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Closed twisted products and $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Lagrangian cones

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We study a construction we call the twisted product; in this construction higher dimensional special Lagrangian (SL) and Hamiltonian stationary cones in $\mathbb{C}^{p+q}$ (equivalently special Legendrian or contact stationary submanifolds in $\mathbb{S}^{2(p+q)-1}$) are constructed by combining such objects in $\mathbb{C}^p$ and $\mathbb{C}^q$ using a suitable Legendrian curve in $\mathbb{S}^3$. We study the geometry of these "twisting" curves and in particular the closing conditions for them. In combination with Carberry–McIntosh's continuous families of special Legendrian 2-tori [3] and the authors' higher genus special Legendrians [13], this yields a constellation of new SL and Hamiltonian stationary cones in $\mathbb{C}^n$ that are topological products. In particular, for all n sufficiently large we exhibit infinitely many topological types of SL and Hamiltonian stationary cone in $\mathbb{C}^n$, which can occur in continuous families of arbitrarily high dimension.

A special case of the twisted product construction yields all $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant SL cones in $\mathbb{C}^{p+q}$. These SL cones are higher-dimensional analogues of the SO(2)-invariant SL cones constructed previously by Haskins [8, 10] and used in our gluing constructions of higher genus SL cones in $\mathbb{C}^3$ [13]. $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant SL cones play a fundamental role as building blocks in gluing constructions of SL cones in high dimensions [14]. We study some basic geometric features of these cones including their closing and embeddedness properties.

1. Introduction

Background. Special Lagrangian (SL) n-folds in Calabi–Yau manifolds have been studied intensively over the past 15 years, thanks in part to their role in Mirror Symmetry [35]. Degenerations of families of smooth SLs and more general singular SLs play a crucial role, but in dimensions 3 and higher are still relatively poorly understood. SL cones in $\mathbb{C}^n$ with isolated singularities form the simplest class of singular SLs, and significant progress on understanding SL cones has been made in the last 10 years. In particular the situation in dimension 3 has been clarified considerably [3, 8, 10, 13, 17, 28]. By comparison the situation in higher dimensions is more complicated and less systematically explored. The current paper constructs a plethora of new higher dimensional SL cones, by combining an ODE-based method, which we call the twisted product construction with the gluing and integrable systems methods developed to construct three-dimensional SL cones [3, 13, 28].

SL cones in dimension 3. SL cones in $\mathbb{C}^3$ or equivalently special Legendrians in $\mathbb{S}^5$ have been studied by a variety of techniques in the last 10 years. The known examples can be summarized and classified roughly as follows:

(1) Homogeneous special Legendrians.

(2) Cohomogeneity one special Legendrians.

(3) Special Legendrians governed by (integrable) nonlinear ODEs.

(4) Special Legendrians obtained by integrable PDE methods.

(5) Special Legendrians obtained by geometric PDE gluing methods.

Equatorial 2-spheres and Legendrian Clifford tori are the only homogeneous special Legendrian surfaces. A special Legendrian submanifold $\Sigma \subset \mathbb{S}^{2n-1}$ is of cohomogeneity one if there is a compact subgroup $G \subset \mathrm{SU}(n)$ so that Σ is the union of a curve of G-orbits. In this case, the special Legendrian PDEs reduce to a system of nonlinear ODEs, which can be interpreted as a flow on the space of (isotropic) G-orbits [17]. These nonlinear ODEs often turn out to be completely integrable.

Cohomogeneity one special Legendrian surfaces have been classified in [8, 10]. Among all the cohomogeneity one examples a distinguished role is played by the SO(2)-invariant SL surfaces described in detail in [13]. They play a special role because they are the only cohomogeneity one examples in which interesting geometric degeneration occurs. This also makes them suitable building blocks for gluing constructions of SL surfaces [13]. More general ODE constructions of SL surfaces also exist [18].

A special feature of special Legendrian surfaces is that all the SL 2-tori can be described by methods from algebraically completely integrable systems [28]. This permits the construction of a zoo of possible special Legendrian 2-tori by building appropriate "spectral data" satisfying certain periodicity conditions. Using these spectral curve methods Carberry–McIntosh construct SL 2-tori that come in continuous families of arbitrarily large dimension [3]. However, it is difficult to control the geometric features of the resulting SL 2-torus from its associated spectral data; see [9] for some results in this direction.

While together classes (2)–(4) are geometrically very rich they permit only the construction of SL 2-tori; to obtain other topological types of closed SL surface the only currently known technique is the gluing method developed by the authors in [13]. By using SO(2)-invariant SL surfaces close to a singular limit as building blocks we were able to construct infinitely many closed SL surfaces of any odd genus (and also of genus 4). For the gluing methods a very precise understanding of the geometry of SO(2)-invariant SL surfaces close to the singular spherical limit is a crucial ingredient in the analysis of the linearization of the SL equation.

Higher dimensional SL cones. SL cones in $\mathbb{C}^n$ for $n \geq 4$ are far less comprehensively understood than SL cones in $\mathbb{C}^3$ and there are new features, which have no analogue in dimension 3, e.g. there is a (homogeneous) non-equatorial special Legendrian 3-sphere in $\mathbb{S}^7$ [6]. Even class (1) — the homogeneous examples — appears not to be completely classified; see [29] for a classification under additional geometric assumptions. Moreover, in higher dimensions it is not clear what the appropriate analogue of class (4) should be since a theory of completely integrable elliptic PDE in higher dimensions is currently lacking. Nevertheless, in this and subsequent papers we will show there is a rich variety of special Legendrians of classes (2), (3) and (5) (cohomogeneity one, general ODE and gluing, respectively).

Scope of the paper. In the current paper, we study a class of higher dimensional special Legendrians that we will call *twisted products*. Although any special Legendrian twisted product is controlled by an ODE system the general twisted product is not of cohomogeneity one (and may have no continuous symmetries). Special Legendrians of this type were first considered by Castro–Li–Urbano [4].

The twisted product construction, despite its simplicity, turns out to be surprisingly powerful when used in combination with the powerful integrable systems and gluing methods already developed in three dimensions. Unlike cohomogeneity one constructions which give only a finite number of topological types in a given dimension the twisted product construction — together with the existing constructions of special Legendrian surfaces — already allows the construction of infinitely many topological types of closed special Legendrians in each dimension; see Theorems A–C later in the introduction for precise statements. Moreover, twisted product special Legendrians show that families of high-dimensional special Legendrians can degenerate in many different ways.

One limitation is that all twisted product special Legendrians are topological products; to obtain infinitely many topological types of special Legendrians, which are not products we need to use gluing methods and these gluing methods need appropriate special Legendrians as building blocks. Suitable building blocks are also constructed in this paper as a special instance of the twisted product construction.

An important subclass of special Legendrian twisted products is the class of $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrians of $\mathbb{S}^{2(p+q)-1}$; this subclass does consist of cohomogeneity one special Legendrians. $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrians share features of the $\mathrm{SO}(2)$-invariant SL surfaces in $\mathbb{S}^5$. These shared features make $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrians appropriate building blocks for our higher-dimensional gluing constructions [11, 12]. In particular, in any given dimension greater than 3 there are multiple possible cohomogeneity one candidates for building blocks in gluing constructions of higher-dimensional SL cones, unlike the situation in dimension 3.

Main results. We now describe the main results of the paper.

Twisted products. At the heart of this paper is a construction of special Legendrian immersions which we call the *twisted product* construction. The twisted product construction (see 2.1) gives a way to combine a pair of lower-dimensional Legendrian immersions $X_1 : \Sigma_1 \to \mathbb{S}^{2p-1}$ and $X_2 : \Sigma_2 \to \mathbb{S}^{2q-1}$ with a Legendrian curve $\mathbf{w} : I \to \mathbb{S}^3$ to produce a new Legendrian immersion $X_1 *_{\mathbf{w}} X_2 : I \times \Sigma_1 \times \Sigma_2 \to \mathbb{S}^{2p+2q-1}$. If the twisting curve $\mathbf{w}$ is appropriately chosen then the cone over $X_1 *_{\mathbf{w}} X_2$ is just the product of the cones over X_1 and X_2, explaining the origin of the term twisted product.

The Lagrangian phase of the twisted product $X_1 *_{\mathbf{w}} X_2$ is determined by the Lagrangian phases of X_1, X_2 and the twisting curve $\mathbf{w}$ (see (2.10)). This formula implies that if the twisting curve $\mathbf{w}$ satisfies a certain Condition depending on p and q (see (2.17)) then the $\mathbf{w}$-twisted product $X_1 *_{\mathbf{w}} X_2$ is special Legendrian whenever both X_1 and X_2 are also special Legendrian. This construction generalizes also to the case of Hamiltonian stationary cones (see 2.28, 2.31). Twisting curves $\mathbf{w}$ satisfying the Condition 2.17 we call (p, q)*-twisted SL curves* in $\mathbb{S}^3$. The bulk of the paper consists of a detailed study of the geometry of all (p, q)-twisted SL curves; understanding all closed (p, q)-twisted SL curves is a particular focus.

Since Condition 2.17 depends on p and q but not on the immersions X_1 and X_2 we can use the twisted product construction to produce special Legendrian immersions from lower-dimensional special Legendrian immersions provided that we can find (p, q)-twisted SL curves. To produce special Legendrian immersions of closed manifolds via (p, q)-twisted SL curves we need to find closed (p, q)-twisted SL curves.

$\mathrm{SO}(p) \times \mathrm{SO}(q)$*-invariant special Legendrians.* The case of $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrians amounts to the special Legendrian twisted product construction when X_1 and X_2 are chosen to be the standard real equatorial embeddings of spheres of dimension $p-1$ and $q-1$, respectively. Thus the study of $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrians in $\mathbb{S}^{2p+2q-1}$ essentially reduces to the study of (p, q)-twisted SL curves in $\mathbb{S}^3$. Moreover, finding $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrian embeddings of closed manifolds is closely related to the problem of finding closed (p, q)-twisted SL curves.

(p, q)-twisted SL curves and ODEs. Key to our study of (p, q)-twisted SL curves in $\mathbb{S}^3$ is Lemma 2.19; this shows that there is a system of first-order ODEs (see (2.20), (2.25)) whose integral curves are (p, q)-twisted SL curves in $\mathbb{S}^3$ and conversely that (p, q)-twisted SL curves in $\mathbb{S}^3$ always admit parameterizations satisfying (2.20). When $(p, q) = (1, 2)$ these ODEs reduce to the fundamental ODEs used to study $\mathrm{SO}(2)$-invariant special Legendrians in [13, equation (3.18)]. For general (p, q) these ODEs first appeared in the work of Castro–Li–Urbano [4]; see the beginning of Section 3 for a more detailed discussion of previous related work by Anciaux, Castro–Li–Urbano and Joyce.

Section 4 studies these ODEs and establishes that up to the action of some obvious symmetries there is a 1-parameter family $\mathbf{w}_\tau$ of inequivalent solutions to (2.25). Via the correspondence between SL twisted products and $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrians the 1-parameter family $\mathbf{w}_\tau$ gives rise to a 1-parameter family of $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrians X_τ.

One of the main tasks of the current paper is to prove existence of closed (p, q)-twisted SL curves and $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrian embeddings of closed manifolds. The former is important for the applications to construct new special Legendrian immersions of the closed manifold $S^1 \times \Sigma_1 \times \Sigma_2$ from a pair of lower-dimensional special Legendrian immersions of Σ_1 and Σ_2. The latter is central to our use of the $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrians as building blocks in gluing constructions of higher-dimensional SL cones.

Section 6 studies the periodicity of $\mathbf{w}_\tau$ and closely related questions. We prove that a single angular period $\widehat{\mathsf{p}}_\tau$ (defined precisely in (6.23)) determines when $\mathbf{w}_\tau$ forms a closed curve in $\mathbb{S}^3$; $\widehat{\mathsf{p}}_\tau$ also controls when the associated $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrian immersion X_τ factors through an embedding of a closed manifold. In Section 7, by studying the dependence of $\widehat{\mathsf{p}}_\tau$ on τ (see 7.7) we prove that for a countably infinite (dense) set of τ, $\mathbf{w}_\tau$ forms a closed curve in $\mathbb{S}^3$ and X_τ factors through a closed embedding as above (see 7.15 and 7.16).

A plethora of new SL and Hamiltonian stationary cones. Our results about closed (p, q)-twisted SL curves in $\mathbb{S}^3$ together with our previous gluing constructions of higher genus SL cones in $\mathbb{C}^3$ [13] allow us to construct a wealth of new topological types of higher-dimensional SL and Hamiltonian stationary cones.

Theorem A.

(i) *For any $n \geq 4$ there are infinitely many topological types of SL cone in $\mathbb{C}^n$, each of which is diffeomorphic to the cone over a product $S^1 \times \Sigma'$ for some smooth closed manifold Σ', and each of which admits infinitely many distinct geometric representatives.*

(ii) *For any $n \geq 4$ there are infinitely many topological types of Hamiltonian stationary cone in $\mathbb{C}^n$ which are not minimal Lagrangian, each of which is diffeomorphic to the cone over a product $S^1 \times \Sigma'$ for some smooth closed manifold Σ', and each of which admits infinitely many distinct geometric representatives.*

Similarly combining our results about (p, q)-twisted SL curves with the work of Carberry–McIntosh [3] on special Legendrian 2-tori via integrable systems methods we obtain the following

Theorem B.

(i) *For $n \geq 3$ there exist special Legendrian immersions of T^{n-1} in $\mathbb{S}^{2n-1}$, which come in continuous families of arbitrarily high dimension.*

(ii) *For $n \geq 4$ there exist contact stationary (and not minimal Legendrian) immersions of T^{n-1} in $\mathbb{S}^{2n-1}$, which come in continuous families of arbitrarily high dimension.*

Finally, by combining the twisted product construction with both integrable systems methods and our gluing methods for special Legendrian surfaces in $\mathbb{S}^5$ we obtain the following striking results

Theorem C.

(i) *For any $n \geq 6$ there are infinitely many topological types of SL cone in $\mathbb{C}^n$ of product type which can occur in continuous families of arbitrarily high dimension.*

(ii) *For each $n \geq 6$ there are infinitely many topological types of Hamiltonian stationary cone in $\mathbb{C}^n$ of product type which are not minimal Lagrangian and which can occur in continuous families of arbitrarily high dimension.*

It is difficult to see how either integrable systems methods or gluing methods by themselves could yield a result like Theorem C.

Forces and torques. Many geometric variational problems admit homological invariants associated with symmetries of the problem. These invariants have played a fundamental role in global structure results including uniqueness questions [25, 30, 31] and also in gluing results [19, 21–24, 32]. For minimal and CMC immersions in Euclidean space or round spheres the invariants associated to translations and rotations are called the forces and torques, respectively.

In this paper, we calculate the torque of the $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrians X_τ in 5.14. An appropriate component of the torque (depending on p and q) is exactly proportional to the parameter τ. This is similar to the case of Delaunay surfaces where (appropriately centred) the torque is zero and the force is a vector along its axis whose magnitude is τ, the parameter of the Delaunay. The torque of X_τ enters into our argument to calculate refined asymptotics of the angular period $\widehat{\mathsf{p}}_\tau$ and its derivative as $\tau \to 0$ and therefore is needed in our work on higher dimensional SL gluing [11, 12, 14]. More generally we expect that the torque will play an important role in controlling aspects of the global geometry of special Legendrians.

The geometry of X_τ and their relatives. We can compare the geometry of X_τ with that of other similar cohomogeneity one geometric objects like the Delaunay surfaces and the Delaunay–Fowler metrics. In all three cases, we can recognize strings of regions which we could call *bulges* connected to their neighbours through *necks*. Important differences exist in the symmetry group of these objects related to the structure of this bulge/neck decomposition. All three families are parameterized by a real parameter that we call τ; the parameter τ can be identified with the value of some properly defined conserved quantity (a component of the force or torque in the Delaunay and in our case, respectively). The parameter τ also controls the size of the smallest orbit.

As $\tau \to 0$ in all three cases the bulges have a round spherical limit, while the necks degenerate either to points or to lower-dimensional equators. Correspondingly for small τ the bulges approximate their spherical limit and the necks approximate standard objects scaled to small size. These standard objects are the catenoid in the Delaunay surface case, the (Riemannian) Schwarzschild metric in the Delaunay–Fowler case, and the Lagrangian catenoid or its product with a unit round sphere of the appropriate dimension in our cases. The fact that the bulges are approximately spherical forms the basis for using these objects as building blocks for gluing constructions [13, 14, 19–21, 32]. A more detailed discussion of these geometric features of X_τ is given in our survey article [14]; full details and proofs appear in [15].

The Lagrangian catenoid belongs to the larger family of Lawlor necks [7, 16, 27]; while the Lagrangian catenoid in $\mathbb{C}^n$ is foliated by round spheres and is $\mathrm{SO}(n)$-invariant, a general Lawlor neck is foliated by ellipsoids and has only discrete symmetries. In a similar way the $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrians X_τ also belong to a larger family of SL cones constructed by evolving quadrics [16]. This construction gives a larger family of special Legendrians still controlled by ODEs; the additional parameters of this family control the distortion of the quadrics in a similar way that the parameters of the Lawlor necks control the maximal eccentricity of its ellipsoidal sections. In this larger class of special Legendrians general Lawlor necks and not just Lagrangian catenoids can appear as appropriate blow-up limits as $\tau \to 0$. We study this and more general degeneration behaviour of these families elsewhere.

Organization of the paper. The paper is organized in seven sections. Section 1 consists of the introduction, this section and some remarks on notation.

In Section 2, we describe how to generate a new special Legendrian immersion from a pair of lower-dimensional special Legendrian immersions and a curve in $\mathbb{S}^3$ satisfying some additional geometric condition. This *twisted product* construction (see Definition 2.1 and Proposition 2.9) is at the heart of the paper. Definition 2.16 introduces the notion of a (p,q)-*twisted special Legendrian (SL) curve* in $\mathbb{S}^3$. Corollary 2.18 explains how to use (p,q)-twisted SL curves in $\mathbb{S}^3$ to construct new special Legendrian immersions from a pair of lower-dimensional special Legendrian immersions via the twisted product construction. Lemma 2.19 reduces the study of (p,q)-twisted SL curves in $\mathbb{S}^3$ to a first-order system of ODEs (2.20).

We also sketch briefly the extension of the twisted product construction to the contact stationary realm. Definition 2.28 introduces (p,q)-*twisted contact stationary (CS) curves* in $\mathbb{S}^3$ and Lemma 2.31 gives the contact stationary analogue of Corollary 2.18. This enables us to construct many new contact stationary (and non minimal Legendrian) immersions of closed manifolds from lower-dimensional special Legendrian immersions. In the rest of the section, assuming results on the existence of countably infinitely many closed (p,q)-twisted SL curves proved later in Theorem 7.15, we prove Theorems A–C quoted above by combining the SL and CS twisted product constructions with our gluing constructions of special Legendrian surfaces of higher genus in $\mathbb{S}^5$ [13] and the integrable systems constructions of special Legendrian tori in $\mathbb{S}^5$ of Carberry–McIntosh [3].

Section 3 establishes the relationship between (p, q)-twisted SL curves in $\mathbb{S}^3$ and $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrians in $\mathbb{S}^{2p+2q-1}$.

Section 4 studies the ODEs (4.8) that control (p, q)-twisted SL curves in $\mathbb{S}^3$. Proposition 4.7 establishes the basic facts about solutions to the (p, q)-twisted SL ODEs (4.8): its conserved quantities, its symmetries, stationary points, local and global existence and dependence on initial data. Following a number of auxiliary results we prove the main result of the section — Proposition 4.26 — which gives a normal form for any solution $\mathbf{w}$ to (4.8).

In Section 5, we use Propositions 4.7 and 4.26 to define a distinguished 1-parameter family $\mathbf{w}_\tau$ of solutions of the fundamental ODE for (p, q)-twisted SL curves by specifying appropriate initial conditions (see (5.2), (5.4) and Proposition 5.1); up to the action of symmetries any solution of (4.8) is equivalent to $\mathbf{w}_\tau$ for some τ. In Definition 5.8, we use the 1-parameter family of solutions $\mathbf{w}_\tau$ to define the 1-parameter family X_τ of $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrian immersions in $\mathbb{S}^{2p+2q-1}$. Proposition 5.9 establishes some basic properties of X_τ. Finally, in Proposition 5.14 we determine the restricted torque for the $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant SL immersions X_τ. The *torque* is a homological invariant of minimal submanifolds of $\mathbb{S}^{2n-1}$ and the *restricted torque* is a variant of the torque for special Legendrian submanifolds of $\mathbb{S}^{2n-1}$.

Section 6 studies the conditions under which $\mathbf{w}_\tau$ forms a closed curve in $\mathbb{S}^3$ or the associated curve of isotropic $\mathrm{SO}(p) \times \mathrm{SO}(q)$-orbits is closed. To this end, we introduce the *periods* and *half-periods* of $\mathbf{w}_\tau$; the periods of $\mathbf{w}_\tau$ control when $\mathbf{w}_\tau$ forms a closed curve in $\mathbb{S}^3$, while the half-periods control when the curve of $\mathrm{SO}(p) \times \mathrm{SO}(q)$ orbits associated with $\mathbf{w}_\tau$ is closed. The half-periods of $\mathbf{w}_\tau$ also control the embedding properties of X_τ (see Proposition 6.32). Fundamental roles are played by the angular period $2\widehat{\mathsf{p}}_\tau$ (defined in (6.23)), the rotational period $\hat{\mathsf{T}}_{2\widehat{\mathsf{p}}_\tau}$ (defined in (6.25)) and by k_0 the order of the rotational period (defined in 6.38). Lemma 6.39 determines the periods and half-periods of $\mathbf{w}_\tau$ in terms of the order of the rotational period k_0 and hence allows us to characterize exactly when $\mathbf{w}_\tau$ forms a closed curve either in $\mathbb{S}^3$ or in the space of isotropic orbits of $\mathrm{SO}(p) \times \mathrm{SO}(q)$.

Section 7 uses the results of Section 6 together with results about the asymptotics of $\widehat{\mathsf{p}}_\tau$ as $\tau \to 0$ and as $\tau \to \tau_{\max}$ to prove for every admissible pair of integers (p, q) the existence of a countable infinite (dense) set of τ for which $\mathbf{w}_\tau$ forms a closed (p, q)-twisted SL curve in $\mathbb{S}^3$ (Theorem 7.15) and a countable infinite (dense) set of τ for which the $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrian immersion X_τ factors through an embedding of a closed manifold (Theorem 7.16).

Notation and conventions. Throughout the paper we use the following notation to express elements of $\mathrm{Isom}(\mathbb{R})$, the isometries of the real line. We denote by T_x, translation by x, $t \mapsto t + x$. We denote by $\underline{\mathsf{T}}$ reflection in the origin $t \mapsto -t$ and reflection in x, $t \mapsto 2x - t$ by $\underline{\mathsf{T}}_x$.

2. Twisted products of Legendrian immersions: new immersions from old

In this section, we describe the twisted product construction; in this construction, given a Legendrian immersion $\mathbf{w} : I \to \mathbb{S}^3$ and a pair of Legendrian immersions $X_1 : \Sigma_1 \to \mathbb{S}^{2p-1}$ and $X_2 : \Sigma_2 \to \mathbb{S}^{2q-1}$ we obtain a new Legendrian immersion $X_1 *_{\mathbf{w}} X_2 : I \times \Sigma_1 \times \Sigma_2 \to \mathbb{S}^{2p+2q-1}$, that we call the $\mathbf{w}$-*twisted product of* X_1 *and* X_2. If the curve $\mathbf{w} : I \to \mathbb{S}^3 \subset \mathbb{C}^2$ is chosen appropriately then the cone over the $\mathbf{w}$-twisted product is precisely the product of the cone over X_1 with the cone over X_2 — hence the name twisted product for the general case. If $\mathbf{w}$ satisfies an appropriate ODE and both X_1 and X_2 are special Legendrian then the $\mathbf{w}$-twisted product $X_1 *_{\mathbf{w}} X_2$ is also special Legendrian. We call solutions of these ODEs, (p, q)-*twisted special Legendrian curves.* To construct new special Legendrian immersions of closed manifolds, the key point is to find closed (p, q)-twisted special Legendrian curves. We achieve a complete understanding of closed (p, q)-twisted SL curves in Sections 6 and 7.

Combining our results on closed (p, q)-twisted SL curves with our earlier work on gluing constructions of special Legendrian immersions in $\mathbb{S}^5$ [13] and constructions of special Legendrian 2-tori via integrable systems methods [3, 28] we are able to prove the existence of a plethora of new special Legendrian immersions with interesting geometric properties in dimensions greater than three. Very minor modifications also allow us to construct a similar variety of contact stationary Legendrian immersions and hence of new Hamiltonian stationary (and not SL) cones. However, all closed special Legendrians constructed via (p, q)-twisted SL curves are topologically products of the form $S^1 \times \Sigma$. We construct infinitely many topological types of higher-dimensional special Legendrians that are not topologically products using gluing methods in [11, 12, 14].

When the immersions X_1 and X_2 are chosen to be the simplest possible special Legendrian immersions, namely the standard totally real equatorial embeddings of $\mathbb{S}^{p-1} \subset \mathbb{R}^p \subset \mathbb{C}^p$ and $\mathbb{S}^{q-1} \subset \mathbb{R}^q \subset \mathbb{C}^q$, then $\mathbf{w}$-twisted special Legendrian immersions $X_1 *_{\mathbf{w}} X_2$ turn out to be suitable building blocks for higher-dimensional gluing constructions of special Legendrian immersions.

When $p=1$ and $q=2$ these are precisely the building blocks used in our previous gluing construction of special Legendrian surfaces in $\mathbb{S}^5$ [8, 10, 13].

Throughout this section, given a Legendrian immersion Y into an odd-dimensional sphere we shall denote its Lagrangian phase by $e^{i\theta_Y}$.

Twisted products of spherical Legendrian immersions

Definition 2.1. Let $I \subseteq \mathbb{R}$ be a connected interval, Σ_1 and Σ_2 be two smooth manifolds of dimensions n_1 and n_2, respectively, and $X_i : \Sigma_i \to \mathbb{S}^{2m_i-1}$ for $i=1,2$ be smooth maps into odd-dimensional spheres. Let $\mathbf{w} = (w_1(t), w_2(t)) : I \to \mathbb{S}^3$ be a smooth immersed curve in $\mathbb{S}^3$. Then the $\mathbf{w}$-twisted product of X_1 and X_2, denoted $X_1 *_{\mathbf{w}} X_2$, is the smooth map

$$X_1 *_{\mathbf{w}} X_2 : I \times \Sigma_1 \times \Sigma_2 \to \mathbb{S}^{2m_1+2m_2-1} \subset \mathbb{C}^{m_1+m_2} = \mathbb{C}^{m_1} \times \mathbb{C}^{m_2},$$

defined by

$$X_1 *_{\mathbf{w}} X_2\,(t, \sigma_1, \sigma_2) = (w_1(t)X_1(\sigma_1),\ w_2(t)X_2(\sigma_2)). \tag{2.2}$$

Remark 2.3. In the definition of a twisted product above, it is also convenient to allow the degenerate case where Σ_1 is 0-dimensional. We will need the case where Σ_1 is a single point p and the map X_1 maps $p \mapsto (1,0) \in \mathbb{S}^1 \subset \mathbb{S}^3 \subset \mathbb{C}^2$. In this case, we will drop the reference to X_1 and Σ_1 and the subscript for X_2 and Σ_2 and write $X_{\mathbf{w}} : I \times \Sigma \to \mathbb{S}^{2m-1}$ for the map defined by

$$X_{\mathbf{w}}(t, \sigma) = (w_1(t), w_2(t)\,X(\sigma)). \tag{2.4}$$

We will still refer to this degenerate case as a twisted product.

The following extended remark explains the origin of the term *twisted product* in Definition 2.1.

Remark 2.5. Let C_1 and C_2 be cones in $\mathbb{C}^{m_1}$ and $\mathbb{C}^{m_2}$ respectively. The product $C_1 \times C_2 \subset \mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \cong \mathbb{C}^{m_1+m_2}$ is also a cone. Suppose now that C_1 and C_2 are both regular cones, i.e., $C_i = C(\Sigma_i)$ is the cone over a smooth closed submanifold $\Sigma_i \subset \mathbb{S}^{2m_i-1}$ and hence has an isolated singularity at $\mathbf{0} \in \mathbb{C}^{m_i}$. Let $\Sigma_{12} \subset \mathbb{S}^{2m_1+2m_2-1}$ denote the link of the product cone

$C_1 \times C_2 \subset \mathbb{C}^{m_1+m_2}$. Clearly

(2.6)
$$\Sigma_{12} = \{(\cos t\,\sigma_1, \sin t\,\sigma_2)\,|\, t \in [0, \tfrac{1}{2}\pi],\ \sigma_1 \in \Sigma_1,\ \sigma_2 \in \Sigma_2\} \subset \mathbb{S}^{2m_1+2m_2-1}.$$

There is an obvious surjective map

$$\Pi : [0, \pi/2] \times \Sigma_1 \times \Sigma_2 \to \Sigma_{12}$$

from the manifold with boundary $[0, \pi/2] \times \Sigma_1 \times \Sigma_2$ to the link of our product cone Σ_{12} defined by

$$\Pi(t, \sigma_1, \sigma_2) = (\cos t\,\sigma_1, \sin t\,\sigma_2). \tag{2.7}$$

Clearly, the map Π can be written as a $\mathbf{w}$-twisted product by taking X_1 and X_2 to be the inclusion maps $i_1 : \Sigma_1 \to \mathbb{S}^{2m_1-1}$ and $i_2 : \Sigma_2 \to \mathbb{S}^{2m_2-1}$, respectively, and $\mathbf{w} : I \to \mathbb{S}^3$ to be the equatorial curve $\mathbf{w} : [0, \pi/2] \to \mathbb{S}^1 \subset \mathbb{S}^3$ defined by

$$\mathbf{w}(t) = (\cos t, \sin t). \tag{2.8}$$

We therefore view the $\mathbf{w}$-twisted product defined in 2.1 as a "twisted" version of taking the product of two regular cones. It "twists" the product construction by allowing a general curve $\mathbf{w} \in \mathbb{S}^3$ instead of the standard equatorial curve $\mathbb{S}^1 \subset \mathbb{S}^3$ defined in (2.8). It is natural therefore to call the curve $\mathbf{w} : I \to \mathbb{S}^3$ the *twisting curve.*

The degenerate case discussed in Remark 2.3 also specializes to a product of cones $C_1 \times C_2$ when the twisting curve is the equatorial curve (2.8) and $C_1 = \mathbb{R}^+ \subset \mathbb{C}$ and $C_2 = C(\Sigma)$. Thus, we can still view $X_{\mathbf{w}}$ (defined in (2.4)) as a twisted version of the product of two cones $\mathbb{R}^+ \times C$ and hence the name twisted product is appropriate even in this degenerate case.

The product cone $C_1 \times C_2$ is not a regular cone even when both C_1 and C_2 are regular cones. Equivalently, the link $\Sigma_{12} \subset \mathbb{S}^{2m_1+2m_2-1}$ is not a smooth submanifold. As a topological space we can think of Σ_{12} as being obtained from the generalized cylinder $[0, \pi/2] \times \Sigma_1 \times \Sigma_2$ by a modified "coning-off the boundary" construction. Namely, at the endpoint $t = 0$ we cone-off Σ_2 inside $\{\mathbf{0}\} \times \Sigma_1 \times \Sigma_2$ but leave Σ_1 untouched, whereas at the endpoint $t = \pi/2$ we instead cone-off Σ_1 but leave Σ_2 alone. Thus, Σ_{12} has two different types of singularities: conical singularities modelled on Σ_2 along a copy of Σ_1 and conical singularities modelled on Σ_1 along a copy of Σ_2.

Π is a smooth embedding away from the endpoints of the interval $[0, \pi/2]$ and induces a Riemannian metric g on $(0, \pi/2) \times \Sigma_1 \times \Sigma_2$ defined by

$$g = dt^2 + \cos^2 t\, g_1 + \sin^2 t\, g_2,$$

where g_1 and g_2 are the Riemannian metrics induced on Σ_1 and Σ_2 by the spherical inclusions i_1 and i_2. In particular, we see that the metric g degenerates at $t = 0$ and $t = \pi/2$ in a manner consistent with the description of the singularities of Σ_{12} we gave in the previous paragraph.

In the exceptional case, where $C_1 = \mathbb{R}^{m_1} \subset \mathbb{C}^{m_1}$ and $C_2 = \mathbb{R}^{m_2} \subset \mathbb{C}^{m_2}$ then obviously $C_1 \times C_2 \cong \mathbb{R}^{m_1+m_2}$ and therefore $\Sigma_{12} = \mathbb{S}^{m_1+m_2-1} \subset \mathbb{S}^{2m_1+2m_2-1}$ is not singular. In this case, the images of the hypersurfaces with t constant under the map

$$\Pi : [0, \tfrac{1}{2}\pi] \times \mathbb{S}^{m_1-1} \times \mathbb{S}^{m_2-1} \to \mathbb{S}^{m_1+m_2-1}$$

give a (singular) codimension one foliation of $\mathbb{S}^{m_1+m_2-1}$ by hypersurfaces isometric to the product of spheres $\mathbb{S}^{m_1-1}(\cos t) \times \mathbb{S}^{m_2-1}(\sin t)$. As $t \to 0$ the second spherical factor shrinks to radius 0, while the first spherical factor shrinks to radius 0 as $t \to \pi/2$. Restricting Π to the open interval $(0, \pi/2)$ gives a foliation of $\mathbb{S}^{m_1+m_2-1} \setminus (\mathbb{S}^{m_1-1}, 0) \cup (0, \mathbb{S}^{m_2-1})$ that omits the two singular leaves corresponding to the endpoints $t = 0$ and $t = \pi/2$. The leaves of this singular foliation of $\mathbb{S}^{m_1+m_2-1}$ are exactly the orbits of the group $\mathrm{SO}(m_1) \times \mathrm{SO}(m_2) \subset \mathrm{SO}(m_1 + m_2)$. When $m_1 = m_2 = 2$ the singular foliation above yields the standard singular foliation of $\mathbb{S}^3$ by an open interval of 2-tori which degenerates at the ends of the interval to the linked Hopf circles $(\mathbb{S}^1, 0) \subset \mathbb{S}^3$ and $(0, \mathbb{S}^1) \subset \mathbb{S}^3$.

Moving from the smooth to the Legendrian category we can refine the notion of twisted product to generate new Legendrian immersions from a pair of lower-dimensional Legendrian immersions, provided the twisting curve itself is Legendrian in $\mathbb{S}^3$.

Proposition 2.9 (Legendrian twisted products [4, Thm 3.1]). *Suppose that the twisting curve* $\mathbf{w}$ *is a Legendrian curve in* $\mathbb{S}^3$, *that* (Σ_1, g_1) *and* (Σ_2, g_2) *are oriented Riemannian manifolds of dimension* $p - 1 > 0$ *and* $q - 1 > 0$, *respectively, and that* $X_1 : \Sigma_1 \to \mathbb{S}^{2p-1}$ *and* $X_2 : \Sigma \to \mathbb{S}^{2q-1}$ *are Legendrian isometric immersions. Away from points where* w_1 *or* w_2 *vanish the* $\mathbf{w}$*-twisted product*

$$X_1 *_{\mathbf{w}} X_2 : I \times \Sigma_1 \times \Sigma_2 \to \mathbb{S}^{2p+2q-1} \subset \mathbb{C}^{p+q} = \mathbb{C}^p \times \mathbb{C}^q,$$

defined in 2.1 is a Legendrian immersion whose Lagrangian phase $e^{i\theta_X}$ satisfies the following twisted product relation

$$e^{i\theta_X} = (-1)^{p-1} e^{i\theta_{X_1}} e^{i\theta_{X_2}} e^{i\theta_{\mathbf{w}} + i(p-1)\arg w_1 + i(q-1)\arg w_2}, \tag{2.10}$$

and the metric g induced by $X_1 *_{\mathbf{w}} X_2$ is

$$g = |\dot{\mathbf{w}}|^2 dt^2 + |w_1|^2 g_1 + |w_2|^2 g_2. \tag{2.11}$$

The analogue of Proposition 2.9 in the degenerate case $p = 1$ considered in (2.4) is

Proposition 2.12. *Suppose that the twisting curve* $\mathbf{w}$ *is a Legendrian curve in $\mathbb{S}^3$, that (Σ, g) is an oriented Riemannian manifold of dimension $n-2$ and that $X : \Sigma \to \mathbb{S}^{2n-3}$ is a Legendrian isometric immersion. Away from points where w_2 vanishes the* $\mathbf{w}$*-twisted product*

$$X_w : I \to \Sigma \to \mathbb{S}^{2n-1} \subset \mathbb{C}^n = \mathbb{C} \times \mathbb{C}^{n-1},$$

defined in (2.4) is a Legendrian immersion whose Lagrangian phase $e^{i\theta}$ satisfies the twisted product relation

$$e^{i\theta} = e^{i\theta_X} e^{i\theta_{\mathbf{w}} + i(n-2)\arg w_2},$$

and the metric induced by $X_{\mathbf{w}}$ is $|\dot{w}|^2 dt^2 + |w_2|^2 g$.

Remark 2.13. $X_1 *_{\mathbf{w}} X_2$ fails to be an immersion at points where either w_1 or w_2 vanish. Away from such points we have $\mathrm{Vol}_g = |\dot{\mathbf{w}}||w_1|^{p-1}|w_2|^{q-1} dt\, \mathrm{Vol}_{g_1} \mathrm{Vol}_{g_2}$, and hence when both Σ_1 and Σ_2 are closed the $\mathbf{w}$-twisted product has volume

$$\mathrm{Vol}\,(X_1 *_{\mathbf{w}} X_2) = \mathrm{Vol}(X_1)\,\mathrm{Vol}(X_2) \int_I |\dot{\mathbf{w}}|\, |w_1|^{p-1} |w_2|^{q-1}\, dt. \tag{2.14}$$

The obvious analogue of (2.14) holds for the degenerate case $p = 1$.

Remark 2.15. Let $\pi\colon \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ denote the Riemannian submersion associated with the Hopf fibration. For any Legendrian immersion $X\colon \Sigma \to \mathbb{S}^{2n+1}$ the map $\pi \circ X\colon \Sigma \to \mathbb{CP}^n$ is a Lagrangian immersion and moreover, locally, any Lagrangian immersion to $\mathbb{CP}^n$ lifts to a Legendrian immersion covering it. Lagrangian immersions $\pi \circ X$ to $\mathbb{CP}^n$ for which the Legendrian immersion X is a twisted product in the sense of 2.9 or 2.12 were termed *warped-product Lagrangian immersions* in [2, Section 1]. Conditions on the second fundamental form of a Lagrangian immersion to $\mathbb{CP}^n$ that characterize when it is of warped-product type are given in [2, Thms 4.4 and 5.1].

Twisted products of special Legendrians and (p,q)-twisted special Legendrian curves. From now on we will always consider the case where the integers p and q satisfy $p \leq q$, $p \geq 1$ and $q \geq 2$. There is no loss of generality in making this assumption. We call such a pair (p,q) of positive integers *admissible*. For each admissible pair of integers (p,q) we define a distinguished class of Legendrian curves in $\mathbb{S}^3$.

Definition 2.16. We call a Legendrian curve $\mathbf{w}$ in $\mathbb{S}^3$ a (p,q)-twisted special Legendrian (SL) curve if the Lagrangian phase of $\mathbf{w}$ satisfies

$$e^{i\theta_{\mathbf{w}}} = (-1)^{p-1} e^{-i(p-1)\arg w_1 - i(q-1)\arg w_2}. \tag{2.17}$$

Proposition 2.9 (and 2.12 for the degenerate case $p=1$) has the following easy corollary, which allows us to generate a new special Legendrian immersion in $\mathbb{S}^{2(p+q)-1}$ from a (p,q)-twisted SL curve in $\mathbb{S}^3$ and a pair of special Legendrian immersions into $\mathbb{S}^{2p-1}$ and $\mathbb{S}^{2q-1}$, respectively.

Corollary 2.18 (Special Legendrian twisted products). *Let X_1, X_2 and $\mathbf{w}$ be as in Proposition 2.9. If additionally, X_1 and X_2 are both special Legendrian then the $\mathbf{w}$-twisted product $X_1 *_{\mathbf{w}} X_2$ is special Legendrian if and only if $\mathbf{w}$ is a (p,q)-twisted SL curve in $\mathbb{S}^3$. Similarly, let X and $\mathbf{w}$ be as in Proposition 2.12. If additionally, X is special Legendrian then the $\mathbf{w}$-twisted product $X_{\mathbf{w}}$ is special Legendrian if and only if $\mathbf{w}$ is a $(1,n-1)$-twisted SL curve in $\mathbb{S}^3$.*

The following characterization of (p,q)-twisted SL curves in $\mathbb{S}^3$ is central to the rest of this paper.

Lemma 2.19 ([4, Cor 1]). *Any curve $\mathbf{w}: I \to \mathbb{C}^2$ satisfying*

$$\overline{w}_1 \dot{w}_1 = -\overline{w}_2 \dot{w}_2 = (-1)^p \overline{w}_1^p \, \overline{w}_2^q, \quad |\mathbf{w}(0)| = 1, \tag{2.20}$$

is a (p,q)-twisted SL curve in $\mathbb{S}^3$. Conversely, any (p,q)-twisted SL curve in $\mathbb{S}^3$ containing no points with $w_1(t)=0$ or $w_2(t)=0$ admits a parameterization satisfying (2.20).

Proof. First, note that the Lagrangian phase $e^{i\theta_{\mathbf{w}}}$ of any Legendrian curve $\mathbf{w}$ in $\mathbb{S}^3$ can be expressed as

$$e^{i\theta_{\mathbf{w}}} = \frac{w_1\dot{w}_2 - \dot{w}_1 w_2}{|\dot{\mathbf{w}}|}, \tag{2.21}$$

since $\mathbf{w}$ has norm 1 and is Hermitian orthogonal to $\dot{\mathbf{w}}$.

Now suppose $\mathbf{w}$ is a curve in $\mathbb{C}^2$ satisfying (2.20). The real part of the equality $\overline{w}_1\dot{w}_1 + \overline{w}_2\dot{w}_2 = 0$ implies that $\frac{d}{dt}|\mathbf{w}|^2 = 0$, and hence $\mathbf{w}$ lies in $\mathbb{S}^3$. The imaginary part of the same equality implies that $\mathbf{w}$ is a Legendrian curve. Straightforward calculation using (2.20) shows that $\mathbf{w}$ satisfies

$$|\dot{\mathbf{w}}| = |w_1|^{p-1}|w_2|^{q-1} \tag{2.22}$$

and

$$w_1\dot{w}_2 - \dot{w}_1 w_2 = (-1)^{p-1}\overline{w}_1^{p-1}\overline{w}_2^{q-1}. \tag{2.23}$$

Combining (2.21), (2.22) and (2.23) it follows that the Lagrangian phase of $\mathbf{w}$ satisfies (2.17) as required.

For the converse, note that any Legendrian curve $\mathbf{w}$ in $\mathbb{S}^3$ satisfies the first and third equalities in (2.20), i.e., $\overline{w}_1\dot{w}_1 = -\overline{w}_2\dot{w}_2$ and $|\mathbf{w}(0)| = 1$. Also we can rewrite (2.17) as

$$e^{i\theta_{\mathbf{w}}} = (-1)^{p-1}\frac{\overline{w}_1^{p-1}\overline{w}_2^{q-1}}{|w_1|^{p-1}|w_2|^{q-1}},$$

and hence using (2.21) also as

$$\frac{w_1\dot{w}_2 - \dot{w}_1 w_2}{|\dot{\mathbf{w}}|} = (-1)^{p-1}\frac{\overline{w}_1^{p-1}\overline{w}_2^{q-1}}{|w_1|^{p-1}|w_2|^{q-1}}.$$

Now if we reparameterize $\mathbf{w}$ so that it satisfies (2.22) then from the previous equality we see that 2.17 is equivalent to equation (2.23). Multiplying (2.23) by $\overline{w}_1\overline{w}_2$ and using the fact that $\mathbf{w}$ satisfies $|\mathbf{w}|^2 = 1$ and $\overline{w}_1\dot{w}_1 = -\overline{w}_2\dot{w}_2$, we get the second equality of (2.20) as required. □

Remark 2.24. By changing the parameter t of the curve $\mathbf{w}$ to $-t$ if necessary one can always absorb the dimension-dependent sign $(-1)^p$ from (2.20) and therefore it suffices to study curves $\mathbf{w}$ in $\mathbb{S}^3$ satisfying

$$\overline{w}_1\dot{w}_1 = -\overline{w}_2\dot{w}_2 = \overline{w}_1^p\overline{w}_2^q,$$

with initial condition $|\mathbf{w}(0)| = 1$. Moreover, away from points where $w_1w_2 = 0$ these ODEs are equivalent to

$$\dot{w}_1 = \overline{w}_1^{p-1}\overline{w}_2^q, \quad \dot{w}_2 = -\overline{w}_1^p\overline{w}_2^{q-1}. \tag{2.25}$$

Equation (2.25) will be the most convenient form of the equations to use since it allows the cleanest treatment of the degenerate solutions where w_1 or w_2 can become zero.

Remark 2.26. If $\mathbf{w}$ is a (p,q)-twisted SL curve in $\mathbb{S}^3$ with $p > 1$, parameterized as in (2.20), then by combining (2.14) and (2.22) we see that when Σ_1 and Σ_2 are both closed

$$\operatorname{Vol}(X_1 *_{\mathbf{w}} X_2) = \operatorname{Vol}(X_1)\operatorname{Vol}(X_2)\int_I |\dot{\mathbf{w}}|^2\, dt. \tag{2.27}$$

Again the obvious analogue of (2.27) holds in the degenerate case $p = 1$. Therefore, there is a close relation between volume of special Legendrian twisted products and the *energy* of (p,q)-twisted SL curves in $\mathbb{S}^3$ when using the parameterization forced by (2.20).

Twisted products of contact stationary immersions. With very little extra effort one can also construct many Hamiltonian stationary cones in $\mathbb{C}^n$ or equivalently contact stationary submanifolds in $\mathbb{S}^{2n-1}$ via the twisted product construction.

To this end we define the following class of Legendrian curves in $\mathbb{S}^3$ generalizing (2.20).

Definition 2.28. We call a curve $\mathbf{w} : I \subset \mathbb{R} \to \mathbb{S}^3$ a (p,q)-twisted contact stationary (CS) curve if it satisfies the ODEs

$$\overline{w}_1\dot{w}_1 = -\overline{w}_2\dot{w}_2 = e^{i(a+bt)}\overline{w}_1^p\overline{w}_2^q, \quad t \in I, \tag{2.29}$$

for some a, $b \in \mathbb{R}$.

Remark 2.30. Note in the degenerate case $p = q = 1$ these ODEs occur as equation (7.1) in Schoen–Wolfson's work on the classification of 2D Hamiltonian stationary cones in $\mathbb{C}^2$ [33]. The system (2.29) is very simple to understand in this case because $\mathbf{w}$ satisfies a system of linear equations. Moreover, by direct differentiation of the equations for $\dot{w}_1$ and $\dot{w}_2$, w_1 and w_2 each satisfy autonomous second-order linear equations.

The reason for making this definition is the following

Lemma 2.31 (Contact stationary twisted products [4, Cor 3.2]). *Let X_1, X_2 and $\mathbf{w}$ be as in Proposition 2.9. If additionally X_1 and X_2 are both oriented contact stationary immersions and $\mathbf{w}$ is a (p,q)-twisted contact stationary curve then the $\mathbf{w}$-twisted product $X_1 *_{\mathbf{w}} X_2 : I \times \Sigma_1 \times \Sigma_2 \to \mathbb{S}^{2(p+q)-1}$ is also a contact stationary immersion away from points where w_1 or w_2 vanish. Moreover, if either X_1 or X_2 is contact stationary but not minimal Legendrian or if $\mathbf{w}$ is a (p,q)-twisted CS curve with $b \neq 0$ then $X_1 *_{\mathbf{w}} X_2$ is contact stationary but not minimal Legendrian.*

Similarly, let X and $\mathbf{w}$ be as in Proposition 2.12. If additionally, X is an oriented contact stationary immersion then the $\mathbf{w}$-twisted product $X_{\mathbf{w}}$ is an oriented contact stationary immersion if $\mathbf{w}$ is a $(1, n-1)$-twisted CS curve in $\mathbb{S}^3$.

Proof. The proof follows from Proposition 2.9 together with the characterization of contact stationary and minimal Legendrian submanifolds of $\mathbb{S}^{2n-1}$ in terms of harmonicity and constancy of the Lagrangian phase $\mathrm{e}^{\mathrm{i}\theta}$, respectively. The proof in the case $p = 1$ follows in the same way using Proposition 2.12 in place of 2.9. $\square$

Remark 2.32. Clearly, (2.20) is a special case of (2.29) where $a = p\pi$ and $b = 0$. If $\mathbf{w}$ is a solution of (2.29) with parameters (a, b) then for any constant $d \in \mathbb{R}$, $\mathbf{w}' = \mathrm{e}^{\mathrm{i}d}\mathbf{w}$ is another solution of (2.29) with parameters $(a', b') = (a + (p+q)d, b)$. Hence if $b = 0$ then by choosing d appropriately we can reduce (2.29) to (2.20). The analysis of (2.29) when $b \neq 0$ is more complicated than that of (2.20) because the system (2.29) is no longer autonomous. In this paper, we will analyse in great detail solutions of (2.20) and say almost nothing further about solutions of (2.29) with $b \neq 0$. However, following [4, equation (13)] we note that for any $c \in (0, \pi/2)$ the Legendrian curve $\mathbf{w} : \mathbb{R} \to \mathbb{S}^3$

$$\mathbf{w}(t) = (\cos c \exp(\mathrm{i}t \sin^p c \cos^{q-2} c), \sin c \exp(-\mathrm{i}t \sin^{p-2} c \cos^q c)), \quad t \in \mathbb{R} \tag{2.33}$$

satisfies (2.29) with $a = \pi/2$ and $b = \sin^{p-2} c \cos^{q-2} c\,(p \sin^2 s - q \cos^2 c)$. Clearly $b = 0$ if and only if $\tan^2 c = q/p$. (This special solution of (2.20) that has $|w_1|^2 \equiv \frac{p}{n}$ and $|w_2|^2 \equiv \frac{q}{n}$ corresponds to the solution $\mathbf{w}_\tau$ described in 5.1 with $|\tau| = \tau_{\max}$; for other values of c, b is nonzero and therefore (2.33) gives no further solutions of (2.20).)

The (p,q)-twisted CS curve (2.33) is closed if and only if $\tan^2 c \in \mathbb{Q}$. In particular given relatively prime positive integers m and n choose the unique value of $c_{m,n} \in (0, \pi/2)$ so that $\tan^2 c_{m,n} = m/n$, and therefore $\cos c_{m,n} = \sqrt{n/(m+n)}$, $\sin c_{m,n} = \sqrt{m/(m+n)}$. Hence, for each fixed (p,q) there is a countable infinite family of closed (p,q)-twisted CS curves $\mathbf{w}_{m,n}$ of the form (2.33) parameterized by the pair of relatively prime positive integers m and n. In the degenerate case when $p = q = 1$ these closed curves $\mathbf{w}_{m,n}$ are (up to a unitary transformation) nothing but the closed contact stationary curves $\gamma_{m,n}$ described in Schoen–Wolfson's work on Hamiltonian stationary cones in $\mathbb{C}^2$ [33].

Remark 2.34. Combining Lemma 2.31 and Remark 2.32 gives us two ways to construct contact stationary submanifolds that are not minimal Legendrian using the twisted product construction: (i) we take at least one of our initial immersions X_i to be contact stationary but not minimal Legendrian and $\mathbf{w}$ to be a (p,q)-twisted SL curve or (ii) we take the twisting Legendrian curve $\mathbf{w}$ to be a (p,q)-twisted CS curve of the form (2.33) with $\tan^2 c \neq q/p$. In the latter case we can allow both X_1 and X_2 to be special Legendrian, yielding a very simple method to generate higher-dimensional contact stationary immersions from a pair of lower-dimensional special Legendrians.

To construct special Legendrian or contact stationary immersions of the closed manifold $S^1 \times \Sigma_1 \times \Sigma_2$ from a pair of immersions of closed manifolds Σ_1 and Σ_2 we need (p,q)-twisted SL or CS curves that are closed. We call Legendrian immersions that arise this way, *closed twisted products*. For each fixed p and q Remark 2.32 exhibited a countably infinite family of closed (p,q)-twisted CS curves $\mathbf{w}_{m,n}$ parameterized by relatively prime positive integers m and n. Moreover, $\mathbf{w}_{m,n}$ is congruent to a (p,q)-twisted SL curve if and only if $m/n = p/q$.

We study closed (p,q)-twisted SL curves in Section 7 by analysing the periodicity conditions for solutions $\mathbf{w}$ of (2.20). We will prove the following result (Theorem 7.15)

For each admissible pair (p,q) of positive integers there exists a countable infinite family of distinct closed (p,q)-twisted SL curves in $\mathbb{S}^3$.

By the SL-twisted product construction of Corollary 2.18, Theorem 7.15 implies that every pair of closed special Legendrian submanifolds Σ_1 and Σ_2 in $\mathbb{S}^{2p-1}$ and $\mathbb{S}^{2q-1}$, respectively, gives rise to a countably infinite family of closed SL twisted products, i.e., special Legendrian immersions of

$S^1 \times \Sigma_1 \times \Sigma_2$ in $\mathbb{S}^{2p+2q-1}$. Similarly, by using closed $(1, n-1)$-twisted SL curves every closed special Legendrian submanifold Σ in $\mathbb{S}^{2n-3}$ gives rise to a countably infinite family of closed special Legendrian submanifolds in $\mathbb{S}^{2n-1}$ with topology $S^1 \times \Sigma$.

By combining the closed twisted product construction with existing constructions of closed special Legendrian immersions we generate a plethora of new closed special Legendrian and contact stationary immersions in essentially all dimensions. For example, we have the following result on topological types of SL and Hamiltonian stationary cones

Theorem A (Infinitely many topological types of SL and HS cones in $\mathbb{C}^n$ for $n \geq 4$).

(i) *For any $n \geq 4$ there are infinitely many topological types of SL cone in $\mathbb{C}^n$, each of which is diffeomorphic to the cone over a product $S^1 \times \Sigma'$ for some closed smooth manifold Σ', and each of which admits infinitely many distinct geometric representatives.*

(ii) *For any $n \geq 4$ there are infinitely many topological types of Hamiltonian stationary cone in $\mathbb{C}^n$, which are not minimal Lagrangian, each of which is diffeomorphic to the cone over a product $S^1 \times \Sigma'$ for some closed smooth manifold Σ', and each of which admits infinitely many distinct geometric representatives.*

Proof. In [13] we proved the existence of infinitely many special Legendrian surfaces in $\mathbb{S}^5$ of every odd genus (and also of genus 4). By Theorem 7.15 there is a countable infinite family of closed $(1, 3)$-twisted SL curves. Appealing to (2.18) using this infinite family of closed $(1, 3)$-twisted SL curves and the infinite number of topological types of SL surfaces in $\mathbb{S}^5$ described above we conclude that there are infinitely many topological types of special Legendrian 3-folds in $\mathbb{S}^7$ of the form $S^1 \times \Sigma$, where Σ is a oriented surface and that each topological type is realised by infinitely many distinct geometric representatives. To prove part (i) for any $n > 4$ we can keep iterating the process using the fact that by Theorem 7.15 for each $n \geq 3$ there is a countable infinite family of closed $(1, n-1)$-twisted SL curves. To prove (ii) we simply substitute Lemma 2.31 on CS-twisted products for Corollary 2.18 and Remark 2.32 for Theorem 7.15 and argue as before using our gluing results for SL surfaces in $\mathbb{S}^5$ as the starting point once again. □

We can also combine the twisted product construction with the SL 2-tori produced by integrable systems methods. McIntosh [28] proved that all SL 2-tori in $\mathbb{S}^5$ can be constructed by integrable systems methods and more specifically by so-called spectral curve methods. Using these methods Carberry–McIntosh [3] produced a very rich variety of special Legendrian 2-tori; in particular they proved the existence of appropriate SL spectral data in which the genus of the spectral curve genus can be any positive even integer. A simple consequence of their result is the remarkable fact that SL 2-tori can come in continuous families of arbitrarily high dimension, by choosing SL spectral data of higher and higher spectral curve genus. We can extend Carberry–McIntosh's result to every dimension and also to contact stationary tori of dimension at least 3 using the closed twisted product construction.

Theorem B (SL/CS tori in $\mathbb{S}^{2n-1}$ occur in families of arbitrarily high dimension).

(i) *For $n \geq 3$ there exist special Legendrian immersions of T^{n-1} in $\mathbb{S}^{2n-1}$ which come in continuous families of arbitrarily high dimension.*

(ii) *For $n \geq 4$ there exist contact stationary (and not minimal Legendrian) immersions of T^{n-1} in $\mathbb{S}^{2n-1}$ which come in continuous families of arbitrarily high dimension.*

Proof. (i) For $n = 3$ we simply appeal to the results of Carberry–McIntosh [3]. For $n = 4$ we use the $(1, 3)$-twisted SL product of a 2-torus coming from the Carberry–McIntosh construction and any closed $(1, 3)$-twisted SL curve. Clearly, the resulting twisted product depends continuously on the input 2-torus. Hence by Carberry–McIntosh's work for any $d \in \mathbb{N}$, we can find a special Legendrian immersion of $S^1 \times T^2$ which moves in a continuous family of dimension at least d. For $n = 5$, we use the $(2, 3)$-twisted product where $X_1 : \mathbb{S}^1 \to \mathbb{S}^3 \subset \mathbb{C}^2$ is the standard totally real equatorial circle, $X_2 : T^2 \to \mathbb{S}^5 \subset \mathbb{C}^3$ is a 2-torus coming from the Carberry–McIntosh construction and $\mathbf{w}$ is any closed $(2, 3)$-twisted SL curve. For $n \geq 6$ we use the twisted $(n-3, 3)$-twisted SL product where $X_1 : T^{n-3} \to \mathbb{S}^{2n-7}$ is the unique SL $n-3$ torus invariant under the diagonal subgroup $T^{n-3} \subset \mathrm{SU}(n-3)$, $X_2 : T^2 \to \mathbb{S}^5$ is a 2-torus coming from the Carberry–McIntosh construction and $\mathbf{w}$ is any closed $(n-3, 3)$-twisted SL curve. Part (ii) is proved in the same way using the twisted CS product construction 2.31 and the closed (p, q)-twisted CS curves exhibited in Remark 2.32. □

Finally, by combining the twisted product construction with both integrable systems constructions and our gluing methods we obtain the following striking results:

Theorem C.

(i) *For any $n \geq 6$, there are infinitely many topological types of SL cone in $\mathbb{C}^n$ of product type which can come in continuous families of arbitrarily high dimension.*

(ii) *For each $n \geq 6$, there are infinitely many topological types of Hamiltonian stationary cone in $\mathbb{C}^n$ of product type, which are not minimal Lagrangian and which can come in continuous families of arbitrarily high dimension.*

Proof. (i) Since $n-3 \geq 3$ by the gluing results of [13] and Theorem A(i) there are infinitely many topological types of SL $n-3$ fold in $\mathbb{S}^{2(n-3)-1}$. The result follows by applying the $(n-3,3)$-twisted SL product construction where X_1 is any of these SL $n-3$ folds, X_2 is a SL 2-torus coming from the Carberry–McIntosh construction and $\mathbf{w}$ is any closed $(n-3,3)$-twisted SL curve.

Part (ii) follows in the same way using the twisted CS product construction and the closed (p,q)-twisted CS curves exhibited in Remark 2.32. □

It is difficult to see how integrable systems methods or gluing methods alone could yield a result like Theorem C.

3. $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrians and (p,q)-twisted SL curves

In this section, we prove that every $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrian in $\mathbb{S}^{2(p+q)-1}$ arises from the special Legendrian twisted product construction (as described in (2.18)); hence the study of $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrians can be reduced to the study of the ODEs (2.19). In Section 4, we will begin an in-depth study of these ODEs.

Relation with work of other authors. $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant SL submanifolds of $\mathbb{C}^n$ are studied in [5, Section 3] and $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrian submanifolds of $\mathbb{S}^{2n-1}$ are studied in [4, Section 3]. The ODEs for $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrian submanifolds of $\mathbb{S}^{2(p+q)-1}$ appear in [5, Lemma 2] and [4, Cor 1]. However, Castro–Li–Urbano did not study closed $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrians in a systematic way.

$\mathrm{SO}(n-1)$-invariant special Legendrians (for $n>3$) were studied recently by Anciaux [1] from a slightly different point-of-view. Anciaux [1, Thm 2] gives the following nice geometric characterization of $\mathrm{SO}(n-1)$-invariant special Legendrians: any minimal Legendrian submanifold of $\mathbb{S}^{2n-1}$ which is foliated by round $n-2$ spheres is either a totally geodesic $\mathbb{S}^{n-1}$ or congruent to an $\mathrm{SO}(n-1)$-invariant special Legendrian. He goes on to study $\mathrm{SO}(n-1)$-invariant special Legendrians in $\mathbb{S}^{2n-1}$ noting that they arise from a Legendrian curve $\mathbf{w}$ in $\mathbb{S}^3$ satisfying (2.17) with $(p,q)=(1,n-1)$. Rather than working directly with this first-order condition and deriving an equation like (2.25) from it, Anciaux differentiates (2.17) and interprets the resulting second-order equation (see [1, equation (3)]) as an equation on the projected curve $\pi(\mathbf{w})\subset\mathbb{CP}^1$ where $\pi:\mathbb{S}^3\to\mathbb{CP}^1$ denotes the Hopf projection. Using this approach he proves the existence of a countable family of closed integral curves in $\mathbb{CP}^1$ and this suffices to prove the existence of closed minimal Lagrangian submanifolds of $\mathbb{CP}^{n-1}$ (see [1, Thm 3]). However, the horizontal lift to $\mathbb{S}^3$ of a closed integral curve in $\mathbb{CP}^1$ is not necessarily closed. In Anciaux's approach an additional period condition must be satisfied for the spherical lift to be closed and because of this his method does not prove the existence of suitable closed curves in $\mathbb{S}^3$ (see his discussion in following Theorem 3).

The key to overcoming this period problem is to work directly with the first-order system (2.25) rather than the second-order system that Anciaux exploits. This approach allows us to prove the existence of countable infinitely many closed (p,q)-twisted special Legendrian curves in $\mathbb{S}^3$ for general p and q. For our gluing constructions [11, 12, 14] it is crucial that we have closed $\mathrm{SO}(p)\times\mathrm{SO}(q)$-invariant special Legendrians at our disposal.

$\mathrm{SO}(2)\times\mathrm{SO}(2)$-invariant SL cones in $\mathbb{C}^4$ can be constructed in a different manner, namely as a special case of Joyce's work on T^{n-2}-invariant SL cones in $\mathbb{C}^n$. To obtain this $\mathrm{SO}(2)\times\mathrm{SO}(2)$ action we should set $n=4$ and take $a_1=a_2=-1,\ a_3=a_4=1$ in [17, Prop. 7.6]. Among all T^2-actions allowed in Joyce's constructions, the $\mathrm{SO}(2)\times\mathrm{SO}(2)$ action is distinguished by having the largest fixed point set.

Isotropic orbits of the $\mathbf{SO}(p)\times\mathbf{SO}(q)$ action on $\mathbb{C}^{p+q}$. As previously we assume that (p,q) is an admissible pair of positive integers, i.e., $p\le q$, $q\ge 2$ and $p\ge 1$, and we set $n=p+q$.

$\mathrm{SO}(p)\times\mathrm{SO}(q)$ acts via isometries on $\mathbb{C}^{p+q}\cong\mathbb{C}^p\times\mathbb{C}^q$ via the product of the standard complex linear actions of $\mathrm{SO}(p)$ and $\mathrm{SO}(q)$ on the $\mathbb{C}^p$ and $\mathbb{C}^q$ factors, respectively. Since $\mathrm{SO}(p)\times\mathrm{SO}(q)\subset\mathrm{SO}(p+q)\subset\mathrm{SU}(n)$ it is natural to look for $\mathrm{SO}(p)\times\mathrm{SO}(q)$-invariant SLs in $\mathbb{C}^{p+q}$ and in particular for SL cones or equivalently special Legendrian submanifolds of $\mathbb{S}^{2n-1}$ invariant under $\mathrm{SO}(p)\times\mathrm{SO}(q)$. If a Legendrian submanifold of $\mathbb{S}^{2n-1}$ is a union of orbits then each orbit $\mathcal{O}$ must be γ-isotropic, i.e., $\gamma|_{\mathcal{O}}=0$, where $\gamma=\iota_X\omega|_{\mathbb{S}^{2n-1}}$ is the standard contact form on $\mathbb{S}^{2n-1}$ (here ω and X denote the standard symplectic form and radial vector field on $\mathbb{C}^n$, respectively). The following simple lemma describes the γ-isotropic orbits $\mathcal{O}$ of $\mathrm{SO}(p) \times \mathrm{SO}(q)$ in $\mathbb{S}^{2n-1}$.

Lemma 3.1 (Isotropic orbits of SO$(p) \times$ SO(q)).

(i) *If $p \geq 2$, $q \geq 2$ then any γ-isotropic $SO(p) \times SO(q)$ orbit $\mathcal{O} \subset \mathbb{S}^{2(p+q)-1}$ has the form*

$$\mathcal{O}_{\mathbf{w}} = (w_1 \cdot \mathbb{S}^{p-1},\, w_2 \cdot \mathbb{S}^{q-1}) \tag{3.2}$$

for some $\mathbf{w} = (w_1, w_2) \in \mathbb{S}^3$. Moreover, if $\mathbf{w}$ and $\mathbf{w}' \in \mathbb{S}^3$ then $\mathcal{O}_{\mathbf{w}} = \mathcal{O}_{\mathbf{w}'}$ if and only if $\mathbf{w}' = \rho_{jk}\mathbf{w}$ for some $(j,k) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ where $\rho : \mathbb{Z}_2 \times \mathbb{Z}_2 \to O(2) \subset U(2)$ is the homomorphism defined by

$$(j,k) \mapsto \rho_{jk} := \begin{pmatrix} (-1)^j & 0 \\ 0 & (-1)^k \end{pmatrix}.$$

In particular, spherical isotropic $SO(p) \times SO(q)$ orbits are in one-to-one correspondence with points in $\mathbb{S}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$.

(ii) *Similarly, for $n \geq 3$ any γ-isotropic $SO(n-1)$ orbit $\mathcal{O} \subset \mathbb{S}^{2n-1}$ has the form*

$$\mathcal{O}_{\mathbf{w}} = (w_1,\, w_2 \cdot \mathbb{S}^{n-2}) \tag{3.3}$$

for some $\mathbf{w} = (w_1, w_2) \in \mathbb{S}^3$. Moreover, if $\mathbf{w}$ and $\mathbf{w}' \in \mathbb{S}^3$ then $\mathcal{O}_{\mathbf{w}} = \mathcal{O}_{\mathbf{w}'}$ if and only if $\mathbf{w}' = \rho_{jk}\mathbf{w}$ for $(j,k) \in \langle(+-)\rangle \cong \mathbb{Z}_2 \leqslant \mathbb{Z}_2 \times \mathbb{Z}_2$. In particular, isotropic $SO(n-1)$ orbits in $\mathbb{S}^{2n-1}$ are in one-to-one correspondence with points in $\mathbb{S}^3/\mathbb{Z}_2$, where $\mathbb{Z}_2 = \langle \rho_{+-} \rangle$.

Proof. We begin with a more general result that applies to isotropic orbits of any connected Lie subgroup of $\mathrm{SU}(n)$. Let G be any connected Lie subgroup of $\mathrm{SU}(n)$, $\mathfrak{g}$ denote the Lie algebra of G and x be any point in $\mathbb{S}^{2n-1}$. Then the orbit $\mathcal{O}_x := G \cdot x$ is contained in $\mathbb{S}^{2n-1}$ and is γ-isotropic if and only if $\gamma(v) = 0$ for all $v \in T_y\mathcal{O}_x$ and $y \in \mathcal{O}_x$. By homogeneity it suffices to check this at x. But since $\mathcal{O}_x$ is a G-orbit we have $T_x\mathcal{O}_x = \mathfrak{g} \cdot x$. Therefore, $\mathcal{O}_x$ is isotropic if and only if $\gamma_x(\mathfrak{g} \cdot x) = 0$. Hence, using the definition of the standard contact form γ on $\mathbb{S}^{2n-1}$ we see that $\mathcal{O}_x$ is isotropic if and only if

$$\operatorname{Im} \langle x, Ax \rangle = 0, \quad \text{for any } A \in \mathfrak{g}, \tag{3.4}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard Hermitian inner product on $\mathbb{C}^n$. In the language of moment maps, (3.4) is equivalent to the condition $x \in \mu^{-1}(\mathbf{0})$, where $\mu : \mathbb{C}^n \to \mathfrak{g}^*$ is the moment map associated to the action of $G \subset \mathrm{SU}(n)$. (For the definition and basic properties of the moment map we refer the reader to Section 4 of [17].)

Specializing to $G = \mathrm{SO}(p) \times \mathrm{SO}(q)$ and $\mathfrak{g} = \mathfrak{so}(p) \times \mathfrak{so}(q)$ (with $p \geq 2$ and $q \geq 2$) we have $\mathcal{O}_x$ is isotropic if and only if

$$\operatorname{Im}\langle x, Ax\rangle = 0, \quad \text{for any } A \in \mathfrak{so}(p) \times \mathfrak{so}(q). \tag{3.5}$$

To analyse (3.5), decompose $x = (x', x'') \in \mathbb{C}^p \times \mathbb{C}^q$ and $A = (A', A'') \in \mathfrak{so}(p) \times \mathfrak{so}(q)$. By considering $x = (x', 0)$ and $A = (A', 0)$ or $x = (0, x'')$ and $A = (0, A'')$ we find it is equivalent to

$$\operatorname{Im}\langle x', A'x'\rangle = \operatorname{Im}\langle x'', A''x''\rangle = 0, \qquad \text{for all } A' \in \mathfrak{so}(p),\ A'' \in \mathfrak{so}(q). \tag{3.6}$$

One can check that $\operatorname{Im}\langle z, Az\rangle = 0$ for all $A \in \mathfrak{so}(m)$ if and only if $z \in \mathbb{C}^m$ has the form $z \in w \cdot \mathbb{S}^{m-1}$ for some $w \in \mathbb{C}$. Applying this to (3.6) twice (for different values of m) we obtain $x' \in w_1 \cdot \mathbb{S}^{p-1}$ and $x'' \in w_2 \cdot \mathbb{S}^{q-1}$ for some $\mathbf{w} = (w_1, w_2) \in \mathbb{C}^2$. But since $\mathcal{O}_x \subset \mathbb{S}^{2(p+q)-1}$ we have $\mathbf{w} \in \mathbb{S}^3$ and hence (3.2) follows. It is straightforward to verify the conditions on $\mathbf{w}$ and $\mathbf{w}'$ under which the orbits $\mathcal{O}_{\mathbf{w}}$ and $\mathcal{O}_{\mathbf{w}'}$ coincide are as stated.

The proof of Lemma 3.1 for $\mathrm{O}(n-1)$ is a minor modification of the proof above and therefore we omit it; the main difference is the condition under which two orbits $\mathcal{O}_{\mathbf{w}}$ and $\mathcal{O}_{\mathbf{w}'}$ coincide. □

By Lemma 3.1 the generic γ-isotropic orbit of $\mathrm{SO}(p) \times \mathrm{SO}(q)$ has dimension $n-2$ and therefore we can look for $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrians that are curves of $\mathrm{SO}(p) \times \mathrm{SO}(q)$ orbits, and these curves will satisfy some first-order system of ODEs.

$\mathbf{SO}(p) \times \mathbf{SO}(q)$-invariant special Legendrians and (p,q)-twisted SL curves. An immediate consequence of Lemma 3.1 is that all $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant Legendrian submanifolds of $\mathbb{S}^{2(p+q)-1}$ arise from the twisted product construction of 2.1.

Corollary 3.7 ($\mathbf{SO}(p) \times \mathbf{SO}(q)$-invariant Legendrians are twisted products).

(i) *For $p \geq 2$, $q \geq 2$ a Legendrian immersion $Y : \Sigma \to \mathbb{S}^{2(p+q)-1}$ is SO(p) × SO(q)-invariant if and only if Y is locally congruent to a twisted product $X_1 *_{\mathbf{w}} X_2$ where $X_1 : \mathbb{S}^{p-1} \to \mathbb{S}^{2p-1}$ and $X_2 : \mathbb{S}^{q-1} \to \mathbb{S}^{2q-1}$ are the standard totally geodesic special Legendrian embeddings.*

(ii) *If* $p = 1$ *a Legendrian immersion* $Y : \Sigma \to \mathbb{S}^{2n-1}$ *is* $SO(n-1)$*-invariant if and only if* Y *is locally congruent to a (degenerate) twisted product* $X_{\mathbf{w}}$ *(as defined in (2.4)) where the immersion* $X : \mathbb{S}^{n-2} \to \mathbb{S}^{2n-3}$ *is the standard totally geodesic special Legendrian embedding.*

In particular, by combining Corollary 3.7 with Corollary 2.18 we obtain

Corollary 3.8 (SO(p) $\times$ SO(q)-invariant special Legendrians and (p, q)-twisted SL curves).

(i) *For* $p,\ q \geq 2$ *any* $SO(p) \times SO(q)$*-invariant special Legendrian immersion is locally congruent to a twisted product with* X_1 *and* X_2 *as in 3.7, where the twisting curve* $\mathbf{w}$ *is a* (p, q)*-twisted SL curve in* $\mathbb{S}^3$*.*

(ii) *For* $p = 1$ *any* $SO(n-1)$*-invariant special Legendrian immersion is locally congruent to a (degenerate) twisted product with* $X : \mathbb{S}^{n-2} \to \mathbb{S}^{2n-3}$ *the standard totally geodesic Legendrian embedding and* $\mathbf{w}$ *a* $(1, n-1)$*-twisted SL curve in* $\mathbb{S}^3$*.*

Corollary 3.7 appears in Castro–Li–Urbano in the statement of Thm 3.1 [4]. Note, however, the assumption $p, q \geq 3$ made in their statement can be relaxed as in our statement. We could also derive these results about SO(p) $\times$ SO(q)-invariant special Legendrians using the methods Joyce developed to study cohomogeneity one SLs and special Legendrians [17].

4. The fundamental ODE system for (p, q)-twisted SL curves

Given an admissible pair of integers p and q (i.e., satisfying $1 \leq p \leq q$ and $q \geq 2$) we set $n = p + q$. This section studies the first order system of complex ODEs (2.25) governing (appropriately parametrized) (p, q)-twisted SL curves. The central result in this section is Proposition 4.26 which establishes a normal form for any solution $\mathbf{w}$ of (2.25) up to the action of certain obvious symmetries. We use (4.26) in Section 5 to define a particular 1-parameter family $\mathbf{w}_\tau$ of (p, q)-twisted SL curves and the associated 1-parameter family of SO(p) $\times$ SO(q)-invariant special Legendrian immersions X_τ. Up to symmetry, every (p, q)-twisted SL curve is equivalent to $\mathbf{w}_\tau$ for some τ.

We begin by discussing the symmetries of (2.25). For any p and q the (p, q)-twisted SL ODEs (2.25) have six obvious types of symmetries:

(1) Time translation invariance $\mathbf{w} \mapsto \mathbf{w} \circ \mathsf{T}_{t_0}$ for any $t_0 \in \mathbb{R}$.

(2) Multiplication by an nth root of unity $\mathbf{w} \mapsto z\mathbf{w}$, where $z^n = 1$.

(3) $\mathbf{w} \mapsto \hat{\mathsf{T}}_x \circ \mathbf{w}$ where $\hat{\mathsf{T}}_x \in \mathrm{U}(1) \times \mathrm{U}(1) \subset \mathrm{U}(2)$ is the 1-parameter subgroup (depending on p and q)

$$\hat{\mathsf{T}}_x = \begin{pmatrix} \mathrm{e}^{\mathrm{i}x/p} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}x/q} \end{pmatrix}. \tag{4.1}$$

(4) Complex conjugation $\mathbf{w} \mapsto \overline{\mathbf{w}}$.

(5) The simultaneous time reflection and spatial rotation given by

$$t \mapsto -t, \quad \mathbf{w} \mapsto z\mathbf{w},$$

where z is any nth root of -1.

(6) The simultaneous time and spatial rescaling given by

$$t \mapsto \lambda^{1-2/n} t, \quad \mathbf{w} \mapsto \lambda^{1/n}\mathbf{w}, \quad \text{for any } \lambda > 0.$$

More precisely, $\mathbf{w}$ is a solution of (2.25) if and only if $\mathbf{w}_\lambda(t) := \lambda^{1/n}\mathbf{w}(\lambda^{1-2/n}t)$ is.

Before establishing the basic facts about solutions to the (p, q)-twisted SL ODES we discuss the geometry of the 1-parameter group of symmetries $\{\hat{\mathsf{T}}_x\}_{x\in\mathbb{R}}$ (which depends on p and q) appearing in symmetry (3) above. As in 3.1, for any $\mathbf{w} \in \mathbb{S}^3$ let $\mathcal{O}_{\mathbf{w}} \subset \mathbb{S}^{2(p+q)-1}$ denote the associated isotropic $\mathrm{SO}(p) \times \mathrm{SO}(q)$ orbit.

Definition 4.2. For fixed integers p and q define a period of the 1-parameter group $\{\hat{\mathsf{T}}_x\}$ by

$$\mathrm{Per}(\{\hat{\mathsf{T}}_x\}) := \{x \in \mathbb{R} \,|\, \hat{\mathsf{T}}_x = \mathrm{Id}\}.$$

Clearly, if $x \in \mathrm{Per}(\{\hat{\mathsf{T}}_x\})$ then $\mathcal{O}_{\hat{\mathsf{T}}_x\mathbf{w}} = \mathcal{O}_{\mathbf{w}}$ for any $\mathbf{w} \in \mathbb{S}^3$. In other words, for any $x \in \mathrm{Per}(\{\hat{\mathsf{T}}_x\})$, $\hat{\mathsf{T}}_x$ leaves invariant all isotropic $\mathrm{SO}(p) \times \mathrm{SO}(q)$ orbits in $\mathbb{S}^{2(p+q)-1}$.

More generally, we call $x \in \mathbb{R}$ a half-period of $\{\hat{\mathsf{T}}_x\}$ if $\hat{\mathsf{T}}_x$ leaves invariant all isotropic $\mathrm{SO}(p) \times \mathrm{SO}(q)$ orbits in $\mathbb{S}^{2(p+q)-1}$. In other words,

$$\mathrm{Per}_{\frac{1}{2}}(\{\hat{\mathsf{T}}_x\}) := \{x \in \mathbb{R} \,|\, \mathcal{O}_{\hat{\mathsf{T}}_x\mathbf{w}} = \mathcal{O}_{\mathbf{w}} \ \forall\ \mathbf{w} \in \mathbb{S}^3\}.$$

A half-period of $\{\hat{\mathsf{T}}_x\}$ that is not a period of $\{\hat{\mathsf{T}}_x\}$ we call a strict half-period of $\{\hat{\mathsf{T}}_x\}$.

Define the finite subgroup $\mathrm{Stab}_{p,q} \subset \mathrm{U}(2)$ by

$$\mathrm{Stab}_{p,q} = \begin{cases} \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \cong \mathbb{Z}_2 \times \mathbb{Z}_2, & \text{if } p > 1; \\ \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \cong \mathbb{Z}_2, & \text{if } p = 1. \end{cases} \tag{4.3}$$

It follows from (3.1) that

$$x \in \mathrm{Per}_{\frac{1}{2}}(\{\hat{\mathsf{T}}_x\}) \iff \hat{\mathsf{T}}_x \in \mathrm{Stab}_{p,q}. \tag{4.4}$$

An immediate consequence of (4.4) is that $2\,\mathrm{Per}_{\frac{1}{2}}(\{\hat{\mathsf{T}}_x\}) \subset \mathrm{Per}(\{\hat{\mathsf{T}}_x\})$; this explains the choice of the terminology half-period. If $\hat{\mathsf{T}}_x = \rho_{jk}$ for some $(j,k) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ with $\rho_{jk} \in \mathrm{Stab}_{p,q}$ as defined in 3.1 we call x *a half-period of type* (jk). If x is a half-period of type (jk) then $\mathrm{e}^{\mathrm{i}x} = (-1)^{jp}$ and $\mathrm{e}^{-\mathrm{i}x} = (-1)^{kq}$ and hence $jp + kq \equiv 0 \mod 2$.

The following lemma describes the periods and half-periods of the 1-parameter group $\{\hat{\mathsf{T}}_x\}$.

Lemma 4.5. *Fix a pair of admissible integers p and q and let $\{\hat{\mathsf{T}}_x\}$ denote the 1-parameter subgroup defined in (4.1).*

(i) *The periods of $\{\hat{\mathsf{T}}_x\}$ are given by*

$$\mathrm{Per}(\{\hat{\mathsf{T}}_x\}) = 2\pi \operatorname{lcm}(p,q)\mathbb{Z}.$$

(ii) *If $p > 1$ then the half-periods of $\{\hat{\mathsf{T}}_x\}$ are given by*

$$\mathrm{Per}_{\frac{1}{2}}(\{\hat{\mathsf{T}}_x\}) = \tfrac{1}{2}\,\mathrm{Per}(\{\hat{\mathsf{T}}_x\}) = \pi \operatorname{lcm}(p,q)\mathbb{Z}.$$

Moreover, any strict half-period of $\{\hat{\mathsf{T}}_x\}$ is of type (jk) where $j = q/\operatorname{hcf}(p,q) \mod 2$ and $k = p/\operatorname{hcf}(p,q) \mod 2$. In particular, for any fixed p and q exactly one type of strict half-period occurs.

(iii) *If $p = 1$ then the half-periods of $\{\hat{\mathsf{T}}_x\}$ are given by*

$$\mathrm{Per}_{\frac{1}{2}}(\{\hat{\mathsf{T}}_x\}) = \begin{cases} \frac{1}{2}\,\mathrm{Per}(\{\hat{\mathsf{T}}_x\}) = \pi \operatorname{lcm}(p,q)\mathbb{Z}, & \textit{if } n \textit{ is odd;} \\ \mathrm{Per}(\{\hat{\mathsf{T}}_x\}) = 2\pi \operatorname{lcm}(p,q)\mathbb{Z}, & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. The proof is a straightforward use of the various definitions, the case $p = 1$ being different because $\mathrm{Stab}_{p,q}$ (defined in (4.3)) is defined differently in this case. $\square$

Remark 4.6. Notice that for j and k defined in 4.5 $jp+kq=2pq/\operatorname{hcf}(p,q)\equiv 0 \mod 2$ as required.

We have the following basic facts about solutions to the (p,q)-twisted SL ODEs.

Proposition 4.7. *(cf. equation (2.25) and Remark 2.24)*

(i) *Solutions to the (p,q)-twisted SL ODEs*

$$\begin{aligned} \dot{w}_1 &= \overline{w}_1^{p-1}\overline{w}_2^{q}, \\ \dot{w}_2 &= -\overline{w}_1^{p}\overline{w}_2^{q-1}, \end{aligned} \tag{4.8}$$

admit two conserved quantities

$$\mathcal{I}_1(\mathbf{w}) := |\boldsymbol{w}|^2 \quad \text{and} \quad \mathcal{I}_2(\mathbf{w}) := \operatorname{Im}(w_1^p w_2^q).$$

The symmetries (1), (2) and (3) preserve both conserved quantities $\mathcal{I}_1$ and $\mathcal{I}_2$. Symmetries (4) and (5) preserve $\mathcal{I}_1$ but send $\mathcal{I}_2 \mapsto -\mathcal{I}_2$. Symmetry (6) sends $(\mathcal{I}_1,\mathcal{I}_2) \mapsto (\lambda^{2/n}\mathcal{I}_1, \lambda\mathcal{I}_2)$. Hence if $\mathbf{w}$ is a solution of (4.8) with $\mathcal{I}_1(\mathbf{w}) \neq 0$ then we may rescale using symmetry (6) to obtain another solution of (4.8) with $\mathcal{I}_1(\mathbf{w}) = 1$. For any solution with $\mathcal{I}_1(\boldsymbol{w}) = 1$, the possible range of values of $\mathcal{I}_2 = \operatorname{Im}(w_1^p w_2^q)$ is $[-2\tau_{max}, 2\tau_{max}]$, where

$$2\tau_{max} = \sqrt{\frac{p^p q^q}{n^n}}. \tag{4.9}$$

(ii) *The stationary points of (4.8) are*

$$\mathbb{C}\times\{0\}\cup\{0\}\times\mathbb{C} \quad \text{if } p>1 \quad \text{or} \quad \mathbb{C}\times\{0\}, \quad \text{if } p=1.$$

(iii) *The initial value problem for (4.8) with any initial data $\boldsymbol{w}(0)\in\mathbb{C}^2$ has a unique real analytic solution $\boldsymbol{w}:\mathbb{R}\to\mathbb{C}^2$ defined for all $t\in\mathbb{R}$, which depends real analytically on the initial data.*

(iv) *For any solution of (4.8) with $\mathcal{I}_1(\mathbf{w}) = 1$ and $\mathcal{I}_2(\mathbf{w}) = \operatorname{Im}(w_1^p w_2^q) = -2\tau$ (and hence by part (i) $\tau\in[-\tau_{max},\tau_{max}]$) the function $y := |w_2|^2$:*

$\mathbb{R} \to [0,1]$ *satisfies the equation*

$$\frac{1}{2}\dot{y} + 2i\tau = -w_1^p w_2^q. \tag{4.10}$$

Therefore y satisfies the energy conservation equation

$$\dot{y}^2 = 4(f(y) - 4\tau^2) = 4y^q(1-y)^p - 16\tau^2, \tag{4.11}$$

and hence also the second-order ODE

$$\ddot{y} = 2f'(y) = 2y^{q-1}(1-y)^{p-1}(q - ny), \tag{4.12}$$

where we define the function $f : \mathbb{R} \to \mathbb{R}$ (depending on p and q) by

$$f(y) = y^q(1-y)^p. \tag{4.13}$$

Remark 4.14. The difference between the stationary points of (4.8) in the case $p > 1$ and the case $p = 1$ reflects the difference in the geometry of the nongeneric isotropic orbits of $\mathrm{SO}(p) \times \mathrm{SO}(q)$ and $\mathrm{SO}(n-1)$, respectively. For $p > 1$ the nongeneric isotropic orbits of $\mathrm{SO}(p) \times \mathrm{SO}(q)$ have the form $(w_1 \cdot \mathbb{S}^{p-1}, 0)$ and $(0, w_2 \cdot \mathbb{S}^{q-1})$. For $p = 1$ the only nongeneric isotropic orbits are of the form $(w_1, 0)$. In particular, the orbits of the form $(0, w_2 \cdot \mathbb{S}^{n-2})$ are generic provided $w_2 \neq 0$.

Proof. (i) Conserved quantities. We verify $\mathcal{I}_1$ and $\mathcal{I}_2$ are conserved by direct calculation. Firstly,

$$\dot{\mathcal{I}}_1 = \tfrac{d}{dt}|\mathbf{w}|^2 = \tfrac{d}{dt}(w_1\overline{w}_1) + \tfrac{d}{dt}(w_2\overline{w}_2) = 2\,\mathrm{Re}\,(\dot{w}_1\overline{w}_1 + \dot{w}_2\overline{w}_2) = 0,$$

where we have used (4.8) in the final equality. Secondly, since

$$\tfrac{d}{dt}(w_1^p w_2^q) = p w_1^{p-1} w_2^q \dot{w}_1 + q w_1^p w_2^{q-1} \dot{w}_2,$$

using (4.8) we obtain

$$\frac{d}{dt}(w_1^p w_2^q) = |w_1|^{2p-2}|w_2|^{2q-2}(p|w_2|^2 - q|w_1|^2) \in \mathbb{R}^{\cdot}. \tag{4.15}$$

Hence $\frac{d}{dt}\mathcal{I}_2 = \frac{d}{dt}\,\mathrm{Im}\,(w_1^p w_2^q) = 0$. It is straightforward to check the action of the symmetries on $\mathcal{I}_1$ and $\mathcal{I}_2$ is as claimed. Define $y = |w_2|^2$. When $\mathcal{I}_1(\mathbf{w}) = 1$

$$|\mathcal{I}_2(\mathbf{w})| = |\mathrm{Im}\, w_1^p w_2^q| \leq |w_1|^p |w_2|^q = \sqrt{y^q(1-y)^p} = \sqrt{f(y)},$$

for the function f defined in (4.13). A short calculation shows that

$$f'(y) = y^{q-1}(1-y)^{p-1}(q-ny), \tag{4.16}$$

and therefore the critical points of f are

$$\operatorname{Crit}(f) = \begin{cases} \{0,\ \frac{q}{n},\ 1\}, & \text{if } p > 1, \\ \{0,\ \frac{q}{n}\}, & \text{if } p = 1. \end{cases} \tag{4.17}$$

Since f is non-negative on $[0,1]$ and vanishes only at the two endpoints, the maximum value of f for $y \in [0,1]$ occurs when $y = \frac{q}{n}$ and hence

$$f_{\max} = f(\tfrac{q}{n}) = \frac{p^p q^q}{n^n} = 4\tau_{\max}^2,$$

where $\tau_{\max}$ is defined in (4.9). Hence $|\mathcal{I}_2(\mathbf{w})| \leq \sqrt{f_{\max}} \leq 2\tau_{\max}$ as claimed. See figure 1 for the graph of the function f on the interval $[0,1]$ for various choices of (p,q).

(ii) Stationary points. Stationary points of (4.8) are given by common zeros of the two polynomials

$$w_1^{p-1}w_2^q = 0 = w_1^p w_2^{q-1}. \tag{4.18}$$

(iii) Global existence, uniqueness and analyticity. The vector field

$$V(\mathbf{w}) = (\overline{w}_1^{p-1}\overline{w}_2^q, \overline{w}_1^p\overline{w}_2^{q-1})$$

on $\mathbb{C}^2$ defining (4.8) is clearly real algebraic. It follows then from the standard local existence and uniqueness results for the initial value problem that locally (4.8) admits a unique real analytic solution for any initial data and this solution depends real analytically on the initial condition. Since $\mathcal{I}_1(\mathbf{w}) = |\mathbf{w}|^2$ is constant, this local solution remains in a compact subset of $\mathbb{C}^2$ and hence global existence follows immediately.

(iv) ODEs for $y := |w_2|^2$. Using (4.8), we have

$$\begin{aligned} \dot{y} &= 2\operatorname{Re}(\dot{w}_2\overline{w}_2) = -2\operatorname{Re}(\overline{w}_1^p\overline{w}_2^q) = -2\operatorname{Re}(w_1^p w_2^q), \\ 2\tau &= \operatorname{Im}(\overline{w}_1\dot{w}_1) = \operatorname{Im}(\overline{w}_1^p\overline{w}_2^q) = -\operatorname{Im}(w_1^p w_2^q). \end{aligned}$$

Hence we obtain (4.10). Taking the modulus squared of both sides of (4.10) proves that $\dot{y}$ satisfies (4.11). Differentiating (4.10) with respect to t and using (4.15) we see that y satisfies the second-order equation (4.12). Note

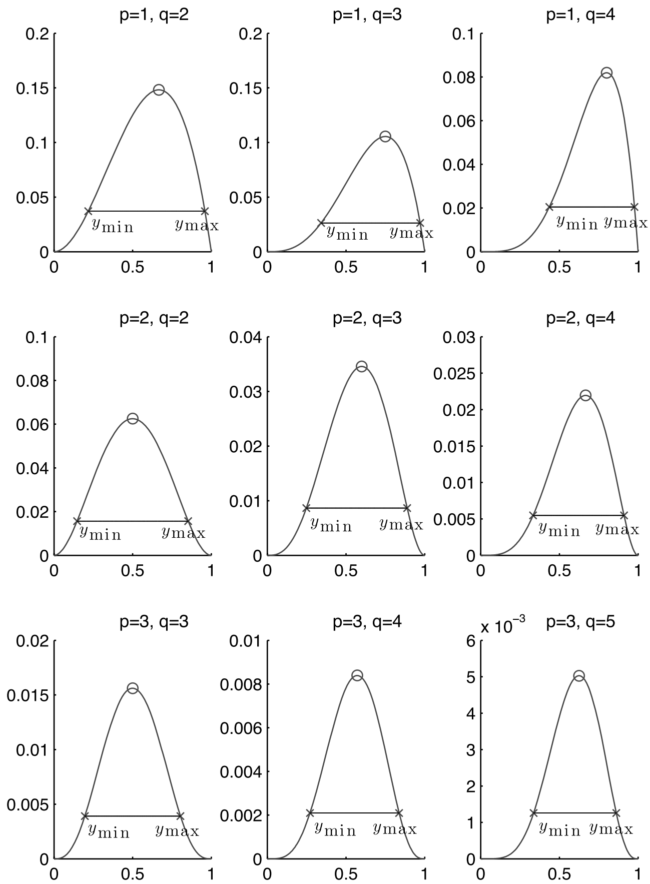

Figure 1: The graph of $f(y) = y^q(1-y)^p$ on the interval $[0,1]$ for various choices of (p,q). $y_{\min}$ and $y_{\max}$ — the two solutions of $f(y) = 4\tau^2$ in the interval $[0,1]$ — are shown for $\tau = \frac{1}{2}\tau_{\max}$. The maximum value $f_{\max} = 4\tau_{\max}^2$ which occurs at $y = \frac{q}{n}$ is marked by $\circ$.

that the stationary points of (4.12) are exactly the critical points of f and hence by (4.17) are 0 and $\frac{q}{n}$ when $p = 1$ and 0, $\frac{q}{n}$ and 1 when $p > 1$. $\square$

To understand the space of solutions to (4.8) modulo the action of the symmetries (1)–(6) we need the following auxiliary result about solutions of (4.11):

Lemma 4.19. *Let* $\mathbf{w}$ *be any solution of (4.8) with* $\mathcal{I}_1(\mathbf{w}) = 1$ *and* $\mathcal{I}_2(\mathbf{w}) = \operatorname{Im}(w_1^p w_2^q) = -2\tau$ *and let* $y := |w_2|^2 : \mathbb{R} \to [0,1]$ *be the associated solution of 4.11.*

(i) *If* $0 < |\tau| < \tau_{max}$*, the following holds:*

a. y *is periodic of period* $2\mathsf{p}_\tau > 0$ *and hence any two solutions of (4.11) with the same value of* τ *differ only by a time translation. Moreover, the period* p_τ *satisfies*

$$\lim_{\tau\to\tau_{max}} 2\mathsf{p}_\tau = \frac{\pi}{\tau_{max}}\sqrt{\frac{pq}{2n^3}}. \tag{4.20}$$

b. *The range of* y *is* $[y_{min}, y_{max}]$*, where* $0 < y_{min} < \frac{q}{n} < y_{max} < 1$ *are the only two solutions of the degree* n *polynomial equation*

$$f(y) = y^q(1-y)^p = 4\tau^2, \tag{4.21}$$

that lie in the interval $[0,1]$.

c. *As* $\tau \to 0$ *we have*

$$y_{min} = (2\tau)^{2/q}(1 + O(\tau^{2/q})), \quad y_{max} = 1 - (2\tau)^{2/p}(1 + O(\tau^{2/p})). \tag{4.22}$$

(ii) *If* $|\tau| = \tau_{max}$*, then* $y \equiv \frac{q}{n}$.

(iii) *If* $\tau = 0$ *and* $p > 1$ *then one of the following holds:*

a. $y \equiv 0$

b. $y \equiv 1$

c. y *is strictly monotone and satisfies*

$$y = \begin{cases} y_0 \circ \mathsf{T}_{t_0}, & \textit{some } \ t_0 \in \mathbb{R}; \quad \textit{if } y \textit{ is decreasing}, \\ y_0 \circ \mathsf{T}_{t_0} \circ \underline{\mathsf{T}}, & \textit{some } \ t_0 \in \mathbb{R}; \quad \textit{if } y \textit{ is increasing}, \end{cases}$$

where $y_0 : \mathbb{R} \to (0,1)$ *denotes the unique (decreasing) solution to the initial value problem*

$$\dot{y} = -2\sqrt{f(y)}, \quad y(0) = \frac{q}{n}.$$

Alternatively, y_0 *can be characterised as the unique solution to (4.12) with initial conditions*

$$y(0) = \frac{q}{n}, \quad \dot{y}(0) = -4\tau_{max}.$$

Moreover, y_0 *satisfies*

$$\lim_{t\to-\infty} y_0(t) = 1 \quad \textit{and} \quad \lim_{t\to\infty} y_0(t) = 0.$$

(iv) *If $\tau = 0$ and $p = 1$ then one of the following holds:*
   a. $y \equiv 0$,
   b. $y = y_0 \circ \mathsf{T}_{t_0}$ *for some* $t_0 \in \mathbb{R}$, *where* $y_0 : \mathbb{R} \to (0,1]$ *is the unique solution to (4.12) with initial conditions*

$$y(0) = 1, \quad \dot{y}(0) = 0.$$

Moreover, y_0 *is even, increasing on* $(-\infty, 0)$ *and satisfies* $\lim_{t\to\pm\infty} y_0(t) = 0$.

Remark 4.23.

(i) Detailed asymptotics for the $\tau \to 0$ limit of the period $2\mathsf{p}_\tau$ are established in Section 7.

(ii) Since y satisfies an equation of the form $\dot{y}^2 = P(y)$, where P is a polynomial of degree n, any solution of (4.11) can be expressed in terms of hyperelliptic functions. When $n = 3$ or 4 y can be expressed in terms of Jacobi elliptic functions — see [10, 13] for such expressions in the $(p, q) = (1, 2)$ case. Moreover, in the $\tau \to 0$ limit the modulus k^2 of the elliptic functions tends to 1. In this limit these elliptic functions, become hyperbolic trigonometric functions. e.g., $y_0 = \operatorname{sech}^2 t$ when $p = 1$, $q = 2$ and $y_0 = \frac{1}{2}(1 - \tanh t)$ when $p = q = 2$.

(iii) Figure 1 shows $y_{\min}$ and $y_{\max}$ on the graph of $f(y)$ for various (p, q) for $\tau = \frac{1}{2}\tau_{\max}$.

Proof. Motivated by (4.11) we define the 2-variable polynomial $P_\tau : \mathbb{R}^2 \to \mathbb{R}$

$$P_\tau(y, z) = z^2 - 4f(y) + 16\tau^2 = z^2 - 4y^q(1 - y)^p + 16\tau^2.$$

Let C_τ denote the real affine curve in $\mathbb{R}^2$ defined by $P_\tau = 0$. We can also view P_τ as a 2-variable complex polynomial and consider the complex affine curve $C_\tau^{\mathbb{C}}$ in $\mathbb{C}^2$ defined by $P_\tau = 0$. We find

$$(y, z) \in \operatorname{Sing}(C_\tau^{\mathbb{C}}) \iff f(y) = 4\tau^2,\ f'(y) = 0,\ z = 0.$$

Hence from (4.16) we have

$$\operatorname{Sing}(C_\tau^{\mathbb{C}}) = \operatorname{Sing}(C_\tau) = \begin{cases} \emptyset, & \text{for } 0 < |\tau| < \tau_{\max}; \\ (\frac{q}{n}\,, 0), & \text{for } |\tau| = \tau_{\max}; \\ (0\,, 0), & \text{for } \tau = 0 \text{ and } p = 1; \\ (0\,, 0) \cup (1\,, 0) & \text{for } \tau = 0 \text{ and } p > 1. \end{cases} \tag{4.24}$$

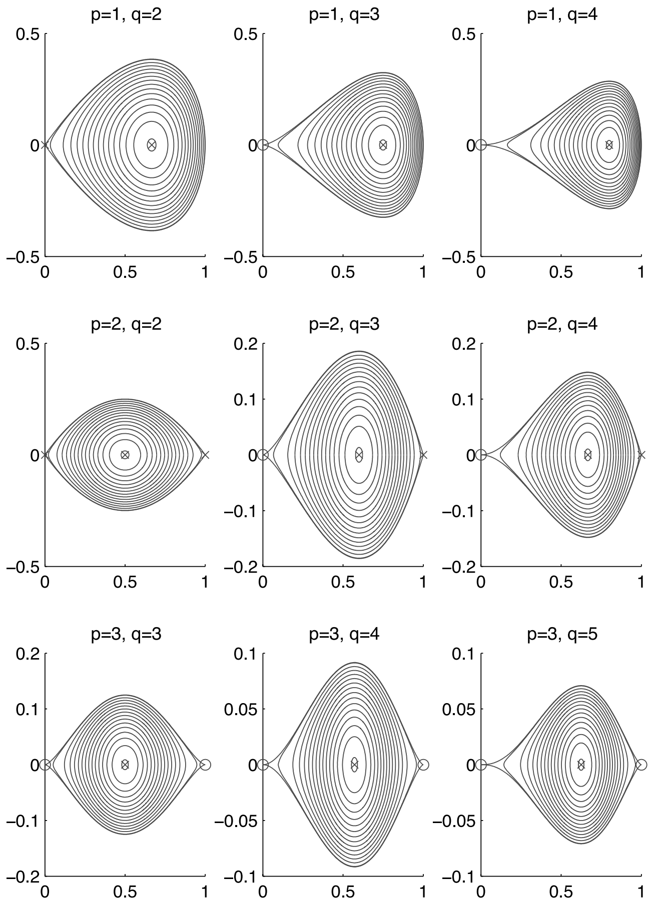

Figure 2: The curves C_τ^0 for $\tau \in [0, \tau_{\max}]$ and various choices of (p, q). Singular points are marked: ordinary double points as $\times$ and nodes as $\circ$.

Since $P_{zz} = 2$, all singular points of $C_\tau^{\mathbb{C}}$ are double point singularities. Further calculation yields:

$(\frac{q}{n}, 0)$ is always an ordinary double point,

$(0\,, 0)$ is an ordinary double point if $q = 2$ but a node if $q \geq 3$,

$(1\,, 0)$ is an ordinary double point if $p = 2$ but a node if $p \geq 3$.

See also figure 2.

(i) : $0 < |\tau| < \tau_{\max}$. If $0 < |\tau| < \tau_{\max}$, (4.24) implies that the real affine curve C_τ is nonsingular. C_τ is not necessarily connected, so let C_τ^0 denote the component containing the point $(\frac{q}{n}, 4\sqrt{\tau_{\max}^2 - \tau^2})$. $(y, z) \in C_\tau$ implies

$f(y) \geq 4\tau^2$. The set $f^{-1}([4\tau^2, \infty)) \subset \mathbb{R}$ is not necessarily connected but the component containing $\frac{q}{n}$ is the closed interval $[y_{\min}, y_{\max}] \subset (0,1)$. Since $f(y) \leq 4\tau_{\max}^2$ for $y \in (0,1)$ any point $(y,z) \in C_\tau^0$ satisfies $(y,z) \in [0,1] \times [-4\tau_{\max}, 4\tau_{\max}]$. In particular, the component C_τ^0 is a compact nonsingular curve and hence is diffeomorphic to S^1. Hence all solutions of (4.11) with $0 < |\tau| < \tau_{\max}$ are nonconstant and periodic with period $2\mathsf{p}_\tau > 0$ depending only on τ. In particular, two solutions of (4.11) with the same values of τ differ only by time translation.

The geometry of the curves C_τ^0 is illustrated for various choices of (p,q) in figure 2. The different types of singular points, which can occur in the $\tau = 0$ energy level are clearly visible in this figure.

a. Asymptotics of p_τ as $\tau \to \tau_{\max}$: we consider the first-order corrections to the stationary point $y \equiv q/n$ when $\tau = \tau_{\max}$. If we write $\tilde{y} = y - q/n$, then (4.12) becomes

$$\ddot{\tilde{y}} = -\omega^2 \tilde{y} + O(\tilde{y}^2),$$

where

$$\omega^2 = 2n \left(\frac{q}{n}\right)^{q-1} \left(\frac{p}{n}\right)^{p-1} = \frac{8n^3}{pq} \tau_{\max}^2.$$

Hence $\lim_{\tau \to \tau_{\max}} 2\mathsf{p}_\tau = 2\pi/\omega$, as claimed.

b. Since $|\mathbf{w}| = 1$ and $y = |w_2|^2$, we have $0 \leq y \leq 1$ for all $t \in \mathbb{R}$. At any critical point of y, (4.11) implies that y satisfies equation (4.21). It follows from the definitions of $y_{\max}$ and $y_{\min}$ in terms of roots of the polynomial (4.21) that the maximum and minimum values of y are therefore $y_{\max}$ and $y_{\min}$, respectively.

c. The stated asymptotics of $y_{\min}$ and $y_{\max}$ as $\tau \to 0$ follow immediately from the characterization of $y_{\min}$ and $y_{\max}$ as the only solutions of (4.21) in the range $[0,1]$.

(ii): $|\tau| = \tau_{\max}$. When $\tau^2 = \tau_{\max}^2$, from (4.11) we have $\dot{y}^2 = 4(f(y) - 4\tau_{\max}^2) \leq 4(f_{\max} - 4\tau_{\max}^2) \leq 0$, with equality if and only if $f(y) = f_{\max}$, i.e., if and only if $y = q/n$. Hence, we have $\dot{y} = 0$ for all $t \in \mathbb{R}$ and $y \equiv q/n$.

(iii): *$\tau = 0$ and $p > 1$.* Recall from (4.17) that for $p > 1$ both $y = 0$ and $y = 1$ are critical points of f and hence give rise to constant solutions $y \equiv 0$ and $y \equiv 1$ of (4.12).

Equation (4.11) implies $\dot{y} = 0$ if and only if $y = 0$ or $y = 1$. Since $y \in [0,1]$ and $\{0,1\} \subset \mathrm{Crit}(f)$ a nonconstant solution y contains no points with $\dot{y} = 0$ and is therefore monotone with $0 < y < 1$ for all t. If y is increasing then $y \circ \underline{\mathsf{T}}$ is decreasing and hence by composing with $\underline{\mathsf{T}}$ if necessary we can assume y satisfies the first-order ODE

$$\dot{y} = -2\sqrt{f(y)}. \tag{4.25}$$

Since y is monotone and bounded it must approach constant values c_- and c_+ as $t \to \pm\infty$. Recall the elementary fact that if γ is an integral curve of a vector field V and $\lim_{t\to\infty} \gamma(t) = \gamma_\infty$, then γ_∞ must be a zero (or stationary point) of the vector field V. Hence, we see that $c_\pm$ must be stationary points of (4.12) which also belong to the zero energy level. Therefore, $c_\pm \in \mathrm{Crit}(f) \cap f^{-1}(0) = \{0, 1\}$. Since y is strictly decreasing we must have $\lim_{t\to-\infty} y(t) = 1$ and $\lim_{t\to\infty} y(t) = 0$. In particular, for any such solution of (4.25) there exists $t_0 \in \mathbb{R}$ so that $y(t_0) = q/n$. Hence $\hat{y} := y \circ \mathsf{T}_{t_0}$ is a solution of (4.25) with $\hat{y}(0) = q/n$, and so by uniqueness of the initial value problem $\hat{y} \equiv y_0$.

(iv): $\tau = 0$ *and* $p = 1$. Recall from (4.17) that for $p = 1$, $y = 0$ (but not $y = 1$) is a critical point of f and so gives rise to the stationary point $y \equiv 0$ of 4.12.

Again from (4.11), $\dot{y} = 0$ if and only if $y = 0$ or $y = 1$. For $p = 1$, $y = 0$ is a stationary point of (4.12) but $y = 1$ is not. If y is nonconstant, then y cannot attain an interior minimum since $\dot{y}(t) = 0$ implies $y(t) = 1$. Therefore, as $\mathrm{Crit}(f) \cap f^{-1}(0) = \{0\}$ for $p = 1$, y must approach 0 as $t \to \pm\infty$. Since $y \in [0, 1]$ is nonconstant and tends to 0 as $t \to \pm\infty$, y attains an interior maximum at some point $t_0 \in \mathbb{R}$. Hence, $\dot{y}(t_0) = 0$ and therefore $y(t_0) = 1$. Then by uniqueness of the initial value problem $y \circ \mathsf{T}_{t_0} = y_0$. Evenness of y_0 follows from the invariance of (4.12) and the initial conditions $y(0) = 1$, $\dot{y}(0) = 0$ under $t \mapsto -t$. $\square$

We use Lemma 4.19 to establish normal forms for solutions of (4.8).

Proposition 4.26. *Fix a pair of admissible integers p and q and let $\mathbf{w}$ be any solution of (4.8) with $\mathcal{I}_1(\mathbf{w}) = 1$ and $\mathcal{I}_2(\mathbf{w}) = -2\tau$ with $0 \le |\tau| \le \tau_{max}$.*

(i) *If $p > 1$ and $0 < |\tau| \le \tau_{max}$ then $\mathbf{w}$ is equivalent under symmetries (1)–(3) to $\mathbf{w}_\tau : \mathbb{R} \to \mathbb{S}^3$ defined as the unique solution to (4.8) with initial value*

$$\mathbf{w}_\tau(0) = \left(\sqrt{\tfrac{p}{n}}\, e^{i\alpha_\tau/2p}, \sqrt{\tfrac{q}{n}}\, e^{i\alpha_\tau/2q}\right),$$

where $\alpha_\tau \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is defined by

$$\alpha_\tau := \arcsin\left(-\frac{\tau}{\tau_{max}}\right).$$

(ii) *If $p > 1$ and $\tau = 0$ then $\mathbf{w}$ is equivalent under symmetries (1)–(3) to the unique solution of 4.7 with one of the following four initial conditions:*
a. $\mathbf{w}(0) = (1, 0)$,
b. $\mathbf{w}(0) = (0, 1)$,
c. $\mathbf{w}(0) = \left(\sqrt{\frac{p}{n}}, \sqrt{\frac{q}{n}}\right)$,
d. $\mathbf{w}(0) = \left(e^{i\pi/2p}\sqrt{\frac{p}{n}}, e^{i\pi/2q}\sqrt{\frac{q}{n}}\right)$.

(iii) *If $p = 1$ and $0 < |\tau| \leq \tau_{max}$ then $\mathbf{w}$ is equivalent under symmetries (1)–(3) to $\mathbf{w}_\tau : \mathbb{R} \to \mathbb{S}^3$ defined as the unique solution to (4.8) with initial value*

$$\mathbf{w}_\tau(0) = (-i \operatorname{sgn} \tau \sqrt{1 - y_{max}}, \sqrt{y_{max}}).$$

(iv) *If $p = 1$ and $\tau = 0$ then $\mathbf{w}$ is equivalent under symmetries (1)–(3) to the unique solution of 4.7 with one of the following two initial conditions:*
a. $\mathbf{w}(0) = (1, 0)$,
b. $\mathbf{w}(0) = (0, 1)$.

Proof. Let $\mathbf{w}$ be any solution of (4.8) with $\mathcal{I}_1(\mathbf{w}) = |\mathbf{w}|^2 = 1$, $\mathcal{I}_2(\mathbf{w}) = -2\tau$ and $0 < |\tau| \leq \tau_{\max}$. Set $y = |w_2|^2$ and write $\mathbf{w}(0) = (\sqrt{1 - y(0)}\, \mathrm{e}^{\mathrm{i}\theta_1}, \sqrt{y(0)}\, e^{i\theta_2})$.

(i) *Case $p > 1$ and $\tau \neq 0$.* If $\tau = \pm\tau_{\max}$ then by 4.19.ii $y \equiv q/n$ and hence $y(0) = q/n$ and $\dot{y}(0) = 0$. If $0 < |\tau| < \tau_{\max}$ then by 4.19.i.a and using the time translation invariance of (4.8) (Symmetry 1) we can arrange that $y(0) = |w_2|^2(0) = \frac{q}{n}$ and that $\dot{y}(0) < 0$. In both cases, we have

$$y(0) = \tfrac{q}{n} \quad \text{and} \quad \dot{y}(0) \leq 0.$$

The former together with (4.10) implies that

$$\sin(p\theta_1 + q\theta_2) = -\tfrac{\tau}{\tau_{\max}},$$

while the latter together with (4.10) implies

$$\cos(p\theta_1 + q\theta_2) \geq 0.$$

Acting with the nth root of unity $z_k = \mathrm{e}^{2\pi k\mathrm{i}/n}$ (symmetry (2)) leaves $y(0)$ invariant and sends $p\theta_1 + q\theta_2 \mapsto p\theta_1 + q\theta_2 + 2k\pi$. Hence by using symmetry

(2), we can arrange that $p\theta_1 + q\theta_2 \in [-\pi, \pi)$. Finally by using symmetry (3), we can arrange that $p\theta_1 = q\theta_2$. Therefore, we have

$$\sin 2p\theta_1 = \sin 2q\theta_2 = -\tfrac{\tau}{\tau_{\max}} = \sin \alpha_\tau, \quad \cos 2p\theta_1 \geq 0 \quad \text{and} \quad 2p\theta_1 \in [-\pi, \pi).$$

Hence $2p\theta_1 = 2q\theta_2 = \alpha_\tau$ as claimed. Note that in this case

$$w_1^p w_2^q(0) = \left(\sqrt{\tfrac{p}{n}}\right)^p \left(\sqrt{\tfrac{q}{n}}\right)^q \mathrm{e}^{\mathrm{i}\alpha_\tau} = 2\tau_{\max}\mathrm{e}^{\mathrm{i}\alpha_\tau}.$$

(ii) *Case* $p > 1$ *and* $\tau = 0$. By 4.19.iii $y = |w_2|^2$ must be one of the following: (a) $y \equiv 0$, (b) $y \equiv 1$, (c) $y = y_0 \circ \mathsf{T}_{t_0}$, (d) $y = y_0 \circ \mathsf{T}_{t_0} \circ \underline{\mathsf{T}}$ for some $t_0 \in \mathbb{R}$, where $y_0 : \mathbb{R} \to (0,1)$ is the function defined in 4.19.iii.c. It is easily seen that (a) implies $\mathbf{w}$ is a stationary point of the form $\mathbf{w} = (\mathrm{e}^{\mathrm{i}\theta_1}, 0)$, while (b) implies $\mathbf{w}$ is a stationary point of the form $\mathbf{w} = (0, \mathrm{e}^{\mathrm{i}\theta_2})$. Hence, $\mathbf{w}$ is equivalent using symmetry (3) to the stationary points $(1,0)$ or $(0,1)$ in cases (a) and (b), respectively. Suppose we are now in case (c) or (d) and hence $y = y_0 \circ \mathsf{T}_{t_0} \circ \underline{\mathsf{T}}^j$ for some $t_0 \in \mathbb{R}$ and $j \in \{0,1\}$. By time translation invariance of (4.8) we can arrange that $y(0) = \frac{q}{n}$ (i.e., that $t_0 = 0$.) Thus, we have

$$y(0) = \tfrac{q}{n} \quad \text{and} \quad \dot{y}(0) = (-1)^{j+1} 4\tau_{\max}.$$

Substituting these initial conditions into (4.10) and simplifying yields

$$\mathrm{e}^{\mathrm{i}(p\theta_1 + q\theta_2)} = (-1)^j.$$

As in case (i) by using symmetry (2), we may arrange that $p\theta_1 + q\theta_2 \in [-\pi, \pi)$ and then use symmetry (3) to arrange that also $p\theta_1 = q\theta_2$. Hence, the previous equality reduces to $\mathrm{e}^{2\mathrm{i}p\theta_1} = (-1)^j$ with $2p\theta_1 \in [-\pi, \pi)$. In case (c) $j = 0$ and so $2p\theta_1 = 2q\theta_2 = 0$ is the unique solution in the required range, whereas in case (d) $j = 1$ and so $2p\theta_1 = 2q\theta_2 = \pi$ as claimed.

(iii) *Case* $p = 1$ *and* $\tau \neq 0$. If $0 < |\tau| < \tau_{\max}$ by using time translation invariance of (4.8) (symmetry (1)) we may assume that $y(0) = y_{\max}$ and therefore also $\dot{y}(0) = 0$. If $\tau = \pm\tau_{\max}$ then by 4.19.ii $y \equiv q/n = y_{\max}$ and $\dot{y}(0) = 0$. Hence, in either case from (4.10) we have

$$\begin{aligned} 2\mathrm{i}\tau &= -w_1 w_2^{n-1}(0) = -\mathrm{i}\sqrt{f(y_{\max})}\sin(\theta_1 + (n-1)\theta_2) \\ &= -2\mathrm{i}|\tau|\sin(\theta_1 + (n-1)\theta_2), \end{aligned}$$

and therefore

$$\sin(\theta_1 + (n-1)\theta_2) = -\frac{\tau}{|\tau|} = -\operatorname{sgn}\tau.$$

As in the previous cases by acting with an nth root of unity, we can arrange that $\theta_1 + (n-1)\theta_2 \in [-\pi, \pi)$ and by acting with symmetry (3) that $\theta_2 = 0$. Therefore, we have $\sin\theta_1 = -\operatorname{sgn}\tau$ with $\theta_1 \in [-\pi, \pi)$. Hence, $\theta_1 = -\operatorname{sgn}\tau \cdot \frac{1}{2}\pi$ as claimed.

(iv) *Case $p = 1$ and $\tau = 0$.* By 4.19.iv $y = |w_2|^2$ must be one of the following: (a) $y \equiv 0$ or (b) $y = y_0 \circ \mathsf{T}_{t_0}$ where $t_0 \in \mathbb{R}$ and $y_0 : \mathbb{R} \to (0, 1]$ is the function defined in 4.19.iv.b. As in (ii), case (a) implies that $\mathbf{w}$ is a stationary point of the form $(\mathrm{e}^{\mathrm{i}\theta_1}, 0)$ and hence is equivalent using symmetry (3) to $(1, 0)$ as claimed. Suppose now that we are in case (b). By time translation invariance we can arrange that $t_0 = 0$ and hence $y(0) = 1$. This implies $w_1(0) = 0$ and $w_2(0) = \mathrm{e}^{\mathrm{i}\theta_2}$ for some $\theta_2 \in \mathbb{R}$. Using symmetry (3) we can arrange that $\theta_2 = 0$ and hence that $\mathbf{w}(0) = (0, 1)$ as claimed. ☐

Remark 4.27. Note that in cases ii.a and ii.b of Proposition 4.26 the initial conditions are stationary points of (4.8) and hence the corresponding solutions with this initial data are $\mathbf{w} \equiv (1, 0)$ and $\mathbf{w} \equiv (0, 1)$, respectively. Let $\mathbf{w}_0$ denote the unique solution to (4.8) with initial condition $\mathbf{w}_0(0) = \left(\sqrt{\frac{p}{n}}, \sqrt{\frac{q}{n}}\right)$ as in (ii.c). Then by uniqueness of the initial value problem for (4.8) we see that

$$\mathbf{w}_0(t) = (\sqrt{1 - y_0(t)}, \sqrt{y_0(t)}\,), \tag{4.28}$$

where $y_0(t) : \mathbb{R} \to (0, 1)$ is the decreasing function defined in 4.19.iii.c.

Note that $(0, 1)$ is not a stationary point of (4.8) for $p = 1$. Let $\mathbf{w}_0$ denote the unique solution of (4.8) with initial condition $\mathbf{w}_0(0) = (0, 1)$ as in (iv.b). Then by the uniqueness of the initial value problem for (4.8) we see that

$$\mathbf{w}_0(t) = (\operatorname{sgn} t\sqrt{1 - y_0(t)}, \sqrt{y_0(t)}\,), \tag{4.29}$$

where $y_0 : \mathbb{R} \to (0, 1]$ is the even function defined in 4.19.iv.b. (Since $w_2(t) = \sqrt{y_0(t)}$ is real and positive for all t, the equation for $\dot{w}_1$ in 4.7 implies that $\dot{w}_1 > 0$ for all t. By 4.19.iv $\sqrt{1 - y_0(t)}$ is decreasing for $t < 0$ and increasing $t > 0$, so $\operatorname{sgn} t\sqrt{1 - y_0(t)}$ is increasing for all t as required.)

Remark 4.30. The argument from 4.26.iii applied in the case $p > 1$ implies that any solution of (4.8) with $\mathcal{I}_1(\mathbf{w}) = 1$ and $\mathcal{I}_2(\mathbf{w}) = -2\tau$ and $0 < |\tau| \leq \tau_{\max}$ is equivalent under symmetries (1)–(3) to

$$\hat{\mathbf{w}}_\tau = \left(-\mathrm{e}^{\mathrm{i}\pi/2p}\ \operatorname{sgn}(\tau)\sqrt{1 - y_{\max}}, \sqrt{y_{\max}}\right).$$

Similarly, the argument from 4.26.i works for $p = 1$, as well as $p > 1$. However, we will only make use of the normal forms stated in 4.26. The difference in our choice of normal form for $p = 1$ and $p > 1$ reflects differences in the geometry of the resulting special Legendrian immersions in these cases as we will explain later.

5. $\mathbf{w}_\tau$ and the $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrian immersions X_τ

We now define the particular 1-parameter family $\mathbf{w}_\tau$ of (p,q)-twisted SL curves we will use throughout the rest of the paper by specifying initial data $\mathbf{w}_\tau(0)$ as in the normal form Proposition 4.26. Associated to the 1-parameter family $\mathbf{w}_\tau$ is the 1-parameter family X_τ of $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrians. Proposition 4.26 implies that any $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrian in $\mathbb{S}^{2p+2q-1}$ is congruent to X_τ for some τ.

Proposition 5.1. *Fix a pair of admissible integers p and q and choose any $\tau \in [-\tau_{max}, \tau_{max}]$. Define $\mathbf{w}_\tau : \mathbb{R} \to \mathbb{S}^3$ as the unique solution of (4.8) with initial data*

$$\mathbf{w}_\tau(0) = \left(\sqrt{\tfrac{p}{n}}e^{i\alpha_\tau/2p}, \sqrt{\tfrac{q}{n}}e^{i\alpha_\tau/2q}\right), \qquad \text{if } p > 1; \tag{5.2}$$

where $\alpha_\tau \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is defined by

$$\alpha_\tau := \arcsin\left(-\frac{\tau}{\tau_{max}}\right), \tag{5.3}$$

or

$$\mathbf{w}_\tau(0) = (-i \operatorname{sgn} \tau \sqrt{1 - y_{max}}, \sqrt{y_{max}}), \quad \text{if } p = 1. \tag{5.4}$$

Then $\mathbf{w}_\tau$ depends real analytically on $\tau \in (-\tau_{max}, \tau_{max})$ and satisfies $\mathbf{w}_{-\tau} = \overline{\mathbf{w}}_\tau$. In particular, $\mathbf{w}_0 : \mathbb{R} \to \mathbb{S}^3 \subset \mathbb{C}^2$ is contained in $\mathbb{R}^2 \subset \mathbb{C}^2$.

Proof. To prove that $\mathbf{w}_\tau$ depends analytically on τ it suffices by 4.8.iii to prove that the initial condition $\mathbf{w}_\tau(0)$ given by (5.2) or (5.4) depends analytically on τ for $\tau \in (-\tau_{\max}, \tau_{\max})$. For $p > 1$ it is clear from (5.3) that α_τ depends real analytically on τ for $|\tau| < \tau_{\max}$. Hence by (5.2) $\mathbf{w}_\tau(0)$ depends analytically on τ for $|\tau| < \tau_{\max}$. For $p = 1$ we have $f'(y_{\max}) = y_{\max}^{n-2}(q - ny_{\max}) \neq 0$ for $|\tau| < \tau_{\max}$. Hence by the real analytic Implicit Function Theorem (see e.g.[26, Thm 2.3.5]) $y_{\max}$ is an analytic function of $\tau \in (-\tau_{\max}, \tau_{\max})$.

Therefore $\sqrt{y_{\max}}$ is also an analytic function of $\tau \in (-\tau_{\max}, \tau_{\max})$ (recall $y_{\max} \geq (n-1)/n$). Write $\mathbf{w}_\tau(0) = (ir_\tau, \sqrt{y_{\max}})$. Because $\mathcal{I}_2(\mathbf{w}_\tau(0)) = \operatorname{Im} w_1 w_2^{n-1}(0) = -2\tau$

$$r_\tau = -\frac{2\tau}{\sqrt{y_{\max}}^{\,n-1}}$$

and hence is an analytic function of $\tau \in (-\tau_{\max}, \tau_{\max})$. From (5.2) or (5.4) we have $\mathbf{w}_{-\tau}(0) = \overline{\mathbf{w}}_\tau(0)$ and hence $\mathbf{w}_{-\tau} = \overline{\mathbf{w}}_\tau$ by uniqueness of the initial value problem for (4.8). □

The associated function $y_\tau := |w_2|^2$ and its initial value characterization For the solution $\mathbf{w}_\tau$ defined in 5.1, define $y_\tau := |w_2|^2$. By 4.7 y_τ satisfies equations (4.11) and (4.12). Analytic dependence of y_τ on $\tau \in (-\tau_{\max}, \tau_{\max})$ follows immediately from analytic dependence of $\mathbf{w}_\tau$.

For $p = 1$, y_τ is the unique solution of (4.12) satisfying the initial conditions

$$y(0) = y_{\max}, \quad \dot{y}(0) = 0. \tag{5.5}$$

In particular, y_0 is the unique solution of (4.12) satisfying $y(0) = 1$, $\dot{y}(0) = 0$ introduced in 4.19.iv.b.

Similarly, for $p > 1$, y_τ is the unique solution of (4.12) satisfying the initial conditions

$$y(0) = \frac{q}{n}, \quad \dot{y}(0) = -4\tau_{\max}\cos\alpha_\tau = -4\sqrt{\tau_{\max}^2 - \tau^2}. \tag{5.6}$$

y_0 coincides with the solution of (4.12) satisfying $y(0) = q/n, \dot{y}(0) = -4\tau_{\max}$ introduced in 4.19.iii.c.

For both $p = 1$ and $p > 1$ it follows from these initial value characterizations of y_τ that $y_{-\tau} = y_\tau$, which is consistent with the fact that $\mathbf{w}_{-\tau} = \overline{\mathbf{w}}_\tau$.

Define round cylinders of type (p, q), $\mathrm{Cyl}_I^{p,q}$, by

$$\mathrm{Cyl}_I^{p,q} := \begin{cases} I \times \mathbb{S}^{p-1} \times \mathbb{S}^{q-1}, & \text{if } p > 1; \\ I \times \mathbb{S}^{n-2}, & \text{if } p = 1, \end{cases} \tag{5.7}$$

where $I \subset \mathbb{R}$ is an interval which we omit in the notation when $I = \mathbb{R}$. In Section 3, we thought about $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrians by treating our special Legendrians as (unparameterized) subsets of $\mathbb{S}^{2(p+q)-1}$. From now on it will be more convenient to deal with special Legendrian immersions $X_\tau : \mathrm{Cyl}^{p,q} \to \mathbb{S}^{2(p+q)-1}$ and to talk about special Legendrian immersions equivariant with respect to the obvious actions of $\mathrm{SO}(p) \times \mathrm{SO}(q)$ on both domain and target.

We now define the 1-parameter family of special Legendrian immersions $X_\tau : \mathrm{Cyl}^{p,q} \to \mathbb{S}^{2(p+q)-1}$ using the 1-parameter family of (p,q)-twisted SL curves $\boldsymbol{w}_\tau$ defined in Proposition 5.1.

Definition 5.8. For $\tau \in [-\tau_{\max}, \tau_{\max}]$ define an immersion $X_\tau : \mathrm{Cyl}^{p,q} \to \mathbb{S}^{2(p+q)-1}$ by

$$X_\tau(t,\sigma_1,\sigma_2) = (w_1(t)\cdot\sigma_1,\ w_2(t)\cdot\sigma_2), \quad \text{for } p>1;$$
$$X_\tau(t,\sigma) = (w_1(t), w_2(t)\cdot\sigma), \quad \text{for } \ p=1,$$

where $t\in\mathbb{R}$, $\sigma_1 \in \mathbb{S}^{p-1}$, $\sigma_2\in\mathbb{S}^{q-1}$, $\sigma\in\mathbb{S}^{n-2}$ and $\mathbf{w}_\tau = (w_1,w_2)$ is the unique solution to (4.8) specified in Proposition 5.1.

We now establish some basic properties of X_τ.

Proposition 5.9. *For $\tau\in[-\tau_{max},\tau_{max}]$ the immersion $X_\tau : \mathrm{Cyl}^{p,q}\to \mathbb{S}^{2(p+q)-1}$ defined in 5.8 has the following properties:*

(i) *X_τ is a smooth special Legendrian immersion depending analytically on τ for $\tau\in(-\tau_{max},\tau_{max})$, and satisfies $X_{-\tau}=\overline{X}_\tau$. In particular, X_0 is contained in $\mathbb{S}^{p+q-1}\subset\mathbb{R}^{p+q}\subset\mathbb{C}^{p+q}$.*

(ii) *For $p>1$, the metric g_τ on $\mathrm{Cyl}^{p,q}$ induced by X_τ is*

$$|\dot{\mathbf{w}}|^2dt^2 + |w_1|^2 g_{\mathbb{S}^{p-1}} + |w_2|^2 g_{\mathbb{S}^{q-1}} = y^{q-1}(1-y)^{p-1}dt^2 + (1-y)g_{\mathbb{S}^{p-1}} + y g_{\mathbb{S}^{q-1}}.$$

For $p=1$, the induced metric g_τ on $\mathrm{Cyl}^{1,n-1}$ is

$$|\dot{\mathbf{w}}|^2dt^2 + |w_2|^2 g_{\mathbb{S}^{n-2}} = y^{n-2}dt^2 + y\, g_{\mathbb{S}^{n-2}}.$$

(iii) *X_τ is $SO(p)\times SO(q)$-equivariant, i.e., for any $\mathsf{M}=(\mathsf{M}_1,\mathsf{M}_2)\in SO(p)\times SO(q)$ we have*

$$\widetilde{\mathsf{M}}\circ X_\tau = X_\tau\circ\mathsf{M},$$

where $\mathsf{M}=(\mathsf{M}_1,\mathsf{M}_2)$ acts on $\mathrm{Cyl}^{p,q}$ by $\mathsf{M}\cdot(t,\sigma_1,\sigma_2)=(t,\mathsf{M}_1\sigma_1,\mathsf{M}_2\sigma_2)$, and

$$\widetilde{\mathsf{M}} = \begin{pmatrix}\mathsf{M}_1 & 0\\ 0 & \mathsf{M}_2\end{pmatrix} \in SO(p)\times SO(q)\subset SO(p+q).$$

(iv) *When $\tau=0$ we have*

$$X_0(\mathrm{Cyl}^{p,q}) = \begin{cases}\mathbb{S}^{p+q-1}\setminus(\mathbb{S}^{p-1},0)\cup(0,\mathbb{S}^{q-1}), & \textit{for } p>1;\\ \mathbb{S}^{n-1}\setminus(\pm1,0)\in\mathbb{R}\oplus\mathbb{R}^{n-1}, & \textit{for } p=1.\end{cases}$$

(v) *When* $\tau = \tau_{max}$, *we have*

(5.10a)
$$X_{\tau_{max}}(t,\sigma) = \left(-i\sqrt{\tfrac{1}{n}}e^{2in\tau t}, \sqrt{\tfrac{n-1}{n}}e^{-2in\tau t/(n-1)}\,\sigma\right), \quad \textit{for } p=1;$$

(5.10b)
$$X_{\tau_{max}}(t,\sigma_1,\sigma_2) = \left(\sqrt{\tfrac{p}{n}}e^{-i\pi/(4p)}e^{2ni\tau t/p}\,\sigma_1, \sqrt{\tfrac{q}{n}}e^{-i\pi/(4q)}e^{-2ni\tau t/q}\,\sigma_2\right),$$
for $p > 1$.

(vi) *If* $X : \mathrm{Cyl}^{p,q} \to \mathbb{S}^{2(p+q)-1}$ *is any non totally geodesic* $SO(p)\times SO(q)$*-invariant special Legendrian immersion then* $X = e^{i\omega}\,\widetilde{\mathsf{T}}_x \circ X_\tau \circ \mathsf{T}_y$ *for some* $x,\ y \in \mathbb{R}$, $0 < |\tau| < \tau_{max}$ *and nth root of unity* $\omega \in \mathbb{S}^1$ *where* $\widetilde{\mathsf{T}}_x \in SU(n)$ *is defined by*

$$\widetilde{\mathsf{T}}_x = \begin{pmatrix} e^{ix/p}\,\mathrm{Id}_p & 0 \\ 0 & e^{-ix/q}\,\mathrm{Id}_q \end{pmatrix}. \tag{5.11}$$

Proof. (i) For $\tau \neq 0$ we have $|w_1|^2 \geq y_{\min} > 0$ and $|w_2|^2 \geq 1 - y_{\max} > 0$. Because there are no points where w_1 or w_2 vanish, 2.9 implies that X_τ is a Legendrian immersion. Since $\mathbf{w}_\tau$ is a solution of (4.8), 2.18 and 2.19 imply that X_τ is special Legendrian. We deal with the exceptional case $\tau = 0$ separately in part (iv). Analytic dependence of X_τ on τ follows from the analytic dependence of $\mathbf{w}_\tau$ on τ proved in Proposition 5.1. The final part follows from the fact that $\mathbf{w}_{-\tau} = \overline{\mathbf{w}}_\tau$ (see 5.1).

(ii) follows immediately from equations (2.11) and (2.22).

(iii) The $\mathrm{SO}(p)\times\mathrm{SO}(q)$-equivariance of X_τ is clear from the definition of X_τ in (5.8).

(iv) $\tau = 0$ limit. From part (i), $X_0(\mathrm{Cyl}^{p,q}) \subset \mathbb{S}^{p+q-1}$.
Consider first the case where $p > 1$. From (4.28)

$$X_0(t,\sigma_1,\sigma_2) = (\sqrt{1-y_0(t)}\,\sigma_1, \sqrt{y_0(t)}\,\sigma_2),$$

where $y_0 : \mathbb{R} \twoheadrightarrow (0,1)$ is the decreasing function defined in 4.19.iii.c. Recall from Remark 2.5 that the map $\Pi : [0,\pi/2]\times\mathbb{S}^{p-1}\times\mathbb{S}^{q-1} \to \mathbb{S}^{p+q-1}$ given by

$$\Pi\,(t,\sigma_1,\sigma_2) = (\cos t\,\sigma_1, \sin t\,\sigma_2),$$

is surjective and on restriction to the interval $(0,\pi/2)$ gives a diffeomorphism between $(0,\pi/2)\times\mathbb{S}^{p-1}\times\mathbb{S}^{q-1}$ and $\mathbb{S}^{p+q-1}\setminus(\mathbb{S}^{p-1},0)\cup(0,\mathbb{S}^{q-1})$. Since by

4.19.iii.c y_0 is decreasing with $\lim_{t\to-\infty} y(t) = 1$ and $\lim_{t\to\infty} y(t) = 0$ we see that X_0 is a reparameterization of this diffeormorphism.

Similarly, from (4.29) for $p = 1$ we have

$$X_0(t,\sigma) = (-\operatorname{sgn} t\sqrt{1-y_0(t)},\, \sqrt{y_0(t)}\,\sigma),$$

where $y_0 : \mathbb{R} \twoheadrightarrow (0,1]$ is the even function defined in 4.19.iv.b. The map $\Pi : [0,\pi] \times \mathbb{S}^{n-2} \twoheadrightarrow \mathbb{S}^{n-1}$ defined by $\Pi\,(t,\sigma) = (\cos t,\, \sin t\,\sigma)$ on restriction to the open interval $(0,\pi)$ gives a diffeomorphism between $(0,\pi)\times\mathbb{S}^{n-2}$ and $\mathbb{S}^{n-1}\setminus(\pm1,0)$. Since by 4.19.iv.b y_0 is even, increasing on $(-\infty,0)$, satisfies $y_0(0) = 1$ and $\lim_{t\to\pm\infty} y_0(t) = 0$ we see that X_0 is a reparameterization of this diffeomorphism.

(v) $\tau = \tau_{\max}$ limit. We leave this as an elementary exercise for the reader.

(vi) follows from 3.8 and the normal form for solutions of (4.8) established in 4.26. $\square$

Torques of X_τ. Many geometric variational problems admit homological invariants associated with symmetries of the problem. These invariants have played a fundamental role in global structure results including uniqueness questions [25, 30, 31] and also in gluing results [19, 21–24, 32]. For minimal and CMC immersions in Euclidean space or round spheres the invariants associated to translations and rotations are called the forces and torques, respectively. We calculate the (restricted) torque of the $\mathrm{SO}(p)\times\mathrm{SO}(q)$-invariant special Legendrians X_τ below in 5.14. An appropriate component of the torque (depending on p and q) is exactly proportional to the parameter τ. This is similar to the case of Delaunay surfaces where (appropriately centred) the torque is zero and the force is a vector along its axis whose magnitude is τ, the parameter of the Delaunay. The torque of X_τ enters into our argument to calculate refined asymptotics of the angular period $\widehat{\mathsf{p}}_\tau$ and its derivative as $\tau\to0$ and therefore is needed in our work on higher-dimensional SL gluing [11, 12, 14]. More generally we expect that the torque will play an important role in controlling aspects of the global geometry of special Legendrians.

Suppose M is an oriented m-dimensional submanifold of the ambient manifold $(\overline{M},\overline{g})$ and $\mathsf{k}\in\mathfrak{iso}(\overline{M},\overline{g})$ is a Killing field on $(\overline{M},\overline{g})$. Given any oriented hypersurface $\Sigma\subset M$ we define the k-flux through Σ by

$$\mathcal{F}_{\mathsf{k}}(\Sigma) := \int_\Sigma \overline{g}(\mathsf{k},\eta)\,d\mathrm{v}_\Sigma\,, \tag{5.12}$$

where η is the unit conformal to Σ, chosen so that the orientation defined by Σ and η agrees with that of M. An immediate consequence of the First Variation of Volume formula [34, 7.6] is

Lemma 5.13. *If M is an oriented m-dimensional minimal submanifold of $(\overline{M}, \overline{g})$, Σ is an oriented hypersurface of M and $\mathsf{k} \in \mathfrak{iso}(\overline{M}, \overline{g})$ then the k-flux through Σ, $\mathcal{F}_{\mathsf{k}}(\Sigma)$, depends only on the homology class $[\Sigma] \in H_{m-1}(M, \mathbb{R})$.*

In other words, when M is a minimal submanifold of $(\overline{M}, \overline{g})$ the k-flux map defined in (5.12) induces a linear map $\mathcal{F} : H_{m-1}(M, \mathbb{R}) \to \mathfrak{iso}(\overline{M}, \overline{g})^*$. In particular, if $(\overline{M}, \overline{g}) = (\mathbb{S}^{2n-1}, g_{\mathrm{std}})$ then $\mathfrak{iso}(\overline{M}, \overline{g}) = \mathfrak{o}(2n)$ and we call the map $\mathcal{F} : H_{m-1}(M, \mathbb{R}) \to \mathfrak{o}(2n)^*$ the *torque of M*. For special Legendrian submanifolds of $\mathbb{S}^{2n-1}$ it is also convenient to define the *restricted torque of M*, which is the restriction of the torque to the subalgebra $\mathfrak{su}(n) \subset \mathfrak{o}(2n)$.

Proposition 5.14. *For $p > 1$ the $\mathfrak{su}(n)$ restricted torque of the $SO(p) \times SO(q)$-invariant special Legendrian immersion $X_\tau : \mathrm{Cyl}^{p,q} \to \mathbb{S}^{2(p+q)-1}$ is given by*

(5.15)
$$\mathcal{F}_{\mathsf{k}}(X_\tau) = \begin{cases} 2\tau\left(\frac{1}{p}\sum_{i=1}^{p}\lambda_i - \frac{1}{q}\sum_{j=1}^{q}\mu_j\right) & \\ \quad \mathrm{Vol}(\mathbb{S}^{p-1})\,\mathrm{Vol}(\mathbb{S}^{q-1}), & \mathsf{k} = i\,\mathrm{diag}(\lambda_1, \dots, \lambda_p, \mu_1, \dots, \mu_q); \\ 0, & \textit{if } \mathsf{k} \in \mathfrak{su}(n) \textit{ is off-diagonal}, \end{cases}$$

where we implicitly use the homology class of any meridian in $\mathrm{Cyl}^{p,q}$.

For $p = 1$ the $\mathfrak{su}(n)$ restricted torque of the $SO(n-1)$-invariant special Legendrian immersion $X_\tau : \mathrm{Cyl}^{1,n-1} \to \mathbb{S}^{2n-1}$ is given by

(5.16)
$$\mathcal{F}_{\mathsf{k}}(X_\tau) = \begin{cases} 2\tau\left(\lambda - \frac{1}{n-1}\sum_{j=1}^{n-1}\mu_j\right)\mathrm{Vol}(\mathbb{S}^{n-2}), & \mathsf{k} = i\,\mathrm{diag}(\lambda, \mu_1, \dots, \mu_{n-1}); \\ 0, & \mathsf{k} \in \mathfrak{su}(n) \textit{ is off-diagonal.} \end{cases}$$

In particular, if we take $\mathsf{k} = \mathsf{t}$ to be the generator of the 1-parameter subgroup $\{\widetilde{\mathsf{T}}_x\}_{x\in\mathbb{R}}$ (defined in (5.11)) associated to the rotational period $\widetilde{\mathsf{T}}_{2\widehat{\mathsf{p}}_\tau}$ of X_τ then we obtain

$$\mathcal{F}_{\mathsf{t}}(X_\tau) = \begin{cases} 2\tau\frac{n}{pq}\,\mathrm{Vol}(\mathbb{S}^{p-1})\,\mathrm{Vol}(\mathbb{S}^{q-1}), & \text{if } p > 1, \\ 2\tau\frac{n}{n-1}\,\mathrm{Vol}(\mathbb{S}^{n-2}), & \text{if } p = 1. \end{cases}$$

Proof. We give the proof in the case $p > 1$. The result in the case $p = 1$ follows by making the obvious adjustments to the argument below.

Case $p > 1$: By the homological invariance of $\mathcal{F}_{\mathsf{k}}(\Sigma)$ we may evaluate the k-flux on any meridian $\{t_0\} \times \mathrm{Mer}^{p,q}$ of $\mathrm{Cyl}^{p,q}$. From 5.9.iii the vector field ∂_t is orthogonal to any meridian $\{t_0\} \times \mathrm{Mer}^{p,q}$. Hence the unit conormal is given by $\eta = \partial_t X_\tau / |\partial_t X_\tau|$. By the definition of X_τ in terms of $\mathbf{w}_\tau$ we have $|\partial_t X_\tau| = |\dot{\mathbf{w}}|$. Using (2.22) and 5.9.ii the volume form induced on the meridian $\{t_0\} \times \mathrm{Mer}^{p,q}$ is

$$|w_1|^{p-1}|w_2|^{q-1} d\mathrm{v}_{\mathbb{S}^{p-1}} \wedge d\mathrm{v}_{\mathbb{S}^{q-1}} = |\partial_t X_\tau| d\mathrm{v}_{\mathbb{S}^{p-1}} \wedge d\mathrm{v}_{\mathbb{S}^{q-1}}.$$

Therefore

$$\mathcal{F}_{\mathsf{k}} = \int_{t=t_0} \mathsf{k} \cdot \frac{\partial_t X_\tau}{|\partial_t X_\tau|} |\partial_t X_\tau| d\mathrm{v}_{\mathbb{S}^{p-1}} \wedge d\mathrm{v}_{\mathbb{S}^{q-1}} = \int_{t=t_0} \mathsf{k} \cdot \partial_t X_\tau d\mathrm{v}_{\mathbb{S}^{p-1}} \wedge d\mathrm{v}_{\mathbb{S}^{q-1}}, \tag{5.17}$$

where $t = t_0$ is a shorthand for the meridian $\{t_0\} \times \mathrm{Mer}^{p,q}$ on which the $\mathbb{R}$ coordinate t equals t_0.

$\mathsf{k} \in \mathfrak{su}(n)$ *is diagonal:* If $\mathsf{k} = i \,\mathrm{diag}(\lambda_1, \dots, \lambda_p, \mu_1, \dots, \mu_q) \in \mathfrak{su}(n)$ a short computation shows that

$$\mathsf{k} \cdot \partial_t X_\tau = \mathrm{Im}(\overline{w}_1 \dot{w}_1) \sum_{i=1}^{p} \lambda_i (\sigma_1^i)^2 + \mathrm{Im}(\overline{w}_2 \dot{w}_2) \sum_{i=1}^{q} \mu_j (\sigma_2^j)^2,$$

where $\sigma_1 = (\sigma_1^1, \dots, \sigma_1^p) \in \mathbb{S}^{p-1} \subset \mathbb{R}^p$ and $\sigma_2 = (\sigma_2^1, \dots, \sigma_2^q) \in \mathbb{S}^{q-1} \subset \mathbb{R}^q$. Hence using (4.8) and the definition of τ, we have

$$\mathsf{k} \cdot \partial_t X_\tau = 2\tau \left(\sum_{i=1}^{p} \lambda_i (\sigma_1^i)^2 - \sum_{j=1}^{q} \mu_j (\sigma_2^j)^2 \right). \tag{5.18}$$

By symmetry we have

$$\int_{\mathbb{S}^{p-1}} (\sigma_1^i)^2 d\mathrm{v}_{\mathbb{S}^{p-1}} = \frac{1}{p} \mathrm{Vol}\, \mathbb{S}^{p-1} \quad \text{and} \quad \int_{\mathbb{S}^{q-1}} (\sigma_2^j)^2 d\mathrm{v}_{\mathbb{S}^{q-1}} = \frac{1}{q} \mathrm{Vol}\, \mathbb{S}^{q-1}, \tag{5.19}$$

for $1 \leq i \leq p$ and $1 \leq j \leq q$. Combining (5.17), (5.18) and (5.19) we obtain

$$\mathcal{F}_{\mathsf{k}} = 2\tau \left(\frac{1}{p} \sum_{i=1}^{p} \lambda_i - \frac{1}{q} \sum_{j=1}^{q} \mu_j \right) \mathrm{Vol}(\mathbb{S}^{p-1}) \,\mathrm{Vol}(\mathbb{S}^{q-1}).$$

$\mathsf{k} \in \mathfrak{su}(n)$ *is off-diagonal:* Off-diagonal elements $\mathsf{k} \in \mathfrak{su}(n)$ can be decomposed as $\mathfrak{so}(n) \oplus \mathrm{i}\,\mathrm{Sym}_{\mathrm{off}}(n, \mathbb{R})$ where $\mathrm{Sym}_{\mathrm{off}}(n, \mathbb{R})$ denotes the off-diagonal real symmetric $n \times n$ matrices. By linearity it suffices to prove $\mathcal{F}_{\mathsf{k}} = 0$ for any $\mathsf{k} \in \mathfrak{so}(n)$ and $\mathsf{k} \in i\,\mathrm{Sym}_{\mathrm{off}}(n, \mathbb{R})$.

First we show that $\mathcal{F}_{\mathsf{k}}$ vanishes for any $\mathsf{k} \in \mathfrak{so}(n) \subset \mathfrak{su}(n)$. Let $e_1, \ldots, e_n$ denote the standard unitary basis of $\mathbb{C}^n$. For $i \neq j \in \{1, \ldots, n\}$ define $\mathsf{R}_{ij} \in \mathfrak{so}(n)$ by

$$\mathsf{R}_{ij}(v) = (e_i \cdot v)\, e_j - (e_j \cdot v)\, e_i, \quad \text{for any } v \in \mathbb{R}^n.$$

$\{\mathsf{R}_{ij}\}$ for $i < j \in \{1, \ldots, n\}$ forms a basis for $\mathfrak{so}(n) \subset \mathfrak{su}(n)$. Using the definition of X_τ and R_{ij} we find

$$\mathsf{R}_{ij} X_\tau = \begin{cases} w_1(\sigma_1^i e_j - \sigma_1^j e_i), & \text{for } i, j \in \{1, \ldots, p\}; \\ w_2(\sigma_2^{i'} e_j - \sigma_2^{j'} e_i), & \text{for } i', j' \in \{1, \ldots, q\}; \\ w_1 \sigma_1^i e_j - w_2 \sigma_2^{j'} e_i, & \text{for } i \in \{1, \ldots, p\}, j' \in \{1, \ldots, q\}, \end{cases}$$

where $i' = i - p$ and $j' = j - p$. Taking the inner product with $\partial_t X_\tau$ we obtain

(5.20)
$$\mathsf{R}_{ij} X_\tau \cdot \partial_t X_\tau = \begin{cases} 0, & \text{for } i, j \in \{1, \ldots, p\}; \\ 0, & \text{for } i', j' \in \{1, \ldots, q\}; \\ \mathrm{Re}(\overline{w}_1 \dot{w}_2 - \overline{w}_2 \dot{w}_1) \sigma_1^i \sigma_2^{j'}, & \text{for } i \in \{1, \ldots, p\}, j' \in \{1, \ldots, q\}. \end{cases}$$

Clearly we have

(5.21)
$$\int_{\mathbb{S}^{p-1} \times \mathbb{S}^{q-1}} \sigma_1^i\, \sigma_2^{j'}\, d\mathrm{v}_{\mathbb{S}^{p-1}} \wedge d\mathrm{v}_{\mathbb{S}^{q-1}} = \int_{\mathbb{S}^{p-1}} \sigma_1^i d\mathrm{v}_{\mathbb{S}^{p-1}} \int_{\mathbb{S}^{q-1}} \sigma_2^{j'} d\mathrm{v}_{\mathbb{S}^{q-1}} = 0.$$

Combining (5.17), (5.20) and (5.21) we conclude $\mathcal{F}_{\mathsf{k}} = 0$ for $\mathsf{k} = \mathsf{R}_{ij}$ and hence by linearity $\mathcal{F}_{\mathsf{k}} = 0$ for all $\mathsf{k} \in \mathfrak{so}(n) \subset \mathfrak{su}(n)$.

Now we show that $\mathcal{F}_{\mathsf{k}} = 0$ for any $\mathsf{k} \in i\mathrm{Sym}_{\mathrm{off}}(n, \mathbb{R})$. For $i < j \in \{1, \ldots, n\}$ define $\mathsf{S}_{ij} \in \mathrm{Sym}_{\mathrm{off}}(n, \mathbb{R})$ by

$$\mathsf{S}_{ij}(v) = (e_i \cdot v)\, e_j + (e_j \cdot v)\, e_i, \quad \text{for any } v \in \mathbb{R}^n.$$

$\{\sqrt{-1}\,\mathsf{S}_{ij}\}$ for $i < j \in \{1, \ldots, n\}$ forms a basis for $i\,\mathrm{Sym}_{\mathrm{off}}(n, \mathbb{R}) \subset \mathfrak{su}(n)$. Using the definition of X_τ and S_{ij} we find

$$\sqrt{-1}\,\mathsf{S}_{ij} X_\tau = \sqrt{-1} \begin{cases} w_1(\sigma_1^i e_j + \sigma_1^j e_i), & \text{for } i, j \in \{1, \ldots, p\}; \\ w_2(\sigma_2^{i'} e_j + \sigma_2^{j'} e_i), & \text{for } i', j' \in \{1, \ldots, q\}; \\ w_1 \sigma_1^i e_j + w_2 \sigma_2^{j'} e_i, & \text{for } i \in \{1, \ldots, p\}, j' \in \{1, \ldots, q\}, \end{cases}$$

where as above $i' = i - p$ and $j' = j - p$. Taking the inner product with $\partial_t X_\tau$ we obtain

(5.22)
$$\mathsf{S}_{ij} X_\tau \cdot \partial_t X_\tau = \begin{cases} 2\,\mathrm{Im}(\overline{w}_1 \dot{w}_1)\sigma_1^i \sigma_1^j, & \text{for } \ i,j \in \{1,\ldots,p\}; \\ 2\,\mathrm{Im}(\overline{w}_2 \dot{w}_2)\sigma_2^{i'} \sigma_2^{j'}, & \text{for } \ i',j' \in \{1,\ldots,q\}; \\ \mathrm{Im}(\overline{w}_1 \dot{w}_2 + \overline{w}_2 \dot{w}_1)\sigma_1^i \sigma_2^{j'}, & \text{for } i \in \{1,\ldots,p\}, j' \in \{1,\ldots,q\}. \end{cases}$$

For any $i \neq j$ we have

$$\int_{\mathbb{S}^{p-1}\times\mathbb{S}^{q-1}} \sigma_1^i \sigma_1^j d\mathrm{v}_{\mathbb{S}^{p-1}} \wedge d\mathrm{v}_{\mathbb{S}^{q-1}} = \mathrm{Vol}(\mathbb{S}^{q-1}) \int_{\mathbb{S}^{p-1}} \sigma_1^i \sigma_1^j d\mathrm{v}_{\mathbb{S}^{p-1}} = 0, \tag{5.23}$$

since for any $i \neq j$, $\sigma_1^i \sigma_1^j$ is an eigenvalue of the Laplacian on $\mathbb{S}^{p-1}$ with eigenvalue $\lambda = 2p$, and hence is L^2-orthogonal to the constant functions. (Alternatively, one can consider the involution mapping $\sigma_1^i \mapsto -\sigma_1^i$ and $\sigma_1^k \mapsto -\sigma_1^k$ for any $k \notin \{i,j\}$ and fixing all other components of σ_1. Clearly this symmetry preserves $d\mathrm{v}_{\mathbb{S}^{p-1}}$ but sends $\sigma_1^i \sigma_1^j \mapsto -\sigma_1^i \sigma_1^j$. Hence the integral in (5.23) is odd under this symmetry and therefore vanishes.) Similarly, we have

$$\int_{\mathbb{S}^{p-1}\times\mathbb{S}^{q-1}} \sigma_2^{i'} \sigma_2^{j'} d\mathrm{v}_{\mathbb{S}^{p-1}} \wedge d\mathrm{v}_{\mathbb{S}^{q-1}} = \mathrm{Vol}(\mathbb{S}^{p-1}) \int_{\mathbb{S}^{q-1}} \sigma_2^{i'} \sigma_2^{j'} d\mathrm{v}_{\mathbb{S}^{q-1}} = 0, \tag{5.24}$$

for any $i \neq j$. For any $i \neq j$, combining (5.17), (5.21–5.24) implies that $\mathcal{F}_\mathsf{k} = 0$ for $\mathsf{k} = \sqrt{-1}\,\mathsf{S}_{ij}$ and hence by linearity $\mathcal{F}_\mathsf{k} = 0$ for all $\mathsf{k} \in i\,\mathrm{Sym}_{\mathrm{off}}(n,\mathbb{R}) \subset \mathfrak{su}(n)$. □

6. Period conditions for $\mathbf{w}_\tau$

In this section we study the conditions under which $\mathbf{w}_\tau$ forms a closed curve in $\mathbb{S}^3$ and also when the curve of isotropic $\mathrm{SO}(p)\times\mathrm{SO}(q)$ orbits determined by $\mathbf{w}_\tau$ is closed; the latter is directly related to understanding when the $\mathrm{SO}(p)\times\mathrm{SO}(q)$-invariant SL immersions X_τ factor through closed SL embeddings.

Symmetries of y_τ. We begin by establishing the symmetries of $y_\tau := |w_2|^2$ in the three cases (i) $p = 1$, (ii) $p > 1$ and $p \neq q$ and (iii) $p > 1$ and $p = q$.

To state these results we need to introduce some notation to describe the basic properties of y_τ. For $p > 1$, recall from (5.6) that y_τ satisfies the

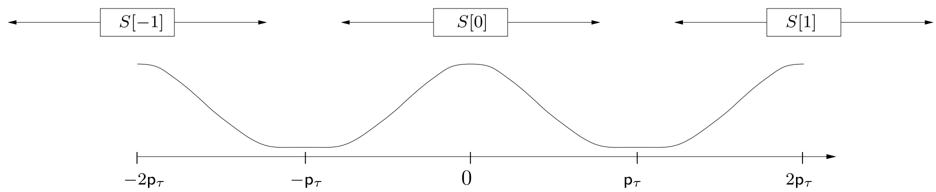

Figure 3: Profile of $y_\tau := |w_2|^2$ for $p = 1$.

initial conditions

$$y(0) = \frac{q}{n}, \quad \dot{y}(0) = -4\tau_{\max} \cos\alpha_\tau = -4\sqrt{\tau_{\max}^2 - \tau^2},$$

whereas for $p = 1$ from (5.5) it satisfies

$$y(0) = y_{\max}, \quad \dot{y}(0) = 0.$$

The different initial conditions for y_τ affect where the $2\mathsf{p}_\tau$-periodic function y_τ attains its maxima and minima in the cases $p = 1$ and $p > 1$. In the case $p > 1$ the choice of initial data for y_τ implies that there exist unique real numbers $\mathsf{p}_\tau^+, \mathsf{p}_\tau^- \in (0, \mathsf{p}_\tau)$ satisfying

$$y_\tau(-\mathsf{p}_\tau^-) = y_{\max}, \quad y_\tau(\mathsf{p}_\tau^+) = y_{\min}, \tag{6.1}$$

and so that y_τ is strictly decreasing on $(-\mathsf{p}_\tau^-, \mathsf{p}_\tau^+)$. We call these two numbers the *partial-periods* of y_τ, since

$$2\mathsf{p}_\tau = 2\mathsf{p}_\tau^+ + 2\mathsf{p}_\tau^-. \tag{6.2}$$

In general, p_τ^+ and p_τ^- are not related except when $p = q$ in which case we will prove shortly that $\mathsf{p}_\tau^+ = \mathsf{p}_\tau^-$. Illustrative plots of y_τ are shown in figures 3 and 4 for $p = 1$ and $p > 1$, $p \neq q$, respectively.

Throughout the following lemma we assume $0 < |\tau| < \tau_{\max}$ and discuss the exceptional cases $\tau = 0$ and $|\tau| = \tau_{\max}$ in Remark 6.11 below. Recall, also the notation for elements in $\mathrm{Isom}(\mathbb{R})$ introduced in Section 1 in Notation and Conventions.

Lemma 6.3 (Symmetries of y_τ).

(i) *For $p = 1$, $q = n - 1$ the symmetries of $y_\tau = |w_2|^2$ are generated by*

$$y_\tau \circ \mathsf{T}_{2\mathsf{p}_\tau} = y_\tau \quad \textit{and} \quad y_\tau \circ \underline{\mathsf{T}} = y_\tau. \tag{6.4}$$

That is, y_τ is an even $2\mathsf{p}_\tau$-periodic function. Moreover, we have

$$y_\tau(0) = y_{max} \quad \textit{and} \quad y_\tau(\mathsf{p}_\tau) = y_{min}. \tag{6.5}$$

(ii) *For $p > 1$ and $p \neq q$ the symmetries of y_τ are generated by*

$$y_\tau \circ \mathsf{T}_{2\mathsf{p}_\tau} = y_\tau, \quad y_\tau \circ \underline{\mathsf{T}}_{\mathsf{p}_\tau^+} = y_\tau \quad \textit{and} \quad y_\tau \circ \underline{\mathsf{T}}_{-\mathsf{p}_\tau^-} = y_\tau. \tag{6.6}$$

(iii) *For $p > 1$ and $p = q$ the symmetries of y_τ are generated by*

$$y_\tau \circ \mathsf{T}_{2\mathsf{p}_\tau} = y_\tau, \quad y_\tau \circ \underline{\mathsf{T}}_{\mathsf{p}_\tau/2} = y_\tau, \quad y_\tau \circ \underline{\mathsf{T}}_{-\mathsf{p}_\tau/2} = y_\tau \quad \textit{and} \quad y_\tau \circ \underline{\mathsf{T}} = 1 - y_\tau, \tag{6.7}$$

and the partial-periods defined in (6.2) satisfy

$$\mathsf{p}_\tau^+ = \mathsf{p}_\tau^- = \tfrac{1}{2}\mathsf{p}_\tau \quad \textit{and} \quad y_\tau(\tfrac{1}{2}\mathsf{p}_\tau) = y_{min}, \quad y_\tau(-\tfrac{1}{2}\mathsf{p}_\tau) = y_{max}. \tag{6.8}$$

Remark 6.9. It follows from the partial-period relation (6.2) that the reflections $\underline{\mathsf{T}}_{\mathsf{p}_\tau^+}$ and $\underline{\mathsf{T}}_{-\mathsf{p}_\tau^-}$ satisfy

$$\underline{\mathsf{T}}_{-\mathsf{p}_\tau^-} \circ \underline{\mathsf{T}}_{\mathsf{p}_\tau^+} = \mathsf{T}_{-2\mathsf{p}_\tau}, \quad \underline{\mathsf{T}}_{\mathsf{p}_\tau^+} \circ \underline{\mathsf{T}}_{-\mathsf{p}_\tau^-} = \mathsf{T}_{2\mathsf{p}_\tau}. \tag{6.10}$$

Hence the first symmetry of y_τ in (6.6) is a consequence of the second and third symmetries.

Similarly, it is straightforward to check that $\underline{\mathsf{T}} \circ \underline{\mathsf{T}}_{\mathsf{p}_\tau/2} \circ \underline{\mathsf{T}} = \underline{\mathsf{T}}_{-\mathsf{p}_\tau/2}$. It follows that the two symmetries $\underline{\mathsf{T}}$ and $\underline{\mathsf{T}}_{\mathsf{p}_\tau/2}$ are sufficient to generate all four symmetries in (6.7).

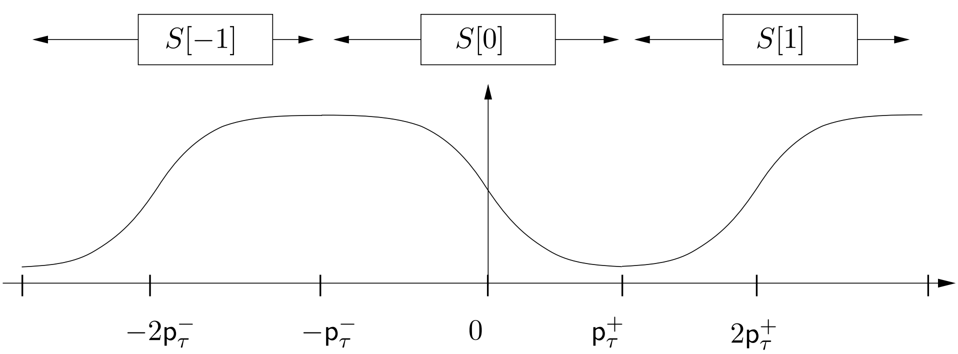

Figure 4: Profile of $y_\tau = |w_2|^2$ for $p > 1$.

Remark 6.11. For $\tau = 0$, y_τ is no longer periodic (the period $2\mathsf{p}_\tau \to \infty$ as $\tau \to 0$; see 7.3 for a more precise statement). For $p = 1$ we have already seen in 4.19.iv.b that y_0 is still even. For $p = q$, $y_0(0)$ is invariant under $y \mapsto 1 - y$, and hence y_0 retains the reflectional symmetry

$$y_0 \circ \underline{\mathsf{T}} = 1 - y_0.$$

When $|\tau| = \tau_{\max}$, y_τ is the constant function q/n, as noted in Proposition 4.19.

Proof of Lemma 6.3. Since the ODE (4.11) is autonomous we have time translation symmetry, i.e., for any solution y of (4.11) and any $t_0 \in \mathbb{R}$, $y \circ \mathsf{T}_{t_0}$ is also a solution of (4.11). Moreover, if y is a solution of (4.11) then so is $y \circ \underline{\mathsf{T}}$. Hence (4.11) is invariant under the whole of $\mathrm{Isom}(\mathbb{R})$. Equation (4.11) is also invariant under $y \mapsto 1 - y$ when $p = q$.
(i) Proof of (6.4): The first equality is immediate since y_τ has period $2\mathsf{p}_\tau$ by Proposition 4.19.i and (5.5). The second symmetry follows from the fact that $y_\tau(0) = y_{\max}$ as in (5.5).
(ii) Proof of (6.6): y_τ is periodic of period $2\mathsf{p}_\tau$ by 4.19.i. Since y_τ has a maximum and a minimum at $-\mathsf{p}_\tau^-$ and p_τ^+ respectively it has the two additional reflection symmetries listed in (6.6).
(iii) We need to prove that y_τ admits the new symmetry $y_\tau \circ \underline{\mathsf{T}} = 1 - y_\tau$. The rest of the claims made will then follow by combining this symmetry with the ones already established in part (ii). Define $\tilde{y} := (1 - y_\tau) \circ \underline{\mathsf{T}}$. $\tilde{y}$ is also a solution of (4.11) and we see from (5.6) that $\tilde{y}$ satisfies the same initial conditions as y_τ. Hence, by the uniqueness of solutions of the initial value problem $y_\tau \equiv (1 - y_\tau) \circ \underline{\mathsf{T}}$ as required. It follows that

$$y_{\max} + y_{\min} = 1, \tag{6.12}$$

and that $y_\tau(\mathsf{p}_\tau^-) = 1 - y_\tau(-\mathsf{p}_\tau^-) = 1 - y_{\max} = y_{\min} = y_\tau(\mathsf{p}_\tau^+)$. Hence $\mathsf{p}_\tau^- = \mathsf{p}_\tau^+ = \frac{1}{2}\mathsf{p}_\tau$. Since $\mathsf{p}_\tau^+ = \frac{1}{2}\mathsf{p}_\tau$, the existing reflectional symmetries $y_\tau \circ \underline{\mathsf{T}}_{\mathsf{p}_\tau^+} = y_\tau$ and $y_\tau \circ \underline{\mathsf{T}}_{-\mathsf{p}_\tau^-} = y_\tau$ become $y_\tau \circ \underline{\mathsf{T}}_{\mathsf{p}_\tau/2} = y_\tau$ and $y_\tau \circ \underline{\mathsf{T}}_{-\mathsf{p}_\tau/2} = y_\tau$, respectively. $\square$

The rotational period of $\mathbf{w}_\tau$. In this section, we study the behaviour of $\mathbf{w}_\tau$ under translation by a period $2\mathsf{p}_\tau$ of y_τ; we call this the *rotational period* of $\mathbf{w}_\tau$. It is fundamental to understanding when $\mathbf{w}_\tau$ forms a closed curve in $\mathbb{S}^3$ or in the space of isotopic $\mathrm{SO}(p) \times \mathrm{SO}(q)$ orbits.

If $\mathbf{w}_\tau = (w_1, w_2)$, $y_\tau = |w_2|^2$ and ψ_1 and ψ_2 denote the arguments of w_1 and w_2 respectively then the equations

$$\mathrm{Im}(\overline{w}_1 \dot{w}_1) = -\,\mathrm{Im}(\overline{w}_2 \dot{w}_2) = 2\tau,$$

are equivalent to

$$(1 - y_\tau)\dot{\psi}_1 = 2\tau, \qquad y_\tau \dot{\psi}_2 = -2\tau. \tag{6.13}$$

It is convenient to write $\mathbf{w}_\tau$ in the form

$$\begin{aligned} w_1(t) &= \begin{cases} \operatorname{sgn} t\sqrt{1 - y_0(t)}, & \text{for } \tau = 0; \\ -i\sqrt{1 - y_\tau(t)}\mathrm{e}^{\mathrm{i}\psi_1}, & \text{for } \tau > 0; \end{cases} \\ w_2(t) &= \begin{cases} \sqrt{y_0(t)}, & \text{for } \tau = 0; \\ \sqrt{y_\tau(t)}\mathrm{e}^{\mathrm{i}\psi_2}, & \text{for } \tau > 0; \end{cases} \end{aligned} \tag{6.14}$$

if $p = 1$ and

$$\begin{aligned} w_1(t) &= \begin{cases} \sqrt{1 - y_0(t)}, & \text{for } \tau = 0; \\ \sqrt{1 - y_\tau(t)}\mathrm{e}^{\mathrm{i}\alpha_\tau/2p}\mathrm{e}^{\mathrm{i}\psi_1}, & \text{for } \tau > 0; \end{cases} \\ w_2(t) &= \begin{cases} \sqrt{y_0(t)}, & \text{for } \tau = 0; \\ \sqrt{y_\tau(t)}\mathrm{e}^{\mathrm{i}\alpha_\tau/2q}\mathrm{e}^{\mathrm{i}\psi_2}, & \text{for } \tau > 0; \end{cases} \end{aligned} \tag{6.15}$$

if $p > 1$, where $\alpha_\tau \in [-\pi/2, \pi/2]$ was defined in (5.3) and where in both cases for $0 < \tau \leq \tau_{\max}$, $\psi_1, \psi_2 : \mathbb{R} \to \mathbb{R}$ are the unique solutions of (6.13) with initial conditions

$$\psi_1(0) = \psi_2(0) = 0. \tag{6.16}$$

The slightly different forms the above w_i take in the cases $p = 1$ and $p > 1$ stem from the fact that we have chosen the initial data $\mathbf{w}(0)$ for $\mathbf{w}_\tau$ differently in these two cases (recall (5.2) and (5.4)).

Define the function Ψ by

$$\Psi := p\psi_1 + q\psi_2. \tag{6.17}$$

Written in terms of y and Ψ the real and imaginary parts of equation (4.10) are equivalent to

$$\dot{y}_\tau = -2\sqrt{f(y)}\sin\Psi, \tag{6.18}$$

$$2\tau = \sqrt{f(y)}\cos\Psi, \tag{6.19}$$

for $p = 1$ and to

$$\dot{y}_\tau = -2\sqrt{f(y)}\cos(\Psi + \alpha_\tau), \tag{6.20}$$

$$-2\tau = \sqrt{f(y)}\sin(\Psi + \alpha_\tau) \tag{6.21}$$

for $p > 1$ with α_τ as defined in (5.3).

Definition 6.22. For any τ with $0 < |\tau| < \tau_{\max}$ we define the *angular period* $\widehat{\mathsf{p}}_\tau$ in terms of ψ_1 by

$$2\widehat{\mathsf{p}}_\tau := p\psi_1(2\mathsf{p}_\tau). \tag{6.23}$$

Lemma 6.24 (Rotational period of $\mathbf{w}_\tau$). *For $0 < \tau < \tau_{max}$ the angular period $\widehat{\mathsf{p}}_\tau$ defined in (6.23) satisfies*

$$\mathbf{w}_\tau \circ \mathsf{T}_{2\mathsf{p}_\tau} = \hat{\mathsf{T}}_{2\widehat{\mathsf{p}}_\tau} \circ \mathbf{w}_\tau, \tag{6.25}$$

where $\hat{\mathsf{T}}_x \in U(2)$ denotes the 1-parameter group defined in (4.1).

We call $\hat{\mathsf{T}}_{2\widehat{\mathsf{p}}_\tau}$ the *rotational period of* $\mathbf{w}_\tau$ since by (6.25) it controls how $\mathbf{w}_\tau$ gets rotated as we move from one domain of periodicity of y_τ to the next.

Proof. The $2\mathsf{p}_\tau$-periodicity of y_τ (recall 6.3) and the definition of ψ_i in terms of y_τ given in (6.13) imply ($i = 1, 2$)

$$\psi_i \circ \mathsf{T}_{2\mathsf{p}_\tau} = \psi_i + \psi_i(2\mathsf{p}_\tau),$$

and hence $\mathbf{w}_\tau \circ \mathsf{T}_{2\mathsf{p}_\tau} = (\mathrm{e}^{\mathrm{i}\psi_1(2\mathsf{p}_\tau)}w_1, \mathrm{e}^{\mathrm{i}\psi_2(2\mathsf{p}_\tau)}w_2) = (\mathrm{e}^{\mathrm{i}2\widehat{\mathsf{p}}_\tau/p}w_1, \mathrm{e}^{\mathrm{i}\psi_2(2\mathsf{p}_\tau)}w_2)$. It remains to prove that

$$\Psi(2\mathsf{p}_\tau) = p\psi_1(2\mathsf{p}_\tau) + q\psi_2(2\mathsf{p}_\tau) = 0.$$

This follows from (6.19) (if $p = 1$) or (6.21) (if $p > 1$), the $2\mathsf{p}_\tau$-periodicity of y_τ and the initial condition $\Psi(0) = 0$. $\square$

In Section 7, we prove that the angular period $2\widehat{\mathsf{p}}_\tau$ is a nonconstant analytic function of τ for $0 < |\tau| < \tau_{\max}$ that satisfies

$$\lim_{\tau \to 0} \widehat{\mathsf{p}}_\tau = \frac{\pi}{2}.$$

Periods and half-periods of $\mathbf{w}_\tau$. We want to understand when the (p, q)-twisted SL curves $\mathbf{w}_\tau$ form closed curves in $\mathbb{S}^3$. Moreover, to understand when the $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrian immersions X_τ

close up we need to understand when $\mathbf{w}_\tau$ gives rise to a closed curve in the space of isotropic $\mathrm{SO}(p)\times\mathrm{SO}(q)$ orbits. As described in Lemma 3.1 this orbit space is $\mathbb{S}^3/\mathrm{Stab}_{p,q}$ where $\mathrm{Stab}_{p,q}\subset \mathrm{U}(2)$ is the finite subgroup defined in (4.3).

To this end we define the periods and half-periods of $\mathbf{w}_\tau$. The periods and half-periods of $\mathbf{w}_\tau$ control when the curve of isotropic orbits $\mathcal{O}_{\mathbf{w}_\tau}$ determined by $\mathbf{w}_\tau$ is a closed curve in the space of $\mathrm{SO}(p)\times\mathrm{SO}(q)$ orbits. Recall from 4.2 the definitions of the periods and half-periods of the 1-parameter group $\{\hat{\mathsf{T}}_x\}$ defined in (4.1). The periods and half-periods of $\mathbf{w}_\tau$ and the periods and half-periods of $\{\hat{\mathsf{T}}_x\}$ are intimately connected because of (6.25).

Definition 6.26. Fix a pair of admissible integers p and q and let $\mathbf{w}_\tau$ be any of the (p,q)-twisted SL curves defined in 5.1. We define the period lattice of $\mathbf{w}_\tau$ by

$$\mathrm{Per}(\mathbf{w}_\tau) := \{x\in\mathbb{R}\,|\,\mathbf{w}_\tau\circ\mathsf{T}_x=\mathbf{w}_\tau\}, \tag{6.27}$$

and the half-period lattice of $\mathbf{w}_\tau$ by

$$\mathrm{Per}_{\frac12}(\mathbf{w}_\tau) := \{x\in\mathbb{R}\,|\,\mathcal{O}_{\mathbf{w}_\tau\circ\mathsf{T}_x(t)}=\mathcal{O}_{\mathbf{w}_\tau(t)}\ \forall\, t\in\mathbb{R}\}, \tag{6.28}$$

where as previously $\mathcal{O}_{\mathbf{w}}\subset\mathbb{S}^{2(p+q)-1}$ denotes the isotropic $\mathrm{SO}(p)\times\mathrm{SO}(q)$ orbit associated with any point $\mathbf{w}\in\mathbb{S}^3$. In other words, x is a half-period of $\mathbf{w}_\tau$ if $\mathbf{w}_\tau\circ\mathsf{T}_x$ and $\mathbf{w}_\tau$ give rise to the same parametrised curve of isotropic $\mathrm{SO}(p)\times\mathrm{SO}(q)$-orbits in $\mathbb{S}^{2(p+q)-1}$. We call elements of $\mathrm{Per}_{\frac12}(\mathbf{w}_\tau)$ the *half-periods of* $\mathbf{w}_\tau$, and elements of $\mathrm{Per}(\mathbf{w}_\tau)$ the *periods of* $\mathbf{w}_\tau$. A *strict half-period* is any half-period which is not a period of $\mathbf{w}_\tau$.

Using 3.1 we see that x is a half-period of $\mathbf{w}_\tau$ if and only if

$$\mathbf{w}_\tau\circ\mathsf{T}_x=\rho_{jk}\circ\mathbf{w}_\tau,\quad \text{for some } \rho_{jk}\in\mathrm{Stab}_{p,q}, \tag{6.29}$$

where as above $\mathrm{Stab}_{p,q}$ is the finite subgroup of $\mathrm{U}(2)$ defined in (4.3). More explicitly, we have

$$\begin{aligned}\mathrm{Per}_{\frac12}(\mathbf{w}_\tau) := \{x\in\mathbb{R}\,|\,&\exists (j,k)\in\langle(+,\pm)\rangle\le\mathbb{Z}_2\times\mathbb{Z}_2 \text{ such that}\\ &\rho_{jk}\circ\mathbf{w}_\tau=\mathbf{w}_\tau\circ\mathsf{T}_x\},\quad \text{if } p=1;\end{aligned} \tag{6.30}$$

or

$$\begin{aligned} \text{(6.31)} \qquad \mathrm{Per}_{\frac{1}{2}}(\mathbf{w}_\tau) := \{x \in \mathbb{R} \,|\, \exists (j,k) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \text{ such that} \\ \rho_{jk} \circ \mathbf{w}_\tau = \mathbf{w}_\tau \circ \mathsf{T}_x\}, \quad \text{if } p > 1. \end{aligned}$$

If x satisfies (6.29) for $(j,k) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ then we call x a *half-period of* $\mathbf{w}_\tau$ *of type* (jk). We see immediately from (6.29) that $2\,\mathrm{Per}_{\frac{1}{2}}(\mathbf{w}_\tau) \subset \mathrm{Per}(\mathbf{w}_\tau)$; this explains the terminology half-period.

The importance of the half-periods of $\mathbf{w}_\tau$ for understanding the geometry of X_τ is explained by the following:

Proposition 6.32. *Suppose* $0 < |\tau| < \tau_{max}$ *and let* X_τ *be one of the* $SO(p) \times SO(q)$*-invariant special Legendrian cylinders defined in 5.8. Suppose there exist triples* $(t_1, \sigma_1, \sigma_2), (t_2, \sigma_1', \sigma_2') \in \mathrm{Cyl}^{p,q}$ *such that*

$$X_\tau(t_1, \sigma_1, \sigma_2) = X_\tau(t_2, \sigma_1', \sigma_2'). \tag{6.33}$$

Then $t_2 - t_1 \in \mathrm{Per}_{\frac{1}{2}}(\mathbf{w}_\tau)$*. Moreover, if* $t_2 - t_1 \in \mathrm{Per}(\mathbf{w}_\tau)$ *then* $\sigma_1 = \sigma_1'$ *and* $\sigma_2 = \sigma_2'$*.*

Proof. From the definition of X_τ in terms of $\mathbf{w}_\tau$ and the isotropic $\mathrm{SO}(p) \times \mathrm{SO}(q)$ orbits $\mathcal{O}_{\mathbf{w}}$ we see that (6.33) implies that $\mathcal{O}_{\mathbf{w}_\tau(t_1)} \cap \mathcal{O}_{\mathbf{w}_\tau(t_2)} \neq \emptyset$ and therefore $\mathcal{O}_{\mathbf{w}_\tau(t_1)} = \mathcal{O}_{\mathbf{w}_\tau(t_2)}$. Hence by 3.1 we have

$$\mathbf{w}_\tau(t_1) = \rho_{jk}\mathbf{w}_\tau(t_2) \quad \text{for some } \rho_{jk} \in \mathrm{Stab}_{p,q}, \tag{6.34}$$

and

$$\sigma_1 = (-1)^j \sigma_1', \quad \sigma_2 = (-1)^k \sigma_2'. \tag{6.35}$$

Using conservation of $\mathcal{I}_2 = \mathrm{Im}(w_1^p w_2^q)$ and (6.34) we have

$$\mathrm{Im}\, w_1^p w_2^q(t_2) = \mathrm{Im}\, w_1^p w_2^q(t_1) = (-1)^{jp+kq}\, \mathrm{Im}\, w_1^p w_2^q(t_2).$$

Hence we have

$$jp + kq \equiv 0 \mod 2. \tag{6.36}$$

Now define $\tilde{\mathbf{w}}$ by

$$\tilde{\mathbf{w}} := \rho_{jk} \circ \mathbf{w}_\tau \circ \mathsf{T}_{t_2 - t_1}.$$

Using the definition of $\tilde{\mathbf{w}}$ and (6.34) we have

$$\tilde{\mathbf{w}}(t_1) = \rho_{jk} \circ \mathbf{w}_\tau(t_2) = \mathbf{w}_\tau(t_1).$$

Because j and k satisfy (6.36) $\tilde{\mathbf{w}}$ is another solution of (4.8) and therefore by uniqueness of the initial value problem $\tilde{\mathbf{w}} \equiv \mathbf{w}_\tau$. It follows that $t_2 - t_1 \in \mathrm{Per}_{\frac{1}{2}}(\mathbf{w}_\tau)$. The final statement in 6.32 follows from (6.35). $\square$

As a simple corollary of 6.32 we have

Corollary 6.37. *Suppose there exist $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^+$ such that $\mathbf{w}_\tau(t_0 + x_0) = \mathbf{w}_\tau(t_0)$, i.e., the curve $\mathbf{w}_\tau$ has a point of self-intersection, then $x_0 \in \mathrm{Per}(\mathbf{w}_\tau)$. Hence either*

(i) $\mathrm{Per}(\mathbf{w}_\tau) = (0)$ *in which case $\mathbf{w}_\tau : \mathbb{R} \to \mathbb{S}^3$ is an injective immersion, or*

(ii) *there exists $T > 0$, such that $T \in \mathrm{Per}(\mathbf{w}_\tau)$ is the smallest nontrivial period of $\mathbf{w}_\tau$ and the restriction $\mathbf{w}_\tau : [0, T] \to \mathbb{S}^3$ is a closed embedded curve.*

In particular, $\mathbf{w}_\tau$ forms a closed curve in $\mathbb{S}^3$ if and only if $\mathrm{Per}(\mathbf{w}_\tau) \neq 0$.

For the rest of this section we always assume $0 < |\tau| < \tau_{\max}$ unless stated otherwise. We now completely determine the periods and half-periods of $\mathbf{w}_\tau$ in terms of the rotational period $\hat{\mathsf{T}}_{2\widehat{\mathsf{p}}_\tau}$ (recall 6.24).

Definition 6.38. Define $k_0 \in \mathbb{N} \cup \{+\infty\}$ to be the order of the rotational period $\hat{\mathsf{T}}_{2\widehat{\mathsf{p}}_\tau} \in \mathrm{U}(2)$. We set $k_0 = +\infty$ if the rotational period has infinite order.

We can completely describe the period lattice $\mathrm{Per}(\mathbf{w}_\tau)$ and the half-period lattice $\mathrm{Per}_{\frac{1}{2}}(\mathbf{w}_\tau)$ in terms of k_0:

Lemma 6.39. *Fix a pair of admissible integers p and q and let $n = p + q$. k_0 the order of the rotational period $\hat{\mathsf{T}}_{2\widehat{\mathsf{p}}_\tau}$ defined in 6.38 can also be characterised as*

$$k_0 = \min\{k \in \mathbb{Z}^+ |\ k\widehat{\mathsf{p}}_\tau \in \pi\,\mathrm{lcm}(p,q)\mathbb{Z}\}, \tag{6.40}$$

and the following are equivalent

$$\widehat{\mathsf{p}}_\tau \notin \pi\mathbb{Q} \quad \Longleftrightarrow \quad k_0 = \infty \quad \Longleftrightarrow \quad \mathrm{Per}_{\frac{1}{2}}(\mathbf{w}_\tau) = \mathrm{Per}(\mathbf{w}_\tau) = (0).$$

If $\widehat{\mathsf{p}}_\tau \in \pi\mathbb{Q}$, *then in all cases we have*

$$\operatorname{Per}(\mathbf{w}_\tau) = 2k_0\mathsf{p}_\tau\mathbb{Z}, \tag{6.41}$$

and

(i) *if* k_0 *is odd then* $\operatorname{Per}_{\frac12}(\mathbf{w}_\tau) = \operatorname{Per}(\mathbf{w}_\tau) = 2k_0\mathsf{p}_\tau\mathbb{Z}$, *i.e.,* $\mathbf{w}_\tau$ *has no strict half-periods.*

(ii) *if* k_0 *is even and* $p > 1$ *then* $\operatorname{Per}_{\frac12}(\mathbf{w}_\tau) = \frac12 \operatorname{Per}(\mathbf{w}_\tau) = k_0\mathsf{p}_\tau\mathbb{Z}$. *Moreover, for fixed* p *and* q *every strict half-period of* $\mathbf{w}_\tau$ *is of type* (jk) *where* $j = q/\operatorname{hcf}(p,q) \mod 2$ *and* $k = p/\operatorname{hcf}(p,q) \mod 2$.

(iii) *a. if* k_0 *is even,* $p = 1$ *and* n *is even then* $\operatorname{Per}_{\frac12}(\mathbf{w}_\tau) = \operatorname{Per}(\mathbf{w}_\tau) = 2k_0\mathsf{p}_\tau\mathbb{Z}$, *i.e.,* $\mathbf{w}_\tau$ *has no strict half-periods.*
b. if k_0 *is even,* $p = 1$ *and* n *is odd then* $\operatorname{Per}_{\frac12}(\mathbf{w}_\tau) = \frac12 \operatorname{Per}(\mathbf{w}_\tau) = k_0\mathsf{p}_\tau\mathbb{Z}$ *(and every strict-half period is necessarily of type* $(+-)$*.)*

Proof. First, we show that

(6.42)
$$x \in \operatorname{Per}_{\frac12}(\mathbf{w}_\tau) \iff x = 2k\mathsf{p}_\tau \text{ for some } k \in \mathbb{Z} \text{ and } 2k\widehat{\mathsf{p}}_\tau \in \operatorname{Per}_{\frac12}(\{\hat{\mathsf{T}}_x\}),$$

and that

(6.43)
$$x \in \operatorname{Per}(\mathbf{w}_\tau) \iff x = 2k\mathsf{p}_\tau \text{ for some } k \in \mathbb{Z} \text{ and } \ 2k\widehat{\mathsf{p}}_\tau \in \operatorname{Per}(\{\hat{\mathsf{T}}_x\}) = 2\pi \operatorname{lcm}(p,q).$$

Proof of 6.42: Suppose $x \in \operatorname{Per}_{\frac12}(\mathbf{w}_\tau)$. From the definition of $\operatorname{Per}_{\frac12}(\mathbf{w}_\tau)$, $w_2 \circ \mathsf{T}_x = \pm w_2$. Since $y_\tau = |w_2|^2$ this implies $y_\tau \circ \mathsf{T}_x = y_\tau$ and hence $x \in \operatorname{Per}(y_\tau) = 2\mathsf{p}_\tau\mathbb{Z}$. Then from (6.25) we have

$$\mathbf{w}_\tau \circ \mathsf{T}_{2k\mathsf{p}_\tau} = \hat{\mathsf{T}}_{2k\widehat{\mathsf{p}}_\tau} \circ \mathbf{w}_\tau.$$

Hence $2k\mathsf{p}_\tau$ is a half-period of $\mathbf{w}_\tau$ of type (jk) if and only if $\hat{\mathsf{T}}_{2k\widehat{\mathsf{p}}_\tau} = \rho_{jk}$. This is equivalent to $2k\widehat{\mathsf{p}}_\tau$ being a half-period of $\{\hat{\mathsf{T}}_x\}$ of type (jk) and 6.42 now follows using 4.5. (6.43) follows from 6.42 by looking only at half-periods of type $(++)$ and using 4.5. The characterization of k_0 given in (6.40) follows immediately from 6.43. The equivalences in the line following (6.40) follow from the characterization of k_0 given in (6.40) together with (6.27).

Now suppose $x \in \mathrm{Per}_{\frac{1}{2}}(\mathbf{w}_\tau)$ and $\widehat{\mathsf{p}}_\tau \in \pi\mathbb{Q}$, so that the rotational period k_0 is finite. Then from 6.42 and (6.41) we have

$$(6.44) \qquad x \in 2\mathsf{p}_\tau\mathbb{Z} \cap \tfrac{1}{2}\,\mathrm{Per}(\mathbf{w}_\tau) = 2\mathsf{p}_\tau\mathbb{Z} \cap k_0\mathsf{p}_\tau\mathbb{Z} = \mathrm{lcm}(2,k_0)\mathsf{p}_\tau\mathbb{Z} = \begin{cases} 2k_0\mathsf{p}_\tau\mathbb{Z} & k_0 \text{ odd;} \\ k_0\mathsf{p}_\tau\mathbb{Z} & k_0 \text{ even.} \end{cases}$$

(i) If k_0 is odd then from (6.44) $x \in 2k_0\mathsf{p}_\tau\mathbb{Z} = \mathrm{Per}(\mathbf{w}_\tau)$ and hence $\mathrm{Per}_{\frac{1}{2}}(\mathbf{w}_\tau) = \mathrm{Per}(\mathbf{w}_\tau)$ as required.
If k_0 is even, then from (6.44) $x \in k_0\mathsf{p}_\tau\mathbb{Z}$. Furthermore, if x is a strict half-period of $\mathbf{w}_\tau$ then $x \in k_0\mathsf{p}_\tau(2\mathbb{Z}+1)$.
(ii) Suppose now that $p > 1$ and hence by (6.31) we should consider all types of half-period. Given any $x \in k_0\mathsf{p}_\tau(2\mathbb{Z}+1)$ notice that $\mathbf{w}_\tau \circ \mathsf{T}_x = \mathbf{w}_\tau \circ \mathsf{T}_{k_0\mathsf{p}_\tau}$ since $2k_0\mathsf{p}_\tau\mathbb{Z} = \mathrm{Per}(\mathbf{w}_\tau)$. Since k_0 is assumed even, $k_0\mathsf{p}_\tau \in \mathrm{Per}(y_\tau)$ and hence $\mathbf{w}_\tau \circ \mathsf{T}_{k_0\mathsf{p}_\tau} = \hat{\mathsf{T}}_{k_0\widehat{\mathsf{p}}_\tau} \circ \mathbf{w}_\tau$. By 6.39 and the definition of k_0, $\hat{\mathsf{T}}_{k_0\widehat{\mathsf{p}}_\tau} \neq \mathrm{Id}$ but $\hat{\mathsf{T}}_{2k_0\widehat{\mathsf{p}}_\tau} = \mathrm{Id}$. Hence from the diagonal form of $\hat{\mathsf{T}}_x$ we must have $\hat{\mathsf{T}}_{k_0\widehat{\mathsf{p}}_\tau} = \rho_{jk} \neq \mathrm{Id}$ for some $(jk) \neq (++)$. Hence x is a strict half-period as claimed. Moreover, since $k_0\widehat{\mathsf{p}}_\tau$ is a strict half-period of $\{\hat{\mathsf{T}}_x\}$ then by 4.5 it must be a half-period of type (jk) with j and k as in 6.39.ii.
(iii) If $p = 1$ the result follows using the structure of $\mathrm{Per}_{\frac{1}{2}}(\{\hat{\mathsf{T}}_x\})$ established in 4.5.iii. □

We define the subgroup $\mathrm{Per}(X_\tau) \subset \mathrm{Diff}(\mathrm{Cyl}^{p,q})$ by

$$(6.45) \qquad \mathrm{Per}(X_\tau) := \{\mathsf{M} \in \mathrm{Diff}(\mathrm{Cyl}^{p,q}) \mid X_\tau \circ \mathsf{M} = X_\tau\}.$$

$\mathrm{Per}(X_\tau)$ is important because the immersion $X_\tau : \mathrm{Cyl}^{p,q} \to \mathbb{S}^{2n-1}$ factors through an embedding of the quotient manifold $\mathrm{Cyl}^{p,q}/\,\mathrm{Per}(X_\tau)$. Combining 6.32 and 6.39 we obtain the following structure result for $\mathrm{Per}(X_\tau)$:

Corollary 6.46. *If k_0 the order of the rotational period $\hat{\mathsf{T}}_{2\widehat{\mathsf{p}}_\tau}$ is infinite then $\mathrm{Per}(X_\tau) = (0)$ and otherwise*

$$\mathrm{Per}(X_\tau) = \begin{cases} \langle(\mathsf{T}_{k_0\mathsf{p}_\tau}, -\mathrm{Id}_{\mathbb{S}^{n-1}})\rangle, & \text{if } p=1 \text{ and } k_0 \text{ is even and } n \text{ is odd;} \\ \langle(\mathsf{T}_{k_0\mathsf{p}_\tau}, (-1)^j\,\mathrm{Id}_{\mathbb{S}^{p-1}}, (-1)^k\,\mathrm{Id}_{\mathbb{S}^{q-1}})\rangle, & \text{if } p>1 \text{ and } k_0 \text{ is even;} \\ \langle\mathsf{T}_{2k_0\mathsf{p}_\tau}\rangle, & \text{otherwise;} \end{cases}$$

where $j = q/\,\mathrm{hcf}(p,q)$ and $k = p/\,\mathrm{hcf}(p,q)$.

In particular the $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant SL immersion X_τ factors through an embedding of a closed manifold if and only if k_0 is finite. In the third case above this closed manifold is diffeomorphic to $S^1 \times S^{p-1} \times S^{q-1}$ if $p > 1$ and to $S^1 \times S^{n-2}$ if $p = 1$. In the first case the manifold is diffeomorphic to a $\mathbb{Z}_2$ quotient of $S^1 \times S^{n-1}$ and in the second case to a $\mathbb{Z}_2$ quotient of $S^1 \times S^{p-1} \times S^{q-1}$.

7. Closed twisted SL curves and closed embedded special Legendrians

In this section, we prove the existence of infinitely many closed (p, q)-twisted SL curves and infinitely many closed embedded $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrian submanifolds.

By 6.37 closure of the curve $\mathbf{w}_\tau$ is determined by the period lattice $\mathrm{Per}(\mathbf{w}_\tau)$ and hence by 6.39 the rationality of the angular period $\widehat{\mathsf{p}}_\tau/\pi$. Therefore, it will suffice to prove that the angular period $\widehat{\mathsf{p}}_\tau$ is a nonconstant real analytic function of $\tau \in (0, \tau_{\max})$. The main point is to study the $\tau \to 0$ asymptotics of the angular period $\widehat{\mathsf{p}}_\tau$.

To obtain the $\tau \to 0$ asymptotics of $\widehat{\mathsf{p}}_\tau$ we will need an auxiliary result describing the $\tau \to 0$ asymptotics of the period $2\mathsf{p}_\tau$. In order to describe these asymptotics it helps to introduce the following notation: We define functions of τ by

$$T_k(\tau) := \begin{cases} \tau^{-1+2/k}, & \text{for } k > 2; \\ \log \tau^{-1}, & \text{for } k = 2, \end{cases} \tag{7.1}$$

and introduce the notation $f_1 \sim f_2$ for functions f_1 and f_2 of τ to mean that

$$\frac{f_2(\tau)}{f_1(\tau)} \to 1, \quad \text{as} \quad \tau \to 0. \tag{7.2}$$

Using this notation we have the following :

Proposition 7.3 (Small τ asymptotics of the period p_τ and partial-periods p^+_τ and p^-_τ).

(i) *For $p > 1$, p^+_τ and p^-_τ are analytic functions of τ for $0 < |\tau| < \tau_{max}$. For $p = 1$, p_τ is an analytic function of τ for $0 < |\tau| < \tau_{max}$.*

(ii) *In the case $p > 1$ we have*

$$\mathsf{p}^+_\tau \sim b_q\, T_q(\tau), \quad \mathsf{p}^-_\tau \sim b_p\, T_p(\tau), \tag{7.4}$$

where

(7.5)
$$b_2 := 1, \quad b_k := 4^{-1+\frac{1}{k}} \int_1^\infty \frac{dz}{\sqrt{z^k - 1}} = 4^{-1+\frac{1}{k}} \frac{\sqrt{\pi}\,\Gamma(\frac{1}{2} - \frac{1}{k})}{\Gamma(-\frac{1}{k})}, \quad \textit{for } k \geq 2,$$

where Γ *is the gamma function. We also have*

(7.6)
$$\mathsf{p}_\tau \sim b_q\, T_q(\tau) \quad \textit{when} \quad 1 \leq p < q, \quad \mathsf{p}_\tau \sim 2b_q\, T_q(\tau), \quad \textit{when} \quad 2 \leq p = q.$$

Proof. For $p = 1$, $\dot{y}_\tau(\mathsf{p}_\tau) = 0$ and for $p > 1$, $\dot{y}_\tau(\mathsf{p}^+_\tau) = \dot{y}_\tau(-\mathsf{p}^-_\tau) = 0$ and locally the vanishing of $\dot{y}$ determines p_τ and p^+_τ and p^-_τ. Moreover, $\ddot{y} = 2f'(y(t))$ is nonzero at $t = \mathsf{p}_\tau$ if $p = 1$ or at either p^+_τ and $-\mathsf{p}^-_\tau$ if $p > 1$ for all $\tau \in (0, \tau_{\max})$. Analyticity of p^+_τ, p^-_τ in the case $p > 1$ (and hence $\mathsf{p}_\tau = \mathsf{p}^+_\tau + \mathsf{p}^-_\tau$) and p_τ in the case $p = 1$ now follows from the real analytic Implicit Function Theorem.

Assume now that $p > 1$. By using (4.11), (5.6) and (6.1), we have that

$$\mathsf{p}^+_\tau = \int_{y_{\min}}^{q/n} \frac{dy}{2\sqrt{y^q(1-y)^p - 4\tau^2}}, \quad \mathsf{p}^-_\tau = \int_{q/n}^{y_{\max}} \frac{dy}{2\sqrt{y^q(1-y)^p - 4\tau^2}}.$$

Clearly if we substitute the limits $y_{\min}$ and $y_{\max}$ in the above integrals by $y_{\min} + \delta$ and $y_{\max} - \delta$ where δ is a small positive number, the integrals we get converge as $\tau \to 0$ to constants, which depend only on δ. Moreover, since for $y \in [y_{\min}, y_{\min} + \delta]$ we have

$$(1 - y_{\min} - \delta)^{p/2} \sqrt{\max(0, y^q - 4(\tau')^2)} \leq \sqrt{y^q(1-y)^p - 4\tau^2} \leq \sqrt{y^q - 4\tau^2},$$

where $\tau' := \tau(1 - y_{\min} - \delta)^{-p/2}$, and for $y \in [y_{\max} - \delta, y_{\max}]$ we have

$$\begin{aligned}(y_{\max} - \delta)^{q/2} \sqrt{\max(0, (1-y)^p - 4(\tau'')^2)} &\leq \sqrt{y^q(1-y)^p - 4\tau^2} \\ &\leq \sqrt{(1-y)^p - 4\tau^2},\end{aligned}$$

where $\tau'' = \tau(y_{\max} - \delta)^{-q/2}$, it is enough to prove

$$\int_{y_{\min}}^{y_{\min}+\delta} \frac{dy}{\sqrt{y^q - 4\tau^2}} \sim 2b_q\, T_q(\tau), \quad \int_{y_{\max}-\delta}^{y_{\max}} \frac{dy}{\sqrt{(1-y)^p - 4\tau^2}} \sim 2b_p\, T_p(\tau).$$

This follows easily by using (4.22) and integration by substitution (substituting $z = y(4\tau^2)^{-1/q}$ or $z = (1-y)(4\tau^2)^{-1/p}$, respectively), and concludes the proof when $p > 1$ (recall also (6.2)).

When $p = 1$ by using (4.11), (5.5) and (6.5), we have

$$\mathsf{p}_\mathcal{T} = \int_{y_{\min}}^{y_{\max}} \frac{dy}{2\sqrt{y^{n-1}(1-y) - 4\tau^2}}$$

and as before the proof reduces to

$$\int_{y_{\min}}^{y_{\min}+\delta} \frac{dy}{\sqrt{y^{n-1} - 4\tau^2}} \sim 2b_{n-1}\, T_{n-1}(\tau).$$

□

Proposition 7.7. *For* $0 < \tau < \tau_{max}$ *the angular period* $\widehat{\mathsf{p}}_\tau$ *(defined in (6.23)) is a nonconstant analytic function, which satisfies*

$$\lim_{\tau \to \tau_{max}} \widehat{\mathsf{p}}_\tau = \pi\sqrt{\frac{2pq}{n}}, \tag{7.8}$$

and

$$\lim_{\tau \to 0} \widehat{\mathsf{p}}_\tau = \frac{\pi}{2}. \tag{7.9}$$

Proof. Real analyticity of $\widehat{\mathsf{p}}_\tau$ for $\tau \in (0, \tau_{\max})$ follows from real analyticity of $\mathbf{w}_\tau$, p_τ^+, p_τ^- and p_τ and the definition of $\widehat{\mathsf{p}}_\tau$ (6.23). To show that $\widehat{\mathsf{p}}_\tau$ is nonconstant it suffices to calculate the asymptotics of $\widehat{\mathsf{p}}_\tau$ as $\tau \to 0$ and $\tau \longrightarrow \tau_{\max}$.

When $\tau = \tau_{\max}$, we have $y \equiv q/n$ and $p\,\dot{\psi}_1 \equiv 2n\tau_{\max}$. Hence we have

$$\lim_{\tau \to \tau_{\max}} 2\widehat{\mathsf{p}}_\tau = \lim_{\tau \to \tau_{\max}} p\,\psi_1(2\mathsf{p}_\tau) = 4n\tau_{\max} \lim_{\tau \to \tau_{\max}} \mathsf{p}_\tau.$$

The asymptotics for $\widehat{\mathsf{p}}_\tau$ now follow from the asymptotics for p_τ established in (4.20).

Now we prove (7.9), dealing first with the case $p = 1$. We define p_τ^* to be the unique $t \in (0, \mathsf{p}_\tau)$ such that $y(\mathsf{p}_\tau^*) = \frac{n-1}{n}$. Also $\dot{y}(\mathsf{p}_\tau^*) = -4\sqrt{\tau_{\max}^2 - \tau^2} \neq 0$ for $\tau \in (-\tau_{\max}, \tau_{\max})$. Hence by the real analytic Implicit Function Theorem p_τ^* is an analytic function of τ in $(-\tau_{\max}, \tau_{\max})$. In particular, p_τ^* approaches a finite limit as $\tau \to 0$. Recall the function Ψ defined in (6.17). The initial condition for y together with (6.18) and (6.19) implies that

$\Psi \in (0, \frac{\pi}{2})$ for $t \in (0, \mathsf{p}_\tau)$. From (6.19) we have $\cos \Psi(\mathsf{p}_\tau^*) = \frac{\tau}{\tau_{\max}}$ and therefore

$$\Psi(\mathsf{p}_\tau^*) = \tfrac{\pi}{2} - \arcsin(\tfrac{\tau}{\tau_{\max}}) = \tfrac{\pi}{2} + \alpha_\tau, \tag{7.10}$$

with $\alpha_\tau = -\arcsin(\tau/\tau_{\max})$ as in (5.3). Equation (7.10) implies that

$$\lim_{\tau \to 0} \Psi(\mathsf{p}_\tau^*) := \psi_1(\mathsf{p}_\tau^*) + (n-1)\psi_2(\mathsf{p}_\tau^*) = \frac{\pi}{2}.$$

For $t \in [0, \mathsf{p}_\tau^*]$ we have $y \in [(n-1)/n, y_{\max}] \subset [(n-1)/n, 1]$ and therefore from (6.13) we have (recall from (6.16) that $\psi_2(0) = 0$)

$$2\tau \mathsf{p}_\tau^* < -\psi_2(\mathsf{p}_\tau^*) < \frac{2n\tau}{n-1}\mathsf{p}_\tau^*.$$

Hence $\psi_2(\mathsf{p}_\tau^*)$ converges to zero as $\tau \to 0$ (since p_τ^* is bounded as $\tau \to 0$). Similarly, for $t \in [\mathsf{p}_\tau^*, \mathsf{p}_\tau]$ we have $1 - y \in [1/n, 1 - y_{\min}] \subset [1/n, 1]$ and therefore from (6.13) we have

$$2\tau(\mathsf{p}_\tau - \mathsf{p}_\tau^*) < \psi_1(\mathsf{p}_\tau) - \psi_1(\mathsf{p}_\tau^*) < 2\tau n(\mathsf{p}_\tau - \mathsf{p}_\tau^*).$$

Hence by the asymptotics for p_τ established in 7.3, we see $\psi_1(\mathsf{p}_\tau) - \psi_1(\mathsf{p}_\tau^*) \to 0$. Therefore, $\widehat{\mathsf{p}}_\tau = \frac{1}{2}\psi_1(2\mathsf{p}_\tau) = \psi_1(\mathsf{p}_\tau)$ converges to $\pi/2$ as desired.

The argument in the case $p > 1$ is very similar. At $t = \mathsf{p}_\tau^+$ or $t = -\mathsf{p}_\tau^-$ we have $\dot{y} = 0$ and $y = y_{\min}$ or $y = y_{\max}$, respectively. Hence (6.20) and (6.21) imply that $\mathrm{e}^{\mathrm{i}(\Psi + \alpha_\tau)} = \mathrm{e}^{-\mathrm{i}\pi/2}$ and therefore we have

$$\Psi(t) = -\tfrac{\pi}{2} - \alpha_\tau \quad \text{at} \quad t = \mathsf{p}_\tau^+ \text{ or } t = -\mathsf{p}_\tau^-. \tag{7.11}$$

Using (6.13) and the symmetries of y_τ from (6.6) one finds (for $i = 1, 2$)

$$\psi_i(2\mathsf{p}_\tau^+) = 2\psi_i(\mathsf{p}_\tau^+) \quad \text{and} \quad \psi_i(-2\mathsf{p}_\tau^-) = 2\psi_i(-\mathsf{p}_\tau^-). \tag{7.12}$$

Using (6.25) (applied with $t = -2\mathsf{p}_\tau^-$) yields (for $i = 1, 2$)

$$\psi_i(2\mathsf{p}_\tau) = \psi_i(2\mathsf{p}_\tau^+) - \psi_i(-2\mathsf{p}_\tau^-). \tag{7.13}$$

Combining (7.11), (7.12) and (7.13) yields

$$\widehat{\mathsf{p}}_\tau = \tfrac{p}{2}\,\psi_1(2\mathsf{p}_\tau) = p(\psi_1(\mathsf{p}_\tau^+) - \psi_1(-\mathsf{p}_\tau^-)) = \tfrac{\pi}{2} + \alpha_\tau + p\,\psi_1(\mathsf{p}_\tau^+) + q\,\psi_2(-\mathsf{p}_\tau^-). \tag{7.14}$$

By analysing the functions ψ_1 on $(0, \mathsf{p}_\tau^+)$ and ψ_2 on $(-\mathsf{p}_\tau^-, 0)$ as above we find

$$2p\tau\mathsf{p}_\tau^+ < p\psi_1(\mathsf{p}_\tau^+) < 2n\tau\mathsf{p}_\tau^+, \quad \text{and} \quad 2q\tau\mathsf{p}_\tau^- < q\psi_2(-\mathsf{p}_\tau^-) < 2n\tau\mathsf{p}_\tau^-.$$

Hence by 7.3 and the definition of α_τ all three nonconstant terms on the RHS of (7.14) converge to zero as $\tau \to 0$. $\square$

Theorem 7.15. *Fix admissible integers p and q. There exists a countable infinite subset $N \subset (0, \tau_{max})$ such that $\tau \in N$ if and only if the (p,q)-twisted SL curve $\mathbf{w}_\tau$ is closed.*

Proof. Define

$$N := \{\tau \in (0, \tau_{\max}) \,|\, \widehat{\mathsf{p}}_\tau \in \pi\mathbb{Q}\}.$$

By Proposition 7.7 $\widehat{\mathsf{p}}_\tau$ is a nonconstant analytic function of τ on the interval $(0, \tau_{\max})$ and hence N is a countable infinite set. By Corollary 6.37 $\mathbf{w}_\tau$ is a closed curve if and only if $\operatorname{Per}(\mathbf{w}_\tau) \neq (0)$ and by Lemma 6.39 $\operatorname{Per}(\mathbf{w}_\tau) \neq 0$ if and only if $\widehat{\mathsf{p}}_\tau \in \pi\mathbb{Q}$. Hence, $\mathbf{w}_\tau$ is closed if and only if $\tau \in N$. $\square$

In fact, with further work one can show that the set N is also dense in $(0, \tau_{\max})$. To prove this requires more precise asymptotics for $\widehat{\mathsf{p}}_\tau$ and $\frac{d\widehat{\mathsf{p}}_\tau}{d\tau}$ as $\tau \to 0$. Refined asymptotics for $\widehat{\mathsf{p}}_\tau$ as $\tau \to 0$ are a key ingredient in our use of X_τ for sufficiently small τ as building blocks in gluing constructions of higher dimensional SL cones and are established in [11, 12].

From 6.39 the condition $\tau \in N$ is equivalent to the condition that the rotational period $\hat{\mathsf{T}}_{2\widehat{\mathsf{p}}_\tau}$ (recall (6.25)) of $\mathbf{w}_\tau$ is of finite order k_0 (recall Definition 6.38) and hence to the condition that X_τ factors through an SL embedding of a closed manifold. Combining 7.15 and 6.46 we obtain the existence of countable infinite families of $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant embeddings of closed manifolds:

Theorem 7.16. *Choose any τ in the countable infinite (dense) set $N \subset (0, \tau_{max})$ and let $k_0 \in \mathbb{N}$ be the order of the rotational period $\hat{\mathsf{T}}_{2\widehat{\mathsf{p}}_\tau}$. The $SO(p) \times SO(q)$-invariant special Legendrian immersion $X_\tau : \mathrm{Cyl}^{p,q} \to \mathbb{S}^{2(p+q)-1}$ factors through a special Legendrian embedding of the closed manifold $\mathrm{Cyl}^{p,q}/\operatorname{Per}(X_\tau)$ where $\operatorname{Per}(X_\tau) \cong \mathbb{Z} \subset \mathrm{Diff}(\mathrm{Cyl}^{p,q})$ is the following infinite cyclic subgroup*

$$\operatorname{Per}(X_\tau) = \begin{cases} \langle (\mathsf{T}_{k_0 \mathsf{p}_\tau}, -\mathrm{Id}_{\mathbb{S}^{n-1}}) \rangle, & \textit{if } p = 1 \textit{ and } k_0 \textit{ is even} \\ & \textit{and } n \textit{ is odd;} \\ \langle (\mathsf{T}_{k_0 \mathsf{p}_\tau}, (-1)^j \,\mathrm{Id}_{\mathbb{S}^{p-1}}, (-1)^k \,\mathrm{Id}_{\mathbb{S}^{q-1}}) \rangle, & \textit{if } p > 1 \textit{ and } k_0 \textit{ is even;} \\ \langle \mathsf{T}_{2k_0 \mathsf{p}_\tau} \rangle, & \textit{otherwise;} \end{cases}$$

where $j = q/\operatorname{hcf}(p,q)$ and $k = p/\operatorname{hcf}(p,q)$.

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