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Scalar curvature rigidity with a volume constraint

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Motivated by Brendle–Marques–Neves' counterexample to the Min-Oo's conjecture, we prove a volume constrained scalar curvature rigidity theorem which applies to the hemisphere.

1. Introduction

Recently, Brendle, Marques and Neves [6] have solved the long-standing Min-Oo's conjecture [15] by constructing a counterexample.

Theorem 1.1 (Brendle, Marques and Neves [6]). *Suppose* $n \geq 3$. *Let* $\bar{g}$ *be the standard metric on the hemisphere* $\mathbb{S}^n_+$. *There exists a smooth metric* g *on* $\mathbb{S}^n_+$, *which can be made to be arbitrarily close to* $\bar{g}$ *in the* C^∞*-topology, satisfying*

- *the scalar curvature of* g *is at least that of* $\bar{g}$ *at each point in* $\mathbb{S}^n_+$,
- g *and* $\bar{g}$ *agree in a neighborhood of* $\partial \mathbb{S}^n_+$,

but g *is not isometric to* $\bar{g}$.

In this paper, we observe that if the metric g in Theorem 1.1 is assumed to satisfy an additional volume constraint, then it must be isometric to $\bar{g}$. Precisely, we have

Theorem 1.2. *Let* $\bar{g}$ *be the standard metric on* $\mathbb{S}^n_+$. *Let* g *be another metric on* $\mathbb{S}^n_+$ *with the properties*

- $R(g) \geq R(\bar{g})$ *in* $\mathbb{S}^n_+$,
- $H(g) \geq H(\bar{g})$ *on* $\partial \mathbb{S}^n_+$,
- g *and* $\bar{g}$ *induce the same metric on* $\partial \mathbb{S}^n_+$,

where $R(g)$, $R(\bar{g})$ *are the scalar curvature of* g, $\bar{g}$, *and* $H(g)$, $H(\bar{g})$ *are the mean curvature of* $\partial \mathbb{S}^n_+$ *in* $(\mathbb{S}^n_+, g), (\mathbb{S}^n_+, \bar{g})$. *Suppose in addition*

$$V(g) \geq V(\bar{g}),$$

where $V(g)$, $V(\bar g)$ are the volume of g, $\bar g$. If $||g-\bar g||_{C^2(\bar{\mathbb{S}}^n_+)}$ is sufficiently small, then there is a diffeomorphism $\varphi : \mathbb{S}^n_+ \to \mathbb{S}^n_+$ with $\varphi|_{\partial \mathbb{S}^n_+} = \mathrm{id}$, the identify map on $\partial \mathbb{S}^n_+$, such that $\varphi^*(g) = \bar g$.

Theorem 1.2 is indeed a special case of a more general result:

Theorem 1.3. *Let $(\Omega, \bar g)$ be an n-dimensional compact Riemannian manifold, of constant sectional curvature 1, with smooth boundary Σ. Suppose $\bar{\mathbb{II}} + \bar H \bar\gamma \geq 0$ (i.e., $\bar{\mathbb{II}} + \bar H \bar\gamma$ is positive semi-definite), where $\bar\gamma$ is the induced metric on Σ and $\bar{\mathbb{II}}$, $\bar H$ are the second fundamental form, the mean curvature of Σ in $(\Omega, \bar g)$. Suppose the first nonzero Neumann eigenvalue μ of $(\Omega, \bar g)$ satisfies $\mu > n - \frac{2}{n+1}$.*

Consider a nearby metric g on Ω with the properties

- *$R(g) \geq n(n-1)$ where $R(g)$ is the scalar curvature of g,*
- *$H(g) \geq \bar H$ where $H(g)$ is the mean curvature of Σ in (Ω, g),*
- *g and $\bar g$ induce the same metric on Σ,*
- *$V(g) \geq V(\bar g)$ where $V(g)$, $V(\bar g)$ are the volumes of g, $\bar g$.*

If $||g - \bar g||_{C^2(\bar\Omega)}$ is sufficiently small, then there is a diffeomorphism φ on Ω with $\varphi|_\Sigma = \mathrm{id}$, such that $\varphi^*(g) = \bar g$.

As a by-product of the method used to derive Theorem 1.3, we obtain a volume estimate for metrics close to the Euclidean metric in terms of the scalar curvature.

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary Σ. Suppose $\bar{\mathbb{II}} + \bar H \bar\gamma > 0$ (i.e., $\bar{\mathbb{II}} + \bar H \bar\gamma$ is positive definite), where $\bar{\mathbb{II}}$, $\bar H$ are the second fundamental form, the mean curvature of Σ in $\mathbb{R}^n$ and $\bar\gamma$ is the metric on Σ induced from the Euclidean metric $\bar g$. Let g be another metric on $\bar\Omega$ satisfying*

- *$H(g) \geq \bar H$, where $H(g)$ is the mean curvature of Σ in (Ω, g)*
- *g and $\bar g$ induce the same metric on Σ.*

Given any point $a \in \mathbb{R}^n$, there exists a constant $\Lambda > \frac{\max_{q \in \bar\Omega} |q-a|^2}{4(n-1)}$, depending only on Ω and a, such that if $||g - \bar g||_{C^3(\bar\Omega)}$ is sufficiently small, then

$$V(g) - V(\bar g) \geq \int_\Omega R(g) \Phi \, d\mathrm{vol}_{\bar g}, \tag{1.1}$$

where $\Phi(x) = -\frac{1}{4(n-1)}|x-a|^2 + \Lambda > 0$ on $\bar\Omega$.

Theorem 1.4 may be compared to a previous theorem of Bartnik [2], which estimates the total mass [1] of an asymptotically flat metric that is a perturbation of the Euclidean metric.

Theorem 1.5 (Bartnik [2]). *Let g be an asymptotically flat metric on $\mathbb{R}^3$. If g is sufficiently close to the Euclidean metric $\bar{g}$ (in certain weighted Sobolev space), then*

$$16\pi\mathfrak{m}(g) \geq \int_{\mathbb{R}^3} R(g)\, d\mathrm{vol}_{\bar{g}}, \tag{1.2}$$

where $\mathfrak{m}(g)$ is the total mass of g.

Our proofs of Theorems 1.2–1.4 follow a recent perturbation analysis of Brendle and Marques in [5], where they established a scalar curvature rigidity theorem for "small" geodesic balls in $\mathbb{S}^n$.

Theorem 1.6 (Brendle and Marques [5]). *Let $\Omega \subset \mathbb{S}^n$ be a geodesic ball of radius δ. Suppose*

$$\cos\delta \geq \frac{2}{\sqrt{n+3}}. \tag{1.3}$$

Let $\bar{g}$ be the standard metric on $\mathbb{S}^n$. Let g be another metric on Ω with the properties

- *$R(g) \geq n(n-1)$ at each point in Ω,*
- *$H(g) \geq \bar{H}$ at each point on $\partial\Omega$,*
- *g and $\bar{g}$ induce the same metric on $\partial\Omega$,*

where $R(g)$ is the scalar curvature of g, and $H(g)$, $\bar{H}$ are the mean curvature of $\partial\Omega$ in (Ω, g), $(\Omega, \bar{g})$. If $g - \bar{g}$ is sufficiently small in the C^2-norm, then $\varphi^*(g) = \bar{g}$ for some diffeomorphism $\varphi : \Omega \to \Omega$ such that $\varphi|_{\partial\Omega} = \mathrm{id}$.

In Theorem 1.6, the condition (1.3) is equivalently to

$$\bar{H} \geq 4\tan\delta \tag{1.4}$$

because the mean curvature $\bar{H}$ of $\partial B(\delta)$ is $(n-1)\frac{\cos\delta}{\sin\delta}$. As another application of the formulas in Section 2, we obtain a generalization of Theorem 1.6 to convex domains in $\mathbb{S}^n$.

Theorem 1.7. *Let $\Omega \subset \mathbb{S}^n$ be a smooth domain contained in a geodesic ball B of radius less than $\frac{\pi}{2}$. Let $\bar{g}$ be the standard metric on $\mathbb{S}^n$. Let $\bar{\mathbb{II}}$, $\bar{H}$ be the second fundamental form, the mean curvature of $\partial\Omega$ in $(\Omega, \bar{g})$. Suppose Ω is convex, i.e., $\bar{\mathbb{II}} \geq 0$. At $\partial\Omega$, suppose*

$$\bar{H} \geq 4 \tan r, \tag{1.5}$$

where r is the $\bar{g}$-distance to the center of B. Then the conclusion of Theorem 1.6 holds on Ω.

Theorem 1.7 is an immediate corollary of Theorem 5.1 in Section 5. In a simpler setting, where the background metric $\bar{g}$ is a flat metric, we have

Theorem 1.8. *Let Ω be a compact manifold with smooth boundary Σ. Suppose there is a flat metric $\bar{g}$ on Ω such that $\bar{\mathbb{II}} + \bar{H}\bar{\gamma} \geq 0$ (i.e., $\bar{\mathbb{II}} + \bar{H}\bar{\gamma}$ is positive semi-definite), where $\bar{\mathbb{II}}$, $\bar{H}$ are the second fundamental form, the mean curvature of Σ, and $\bar{\gamma}$ is the induced metric on Σ. Given another metric g on Ω such that*

- *$R(g) \geq 0$ on Ω,*
- *$H(g) \geq \bar{H}$ at Σ,*
- *g and $\bar{g}$ induce the same metric on Σ,*

if $||g - \bar{g}||_{C^2(\bar{\Omega})}$ is sufficiently small, then $\varphi^*(g) = \bar{g}$ for some diffeomorphism $\varphi : \Omega \to \Omega$ with $\varphi|_\Sigma = \mathrm{id}$.

Similar calculation at the infinitesimal level provides examples of compact 3-manifolds of nonnegative scalar curvature whose boundary surface does not have positive Gaussian curvature but still has positive Brown–York mass [7, 8]. We include this in the end of the paper to compare with known results in [17].

Theorem 1.9. *Let $\Sigma \subset \mathbb{R}^n$ be a connected, closed hypersurface satisfying $\bar{\mathbb{II}} + \bar{H}\bar{\gamma} \geq 0$, where $\bar{\mathbb{II}}$, $\bar{H}$ are the second fundamental form, the mean curvature of Σ, and $\bar{\gamma}$ is the induced metric on Σ. Let Ω be the domain enclosed by Σ in $\mathbb{R}^n$. Let h be any nontrivial $(0,2)$ symmetric tensor on Ω satisfying*

$$\mathrm{div}_{\bar{g}} h = 0, \quad \mathrm{tr}_{\bar{g}} h = 0, \quad h|_{T\Sigma} = 0. \tag{1.6}$$

Let $\{g(t)\}_{|t|<\epsilon}$ be a 1-parameter family of metrics on Ω satisfying

$$g(0) = \bar{g}, \quad g'(0) = h, \quad R(g(t)) \geq 0, \quad g(t)|_{T\Sigma} = \bar{g}|_{T\Sigma}. \tag{1.7}$$

Then

$$\int_{\Sigma} \bar{H} d\sigma_{\bar{g}} > \int_{\Sigma} H(g(t)) d\sigma_{\bar{g}} \tag{1.8}$$

for small $t \neq 0$, *where* $H(g(t))$ *is the mean curvature of* Σ *in* $(\Omega, g(t))$.

This paper is organized as follows. In Section 2, we derive a basic formula concerning a perturbed metric (Theorem 2.1), which corresponds to [5, Theorem 10] of Brendle and Marques. In Section 3, we prove Theorem 1.3, which implies Theorem 1.2. In Section 4, we give a proof of Theorem 1.4. In Section 5, we consider other applications of the formulas in Section 2 and prove Theorem 1.7–1.9.

2. Basic formulas for a perturbed metric

Let Ω be an n-dimensional, smooth, compact manifold with boundary Σ. Let $\bar{g}$ be a fixed smooth Riemannian metric on Ω. Given a tensor η, let "$|\eta|$" denote the length of η measured with respect to $\bar{g}$. Denote the covariant derivative with respect to $\bar{g}$ by $\overline{\nabla}$. Indices of tensors are raised by $\bar{g}$. Let $\bar{R}_{ikjl}$ denote the curvature tensor of $\bar{g}$ such that if $\bar{g}$ has constant sectional curvature κ, then $\bar{R}_{ikjl} = \kappa(g_{ij}g_{kl} - g_{il}g_{kj})$. Consider a nearby Riemannian metric $g = \bar{g} + h$ where h is a symmetric $(0,2)$ tensor with $|h|$ very small, say $|h| \leq \frac{1}{2}$.

The following pointwise estimates of the scalar curvature of g and the mean curvature of Σ were derived by Brendle and Marques in [5].

Proposition 2.1 (Brendle and Marques [5])**.** *The scalar curvatures* $R(g)$, $R(\bar{g})$ *of the metrics* g, $\bar{g}$ *satisfy*

$$\begin{aligned}
&\Big|R(g) - R(\bar{g}) + \langle \mathrm{Ric}(\bar{g}), h\rangle - \langle \mathrm{Ric}(\bar{g}), h^2\rangle + \frac{1}{4}|\overline{\nabla} h|^2 - \frac{1}{2}\bar{g}^{ij}\bar{g}^{kl}\bar{g}^{pq}\overline{\nabla}_i h_{kp}\overline{\nabla}_l h_{jq} \\
&\quad + \frac{1}{4}|\overline{\nabla}(\mathrm{tr}_{\bar{g}} h)|^2 + \overline{\nabla}_i[g^{ik}g^{jl}(\overline{\nabla}_k h_{jl} - \overline{\nabla}_l h_{jk})]\Big| \\
&\qquad \leq C\left(|h||\overline{\nabla} h|^2 + |h|^3\right),
\end{aligned}$$

where $\mathrm{Ric}(\bar{g})$ *is the Ricci curvature of* $\bar{g}$, h^2 *is the* $\bar{g}$*-square of* h, *i.e.,* $(h^2)_{ik} = \bar{g}^{jl}h_{ij}h_{kl}$, $\langle\cdot,\cdot\rangle$ *is taken with respect to* $\bar{g}$, *and* C *is a positive constant depending only on* n.

Remark 2.1. If the background metric $\bar{g}$ is Ricci flat, i.e., $\bar{R}_{ik} = 0$, then there will be no $|h|^3$ term in the above estimate. That is because

$$R(g) = g^{ik}\bar{R}_{ik} - g^{ik}g^{lj}\left(\overline{\nabla}_{i,k}h_{jl} - \overline{\nabla}_{i,l}h_{jk}\right) + g^{ik}g^{jl}g_{pq}\left(\Gamma^q_{il}\Gamma^p_{jk} - \Gamma^q_{jl}\Gamma^p_{ik}\right),$$

where each term on the right, except $g^{ik}\bar{R}_{ik}$, involves derivatives of h.

Proposition 2.2 (Brendle and Marques [5])**.** *Assume that g and $\bar{g}$ induce the same metric on Σ, i.e., $h|_{T\Sigma} = 0$ where $T\Sigma$ is the tangent bundle of Σ. Then the mean curvatures $H(g)$, $H(\bar{g})$ of Σ in (Ω, g), $(\Omega, \bar{g})$, each with respect to the outward normals, satisfy*

$$\begin{aligned}\Bigg| 2\left[H(g) - H(\bar{g})\right] &- \left(h(\overline{\nu},\overline{\nu}) - \frac{1}{4}h(\overline{\nu},\overline{\nu})^2 + \sum_{\alpha=1}^{n-1} h(e_\alpha,\overline{\nu})^2\right) H(\bar{g}) \\ &+ \left(1 - \frac{1}{2}h(\overline{\nu},\overline{\nu})\right)\sum_{\alpha=1}^{n-1}\left[2\overline{\nabla}_{e_\alpha}h(e_\alpha,\overline{\nu}) - \overline{\nabla}_{\overline{\nu}}h(e_\alpha,e_\alpha)\right]\Bigg| \\ &\le C\left(|h|^2|\overline{\nabla}h| + |h|^3\right),\end{aligned}$$

where $\{e_\alpha \mid 1 \le \alpha \le n-1\}$ is a local orthonormal frame on Σ, $\overline{\nu}$ is the $\bar{g}$-unit outward normal vector to Σ, and C is a positive constant depending only on n.

To derive the main formula (2.23) in this section, we let

$$DR_{\bar{g}}(h) = -\Delta_{\bar{g}}(\mathrm{tr}_{\bar{g}}h) + \mathrm{div}_{\bar{g}}\mathrm{div}_{\bar{g}}h - \langle \mathrm{Ric}(\bar{g}), h\rangle \tag{2.1}$$

be the linearization of the scalar curvature at $\bar{g}$ along h. Here "$\Delta_{\bar{g}}$, $\mathrm{div}_{\bar{g}}$" denote the Laplacian, the divergence with respect to $\bar{g}$.

Lemma 2.1. *With the same notations in Proposition 2.1, assume in addition $\mathrm{div}_{\bar{g}}h = 0$, then*

$$\begin{aligned}R(g) - R(\bar{g}) = DR_{\bar{g}}(h) - \frac{1}{2}DR_{\bar{g}}(h^2) + \langle h, \overline{\nabla}^2\mathrm{tr}_{\bar{g}}h\rangle - \frac{1}{4}\left(|\overline{\nabla}h|^2 + |\overline{\nabla}(\mathrm{tr}_{\bar{g}}h)|^2\right) \\ + \frac{1}{2}h^{ij}h^{kl}\overline{R}_{ikjl} + E(h) + \overline{\nabla}_i(E_1^i(h)),\end{aligned}$$

where $E(h)$ is a function and $E_1(h)$ is a vector field on Ω satisfying

$$|E(h)| \leq C(|h||\overline{\nabla} h|^2 + |h|^3), \quad |E_1(h)| \leq C|h|^2|\overline{\nabla} h|$$

for a positive constant C depending only on n.

Proof. First note that

$$-\overline{\nabla}_i \left[\bar{g}^{ik}\bar{g}^{jl}\left(\overline{\nabla}_k h_{jl} - \overline{\nabla}_l h_{jk}\right)\right] - \langle \mathrm{Ric}(\bar{g}), h\rangle = DR_{\bar{g}}(h). \tag{2.2}$$

Suppose $g^{ik} = \bar{g}^{ik} + \tau^{ik}$. Then $\tau^{ik} = -h^{ik} + E_2^{ik}(h)$ where $h^{ik} = \bar{g}^{ij}h_{jl}\bar{g}^{lk}$ and $|E_2(h)| \leq C|h|^2$. Hence,

$$g^{ik}g^{jl} - \bar{g}^{ik}\bar{g}^{jl} = -\bar{g}^{ik}h^{jl} - \bar{g}^{jl}h^{ik} + E_3^{ikjl}(h),$$

where $|E_3(h)| \leq C|h|^2$. Therefore,

$$\begin{aligned} & -\overline{\nabla}_i[(g^{ik}g^{jl} - \bar{g}^{ik}\bar{g}^{jl})(\overline{\nabla}_k h_{jl} - \overline{\nabla}_l h_{jk})] \\ &= \overline{\nabla}_i[(\bar{g}^{ik}h^{jl} + \bar{g}^{jl}h^{ik} - E_3^{ikjl}(h))(\overline{\nabla}_k h_{jl} - \overline{\nabla}_l h_{jk})] \\ &= \frac{1}{2}\Delta_{\bar{g}}|h|^2 + \langle h, \nabla^2 \mathrm{tr}_{\bar{g}}(h)\rangle_{\bar{g}} - \mathrm{div}_{\bar{g}}\mathrm{div}_{\bar{g}}(h^2) \\ &\quad - \overline{\nabla}_i(E_3^{ikjl}(\overline{\nabla}_k h_{jl} - \overline{\nabla}_l h_{jk})). \end{aligned} \tag{2.3}$$

Applying the Ricci identity, one has

$$\begin{aligned} \frac{1}{2}\bar{g}^{ij}\bar{g}^{kl}\bar{g}^{pq}\overline{\nabla}_i h_{kp}\overline{\nabla}_l h_{jq} = {} & \frac{1}{2}\mathrm{div}_{\bar{g}}\mathrm{div}_{\bar{g}}(h^2) - \frac{1}{2}\langle \mathrm{Ric}(\bar{g}), h^2\rangle \\ & + \frac{1}{2}h^{ij}h^{kl}\overline{R}_{ikjl}. \end{aligned} \tag{2.4}$$

The lemma follows from Proposition 2.1, (2.2), (2.3) and (2.4). □

Next, let $DH_{\bar{g}}(h)$ denote the linearization of the mean curvature at $\bar{g}$ along h. Proposition 2.2 implies

$$DH_{\bar{g}}(h) = \frac{1}{2}\left[h(\bar{\nu},\bar{\nu})H(\bar{g}) - \sum_{\alpha=1}^{n-1}\left(2\overline{\nabla}_{e_\alpha}h(e_\alpha,\bar{\nu}) - \overline{\nabla}_{\bar{\nu}}h(e_\alpha,e_\alpha)\right)\right]. \tag{2.5}$$

For later use, we note the following equivalent expression of $DH_{\bar g}(h)$ (see [13, (34)] for instance)

$$DH_{\bar g}(h) = \frac{1}{2}\left\{[d(\mathrm{tr}_{\bar g} h) - \mathrm{div}_{\bar g} h](\overline{\nu}) - \mathrm{div}_\Sigma X\right\}, \tag{2.6}$$

where X is the vector field on Σ dual to the 1-form $h(\overline{\nu}, \cdot)|_{T\Sigma}$.

Let $DR^*_{\bar g}(\cdot)$ denote the formal L^2 $\bar g$-adjoint of $DR_{\bar g}(\cdot)$, i.e.,

$$DR^*_{\bar g}(\lambda) = -(\Delta_{\bar g}\lambda)\bar g + \nabla^2_{\bar g}\lambda - \lambda \mathrm{Ric}(\bar g) \tag{2.7}$$

where λ is a function and $\nabla^2_{\bar g}\lambda$ denotes the Hessian of λ with respect to $\bar g$. The content of the following lemma had been used in [13].

Lemma 2.2. *Let p be any smooth $(0,2)$ symmetric tensor on Ω, then*

$$\begin{aligned}\int_\Omega DR_{\bar g}(p)\lambda\, d\mathrm{vol}_{\bar g} = & \int_\Omega \langle DR^*_{\bar g}(\lambda), p\rangle\, d\mathrm{vol}_{\bar g} - \int_\Sigma 2DH_{\bar g}(p)\lambda\, d\sigma_{\bar g} \\ & + \int_\Sigma \lambda_{\overline{\nu}}\left(\mathrm{tr}_{\bar g}(p) - p(\overline{\nu},\overline{\nu})\right) d\sigma_{\bar g},\end{aligned} \tag{2.8}$$

where $\lambda_{\overline{\nu}} = \partial_{\overline{\nu}}\lambda$ denotes the directional derivative of λ along $\overline{\nu}$.

Proof. Let Y be the vector field on Σ dual to the 1-form $p(\overline{\nu}, \cdot)|_{T\Sigma}$. Integrating by parts, one has

$$\begin{aligned}&\int_\Omega DR_{\bar g}(p)\lambda\, d\mathrm{vol}_{\bar g} - \int_\Omega \langle DR^*_{\bar g}(\lambda), p\rangle\, d\mathrm{vol}_{\bar g} \\ &= \int_\Sigma -\lambda\partial_{\overline{\nu}}(\mathrm{tr}_{\bar g} p) + (\mathrm{tr}_{\bar g} p)\partial_{\overline{\nu}}\lambda + \lambda \mathrm{div}_{\bar g} p(\overline{\nu}) - p(\overline{\nu}, \overline{\nabla}\lambda)\, d\sigma_{\bar g} \\ &= \int_\Sigma \lambda[-\partial_{\overline{\nu}}(\mathrm{tr}_{\bar g} p) + \mathrm{div}_{\bar g} p(\overline{\nu})] - \langle Y, \overline{\nabla}^\Sigma\lambda\rangle\, d\sigma_{\bar g} \\ &\quad + \int_\Sigma \lambda_{\overline{\nu}}\left(\mathrm{tr}_{\bar g}(p) - p(\overline{\nu},\overline{\nu})\right) d\sigma_{\bar g} \\ &= \int_\Sigma \lambda[-\partial_{\overline{\nu}}(\mathrm{tr}_{\bar g} p) + \mathrm{div}_{\bar g} p(\overline{\nu}) + \mathrm{div}_\Sigma Y]\, d\sigma_{\bar g} \\ &\quad + \int_\Sigma \lambda_{\overline{\nu}}\left(\mathrm{tr}_{\bar g}(p) - p(\overline{\nu},\overline{\nu})\right) d\sigma_{\bar g},\end{aligned} \tag{2.9}$$

where $\overline{\nabla}^\Sigma(\cdot)$ denotes the gradient on Σ with respect to the induced metric. From this and (2.6) the Lemma follows. □

Using Lemma 2.2, we can estimate $\int_\Omega [R(g) - R(\bar g)]\lambda\, d\mathrm{vol}_{\bar g}$.

Proposition 2.3. *Suppose g and $\bar{g}$ induce the same metric on Σ and h satisfies* $\mathrm{div}_{\bar{g}} h = 0$. *Given any C^2 function λ on Ω, one has*

$$
\begin{aligned}
&\int_\Omega [R(g) - R(\bar{g})]\,\lambda\, d\mathrm{vol}_{\bar{g}} \\
&\quad = \int_\Omega \langle h, DR_{\bar{g}}^*(\lambda)\rangle\, d\mathrm{vol}_{\bar{g}} - \frac{1}{2}\int_\Omega \langle h^2, DR_{\bar{g}}^*(\lambda)\rangle\, d\mathrm{vol}_{\bar{g}} \\
&\quad + \int_\Omega \left[(\mathrm{tr}_{\bar{g}} h)\langle h, \nabla_{\bar{g}}^2 \lambda\rangle + \frac{1}{2} h^{ij}h^{kl}\bar{R}_{ikjl}\lambda - \frac{1}{4}(|\overline{\nabla} h|^2 + |\overline{\nabla}(\mathrm{tr}_{\bar{g}} h)|^2)\lambda\right] d\mathrm{vol}_{\bar{g}} \\
&\quad + \int_\Sigma \left[-(h_{nn})^2 - \frac{1}{2}|X|^2\right]\lambda_{;n}\, d\sigma_{\bar{g}} - \int_\Sigma h_{nn}\langle X, \overline{\nabla}^\Sigma \lambda\rangle\, d\sigma_{\bar{g}} \\
&\quad + \int_\Sigma \left[-\frac{1}{2}(h_{nn})^2 H(\bar{g}) - \frac{1}{2}\overline{\mathrm{II}}(X, X) - \frac{3}{2}|X|^2 H(\bar{g})\right]\lambda\, d\sigma_{\bar{g}} \\
&\quad - \int_\Sigma (2 - 2\mathrm{tr}_{\bar{g}} h) DH_{\bar{g}}(h)\lambda\, d\sigma_{\bar{g}} + \int_\Omega E(h)\lambda\, d\mathrm{vol}_{\bar{g}} \\
&\quad - \int_\Omega E_1^i(h)\overline{\nabla}_i \lambda\, d\mathrm{vol}_{\bar{g}} + \int_\Sigma F_1(h)\lambda\, d\sigma_{\bar{g}},
\end{aligned}
$$

where $\overline{\mathrm{II}}$ is the second fundamental form of Σ in $(\Omega, \bar{g})$ with respect to $\bar{\nu}$, X is the vector field on Σ that is dual to the 1-form $h(\bar{\nu}, \cdot)|_{T\Sigma}$, $E(h)$ and $E_1^i(h)$ are as in Lemma 2.1, and $F_1(h)$ is a function on Σ satisfying

$$|F_1(h)| \le C|h|^2|\overline{\nabla} h|$$

for a positive constant C depending only on n.

Proof. By (2.8) with $p = h$, using the fact that $h|_{T(\Sigma)} = 0$, we have

$$\int_\Omega DR_{\bar{g}}(h)\lambda\, d\mathrm{vol}_{\bar{g}} = \int_\Omega \langle DR_{\bar{g}}^*(\lambda), h\rangle\, d\mathrm{vol}_{\bar{g}} - \int_\Sigma 2DH_{\bar{g}}(h)\lambda\, d\sigma_{\bar{g}}. \tag{2.10}$$

By the second line in (2.9) with $p = h^2$, and integrating by parts, we also have

$$
\begin{aligned}
&\int_\Omega -\frac{\lambda}{2} DR_{\bar{g}}(h^2) + \lambda\langle h, \overline{\nabla}^2 \mathrm{tr}_{\bar{g}} h\rangle\, d\mathrm{vol}_{\bar{g}} \\
&\quad = \int_\Omega -\frac{1}{2}\langle DR_{\bar{g}}^*(\lambda), h^2\rangle + \mathrm{tr}_{\bar{g}} h\langle h, \overline{\nabla}^2 \lambda\rangle\, d\mathrm{vol}_{\bar{g}} + \mathcal{B},
\end{aligned} \tag{2.11}
$$

where

$$\mathcal{B} = \int_\Sigma \frac{1}{2}\left[\lambda\partial_{\overline{\nu}}(|h|^2) - |h|^2\partial_{\overline{\nu}}\lambda - \lambda(\mathrm{div}_{\bar g}h^2)(\overline{\nu}) + (h^2)(\overline{\nu},\overline{\nabla}\lambda)\right] d\sigma_{\bar g} + \int_\Sigma \left[\lambda h(\overline{\nu},\overline{\nabla}\mathrm{tr}_{\bar g}h) - \mathrm{tr}_{\bar g}h h(\overline{\nu},\overline{\nabla}\lambda)\right] d\sigma_{\bar g}. \tag{2.12}$$

To compute $\mathcal{B}$, let $\{e_\alpha \mid 1 \le \alpha \le n-1\}$ be an orthonormal frame on Σ and let $e_n = \overline{\nu}$. Denote $\overline{\nabla}$ also by ";", thus $h_{ij;k} = \overline{\nabla}_k h_{ij}$. The assumptions $h|_{T\Sigma} = 0$ and $\mathrm{div}_{\bar g}h = 0$ imply the following facts on Σ:

$$|h|^2 = (h_{nn})^2 + 2|X|^2,\ (h^2)_{nn} = (h_{nn})^2 + |X|^2,\ (h^2)_{n\alpha} = h_{nn}h_{n\alpha}, \tag{2.13}$$

$$(h^2)(\overline{\nu},\overline{\nabla}\lambda) = [(h_{nn})^2 + |X|^2]\lambda_{;n} + h_{nn}\langle X, \overline{\nabla}^\Sigma\lambda\rangle, \tag{2.14}$$

$$h_{\beta\gamma;\alpha} = h_{\beta n}\overline{\mathrm{II}}_{\gamma\alpha} + h_{n\gamma}\overline{\mathrm{II}}_{\beta\alpha}, \tag{2.15}$$

$$h_{nn;\alpha} = (\mathrm{tr}_{\bar g}h)_{;\alpha} - \sum_{\beta=1}^{n-1} h_{\beta\beta;\alpha} = (\mathrm{tr}_{\bar g}h)_{;\alpha} - 2\overline{\mathrm{II}}(X, e_\alpha), \tag{2.16}$$

$$0 = (\mathrm{div}h)_\alpha = h_{\alpha n;n} + \sum_{\beta=1}^{n-1} h_{\alpha\beta;\beta} = h_{\alpha n;n} + h_{n\alpha}H(\bar g) + \overline{\mathrm{II}}(X, e_\alpha), \tag{2.17}$$

$$0 = (\mathrm{div}_{\bar g}h)_n = h_{nn;n} + \sum_{\alpha=1}^{n-1} h_{n\alpha;\alpha} = h_{nn;n} + \mathrm{div}_\Sigma X + h_{nn}H(\bar g), \tag{2.18}$$

$$2DH_{\bar g}(h) = (\mathrm{tr}_{\bar g}h)_{;n} - \mathrm{div}_\Sigma X, \tag{2.19}$$

where (2.19) follows from (2.6). By (2.16)–(2.18), we have

$$\begin{aligned} \partial_{\overline{\nu}}(|h|^2) - (\mathrm{div}_{\bar g}h^2)(\overline{\nu}) &= 3h_{n\alpha}h_{n\alpha;n} + h_{nn}h_{nn;n} - h_{n\alpha}h_{nn;\alpha} \\ &= -\overline{\mathrm{II}}(X,X) - 3H(\bar g)|X|^2 - H(\bar g)(h_{nn})^2 \\ &\quad - h_{nn}\mathrm{div}_\Sigma X - \langle X, \overline{\nabla}^\Sigma \mathrm{tr}_{\bar g}h\rangle. \end{aligned} \tag{2.20}$$

By (2.12), (2.13), (2.14), (2.20) and integration by parts, we have

(2.21)

$$\begin{aligned} \mathcal{B} = &\int_\Sigma \left[-(h_{nn})^2 - \frac{1}{2}|X|^2\right]\lambda_{;n} - \int_\Sigma h_{nn}\langle X, \overline{\nabla}^\Sigma\lambda\rangle \\ &+ \int_\Sigma \left[-\frac{1}{2}\overline{\mathrm{II}}(X,X) - \frac{3}{2}H(\bar g)|X|^2 - \frac{1}{2}H(\bar g)(h_{nn})^2 + 2h_{nn}DH_{\bar g}(h)\right]\lambda d\sigma_{\bar g}. \end{aligned}$$

Note that

$$(2.22)\qquad \int_\Omega (\overline{\nabla}_i E_1^i(h))\lambda \, d\mathrm{vol}_{\bar g} = -\int_\Omega E_1^i(h)\overline{\nabla}_i\lambda \, d\mathrm{vol}_{\bar g} + \int_\Sigma \lambda F_1(h)\, d\sigma_{\bar g},$$

where $|F_1(h) = \langle E_1(h), \overline{\nu}\rangle| \leq C|h|^2|\overline{\nabla} h|$. Proposition 2.3 now follows from Lemma 2.1, (2.10), (2.11), (2.21) and (2.22). □

The formula (2.23) below is a general form of [5, Theorem 10], which Brendle and Marques derived for geodesic balls in $\mathbb{S}^n$.

Theorem 2.1. *Suppose g and $\bar g$ induce the same metric on Σ and h satisfies* $\mathrm{div}_{\bar g} h = 0$. *Given any C^2 function λ on Ω, one has*

(2.23)

$$\begin{aligned}
&\int_\Omega [R(g) - R(\bar g)]\,\lambda \, d\mathrm{vol}_{\bar g} + \int_\Sigma (2 - \mathrm{tr}_{\bar g} h)\,[H(g) - H(\bar g)]\,\lambda \, d\sigma_{\bar g} \\
&\quad = \int_\Omega \langle h, DR_{\bar g}^*(\lambda)\rangle \, d\mathrm{vol}_{\bar g} - \frac{1}{2}\int_\Omega \langle h^2, DR_{\bar g}^*(\lambda)\rangle \, d\mathrm{vol}_{\bar g} \\
&\qquad + \int_\Omega \left[(\mathrm{tr}_{\bar g} h)\langle h, \nabla_{\bar g}^2 \lambda\rangle + \frac{1}{2} h^{ij} h^{kl} \bar R_{ikjl}\lambda - \frac{1}{4}(|\overline{\nabla} h|^2 + |\overline{\nabla}(\mathrm{tr}_{\bar g} h)|^2)\lambda \right] d\mathrm{vol}_{\bar g} \\
&\qquad + \int_\Sigma \left[-\frac{1}{4}(h_{nn})^2 H(\bar g) - \frac{1}{2}(\mathbb{II}(X,X) + H(\bar g)|X|^2) \right] \lambda \, d\sigma_{\bar g} \\
&\qquad + \int_\Sigma \lambda_{;n} \left[-(h_{nn})^2 - \frac{1}{2}|X|^2 \right] d\sigma_{\bar g} + \int_\Sigma (-1) h_{nn} \langle X, \overline{\nabla}^\Sigma \lambda\rangle \, d\sigma_{\bar g} \\
&\qquad + \int_\Omega E(h)\lambda \, d\mathrm{vol}_{\bar g} + \int_\Omega Z^i(h)\overline{\nabla}_i \lambda \, d\mathrm{vol}_{\bar g} + \int_\Sigma F(h)\lambda \, d\sigma_{\bar g},
\end{aligned}$$

where $E(h)$ is a function and $Z(h)$ is a vector field on Ω satisfying

$$|E(h)| \leq C(|h||\overline{\nabla} h|^2 + |h|^3), \quad |Z(h)| \leq C|h|^2|\overline{\nabla} h|,$$

and $F(h)$ is some function on Σ satisfying

$$|F(h)| \leq C(|h|^2|\overline{\nabla} h| + |h|^3).$$

Proof. Proposition 2.2 implies

$$(2.24)\qquad 2[H(g) - H(\bar g)] = 2DH_{\bar g}(h) + J(h) + F_2(h)$$

where

$$J(h) = \left[\frac{1}{4}(h_{nn})^2 + |X|^2\right] H(\bar g) - h_{nn} DH_{\bar g}(h)$$

and $F_2(h)$ is some function on Σ satisfying $|F_2(h)| \le C(|h|^2|\overline{\nabla} h| + |h|^3)$. Therefore

$$\begin{aligned}(2-h_{nn})[H(g) - H(\bar g)] = {} & (2 - 2h_{nn}) DH_{\bar g}(h) \\ & + \left[\frac{1}{4}(h_{nn})^2 + |X|^2\right] H(\bar g) \\ & + F_2(h) - \frac{1}{2} h_{nn}[J(h) + F_2(h)].\end{aligned} \tag{2.25}$$

(2.23) now follows readily from Proposition 2.3 and (2.25). □

The term $DR^*_{\bar g}(\lambda)$ in (2.23) may suggest that one consider a background metric $\bar g$ which admits a nontrivial function λ such that $DR^*_{\bar g}(\lambda) = 0$ (such metrics are known as *static metrics* [10].) For instance, if Ω is a geodesic ball B in $\mathbb{S}^n$, $\bar g$ is the standard metric on $\mathbb{S}^n$ and $\lambda = \cos r$, where r is the $\bar g$-distance to the center of B, then (2.23) reduces to the formula in [5, Theorem 10].

Besides static metrics, one can also consider those metrics $\bar g$ with the property that there exists a function λ such that

$$DR^*_{\bar g}(\lambda) = \bar g. \tag{2.26}$$

These metrics were studied by the authors in [13, 14]. In this case, the terms

$$\int_\Omega \langle h, DR^*_{\bar g}(\lambda)\rangle \, d\mathrm{vol}_{\bar g} - \frac{1}{2}\int_\Omega \langle h^2, DR^*_{\bar g}(\lambda)\rangle \, d\mathrm{vol}_{\bar g}$$

in (2.23) become

$$\int_\Omega \mathrm{tr}_{\bar g} h \, d\mathrm{vol}_{\bar g} - \frac{1}{2}\int_\Omega |h|^2 \, d\mathrm{vol}_{\bar g}.$$

To compensate these terms, one can include the difference between the volumes of g and $\bar g$ into (2.23).

Corollary 2.1. *Suppose $\bar g$ is a metric on Ω with the property that there exists a function λ satisfying $DR^*_{\bar g}(\lambda) = \bar g$. Let $g = \bar g + h$ be a nearby metric such that g and $\bar g$ induce the same metric on Σ and h satisfies* $\mathrm{div}_{\bar g} h = 0$.

Let $V(g)$, $V(\bar g)$ denote the volume of (Ω, g), $(\Omega, \bar g)$. Then

$$
\begin{aligned}
&-2(V(g) - V(\bar g)) + \int_\Omega [R(g) - R(\bar g)]\,\lambda\,d\mathrm{vol}_{\bar g} \\
&\quad + \int_\Sigma (2 - \mathrm{tr}_{\bar g} h)\,[H(g) - H(\bar g)]\,\lambda\,d\sigma_{\bar g} \\
&\quad = \int_\Omega \left[-\frac{1}{4} - \frac{1}{n-1}\right] (\mathrm{tr}_{\bar g} h)^2\,d\mathrm{vol}_{\bar g} \\
&\quad + \int_\Omega \left[-\frac{1}{4}(|\overline{\nabla} h|^2 + |\nabla_{\bar g}(\mathrm{tr}_{\bar g} h)|^2)\lambda\right] d\mathrm{vol}_{\bar g} \\
&\quad + \int_\Omega \left[\frac{1}{1-n} R(\bar g)(\mathrm{tr}_{\bar g} h)^2 + \langle h, \mathrm{Ric}(\bar g)\rangle(\mathrm{tr}_{\bar g} h) + \frac{1}{2} h_{ij} h_{kl} R_{ikjl}\right] \lambda\,d\mathrm{vol}_{\bar g} \\
&\quad + \int_\Sigma \left[-\frac{1}{4}(h_{nn})^2 H(\bar g) - \frac{1}{2}(\overline{\mathrm{II}}(X, X) + H(\bar g)|X|^2)\right] \lambda\,d\sigma_{\bar g} \\
&\quad + \int_\Sigma \lambda_{;n} \left[-(h_{nn})^2 - \frac{1}{2}|X|^2\right] d\sigma_{\bar g} + \int_\Sigma (-1) h_{nn} \langle X, \overline{\nabla}^\Sigma \lambda\rangle\,d\sigma_{\bar g} \\
&\quad + \int_\Omega G(h)\,d\mathrm{vol}_{\bar g} + \int_\Omega E(h)\lambda\,d\mathrm{vol}_{\bar g} + \int_\Omega Z^i(h)\overline{\nabla}_i \lambda\,d\mathrm{vol}_{\bar g} \\
&\quad + \int_\Sigma F(h)\lambda\,d\sigma_{\bar g},
\end{aligned}
\tag{2.27}
$$

where $G(h)$ and $E(h)$ are functions on Ω satisfying

$$|G(h)| \le C|h|^3, \quad |E(h)| \le C(|h||\overline{\nabla} h|^2 + |h|^3),$$

$Z(h)$ is a vector field on Ω satisfying

$$|Z(h)| \le C|h|^2|\overline{\nabla} h|,$$

and $F(h)$ is a function on Σ satisfying

$$|F(h)| \le C(|h|^2|\overline{\nabla} h| + |h|^3).$$

Proof. The difference between the volumes of $\bar g$ and $g = \bar g + h$ is

$$V(g) - V(\bar g) = \int_\Omega \frac{1}{2}(\mathrm{tr}_{\bar g} h) + \left[\frac{1}{8}(\mathrm{tr}_{\bar g} h)^2 - \frac{1}{4}|h|^2\right] + G(h)\,d\mathrm{vol}_{\bar g}, \tag{2.28}$$

where $G(h)$ is a function satisfying $|G(h)| \leq C|h|^3$ for a constant C depending only on n. Suppose $DR^*_{\bar g}(\lambda) = \bar g$, i.e.,

$$-(\Delta_{\bar g}\lambda)\bar g + \nabla^2_{\bar g}\lambda - \lambda \mathrm{Ric}(\bar g) = \bar g.$$

Taking trace, one has $\Delta_{\bar g}\lambda = \frac{1}{1-n}[R(\bar g)\lambda + n]$. Thus,

$$\nabla^2_{\bar g}\lambda = \frac{1}{1-n}[R(\bar g)\lambda + 1]\bar g + \lambda \mathrm{Ric}(\bar g). \tag{2.29}$$

(2.27) follows from (2.23), (2.28) and (2.29). □

3. Volume constrained rigidity

We prove Theorem 1.3 in this section. First, we recall its statement:

Theorem 3.1. *Let $(\Omega, \bar g)$ be an n-dimensional compact Riemannian manifold, of constant sectional curvature 1, with smooth boundary Σ. Suppose $\bar{\mathrm{II}} + \bar H \bar\gamma \geq 0$ (i.e., $\bar{\mathrm{II}} + \bar H \bar\gamma$ is positive semi-definite), where $\bar\gamma$ is the induced metric on Σ and $\bar{\mathrm{II}}$, $\bar H$ are the second fundamental form, the mean curvature of Σ in $(\Omega, \bar g)$. Suppose the first nonzero Neumann eigenvalue μ of $(\Omega, \bar g)$ satisfies $\mu > n - \frac{2}{n+1}$.*

Consider a nearby metric g on Ω with the properties

- *$R(g) \geq n(n-1)$ where $R(g)$ is the scalar curvature of g,*
- *$H(g) \geq \bar H$ where $H(g)$ is the mean curvature of Σ in (Ω, g),*
- *g and $\bar g$ induce the same metric on Σ,*
- *$V(g) \geq V(\bar g)$ where $V(g)$, $V(\bar g)$ are the volumes of g, $\bar g$.*

If $||g - \bar g||_{C^2(\bar\Omega)}$ is sufficiently small, then there is a diffeomorphism φ on Ω with $\varphi|_\Sigma = \mathrm{id}$, which is the identity map on Σ, such that $\varphi^*(g) = \bar g$.

Proof. Fix a real number $p > n$. By [5, Proposition 11], if $||g - \bar g||_{W^{2,p}(\Omega)}$ is sufficiently small, there exists a $W^{3,p}$ diffeomorphism φ on Ω with $\varphi|_\Sigma = \mathrm{id}$ such that $h = \varphi^*(g) - g$ is divergence free with respect to $\bar g$, and $||h||_{W^{2,p}(\Omega)} \leq N||g - \bar g||_{W^{2,p}(\Omega)}$ for some positive constant N depending only on $(\Omega, \bar g)$. Replacing g by $\varphi^*(g)$, we may assume $g = \bar g + h$ with $\mathrm{div}_{\bar g} h = 0$. We want to prove that if $||h||_{C^1(\bar\Omega)}$ is sufficiently small and g satisfies the conditions in the theorem, then h must be zero.

Since $\bar{g}$ has constant sectional curvature 1, we choose $\lambda = -\frac{1}{n-1}$ such that $DR^*_{\bar{g}}(\lambda) = \bar{g}$. Corollary 2.1 then shows

(3.1)
$$\begin{aligned}
&-2(V(g)-V(\bar{g})) - \frac{1}{n-1}\int_\Omega [R(g)-R(\bar{g})]\, d\mathrm{vol}_{\bar{g}} \\
&\quad - \frac{1}{n-1}\int_\Sigma (2-\mathrm{tr}_{\bar{g}}h)\,[H(g)-H(\bar{g})]\, d\sigma_{\bar{g}} \\
&\quad \geq \frac{1}{4(n-1)}\int_\Omega \left[-(n+1)(\mathrm{tr}_{\bar{g}}h)^2 + 2|h|^2 + |\overline{\nabla}h|^2 + |\overline{\nabla}(\mathrm{tr}_{\bar{g}}h)|^2\right] d\mathrm{vol}_{\bar{g}} \\
&\qquad + \frac{1}{4(n-1)}\int_\Sigma \left[(h_{nn})^2 H(\bar{g}) + 2(\overline{\mathrm{II}}(X,X) + H(\bar{g})|X|^2)\right] d\sigma_{\bar{g}} \\
&\qquad - C||h||_{C^1(\bar{\Omega})}\left[\int_\Omega (|h|^2 + |\overline{\nabla}h|^2)\, d\mathrm{vol}_{\bar{g}} + \int_\Sigma |h|^2\, d\sigma_{\bar{g}}\right]
\end{aligned}$$

for a constant C depending only on $(\Omega, \bar{g})$.

Using the variational property of μ, we have

(3.2)
$$\int_\Omega |\overline{\nabla}(\mathrm{tr}_{\bar{g}}h)|^2\, d\mathrm{vol}_{\bar{g}} \geq \mu\left[\left(\int_\Omega (\mathrm{tr}_{\bar{g}}h)^2\, d\mathrm{vol}_{\bar{g}}\right) - \frac{1}{V(\bar{g})}\left(\int_\Omega \mathrm{tr}_{\bar{g}}h\, d\mathrm{vol}_{\bar{g}}\right)^2\right].$$

By (2.28), $\int_\Omega \mathrm{tr}_{\bar{g}}h\, d\mathrm{vol}_{\bar{g}}$ is related to $(V(g)-V(\bar{g}))$ by

(3.3)
$$\int_\Omega \mathrm{tr}_{\bar{g}}h\, d\mathrm{vol}_{\bar{g}} = 2(V(g)-V(\bar{g})) - \int_\Omega \left\{\left[\frac{1}{4}(\mathrm{tr}_{\bar{g}}h)^2 - \frac{1}{2}|h|^2\right] + 2G(h)\right\} d\mathrm{vol}_{\bar{g}},$$

where $G(h) \leq C|h|^3$.

Given any constant $0 < \epsilon < 1$, using (3.2) and the fact $|h|^2 \geq \frac{1}{n}(\mathrm{tr}_{\bar{g}}h)^2$ and $|\overline{\nabla}h|^2 \geq \frac{1}{n}|\overline{\nabla}(\mathrm{tr}_{\bar{g}}h)|^2$, we have

(3.4)
$$\begin{aligned}
&\int_\Omega \left[-(n+1)(\mathrm{tr}_{\bar{g}}h)^2 + 2|h|^2 + |\overline{\nabla}h|^2 + |\nabla_{\bar{g}}(\mathrm{tr}_{\bar{g}}h)|^2\right] d\mathrm{vol}_{\bar{g}} \\
&\quad \geq \int_\Omega \left[\epsilon|h|^2 + \epsilon|\overline{\nabla}h|^2 + \left[-(n+1) + \frac{2-\epsilon}{n}\right](\mathrm{tr}_{\bar{g}}h)^2 \right.\\
&\qquad \left. + \left[\frac{(1-\epsilon)}{n} + 1\right]|\overline{\nabla}(\mathrm{tr}_{\bar{g}}h)|^2\right] d\mathrm{vol}_{\bar{g}}
\end{aligned}$$

$$\geq \int_\Omega \left[\epsilon|h|^2 + \epsilon|\overline{\nabla} h|^2 + \left[-(n+1) + \frac{2-\epsilon}{n} + \frac{(1-\epsilon)}{n}\mu + \mu\right] (\mathrm{tr}_{\bar g} h)^2\right] d\mathrm{vol}_{\bar g}$$
$$-\mu \left[\frac{(1-\epsilon)}{n} + 1\right] \frac{1}{V(\bar g)} \left(\int_\Omega \mathrm{tr}_{\bar g} h \, d\mathrm{vol}_{\bar g}\right)^2 .$$

Since $\mu > n - \frac{2}{n+1}$, we can chose ϵ (depending only on μ and n) such that

$$\left[-(n+1) + \frac{2-\epsilon}{n} + \frac{(1-\epsilon)}{n}\mu + \mu\right] \geq 0. \tag{3.5}$$

Then it follows from (3.3), (3.4) and (3.5) that

$$\begin{aligned} &\int_\Omega \left(-(n+1)(\mathrm{tr}_{\bar g} h)^2 + 2|h|^2 + |\overline{\nabla} h|^2 + |\overline{\nabla}(\mathrm{tr}_{\bar g} h))|^2\right) d\mathrm{vol}_{\bar g} \\ &\quad \geq \epsilon \int_\Omega \left(|h|^2 + |\overline{\nabla} h|^2\right) d\mathrm{vol}_{\bar g} - C_1 (V(g) - V(\bar g))^2 - C_1 \int_\Omega |h|^4 \, d\sigma_{\bar g}, \end{aligned} \tag{3.6}$$

where C_1 is a positive constant depending only on $(\Omega, \bar g)$.

At the boundary Σ, the assumption $\overline{\mathbb{II}} + H(\bar g)\bar\gamma \geq 0$ implies $H(\bar g) \geq 0$, therefore

$$\int_\Sigma \left[(h_{nn})^2 H(\bar g) + 2(\overline{\mathbb{II}}(X,X) + H(\bar g)|X|^2)\right] d\sigma_{\bar g} \geq 0 \tag{3.7}$$

for any h. By (3.1), (3.6) and (3.7), we have

$$\begin{aligned} &-8(n-1)(V(g) - V(\bar g)) - 4\int_\Omega [R(g) - R(\bar g)] \, d\mathrm{vol}_{\bar g} \\ &-4 \int_\Sigma (2 - \mathrm{tr}_{\bar g} h) [H(g) - H(\bar g)] \, d\sigma_{\bar g} \\ &\quad \geq \epsilon \int_\Omega \left(|h|^2 + |\overline{\nabla} h|^2\right) d\mathrm{vol}_{\bar g} \\ &- C(V(g) - V(\bar g))^2 - C \int_\Omega |h|^4 \, d\mathrm{vol}_{\bar g} \\ &- C||h||_{C^1(\bar\Omega)} \left[\int_\Omega (|h|^2 + |\overline{\nabla} h|^2) \, d\mathrm{vol}_{\bar g} + \int_\Sigma |h|^2 \, d\sigma_{\bar g}\right] \end{aligned} \tag{3.8}$$

for some positive constant C depending only on $(\Omega, \bar g)$.

Finally, we note that

$$(V(g) - V(\bar{g}))^2 \leq C \left(\int_\Omega |h| \, d\mathrm{vol}_{\bar{g}} \right) (V(g) - V(\bar{g})) \tag{3.9}$$

by (3.3) and the assumption $V(g) \geq V(\bar{g})$. Also, by the trace theorem,

$$||h||_{L^2(\Sigma)} \leq C ||h||_{W^{1,2}(\Omega)} \tag{3.10}$$

for a constant C only depending on Ω. Therefore, by (3.8), (3.9), (3.10) and the assumptions $V(g) \geq V(\bar{g})$, $R(g) \geq R(\bar{g})$ and $H(g) \geq H(\bar{g})$, we conclude that if $||h||_{C^1(\bar{\Omega})}$ is sufficiently small, then

$$0 \geq \frac{\epsilon}{2} \int_\Omega (|h|^2 + |\overline{\nabla} h|^2) \, d\mathrm{vol}_{\bar{g}}, \tag{3.11}$$

which implies h must be identically zero. This completes the proof. □

Remark 3.1. In Theorem 3.1, if Σ is indeed empty, i.e., $(\Omega, \bar{g})$ is a closed space form, its first nonzero Neumann eigenvalue satisfies $\mu \geq n$ as $(\Omega, \bar{g})$ is covered by $\mathbb{S}^n$. In this case, Theorem 3.1 says that $V(g) \geq V(\bar{g})$ implies g is isometric to $\bar{g}$ for a nearby metrics g with $R(g) \geq R(\bar{g})$. This could be compared to a more profound theorem known in three-dimension: "*If (M, g) is closed 3-manifold with $R(g) \geq 6$, $\mathrm{Ric}(g) \geq g$ and $V(g) \geq V(\mathbb{S}^3)$, then (M, g) is isometric to $\mathbb{S}^3$.*" (See [4, Corollary 5.4] and earlier reference of [3, 11])

When $\Sigma \neq \emptyset$, the boundary assumption $\overline{\mathbb{II}} + \bar{H}\bar{\gamma} \geq 0$ in Theorem 3.1 can be relaxed in certain circumstances. A detailed examination of the above proof shows, if

$$\overline{\mathbb{II}}(v, v) + \bar{H}\bar{\gamma} \geq -\beta\bar{\gamma} \tag{3.12}$$

for some positive constant β, where β is sufficiently small comparing to the constant ϵ in (3.5) and the constant C in (3.10), then the conclusion of Theorem 3.1 still holds on such an $(\Omega, \bar{g})$. In particular, this shows

Corollary 3.1. *Let $(M, \bar{g})$ be an n-dimensional Riemannian manifold of constant sectional curvature 1. Suppose $\Omega \subset M$ is a bounded domain with smooth boundary Σ, satisfying the assumptions in Theorem 3.1, i.e., $\mu > n - \frac{2}{n+1}$ and $\overline{\mathbb{II}} + \bar{H}\bar{\gamma} \geq 0$ on Σ. Let $\tilde{\Omega} \subset M$ be another bounded domain with smooth boundary $\tilde{\Sigma}$. If $\tilde{\Sigma}$ is sufficiently close to Σ in the C^2 norm, then the conclusion of Theorem 3.1 holds on $\tilde{\Omega}$.*

It is known that the fist nonzero Neumann eigenvalue of $\mathbb{S}^n_+$ is n (see [9, Theorem 3]). Therefore, Theorem 1.2 follows from Theorem 3.1. Moreover, by Corollary 3.1, Theorem 3.1 holds on a geodesic ball in $\mathbb{S}^n$ whose radius is slightly larger than $\frac{\pi}{2}$.

By the next lemma, we know Theorem 3.1 also holds on any geodesic ball in $\mathbb{S}^n$ that is strictly contained in $\mathbb{S}^n_+$.

Lemma 3.1. *Let $B(\delta) \subset \mathbb{S}^n$ be a geodesic ball of radius δ. Let $\mu(\delta)$ be the first nonzero Neumann eigenvalue of $B(\delta)$.*

(i) *$\mu(\delta)$ is a strictly decreasing function of δ on $(0, \frac{\pi}{2}]$.*

(ii) *For any $0 < \delta < \frac{\pi}{2}$,*

$$\mu(\delta) > n + \frac{(\sin\delta)^{n-2}\cos\delta}{\int_0^\delta (\sin t)^{n-1} dt} > \frac{n}{(\sin\delta)^2}.$$

Proof. By [9, Theorem 2, p.44], $\mu(\delta)$ is characterized by the fact that

$$\left\{(\sin t)^{n-1} J'\right\}' + [\mu(\delta) - (n-1)(\sin t)^{-2}](\sin t)^{n-1} J = 0 \tag{3.13}$$

has a solution $J = J(t)$ on $[0, \delta]$ satisfying

$$J(0) = 0, \quad J'(\delta) = 0, \quad J'(t) \neq 0, \quad \forall t \in [0, \delta). \tag{3.14}$$

Given $0 < \delta_1 < \delta_2 \leq \frac{\pi}{2}$, let $J_i = J_i(t)$ be a solution to (3.13) with $\mu(\delta)$ replaced by $\mu(\delta_i)$, satisfying (3.14) on $[0, \delta_i]$, $i = 1, 2$. Replacing J_i by $-J_i$ if necessary, we may assume that $J_i' > 0$ on $[0, \delta_i)$, hence $J_i > 0$ on $(0, \delta_i]$. Define

$$f_i = \frac{(\sin t)^{n-1} J_i'}{J_i}, \quad \beta_i(t) = \left[\mu(\delta_i) - \frac{n-1}{(\sin t)^2}\right](\sin t)^{n-1}.$$

By (3.13), f_i satisfies

$$f_i' = -\beta_i - \frac{1}{(\sin t)^{n-1}} f_i^2.$$

Therefore, on $(0, \delta_1]$,

$$(f_1 - f_2)' = \frac{1}{(\sin t)^{n-1}}(f_2^2 - f_1^2) + [\mu(\delta_2) - \mu(\delta_1)](\sin t)^{n-1}. \tag{3.15}$$

Note that $f_1(t)$, $f_2(t)$ can be extended continuously to 0 such that $f_1(0) = f_2(0)$. Moreover, $f_1 > 0$, $f_2 > 0$ on $(0, \delta_1)$, $f_2(\delta_1) > 0 = f_1(\delta_1)$. Let $0 \leq t_0 <$

δ_1 be such that $f_1 = f_2$ at t_0 and $f_2 > f_1$ for $t_0 < t \le \delta_1$. On $(t_0, \delta_1]$, one would have $(f_1 - f_2)' > 0$ if $\mu(\delta_2) \ge \mu(\delta_1)$, which is a contradiction to $f_2 > f_1$. Therefore, $\mu(\delta_2) < \mu(\delta_1)$. This proves (i).

To prove (ii), we further claim that $t_0 = 0$, i.e., $f_2 > f_1$ on $(0, \delta_1]$. If not, there would be a nonpositive local minimum of $(f_2 - f_1)$ at some $\tilde{t}_0 \in (0, t_0]$. At $\tilde{t}_0$, (3.15) implies

$$0 = (f_1 - f_2)' \le [\mu(\delta_2) - \mu(\delta_1)](\sin \tilde{t}_0)^{n-1} < 0 \tag{3.16}$$

because $0 < f_2(\tilde{t}_0) \le f_1(\tilde{t}_0)$ and $\mu(\delta_2) < \mu(\delta_1)$. Hence $f_2 > f_1$ on $(0, \delta_1]$. Integrating (3.15) on $[0, \delta_1]$, we have

$$-f_2(\delta_1) = \int_0^{\delta_1} (f_1 - f_2)' dt > [\mu(\delta_2) - \mu(\delta_1)] \int_0^{\delta_1} (\sin t)^{n-1} dt. \tag{3.17}$$

Therefore

$$\mu(\delta_1) > \mu(\delta_2) + \frac{f_2(\delta_1)}{\int_0^{\delta_1} (\sin t)^{n-1} dt}. \tag{3.18}$$

Now let $\delta_1 = \delta \in (0, \frac{\pi}{2})$ and $\delta_2 = \pi/2$. Applying the fact that $\mu(\frac{\pi}{2}) = n$, $J_2 = \sin t$, and

$$f_2 = (\sin t)^{n-2} \cos t,$$

we have

$$\begin{aligned} \mu(\delta) &> n + \frac{(\sin \delta)^{n-2} \cos \delta}{\int_0^{\delta} (\sin t)^{n-1} dt} \\ &> n + \frac{(\sin \delta)^{n-2} \cos^2 \delta}{\int_0^{\delta} \cos t (\sin t)^{n-1} dt} \\ &= \frac{n}{\sin^2 \delta}. \end{aligned} \tag{3.19}$$

Therefore, (ii) is proved. □

4. A volume estimate on domains in $\mathbb{R}^n$

On $\mathbb{R}^n$, the standard Euclidean metric $\bar{g}$ satisfies $DR^*_{\bar{g}}(\lambda) = \bar{g}$ with

$$\lambda(x) = -\frac{1}{2(n-1)} |x - a|^2 + L \tag{4.1}$$

where $|\cdot|$ denotes the Euclidean length, $a \in \mathbb{R}^n$ is any fixed point and L is an arbitrary constant. In this section, we use this fact and Corollary 2.1 to prove Theorem 1.4 in the introduction. First we need some lemmas.

Lemma 4.1. *On a compact Riemannian manifold $(\Omega, \bar{g})$ with smooth boundary Σ, there exists a positive constant C depending only on $(\Omega, \bar{g})$ such that, for any Lipschitz function ϕ on Σ, there is an extension of ϕ to a Lipschitz function $\widetilde{\phi}$ on Ω such that*

$$\int_\Omega \left(|\widetilde{\phi}|^2 + |\overline{\nabla} \widetilde{\phi}|^2 \right) d\mathrm{vol}_{\bar{g}} \leq C \int_\Sigma \left(\phi^2 + |\overline{\nabla}^\Sigma \phi|^2 \right) d\sigma_{\bar{g}}, \tag{4.2}$$

where $\overline{\nabla}$, $\overline{\nabla}^\Sigma$ denote the gradient on Ω, Σ respectively.

Proof. Let $d(\cdot, \Sigma)$ be the distance to Σ. Let $\delta > 0$ be a small constant such that the tubular neighborhood $U_{2\delta} = \{x \in \Omega | \ d(x, \Sigma) < 2\delta\}$ can be parametrized by $F : \Sigma \times [0, 2\delta) \to U_{2\delta}$, with $F(y,t) = \exp_y(t\nu(y))$ where $\exp_y(\cdot)$ is the exponential map at $y \in \Sigma$ and $\nu(y)$ is the inward unit normal at y. In $U_{2\delta}$, the metric $\bar{g}$ takes the form $dt^2 + \sigma^t$, where $\{\sigma^t\}_{0 \leq t < 2\delta}$ is a family of metrics on Σ. By choosing δ sufficiently small, one can assume σ^t is equivalent to σ^0 in the sense that $\frac{1}{2} \leq \sigma^t(v,v) \leq 2$ for any tangent vector v with $\sigma^0(v,v) = 1$, $\forall 0 \leq t < 2\delta$.

Let $\rho = \rho(t)$ be a fixed smooth cut-off function on $[0, \infty)$ such that $0 \leq \rho \leq 1$, $\rho(t) = 1$ for $0 \leq t \leq \delta$ and $\rho(t) = 0$ for $t \geq \frac{3}{2}\delta$. On $U_{2\delta}$, consider the function $\widetilde{\phi}(y,t) = \phi(y)\rho(t)$. Since $\widetilde{\phi}$ is identically zero outside $U_{\frac{3}{2}\delta} = \{x \in \Omega | \ d(x, \Sigma) < \frac{3}{2}\delta\}$, $\tilde{\phi}$ can be viewed as an extension of ϕ on Ω. For such an $\widetilde{\phi}$, one has

$$\int_\Omega |\widetilde{\phi}|^2 d\mathrm{vol}_{\bar{g}} \leq \int_0^{2\delta} \left(\int_\Sigma |\phi|^2 d\sigma^t \right) dt \leq C\delta \int_\Sigma |\phi|^2 d\sigma_{\bar{g}} \tag{4.3}$$

and

$$\begin{aligned} \int_\Omega |\overline{\nabla} \widetilde{\phi}|^2 d\mathrm{vol}_{\bar{g}} &\leq 2 \int_{U_{2\delta}} \left(|\overline{\nabla} \rho|^2 \phi^2 + |\overline{\nabla} \phi|^2 \rho^2 \right) d\mathrm{vol}_{\bar{g}} \\ &\leq C\delta \int_\Sigma |\phi|^2 d\sigma_{\bar{g}} + 2 \int_0^{2\delta} \left(\int_\Sigma |\overline{\nabla}_t^\Sigma \phi|^2 d\sigma^t \right) dt \\ &\leq C \left[\int_\Sigma |\phi|^2 d\sigma_{\bar{g}} + \int_\Sigma |\overline{\nabla}^\Sigma \phi|^2 d\sigma_{\bar{g}} \right], \end{aligned} \tag{4.4}$$

where $\overline{\nabla}_t^\Sigma$ denotes the gradient on (Σ, σ^t) and C is a positive constant depending only on $(\Omega, \bar{g})$. (4.2) now follows from (4.3) and (4.4). □

Lemma 4.2. *On a compact Riemannian manifold* $(\Omega, \bar{g})$ *with smooth boundary* Σ*, there exists a positive constant* C *depending only on* $(\Omega, \bar{g})$ *such that, for any smooth* $(0,2)$ *symmetric tensor* h *on* Ω*, one has*

(4.5)
$$\int_\Omega |h|^3 d\mathrm{vol}_{\bar{g}} \le C\left(\int_\Sigma |h|^3 d\sigma_{\bar{g}} + ||h||_{C^2(\Omega)} \int_\Sigma |h|^2 d\sigma_{\bar{g}} + \int_\Omega |h||\overline{\nabla} h|^2 d\mathrm{vol}_{\bar{g}} \right).$$

Proof. On Ω, let $\phi = |h|^{\frac{3}{2}}$. By lemma 4.1, there exists a Lipschitz function $\widetilde{\phi}$ on Ω such that $\widetilde{\phi}|_\Sigma = \phi|_\Sigma$ and

$$\int_\Omega \left(|\widetilde{\phi}|^2 + |\overline{\nabla}\widetilde{\phi}|^2 \right) d\mathrm{vol}_{\bar{g}} \le C \int_\Sigma \left(\phi^2 + |\overline{\nabla}^\Sigma \phi|^2 \right) d\sigma_{\bar{g}}.$$

Let $\lambda_1 > 0$ be the first Dirichlet eigenvalue of $(\Omega, \bar{g})$, then

$$\begin{aligned} \int_\Omega \phi^2 \, d\mathrm{vol}_{\bar{g}} &\le 2 \int_\Omega \left[\widetilde{\phi}^2 + (\phi - \widetilde{\phi})^2 \right] d\mathrm{vol}_{\bar{g}} \\ &\le 2 \int_\Omega \widetilde{\phi}^2 \, d\mathrm{vol}_{\bar{g}} + 2\lambda_1^{-1} \int_\Omega |\overline{\nabla}(\phi - \widetilde{\phi})|^2 d\mathrm{vol}_{\bar{g}} \\ &\le C \left[\int_\Sigma \left(\phi^2 + |\overline{\nabla}^\Sigma \phi|^2 \right) d\sigma_{\bar{g}} + \int_\Omega |\overline{\nabla}\phi|^2 d\mathrm{vol}_{\bar{g}} \right], \end{aligned} \tag{4.6}$$

where

$$\int_\Omega |\overline{\nabla}\phi|^2 d\mathrm{vol}_{\bar{g}} = \int_\Omega |\overline{\nabla}|h|^{\frac{3}{2}}|^2 d\mathrm{vol}_{\bar{g}} \le \frac{9}{4} \int_\Omega |h||\overline{\nabla} h|^2 d\mathrm{vol}_{\bar{g}}. \tag{4.7}$$

To handle the boundary term $\int_\Sigma |\overline{\nabla}^\Sigma \phi|^2 d\sigma_{\bar{g}}$, given any constant $\epsilon > 0$, one considers

$$\int_\Sigma |\overline{\nabla}^\Sigma (|h|^2 + \epsilon)^{\frac{3}{4}}|^2 d\sigma_{\bar{g}} = - \int_\Sigma (|h|^2 + \epsilon)^{\frac{3}{4}} \Delta_\Sigma (|h|^2 + \epsilon)^{\frac{3}{4}} d\sigma_{\bar{g}}, \tag{4.8}$$

where Δ_Σ denotes the Laplacian on Σ. Let $\{e_\alpha \mid \alpha = 1, \ldots, n-1\}$ be a local orthonormal frame on Σ and e_n be the outward unit normal to Σ. Let $\bar{H}$ be the mean curvature of Σ with respect to e_n. Denote covariant differentiation

Ω by ";". Let i, j run through $\{1,\ldots,n\}$. One has

$$\begin{aligned}\Delta_\Sigma |h|^2 &= \sum_\alpha (|h|^2)_{;\alpha\alpha} - \bar{H}(|h|^2)_{;n} \\ &= \sum_{\alpha,i,j,} 2(h_{ij}h_{ij;\alpha\alpha} + h^2_{ij;\alpha}) - \bar{H}\sum_{i,j} 2h_{ij}h_{ij;n} \\ &\geq -C||h||_{C^2(\bar{\Omega})}|h|.\end{aligned} \tag{4.9}$$

Therefore,

$$\begin{aligned}\Delta_\Sigma (|h|^2+\epsilon)^{\frac{3}{4}} &= \frac{3}{4}(|h|^2+\epsilon)^{-\frac{1}{4}}\Delta_\Sigma|h|^2 - \frac{3}{16}(|h|^2+\epsilon)^{-\frac{5}{4}}|\overline{\nabla}^\Sigma|h|^2|^2 \\ &\geq -C||h||_{C^2(\bar{\Omega})}(|h|^2+\epsilon)^{-\frac{1}{4}}|h| - \frac{3}{16}(|h|^2+\epsilon)^{-\frac{5}{4}}|\overline{\nabla}^\Sigma|h|^2|^2.\end{aligned} \tag{4.10}$$

It follows from (4.8) and (4.10) that

$$\begin{aligned}\int_\Sigma |\overline{\nabla}^\Sigma (|h|^2+\epsilon)^{\frac{3}{4}}|^2 d\sigma_{\bar{g}} &\leq C||h||_{C^2(\bar{\Omega})}\int_\Sigma (|h|^2+\epsilon)^{\frac{1}{2}}|h| d\sigma_{\bar{g}} \\ &\quad + \frac{1}{3}\int_\Sigma |\overline{\nabla}^\Sigma (|h|^2+\epsilon)^{\frac{3}{4}}|^2 d\sigma_{\bar{g}}.\end{aligned} \tag{4.11}$$

Letting $\epsilon \to 0$, one has

$$\int_\Sigma |\overline{\nabla}^\Sigma |h|^{\frac{3}{2}}|^2 d\sigma_{\bar{g}} \leq C||h||_{C^2(\bar{\Omega})}\int_\Sigma |h|^2 d\sigma_{\bar{g}}. \tag{4.12}$$

(4.5) now follows from (4.6), (4.7) and (4.12). □

We recall the statement of Theorem 1.4 and give its proof.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary Σ. Suppose $\bar{\mathrm{II}} + \bar{H}\bar{\gamma} > 0$ (i.e., $\bar{\mathrm{II}} + \bar{H}\bar{\gamma}$ is positive definite), where $\bar{\mathrm{II}}$, $\bar{H}$ are the second fundamental form, the mean curvature of Σ in $\mathbb{R}^n$ and $\bar{\gamma}$ is the metric on Σ induced from the Euclidean metric $\bar{g}$. Let g be another metric on $\bar{\Omega}$ satisfying*

- *g and $\bar{g}$ induce the same metric on Σ.*
- *$H(g) \geq \bar{H}$, where $H(g)$ is the mean curvature of Σ in (Ω, g).*

Given any point $a \in \mathbb{R}^n$, there exists a constant $\Lambda > \frac{\max_{q\in\bar{\Omega}}|q-a|^2}{4(n-1)}$, which depends only on Ω and a, such that if $||g-\bar{g}||_{C^3(\bar{\Omega})}$ is sufficiently small,

then

$$V(g) - V(\bar g) \geq \int_\Omega R(g)\Phi \, d\mathrm{vol}_{\bar g}, \tag{4.13}$$

where $\Phi = -\frac{1}{4(n-1)}|x-a|^2 + \Lambda > 0$ *on* $\bar\Omega$.

Proof. Fix a number $p > n$. By the proof of [5, Proposition 11], one knows if $||g-\bar g||_{W^{3,p}(\Omega)}$ is sufficiently small, then there exists a $W^{4,p}$ diffeomorphism $\varphi : \Omega \to \Omega$ such that $\varphi|_\Sigma = \mathrm{id}$, $h = \varphi^*(g) - \bar g$ is divergence free with respect to $\bar g$, and $||h||_{W^{3,p}(\Omega)} \leq N||g-\bar g||_{W^{3,p}(\Omega)}$ for a positive constant N depending only on $(\Omega, \bar g)$. In what follows, we will work with $\phi^*(g)$. For convenience, we still denote $\phi^*(g)$ by g.

Given $a \in \mathbb{R}^n$, consider $\lambda(x) = -\frac{1}{2(n-1)}|x-a|^2 + L$ where L is a constant to be determined. First, we require $L > \frac{1}{2(n-1)} \max_{q\in\bar\Omega}|q-a|^2$ so that $\lambda > 0$ on $\bar\Omega$. Since λ satisfies $DR^*_{\bar g}(\lambda) = \bar g$, Corollary 2.1 shows

(4.14)
$$\begin{aligned}
&-2(V(g)-V(\bar g)) + \int_\Omega R(g)\lambda \, d\mathrm{vol}_{\bar g} + \int_\Sigma (2 - \mathrm{tr}_{\bar g} h)\left[H(g) - \bar H\right]\lambda \, d\sigma_{\bar g} \\
&\quad \leq -\int_\Omega \frac14 |\overline\nabla h|^2 \lambda \, d\mathrm{vol}_{\bar g} + \int_\Sigma \left[-\frac14 (h_{nn})^2 \bar H - \frac12(\bar{\mathrm{I\!I}}(X,X) + \bar H |X|^2)\right]\lambda \, d\sigma_{\bar g} \\
&\qquad + \int_\Sigma \lambda_{;n}\left[-(h_{nn})^2 - \frac12|X|^2\right] d\sigma_{\bar g} + \int_\Sigma (-1) h_{nn}\langle X, \overline\nabla^\Sigma \lambda\rangle \, d\sigma_{\bar g} \\
&\qquad + \int_\Omega G(h)\, d\mathrm{vol}_{\bar g} + \int_\Omega E(h)\lambda \, d\mathrm{vol}_{\bar g} + \int_\Omega Z^i(h)\overline\nabla_i \lambda \, d\mathrm{vol}_{\bar g} \\
&\qquad + \int_\Sigma F(h)\lambda \, d\sigma_{\bar g},
\end{aligned}$$

where $|G(h)| \leq C|h|^3$, $|E(h)| \leq C(|h||\overline\nabla h|^2 + |h|^3)$, $|Z(h)| \leq C|h|^2|\overline\nabla h|$, $|F(h)| \leq C(|h|^2|\overline\nabla h| + |h|^3)$ for some constant C depending only on Ω.

At Σ, $\lambda_{;n}$ and $\overline\nabla^\Sigma \lambda$ are determined solely by Ω and a (in particular they are independent on L). Apply the assumption $\bar{\mathrm{I\!I}} + \bar H \bar\gamma > 0$ (which implies $\bar H > 0$) and the fact $|h|^2 = (h_{nn})^2 + 2|X|^2$, we have

$$\begin{aligned}
&\left[-\frac14 (h_{nn})^2 \bar H - \frac12(\bar{\mathrm{I\!I}}(X,X) + \bar H|X|^2)\right]\lambda \\
&\quad + \lambda_{;n}\left[-(h_{nn})^2 - \frac12 |X|^2\right] + (-1)h_{nn}\langle X, \overline\nabla^\Sigma\lambda\rangle \\
&\quad \leq -LC_1|h|^2 + C_2|h|^2,
\end{aligned} \tag{4.15}$$

where C_1, C_2 are positive constants depending only on Ω and a. We fix L such that

$$LC_1 - C_2 > 0 \tag{4.16}$$

and let $m = \frac{1}{4}\min_{\bar\Omega}\lambda$ (note that λ is fixed now). (4.14)–(4.16) imply

$$\begin{aligned} &-2(V(g)-V(\bar g)) + \int_\Omega R(g)\lambda\, d\mathrm{vol}_{\bar g} + \int_\Sigma (2-\mathrm{tr}_{\bar g}h)\left[H(g)-\bar H\right]\lambda\, d\sigma_{\bar g} \\ &\quad \le -m\int_\Omega |\overline\nabla h|^2\, d\mathrm{vol}_{\bar g} - (LC_1 - C_2)\int_\Sigma |h|^2 d\sigma_{\bar g} \\ &\qquad + C_3\left(\int_\Omega (|h||\overline\nabla h|^2 + |h|^3) d\mathrm{vol}_{\bar g} + \int_\Sigma (|h|^2|\overline\nabla h| + |h|^3)\, d\sigma_{\bar g}\right), \end{aligned} \tag{4.17}$$

where C_3 depends only on Ω, a and L. Apply Lemma 4.2 to the term $\int_\Omega |h|^3\, d\mathrm{vol}_{\bar g}$ on the right side of (4.17), we have

$$\begin{aligned} &-2(V(g)-V(\bar g)) + \int_\Omega R(g)\lambda\, d\mathrm{vol}_{\bar g} + \int_\Sigma (2-\mathrm{tr}_{\bar g}h)\left[H(g)-\bar H\right]\lambda\, d\sigma_{\bar g} \\ &\quad \le -m\int_\Omega |\overline\nabla h|^2\, d\mathrm{vol}_{\bar g} - (LC_1 - C_2)\int_\Sigma |h|^2 d\sigma_{\bar g} \\ &\qquad + C||h||_{C^2(\bar\Omega)}\left(\int_\Omega |\overline\nabla h|^2 d\mathrm{vol}_{\bar g} + \int_\Sigma |h|^2\, d\sigma_{\bar g}\right), \end{aligned}$$

where C is independent on h. From this, we conclude that if $||h||_{C^2(\bar\Omega)}$ is sufficiently small, then (4.13) holds with $\Phi = \frac{1}{2}\lambda$. This completes the proof. □

Remark 4.1. When $\Omega \subset \mathbb{R}^n$ is a ball of radius R, one can take a to be the center of Ω. In this case, by computing $\bar H$, $\overline{\mathrm{II}}$ and $\lambda_{;n}$ explicitly in (4.16), the constant L can be chosen to be any constant satisfying

$$L > \left[\frac{1}{2(n-1)} + \frac{4}{(n-1)^2}\right] R^2.$$

Remark 4.2. By the results in [12, 17] based on the positive mass theorem [16, 18], a metric g on Ω satisfying the boundary conditions in Theorem 4.1 must be isometric to the Euclidean metric if $R(g) \ge 0$. Therefore, a nontrivial metric g in Theorem 4.1 necessarily has negative scalar curvature somewhere. For such a g, Theorem 4.1 shows if the weighted integral $\int_\Omega R(g)\Phi\, d\mathrm{vol}_{\bar g}$ is nonnegative, then $V(g) \ge V(\bar g)$.

5. Other related results

In this section, we collect some other by-products of the formulas derived in Section 2. First, we discuss a scalar curvature rigidity result for general domains in $\mathbb{S}^n$.

Theorem 5.1. *Let $\Omega \subset \mathbb{S}^n$ be a smooth domain contained in a geodesic ball B of radius less than $\frac{\pi}{2}$. Let $\bar{g}$ be the standard metric on $\mathbb{S}^n$. Let $\bar{\mathbb{II}}$, $\bar{H}$ be the second fundamental form, the mean curvature of $\Sigma = \partial\Omega$ in $(\Omega, \bar{g})$ with respect to the outward unit normal $\bar{\nu}$. Suppose $\bar{\mathbb{II}} \geq -c\bar{\gamma}$, where $c \geq 0$ is a function on Σ and $\bar{\gamma}$ is the induced metric on Σ. Let q be the center of B. Suppose at $\Sigma \setminus \{q\}$,*

$$\bar{H} - c \geq \left[\frac{5\cos\theta + \sqrt{\cos^2\theta + 8}}{2}\right] \tan r, \tag{5.1}$$

where r is the $\bar{g}$-distance to q and θ is the angle between $\bar{\nu}$ and $\overline{\nabla} r$. Then the conclusion of Theorem 1.6 holds on Ω.

Proof. As before, replacing g by $\varphi^*(g)$ for some diffeomorphism φ, we may assume $\mathrm{div}_{\bar{g}} h = 0$ where $h = g - \bar{g}$. On Ω, let $\lambda = \cos r > 0$, where r is the $\bar{g}$-distance to q. At $\Sigma \setminus \{q\}$, we have

$$\lambda_{;n} = -\sin r \cos\theta, \quad |\overline{\nabla}^{\Sigma} \lambda| = \sin r \sin\theta. \tag{5.2}$$

Apply Theorem 2.1, using the fact $DR^*_{\bar{g}}(\lambda) = 0$ and the assumptions on $R(g)$ and $H(g)$, we have

$$\begin{aligned}
&\int_\Omega \left[\frac{1}{4}(|\overline{\nabla} h|^2 + |\overline{\nabla}(\mathrm{tr}_{\bar{g}} h)|^2) + \frac{1}{2}\left(|h|^2 + (\mathrm{tr}_{\bar{g}} h)^2\right)\right] \cos r \, d\mathrm{vol}_{\bar{g}} \\
&\leq \int_\Sigma \left[-\frac{1}{4}(h_{nn})^2 \bar{H} - \frac{1}{2}(\bar{\mathbb{II}}(X,X) + \bar{H}|X|^2)\right] \cos r \, d\sigma_{\bar{g}} \\
&\quad + \int_{\Sigma\setminus\{q\}} \left[(h_{nn})^2 + \frac{1}{2}|X|^2\right] (\sin r \cos\theta) \, d\sigma_{\bar{g}} \\
&\quad + \int_{\Sigma\setminus\{q\}} |h_{nn}||X| (\sin r \sin\theta) \, d\sigma_{\bar{g}} \\
&\quad + C||h||_{C^1(\bar{\Omega})} \left\{\int_\Omega (|h|^2 + |\overline{\nabla} h|^2) \, d\mathrm{vol}_{\bar{g}} + \int_\Sigma |h|^2 \, d\sigma_{\bar{g}}\right\}
\end{aligned} \tag{5.3}$$

$$
\begin{aligned}
&\leq -\int_{\Sigma\setminus\{q\}} \Bigg[\left(\frac{1}{4}(\bar{H}-c)\cos r - \sin r\cos\theta \right) (h_{nn})^2 \\
&\quad + \frac{1}{2}\big((\bar{H}-c)\cos r - \sin r \cos\theta \big) |X|^2 \\
&\quad - |h_{nn}||X|(\sin r \sin\theta) \Bigg] \, d\sigma_{\bar{g}} \\
&\quad + C||h||_{C^1(\bar{\Omega})} \left\{ \int_{\Omega} (|h|^2 + |\overline{\nabla} h|^2)\, d\mathrm{vol}_{\bar{g}} + \int_{\Sigma} |h|^2 \, d\sigma_{\bar{g}} \right\}
\end{aligned}
$$

for some positive constant C independent on h.

Note that the assumption (5.1) implies

$$
\frac{1}{4}(\bar{H}-c)\cos r - (\sin r\cos\theta) \geq 0 \tag{5.4}
$$

and

$$
(\bar{H}-c)\cos r - (\sin r \cos\theta) \geq 0. \tag{5.5}
$$

By (5.1), (5.4) and (5.5), we have

$$
\begin{aligned}
0 \leq &\left(\frac{1}{4}(\bar{H}-c)\cos r - \sin r\cos\theta \right) (h_{nn})^2 - |h_{nn}||X|(\sin r\sin\theta) \\
&+ \frac{1}{2}\big((\bar{H}-c)\cos r - \sin r\cos\theta \big) |X|^2
\end{aligned} \tag{5.6}
$$

for any h_{nn} and X. The result now follows from (5.3) and (5.6). □

Remark 5.1. It is clear from the proof of Theorem 5.1 that the center q of B does not need to be inside Ω.

Theorem 5.1 directly implies Theorem 1.7 in the introduction.

Proof of Theorem 1.7. Choose $c = 0$ in Theorem 5.1. Since

$$
4 \geq \frac{5\cos\theta + \sqrt{\cos^2\theta + 8}}{2}
$$

for any θ, the result follows from Theorem 5.1. □

Next, we consider a corresponding scalar curvature rigidity result when the background metric $\bar{g}$ is a flat metric.

Theorem 5.2. *Let Ω be a compact manifold with smooth boundary Σ. Suppose $\bar{g}$ is a smooth Riemannian metric on Ω such that $\bar{g}$ has zero sectional curvature and $\overline{\mathrm{II}} + \bar{H}\bar{\gamma} \geq 0$ on Σ, where $\overline{\mathrm{II}}$, $\bar{H}$ are the second fundamental form, the mean curvature of Σ, and $\bar{\gamma}$ is the induced metric on Σ. Suppose g is another metric on Ω satisfying*

- *$R(g) \geq 0$ where $R(g)$ is the scalar curvature of g,*
- *g and $\bar{g}$ induce the same metric on Σ,*
- *$H(g) \geq \bar{H}$ where $H(g)$ is the mean curvature of Σ in (Ω, g).*

If $||g - \bar{g}||_{C^2(\bar{\Omega})}$ is sufficiently small, then there is a diffeomorphism φ on Ω with $\varphi|_\Sigma = \mathrm{id}$ such that $\varphi^*(g) = \bar{g}$.

Proof. As before, we may assume $\mathrm{div}_{\bar{g}} h = 0$ where $h = g - \bar{g}$. Choose $\lambda = 1$ in (2.23), one has

$$\begin{aligned}
(5.7) \qquad & \int_\Omega \left[\frac{1}{4}(|\overline{\nabla} h|^2 + |\overline{\nabla}(\mathrm{tr}_{\bar{g}} h)|^2)\right] d\mathrm{vol}_{\bar{g}} \\
& + \int_\Sigma \left[\frac{1}{4}(h_{nn})^2 H(\bar{g}) + \frac{1}{2}(\overline{\mathrm{II}}(X,X) + H(\bar{g})|X|^2)\right] d\sigma_{\bar{g}} \\
& \leq \int_\Omega E(h)\, d\mathrm{vol}_{\bar{g}} + \int_\Sigma F(h)\, d\sigma_{\bar{g}},
\end{aligned}$$

where $|F(h)| \leq C(|h|^2|\overline{\nabla} h| + |h|^3)$ and $|E(h)| \leq C|h||\overline{\nabla} h|^2$ by Remark 2.1. The result follows from (5.7). □

To finish, we mention that the positive Gaussian curvature condition of the boundary surface in [17] is not a necessary condition for the positivity of its Brown–York mass.

Theorem 5.3. *Let $\Sigma \subset \mathbb{R}^n$ be a connected, closed hypersurface satisfying $\overline{\mathrm{II}} + \bar{H}\bar{\gamma} \geq 0$, where $\overline{\mathrm{II}}$, $\bar{H}$ are the second fundamental form, the mean curvature of Σ, and $\bar{\gamma}$ is the induced metric on Σ. Let Ω be the domain enclosed by Σ in $\mathbb{R}^n$. Let h be any nontrivial $(0,2)$ symmetric tensor on Ω satisfying*

$$(5.8) \qquad \mathrm{div}_{\bar{g}} h = 0, \quad \mathrm{tr}_{\bar{g}} h = 0, \quad h|_{T\Sigma} = 0.$$

Let $\{g(t)\}_{|t|<\epsilon}$ be a 1-parameter family of metrics on Ω satisfying

$$(5.9) \qquad g(0) = \bar{g}, \quad g'(0) = h, \quad R(g(t)) \geq 0, \quad g(t)|_{T\Sigma} = \bar{g}|_{T\Sigma}.$$

Then

$$\int_\Sigma \bar{H} d\sigma_{\bar{g}} > \int_\Sigma H(g(t)) d\sigma_{\bar{g}} \tag{5.10}$$

for small $t \neq 0$, where $H(g(t))$ is the mean curvature of Σ in $(\Omega, g(t))$.

Proof. By Lemma 2.2, one knows

$$\frac{d}{dt}\left(\int_\Omega [R(g(t)) - R(\bar{g})] \, d\mathrm{vol}_{\bar{g}} - 2\int_\Sigma [\bar{H} - H(g(t))] \, d\sigma_{\bar{g}}\right)\Big|_{t=0} = 0.$$

Direct calculation using Lemma 2.2, (2.17) and (5.8) shows

$$\begin{aligned} &\frac{d^2}{dt^2}\left(\int_\Omega [R(g(t)) - R(\bar{g})] \, d\mathrm{vol}_{\bar{g}} - 2\int_\Sigma [\bar{H} - H(g(t))] \, d\sigma_{\bar{g}}\right)\Big|_{t=0} \\ &= -\frac{1}{2}\int_\Omega |\overline{\nabla} h|^2 \, d\mathrm{vol}_{\bar{g}} - \int_\Sigma \left[(\overline{\mathrm{I\!I}}(X,X) + H(\bar{g})|X|^2)\right] \, d\sigma_{\bar{g}}, \end{aligned} \tag{5.11}$$

which is negative by the assumption on $\overline{\mathrm{I\!I}} + \bar{H}\bar{\gamma}$. Thus, for small t,

$$2\int_\Sigma [\bar{H} - H(g(t))] \, d\sigma_{\bar{g}} > \int_\Omega [R(g(t)) - R(\bar{g})] \, d\mathrm{vol}_{\bar{g}} \geq 0. \tag{5.12}$$

□

Given an h satisfying (5.8), a family of deformation $\{g(t)\}$ satisfying (5.9) is given by $g(t) = u(t)^{\frac{4}{n-2}}(\bar{g} + th)$ for small t, where $u(t) > 0$ is a conformal factor such that $R(g(t)) = 0$ (see [13, Lemma 4]).

An example of a non-convex surface $\Sigma \subset \mathbb{R}^3$, which is topologically a 2-sphere and satisfies the condition $\overline{\mathrm{I\!I}} + \bar{H}\bar{\gamma} \geq 0$, is given by a capsule-shaped surface with its middle slightly pinched.

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