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# New invariants for complex manifolds and isolated singularities

Rong Du, Hing Sun Luk and Stephen Yau

In this paper, we introduce some new invariants for complex manifolds. These invariants measure in some sense how far the complex manifolds are away from having global complex coordinates. For applications, we introduce two new invariants $f^{(1,1)}$ and $g^{(1,1)}$ for isolated surface singularities. We show that $f^{(1,1)} = g^{(1,1)} = 1$ for rational double points and cyclic quotient singularities.

Dedicated to Professor Michael Artin on the occasion of his 78th Birthday.

## 1. Introduction

Let $M$ be a complex manifold of dimension $n$. It is a natural question to ask how far this complex manifold is away from having global complex coordinates. In this paper, we introduce some new biholomorphic invariants which give some measurements for this purpose.

**Definition 1.1.** For any $1 \leq p \leq n$, let $\Omega_M^p$ be the sheaf of germs of holomorphic $p$-forms on $M$. Denote by $< \Lambda^p\Gamma(M, \Omega_M^1) >$ the linear span of all $p$th wedge products of global holomorphic one-forms on $M$. Define $\gamma^{(p)}(M) := \dim \Gamma(M, \Omega_M^p)/ < \Lambda^p\Gamma(M, \Omega_M^1) >$. Then $\gamma^{(p)}$ is a biholomorphic invariant of $M$.

If $M$ is a complex submanifold in $\mathbb{C}^N$, then given any global holomorphic $p$-form $\alpha$ on $M$, there exists a holomorphic $p$-form $\widetilde{\alpha}$ on $\mathbb{C}^N$ such that the restriction of $\widetilde{\alpha}$ on $M$ is $\alpha$. Obviously, $\widetilde{\alpha}$ is a $p$th wedge product of holomorphic one-forms. We see that $\gamma^{(p)}(M) = 0$. If $M$ is a compact complex torus, then it is easy to see that $\gamma^{(p)}(M) = 0$. It is an interesting question to classify those compact complex manifolds with $\gamma^{(p)} = 0$.

One of the most fundamental questions in complex geometry is the complex Plateau problem. Given a strongly pseudoconvex CR manifold $X$ in $\mathbb{C}^N$, the problem asks when $X$ is the boundary of a complex manifold $V$ in $\mathbb{C}^N$. By the beautiful work of Harvey–Lawson [5], the works of Yau [20] and Luk and Yau [8], $X$ is a boundary of a complex variety $V$ with only isolated singularities if $X$ is contained in the boundary of a strictly pseudoconvex domain in $\mathbb{C}^N$. Thus from the complex Plateau problem point of view, it is very desirable to introduce a numerical invariant for isolated singularities which never vanishes. Hopefully this numerical invariant is computable in terms of $X$. The purpose of this paper is to use the above idea to study singularities. Specifically, we introduce two new invariants $f^{(1,1)}$ and $g^{(1,1)}$ for isolated surface singularities. Previously, numerical invariants are used for the classification of surface singularities. The fundamental invariants are the geometric genus $p_{\mathrm{g}}$ and the arithmetic genus $p_{\mathrm{a}}$. In [1], Artin introduced a definition for singularity to be rational, i.e., those singularities with $p_{\mathrm{g}} = 0$. Wagreich [13] introduced a definition for singularity to be weakly elliptic, i.e., for those singularities with $p_{\mathrm{a}} = 1$. Later, Laufer [6] studied the so-called minimal elliptic singularities, i.e. for those Gorenstein singularities with $p_{\mathrm{g}} = 1$. In a series of papers [15–19], Yau developed a novel theory of elliptic sequence to study weakly elliptic singularities which may have $p_{\mathrm{g}}$ arbitrarily large. In particular, he classified all weighted dual graphs of hypersurface singularities with $p_{\mathrm{g}} = 2$ [15]. In 1982, he considered another invariant, namely irregularity $q$ [21–23]. Later Wahl [14], Straten and Steenbrink [12] studied this invariant further. Unfortunately, all these numerical invariants vanish on rational singularities. In this paper, we shall give a detailed study of $f^{(1,1)}$. We also give explicit calculation for $f^{(1,1)}$ and $g^{(1,1)}$ for rational double points and cyclic quotient singularities and prove that they do not vanish. The following are our main results.

**Theorem A:** *Let* $(V, 0)$ *be a two-dimensional normal Stein space with* $\mathbb{C}^*$-*action and with an isolated singularity at* 0. *Then* $f^{(1,1)} \geq 1$.

**Theorem B:** *Let* $(V, 0)$ *be a two-dimensional Stein space with* 0 *as its only singular point. If* 0 *is a rational double point or cyclic quotient singularity, then* $f^{(1,1)} = g^{(1,1)} = 1$.

The invariant $g^{(1,1)}$ studied in this paper was used by Du and Yau [4] to solve the regularity problem of the Harvey–Lawson solution to the complex Plateau problem.

## 2. Invariants of singularities

Let $V$ be a $n$-dimensional complex analytic subvariety in $\mathbb{C}^N$ with only isolated singularities. In [22], Yau considered four kinds of sheaves of germs of holomorphic $p$-forms

(1) $\bar{\Omega}_V^p := \pi_*\Omega_M^p$, where $\pi : M \longrightarrow V$ is a resolution of singularities of $V$.

(2) $\bar{\bar{\Omega}}_V^p := \theta_*\Omega_{V\backslash V_{\text{sing}}}^p$ where $\theta : V\backslash V_{\text{sing}} \longrightarrow V$ is the inclusion map and $V_{\text{sing}}$ is the singular set of $V$.

(3) $\Omega_V^p := \Omega_{\mathbb{C}^N}^p/\mathscr{K}^p$, where $\mathscr{K}^p = \{f\alpha + dg \wedge \beta : \alpha \in \Omega_{\mathbb{C}^N}^p; \beta \in \Omega_{\mathbb{C}^N}^{p-1}; f, g \in \mathscr{I}\}$ and $\mathscr{I}$ is the ideal sheaf of $V$ in $\mathbb{C}^N$.

(4) $\widetilde{\Omega}_V^p := \Omega_{\mathbb{C}^N}^p/\mathscr{H}^p$, where $\mathscr{H}^p = \{\omega \in \Omega_{\mathbb{C}^N}^p : \omega|_{V\backslash V_{\text{sing}}} = 0\}$.

$\Omega_V^p$ is Grauert–Grothendieck sheaf of germs of holomorphic $p$-form on $V$. In case $V$ is a normal variety, the dualizing sheaf $\omega_V$ of Grothendieck is actually the sheaf $\bar{\bar{\Omega}}_V^n$. Clearly $\Omega_V^p$, $\widetilde{\Omega}_V^p$ are coherent. $\bar{\Omega}_V^p$ is a coherent sheaf because $\pi$ is a proper map. $\bar{\bar{\Omega}}_V^p$ is also a coherent sheaf by a theorem of Siu (cf. Theorem A of [11]).

**Definition 2.1.** The Siu complex is a complex of coherent sheaves $J^\bullet$ supported on the singular points of $V$ which is defined by the following exact sequence:

$$0 \longrightarrow \bar{\Omega}^\bullet \longrightarrow \bar{\bar{\Omega}}^\bullet \longrightarrow J^\bullet \longrightarrow 0. \tag{2.1}$$

**Definition 2.2.** Let $V$ be a $n$-dimensional Stein space with 0 as its only singular point. Let $\pi : (M, A) \rightarrow (V, 0)$ be a resolution of the singularity with $A$ as exceptional set. The geometric genus $p_g$, the irregularity $q$ and the $g^{(p)}$-invariant of the singularity are defined as follows (cf. [Ya7,St-St]):

$$p_g := \dim \Gamma(M\backslash A, \Omega^n)/\Gamma(M, \Omega^n), \tag{2.2}$$

$$q := \dim \Gamma(M\backslash A, \Omega^{n-1})/\Gamma(M, \Omega^{n-1}), \tag{2.3}$$

$$g^{(p)} := \dim \Gamma(M, \Omega_M^p)/\pi^*\Gamma(V, \Omega_V^p). \tag{2.4}$$

The $s$-invariant of the singularity is defined in [9] as follows:

$$s := \dim \Gamma(M\backslash A, \Omega^n)/[\Gamma(M, \Omega^n) + d\Gamma(M\backslash A, \Omega^{n-1})]. \tag{2.5}$$

The following lemma follows from a deep theorem of Straten and Steenbrink.

**Lemma 2.1.** [12]. *Let $V$ be a $n$-dimensional Stein space with* 0 *as its only singular point. Let $J^\bullet$ be the Siu complex of coherent sheaves supported on* 0. *Then*

(1) $\dim J^n = p_g$,

(2) $\dim J^{n-1} = q$,

(3) $\dim J^i = 0$, *for* $1 \leq i \leq n-2$.

**Proposition 2.1.** [12]. *Let* $V$ *be a* $n$*-dimensional Stein space with* 0 *as its only singular point. Let* $J^\bullet$ *be the Siu complex of coherent sheaves supported on* 0*. Then the* $s$*-invariant is given by*

$$s := \dim H^n(J^\bullet) = p_g - q \tag{2.6}$$

*and*

$$\dim H^{n-1}(J^\bullet) = 0. \tag{2.7}$$

**Definition 2.3.** Let $(V, 0)$ be a two-dimensional Stein analytic space with an isolated singularity at 0. Let $\pi : (M, A) \to (V, 0)$ be a resolution of the singularity with $A$ as exceptional set. Define a sheaf of germs $\bar{\Omega}_V^{1,1}$ by the sheaf associated to the presheaf

$$U \mapsto < \Gamma(\pi^{-1}(U), \Omega_M^1) \wedge \Gamma(\pi^{-1}(U), \Omega_M^1) >,$$

where $U$ is an open set of $V$ and $< \Gamma(\pi^{-1}(U), \Omega_M^1) \wedge \Gamma(\pi^{-1}(U), \Omega_M^1) >$ means that it is generated by elements in $\Gamma(\pi^{-1}(U), \Omega_M^1) \wedge \Gamma(\pi^{-1}(U), \Omega_M^1)$ over the ring $\Gamma(\pi^{-1}(U), \mathscr{O}_M)$.

**Lemma 2.2.** *Let* $(V, 0)$ *be a two-dimensional Stein analytic space with an isolated singularity at* 0*. Let* $\pi : (M, A) \to (V, 0)$ *be a resolution of the singularity with* $A$ *as exceptional set. Then* $\bar{\Omega}_V^{1,1}$ *is coherent and there is a short exact sequence*

$$0 \longrightarrow \bar{\Omega}_V^{1,1} \longrightarrow \bar{\Omega}_V^2 \longrightarrow \mathscr{F}^{(1,1)} \longrightarrow 0, \tag{2.8}$$

*where* $\mathscr{F}^{(1,1)}$ *is a sheaf supported on the singular point of* $V$*. Let*

$$F^{(1,1)}(M) := \Gamma(M, \Omega_M^2) / < \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) >, \tag{2.9}$$

*then* $\dim \mathscr{F}_0^{(1,1)} = \dim F^{(1,1)}(M)$.

*Proof.* Since the sheaf of germ $\bar{\Omega}_V^1$ is coherent by the direct image theorem, for any point $w \in V$ there exists an open neighborhood $U$ of $w$ in $V$ such that $\Gamma(\pi^{-1}(U), \Omega_M^1)$ is finitely generated over $\Gamma(\pi^{-1}(U), \mathscr{O}_M)$. So $\Gamma(\pi^{-1}(U), \Omega_M^1)$

$\wedge\,\Gamma(\pi^{-1}(U),\Omega^1_M)$ is also finitely generated over $\Gamma(\pi^{-1}(U),\ \mathscr{O}_M)$, which means $\bar{\Omega}^{1,1}_V$ is a sheaf of finite type. It is obvious that $\bar{\Omega}^{1,1}_V$ is a subsheaf of $\bar{\Omega}^2_V$ which is also coherent. So $\bar{\Omega}^{1,1}_V$ is also coherent.

Notice that the stalk of $\bar{\Omega}^{1,1}_V$ and $\bar{\Omega}^2_V$ coincide at each point different from the singular point 0, $\mathscr{F}^{(1,1)}$ is supported at 0. It follows from Cartan Theorem B that:

$$\dim \mathscr{F}^{(1,1)}_0 = \dim \Gamma(V,\bar{\Omega}^2_V)/\Gamma(V,\bar{\Omega}^{1,1}_V) = \dim F^{(1,1)}(M).$$

□

Observe that $\Gamma(M,\Omega^2_M)$ and $<\Gamma(M,\Omega^1_M)\wedge\Gamma(M,\Omega^1_M)>$ are birational invariants. Thus, from Lemma 2.2, we can define a local invariant of a singularity which is independent of resolution.

**Definition 2.4.** Let $V$ be a two-dimensional Stein space with 0 as its only singular point. Let $\pi:(M,A)\to(V,0)$ be a resolution of the singularity with $A$ as exceptional set. Let

$$f^{(1,1)}(0) := \dim \mathscr{F}^{(1,1)}_0 = \dim F^{(1,1)}(M). \tag{2.10}$$

We will omit 0 in $f^{(1,1)}(0)$ if there is no confusion from the context.

$f^{(1,1)}$ is independent of the resolution of $V$.

The following proposition is to show that $f^{(1,1)}$ is bounded above by $g^{(2)}$.

**Proposition 2.2.** *Let $V$ be a two-dimensional Stein space with 0 as its only singular point. Then $f^{(1,1)}\le g^{(2)}$.*

*Proof.* Since

$$f^{(1,1)} = \dim \Gamma(M,\Omega^2_M)/<\Gamma(M,\Omega^1_M)\wedge\Gamma(M,\Omega^1_M)>,$$

$$g^{(2)} := \dim \Gamma(M,\Omega^2)/\pi^*\Gamma(V,\Omega^2_V),$$

and

$$\pi^*\Gamma(V,\Omega^2_V) = <\pi^*\Gamma(V,\Omega^1_V)\wedge\pi^*\Gamma(V,\Omega^1_V)> \subseteq <\Gamma(M,\Omega^1_M)\wedge\Gamma(M,\Omega^1_M)>,$$

the result follows. □

**Definition 2.5.** Let $(V, 0)$ be a Stein germ of a 2-dimensional analytic space with an isolated singularity at 0. Define a sheaf of germs $\bar{\bar{\Omega}}_V^{1,1}$ by the sheaf associated to the presheaf

$$U \mapsto < \Gamma(U, \bar{\bar{\Omega}}_V^1) \wedge \Gamma(U, \bar{\bar{\Omega}}_V^1) >,$$

where $U$ is an open set of $V$.

**Lemma 2.3.** *Let $V$ be a two-dimensional Stein space with 0 as its only singular point in $\mathbb{C}^N$. Let $\pi : (M, A) \to (V, 0)$ be a resolution of the singularity with $A$ as exceptional set. Then $\bar{\bar{\Omega}}_V^{1,1}$ is coherent and there is a short exact sequence*

$$0 \longrightarrow \bar{\bar{\Omega}}_V^{1,1} \longrightarrow \bar{\bar{\Omega}}_V^2 \longrightarrow \mathscr{G}^{(1,1)} \longrightarrow 0, \tag{2.11}$$

*where $\mathscr{G}^{(1,1)}$ is a sheaf supported on the singular point of $V$. Let*

$$G^{(1,1)}(M\backslash A) := \Gamma(M\backslash A, \Omega_M^2)/ < \Gamma(M\backslash A, \Omega_M^1) \wedge \Gamma(M\backslash A, \Omega_M^1) >, \tag{2.12}$$

*then* $\dim \mathscr{G}_0^{(1,1)} = \dim G^{(1,1)}(M\backslash A)$.

*Proof.* This proof is the same as the proof of Lemma 2.2 with symbols replaced according to the statement. □

**Definition 2.6.** Let $V$ be a two-dimensional Stein space with 0 as its only singular point. Let $\pi : (M, A) \to (V, 0)$ be a resolution of the singularity with $A$ as exceptional set. Let

$$g^{(1,1)}(0) := \dim \mathscr{G}_0^{(1,1)} = \dim G^{(1,1)}(M\backslash A). \tag{2.13}$$

We will omit 0 in $g^{(1,1)}(0)$ if there is no confusion from the context.
The following proposition is to show that $g^{(1,1)}$ is bounded above.

**Proposition 2.3.** *Let $V$ be a two-dimensional Stein space with 0 as its only singular point. Then $g^{(1,1)} \leq p_{\mathrm{g}} + g^{(2)}$.*

*Proof.* Since

$$g^{(1,1)} = \dim \Gamma(M\backslash A, \Omega_M^2)/ < \Gamma(M\backslash A, \Omega_M^1) \wedge \Gamma(M\backslash A, \Omega_M^1) >,$$
$$p_{\mathrm{g}} = \dim \Gamma(M\backslash A, \Omega_M^2)/\Gamma(M, \Omega_M^2),$$
$$g^{(2)} := \dim \Gamma(M, \Omega^2)/\pi^*\Gamma(V, \Omega_V^2).$$

and

$$\begin{aligned}\pi^*\Gamma(V,\Omega_V^2) &=< \pi^*\Gamma(V,\Omega_V^1)\wedge\pi^*\Gamma(V,\Omega_V^1) > \\ &\subseteq \Gamma(M,\Omega_M^1)\wedge\Gamma(M,\Omega_M^1) \\ &\subseteq \Gamma(M\backslash A,\Omega_M^1)\wedge\Gamma(M\backslash A,\Omega_M^1),\end{aligned} \tag{2.14}$$

the result follows. □

The next theorem is one of our main theorems (Theorem A).

**Theorem 2.1.** *Let $V$ be a two-dimensional Stein space with $0$ as its only normal singular point with $\mathbb{C}^*$-action. Then $f^{(1,1)} \geq 1$.*

*Proof.* It suffices to prove there is a holomorphic two-form $\omega$ on $M$ which is not contained in $< \Gamma(M,\Omega_M^1)\wedge\Gamma(M,\Omega_M^1) >$. Embed (V, 0) locally into $\mathbb{C}^m$ and let $z_1,\dots,z_m$ be coordinate function of $\mathbb{C}^m$. We are going to prove this theorem by local calculation.

If $p_g > 0$, there exists a holomorphic two-form $\omega_0$ on $M\backslash A$ but not on $M$, i.e. $\omega_0 \in \Gamma(M\backslash A,\Omega_M^2)\backslash\Gamma(M,\Omega_M^2)$. So $\omega_0$ must have pole along some irreducible component $A_1$ of $A$. Suppose $\omega_1$ has the lowest order of pole along $A_1$ such that $\omega_1 \in \Gamma(M\backslash A,\Omega_M^2)\backslash\Gamma(M,\Omega_M^2)$. Then $\pi^*(z_j)\omega_1$ is holomorphic along $A_1$ for all $j$, $1\leq j\leq m$. If $\pi^*(z_j)\omega_1 \notin \Gamma(M,\Omega_M^2)$, it must has pole along another irreducible component $A_2$ of $A$. Suppose $\omega_2 \in \Gamma(M\backslash A,\Omega_M^2)$ has the lowest order of poles along $A_2$ and holomorphic along $A_1$. Then $\pi^*(z_j)\omega_2$ is holomorphic along $A_1$ and $A_2$, for all $j$, $1\leq j\leq m$. Since the number of irreducible components of $A$ is finite, by induction, there exists a non-empty set $W_k = \{\omega \in \Gamma(M\backslash A,\Omega_M^2)\backslash\Gamma(M,\Omega_M^2) : \omega$ has pole along some irreducible component $A_k$ and holomorphic along $A_1,\dots,A_{k-1}$ such that $\pi^*(z_j)\omega \in \Gamma(M,\Omega_M^2)$ for all $j$, $1\leq j\leq m\}$. Suppose $\omega_k \in W_k$, then choose a point $b$ in $A_k$ which is a smooth point of $A$. Let$(x_1,x_2)$ be a coordinate system centered at $b$ such that $A_k$ is given locally by $x_1 = 0$ at $b$. Take the power series expansion of $\pi^*(z_j)$ around $b$:

$$\pi^*(z_j) \mathrel{\mathring{=}} x_1^{r_j} f_j, 1\leq j\leq m, \tag{2.15}$$

where $f_j$ is holomorphic function such that $f_j(0,x_2)\neq 0$ and "$\mathring{=}$" means local equality around $b$. Without loss of generality, we can suppose $r_1 = \min\{r_1,\dots,r_m\}$. By local calculation, we know that for any element $\psi$ in $\pi^*\Gamma(V,\Omega_V^2)$, the vanishing order of $\psi$ along $A_k$, which is denoted by $\mathrm{Ord}_{A_k}\psi$, is at least $2r_1-1$. However, $\mathrm{Ord}_{A_k}\pi^*(z_1)\omega_k$ is at most $r_1-1$. So $\pi^*(z_1)\omega_k \in \Gamma(M,\Omega_M^2)\backslash\pi^*\Gamma(V,\Omega_V^2)$. We pick such kind $\omega \in \Gamma(M,\Omega_M^2)\backslash\pi^*\Gamma(V,\Omega_V^2)$ which has the lowest order of zeros, $r$, along $A_k$, i.e., $\mathrm{Ord}_{A_k}\omega = r$. So $r < r_1$.

Let $\xi_V \in \Gamma(V, \Theta_V)$, where $\Theta_V := \mathscr{H}om_{\mathscr{O}_V}(\Omega^1_V, \mathscr{O}_V)$, denote the generating vector field of the $\mathbb{C}^*$-action and let $i_{\xi_V}$ denote the contraction map. For every $\alpha \in \Gamma(V, \bar{\Omega}^1_V)$, write $\alpha$ as a sum $\sum \alpha^j$ of quasi-homogeneous elements where $\alpha^j$ is a quasi-homogeneous element of degree $l_j > 0$. Let $L_{\xi_V} = i_{\xi_V} d + d i_{\xi_V}$ be the Lie derivation. Then

$$l_j \alpha^j = L_{\xi_V} \alpha^j = i_{\xi_V} d(\alpha^j) + d i_{\xi_V}(\alpha^j).$$

So

$$\Gamma(V, \bar{\Omega}^1_V) = d(\Gamma(V, \mathscr{O}_V)) + i_{\xi_V}(\Gamma(V, \bar{\Omega}^2_V)). \tag{2.16}$$

Since for minimal good resolution, we have $\pi_* \Theta_M = \Theta_V$ (cf [2]), where $\Theta_M$ is the vector field on $M$. Thus, there exists $\xi_M$ which is a lift of $\xi_V$, i.e., $\pi_* \xi_M = \xi_V$. We know that $\xi_M$ is tangential to the exceptional set, so

$$\xi_M \mathrel{\mathring{=}} x_1^{a_1} p \frac{\partial}{\partial x_1} + x_1^{a_2} q \frac{\partial}{\partial x_2}, a_1 \geq 1, a_2 \geq 0,$$

where $p$ and $q$ are holomorphic functions.

Let $i_{\xi_M} : \Gamma(M, \Omega^2_M) \longrightarrow \Gamma(M, \Omega^1_M)$ be the contraction map corresponding to $i_{\xi_V}$. If $\zeta \in \Gamma(M, \Omega^2_M)$ and $\zeta \mathrel{\mathring{=}} x_1^u g dx_1 \wedge dx_2$, then

$$i_{\xi_M}(\zeta) \mathrel{\mathring{=}} i_{\xi_M}(x_1^u g dx_1 \wedge dx_2) = -x_1^{u+a_2} q g dx_1 + x_1^{u+a_1} p g dx_2.$$

From (2.16),

$$\Gamma(M, \Omega^1_M) = d(\Gamma(M, \mathscr{O}_M)) + i_{\xi_M}(\Gamma(M, \Omega^2_M)).$$

Since $V$ is normal , $g^{(0)} = 0$, i.e., $\Gamma(M, \mathscr{O}_M) = \pi^*(\Gamma(V, \mathscr{O}_V))$.

We now prove that $\omega$ is not contained in $< \Gamma(M, \Omega^1_M) \wedge \Gamma(M, \Omega^1_M) >$. Consider $\eta, \varphi \in \Gamma(M, \Omega^1_M)$ locally around $b$.

Suppose $\eta = \eta_1 + \eta_2$ and $\varphi = \varphi_1 + \varphi_2$, where $\eta_1$, $\varphi_1 \in d(\Gamma(M, \mathscr{O}_M))$, $\eta_2$, $\varphi_2 \in i_{\xi_M}(\Gamma(M, \Omega^2_M))$. Let

$$\eta_2 = i_{\xi_M}(\zeta), \quad \zeta \mathrel{\mathring{=}} x_1^u g dx_1 \wedge dx_2, \quad g(0, x_2) \neq 0,$$

and

$$\varphi_2 = i_{\xi_M}(\varsigma), \quad \varsigma \mathrel{\mathring{=}} x_1^v h dx_1 \wedge dx_2, \quad h(0, x_2) \neq 0.$$

Then

$$\eta \wedge \varphi = \eta_1 \wedge \varphi_1 + (\eta_1 \wedge \varphi_2 + \eta_2 \wedge \varphi_1) + \eta_2 \wedge \varphi_2.$$

Since

$$d\pi^*(z_i) \wedge d\pi^*(z_j) = \left( r_i x_1^{r_i+r_j-1} f_i \frac{\partial f_j}{\partial x_2} - r_j x_1^{r_i+r_j-1} f_j \frac{\partial f_i}{\partial x_2} \right) dx_1 \wedge dx_2,$$

$\mathrm{Ord}_{A_k} \eta_1 \wedge \varphi_1 \geq 2 \cdot r_1 - 1 > r$.

Write $\eta_2$ and $\varphi_2$ locally around b:

$$\eta_2 \overset{\circ}{=} -x_1^{u+a_2} qg dx_1 + x_1^{u+a_1} pg dx_2,$$
$$\varphi_2 \overset{\circ}{=} -x_1^{v+a_2} qh dx_1 + x_1^{v+a_1} ph dx_2.$$

So $\eta_2 \wedge \varphi_2 \overset{\circ}{=} 0$.

Also notice that

$$d\pi^*(z_j) = r_j x_1^{r_j-1} f_j dx_1 + x_1^{r_j} \frac{\partial f_j}{\partial x_2} dx_2.$$

So

$$\mathrm{Ord}_{A_k} \eta_1 \wedge \varphi_2 \geq r_1 + v > r$$

and

$$\mathrm{Ord}_{A_k} \eta_2 \wedge \varphi_1 \geq r_1 + u > r.$$

From the discussion above, we can get $\mathrm{Ord}_{A_k} \eta \wedge \varphi > r$.

Therefore $\omega$ is not a linear combination of elements in $< \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) >$.

If $p_g = 0$, from [24], the canonical bundle $K_M$ is generated by its global sections in a neighborhood of the exceptional set. So there exists $\omega \in \Gamma(M, \Omega_M^2)$ such that $\omega$ does not vanish along some irreducible component $A_k$ of $A$. The rest of the argument is same as those in the case of $p_g > 0$, with $r$ is 0. □

The following theorem is the crucial part for the solution of the classical complex Plateau problem.

**Theorem 2.2.** ([Du-Ya]) *Let $V$ be a two-dimensional Stein space with 0 as its only normal singular point with $\mathbb{C}^*$-action, then $g^{(1,1)} \geq 1$.*

In the next section we will show that this bound is sharp.

## 3. Explicit calculation of new invariants for special rational singularities

In this section, we suppose that $V$ is a two-dimensional Stein space with 0 as its only normal singularity and $V$ is contractible to 0. It is well known that the singularities of type $A_n$, $D_n$, $E_6$, $E_7$, $E_8$ may be given by the following equations in $\mathbb{C}^3$, with singularities at the origin.

$$\begin{array}{lll} A_n & F(x,y,z) = xy - z^{n+1} = 0, & n \geq 1, \\ D_n & F(x,y,z) = x^2 z + y^2 - z^{n-1} = 0, & n \text{ even} \geq 4, \\ & F(x,y,z) = x^2 + y^2 z - z^{n-1} = 0, & n \text{ odd} \geq 5, \\ E_6 & F(x,y,z) = x^2 - y^3 - z^4 = 0, & \\ E_7 & F(x,y,z) = x^2 + y^3 - yz^3 = 0, & \\ E_8 & F(x,y,z) = x^2 - y^3 + z^5 = 0. & \end{array}$$

The cyclic quotient singularities are also well understood (see [7]). In the following computation, we shall use explicit resolutions $\pi : M \to V$ of $A_n$, $D_n$, $E_6$, $E_7$, $E_8$ (see [10], where he splits Type $D_n$ into two cases for calculation) and cyclic quotient singularities to compute our new invariants.

**Proposition 3.1.** [3]. *If $(V, 0)$ is rational isolated singularity of dimension $n \geq 2$, then any closed holomorphic $p$-form $\eta$ on $V\backslash\{0\}$ with $1 \leq p \leq 2$ is exact, i.e. after shrinking $V$ as a neighborhood of 0, there exists a $p-1$-form $\xi$ on $V\backslash\{0\}$ with $d(\xi) = \eta$.*

**Corollary 3.1.** *If $(V, 0)$ is rational isolated singularity of dimension 2, $M$ is a resolution of the singularity, then $H^1_h(M) = H^2_h(M) = 0$.*

Note that for these rational singularities of dimension 2, the irregularity $q = 0$ (cf.[21]), so $f^{(1,1)} = g^{(1,1)}$. In order to calculate our new invariants for these rational singularities, we must know all the holomorphic one-forms and holomorphic two-forms on $M$. Usually, it is not easy to calculate holomorphic one-forms, but for rational singularities, we have the following lemma.

**Lemma 3.1.** *If $(V, 0)$ is rational isolated singularity of dimension 2 and $\pi : M \to V$ is a resolution then for any $\xi \in \Gamma(M, \Omega^1_M)$ and $\zeta \in d^{-1}(d\xi)$, there exists an $f \in \Gamma(M, \mathcal{O}_M)$, such that $\xi = \zeta + d(f)$, where $d$ is the exterior differential operator.*

*Proof.* From the corollary above, we have the following exact sequence:

$$0 \to \Gamma(M, \mathscr{O}_M) \xrightarrow{d} \Gamma(M, \Omega^1_M) \xrightarrow{d} \Gamma(M, \Omega^2_M) \to 0.$$

For $\xi \in \Gamma(M, \Omega^1_M)$ and any $\zeta \in d^{-1}(d\xi)$, $d(\xi - \zeta) = 0$. So there exist $f \in \Gamma(M, \mathscr{O}_M)$ such that $\xi = \zeta + d(f)$. □

From the lemma above, we see that in order to get holomorphic one-forms on $M$, we only need to calculate holomorphic functions and holomorphic two-forms on $M$.

For rational double points, we may, without loss of generality, suppose $V$ contains $\{(x, y, z) : F(x, y, z) = 0, |x|^2 + |y|^2 + |z|^2 < 1\}$, $F(x, y, z)$ as above. By abusing of notation, we denote local resolutions of the rational double points at 0 by $\pi : M \to V$.

**Type $A_n$:**

An explicit resolution $\pi : M \to V$ can be given in terms of coordinates and transition functions on $M$ as follows:

Coordinates charts: $W_k = \{(u_k, v_k)\}$, $k = 0, 1, \dots, n$

Transition functions:

$$\begin{cases} u_{k+1} = \dfrac{1}{v_k} \\ v_{k+1} = u_k v_k^2 \end{cases} \quad \text{or} \quad \begin{cases} u_k = u_{k+1}^2 v_{k+1} \\ v_k = \dfrac{1}{u_{k+1}} \end{cases}.$$

Projection map: $\pi(u_k, v_k) = (u_k^{k+1} v_k^k, u_k^{n-k} v_k^{n-k+1}, u_k v_k)$

Exceptional set: $A = \pi^{-1}(0) = C_1 \cup \cdots \cup C_n$, where $C_k = \{u_{k-1} = 0\} \cup \{v_k = 0\}$

Dual graph: 

$C_1 \quad C_2 \quad C_{n-1} C_n.$

Holomorphic functions on $M$:

Any holomorphic function on $M$ has power series expansion $\sum_{\alpha,\beta \geq 0} c_{\alpha\beta} u_0^\alpha v_0^\beta$ which converges for all $(u_0, v_0) \in W_0$. Under changes of charts,

$$\begin{aligned} \sum_{\alpha,\beta \geq 0} c_{\alpha\beta} u_0^\alpha v_0^\beta = \cdots &= \sum_{\alpha,\beta \geq 0} c_{\alpha\beta} u_k^{(k+1)\alpha - k\beta} v_k^{k\alpha - (k-1)\beta} \\ = \cdots &= \sum_{\alpha,\beta \geq 0} c_{\alpha\beta} u_n^{(n+1)\alpha - n\beta} v_n^{n\alpha - (n-1)\beta}. \end{aligned}$$

The $k$th power series has to converge for all $(u_k, v_k) \in W_k$. This occurs for all $k$ if only if the indices $\alpha$, $\beta$ in each sum satisfy $(n+1)\alpha - n\beta \geq 0$. Thus any holomorphic function on $M$ can be given by a convergent power series

$$\sum_{\substack{\alpha,\beta\geq 0\\(n+1)\alpha-n\beta\geq 0}} c_{\alpha\beta}u_0^\alpha v_0^\beta$$

on $W_0$.

Conversely, any such convergent power series in the $(u_0, v_0)$ chart defines a holomorphic function on $M$.

Holomorphic two-forms on $M$:

The holomorphic two-form $du_0 \wedge dv_0 = \cdots = du_n \wedge dv_n (= \pi^*(\frac{dx\wedge dy}{F_z}))$ is nowhere vanishing on $M$. It follows that any holomorphic two-form on $M$ can be given in the $(u_0, v_0)$ chart by a two-form

$$\sum_{\substack{\alpha,\beta\geq 0\\(n+1)\alpha-n\beta\geq 0}} c_{\alpha\beta}u_0^\alpha v_0^\beta du_0 \wedge dv_0,$$

whose power series coefficient converges on $W_0$.

Conversely, any such two-forms in the $(u_0, v_0)$ chart defines a holomorphic two-form on $M$.

**Proposition 3.2.** *With the above notation for $A_n$ singularities, $f^{(1,1)} = 1$.*

*Proof.* From the above calculation, we know that the holomorphic functions on $M$ are generated by a base $\{u_0^\alpha v_0^\beta\}_{(n+1)\alpha-n\beta\geq 0}$ and holomorphic two-forms are generated by a base $\{u_0^\alpha v_0^\beta du_0 \wedge dv_0\}_{(n+1)\alpha-n\beta\geq 0}$. For every holomorphic two-form $\omega = u_0^\alpha v_0^\beta du_0 \wedge dv_0$ on $M$, we consider $\xi = -\frac{u_0^\alpha v_0^{\beta+1}}{\beta+1} du_0$. $\xi$ defines a holomorphic one-form on $W_0$ and $d\xi = \omega$. It remains to check that under all changes of charts, $\xi$ transforms to define a holomorphic one-form in each coordinate chart. In fact, changed to $(u_k, v_k)$ chart, for $k = 1, \ldots, n$,

$$\xi = -\frac{1}{\beta+1} u_k^{(k+1)\alpha-k\beta} v_k^{k\alpha-(k-1)\beta}((k+1)v_k du_k + k u_k dv_k) \quad \text{on } W_k,$$

which is holomorphic.

We also know that $d(\Gamma(M, \mathscr{O}_M))$ is generated by

$$\{\alpha u_0^{\alpha-1} v_0^\beta du_0 + \beta u_0^\alpha v_0^{\beta-1} dv_0\}_{(n+1)\alpha-n\beta\geq 0,\ \alpha\geq 1}.$$

By Lemma 3.1, $\Gamma(M, \Omega^1_M)$ is generated by

$$\{\alpha u_0^{\alpha-1} v_0^{\beta} du_0 + \beta u_0^{\alpha} v_0^{\beta-1} dv_0\}_{(n+1)\alpha - n\beta \geq 0,\ \alpha \geq 1} \cup \{u_0^{\alpha} v_0^{\beta+1} du_0\}_{(n+1)\alpha - n\beta \geq 0}.$$

So by easy calculation $\Gamma(M, \Omega^1_M) \wedge \Gamma(M, \Omega^1_M)$ is generated by

$$\{u_0^{\alpha} v_0^{\beta} du_0 \wedge dv_0\}_{(n+1)\alpha - n\beta \geq 0,\ \alpha \geq 1}.$$

Therefore

$$\frac{\Gamma(M, \Omega^2_M)}{< \Gamma(M, \Omega^1_M) \wedge \Gamma(M, \Omega^1_M) >} = < u_0 \wedge v_0 >,$$

and

$$f^{(1,1)} = dim \frac{\Gamma(M, \Omega^2_M)}{< \Gamma(M, \Omega^1_M) \wedge \Gamma(M, \Omega^1_M) >} = 1.$$

□

**Type $D_n$, $n \geq 4$ and even:**

An explicit resolution $\pi : M \to V$ can be given in terms of coordinate charts and transition functions on $M$ as follows.

Coordinate charts:

$$W_k = \{(u_k, v_k) : u_k^{n-k-3} v_k^{n-k-2} \neq 1\}, \quad 0 \leq k \leq n-4$$
$$W_k = \{(u_k, v_k)\}, \quad k = n-3, n-2$$
$$W_k = \{(u_k, v_k) : u_k^2 v_k \neq -1\}, \quad k = n-1, n.$$

Transition functions:

$$\begin{cases} u_k = u_{k+1}^2 v_{k+1}, \\ v_k = \dfrac{1}{u_{k+1}}, \end{cases} \quad 0 \leq k \leq n-3$$

$$\begin{cases} u_{n-2} = \dfrac{1}{1 + u_{n-1}^2 v_{n-1}} \\ v_{n-2} = \dfrac{1 + u_{n-1}^2 v_{n-1}}{u_{n-1}} \end{cases} \quad \text{and} \quad \begin{cases} u_{n-2} = \dfrac{u_n^2 v_n}{1 + u_n^2 v_n} \\ v_{n-2} = \dfrac{1 + u_n^2 v_n}{u_n} \end{cases}.$$

Projection map: $\pi(u_k, v_k) = (x, y, z)$, where

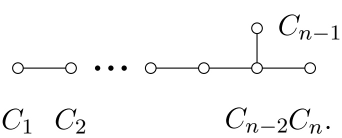
$$\begin{aligned}
x &= u_0(u_0^{n-3}v_0^{n-2}-1)^{\frac{n}{2}-1}(u_0^{n-3}v_0^{n-2}-2) = \cdots \\
&= u_{n-2}^{\frac{n}{2}-1}v_{n-2}^{n-2}(1-u_{n-2})^{\frac{n}{2}-1}(1-2u_{n-2}) = \frac{v_{n-1}^{\frac{n}{2}-1}(u_{n-1}^2 v_{n-1}-1)}{1+u_{n-1}^2 v_{n-1}} \\
&= \frac{v_n^{\frac{n}{2}-1}(1-u_n^2 v_n)}{1+u_n^2 v_n} \\
y &= 2u_0^2 v_0(u_0^{n-3}v_0^{n-2}-1)^{\frac{n}{2}} = \cdots \\
&= 2u_{n-2}^{\frac{n}{2}}v_{n-2}^{n-1}(1-u_{n-2})^{\frac{n}{2}} = \frac{2u_{n-1}v_{n-1}^{\frac{n}{2}}}{1+u_{n-1}^2 v_{n-1}} = \frac{2u_n v_n^{\frac{n}{2}}}{1+u_n^2 v_n} \\
z &= u_0^2 v_0^2(u_0^{n-3}v_0^{n-2}-1) = \cdots = u_{n-2}v_{n-2}^2(1-u_{n-2}) = v_{n-1} = v_n.
\end{aligned}$$

Exceptional set: $A = \pi^{-1}(0) = C_1 \cup \cdots \cup C_n$, where

$$C_k = \{u_{k-1} = 0\} \cup \{v_k = 0\} \quad 1 \le k \le n-2$$

$$C_{n-1} = \{v_{n-3} = 1\} \cup \{u_{n-2} = 1\} \cup \{v_{n-1} = 0\},$$
$$C_n = \{u_{n-2} = 0\} \cup \{v_n = 0\}.$$

Dual graph:




Holomorphic functions on $M$:

Any holomorphic function $f$ on $M$ has a series expansion of the form $\sum_{\alpha \ge 0} f_\alpha(u_{n-2})v_{n-2}^\alpha$ on $W_{n-2}$, where

$$f_\alpha(u_{n-2}) = \frac{1}{2\pi i}\int_{|v|=r} \frac{f(u_{n-2}, v)}{v^{\alpha+1}} dv,$$

provided $\{u_{n-2}\} \times \{|v_{n-3}| \le r\} \subset W_{n-2}$. $\sum_{\alpha \ge 0} f_\alpha(u_{n-2})v_{n-2}^\alpha$ converges absolutely and uniformly on any subset of $W_{n-2}$ of the form (compact set) $\times$ (closed disc centered at 0), while $f_\alpha(u_{n-2})$ is holomorphic for all $u_2$. Then $f_\alpha(u_{n-2})$ has an expansion $f_\alpha(u_{n-2}) = \sum_{\beta \ge 0} c_{\alpha\beta}u_{n-2}^\beta$ on $\mathbb{C}$ and $f$ has an expansion $\sum_{\alpha \ge 0}(\sum_{\beta \ge 0} c_{\alpha\beta}u_{n-2}^\beta)v_{n-2}^\alpha$ on $W_{n-2}$. Note that the expansion rearranges into a convergent power series near $(0,0)$, but not necessarily on all $W_{n-2}$.

Under changes of charts, the expressions of $f$ take on the following forms:

$$\sum_\alpha\left(\sum_\beta c_{\alpha\beta}u_{n-2}^\beta v_{n-2}^\alpha\right)=\sum_\alpha\left(\sum_\beta c_{\alpha\beta}u_{n-3}^\alpha v_{n-3}^{2\alpha-\beta}\right)=\cdots$$
$$=\sum_\alpha\left(\sum_\beta c_{\alpha\beta}u_0^{(n-2)\alpha-(n-3)\beta}v_0^{(n-1)\alpha-(n-2)\beta}\right).$$

Since all these series have to be convergent power series in respective neighborhoods of $(0,0)$, the indices $\alpha$, $\beta$ restrict to $(n-1)\alpha\geq(n-2)\beta$. Thus

$$f=\sum_{\alpha\geq 0}\left(\sum_{0\leq\beta\leq\frac{n-1}{n-2}\alpha}c_{\alpha\beta}u_{n-2}^\beta\right)v_{n-2}^\alpha\quad\text{on } W_{n-2}.$$

**Lemma 3.2.** *With the above notation for $D_n$ ($n$ even), any holomorphic function $f$ on $M$ has the expansions*

$$f=\sum_{\alpha\geq 0}\left(\sum_{\frac{1}{2}\alpha\leq\beta\leq\frac{n-1}{n-2}\alpha}c_{\alpha\beta}u_k^{(n-k-2)\alpha-(n-k-3)\beta}v_k^{(n-k-1)\alpha-(n-k-2)\beta}\right)\tag{3.1}$$

*on $W_k$ for $k=0,1,\ldots,n-2$,*

$$f=\sum_{\alpha\geq 0}\left(\sum_{\frac{1}{2}\alpha\leq\beta\leq\frac{n-1}{n-2}\alpha}c_{\alpha\beta}u_{n-1}^{-\alpha}(1+u_{n-1}^2v_{n-1})^{\alpha-\beta}\right)\quad\text{on } W_{n-1}\tag{3.2}$$

*and*

$$f=\sum_{\alpha\geq 0}\left(\sum_{\frac{1}{2}\alpha\leq\beta\leq\frac{n-1}{n-2}\alpha}c_{\alpha\beta}u_n^{2\beta-\alpha}v_n^\beta(1+u_n^2v_n)^{\alpha-\beta}\right)\quad\text{on } W_n.\tag{3.3}$$

*In each expansion, the sum over $\beta$ is holomorphic on the corresponding $W_k$ for each $\alpha\geq 0$, and the sum over $\alpha$ is absolutely convergent.*

*Proof.* 1° With $f = \sum_\alpha(\sum_\beta c_{\alpha\beta} u_{n-2}^\beta v_{n-2}^\alpha)$ on $W_{n-2}$, changing from $(u_{n-2}, v_{n-2})$ to $(u_{n-3}, v_{n-3})$ gives $f = \sum_\alpha(\sum_\beta c_{\alpha\beta} u_{n-3}^\alpha v_{n-3}^{2\alpha-\beta})$ on $W_{n-3}\backslash\{v_{n-3} = 0\}$. For each $\alpha \geq 0$, the sum over $\beta$ is a polynomial, say $\varphi_\alpha(u_{n-3}, v_{n-3})$. To show that the expansion of $f$ also holds on $v_{n-3} = 0$, consider any $(c, 0) \in W_{n-3}$ and take any $\epsilon > 0$ such that $\{c\} \times \{|v_{n-3}| \leq \epsilon\} \subset W_{n-3}$. Then, on the circle $\{c\} \times \{|v_{n-3}| = \epsilon\}$,

$$\sum_\alpha \varphi_\alpha(c, v_{n-3}) = \sum_\alpha \left(\sum_\beta c_{\alpha\beta} u_{n-2}^\beta v_{n-2}^\alpha\right), \tag{3.4}$$

where $(u_{n-2}, v_{n-2}) = (v_{n-3}^{-1}, c v_{n-3}^2)$ lies on $\{|u_{n-2}| = \epsilon^{-1}\} \times \{|v_{n-2}| = |c|\epsilon^2\}$ in $W_{n-2}$. Since the right hand side of (3.4) converges uniformly on the indicated subset of $W_{n-2}$, so does the left hand side on $\{c\} \times \{|v_{n-3}| = \epsilon\}$. Then $\sum_\alpha \varphi_\alpha(c, v_{n-3})$ is holomorphic on $\{c\} \times \{|v_{n-3}| \leq \epsilon\}$, hence coincides with $f$, in particular at $(c, 0)$. The absolute convergence of $\sum_\alpha \varphi_\alpha(u_{n-3}, v_{n-3})$ follows from that of $\sum_\alpha(\sum_\beta c_{\alpha\beta} u_{n-2}^\beta v_{n-2}^\alpha)$ over $\alpha$.

By the same argument, changing from $(u_{n-2}, v_{n-2})$ to $(u_k, v_k)$ via

$$u_{n-2} = \frac{1}{u_k^{n-k-3} v_k^{n-k-2}}, \quad v_{n-2} = u_k^{n-k-2} v_k^{n-k-1}$$

gives (3.1) first on $W_k\backslash\{u_k v_k = 0\}$ and then on $W_k$, for $k = 0, 1, \ldots, n-4$. It suffices to remark that for $c \neq 0$, small circle $\{c\} \times \{|v_k| = \epsilon\}$ (resp. $\{|u_k| = \epsilon\} \times \{c\}$) in $W_k$ correspond to $(u_{n-2}, v_{n-2})$ with $|u_{n-2}| = \frac{1}{|c|^{n-k-3}\epsilon^{n-k-2}}$, $|v_{n-2}| = |c|^{n-k-2}\epsilon^{n-k-1}$ (resp. $|u_{n-2}| = \frac{1}{\epsilon^{n-k-3}|c|^{n-k-2}}$, $|v_{n-2}| = \epsilon^{n-k-2}|c|^{n-k-1}$) in $W_{n-2}$.

2° Changing from $(u_{n-2}, v_{n-2})$ to $(u_{n-1}, v_{n-1})$ gives (3.2) on $W_{n-1}\backslash\{u_{n-1} = 0\}$, where $1 + u_{n-1}^2 v_{n-1} \neq 0$. We need a trick to ensure that for each $\alpha$, the sum over $\beta$ is holomorphic on $W_{n-1}$. Since $(V, 0)$ is normal, $f$ is the pullback under $\pi$ of some power series $\sum_{i,j,k \geq 0} \widetilde{c}_{ijk} x^i y^j z^k$ which converges in some neighborhood $\widetilde{U}$ of 0 in $\mathbb{C}^3$. On $\pi^{-1}(\widetilde{U}) \cap W_{n-2}$,

$$\begin{aligned}\sum_\alpha \left(\sum_\beta c_{\alpha\beta} u_{n-2}^\beta\right) v_{n-2}^\alpha \tag{3.5}\\ = \sum \widetilde{c}_{ijk}(u_{n-2}^{\frac{n}{2}-1} v_{n-2}^{n-2}(1-u_{n-2})^{\frac{n}{2}-1}(1-2u_{n-2}))^i \\ \cdot (2u_{n-2}^{\frac{n}{2}} v_{n-2}^{n-1}(1-u_{n-2})^{\frac{n}{2}})^j (u_{n-2} v_{n-2}^2(1-u_{n-2}))^k,\end{aligned}$$

which implies that for each $\alpha \geq 0$,

$$\begin{aligned}(3.6)\qquad &\sum_{0\leq\beta\leq\frac{n-1}{n-2}\alpha} c_{\alpha\beta}u_{n-2}^{\beta}v_{n-2}^{\alpha}\\ &= \sum_{(n-2)i+(n-1)j+2k=\alpha} \widetilde{c}_{ijk}(u_{n-2}^{\frac{n}{2}-1}v_{n-2}^{n-2}(1-u_{n-2})^{\frac{n}{2}-1}(1-2u_{n-2}))^{i}\\ &\quad\cdot(2u_{n-2}^{\frac{n}{2}}v_{n-2}^{n-1}(1-u_{n-2})^{\frac{n}{2}})^{j}(u_{n-2}v_{n-2}^{2}(1-u_{n-2}))^{k}.\end{aligned}$$

Take a neighborhood $U_{n-1}$ of $(0,0)$ in $W_{n-1}$ such that $U_{n-1} \subset \pi^{-1}(\widetilde{U}) \cap W_{n-1}$. Changing the finite sums on both side of (3.6) to $(u_{n-1}, v_{n-1})$ gives, on $U_{n-1}\backslash\{u_{n-1}=0\}$,

(3.7)

$$\begin{aligned}&\sum_{0\leq\beta\leq\frac{n-1}{n-2}\alpha} c_{\alpha\beta}u_{n-1}^{-\alpha}(1+u_{n-1}^{2}v_{n-1})^{\alpha-\beta}\\ &= \sum_{(n-2)i+(n-1)j+2k=\alpha} \widetilde{c}_{ijk}\left(\frac{v_{n-1}^{\frac{n}{2}-1}(u_{n-1}^{2}v_{n-1}-1)}{1+u_{n-1}^{2}v_{n-1}}\right)^{i}\left(\frac{2u_{n-1}v_{n-1}^{\frac{n}{2}}}{1+u_{n-1}^{2}v_{n-1}}\right)^{j}v_{n-1}^{k}.\end{aligned}$$

The two sides of (3.7) being rational functions of $(u_{n-1}, v_{n-1})$, they must be identical. Since the right-hand side is holomorphic on $W_{n-1}$, so is the left hand side.

Denote $\sum_{0\leq\beta\leq\frac{n-1}{n-2}\alpha} c_{\alpha\beta}u_{n-1}^{-\alpha}(1+u_{n-1}^{2}v_{n-1})^{\alpha-\beta}$ by $\psi_{\alpha}(u_{n-1}, v_{n-1})$. Since all $\psi_{\alpha}$ are holomorphic, to prove (3.2) also at $u_{n-1}=0$, say at any $(0,c) \in W_{n-1}$, it suffices, as before, to find some $\epsilon > 0$ such that $(i)$ $\{|u_{n-1}| \leq \epsilon\} \times \{c\} \subset W_{n-1}$ and $(ii)$ $\sum_{\alpha}\psi_{\alpha}(u_{n-1}, c)$ converges uniformly on the circle $\{|u_{n-1}| = \epsilon\} \times \{c\}$. We can clearly take $\epsilon > 0$ satisfying $(i)$. Then $(u_{n-1}, v_{n-1}) = (\epsilon \mathrm{e}^{\mathrm{i}\theta}, c)$, $\theta \in \mathbb{R}$, corresponds to $(u_{n-2}, v_{n-2}) = (\frac{1}{1+c\epsilon^2\,\mathrm{e}^{2\mathrm{i}\theta}}, \frac{1+c\epsilon^2\,\mathrm{e}^{2\mathrm{i}\theta}}{\epsilon\,\mathrm{e}^{\mathrm{i}\theta}}) \in W_{n-2}$. If we can bound these $(u_{n-2}, v_{n-2})$ within some $S =$(compact set) $\times$ (closed disc centered at 0) $\subset W_{n-2}$, then the uniform convergence of $\sum_{\alpha}$ $(\sum_{\beta} c_{\alpha\beta}u_{n-2}^{\beta}v_{n-2}^{\alpha})$ on $S$ gives $(ii)$. To get $S$, it suffices to choose a sufficiently smaller $\epsilon > 0$ such that for all $\theta$, $(\frac{1}{1+c\epsilon^2\,\mathrm{e}^{2\mathrm{i}\theta}}, \frac{\max_{\phi}|1+c\epsilon^2\,\mathrm{e}^{2\mathrm{i}\phi}|}{\epsilon}) \in W_{n-2}$.

Since $(u_{n-1}, v_{n-1}) = (\epsilon \mathrm{e}^{\mathrm{i}\theta}, c) \in W_{n-1}$ for all $\theta$, $|x|^2+|y|^2+|z|^2 < 1$ on $W_{n-1}$ gives

$$(3.8)\qquad |c|^{n-2}\left|\frac{1-c\epsilon^2\,\mathrm{e}^{2\mathrm{i}\theta}}{1+c\epsilon^2\,\mathrm{e}^{2\mathrm{i}\theta}}\right|^2 + \frac{4|c|^n\varepsilon^2}{|1+c\epsilon^2\,\mathrm{e}^{2\mathrm{i}\theta}|^2} + |c|^2 < 1 \quad \text{for all } \theta.$$

Let $|1+c\epsilon^2 e^{2i\theta_0}| = \max_\theta |1+c\epsilon^2 e^{2i\theta}|$. Then $(\frac{1}{1+c\epsilon^2 e^{2i\theta}}, \frac{|1+c\epsilon^2 e^{2i\theta_0}|}{\epsilon}) \in W_{n-2}$ if the corresponding inequality $|x|^2+|y|^2+|z|^2<1$ is satisfied, namely,

$$(3.9)\qquad |c|^{n-2}\left|\frac{1-c\epsilon^2 e^{2i\theta}}{1+c\epsilon^2 e^{2i\theta}}\right|^2\left|\frac{1+c\epsilon^2 e^{2i\theta_0}}{1+c\epsilon^2 e^{2i\theta}}\right|^{2n-4} + \frac{4|c|^n\varepsilon^2}{|1+c\epsilon^2 e^{2i\theta}|^2}\left|\frac{1+c\epsilon^2 e^{2i\theta_0}}{1+c\epsilon^2 e^{2i\theta}}\right|^{2n-2} + |c|^2\left|\frac{1+c\epsilon^2 e^{2i\theta_0}}{1+c\epsilon^2 e^{2i\theta}}\right|^4 < 1.$$

In view of (3.8), a sufficiently smaller $\epsilon > 0$ can be chosen such that (3.9) holds for all $\theta$. With such choice of $\epsilon$, the lemma for $k = n-1$ follows.

3° Changing from $(u_{n-2}, v_{n-2})$ to $(u_n, v_n)$ gives (3.3) on $W_n\backslash\{u_n = 0\}$, where $1+u_n^2 v_n \neq 0$. To see that for each $\alpha$, the sum over $\beta$ is holomorphic on $W_n$, it suffices to check that $2\beta - \alpha \geq 0$. By (3.6),

$$(3.10)\qquad \sum_{0\leq\beta\leq\frac{n-1}{n-2}\alpha} c_{\alpha\beta}u_{n-2}^\beta = \sum_{(n-2)i+(n-1)j+2k=\alpha} \widetilde{c}_{ijk}(u_{n-2}^{\frac{n}{2}-1}(1-u_{n-2})^{\frac{n}{2}-1}(1-2u_{n-2}))^i \cdot (2u_{n-2}^{\frac{n}{2}}(1-u_{n-2})^{\frac{n}{2}})^j(u_{n-2}(1-u_{n-2}))^k.$$

Equation (3.10) implies that for $(n-2)i+(n-1)j+2k = \alpha$, $\beta \geq (\frac{n}{2}-1)$ $i + \frac{n}{2}j + k$. Hence $2\beta - \alpha \geq 0$.

To check (3.3) holds at any $(0, c) \in W_n$, we repeat the argument for $W_{n-1}$. Then it suffices to find some $\epsilon > 0$ such that $(i)'$ $(u_n, v_n) = (\epsilon e^{i\theta}, c)$ and $(ii)'$ $(u_{n-2}, v_{n-2}) = (\frac{c\epsilon^2 e^{2i\theta}}{1+c\epsilon^2 e^{2i\theta}}, \frac{1+c\epsilon^2 e^{2i\theta}}{\epsilon e^{i\theta}}) \in W_{n-2}$ lies in some $S' =$ (compact set) $\times$ (closed disc centered at 0) $\subset W_{n-2}$, for all $\theta \in \mathbb{R}$. We first take $\epsilon > 0$ satisfying $(i)'$. Then, letting $|1+c\epsilon^2 e^{2i\theta_0}| = \max_\theta|1+c\epsilon^2 e^{2i\theta}|$, we compare the inequality $|x|^2+|y|^2+|z|^2 < 1$ on $W_n$ guaranteed by $(i)'$ with the inequality $|x|^2+|y|^2+|z|^2<1$ on $W_{n-2}$ required for $(\frac{c\epsilon^2 e^{2i\theta}}{1+c\epsilon^2 e^{2i\theta}}, \frac{1+c\epsilon^2 e^{2i\theta}}{\epsilon e^{i\theta}}) \in W_{n-2}$, for all $\theta \in \mathbb{R}$. It turns out that the inequalities are exactly the same as (3.8) and (3.9) respectively. Thus we can choose $\epsilon$, $S'$ and finish the proof. We remark that the first factor of $S'$ is a compact neighborhood of 0 while the first factor of $S$ in the previous case is a compact neighborhood of 1. □

Holomorphic two-forms on $M$:

The holomorphic two-form $\varphi_0 = du_0 \wedge dv_0 = du_1 \wedge dv_1 = \cdots = du_{n-2} \wedge dv_{n-2} = -\frac{du_{n-1}\wedge dv_{n-1}}{1+u_{n-1}^2 v_{n-1}} = \frac{du_n \wedge dv_n}{1+u_n^2 v_n} (= \pi^*(\frac{dx\wedge dy}{F_z}))$ is nowhere zero on $M$. Hence any holomorphic two-form on $M$ is of the form $f\varphi_0$, where $f$ is a holomorphic function on $M$.

**Type $D_n$, $n \geq 5$ and odd:**

A resolution $\pi : M \to V$ can be given by the same charts and transition functions as for even $n$, but with a different projection

$$
\begin{aligned}
x &= u_0^2 v_0 (u_0^{n-3} v_0^{n-2} - 1)^{\frac{n-1}{2}} (u_0^{n-3} v_0^{n-2} - 2) = \cdots \\
&= u_{n-2}^{\frac{n-1}{2}} v_{n-2}^{n-1} (1 - u_{n-2})^{\frac{n-1}{2}} (1 - 2u_{n-2}) = \frac{v_{n-1}^{\frac{n-1}{2}} (u_{n-1}^2 v_{n-1} - 1)}{1 + u_{n-1}^2 v_{n-1}} \\
&= \frac{v_n^{\frac{n-1}{2}} (1 - u_n^2 v_n)}{1 + u_n^2 v_n} \\
y &= 2u_0 (u_0^{n-3} v_0^{n-2} - 1)^{\frac{n-1}{2}} = \cdots \\
&= 2u_{n-2}^{\frac{n-1}{2}} v_{n-2}^{n-2} (1 - u_{n-2})^{\frac{n-1}{2}} = \frac{2u_{n-1} v_{n-1}^{\frac{n-1}{2}}}{1 + u_{n-1}^2 v_{n-1}} = \frac{2u_n v_n^{\frac{n-1}{2}}}{1 + u_n^2 v_n} \\
z &= u_0^2 v_0^2 (u_0^{n-3} v_0^{n-2} - 1) = \cdots = u_{n-2} v_{n-2}^2 (1 - u_{n-2}) = v_{n-1} = v_n.
\end{aligned}
$$

The exceptional set is given in the same way as for even $n$.

Holomorphic function on $M$ is again given by (3.1), (3.2), (3.3). The proof is similar as for odd $n$.

Any holomorphic two-form on $M$ also has form $f\varphi_0$ where $f$ is a holomorphic function on $M$ and $\varphi_0$ is the same as for even $n$.

**Proposition 3.3.** *With the above notation for $D_n$ singularities, $f^{(1,1)} = 1$.*

*Proof.* From the above calculation, we know that the holomorphic functions on $M$ are generated by a base $\{u_{n-2}^{\beta} v_{n-2}^{\alpha}\}_{\frac{1}{2}\alpha \leq \beta \leq \frac{n-1}{n-2}\alpha}$ and holomorphic two-forms are generated by a base $\{u_{n-2}^{\beta} v_{n-2}^{\alpha} du_{n-2} \wedge dv_{n-2}\}_{\frac{1}{2}\alpha \leq \beta \leq \frac{n-1}{n-2}\alpha}$. For every holomorphic two-form $\omega = u_{n-2}^{\beta} v_{n-2}^{\alpha} du_{n-2} \wedge dv_{n-2}$ on $M$, we consider $\xi = -\frac{u_{n-2}^{\beta} v_{n-2}^{\alpha+1}}{\alpha+1} du_{n-2}$. $\xi$ defines a holomorphic one-form on $W_{n-2}$ and $d\xi = \omega$. It remains to check that under all changes of charts, $\xi$ transforms to define a holomorphic one-form in each coordinate chart. In fact, changed to $(u_k, v_k)$ chart, for $k = 1, \ldots, n-2$, on $z_k$,

$$
\begin{aligned}
\xi = -\frac{1}{\alpha+1} & u_k^{(n-k-2)\alpha - (n-k-3)\beta} v_k^{(n-k-1)\alpha - (n-k-2)\beta} \\
& \cdot ((n-k-3) v_k du_k + (n-k-2) u_k dv_k)
\end{aligned}
$$

which is holomorphic. And changing charts to $(u_{n-2}, v_{n-3})$, we see by Lemma 3.2 that

$$\xi = -\frac{v_{n-3}^{2\alpha-\beta} u_{n-3}^{\alpha+1}}{\alpha+1} dv_{n-3}$$

defines a holomorphic one-form on $W_{n-3}$. Finally, changed to $(u_{n-1}, v_{n-1})$ and $(u_n, v_n)$ charts

$$\xi = \left( \frac{u_{n-1}^{-\alpha}(1 + u_{n-1}^2 v_{n-1})^{\alpha-\beta}}{\alpha+1} \right) \left( \frac{2v_{n-1} du_{n-1} + u_{n-1} dv_{n-1}}{1 + u_{n-1}^2 v_{n-1}} \right) \quad \text{on } W_{n-1}$$

and

$$\xi = \left( \frac{u_n^{2\beta-\alpha} v_n^{\beta}(1 + u_n^2 v_n)^{\alpha-\beta}}{\alpha+1} \right) \left( \frac{2v_n du_n + u_n dv_n}{1 + u_n^2 v_n} \right) \quad \text{on } W_n.$$

Again by Lemma 3.2, $\xi$ is holomorphic on $W_{n-1}$ and $W_n$, respectively.

We also know that $d(\Gamma(M, \mathscr{O}_M))$ is generated by

$$\{\beta u_{n-2}^{\beta-1} v_{n-2}^{\alpha} du_{n-2} + \alpha u_{n-2}^{\beta} v_{n-2}^{\alpha-1} dv_{n-2}\}_{\frac{1}{2}\alpha \le \beta \le \frac{n-1}{n-2}\alpha}.$$

By Lemma 3.1, $\Gamma(M, \Omega_M^1)$ is generated by

$$\{\beta u_{n-2}^{\beta-1} v_{n-2}^{\alpha} du_{n-2} + \alpha u_{n-2}^{\beta} v_{n-2}^{\alpha-1} dv_{n-2}\}_{\frac{1}{2}\alpha \le \beta \le \frac{n-1}{n-2}\alpha} \cup \{u_{n-2}^{\beta} v_{n-2}^{\alpha+1} du_{n-2}\}_{\frac{1}{2}\alpha \le \beta \le \frac{n-1}{n-2}\alpha}.$$

So by easy calculation, $\Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1)$ is generated by

$$\{u_{n-2}^{\beta} v_{n-2}^{\alpha} du_{n-2} \wedge dv_{n-2}\}_{\frac{1}{2}\alpha \le \beta \le \frac{n-1}{n-2}\alpha,\ \alpha \ge 1}.$$

Therefore

$$\frac{\Gamma(M, \Omega_M^2)}{< \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) >} = < u_{n-2} \wedge v_{n-2} >,$$

and

$$f^{(1,1)} = \dim \frac{\Gamma(M, \Omega_M^2)}{< \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) >} = 1.$$

□

**Type** $E_n,\ n = 6, 7, 8$**:**
Resolutions $\pi : M \to V$ for $E_n,\ n = 6, 7, 8$, can be given as follows.
Coordinate charts:

$$W_k = \{(u_k, v_k) : u_k^{2-k} v_k^{3-k} \neq 1\}, \quad k = 0, 1$$

$$W_k = \{(u_k, v_k)\}, \quad k = 2, 3$$

$$W_4 = \{(u_4, v_4) : u_4^2 v_4 \neq 1\}$$

$$W_k = \{(u_k, v_k) : u_k^{k-3} v_k^{k-4} \neq -1\}, \quad 5 \leq k \leq n$$

Transition functions:

$$\begin{cases} u_k = u_{k+1}^2 v_{k+1} \\ v_k = \dfrac{1}{u_{k+1}} \end{cases} \qquad 0 \leq k \leq 2 \text{ and } 5 \leq k \leq n-1$$

$$\begin{cases} u_3 = \dfrac{1}{1 + u_4^2 v_4} \\ v_3 = \dfrac{1 + u_4^2 v_4}{u_4} \end{cases} \quad \text{and} \quad \begin{cases} u_3 = \dfrac{u_5^2 v_5}{1 + u_5^2 v_5} \\ v_3 = \dfrac{1 + u_5^2 v_5}{u_5} \end{cases}$$

Projection map: $\pi(u_k, v_k) = (x, y, z)$, where
For $E_6$:

$$\begin{cases} x = 4u_0^2(u_0^2 v_0^3 - 1)^3(u_0^2 v_0^3 + 1) = \cdots \\ \quad = 4u_3^4 v_3^6 (1 - u_3)^3 (1 + u_3) = \cdots = \dfrac{4v_6^2(1 + 2u_6^3 v_6^2)}{(1 + u_6^3 v_6^2)^2} \\ y = 4u_0^2 v_0 (u_0^2 v_0^3 - 1)^2 = \cdots \\ \quad = 4u_3^3 v_3^4 (1 - u_3)^2 = \cdots = \dfrac{4u_6 v_6^2}{1 + u_6^3 v_6^2} \\ z = 2u_0(u_0^2 v_0^3 - 1)^2 = \cdots = 2u_3^2 v_3^3 (1 - u_3)^2 = \cdots = \dfrac{2v_6}{1 + u_6^3 v_6^2}. \end{cases}$$

For $E_7$:

$$\begin{cases} x = u_0^3(u_0^2v_0^3-1)^5 = \cdots \\ \quad = u_3^7v_3^9(1-u_3)^5 = \cdots = \dfrac{u_7v_7^3}{(1+u_7^4v_7^3)^3} \\ y = u_0^2(u_0^2v_0^3-1)^3 = \cdots \\ \quad = u_3^5v_3^6(1-u_3)^3 = \cdots = \dfrac{u_7^2v_7^3}{(1+u_7^4v_7^3)^3} \\ z = u_0^2v_0(u_0^2v_0^3-1)^2 = \cdots = u_3^3v_3^4(1-u_3)^2 = \cdots = \dfrac{v_7}{(1+u_7^4v_7^3)^3}. \end{cases}$$

For $E_8$:

$$\begin{cases} x = u_0^5(u_0^2v_0^3-1)^8 = \cdots \\ \quad = u_3^{12}v_3^{15}(1-u_3)^8 = \cdots = \dfrac{v_8^3}{(1+u_8^5v_8^4)^5} \\ y = u_0^4v_0(u_0^2v_0^3-1)^5 = \cdots \\ \quad = u_3^8v_3^{10}(1-u_3)^5 = \cdots = \dfrac{v_8^2}{(1+u_8^5v_8^4)^3} \\ z = u_0^2(u_0^2v_0^3-1)^3 = \cdots = u_3^5v_3^6(1-u_3)^3 = \cdots = \dfrac{u_8v_8^2}{(1+u_8^5v_8^4)^2}. \end{cases}$$

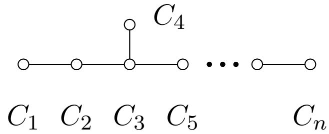
Exceptional set: $A = \pi^{-1}(0) = C_1 \cup \cdots \cup C_k$, where

$$C_k = \{u_{k-1} = 0\} \cup \{v_k = 0\}, \quad 1 \le k \le 3 \text{ and } 6 \le k \le n$$

$$C_4 = \{v_2 = 1\} \cup \{u_3 = 1\} \cup \{v_4 = 0\}$$

$$C_5 = \{u_3 = 0\} \cup \{v_5 = 0\}.$$

Dual graph:




Holomorphic function on $M$:

$W_3$ (resp. $W_2$) has the property that it intersects each plane $\{u_3 = \text{constant}\}$ (resp. $\{v_2 = \text{constant}\}$) in a disc centered at the point $(u_3, 0)$

(resp. $(0, v_2)$) which belongs to $A$. Any holomorphic function of $M$ has an expression $f = \sum_{\alpha\geq 0}(\sum_{\beta\geq 0} c_{\alpha\beta}u_3^\beta)v_3^\alpha$ on $W_3$.

**Lemma 3.3.** *With the above notation for $E_n$ $(n = 6, 7, 8)$, any holomorphic function $f$ on $M$ has the expansions*

$$f = \sum_{\alpha\geq 0}\left(\sum_{\frac{n-4}{n-3}\alpha\leq\beta\leq\frac{4}{3}\alpha} c_{\alpha\beta}u_k^{(3-k)\alpha-(2-k)\beta}v_k^{(4-k)\alpha-(3-k)\beta}\right), \tag{3.11}$$

*on $W_k$ for $0 \leq k \leq 3$,*

$$f = \sum_{\alpha\geq 0}\left(\sum_{\frac{n-4}{n-3}\alpha\leq\beta\leq\frac{4}{3}\alpha} c_{\alpha\beta}u_4^{-\alpha}(1 + u_4^2v_4)^{\alpha-\beta}\right) \quad \text{on } W_4 \tag{3.12}$$

*and*

(3.13)

$$f = \sum_{\alpha\geq 0}\left(\sum_{\frac{n-4}{n-3}\alpha\leq\beta\leq\frac{4}{3}\alpha} c_{\alpha\beta}u_k^{(k-3)\beta-(k-4)\alpha}v_k^{(k-4)\beta-(k-5)\alpha}(1 + u_k^{k-3}v_k^{k-4})^{\alpha-\beta}\right)$$
$$\text{on } W_k,\ 5 \leq k \leq n.$$

*In each expansion, the sum over $\beta$ is holomorphic on the corresponding $W_k$ for each $\alpha \geq 0$, and the sum over $\alpha$ is absolutely convergent.*

*Proof.* The charts $(u_k, v_k)$, $0 \leq k \leq 5$, and their transition functions are the same as those of $D_5$. The assertions for $0 \leq k \leq 5$ can be proved as in Lemma 3.2, except that we need to check the different projection.

Case $E_6$. The formulas corresponding to (3.7), (3.10) are

$$\sum_\beta c_{\alpha\beta}u_4^{-\alpha}(1 + u_4^2v_4)^{\alpha-\beta}) = \sum_{6i+4j+3k=\alpha} \widetilde{c}_{ijk}\left(\frac{4v_4^3(2 + u_4^2v_4)}{(1 + u_4^2v_4)^2}\right)^i\left(\frac{4v_4}{1 + u_4^2v_4}\right)^j\left(\frac{2u_4v_4^2}{1 + u_4^2v_4}\right)^k, \tag{3.14}$$

$$\sum_\beta c_{\alpha\beta}u_3^\beta = \sum_{6i+4j+3k=\alpha} \widetilde{c}_{ijk}(4u_3^4(1 - u_3)^3(1 + u_3))^i \cdot (4u_3^3(1 - u_3)^2)^j(2u_3^2(1 - u_3)^2)^k, \tag{3.15}$$

because $v_3$ appears as powers of $6, 4, 3$ in $x, y, z$, respectively.

(3.14) shows that the sum over $\beta$ is holomorphic on $W_4$. Equation (3.15) shows that $3\beta - 2\alpha \geq 0$, hence the sums over $\beta$ in (3.13) are holomorphic on each $W_k$.

To prove (3.12) at $(0, c) \in W_4$ (resp. $(0, c) \in W_5$, $(0, c)$ and $(c, 0) \in W_6$), we assume the inequality $|x|^2 + |y|^2 + |z|^2 < 1$ at $(u_4, v_4) = (\epsilon \mathrm{e}^{\mathrm{i}\theta}, c)$ (resp. $(u_5, v_5) = (\epsilon \mathrm{e}^{\mathrm{i}\theta}, c)$, $(u_6, v_6) = (\epsilon \mathrm{e}^{\mathrm{i}\theta}, c)$ and $(c, \epsilon \mathrm{e}^{\mathrm{i}\theta})$) for all $\theta \in \mathbb{R}$. Under coordinate changes, we would get the same inequality for $(u_3, v_3) = (\frac{1}{1+c\epsilon^2 \mathrm{e}^{2\mathrm{i}\theta}}, \frac{1+c\epsilon^2 \mathrm{e}^{2\mathrm{i}\theta}}{\epsilon \mathrm{e}^{\mathrm{i}\theta}})$ (resp. $(\frac{c\epsilon^2 \mathrm{e}^{2\mathrm{i}\theta}}{1+c\epsilon^2 \mathrm{e}^{2\mathrm{i}\theta}}, \frac{1+c\epsilon^2 \mathrm{e}^{2\mathrm{i}\theta}}{\epsilon \mathrm{e}^{\mathrm{i}\theta}})$, $(\frac{c^2\epsilon^3 \mathrm{e}^{3\mathrm{i}\theta}}{1+c^2\epsilon^3 \mathrm{e}^{3\mathrm{i}\theta}}, \frac{1+c^2\epsilon^3 \mathrm{e}^{3\mathrm{i}\theta}}{c\epsilon^2 \mathrm{e}^{2\mathrm{i}\theta}})$ and $(\frac{c^3\epsilon^2 \mathrm{e}^{2\mathrm{i}\theta}}{1+c^3\epsilon^2 \mathrm{e}^{2\mathrm{i}\theta}}, \frac{1+c^3\epsilon^2 \mathrm{e}^{2\mathrm{i}\theta}}{c^2\epsilon \mathrm{e}^{\mathrm{i}\theta}})$). If for all $\theta$, we change $v_3$ to $v_3^\circ = \frac{\max_\theta |1+c\epsilon^2 \mathrm{e}^{2\mathrm{i}\theta}|}{\epsilon}$ (resp. $\frac{\max_\theta |1+c\epsilon^2 \mathrm{e}^{2\mathrm{i}\theta}|}{\epsilon}$, $\frac{\max_\theta |1+c^2\epsilon^3 \mathrm{e}^{3\mathrm{i}\theta}|}{|c|\epsilon^2}$ and $\frac{\max_\theta |1+c^3\epsilon^2 \mathrm{e}^{2\mathrm{i}\theta}|}{|c|^2\epsilon}$), then the inequality corresponding to (3.9) becomes

$$|x|^2 \left|\frac{v_3^\circ}{v_3}\right|^{12} + |y|^2 \left|\frac{v_3^\circ}{v_3}\right|^8 + |z|^2 \left|\frac{v_3^\circ}{v_3}\right|^6 < 1 \tag{3.16}$$

in all cases, because of the way $v_3$ appears in $x, y, z$. It is clear that $\epsilon > 0$ can be found such that (3.16) holds for all $\theta \in \mathbb{R}$. Then the proof for $E_6$ can be finished as for Lemma 3.2.

Case $E_7$. The formulas corresponding to (3.7), (3.10) are

(3.17)
$$\sum_\beta c_{\alpha\beta} u_4^{-\alpha} (1 + u_4^2 v_4)^{\alpha-\beta)} = \sum_{9i+6j+4k=\alpha} \widetilde{c}_{ijk} \left(\frac{u_4 v_4^5}{(1+u_4^2 v_4)^3}\right)^i \left(\frac{v_4^3}{(1+u_4^2 v_4)^2}\right)^j \left(\frac{v_4^2}{1+u_4^2 v_4}\right)^k,$$

(3.18)
$$\sum_\beta c_{\alpha\beta} u_3^\beta = \sum_{9i+6j+4k=\alpha} \widetilde{c}_{ijk} (u_3^7 (1-u_3)^5)^i (u_3^5 (1-u_3))^j (u_3^3 (1-u_3)^2)^k.$$

Again 3.18, implies $4\beta - 3\alpha (\geq i + 2j) \geq 0$.

The inequality corresponding to (3.9) for proving the various cases of (3.12), (3.13) is

$$|x|^2 \left|\frac{v_3^\circ}{v_3}\right|^{18} + |y|^2 \left|\frac{v_3^\circ}{v_3}\right|^{12} + |z|^2 \left|\frac{v_3^\circ}{v_3}\right|^8 < 1 \tag{3.19}$$

under a corresponding assumption $|x|^2 + |y|^2 + |z|^2 < 1$. The rest of the proof is similar.

Case $E_8$. The formulas corresponding to (3.7), (3.10) are

(3.20)
$$\sum_\beta c_{\alpha\beta} u_4^{-\alpha}(1+u_4^2 v_4)^{\alpha-\beta}) = \sum_{15i+10j+6k=\alpha} \widetilde{c}_{ijk} \left(\frac{u_4 v_4^8}{(1+u_4^2 v_4)^5}\right)^i \left(\frac{v_4^5}{(1+u_4^2 v_4)^3}\right)^j \left(\frac{v_4^3}{(1+u_4^2 v_4)^2}\right)^k,$$

(3.21)
$$\sum_\beta c_{\alpha\beta} u_3^\beta = \sum_{15i+10j+6k=\alpha} \widetilde{c}_{ijk} (u_3^{12}(1-u_3)^8)^i (u_3^8(1-u_3)^5)^j (u_3^5(1-u_3)^3)^k.$$

Again 3.21, implies $5\beta - 4\alpha(\geq k) \geq 0$.

The inequality corresponding to (3.9) takes the form

$$|x|^2 \left|\frac{v_3^\circ}{v_3}\right|^{30} + |y|^2 \left|\frac{v_3^\circ}{v_3}\right|^{20} + |z|^2 \left|\frac{v_3^\circ}{v_3}\right|^{12} < 1. \tag{3.22}$$

The proof is then clear. □

Holomorphic two-forms on $M$:

The holomorphic two-form $\varphi_0 = \pi^{-1}(\frac{dx\wedge dy}{F_z}) = du_0 \wedge dv_0 = \cdots = du_3 \wedge dv_3 = du_{n-2} \wedge dv_{n-2} = -\frac{du_4 \wedge dv_4}{1+u_4^2 v_4} = \frac{du_k \wedge dv_k}{1+u_k^{k-3} v_k^{k-4}}$, $5 \leq k \leq n$, is nowhere zero on $M$. Hence any holomorphic two-form on $M$ is of the form $f\varphi_0$, where $f$ is a holomorphic function on $M$.

**Proposition 3.4.** *With the above notation for $E_n$ singularities, $n = 6$, 7, 8, $f^{(1,1)} = 1$.*

*Proof.* From the above calculation, we know that the holomorphic functions on $M$ are generated by a base $\{u_3^\beta v_3^\alpha\}_{\frac{n-4}{n-3}\alpha \leq \beta \leq \frac{4}{3}\alpha}$ and holomorphic two-forms are generated by a base $\{u_3^\beta v_3^\alpha du_3 \wedge dv_3\}_{\frac{n-4}{n-3}\alpha \leq \beta \leq \frac{4}{3}\alpha}$. For every holomorphic two-form $\omega = u_3^\beta v_3^\alpha du_3 \wedge dv_3$ on $M$, we consider $\xi = -\frac{u_3^\beta v_3^{\alpha+1}}{\alpha+1} du_3$. $\xi$ defines a holomorphic one-form on $W_3$ and $d\xi = \omega$. It remains to check that under all changes of charts, $\xi$ transforms to define a holomorphic one-form in each coordinate chart. In fact, changed to $(u_k, v_k)$ chart, for $k = 0, 1, 2$, on $W_k$,

$$\xi = -\frac{1}{\alpha+1} u_k^{(3-k)\alpha-(2-k)\beta} v_k^{(4-k)\alpha-(3-k)\beta}((2-k)v_k du_k + (3-k)u_k dv_k),$$

which is holomorphic. And changing charts to $(u_4, v_4)$ chart, we see by Lemma 3.3 that

$$\xi = -\frac{u_4^{-\alpha}}{\alpha+1}(1+u_4^2v_4)^{\alpha-\beta}\left(\frac{2v_4du_4+u_4dv_4}{1+u_4^2v_k}\right)$$

defines a holomorphic one-form on $W_4$. Finally, changed to $(u_k, v_k)$ chart, for $5 \le k \le n$, charts

$$\xi = -\frac{u_k^{(k-3)\beta-(k-4)\alpha}v_k^{(k-4)\beta-(k-5)\alpha}}{\alpha+1}(1+u_k^{k-3}v_k^{k-4})^{\alpha-\beta} \times \left(\frac{(k-3)v_kdu_k+(k-4)u_kdv_k}{1+u_k^{k-3}v_k^{k-4}}\right)$$

Again by Lemma 3.3, $\xi$ is holomorphic on $W_k$, for $5 \le k \le n$.

We also know that $d(\Gamma(M, \mathscr{O}_M))$ is generated by

$$\{\beta u_3^{\beta-1}v_3^{\alpha}du_3 + \alpha u_3^{\beta}v_3^{\alpha-1}dv_3\}_{\frac{n-4}{n-3}\alpha\le\beta\le\frac{4}{3}\alpha}.$$

By Lemma 3.1, $\Gamma(M, \Omega_M^1)$ is generated by

$$\{\beta u_3^{\beta-1}v_3^{\alpha}du_3 + \alpha u_3^{\beta}v_3^{\alpha-1}dv_3\}_{\frac{n-4}{n-3}\alpha\le\beta\le\frac{4}{3}\alpha} \cup \{u_3^{\beta}v_3^{\alpha+1}du_3\}_{\frac{n-4}{n-3}\alpha\le\beta\le\frac{4}{3}\alpha}.$$

So by easy calculation $\Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1)$ is generated by

$$\{u_3^{\beta}v_3^{\alpha}du_3 \wedge dv_3\}_{\frac{n-4}{n-3}\alpha\le\beta\le\frac{4}{3}\alpha,\ \alpha\ge1}.$$

Therefore

$$\frac{\Gamma(M, \Omega_M^2)}{<\Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1)>} = < u_3 \wedge v_3 >,$$

and

$$f^{(1,1)} = \dim \frac{\Gamma(M, \Omega_M^2)}{<\Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1)>} = 1.$$

□

**Cyclic quotient singularities:**

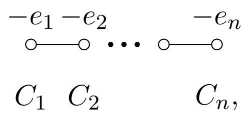
An explicit resolution $\pi : M \to V$ can be given in terms of coordinates and transition functions on $M$ as follows:

Coordinates charts: $W_k = \mathbb{C}^2 = \{(u_k, v_k)\},\ k = 0, 1, \dots, n$
Transition functions :

$$\begin{cases} u_k = u_{k+1}^{e_{k+1}} v_{k+1} \\ v_k = \dfrac{1}{u_{k+1}} \end{cases}.$$

Exceptional set: $A = \pi^{-1}(0) = C_1 \cup \cdots \cup C_n$, where $C_k = \{u_{k-1} = 0\} \cup \{v_k = 0\}$

Dual graph:




where $e_i \geq 2$ is the self-intersection number, $i = 1, 2, \dots, n$.
Holomorphic functions on $M$:
Let

$$q_0(\alpha, \beta) = \alpha, \quad q_1(\alpha, \beta) = e_1 q_0(\alpha, \beta) - \beta = e_1 \alpha - \beta,$$
$$q_i(\alpha, \beta) = e_i q_{i-1}(\alpha, \beta) - q_{i-2}(\alpha, \beta), \quad i = 2, 3, \dots, n.$$

Any holomorphic function on $M$ has power series expansion $\sum_{\alpha,\beta \geq 0} c_{\alpha\beta} u_0^\alpha v_0^\beta$, which converges for all $(u_0, v_0) \in W_0$. Under changes of charts,

$$\sum_{\alpha,\beta \geq 0} c_{\alpha\beta} u_0^\alpha v_0^\beta = \sum_{\alpha,\beta \geq 0} c_{\alpha\beta} u_1^{q_1(\alpha,\beta)} v_1^{q_0(\alpha,\beta)} = \sum_{\alpha,\beta \geq 0} c_{\alpha\beta} u_2^{q_2(\alpha,\beta)} v_2^{q_1(\alpha,\beta)}$$
$$= \cdots = \sum_{\alpha,\beta \geq 0} c_{\alpha\beta} u_n^{q_n(\alpha,\beta)} v_n^{q_{n-1}(\alpha,\beta)}.$$

The $k$th power series has to converge for all $(u_k, v_k) \in W_k$. This occurs for all $k$ if only if the indices $\alpha$, $\beta$ in each sum satisfy $q_i(\alpha, \beta) \geq 0$, $\beta \geq 0$, $i = 0, 1, \dots, n$. Thus any holomorphic function on $M$ can be generated by

$$\{u_0^\alpha v_0^\beta\}_{q_i(\alpha,\beta) \geq 0,\ \beta \geq 0,\ i = 0,1,\dots,n},$$

on $W_0$.

Conversely, any such convergent power series in the $(u_0, v_0)$ chart defines a holomorphic function on $M$.

Holomorphic two-forms on $M$:
We know

$$\begin{cases} du_i = e_{i+1} u_{i+1}^{e_{i+1}-1} v_{i+1} du_{i+1} + u_{i+1}^{e_{i+1}} dv_{i+1} \\ dv_i = -\dfrac{1}{u_{i+1}^2} du_{i+1}. \end{cases}$$

The holomorphic two-form

$$\begin{aligned} du_0 \wedge dv_0 &= u_1^{q_1(1,1)-1} du_1 \wedge dv_1 = u_2^{q_2(1,1)-1} v_2^{q_1(1,1)-1} du_2 \wedge dv_2 = \cdots \\ &= u_n^{q_n(1,1)-1} v_n^{q_{n-1}(1,1)-1} du_n \wedge dv_n. \end{aligned}$$

So

$$\begin{aligned} u_0^\alpha v_0^\beta du_0 \wedge dv_0 &= u_1^{q_1(\alpha+1,\beta+1)-1} v_1^{q_0(\alpha+1,\beta+1)-1} du_1 \wedge dv_1 \\ &= u_2^{q_2(\alpha+1,\beta+1)-1} v_2^{q_1(\alpha+1,\beta+1)-1} du_2 \wedge dv_2 = \cdots \\ &= u_n^{q_n(\alpha+1,\beta+1)-1} v_n^{q_{n-1}(\alpha+1,\beta+1)-1} du_n \wedge dv_n. \end{aligned}$$

It follows that any holomorphic 2-form on $M$ can be generated in the $(u_0, v_0)$ chart by 2-forms

$$\{u_0^\alpha v_0^\beta du_0 \wedge dv_0\}_{q_i(\alpha+1,\beta+1)\geq 1,\ \beta\geq 0,\ i=0,1,\ldots,n}.$$

**Proposition 3.5.** *With the above notation for cyclic quotient singularities,* $f^{(1,1)} = 1$.

*Proof.* From the above calculation, we know that the holomorphic functions on $M$ are generated by a base

$$\{u_0^\alpha v_0^\beta\}_{q_i(\alpha,\beta)\geq 0,\ \beta\geq 0,\ i=0,1,\ldots,n}$$

and holomorphic two-forms are generated by a base

$$\{u_0^\alpha v_0^\beta du_0 \wedge dv_0\}_{q_i(\alpha+1,\beta+1)\geq 1,\ \beta\geq 0,\ i=0,1,\ldots,n}.$$

For every holomorphic two-form $\omega = u_0^\alpha v_0^\beta du_0 \wedge dv_0$ on $M$, we consider $\xi = -\frac{u_0^\alpha v_0^{\beta+1}}{\beta+1} du_0$. $\xi$ defines a holomorphic one-form on $W_0$ and $d\xi = \omega$. It remains to check that under all changes of charts, $\xi$ transforms to define a holomorphic one-form in each coordinate chart. In fact, changed to $(u_k, v_k)$ chart, for $k = 1, \ldots, n$, on $W_k$,

$$\xi = -\frac{1}{\beta+1} u_k^{q_k(\alpha+1,\beta+1)-1} \times v_k^{q_{k-1}(\alpha+1,\beta+1)-1}(q_k(0,-1)v_k du_k + q_{k-1}(0,-1)u_k dv_k),$$

which is holomorphic.

We also know that $d(\Gamma(M, \mathscr{O}_M))$ is generated by

$$\{u_0^\alpha v_0^\beta\}_{q_i(\alpha,\beta)\geq 0,\ \beta\geq 0,\ \alpha\geq 1,\ i=0,1,\ldots,n}.$$

By Lemma 3.1, $\Gamma(M, \Omega_M^1)$ is generated by

$$\{u_0^\alpha v_0^\beta\}_{q_i(\alpha,\beta)\geq 0,\ \beta\geq 0,\ \alpha\geq 1,\ i=0,1,\ldots,n} \cup \{u_0^\alpha v_0^{\beta+1} du_0\}_{q_i(\alpha+1,\beta+1)\geq 1,\ \beta\geq 0,\ i=0,1,\ldots,n}.$$

So by easy calculation $\Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1)$ is generated by

$$\{u_0^\alpha v_0^\beta du_0 \wedge dv_0\}_{q_i(\alpha+1,\beta+1)\geq 1,\ \beta\geq 0,\ \alpha\geq 1\ i=0,1,\ldots,n}.$$

Therefore

$$\frac{\Gamma(M, \Omega_M^2)}{< \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) >} = < u_0 \wedge v_0 >,$$

and

$$f^{(1,1)} = \dim \frac{\Gamma(M, \Omega_M^2)}{< \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) >} = 1.$$

□

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## References

[1] M. Artin, *On isolated rational singularities of surfaces*, Amer. J. Math. **88**(1) (1963), 8–28.

[2] D.M. Jr. Burns and J.M. Wahl, *Local contributions to global deformations of surfaces*, Invent. Math. **26** (1974), 67–88.

[3] F. Campana and H. Flenner, *Contact singularities*, Manuscripta Math. **108**(4) (2002), 529–541.

[4] R. Du [Du-Ya] and S.S.-T. Yau, *Kohn-Rossi cohomology and its application to the complex Plateau problem, III* (to appear, J. Differential Geom.).

[5] R. Harvey and B. Lawson, *On boundaries of complex analytic varieties* I, Ann. Math. **102** (1975), 233–290.

[6] H. Laufer, *On minimally elliptic singularities*, Amer. J. Math. **99**(6) (1977), 1257–1295.

[7] H. Laufer, *Normal Two-Dimensional Singularities*, Ann. Math. Studies **71**, Princeton University Press, Princeton, NJ, 1971.

[8] H.S. Luk and S.S.-T. Yau, *Counterexample to boundary regularity of a strongly pseudoconvex CR manifold: an addendum to the paper of Harvey–Lawson*, Ann. Math. **148** (1998), 1153–1154.

[9] H.S. Luk and S.S.-T. Yau, *Kohn-Rossi cohomology and its application to the complex Plateau problem, II*, J. Differential Geom. **77** (2007), 135–148.

[10] R. Morrison David, *The birational geometry of surfaces with rational double points*, Math. Ann. **271**(3) (1985), 415–438.

[11] Y.-T. Siu, *Analytic sheaves of local cohomology*, Trans. AMS **148**(2) (1970), 347–366.

[12] D.V. Straten and J. Steenbrink, *Extendability of holomorphic differential forms near isolated hypersurface singularities*, Abh. Math. Sem. Univ. Hamburg **55** (1985), 97–110.

[13] P. Wagreich, *Elliptic singularities of surfaces*, Amer. J. Math. **92**(2) (1970), 419–454.

[14] W.M. Jonathan, *A characterization of quasi-homogeneous Gorenstein surface singularities*, Compos. Math. **55** (1985), 269–288.

[15] S.S.-T. Yau, *Hypersurface weighted dual graph of normal singularities of surfaces*, Amer. J. Math. **101**(4) (1979), 761–812.

[16] S.S.-T. Yau, *Gorenstein singularities with geometric genus equal to two*, Amer. J. Math. **101**(4) (1979), 813–854.

[17] S.S.-T. Yau, *On strongly elliptic singularities*, Amer. J. Math. **101**(4) (1979), 855–884.

[18] S.S.-T. Yau, *Normal two-dimensional elliptic singularities*, Trans. Amer. Math. Soc. **254** (1979), 117–134.

[19] S.S.-T. Yau, *On maximally elliptic singularities*, Trans. AMS **257**(2) (1980), 269–329.

[20] S.S.-T. Yau, *Kohn-Rossi cohomology and its application to the complex Plateau problem, I*, Ann. Math. **113**(1) (1981), 67–110.

[21] S.S.-T. Yau, $s^{(n-1)}$ *invariant for isolated n-dimensional singularities and its application to moduli problem*, Amer. J. Math. **104**(4) (1982), 829–841.

[22] S.S.-T. Yau, *Various numerical invariants for isolated singularities*, Amer. J. Math. **104**(5) (1982), 1063–1110.

[23] S.S.-T. Yau, *On irregularity and geometric genus of isolated singularities*, Proc. Symp. Pure Math. **40**, Part 2, (1983), 653–662.

[24] S.S.-T. Yau, *Existence of $L^2$-integrable holomorphic forms and low estimates of $T^1_V$*, Duke Mthe. J. **48**(3) (1981), 537–547.

Department of Mathematics
East China Normal University
No. 500 Dongchuan Road
Shanghai 200241, People's Republic of China
*E-mail address*: `rdu@math.ecnu.edu.cn`

Department of Mathematics
The Chinese University of Hong Kong
Shatin, N. T.
Hong Kong
*E-mail address*: `hsluk@math.cuhk.edu.hk`

Department of Mathematical Sciences
Tsinghua University
Beijing, 100084, People's Republic of China
*E-mail address*: `yau@uic.edu`

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