathrm{loc}}(\Sigma \backslash \mathcal{S})$, where f has induced metric $g = e^{2u} g_0$ and $u \in L^\infty_{\mathrm{loc}}(\Sigma \backslash \mathcal{S})$. If $\limsup_{k\to\infty} \mu_{g_k}(\Sigma) < \infty$, then $\mu_g(\Sigma) < \infty$ by Fatou's lemma, and the main statement of Proposition 4.1 follows from Theorem 3.1.

To prove the statement also in the case $\mu_{g_k}(\Sigma) \to \infty$, suppose that there is a ball $B_1(x_0)$ with $f_k(\Sigma) \cap B_1(x_0) = \emptyset$ for all k. Then $\hat{f}_k = I_{x_0} \circ f_k$ converges to $\hat{f} = I_{x_0} \circ f$ weakly in $W^{2,2}_{\mathrm{loc}}(\Sigma \backslash \mathcal{S})$, and $\hat{f}$ has induced metric $\hat{g} = \mathrm{e}^{2\hat{u}} g_0$ where $\hat{u} = u - \log |f - x_0|^2 \in L^\infty_{\mathrm{loc}}(\Sigma \backslash \mathcal{S})$. Moreover, Lemma 1.1 in [18] yields that

$$\mu_{\hat{g}_k}(\Sigma) \leq \Lambda \big(\operatorname{diam} \hat{f}_k(\Sigma)\big)^2 \leq 2\Lambda.$$

Thus $\mu_{\hat{g}}(\Sigma) < \infty$ and the result follows as above. To find the ball $B_1(x_0)$ we employ an argument from [12]. For $\mu_k = f_k(\mu_{g_k})$ we have by equation (1.3) in [18]

$$\mu_k(B_R(0)) \leq CR^2, \qquad \text{for all } R > 0.$$

Thus $\mu_k \to \mu$ and $f_k(\mu_{g_k} \llcorner |H_k|^2) \to \nu$ as Radon measures after passing to a subsequence. Equation 1.4 in [18] implies in the limit

$$\frac{\mu(B_\varrho(x))}{\varrho^2} + \nu(B_\varrho(x)) \geq c > 0, \qquad \text{for all } x \in \operatorname{spt} \mu,\ \varrho > 0.$$

As shown in [18], p. 310, the set of accumulation points of the sets $f_k(\Sigma)$ is just $\operatorname{spt} \mu$. For $R > 0$ to be chosen, let $B_2(x_j)$, $j = 1, \ldots, N$, be a maximal collection of 2-balls with centers $x_j \in B_R(0)$, hence $N \geq R^n / 4^n$. Now if $\operatorname{spt} \mu \cap B_1(x_j) \neq \emptyset$ for all j, then summation of the inequality over the balls yields

$$cN \leq \sum_{j=1}^N \Big(\mu\big(B_2(x_j)\big) + \nu\big(B_2(x_j)\big) \Big) \leq C(\Lambda, n)(R^2 + 1).$$

Therefore $\operatorname{spt} \mu \cap B_1(x_j) = \emptyset$ for some j, if $R = R(\Lambda, n)$ is sufficiently large. The additional conclusions in the case $\Lambda < 8\pi$ are clear from formula (3.1) and Theorem 3.1. □

The following existence result is proved independently in a recent preprint by Rivière [16]. It extends previous work of Kuwert and Schätzle [13]. In their paper, it is shown that the minimizers are actually smooth.

Corollary 4.1. *Let Σ be a closed Riemann surface such that*

$$\beta^n_\Sigma = \inf\{\mathcal{W}(f) : f \in W^{2,2}_{\mathrm{conf}}(\Sigma, \mathbb{R}^n)\} < 8\pi.$$

Then the infimum β^n_Σ is attained.

We now generalize Proposition 4.1 to the case of varying Riemann surfaces. The following standard lemma will be useful, see [5] for a proof.

Lemma 4.2. *Let g_k, g be smooth Riemannian metrics on a surface M, such that $g_k \to g$ in $C^{s,\alpha}(M)$, where $s \in \mathbb{N}$, $\alpha \in (0,1)$. Then for each $p \in M$ there exist neighborhoods U_k, U and smooth conformal diffeomorphisms $\varphi_k : D \to (U_k, g_k)$, $\varphi : D \to (U, g)$, such that $\varphi_k \to \varphi$ in $C^{s+1,\alpha}(\overline{D}, M)$.*

Theorem 4.1. *Let $f_k \in W^{2,2}(\Sigma_k, \mathbb{R}^n)$ be conformal immersions of compact Riemann surfaces of genus p. Assume that the Σ_k converge to Σ in moduli space, i.e., $\phi_k^*(\Sigma_k) \to \Sigma$ as complex structures for suitable diffeomorphisms ϕ_k, and that*

$$\mathcal{W}(f_k) \leq \Lambda < \infty.$$

Then there exist a branched conformal immersion $f : \Sigma \to \mathbb{R}^n$ with square integrable second fundamental form, a finite set $\mathcal{S} \subset M$ and Möbius transformations σ_k, such that for a subsequence

$$\sigma_k \circ f_k \circ \phi_k \to f, \quad \text{weakly in } W^{2,2}(\Sigma \backslash \mathcal{S}, \mathbb{R}^n).$$

The convergence of the complex structures implies that $\phi_k^* g_{0,k} \to g_0$, where $g_{0,k}$, g are the suitably normalized, constant curvature metrics in Σ_k, Σ; see chapter 2.4 in [19]. The proof is now along the lines of Proposition 4.1, using the local conformal charts from Lemma 4.2.

5. The energy of surfaces diverging in moduli space

5.1. Hélein's convergence theorem

The following result, with constant $8\pi/3$ instead of γ_n, is due to Hélein; see Theorem 5.1.1 in [6]. To obtain the constant γ_n, one combines with the estimate of [15]. For convenience of the reader, we include the proof. At the end of the subsection we will show that γ_n is in fact optimal.

Theorem 5.1. *Let $f_k \in W^{2,2}_{\mathrm{conf}}(D, \mathbb{R}^n)$ be a sequence of conformal immersions with induced metrics $(g_k)_{ij} = \mathrm{e}^{2u_k}\delta_{ij}$, and assume*

$$\int_D |A_{f_k}|^2 \, d\mu_{g_k} \leq \gamma < \gamma_n = \begin{cases} 8\pi, & \text{for } n = 3, \\ 4\pi, & \text{for } n \geq 4. \end{cases}$$

Assume also that $\mu_{g_k}(D) \leq C$ and $f_k(0) = 0$. Then f_k is bounded in $W^{2,2}(D_r, \mathbb{R}^n)$ for any $r \in (0,1)$, and there is a subsequence such that one of the following two alternatives holds:

(a) *u_k is bounded in $L^\infty(D_r)$ for any $r \in (0,1)$, and f_k converges weakly in $W^{2,2}_{\mathrm{loc}}(D, \mathbb{R}^n)$ to a conformal immersion $f \in W^{2,2}_{\mathrm{conf,loc}}(D, \mathbb{R}^n)$.*

(b) *$u_k \to -\infty$ and $f_k \to 0$ locally uniformly on D.*

Proof. By Corollary 2.1 there is a solution v_k of the equation $-\Delta v_k = K_{g_k} \mathrm{e}^{2u_k}$ satisfying

$$\|v_k\|_{L^\infty(D)} + \|Dv_k\|_{L^2(D)} \leq C(\gamma) \int_D |A_{f_k}|^2 \, d\mu_{g_k}.$$

Clearly $h_k = u_k - v_k$ is harmonic on D. Now

$$\int_D \mathrm{e}^{2u_k^+} = |\{u_k \leq 0\}| + \int_{\{u_k > 0\}} \mathrm{e}^{2u_k} \leq C,$$

and hence

$$\int_D u_k^+ \leq C.$$

For dist $(z, \partial D) \geq r$ where $r \in (0,1)$ we get

$$h_k(z) = \frac{1}{\pi r^2} \int_{D_r(z)} (u_k - v_k) \leq \frac{1}{\pi r^2} \int_D u_k^+ + \|v_k\|_{L^\infty(D)} \leq C(\gamma, r).$$

Thus $u_k = v_k + h_k$ is locally bounded from above, which implies that the sequence f_k is bounded in $W^{1,\infty}_{\mathrm{loc}}(D, \mathbb{R}^n)$. As $\Delta f_k = \mathrm{e}^{2u_k} H_{f_k}$, we further have for $\Omega = D_{1-r}(0)$

$$\int_\Omega |\Delta f_k|^2 = \int_\Omega \mathrm{e}^{2u_k} |H_{f_k}|^2 \, d\mu_{g_k} \leq C(\gamma, r) \int_\Omega |A_{f_k}|^2 \, d\mu_{g_k} \leq C(\gamma, r).$$

Thus f_k is also bounded in $W^{2,2}_{\mathrm{loc}}(D, \mathbb{R}^n)$ and converges, after passing to a subsequence, $W^{2,2}$-weakly to some $f \in W^{2,2}_{\mathrm{loc}} \cap W^{1,\infty}_{\mathrm{loc}}(D, \mathbb{R}^n)$. Now if $\int_D u_k^- \leq C$, then for dist $(z, \partial D) \geq r$

$$h_k(z) = \frac{1}{\pi r^2} \int_{D_r(z)} (u_k - v_k) \geq -\frac{1}{\pi r^2} \int_D u_k^- - \|v_k\|_{L^\infty(D)} \geq -C(\gamma, r).$$

Thus $u_k = v_k + h_k$ is bounded in $L^\infty_{\text{loc}} \cap W^{1,2}_{\text{loc}}(D)$, and u_k converges pointwise to a function $u \in L^\infty_{\text{loc}}(D)$. We conclude

$$g_{ij} = \langle \partial_i f, \partial_j f \rangle = \mathrm{e}^{2u} \delta_{ij},$$

which means that f is a conformal immersion as claimed in case (a). We will now show that $\int_D u_k^- \to \infty$ implies alternative (b). Namely, we then have

$$h_k(0) = \frac{1}{\pi} \int_D (u_k - v_k) \to -\infty.$$

As $C(\gamma, r) - h_k \geq 0$ on Ω, we get by the Harnack inequality

$$\sup_{\Omega'} h_k \leq \frac{1}{C(r)} \inf_{\Omega'} h_k + C(\gamma, r) \to -\infty, \qquad \text{where } \Omega' = D_{1-2r}(0).$$

Thus $u_k = v_k + h_k \to -\infty$ and $f_k \to 0$ locally uniformly on D. □

Applying Lemma 4.2, we get a version of Hélein's theorem for conformal immersions with respect to a convergent sequence of metrics.

Corollary 5.1. *The statement of Theorem 5.1 continues to hold for immersions $f_k \in W^{2,2}(D, \mathbb{R}^n)$ with induced metric $g_k = \mathrm{e}^{2u_k} g_{0,k}$, if the $(g_{0,k})_{ij}$ converge to δ_{ij} smoothly on $\overline{D}$.*

Relating to Remark 5.1.3 in [6], we now show that the constant 4π in Theorem 5.1 is optimal for $n \geq 4$. For $\varepsilon > 0$, we consider the conformally immersed minimal discs

$$f_\varepsilon : D \to \mathbb{C}^2,\ f_\varepsilon(z) = \left(\frac{1}{2} z^2, \varepsilon z \right).$$

We compute $(g_\varepsilon)_{ij} = \mathrm{e}^{2u_\varepsilon} \delta_{ij}$ where $u_\varepsilon(z) = \frac{1}{2} \log(|z|^2 + \varepsilon^2)$, and further

$$\int_D |A_{f_\varepsilon}|^2 \, d\mu_{g_\varepsilon} = -2 \int_D K_{g_\varepsilon} \, d\mu_{g_\varepsilon} = 2 \int_{\partial D} \frac{\partial u_\varepsilon}{\partial r} \, ds = \frac{4\pi}{1 + \varepsilon^2} < 4\pi.$$

As $f_\varepsilon(z) \to (\frac{1}{2} z^2, 0)$ for $\varepsilon \searrow 0$, none of the two alternatives (a) or (b) is satisfied. For the optimality of $\gamma_3 = 8\pi$ we also follow [15] and consider

Enneper's minimal surface

$$f : \mathbb{C} \to \mathbb{R}^3,\ f(z) = \frac{1}{2}\,\mathrm{Re}\,\Big(z - \frac{1}{3}z^3, i\big(z + \frac{1}{3}z^3\big), z^2\Big).$$

We have $f_\lambda(z) = \frac{1}{\lambda^3} f(\lambda z) \to -\frac{1}{6}(z^3, 0) \in \mathbb{C} \times \mathbb{R} = \mathbb{R}^3$ as $\lambda \nearrow \infty$. Restricting f_λ to D yields conformally immersed discs with $\int_D |A_{f_\lambda}|^2\, d\mu_{g_\lambda} < 8\pi$.

5.2. The case of tori

The following was proved in [12] for $n = 3$, and for $n = 4$ with bound $\min(8\pi, \beta_1^4 + \frac{8\pi}{3})$.

Theorem 5.2. *Let Σ_k be tori which diverge in moduli space. Then for any sequence of conformal immersions $f_k \in W^{2,2}_{\mathrm{conf}}(\Sigma_k, \mathbb{R}^n)$ we have*

$$\liminf_{k\to\infty} \mathcal{W}(f_k) \geq 8\pi.$$

Proof. We may assume that $\Sigma_k = \mathbb{C}/\Gamma_k$ where $\Gamma_k = \mathbb{Z} \oplus \mathbb{Z}(a_k + \mathrm{i}b_k)$ is normalized by $0 \leq a_k \leq \frac{1}{2}$, $a_k^2 + b_k^2 \geq 1$ and $b_k > 0$. We also assume that the $f_k : \Sigma_k \to \mathbb{R}^n$ satisfy

$$\frac{1}{4} \limsup_{k\to\infty} \int_{\Sigma_k} |A_{f_k}|^2\, d\mu_{g_k} = \limsup_{k\to\infty} \mathcal{W}(f_k) \leq \Lambda < \infty.$$

We lift the f_k to Γ_k-periodic maps from $\mathbb{C}$ into $\mathbb{R}^n$. Theorem 3.1 shows that f_k is not constant on any circle $C_v = [0,1] \times \{v\}$, $v \in \mathbb{R}$. Hence by passing to $\frac{1}{\lambda_k}(f_k(u, v + v_k) - f_k(0, v_k))$ for suitable $\lambda_k > 0$, $v_k \in [0, b_k)$, we may assume that

$$1 = \operatorname{diam} f_k(C_0) \leq \operatorname{diam} f_k(C_v) \quad \text{for all } v \in \mathbb{R}, \quad \text{and} \quad f_k(0,0) = 0.$$

Arguing as in the proof of Proposition 4.1, we obtain $B_1(x_0) \subset \mathbb{R}^n$ such that $f_k(\Sigma_k) \cap B_1(x_0) = \emptyset$ for all k. For $\hat{f}_k = I_{x_0} \circ f_k$ we have $\hat{f}_k(\Sigma_k) \subset \overline{B_1(x_0)}$, and Lemma 1.1 in [18] implies an area bound $\mu_{\hat{g}_k}(\Sigma_k) \leq C$. Up to a subsequence, we have $\mu_{\hat{g}_k} \llcorner |A_{\hat{f}_k}|^2 \to \alpha$ as Radon measures on the cylinder $C = [0,1] \times \mathbb{R}$. The set $\mathcal{S} = \{w \in C : \alpha(\{w\}) \geq \gamma_n\}$ is discrete, and

$$\varrho(w) = \inf\{\varrho > 0 : \alpha(D_\varrho(w)) \geq \gamma_n\} > 0, \quad \text{for } w \in \Omega = C \backslash \mathcal{S}.$$

Now $\hat{f}_k$ converges locally uniformly in Ω either to a conformal immersion, or to a point $x_1 \in \mathbb{R}^n$. This follows from Theorem 5.1 together with a continuation argument, using that $\varrho(w)$ is lower semicontinuous and hence locally bounded from below. Note

$$\hat{f}_k(C_0) \subset I_{x_0}\big(\overline{B_1(0)}\big) \subset \mathbb{R}^n \backslash B_\theta(x_0), \qquad \text{where } \theta = \frac{1}{|x_0|+1} > 0.$$

In the second alternative we thus get $|x_1 - x_0| \geq \theta > 0$, and $f_k|_{C_v}$ converges uniformly to the point $I_{x_0}(x_1)$ for any $C_v \subset \Omega$, in contradiction to $\operatorname{diam} f_k(C_v) \geq 1$. Therefore $\hat{f}_k$ converges to a conformal immersion $\hat{f} : \Omega \to \mathbb{R}^n$. Now the assumption that Σ_k diverges in moduli space yields that $b_k \to \infty$, so that $\hat{f} : \Omega \to \mathbb{R}^n$ has second fundamental form in $L^2(C)$ and finite area. Applying Theorem 3.1 to the points at $v = \pm\infty$ we see that $\hat{f}(C_v) \to x_\pm \in \mathbb{R}^n$ for $v \to \pm\infty$. Let us assume that $x_+ \neq x_0$. Then for any $\varepsilon > 0$ we find a $\delta > 0$ with $I(B_\delta(x_+)) \subset B_\varepsilon(I(x_+))$. Choosing $v < \infty$ large such that $\hat{f}(C_v) \subset B_{\frac{\delta}{2}}(x_+)$, we get for sufficiently large k

$$f_k(C_v) = I\big(\hat{f}_k(C_v)\big) \subset I\big(B_\delta(x_+)\big) \subset B_\varepsilon(I(x_+)).$$

Taking $\varepsilon = \frac{1}{3}$ yields a contradiction to $\operatorname{diam} f_k(C_v) \geq 1$. This shows $x_\pm = x_0$ and in particular $\theta^2(\hat{\mu}, x_0) \geq 2$ where $\hat{\mu} = \hat{f}(\mu_{\hat{g}})$. We conclude from the Li–Yau inequality, see Section 2.2,

$$\liminf_{k\to\infty} \mathcal{W}(f_k) = \liminf_{k\to\infty} \mathcal{W}(\hat{f}_k) \geq \mathcal{W}(\hat{f}) \geq 8\pi.$$

□

5.3. The case of genus $p \geq 2$

We first collect some facts about degenerating Riemann surfaces from [3, 7]. By definition, a compact Riemann surface with nodes is a compact, connected Hausdorff space Σ together with a finite subset N, such that $\Sigma\backslash N$ is locally homeomorphic to D, while each $a \in N$ has a neighborhood homeomorphic to $\{(z,w) \in \mathbb{C}^2 : zw = 0,\ |z|, |w| < 1\}$. Moreover, all transition functions are required to be holomorphic. The points in N are called nodes. Each component Σ^i of $\Sigma\backslash N$ is contained in a compact Riemann surface $\overline{\Sigma}^i$, which is given by adding points to the punctured coordinate discs at the nodes. We have $q \leq \nu + 1$, where q and ν are the number of the components and the nodes, respectively. We denote by p_i the genus of $\overline{\Sigma}^i$ and ν_i the number of punctures of Σ^i. If $2p_i + \nu_i \geq 3$ or equivalently $\chi(\Sigma^i) < 0$, then Σ^i carries a unique conformal, complete metric having constant curvature -1. With respect to this metric, the surface has cusps at the punctures and area $4\pi(p_i - 1 + \nu_i)$.

Next let Σ_k be a sequence of compact Riemann surfaces of fixed genus $p \geq 2$, with hyperbolic metrics h_k. By Proposition 5.1 in [7], there exists a compact Riemann surface Σ with nodes $N = \{a_1, \ldots, a_r\}$, and for each k a maximal collection $\Gamma_k = \{\gamma_k^1, \ldots, \gamma_k^r\}$ of pairwise disjoint, simply closed geodesics in Σ_k with $\ell_k^j = L(\gamma_k^j) \to 0$, such that after passing to a subsequence the following holds:

(1) $p - \sum_{i=1}^q p_i = \nu + 1 - q \geq 0$.

(2) There are maps $\varphi_k \in C^0(\Sigma_k, \Sigma)$, such that $\varphi_k : \Sigma_k \backslash \Gamma_k \to \Sigma \backslash N$ is diffeomorphic and $\varphi_k(\gamma_k^j) = a_j$ for $j = 1, \ldots, r$.

(3) For the inverse diffeomorphisms $\psi_k : \Sigma \backslash N \to \Sigma_k \backslash \Gamma_k$, we have $\psi_k^* h_k \to h$ in $C^\infty_{\mathrm{loc}}(\Sigma \backslash N)$.

In the following, we consider a sequence of conformal immersions $f_k \in W^{2,2}(\Sigma_k, \mathbb{R}^n)$ with $\mathcal{W}(f_k) \leq \Lambda$, and we assume that the hyperbolic surfaces (Σ_k, h_k) converge to a surface with nodes (Σ, N) as described above.

Lemma 5.1. *There exist branched conformal immersions $f^i : \overline{\Sigma}^i \to \mathbb{R}^n$, finite sets $\mathcal{S}_i \subset \Sigma^i$ and Möbius transformations σ_k^i, such that*

$$\sigma_k^i \circ f_k \circ \psi_k|_{\Sigma^i} \to f^i, \quad \text{weakly in } W^{2,2}_{\mathrm{loc}}(\Sigma^i \backslash \mathcal{S}_i, \mathbb{R}^n) \text{for } i = 1, \ldots, q.$$

Replacing f_k by $\sigma_k \circ f_k$ for suitable Möbius transformations σ_k we can take $\sigma_k^1 = \mathrm{id}$ and

$$\sigma_k^i(y) = I_{x_i}\left(\frac{y - y_k^i}{\lambda_k^i}\right), \quad \text{where } x_i \in \mathbb{R}^n,\ y_k^i = (f_k \circ \psi_k)(b_i)$$
$$\text{for } b_i \in \Sigma^i \text{and } \lambda_k^i > 0,$$

for $i = 2, \ldots, q$. Further the maps $\sigma_k^i \circ f_k$ are uniformly bounded, and $\mathcal{W}(f^i) \geq \beta^n_{p_i}$.

Proof. By the Gauß–Bonnet formula, the second fundamental form is bounded by

$$\int_{\Sigma_k} |A_{f_k}|^2 \, d\mu_{f_k} \leq C(\Lambda, p) < \infty.$$

The maps $f_k \circ \psi_k : \Sigma \backslash N \to \mathbb{R}^n$ are conformal immersions with respect to the metric $\psi_k^* h_k$, which converges to h in $C^\infty_{\mathrm{loc}}(\Sigma \backslash N)$. Let $\xi_i \subset \Sigma^i$ be an embedded arc, which is subdivided into $\xi_i^1, \ldots, \xi_i^m$. We can choose a subsequence and

$j_0 \in \{1, \dots, m\}$ with

$$\operatorname{diam}(f_k \circ \psi_k)(\xi_i^{j_0}) = \min_{1 \le j \le m} \operatorname{diam}(f_k \circ \psi_k)(\xi_i^j) =: \lambda_k^i.$$

We have $\lambda_k^i > 0$ by Theorem 3.1. Select $b_i \in \xi_i^{j_0}$, and define the maps

$$f_k^i : \Sigma_k \to \mathbb{R}^n,\ f_k^i(p) = \frac{f_k(p) - y_k^i}{\lambda_k^i}, \qquad \text{where } y_k^i = (f_k \circ \psi_k)(b_i).$$

As in Proposition 4.1, we can choose $B_1(x_i) \subset \mathbb{R}^n$ with $f_k^i(\Sigma_k) \cap B_1(x_i) = \emptyset$ for all k. Applying (2.1) to $I_{x_i} \circ f_k^i$ yields

$$\mu_{I_{x_i} \circ f_k^i}(\Sigma_k) \le C < \infty.$$

Now consider the maps

$$\hat{f}_k^i = I_{x_i} \circ f_k^i \circ \psi_k|_{\Sigma^i} : \Sigma^i \to \mathbb{R}^n.$$

We can assume that $\mu_{\hat{f}_k^i} \llcorner |A_{\hat{f}_k^i}|^2$ converges to α as Radon measures, and put

$$\mathcal{S}_i = \{p \in \Sigma^i : \alpha(\{p\}) \ge \gamma_n\}.$$

Corollary 5.1 implies that, away from $\mathcal{S}_i$, the $\hat{f}_k^i$ subconverge locally uniformly either to a conformal immersion, or to a point $x_1 \in \mathbb{R}^n$. As in Theorem 5.2

$$\hat{f}_k^i(\xi_i^{j_0}) \subset I_{x_i}(\overline{B_1(0)}) \subset \overline{B_1(x_i)} \backslash B_{\theta_i}(x_i), \qquad \text{where } \theta_i = \frac{1}{|x_i| + 1} > 0.$$

Therefore in the second alternative we get $|x_1 - x_i| \ge \theta_i$, and $f_k^i \circ \psi_k$ converges to $I_{x_i}(x_1)$ locally uniformly on $\Sigma^i \backslash \mathcal{S}_i$. But for $m > \frac{C(\Lambda,p)}{\gamma_n}$ there is a $j \in \{1, \dots, m\}$ with $\xi_i^j \cap \mathcal{S}_i = \emptyset$, and we conclude $1 \le \operatorname{diam}(f_k^i \circ \psi_k)(\xi_i^j) \to 0$, a contradiction. Therefore $\hat{f}_k^i$ converges locally uniformly and weakly in $W^{2,2}_{\mathrm{loc}}(\Sigma^i \backslash \mathcal{S}_i, \mathbb{R}^n)$ to $f^i \in W^{2,2}_{\mathrm{conf,loc}}((\Sigma^i \backslash \mathcal{S}_i, \mathbb{R}^n)$. Furthermore, Theorem 3.1 shows that f^i extends as a branched conformal immersion to $\overline{\Sigma}^i$. Applying the argument for $i = 2, \dots, q$ with f_k replaced by $\sigma_k^1 \circ f_k$ yields the second statement of the lemma. Finally, the inequality $\mathcal{W}(f^i) \ge \beta_{p_i}^n$ is clear when f^i is unbranched, otherwise we get $\mathcal{W}(f^i) \ge 8\pi > \beta_{p_i}^n$ from the Li–Yau inequality (3.1) in connection with [8]. $\square$

For our last result we need more details on degenerating hyperbolic surfaces. For $\ell > 0$ we define a reference cylinder $C(\ell) = [0,1] \times [-T(\ell), T(\ell)]$ with metric g_ℓ, where

$$T(\ell) = \frac{1}{\ell} \operatorname{arccot}\Big(\sinh \frac{\ell}{2}\Big) \quad \text{and} \quad g_\ell(s,t) = \frac{\ell^2}{\cos^2 \ell t}\,(ds^2 + dt^2).$$

The map $(s,t) \mapsto \mathrm{i}e^{\ell(s+\mathrm{i}t)}$ yields an isometry between $(C(\ell), g_\ell)$ and the sector in the upper half-plane given by $1 \leq r \leq e^\ell$, $|\theta - \frac{\pi}{2}| \leq \operatorname{arccot}\big(\sinh \frac{\ell}{2}\big)$. The circles $c_t = \{(s,t) : s \in [0,1]\}$ have constant geodesic curvature $\varkappa_{g_\ell}(t) = \sin \ell t$ and length $L_{g_\ell}(t) = \ell / \cos \ell t$. We note

$$\lim_{\ell \searrow 0} \varkappa_{g_\ell}\big(\pm(T(\ell) - t)\big) = 1 \quad \text{and} \quad \lim_{\ell \searrow 0} L_{g_\ell}\big(\pm(T(\ell) - t)\big) = \frac{1}{t + \frac{1}{2}},$$
for any $t > 0$.

Now let $\gamma_k \subset \Sigma_k$ be a sequence of geodesics with length $\ell_k \to 0$, corresponding to the node $a \in \Sigma$. By the collar lemma, see [7], there is an isometric embedding

$$\big(C(\ell_k), g_{\ell_k}\big) \subset (\Sigma_k, h_k),$$

with c_0 corresponding to γ_k. Clearly $T_k = T(\ell_k) \to \infty$. We will need the following property of the construction in [7]: for any $t \in [0, \infty)$ there is a compact set $K_t \subset \Sigma \backslash N$ such that

$$\varphi_k([0,1] \times [T_k - t, T_k]) \subset K_t, \text{ for all } k \in \mathbb{N}. \tag{5.1}$$

For this we refer to Section 4 in [22].

Theorem 5.3. *Let Σ_k be sequence of compact Riemann surfaces of genus $p \geq 2$, which diverges in moduli space. Then for any sequence of conformal immersions $f_k \in W^{2,2}_{\mathrm{conf}}(\Sigma_k, \mathbb{R}^n)$ we have*

$$\liminf_{k \to \infty} \mathcal{W}(f_k) \geq \min(8\pi, \omega_p^n).$$

Proof. We first consider the case $q = \nu + 1$, hence $p = p_1 + \cdots + p_q$. By Lemma 5.1 we have, away from a finite set of points, $f_k \circ \psi_k \to f^1$ weakly on Σ^1 and

$$\frac{f_k \circ \psi_k - y_k^i}{\lambda_k^i} \to I_{x_i} \circ f^i, \qquad \text{weakly on } \Sigma^i \text{ for } i = 2, \dots, q.$$

Now if f^i attains x_i with multiplicity two or more, then the Li–Yau inequality (3.1) yields

$$\liminf_{k\to\infty} \mathcal{W}(f_k) \geq \mathcal{W}(f^i) \geq 8\pi.$$

Otherwise we obtain, again by (3.1),

$$\lim_{k\to\infty} \mathcal{W}(f_k) \geq \mathcal{W}(f^1) + \sum_{i=2}^{q} \mathcal{W}(I_{x_i} \circ f^i) \geq \beta^n_{p_1} + \sum_{i=2}^{q} (\beta^n_{p_i} - 4\pi) \geq \omega^n_p.$$

In the case $q < \nu + 1$ there must be a node which does not disconnect Σ. After renumbering we can chose components $\Sigma^1, \dots, \Sigma^s$, and for each Σ^i two punctures $a_i^{\pm}$ such that a_i^+, a_{i+1}^- correspond to the same node a_i for $i = 1, \dots, s$; here $a_{s+1}^- = a_1^-$. We say that a puncture $a_i^{\pm}$ is good, if either $i = 1$ or $f^i(a_i^{\pm}) \neq x_i$. If both a_i^+ and a_i^- are not good, then the theorem follows with lower bound 8π by the Li–Yau inequality (3.1). Therefore, omitting subscripts we can assume that there is a node a at which both punctures are good.

Using the collar embedding we now choose $\tau_k \in [-T_k, T_k]$ with

$$\operatorname{diam} f_k(c_{\tau_k}) = \min_{t\in[-T_k,T_k]} \operatorname{diam} f_k(c_t) =: \delta_k.$$

The result follows as in Theorem 5.2, once we can show that for a subsequence

$$\lim_{k\to\infty} |T_k \pm \tau_k| = \infty. \tag{5.2}$$

For fixed $t \in [0, \infty)$ the curves $\varphi_k(c_{T_k - t})$ are contained in the compact set $K_t \subset \Sigma \backslash N$. Since $\psi_k^* h_k$ converges to h smoothly on K_t, we can assume that the curves converge smoothly to a limiting curve β_t in K_t with length $L_h(\beta_t) = (t + \frac{1}{2})^{-1}$. Now if $\varphi_k(c_{T_k - t}) \subset \Sigma^1$ we have

$$\operatorname{diam} f_k(c_{T_k - t}) = \operatorname{diam}\,(f_k \circ \psi_k)(\varphi_k(c_{T_k - t})) \to \operatorname{diam} f^1(\beta_t).$$

By Theorem 3.1, we see $\operatorname{diam} f^1(\beta_t) > 0$ for any $t \in [0,\infty)$. On the other hand,

$$\limsup_{k\to\infty} \delta_k \leq \limsup_{k\to\infty} \big(\operatorname{diam} f_k(c_{T_k - t})\big) = \operatorname{diam} f^1(\beta_t).$$

Letting $t \to \infty$ we conclude $\lim_{k\to\infty} \delta_k = 0$ by continuity of f^1, which proves claim (5.2). In the remaining case $\varphi_k(c_{T_k-t}) \subset \Sigma^i$ for some $i \geq 2$, we compute similarly

$$\frac{\operatorname{diam} f_k(c_{T_k-t})}{\lambda_k^i} = \operatorname{diam}\,(I_{x_i} \circ f_k \circ \psi_k)(\varphi_k(c_{T_k-t})) \to \operatorname{diam}(I_{x^i} \circ f^i)(\beta_t) > 0,$$

and further

$$\limsup_{k\to\infty} \frac{\delta_k}{\lambda_k^i} \leq \limsup_{k\to\infty} \frac{\operatorname{diam} f_k(c_{T_k-t})}{\lambda_k^i} = \operatorname{diam}(I_{x^i} \circ f^i)(\beta_t).$$

Again letting $t \to \infty$ we see that $\delta_k/\lambda_k^i \to 0$, using the fact that the puncture is good, i.e. $f^i(a) \neq x_i$. Thus (5.2) holds also for $i \geq 2$, and the theorem is proved. □

The constants β_p^n and hence ω_p^n are not known explicitly. The Willmore conjecture in $\mathbb{R}^n$ would imply that $\omega_2^n = 4\pi(\pi-1) > 8\pi$. The inequality $\omega_p^n > 8\pi$ holds at least for large p, since $\beta_p^n \to 8\pi$ as $p \to \infty$ by [10], as noted in the introduction.

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