

Coordinate-free characterization of homogeneous polynomials with isolated singularities

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The Durfee conjecture, proposed in 1978, relates two important invariants of isolated hypersurface singularities by a famous inequality; however, the inequality in this conjecture is not sharp. In 1995, Yau announced his conjecture which proposed a sharp inequality. The Yau conjecture characterizes the conditions under which an affine hypersurface with an isolated singularity at the origin is a cone over a nonsingular projective hypersurface; in other words, the conjecture gives a coordinate-free characterization of when a convergent power series is a homogeneous polynomial after a biholomorphic change of variables. In this paper, we have proved that if $p_g > 0$, then $5!p_g \leq \mu - p(v)$, $p(v) = (v - 1)^5 - v(v - 1) \dots (v - 4)$ and p_g , μ and v are, respectively, the geometric genus, the Milnor number, and the multiplicity of the isolated singularity at the origin of a weighted homogeneous polynomial. As a consequence, we prove that the Yau conjecture holds for $n = 5$ if $p_g > 0$. In the process, we have also defined yet another sharp upper bound for the number of positive integral points within a five-dimensional simplex.

1. Introduction

Let Δ_n be an n -dimensional real right-angled simplex defined by the inequality

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \dots + \frac{x_n}{a_n} \leq 1,$$

where $x_1 \geq 0, \dots, x_n \geq 0$ and $a_1 \geq a_2 \geq \dots \geq a_n \geq 1$. Define P_n to be the number of positive integral points in Δ_n , as shown below:

$$P_n = \# \left\{ (x_1, x_2, \dots, x_n) \in \mathbb{Z}_+^n \mid \frac{x_1}{a_1} + \frac{x_2}{a_2} + \dots + \frac{x_n}{a_n} \leq 1 \right\}.$$

Define Q_n to be the number of nonnegative integral points in Δ_n , as shown below:

$$Q_n = \# \left\{ (x_1, x_2, \dots, x_n) \in (\mathbb{Z}_+ \cup \{0\})^n \mid \frac{x_1}{a_1} + \frac{x_2}{a_2} + \dots + \frac{x_n}{a_n} \leq 1 \right\}.$$

The problem of obtaining the numbers P_n and Q_n has occupied mathematicians for decades, simply because a sharp upper estimate of the former would benefit those in singularity theory and a sharp upper estimate of the latter would benefit those in number theory. Granville [3] has stated that an estimate of Q_n would help with finding large gaps between primes, and research done by Xu and Yau [19, 21] on the Durfee conjecture has shown that an estimate of P_n does aid mathematicians in singularity theory. These two different numbers are tied together through the equation

$$P_n(a_1, a_2, \dots, a_n) = Q_n(a_1(1-a), a_2(1-a), \dots, a_n(1-a)),$$

where $a = \frac{1}{a_1} + \dots + \frac{1}{a_n}$. Thus, if one discovers a new estimate for P_n , a new estimate for Q_n will also be present.

In 1899, Pick [15] discovered a formula for Q_2 :

$$Q_2 = \text{area}(\Delta) + \frac{|\partial\Delta \cap \mathbb{Z}^2|}{2} + 1,$$

where $\partial\Delta$ is the boundary of the simplex and $|\partial\Delta \cap \mathbb{Z}^n|$ is the number of integral points on the boundary. In 1951, Mordell [13] discovered the formula for Q_3 using Dedekind sums, but the real breakthrough occurred when Ehrhart [2] proved a polynomial of degree n could calculate the number of nonnegative lattice points in n -dimensional simplex. However, his formula is only effective when the coefficients of every term are known. Starting in 1939, attempts were made to find lower and upper bounds for Q_n instead of a precise formula. It was later discovered by Lehmer [5] that if $a = a_1 = a_2 = \dots = a_n$,

$$Q_n = \binom{[a] + n}{n}.$$

From that formula, it is very natural to introduce the following definition for a sharp estimate R_n of Q_n [18]. The estimate is sharp if

$$R_n|_{a_1=\dots=a_n=a \in \mathbb{Z}} = \binom{a + n}{n}.$$

In other words, any upper or lower bound is only considered to be a sharp estimate if and only if the exact number of lattice points in n -dimensional simplex is attained when $a = a_1 = a_2 = \dots = a_n \in \mathbb{Z}$.

Let $(V, 0)$ be an isolated hypersurface singularity defined by holomorphic function $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$. Let $\pi : M \rightarrow V$ be a resolution of the singularity at 0. Define the geometric genus of the singularity $(V, 0)$ to be $p_g = \dim H^{n-1}(M, \mathcal{O})$. Let ω be a holomorphic n forms on $V - \{0\}$. ω is said to be L^2 integrable if $\int_{W-\{0\}} \omega \wedge \bar{\omega} < \infty$ for sufficiently small relatively compact neighborhood W of 0 in V . Let $L^2(V - \{0\}, \Omega^n)$ be the set of all L^2 integral holomorphic n forms $V - \{0\}$, which is a linear subspace of $\Gamma(V - \{0\}, \Omega^n)$. Then it was proved that $p_g = \dim \Gamma(V - \{0\}, \Omega^n) / L^2(V - \{0\}, \Omega^n)$ (see [23]). The Milnor number of the singularity $(V, 0)$ is defined as

$$\mu = \dim \mathbb{C}\{z_0, z_1, \dots, z_n\} / (f_{z_0}, f_{z_1}, \dots, f_{z_n}).$$

The multiplicity of the singularity is defined to be the order of the lowest nonvanishing term in the power series Taylor expansion of f at 0. In singularity theory, Durfee [1] formed his famous conjecture. The conjecture states that for an isolated singularity $(V, 0)$ defined by a weighted homogeneous polynomial $f(z_0, z_1, z_2, \dots, z_n)$,

$$(n + 1)!p_g \leq \mu,$$

where p_g is the geometric genus of V , μ is the Milnor number, and equality holds if and only if $\mu = 0$. A polynomial $f(z_0, z_1, z_2, \dots, z_n)$ is a weighted homogeneous polynomial of the type $(w_0, w_1, w_2, \dots, w_n)$, where $w_0, w_1, w_2, \dots, w_n$ are fixed positive rational numbers, if f can be expressed as a linear combination of monomials $z_0^{i_0} z_1^{i_1} \dots z_n^{i_n}$ for which $\frac{i_0}{w_0} + \frac{i_1}{w_1} + \dots + \frac{i_n}{w_n} = 1$. Furthermore, the Milnor number μ [12] can be expressed as $(w_0 - 1)(w_1 - 1) \dots (w_n - 1)$. As a consequence of the theorem of Merle–Teissier [14], we know that in case of isolated singularity defined by a weighted homogeneous polynomial, computing the geometric genus is equivalent to counting the number of positive integral points in a tetrahedron.

The next sharp estimate to be constructed was the GLY conjecture, formulated by Lin *et al.* [9]. It has two different parts: the sharp estimate and the rough estimate. However, before we state it, it is convenient to introduce the Sterling numbers first:

$$S_k^{n-1} = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n-1} i_1 i_2 \dots i_k, S_0^{n-1} = 1, S_{n-1}^{n-1} = 1 \cdot 2 \dots (n - 1),$$

where i_1, i_2, \dots, i_k are integers. It has the following property:

$$\begin{aligned} & x(x-1)(x-2)\dots(x-n+1) \\ &= x^n - \binom{n-1}{i_1=1} x^{n-1} + (-1)^2 \binom{\sum_{1 \leq i_1 < i_2 \leq n-1} i_1 i_2}{} x^{n-2} \\ &+ \dots + (-1)^k \binom{\sum_{i \leq i_1 < i_2 < \dots < i_k \leq n-1} i_1 i_2 \dots i_k}{} x^{n-k} \\ &+ \dots + (-1)^{n-1} \left(\prod_{i=1}^{n-1} i \right) x \\ &= x^n + (-1)S_1^{n-1}x^{n-1} + (-1)^2S_2^{n-1}x^{n-2} \\ &+ \dots + (-1)^k S_k^{n-1}x^{n-k} + \dots + (-1)^{n-1}S_{n-1}^{n-1}x. \end{aligned}$$

Let a_1, a_2, \dots, a_n be positive real numbers. We shall denote

$$\begin{aligned} A_{n-k}^n &= \binom{n}{i=1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \frac{1}{a_{i_1} a_{i_2} \dots a_{i_k}}, \\ A_1^n &= \sum_{i=1}^n a_i, A_0^n = 1. \end{aligned}$$

Observe that A_{n-k}^n is a polynomial in a_1, a_2, \dots, a_n of degree $n - k$.

Conjecture 1.1 (Granville–Lin–Yau (GLY) conjecture [9, 18]). *Let $P_n = \#\{(x_1, x_2, \dots, x_n) \in \mathbb{Z}_+^n : \frac{x_1}{a_1} + \frac{x_2}{a_2} + \dots + \frac{x_n}{a_n} \leq 1\}$, where $a_1 \geq a_2 \geq \dots \geq a_n \geq 1$ are real numbers. If $n \geq 3$, then*

- (1) *Rough (general) upper estimate: For all $a_n \geq 1$,*

$$n!P_n \leq q_n := \prod_{i=1}^n (a_i - 1),$$

where equality holds if and only if $a_1 = a_2 = \dots = a_n =$ integer.

(2) *Sharp upper estimate: For a_n sufficiently large, there exists an integer $\beta(n)$ that depends on n such that when $a_n \geq \beta(n)$, then*

$$n!P_n \leq A_n^n + (-1) \frac{S_1^{n-1}}{n} A_{n-1}^n + (-1)^2 \frac{S_2^{n-1}}{\binom{n-1}{1}} A_{n-2}^n + (-1)^3 \frac{S_3^{n-1}}{\binom{n-1}{2}} A_{n-3}^n + \dots + (-1)^{k-1} \frac{S_{k+1}^{n-1}}{\binom{n-1}{k}} A_{n-k-1}^n + \dots + (-1)^{n-1} \frac{S_{n-1}^{n-1}}{\binom{n-1}{n-2}} A_1^{n-1},$$

and equality holds if and only if $a_1 = a_2 = \dots = a_n \in \mathbb{Z}_+$.

The GLY conjecture was proven by Xu and Yau for $n = 3$ [20] and $n = 4$ [22], for $n = 5$, see [4], [6] and [8], Wang and Yau for $3 \leq n \leq 6$ [18], where $\beta(n) = n - 1$ for $3 \leq n \leq 6$. The rough GLY upper estimate for all n was proven in [24].

In particular, the following theorems will be used frequently.

Theorem 1.1. *Let $a \geq b \geq c \geq d \geq 2$, and P_4 be the number of positive integral solutions of $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leq 1$, i.e., $P_4 = \#\{(x, y, z, w) \in \mathbb{Z}_+^4 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leq 1\}$. If $P_4 > 0$, then*

$$24P_4 \leq f_4(a, b, c, d) := abcd - \frac{3}{2}(abc + abd + acd + bcd) + \frac{11}{3}(ab + ac + bc) - 2(a + b + c)$$

and equality is attained if and only if $a = b = c = d = \text{integer}$.

If $a \geq b \geq c \geq d \geq 3$ or $a \geq 8, b \geq 6, c \geq 4, d \geq 2$, then Theorem 1.1 is true without the condition $P_4 > 0$ and equality is attained if and only if $a = b = c = d = \text{integer}$.

Theorem 1.2. *Let $a \geq b \geq c \geq d \geq 1$, and P_4 be the number of positive integral solutions of $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leq 1$, i.e., $P_4 = \#\{(x, y, z, w) \in \mathbb{Z}_+^4 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leq 1\}$. Define $\mu = (a - 1)(b - 1)(c - 1)(d - 1)$, then*

$$24P_4 \leq \mu = abcd - (abc + abd + acd + bcd) + (ab + ac + ad + bc + bd + cd) - (a + b + c + d) + 1.$$

Theorem 1.3 (Lu–Ya–Zu [11]). *Let $a \geq b \geq c \geq d \geq 1$, and P_4 be the number of positive integral solutions of $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leq 1$, i.e., $P_4 = \#\{(x, y, z, w) \in \mathbf{Z}_+^4 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leq 1\}$. Define $\mu = (a - 1)(b - 1)(c - 1)(d - 1)$, if $P_4 > 0$, then*

$$24P_4 \leq g_4(a, b, c, d) := (a - 1)(b - 1)(c - 1)(d - 1) - (d - 1)^4 + d(d - 1)(d - 2)(d - 3)$$

and equality is attained if and only if $a = b = c = d = \text{integer}$.

The GLY conjecture was the first major step towards proving the following conjecture made by Yau in 1995 [24]:

Conjecture 1.2. *Let $f: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be a weighted homogeneous polynomial with an isolated singularity at the origin. Let μ , p_g and v be the Milnor number, geometric genus and multiplicity of the singularity $V = \{z: f(z) = 0\}$, then*

$$\mu - p(v) \geq n!p_g,$$

where $p(v) = (v - 1)^n - v(v - 1) \dots (v - n + 1)$, and equality holds if and only if f is a homogeneous polynomial.

This conjecture is a sharp estimate that holds without the restriction of the sharp GLY estimate, $a_n \geq \beta(n)$, and it also has some important applications in geometry.

The Yau conjecture was already proven for the cases $n = 3$ [20] and $n = 4$ [10]. In this paper, we aim to prove the Yau conjecture for $n = 5$, which is stated below.

Theorem 1.4. *Let $f: (\mathbb{C}^5, 0) \rightarrow (\mathbb{C}, 0)$ be a weighted homogeneous polynomial with an isolated singularity at the origin. Let μ , p_g and v be the Milnor number, geometric genus and multiplicity of the singularity $V = \{z: f(z) = 0\}$. If $p_g > 0$ then*

$$\mu - p(v) \geq 5!p_g,$$

where $p(v) = (v - 1)^5 - v(v - 1) \dots (v - 4)$, and equality holds if and only if f is a homogeneous polynomial.

As a corollary of Theorem 1.4, we have the following coordinate-free characterization of homogeneous polynomials with isolated singularities.

Corollary. *Let $f: (\mathbb{C}^5, 0) \rightarrow (\mathbb{C}, 0)$ be a holomorphic function with an isolated singularity at the origin. Let μ , p_g and v be the Milnor number, the geometric genus and multiplicity of the singularity $V = \{z: f(z) = 0\}$ at the origin. Let τ be the dimension of the semi-universal deformation space of $(V, 0)$ which is $\dim \mathbb{C}\{z_1, \dots, z_5\}/(f, f_{z_1}, \dots, f_{z_5})$. Assume that $p_g > 0$, then, after a biholomorphic change of coordinates, f is a homogeneous polynomial if and only if $\mu - (v - 1)^5 + v(v - 1)(v - 2)(v - 3)(v - 4) = 5!P_g$ and $\mu = \tau$.*

It was observed by Saeki [16] (also see [20]) that the multiplicity v is $\inf\{n \in \mathbb{Z}_+ : n\} \geq \inf\{w_0, w_1, w_2, w_3, w_4\}$, where w_i is a weight of x_i in f . Therefore, proving our Main Theorem, Theorem 1.4 above, is akin to proving the following theorem about the number of integral points within a five-dimensional simplex, in view of the Theorem 2.5 in Section 2.5 of this paper.

Theorem 1.5. *Let $a \geq b \geq c \geq d \geq e > 1$ be real numbers and let P_5 be the number of positive integral solutions of $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1$, i.e., $P_5 = \#\{(x, y, z, v, w) \in \mathbb{Z}_+^5 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1\}$. Define $\mu = (a - 1)(b - 1)(c - 1)(d - 1)(e - 1)$. If $P_5 > 0$, then*

$$5!P_5 \leq \mu - (5v^4 - 25v^3 + 40v^2 - 19v - 1),$$

where v is the multiplicity calculated by $v = e$, if e is an integer, or by $v = [e] + 1 = e - \beta + 1$, if e is not an integer and β is either $\frac{e}{a}$, $\frac{e}{b}$, $\frac{e}{c}$ or $\frac{e}{d}$, where $[e] = e - \beta$ is the integral part of e . Equality holds if and only if $a = b = c = d = e$ are all integers.

Although the idea of the proof of our theorem is very simple, our proof is quite delicate. We try to estimate P_4 on hyperplanes parallel to the $xyzv$ -plane by using the upper bounds in the four-dimensional case provided by Theorems 1.1 to 1.3 above and sum these estimates up to get P_5 . In order to avoid the negative amount difficulty in $f_4(a, b, c, d)$ and $g_4(a, b, c, d)$ which may happen when $d < 3$, we need a careful analysis of the last two or three hyperplanes. For this reason, Theorems 1.1 to 1.3 are needed to deal with the problem. Our main theorem follows from a careful analysis of this sum. All the computations in this paper are done by Maple V.

By level $w = k$, we shall mean the intersection of the tetrahedron in Theorem 1.5 with the hyperplane $w = k$. In our tetrahedron, $w = k$ points are in following four-dimensional tetrahedron

$$\frac{x}{a(1 - \frac{k}{e})} + \frac{y}{b(1 - \frac{k}{e})} + \frac{z}{c(1 - \frac{k}{e})} + \frac{v}{d(1 - \frac{k}{e})} \leq 1.$$

We can use Theorem 1.1 to bound the number of positive integral solutions at level $w = k$ if $d(1 - \frac{k}{e}) \geq 3$ or $d(1 - \frac{k}{e}) \geq 2$ with the condition there exist positive integral solutions at this level. We can use Theorem 1.2 to bound the number of positive integral solutions at level $w = k$ if $d(1 - \frac{k}{e}) > 1$ without the condition there exist positive integral solutions at this level. We can use Theorem 1.3 to bound the number of positive integral solutions at level $w = k$ if $d(1 - \frac{k}{e}) > 1$ with the condition there exist positive integral solutions at this level. In order to do this, we will split up the proof into five main cases, depending on the value of e , and utilize Theorems 1.1 to 1.3 carefully in each case:

- (I) $e \geq 5, e \in \mathbb{Z}_+,$
- (II) $e > 4, e \notin \mathbb{Z}_+,$
- (III) $4 \geq e > 3,$
- (IV) $3 \geq e > 2,$
- (V) $2 \geq e > 1.$

We slice the five-dimensional simplex into the hyperplanes, $w = 1, w = 2, \dots, w = e - \beta - 1$ and $w = e - \beta$. It is obvious there are no positive integral solutions at level $w = e - \beta$. At level $w = e - \beta - 2$, the defining inequality of the simplex becomes $\frac{x}{\frac{a}{e}(\beta+2)} + \frac{y}{\frac{b}{e}(\beta+2)} + \frac{z}{\frac{c}{e}(\beta+2)} + \frac{v}{\frac{d}{e}(\beta+2)} \leq 1$. If $\frac{d}{e}(\beta + 2) > 3$, then Theorem 1.1 can be used. If $\frac{d}{e}(\beta + 2) < 3$ with the condition there are positive integral solutions at this level, then Theorem 1.3 can be used. If $\frac{d}{e}(\beta + 2) < 3$ with the condition there are no positive integral solutions at this level, then Theorem 1.2 can be used. At level $w = e - \beta - 1$, the defining inequality of the simplex becomes $\frac{x}{\frac{a}{e}(\beta+1)} + \frac{y}{\frac{b}{e}(\beta+1)} + \frac{z}{\frac{c}{e}(\beta+1)} + \frac{v}{\frac{d}{e}(\beta+1)} \leq 1$. In Case (II), we divide our proof into two cases depending on whether the tetrahedron

$$\frac{x}{\frac{a}{e}(\beta + 1)} + \frac{y}{\frac{b}{e}(\beta + 1)} + \frac{z}{\frac{c}{e}(\beta + 1)} + \frac{v}{\frac{d}{e}(\beta + 1)} \leq 1$$

at level $w = e - \beta - 1$ has positive integral solutions or not.

Case (A). $\frac{a}{e}(1 + \beta) < 4$ or $\frac{b}{e}(1 + \beta) \leq 3$ or $\frac{c}{e}(1 + \beta) \leq 2$ or $\frac{d}{e}(1 + \beta) \leq 1$.

In this case, there are no positive integral solutions on level $w = e - \beta - 1$.

Case (B). $\frac{a}{e}(1 + \beta) \geq 4, \frac{b}{e}(1 + \beta) > 3, \frac{c}{e}(1 + \beta) > 2$ and $\frac{d}{e}(1 + \beta) > 1$, where $0 < \beta < 1$.

In this case, there may exist positive integral solutions on level $w = e - \beta - 1$.

2. Proof of the main Theorem 4

2.1. Case I

We will first analyze the case that occurs when $e \geq 5$ and e is an integer.

Proposition 2.1. *Let $a \geq b \geq c \geq d \geq e \geq 5$ be real numbers. Consider $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1$. Let P_5 be the number of positive integral solutions of the above inequality, i.e., $P_5 = \#\{(x, y, z, v, w) \in \mathbb{Z}_+^5 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1\}$. Then,*

$$120P_5 \leq abcde - 2(abcd + abce + abde + acde + bcde) + \frac{35}{4}(abc + abd + acd + bcd) - \frac{25}{3}(ab + ac + ad + bc + bd + cd) + 6(a + b + c + d).$$

The inequality above has been taken from the main theorem of [7].

Before we go any further, we should note the interesting properties of P_4 that the lemma below points out.

Lemma 2.1. *Let $a \geq b \geq c \geq d \geq 1$ be real numbers and let $P_4 = \#\{(x, y, z, v) \in \mathbb{Z}_+^4 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} \leq 1\}$. Then, the following statements hold:*

- (1) *if $b \leq 3$, then $P_4 = 0$,*
- (2) *if $c \leq 2$, then $P_4 = 0$.*

The theorem below is basically the Yau conjecture for $n = 5$ with the property that e is an integer.

Theorem 2.1. *Let $a \geq b \geq c \geq d \geq e \geq 4$ be real numbers with e an integer. Consider $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1$. Let P_5 be the number of positive integral solutions of $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1$, i.e., $P_5 = \#\{(x, y, z, v, w) \in \mathbb{Z}_+^5 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1\}$. Define $\mu = (a - 1)(b - 1)(c - 1)(d - 1)(e - 1)$. Then,*

$$120P_5 \leq \mu - (5v^4 - 25v^3 + 40v^2 - 19v - 1)|_{v=e} = (a - 1)(b - 1)(c - 1)(d - 1)(e - 1) - (5e^4 - 25e^3 + 40e^2 - 19e - 1)$$

with equality if and only if $a = b = c = d = e = \text{integer}$.

Proof. Because Proposition 2.1 is a previously proven upper estimate for P_5 , we only need to show that the right-hand side (RHS) of Theorem 2.1 is larger than the RHS of Proposition 2.1. Let Δ_1 be the difference by subtracting the RHS of Proposition 2.1 from the RHS of Theorem 2.1. We only need to show $\Delta_1 \geq 0$.

$$\begin{aligned}\Delta_1 &= abcd + abce + abde + acde + bcde - \frac{31}{4}(abc + abd + acd + bcd) \\ &\quad + abe + ace + bce + ade + bde + cde + \frac{22}{3}(ab + ac + bc \\ &\quad + ad + bd + cd) - ae - be - ce - de - 5(a + b + c + d) \\ &\quad - 5e^4 + 25e^3 - 40e^2 + 20e.\end{aligned}$$

Let $A = \frac{a}{e}$, $B = \frac{b}{e}$, $C = \frac{c}{e}$ and $D = \frac{d}{e}$; we can now rewrite Δ_1 .

$$\begin{aligned}\Delta_1 &= (ABCD + ABC + ABD + ACD + BCD - 5)e^4 \\ &\quad - \frac{31}{4}e^3(ABC + ABD + ACD + BCD) \\ &\quad + e^3(AB + AC + BC + AD + BD + CD + 25) \\ &\quad + \frac{22}{3}e^2(AB + AC + BC + AD + BD + CD) \\ &\quad - e^2(A + B + C + D + 40) - 5e(A + B + C + D - 4).\end{aligned}$$

Note that Δ_1 is symmetric with respect to A , B , C and D . We can then apply a method that we call the “partial differentiation test”: we calculate the partial derivative with respect to all the variables first and then partial differentiate with respect to one less variable for each consecutive step until we have the expressions for the first-order partials, which are $\frac{\partial}{\partial A}\Delta_1$, $\frac{\partial}{\partial B}\Delta_1$, $\frac{\partial}{\partial C}\Delta_1$ and $\frac{\partial}{\partial D}\Delta_1$ in this case. As long as we show that those first-order partials are positive throughout the domain and that the function is positive at the minimum, we can conclude that the function, and therefore the difference, is positive throughout the domain.

First,

$$\frac{\partial^4 \Delta_1}{\partial A \partial B \partial C \partial D} = e^4 > 0,$$

for all $e > 1$. Then, it follows that $\frac{\partial^3 \Delta_1}{\partial A \partial B \partial C}$ is an increasing function of D for $e > 1$, $D \geq 1$, and at the minimum $D = 1$,

$$\left. \frac{\partial^3 \Delta_1}{\partial A \partial B \partial C} \right|_{D=1} = e^3 \left(2e - \frac{31}{4} \right) > 0,$$

for $e > \frac{31}{8}$, so $\frac{\partial^3 \Delta_1}{\partial A \partial B \partial C}$ must be positive throughout the domain $D \geq 1$, $e > \frac{31}{8}$. Because $\frac{\partial^2 \Delta_1}{\partial A \partial B}$ is symmetric with respect to C and D , $\frac{\partial^3 \Delta_1}{\partial A \partial B \partial D}$ must also be positive on the domain $C \geq 1$, $e > \frac{31}{8}$. Then, from knowing that both $\frac{\partial^3 \Delta_1}{\partial A \partial B \partial C}$ and $\frac{\partial^3 \Delta_1}{\partial A \partial B \partial D}$ are positive, $\frac{\partial^2 \Delta_1}{\partial A \partial B}$ is increasing with respect to C and D for $C \geq D \geq 1$ and $e > \frac{31}{8}$. At the minimum $C = D = 1$,

$$\left. \frac{\partial^2 \Delta_1}{\partial A \partial B} \right|_{C=D=1} = e^2 \left(3e^2 - \frac{29}{2}e + \frac{22}{3} \right) > 0,$$

for $e > \frac{29 + \sqrt{489}}{12}$ or $e \geq 5$, so $\frac{\partial^2 \Delta_1}{\partial A \partial B} > 0$ for $C \geq 1$, $D \geq 1$ and $e \geq 5$. Because $\frac{\partial \Delta_1}{\partial A}$ is symmetric with respect to B , C and D , both $\frac{\partial^2 \Delta_1}{\partial A \partial C}$ and $\frac{\partial^2 \Delta_1}{\partial A \partial D}$ must also be positive on the respective domains of $B \geq 1$, $D \geq 1$, $e \geq 5$ and $B \geq 1$, $C \geq 1$, $e \geq 5$ and $\frac{\partial \Delta_1}{\partial A}$ is an increasing function of B , C and D for $B \geq 1$, $C \geq 1$, $D \geq 1$ and $e \geq 5$. At the minimum of $B = C = D = 1$,

$$\left. \frac{\partial \Delta_1}{\partial A} \right|_{B=C=D=1} = e \left(4e^3 - \frac{81}{4}e^2 + 21e - 5 \right) > 0,$$

for $e \geq 5$. Then, we know that Δ_1 is increasing with respect to A for $B \geq 1$, $C \geq 1$, $D \geq 1$ and $e \geq 5$. Because Δ_1 is symmetric with respect to A , B , C and D , Δ_1 must also be increasing with respect to B , C and D for the respective domains of $A \geq 1$, $C \geq 1$, $D \geq 1$, $e \geq 5$; $A \geq 1$, $B \geq 1$, $D \geq 1$, $e \geq 5$; and $A \geq 1$, $B \geq 1$, $C \geq 1$, $e \geq 5$. Evaluated at the minimum of $A = B = C = D = 1$,

$$\Delta_1|_{A=B=C=D=1} = 0,$$

hence Δ_1 is nonnegative on the domain $A \geq 1$, $B \geq 1$, $C \geq 1$, $D \geq 1$ and $e \geq 5$, and therefore the RHS of Theorem 2.1 is greater than or equal to the RHS of Proposition 2.1. \square

2.2. Case V

Theorem 2.2. *Let $a \geq b \geq c \geq d \geq e$ and $e \in (1, 2]$ be real numbers. Consider $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1$. Let P_5 be the number of positive integral solutions of $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1$, i.e., $P_5 = \#\{(x, y, z, v, w) \in \mathbb{Z}_+^5 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1\}$. Define $\mu = (a - 1)(b - 1)(c - 1)(d - 1)(e - 1)$. Then, if $P_5 > 0$,*

$$\begin{aligned} 120P_5 &\leq \mu - (5v^4 - 25v^3 + 40v^2 - 19v - 1)|_{v=2} \\ &= (a - 1)(b - 1)(c - 1)(d - 1)(e - 1) - 1 \end{aligned}$$

with equality if and only if $a = b = c = d = e = \text{integer}$.

Proof. In this case, $e \in (1, 2]$ and there are two levels to consider, $k = 1$ and $k = 2$. When $k = 2$, the inequality $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{k}{e} \leq 1$ has no positive solutions, and when $k = 1$, the set of positive solutions to the inequality contains the point $(1, 1, 1, 1, 1)$ if we assume $P_5 > 0$. If $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq 1 - \frac{1}{e} \triangleq \alpha$, then $\alpha \in (0, \frac{1}{2}]$ since $e \in (1, 2]$. Let $A = a\alpha$, $B = b\alpha$, $C = c\alpha$, and $D = d\alpha$ and the following restrictions hold:

$$A \geq 4, B \geq 3, C \geq 2, D > 1,$$

since $\frac{1}{D} < 1, \frac{2}{C} \leq \frac{1}{C} + \frac{1}{D} \leq 1, \frac{3}{B} \leq \frac{1}{B} + \frac{1}{C} + \frac{1}{D} \leq 1$ and $\frac{4}{A} \leq \frac{1}{A} + \frac{1}{B} + \frac{1}{C} + \frac{1}{D} \leq 1$. Then, we have

$$5!P_4(k = 1) \leq 5[(A - 1)(B - 1)(C - 1)(D - 1) - (D - 1)^4 + D(D - 1)(D - 2)(D - 3)].$$

If we let Δ_2 be the difference by subtracting the RHS of Theorem 2.2 from the RHS of the above inequality; then substituting $a = \frac{A}{\alpha}, b = \frac{B}{\alpha}, c = \frac{C}{\alpha}, d = \frac{D}{\alpha}, e = \frac{1}{1-\alpha}, \nu = 2$, into Δ_2 , then all we have to do is apply the partial differentiation test for the expression

$$\begin{aligned} \Delta_2 &= \frac{1}{\alpha^3(1 - \alpha)}(ABCD - \alpha(ABC + ABD + ACD + BCD) \\ &\quad + \alpha^2(AB + AC + BC + AD + BD + CD) \\ &\quad - \alpha^3(1 - 4A - 4B - 4C + 5AB + 5AC + 5BC \\ &\quad - 5ABC - 14D + 5AD + 5BD - 5ABD + 5CD \\ &\quad - 5ACD - 5BCD + 5ABCD + 25D^2 - 10D^3) \\ &\quad + \alpha^4(2 - 5A - 5B + 5AB - 5C + 5AC + 5BC \\ &\quad - 5ABC - 15D + 5AD + 5BD - 5ABD + 5CD \\ &\quad - 5ACD - 5BCD + 5ABCD + 25D^2 - 10D^3)) \\ &= \frac{1}{\alpha^3(1 - \alpha)}\Delta_3 \end{aligned}$$

on the domain $A \geq 4, B \geq 3, C \geq 2, D > 1$, and $\alpha \in (0, \frac{1}{2}]$. We see that

$$\frac{\partial^4 \Delta_3}{\partial A \partial B \partial C \partial D} = 1 - 5\alpha^3 + 5\alpha^4 > 0$$

for all $\alpha \in (0, 1)$. Then, $\frac{\partial^3 \Delta_3}{\partial A \partial B \partial C}$ is an increasing function of D on $D \geq 1$, $\alpha \in (0, 1)$ with a minimum at $D = 1$.

$$\frac{\partial^3 \Delta_3}{\partial A \partial B \partial C} \Big|_{D=1} = 1 - \alpha > 0,$$

for all $\alpha \in (0, 1)$. It follows that $\frac{\partial^3 \Delta_3}{\partial A \partial B \partial C} > 0$ for all $D \geq 1$, $\alpha \in (0, 1)$, and because $\frac{\partial^2 \Delta_3}{\partial A \partial B}$ is symmetric with respect to C and D , we also know $\frac{\partial^3 \Delta_3}{\partial A \partial B \partial D} > 0$ for all $C \geq 1$, $\alpha \in (0, 1)$. Therefore, $\frac{\partial^2 \Delta_3}{\partial A \partial B}$ is an increasing function with respect to C, D for all $C \geq 1$, $D \geq 1$, $\alpha \in (0, 1)$, and its minimum occurs at $C = D = 1$.

$$\frac{\partial^2 \Delta_3}{\partial A \partial B} \Big|_{C=D=1} = 1 - 2\alpha + \alpha^2 = (1 - \alpha)^2 > 0,$$

for all $\alpha \in (0, 1)$. It follows that $\frac{\partial^2 \Delta_3}{\partial A \partial B} > 0$ for all $C \geq 1$, $D \geq 1$, $\alpha \in (0, 1)$, and because $\frac{\partial \Delta_3}{\partial A}$ is symmetric with respect to B, C and D , we also know $\frac{\partial^2 \Delta_3}{\partial A \partial C} > 0$ for all $B \geq 1$, $D \geq 1$, $\alpha \in (0, 1)$ and $\frac{\partial^2 \Delta_3}{\partial A \partial D} > 0$ for all $B \geq 1$, $C \geq 1$, $\alpha \in (0, 1)$. Therefore, $\frac{\partial \Delta_3}{\partial A}$ is an increasing function with respect to B, C, D for all $B \geq 1$, $C \geq 1$, $D \geq 1$, $\alpha \in (0, 1)$ with a minimum at $B = C = D = 1$.

$$\frac{\partial \Delta_3}{\partial A} \Big|_{B=C=D=1} = 1 - 3\alpha + 3\alpha^2 + 3\alpha^3 > (1 - \alpha)^3 > 0,$$

for all $\alpha \in (0, 1)$, so $\frac{\partial \Delta_3}{\partial A} > 0$ for all $B \geq 1, C \geq 1, D \geq 1, \alpha \in (0, 1)$. By the property that Δ_3 is symmetric with respect to A, B and C , we also have $\frac{\partial \Delta_3}{\partial B} > 0$ for all $A \geq 1, C \geq 1, D \geq 1, \alpha \in (0, 1)$ and $\frac{\partial \Delta_3}{\partial C} > 0$ for all $A \geq 1, B \geq 1, D \geq 1, \alpha \in (0, 1)$, and therefore, Δ_3 is an increasing function with respect to A, B and C . Meanwhile,

$$\frac{\partial^3 \Delta_3}{\partial D^3} = 60\alpha^3(1 - \alpha) > 0,$$

for all $\alpha \in (0, 1)$, so $\frac{\partial^2 \Delta_3}{\partial D^2}$ is an increasing function of D for all $D \geq 1$, $\alpha \in (0, 1)$ with a minimum at $D = 1$.

$$\frac{\partial^2 \Delta_3}{\partial D^2} \Big|_{D=1} = 60\alpha^3(1 - \alpha) > 0,$$

for all $\alpha \in (0, 1)$, so $\frac{\partial^2 \Delta_3}{\partial D^2} > 0$ on the domain of $D \geq 1$, $\alpha \in (0, 1)$, and therefore, $\frac{\partial \Delta_3}{\partial D}$ is an increasing function of D for all $D \geq 1$, $\alpha \in (0, 1)$. Furthermore, $\frac{\partial \Delta_3}{\partial D}$ is an increasing function with respect to A, B, C and D for

$A \geq B \geq C \geq D \geq 1$, $\alpha \in (0, 1)$, because $\frac{\partial \Delta_3}{\partial D}$ is symmetric with respect to A, B and C . Then, the minimum of $\frac{\partial \Delta_3}{\partial D}$ occurs at $A = 4, B = 3, C = 2, D = 1$.

$$\left. \frac{\partial \Delta_3}{\partial D} \right|_{A=4, B=3, C=2, D=1} = 24 - 26\alpha + 9\alpha^2 - 41\alpha^3 + 40\alpha^4 > 0,$$

for all $\alpha \in (0, 1)$, and therefore, $\frac{\partial \Delta_3}{\partial D} > 0$ for $A \geq 4, B \geq 3, C \geq 2, D \geq 1$, $\alpha \in (0, 1)$, and Δ_3 is an increasing function of A, B, C, D for $A \geq 4, B \geq 3, C \geq 2, D \geq 1, \alpha \in (0, 1)$. Thus, the minimum of Δ_3 occurs at $A = 4, B = 3, C = 2, D = 1$:

$$\Delta_3|_{A=4, B=3, C=2, D=1} = 24 - 50\alpha + 35\alpha^2 - 11\alpha^3 + 2\alpha^4 > 0$$

for all $\alpha \in (0, 1)$. Then, it follows that $\Delta_3 > 0$ for $A \geq 4, B \geq 3, C \geq 2, D \geq 1$ and $\alpha \in (0, \frac{1}{2}]$. □

2.3. Case IV

Theorem 2.3. *Let $a \geq b \geq c \geq d \geq e$ and $e \in (2, 3]$ be real numbers. Consider $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1$. Let P_5 be the number of positive integral solutions of $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1$, i.e., $P_5 = \#\{(v, w, x, y, z) \in \mathbb{Z}_+^5 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1\}$. Define $\mu = (a - 1)(b - 1)(c - 1)(d - 1)(e - 1)$. Then, if $P_5 > 0$,*

$$\begin{aligned} 120P_5 &\leq \mu - (5v^4 - 25v^3 + 40v^2 - 19v - 1)|_{v=3} \\ &= (a - 1)(b - 1)(c - 1)(d - 1)(e - 1) - 32 \end{aligned}$$

with equality if and only if $a = b = c = d = e = \text{integer}$.

Proof. In this case, $e \in (2, 3]$, and there are two levels to consider: $k = 1$, and $k = 2$. Because of the criterion $P_5 > 0$, we know the level $k = 1$ contains positive integral solutions, but due to the fact that we cannot draw any conclusions about the level $k = 2$, we have two different subcases to consider:

- (a) $P_4(2) = 0$,
- (b) $P_4(2) > 0$.

For subcase (a), the proof is almost exactly the same as the proof presented in Case V. Because $P_5 > 0$, the set of positive integral solutions includes the point $(1, 1, 1, 1, 1)$, and let $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq 1 - \frac{1}{e} \triangleq \alpha, \alpha \in$

$(\frac{1}{2}, \frac{2}{3}]$, since $e \in (2, 3]$. Let $A = a\alpha$, $B = b\alpha$, $C = c\alpha$ and $D = d\alpha$; then, the following restrictions apply:

$$A \geq 4, B \geq 3, C \geq 2, D \geq \frac{\alpha}{1 - \alpha},$$

because $D = d\alpha \geq e\alpha = \frac{\alpha}{1 - \alpha}$. With $\alpha \in (\frac{1}{2}, \frac{2}{3}]$, $\frac{\alpha}{1 - \alpha} \in (1, 2]$, so we only need to show that the difference by subtracting the estimate provided in Case V by [11] from the RHS of Theorem 2.3, which is equal to $\frac{\Delta_3 - 31\alpha^3(1 - \alpha)}{\alpha^3(1 - \alpha)}$ is positive for $A \geq 4, B \geq 3, C \geq 2, D \geq \frac{\alpha}{1 - \alpha}$ and $\alpha \in (\frac{1}{2}, \frac{2}{3}]$. Notice that in Case V, we showed that all the first-order partial derivatives are positive for $A \geq 4, B \geq 3, C \geq 2, D \geq 1$ and $\alpha \in (0, 1)$; the only difference was the minimum of Δ_3 . Note that the constant terms in Theorems 2.2 and 2.3 differ, so we should subtract $31\alpha^3(1 - \alpha)$ from Δ_3 to make sure we are still proving Theorem 2.3 instead of Theorem 2.2. Then, we can substitute our new restrictions into the calculation of the minimum value of $\Delta_3 - 31\alpha^3(1 - \alpha)$:

$$\begin{aligned} & \Delta_3 - 31\alpha^3(1 - \alpha)|_{A=4, B=3, C=2, D=\frac{\alpha}{1-\alpha}} \\ &= \frac{\alpha^2(24 - 47\alpha + 6\alpha^2 + 14\alpha^3 + 13\alpha^4)}{(\alpha - 1)^2} > 0 \end{aligned}$$

for $\alpha \in (\frac{1}{2}, \frac{2}{3}]$. Then, it follows that $\Delta_3 - 31\alpha^3(1 - \alpha) > 0$ for $A \geq 4, B \geq 3, C \geq 2, D \geq \frac{\alpha}{1 - \alpha}$ and $\alpha \in (\frac{1}{2}, \frac{2}{3}]$.

For subcase (b), $P_4(2) > 0$, which implies that $(1, 1, 1, 1, 2)$ is the smallest positive integer solution to the level $k = 2$. Then, we have $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq 1 - \frac{2}{e} \triangleq \alpha_1, \alpha_1 \in (0, \frac{1}{3}]$, since $e \in (2, 3]$. Let $A = a\alpha_1, B = b\alpha_1, C = c\alpha_1$ and $D = d\alpha_1$, and then we know $A \geq 4, B \geq 3, C \geq 2, D > 1$ for the same reasons listed in Case V. If we utilize the estimate provided by Luo *et al.* [11], we get the following:

$$\begin{aligned} 5!P_5 &= 5!(P_4(1) + P_4(2)) \\ &\leq 5 \left[\left(A \left(\frac{1 - \alpha_1}{2\alpha_1} + 1 \right) - 1 \right) \left(B \left(\frac{1 - \alpha_1}{2\alpha_1} + 1 \right) - 1 \right) \right. \\ &\quad \cdot \left(C \left(\frac{1 - \alpha_1}{2\alpha_1} + 1 \right) - 1 \right) \cdot \left(D \left(\frac{1 - \alpha_1}{2\alpha_1} + 1 \right) - 1 \right) \\ &\quad - \left(D \left(\frac{1 - \alpha_1}{2\alpha_1} + 1 \right) - 1 \right)^4 + D \left(\frac{1 - \alpha_1}{2\alpha_1} + 1 \right) \\ &\quad \left. + \left(D \left(\frac{1 - \alpha_1}{2\alpha_1} + 1 \right) - 1 \right) \left(D \left(\frac{1 - \alpha_1}{2\alpha_1} + 1 \right) - 2 \right) \right] \end{aligned}$$

$$\begin{aligned} & \times \left(D \left(\frac{1 - \alpha_1}{2\alpha_1} + 1 \right) - 3 \right) + (A - 1)(B - 1)(C - 1)(D - 1) \\ & - (D - 1)^4 + D(D - 1)(D - 2)(D - 3). \end{aligned}$$

It is sufficient to show that the RHS of the inequality above is strictly less than the RHS of Theorem 2.3. Let Δ_4 be the difference between the latter, substituting $a = \frac{A}{\alpha_1}, b = \frac{B}{\alpha_1}, c = \frac{C}{\alpha_1}, d = \frac{D}{\alpha_1}, e = \frac{2}{1 - \alpha_1}, \nu = 3$ and the former.

$$\begin{aligned} \Delta_4 &= \frac{1}{16(1 - \alpha_1)\alpha_1^4} (11ABCD - \alpha_1(6ABC + 6ABD + 6ACD \\ &+ 6BCD - ABCD\alpha_1 - 20D^3) \\ &- \alpha_1^2(4AB + 4AC + 4BC - 4ABC + 4AD + 4BD - 4ABD \\ &+ 4CD - 4ACD - 4BCD + 10ABCD + 100D^2 - 40D^3 \\ &+ \alpha_1^3(24A + 24B - 4AB + 24C - 4AC - 4BC + 104D - 4AD \\ &- 4BD - 4CD + 10ABCD - 100D^2) \\ &+ \alpha_1^4(-496 + 64A + 64B - 60AB + 64C - 60AC - 60BC + 60ABC \\ &+ 224D - 60AD - 60BD + 60ABD - 60CD + 60ACD + 60BCD \\ &- 65ABCD - 300D^2 + 120D^3) \\ &+ \alpha_1^5(528 - 120A - 120B + 100AB - 120C + 100AC + 100BC \\ &- 90ABC - 360D + 100AD + 100BD - 90ABD + 100CD - 90ACD \\ &- 90BCD + 85ABCD + 500D^2 - 180D^3)). \\ &= \frac{1}{16(1 - \alpha_1)\alpha_1^4} \Delta_5. \end{aligned}$$

Now, we apply the partial differentiation test to Δ_5 on the domain $A \geq 4, B \geq 3, C \geq 2, D \geq 1$ and $\alpha_1 \in (0, \frac{1}{3}]$.

$$\frac{\partial^4 \Delta_5}{\partial A \partial B \partial C \partial D} = 11 + \alpha_1 - 10\alpha_1^2 + 10\alpha_1^3 - 65\alpha_1^4 + 85\alpha_1^5 > 0$$

for $\alpha_1 \in (0, 1)$, so $\frac{\partial^3 \Delta_5}{\partial A \partial B \partial C}$ is an increasing function of D on $D \geq 1$ and $\alpha_1 \in (0, 1)$ with its minimum at $D = 1$.

$$\left. \frac{\partial^3 \Delta_5}{\partial A \partial B \partial C} \right|_{D=1} = 11 - 5\alpha_1 - 6\alpha_1^2 + 10\alpha_1^3 - 5\alpha_1^4 - 5\alpha_1^5 > 0$$

so $\frac{\partial^3 \Delta_5}{\partial A \partial B \partial C} > 0$ for $D \geq 1$ and $\alpha_1 \in (0, 1)$. Because $\frac{\partial^2 \Delta_5}{\partial A \partial B}$ is symmetric with respect to C and D , $\frac{\partial^3 \Delta_5}{\partial A \partial B \partial D} > 0$ for $C \geq 1$ and $\alpha_1 \in (0, 1)$, and $\frac{\partial^2 \Delta_5}{\partial A \partial B}$ is

an increasing function of C and D for $C \geq D \geq 1$ and $\alpha_1 \in (0, 1)$ with its minimum at $C = D = 1$.

$$\frac{\partial^2 \Delta_5}{\partial A \partial B} \Big|_{C=D=1} = (\alpha_1 - 1)^2(1 + \alpha_1)(11 + 5\alpha_1^2) > 0$$

for $\alpha_1 \in (0, 1)$, so $\frac{\partial^2 \Delta_5}{\partial A \partial B} > 0$ on the interval $C \geq 1, D \geq 1$ and $\alpha_1 \in (0, 1)$. Because $\frac{\partial \Delta_5}{\partial A}$ is symmetric with respect to B, C and D , $\frac{\partial^2 \Delta_5}{\partial A \partial C} > 0$ on the interval $B \geq 1, D \geq 1$ and $\alpha_1 \in (0, 1)$, and $\frac{\partial^2 \Delta_5}{\partial A \partial D} > 0$ on the interval $B \geq 1, C \geq 1$ and $\alpha_1 \in (0, 1)$. Then, we know that $\frac{\partial \Delta_5}{\partial A}$ is an increasing function of B, C and D for $B \geq C \geq D \geq 1$ and $\alpha_1 \in (0, 1)$ with its minimum at $B = C = D = 1$.

$$\frac{\partial \Delta_5}{\partial A} \Big|_{B=C=D=1} = -(\alpha_1 - 1)^3(\alpha_1 + 1)(11 + 5\alpha_1) > 0$$

for $\alpha_1 \in (0, \frac{3}{5})$, so $\frac{\partial \Delta_5}{\partial A} > 0$ on the interval $B \geq 1, C \geq 1, D \geq 1$ and $\alpha_1 \in (0, \frac{3}{5})$. Due to the fact that Δ_5 is symmetric with respect to A, B and C , we also have $\frac{\partial \Delta_5}{\partial B} > 0$ on the interval $A \geq 1, C \geq 1, D \geq 1, \alpha_1 \in (0, \frac{3}{5})$ and $\frac{\partial \Delta_5}{\partial C} > 0$ on the interval $A \geq 1, B \geq 1, D \geq 1, \alpha_1 \in (0, \frac{3}{5})$. In the meantime,

$$\frac{\partial^3 \Delta_5}{\partial D^3} = 120(1 - \alpha_1)\alpha_1(1 + 3\alpha_1)(1 + 3\alpha_1^2) > 0$$

for $\alpha_1 \in (0, 1)$, so $\frac{\partial^2 \Delta_5}{\partial D^2}$ is an increasing function of D for $D \geq 1$ and $\alpha_1 \in (0, 1)$ with its minimum at $D = 1$.

$$\frac{\partial^2 \Delta_5}{\partial D^2} \Big|_{D=1} = 40\alpha_1(1 - \alpha_1)(3 + 4\alpha_1 - \alpha_1^2 + 2\alpha_1^3) > 0$$

for $\alpha_1 \in (0, 1)$, which means $\frac{\partial^2 \Delta_5}{\partial D^2} > 0$ for $D \geq 1$ and $\alpha_1 \in (0, 1)$ hence $\frac{\partial \Delta_5}{\partial D}$ is an increasing function of A, B, C and D , on the interval $A \geq B \geq C \geq D \geq 1, \alpha_1 \in (0, 1)$ with its minimum at $A = B = C = D = 1$.

$$\frac{\partial \Delta_5}{\partial D} \Big|_{A=4, B=3, C=2, D=1} = 4(66 - 18\alpha_1 - 63\alpha_1^2 + 27\alpha_1^3 - 139\alpha_1^4 + 175\alpha_1^5) > 0$$

for $\alpha_1 \in (0, 1)$, and thus $\frac{\partial \Delta_5}{\partial D} > 0$ for $A \geq 4, B \geq 3, C \geq 2, D \geq 1$ and $\alpha_1 \in (0, 1]$. Hence, Δ_5 is an increasing function of A, B, C, D for $A \geq 4, B \geq 3,$

$C \geq 2, D \geq 1$ and $\alpha_1 \in (0, 1]$ with its minimum at $A = 4, B = 3, C = 2, D = 1$.

$$\Delta_5|_{A=4,B=3,C=2,D=1} = 8(\alpha_1 - 1)(-33 - \alpha_1 + 29\alpha_1^2 - 11\alpha_1^3 + 56\alpha_1^4) \geq 0$$

for $\alpha_1 \in (0, \frac{3}{5})$, $\Delta_5 > 0$ for $A \geq 4, B \geq 3, C \geq 2, D \geq 1$ and $\alpha_1 \in (0, \frac{3}{5})$, thus ending this proof. □

2.4. Case III

Theorem 2.4. *Let $a \geq b \geq c \geq d \geq e$ and $e \in (3, 4]$ be real numbers. Consider $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1$. Let P_5 be the number of positive integral solutions of $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1$, i.e., $P_5 = \#\{(x, y, z, v, w) \in \mathbb{Z}_+^5 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1\}$. Define $\mu = (a - 1)(b - 1)(c - 1)(d - 1)(e - 1)$. Then, if $P_5 > 0$,*

$$\begin{aligned} 120P_5 &\leq \mu - (5v^4 - 25v^3 + 40v^2 - 19v - 1)|_{v=4} \\ &= (a - 1)(b - 1)(c - 1)(d - 1)(e - 1) - (243) \end{aligned}$$

with equality if and only if $a = b = c = d = e = \text{integer}$.

Proof. In this case, there are four levels that might contain positive integral solutions: $k = 1, k = 2, k = 3$ and $k = 4$. When $k = 4$, the defining inequality of the five-dimensional simplex becomes $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1$, which has no positive integral solutions. Because $P_5 > 0$, we know $k = 1$ must have points, but the nature of the levels $k = 2$ and $k = 3$ are unknown, and therefore, we have to consider three different subcases:

- (a) $P_4(3) = P_4(2) = 0$,
- (b) $P_4(3) = 0, P_4(2) > 0$,
- (c) $P_4(3) > 0, P_4(2) > 0$.

In subcase (a), we will apply the GLY conjecture for $n = 4$ on the level $k = 1$ to estimate $P_4(1)$. This estimate is given by the expression below, which is also present as Theorem 2.1 of [8]:

$$4!P_4 \leq abcd - \frac{3}{2}(abc + abd + acd + bcd) + \frac{11}{3}(ab + ac + bc) - 2(a + b + c),$$

as long as $a \geq b \geq c \geq d \geq 2$ and $P_4 > 0$.

Let α be defined by $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq 1 - \frac{1}{e} \triangleq \alpha, \alpha \in (\frac{2}{3}, \frac{3}{4}]$ since $e \in (3, 4]$, and let $A = a\alpha, B = b\alpha, C = c\alpha$ and $D = d\alpha$. Then, the following restrictions apply:

$$A \geq 4, B \geq 3, C \geq D \geq \frac{\alpha}{1-\alpha} > 2.$$

We can apply the GLY conjecture to the above region because $D \geq \frac{\alpha}{1-\alpha} \in (2, 3]$, satisfying the condition of Theorem 2.1 from [8] that the smallest weight has to be greater than or equal to 2.

$$5!P_4 \leq 5[ABCD - \frac{3}{2}(ABC + ABD + ACD + BCD) + \frac{11}{3}(AB + AC + BC) - 2(A + B + C)].$$

Let Δ_6 now be defined as the difference by subtracting the RHS of the above inequality from the RHS of Theorem 2.4.

$$\begin{aligned} \Delta_6 &= \frac{1}{6(1-\alpha)\alpha^3} ((1464 + 110AC + 110BC - 60A - 60B - 60C \\ &\quad - 45ABC + 30ABCD - 45ABD - 45ACD - 45BCD + 110AB)\alpha^4 \\ &\quad + (-110AC - 110BC - 110AB - 1458 + 45ABC + 54A + 54B \\ &\quad + 45BCD + 45ABD + 45ACD - 6D + 54C - 30ABCD)\alpha^3 \\ &\quad + (6BC + 6AD + 6BD + 6CD + 6AC + 6AB)\alpha^2 \\ &\quad + (-6ABC - 6ABD - 6ACD - 6BCD)\alpha + 6ABCD) \\ &= \frac{1}{6(1-\alpha)\alpha^3} \Delta_7. \end{aligned}$$

Then, we only need to show that $\Delta_7 > 0$ by the partial differentiation test.

$$\frac{\partial^4 \Delta_7}{\partial A \partial B \partial C \partial D} = 6 - 30\alpha^3 + 30\alpha^4 > 0$$

for all $\alpha \in (\frac{2}{3}, \frac{3}{4}]$. Then, $\frac{\partial^3 \Delta_7}{\partial A \partial B \partial C}$ is an increasing function of D on $D \geq 1, \alpha \in (\frac{2}{3}, \frac{3}{4}]$ with a minimum at $D = 2$.

$$\left. \frac{\partial^3 \Delta_7}{\partial A \partial B \partial C} \right|_{D=2} = 15\alpha^4 - 15\alpha^3 - 6\alpha + 12 > 0$$

for all $\alpha \in (\frac{2}{3}, \frac{3}{4}]$, so $\frac{\partial^3 \Delta_7}{\partial A \partial B \partial C} > 0$ for all $D \geq 2, \alpha \in (\frac{2}{3}, \frac{3}{4}]$, and because $\frac{\partial^2 \Delta_7}{\partial A \partial B}$ is symmetric with respect to C and D , we also know $\frac{\partial^3 \Delta_7}{\partial A \partial B \partial D} > 0$ for all $C \geq 2, \alpha \in (\frac{2}{3}, \frac{3}{4}]$. (From symmetry, we also know that $\frac{\partial^3 \Delta_7}{\partial A \partial C \partial D} > 0$ for all $B \geq 2$,

$\alpha \in (\frac{2}{3}, \frac{3}{4}]$, a fact that will come in handy later in the proof.) Therefore, $\frac{\partial^2 \Delta_7}{\partial A \partial B}$ is an increasing function with respect to C and D for all $C \geq \frac{\alpha}{1-\alpha}$, $D \geq \frac{\alpha}{1-\alpha}$ and $\alpha \in (\frac{2}{3}, \frac{3}{4}]$ with a minimum at $C = D \geq \frac{\alpha}{1-\alpha} > 2$.

$$\frac{\partial^2 \Delta_7}{\partial A \partial B} \Big|_{C=D=2} = 50\alpha^4 - 50\alpha^3 + 6\alpha^2 - 24\alpha + 24 > 0$$

for all $\alpha \in (\frac{2}{3}, \frac{3}{4}]$. It follows that $\frac{\partial^2 \Delta_7}{\partial A \partial B} > 0$ for all $C \geq \frac{\alpha}{1-\alpha} > 2$, $D \geq \frac{\alpha}{1-\alpha} > 2$ and $\alpha \in (\frac{2}{3}, \frac{3}{4}]$, and because $\frac{\partial \Delta_7}{\partial A}$ is symmetric with respect to B and C , we also know $\frac{\partial^2 \Delta_7}{\partial A \partial C} > 0$ for all $B \geq \frac{\alpha}{1-\alpha} > 2$, $D \geq \frac{\alpha}{1-\alpha} > 2$, $\alpha \in (\frac{2}{3}, \frac{3}{4}]$. We now need to show that $\frac{\partial^2 \Delta_7}{\partial A \partial D} > 0$ for all $B \geq \frac{\alpha}{1-\alpha}$, $C \geq \frac{\alpha}{1-\alpha}$, $\alpha \in (\frac{2}{3}, \frac{3}{4}]$; we already know that it is an increasing function on the domain because $\frac{\partial^3 \Delta_7}{\partial A \partial B \partial D} > 0$ and $\frac{\partial^3 \Delta_7}{\partial A \partial C \partial D} > 0$ for their respective domains so we only need to check the value of $\frac{\partial^2 \Delta_7}{\partial A \partial D}$ at its minimum of $B = 3$, $C = \frac{\alpha}{1-\alpha} > 2$.

$$\frac{\partial^2 \Delta_7}{\partial A \partial D} \Big|_{B=3, C=2} = -45\alpha^4 + 45\alpha^3 + 6\alpha^2 - 30\alpha + 36 > 0$$

for all $\alpha \in (\frac{2}{3}, \frac{3}{4}]$, so $\frac{\partial^2 \Delta_7}{\partial A \partial D} > 0$ on the domain. (By symmetry, we also know $\frac{\partial^2 \Delta_7}{\partial B \partial D} > 0$ and $\frac{\partial^2 \Delta_7}{\partial C \partial D} > 0$ on their respective domains, which will help later in the proof.) Therefore, $\frac{\partial \Delta_7}{\partial A}$ is an increasing function with respect to B , C , D for all $B \geq 3$, $C \geq \frac{\alpha}{1-\alpha}$, $D \geq \frac{\alpha}{1-\alpha}$, $\alpha \in (\frac{2}{3}, \frac{3}{4}]$ with a minimum at $B = 3$, $C = D = \frac{\alpha}{1-\alpha} > 2$.

$$\frac{\partial \Delta_7}{\partial A} \Big|_{B=3, C=D=2} = 130\alpha^4 - 136\alpha^3 + 42\alpha^2 - 96\alpha + 72 > 0$$

for all $\alpha \in (\frac{2}{3}, \frac{3}{4}]$ so $\frac{\partial \Delta_7}{\partial A} > 0$ for $B \geq 3$, $C \geq \frac{\alpha}{1-\alpha}$, $D \geq \frac{\alpha}{1-\alpha}$, $\alpha \in (\frac{2}{3}, \frac{3}{4}]$. By the property that Δ_7 is symmetric with respect to A, B, C , we also have $\frac{\partial \Delta_7}{\partial B} > 0$ for $A \geq 3$, $C \geq \frac{\alpha}{1-\alpha}$, $D \geq \frac{\alpha}{1-\alpha}$, $\alpha \in (\frac{2}{3}, \frac{3}{4}]$ and $\frac{\partial \Delta_7}{\partial C} > 0$ for $A \geq 3$, $B \geq \frac{\alpha}{1-\alpha}$, $D \geq \frac{\alpha}{1-\alpha}$, $\alpha \in (\frac{2}{3}, \frac{3}{4}]$, and therefore, Δ_7 is an increasing function with respect to A, B, C . Meanwhile, because we know $\frac{\partial^2 \Delta_7}{\partial A \partial D} > 0$, $\frac{\partial^2 \Delta_7}{\partial B \partial D} > 0$, $\frac{\partial^2 \Delta_7}{\partial C \partial D} > 0$ on their respective domains, we know that $\frac{\partial \Delta_7}{\partial D}$ is an increasing function with respect to A, B, C for $A \geq 3$, $B \geq 3$, $C \geq \frac{\alpha}{1-\alpha}$, $\alpha \in (\frac{2}{3}, \frac{3}{4}]$ with its minimum at $A = 4$, $B = 3$, $C = \frac{\alpha}{1-\alpha} > 2$.

$$\frac{\partial \Delta_7}{\partial D} \Big|_{A=4, B=3, C=2} = -450\alpha^4 + 444\alpha^3 + 54\alpha^2 - 156\alpha + 144 > 0$$

for all $\alpha \in (\frac{2}{3}, \frac{3}{4}]$ so $\frac{\partial \Delta_7}{\partial D} > 0$ for $A \geq 4, B \geq 3, C \geq \frac{\alpha}{1-\alpha}, \alpha \in (\frac{2}{3}, \frac{3}{4}]$. It follows that Δ_7 is an increasing function with respect to D , too. We separate the proof over the region considered into the following cases.

Case (1). $A \geq 4, B \geq 3, C \geq D \geq 2.76$. It's easy to verify $\Delta_7 \geq 0$ in this subcase.

Case (2). $2.76 \geq D \geq \frac{\alpha}{1-\alpha}$. Hence $\frac{1}{A} + \frac{1}{B} + \frac{1}{C} \leq 1 - \frac{1}{D} \leq 1 - \frac{1}{2.76} = \frac{44}{69}$. $\frac{44}{69}A + \frac{44}{69}B + \frac{44}{69}C \leq 1$. Hence $\frac{44}{69}A \geq 3, \frac{44}{69}B \geq 2$. We obtain $A \geq \frac{207}{44}, B \geq \frac{69}{22}$. It is easy to verify $\Delta_7 \geq 0$ for $A \geq \frac{207}{44}, B \geq C \geq 3.41$ and $2.76 \geq D \geq \frac{\alpha}{1-\alpha}$.

Case (3). $2.76 \geq D \geq \frac{\alpha}{1-\alpha}$, and $D \leq C \leq 3.41$.

Hence $\frac{1}{A} + \frac{1}{B} \leq 1 - \frac{1}{C} - \frac{1}{D} \leq 1 - \frac{1}{2.76} - \frac{1}{3.41} = \frac{8104}{23529}$. Hence $\frac{8104}{23529}A \geq 2$. We obtain $A \geq \frac{23529}{4052}$.

It's easy to verify $\Delta_7 \geq 0$ for $A \geq \frac{23529}{4052}, B \geq 5.59, 2.76 \geq D \geq \frac{\alpha}{1-\alpha}$ and $D \leq C \leq 3.41$.

Case (4). When $2.76 \geq D \geq \frac{\alpha}{1-\alpha}, D \leq C \leq 3.41$ and $5.59 \geq B \geq \frac{69}{22}$.

$\frac{1}{A} \leq 1 - \frac{1}{B} - \frac{1}{C} - \frac{1}{D} \leq 1 - \frac{1}{5.59} - \frac{1}{3.41} - \frac{1}{2.76} = \frac{2177236}{13152711}$. Hence $A \geq \frac{13152711}{2177236}$. It is easy to verify $\Delta_7 \geq 0$ for $A \geq 11.5, \frac{69}{22} \leq B \leq 5.59, 2.76 \geq D \geq \frac{\alpha}{1-\alpha}$ and $D \leq C \leq 3.41$.

Case (5). We still need to consider $11.5 \geq A \geq \frac{13152711}{2177236}, 5.59 \geq B \geq \frac{69}{22}, 2.76 \geq D \geq \frac{\alpha}{1-\alpha}$, and $D \leq C \leq 3.41$.

From $\frac{1}{A} + \frac{1}{B} + \frac{1}{C} + \frac{1}{D} \leq 1$ we have the following:

$$\frac{1}{D} \leq 1 - \frac{1}{A} - \frac{1}{B} - \frac{1}{C} \leq 1 - \frac{1}{11.5} - \frac{1}{5.59} - \frac{1}{3.41} = \frac{1932999}{4384237} \text{ Hence } D \geq \frac{4384237}{1932999}.$$

$$\frac{1}{B} \leq 1 - \frac{1}{A} - \frac{1}{C} - \frac{1}{D} \leq 1 - \frac{1}{11.5} - \frac{1}{3.41} - \frac{1}{2.76} = \frac{6058}{23529}. \text{ Hence } B \geq \frac{23529}{6058}.$$

$$\frac{1}{C} \leq 1 - \frac{1}{A} - \frac{1}{B} - \frac{1}{D} \leq 1 - \frac{1}{11.5} - \frac{1}{5.59} - \frac{1}{2.76} = \frac{14342}{38571}. \text{ Hence } C \geq \frac{38571}{14342}.$$

It is easy to verify $\Delta_7 \geq 0$ for $11.5 \geq A \geq \frac{13152711}{2177236}, 5.59 \geq B \geq \frac{23529}{6058}, 3.41 \geq C \geq \frac{38571}{14342}$ and $2.76 \geq D \geq \frac{4384237}{1932999}$. Then, it follows that $\Delta_7 > 0$ for $A \geq 4, B \geq 3, C \geq D \geq \frac{\alpha}{1-\alpha}$ and $\alpha \in (\frac{2}{3}, \frac{3}{4}]$.

For subcase (b), the proof is similar to the proof in subcase IV(b). Because $P_4(2) > 0, (1, 1, 1, 1, 2)$ is the smallest positive solution for the level $k = 2$, and so, let α_1 be defined by $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq 1 - \frac{2}{e} \triangleq \alpha_1, \alpha_1 \in (\frac{1}{3}, \frac{1}{2}]$, since $e \in (3, 4]$. Let $A = a\alpha_1, B = b\alpha_1, C = c\alpha_1$ and $D = d\alpha_1$; then the following restrictions apply:

$$A \geq 4, B \geq 3, C \geq 2, D \geq \frac{2\alpha_1}{1 - \alpha_1},$$

since $D = d\alpha_1 \geq e\alpha_1 = \frac{2\alpha_1}{1-\alpha_1}$. With $\alpha_1 \in (\frac{1}{3}, \frac{1}{2}], \frac{2\alpha_1}{1-\alpha_1} \in (1, 2]$, so it is sufficient to show $\Delta_4 - 211 = \frac{1}{16(1-\alpha_1)\alpha_1^4}(\Delta_5 - 211(16)\alpha_1^4(1 - \alpha_1)) > 0$, hence to show that $\Delta_5 - 211(16)\alpha_1^4(1 - \alpha_1) > 0$ for $A \geq 4, B \geq 3, C \geq 2, D \geq \frac{2\alpha_1}{1-\alpha_1}$ and $\alpha_1 \in (\frac{1}{3}, \frac{1}{2}]$. Note that the term $-211(16)\alpha_1^4(1 - \alpha_1)$ only affects the

value of the function at the minimum, and therefore, the proof of this subcase is exactly the same as the one in subcase IV(b) except for the very last calculation, the calculation of the value of the function's minimum.

$$\begin{aligned} &\Delta_5 - 211(16)\alpha_1^4(1 - \alpha_1)|_{A=4, B=3, C=2, D=\frac{2\alpha_1}{1-\alpha_1}} \\ &= \frac{16\alpha_1(24 - 32\alpha_1 - 6\alpha_1^2 - 183\alpha_1^3 + 565\alpha_1^4 - 597\alpha_1^5 + 389\alpha_1^6)}{(\alpha_1 - 1)^2} > 0, \end{aligned}$$

for $\alpha_1 \in (\frac{1}{3}, \frac{1}{2}]$. Then, it follows that $\Delta_5 - 211(16)\alpha_1^4(1 - \alpha_1) > 0$ for $A \geq 4$, $B \geq 3$, $C \geq 2$, $D \geq \frac{2\alpha_1}{1-\alpha_1}$ and $\alpha_1 \in (\frac{1}{3}, \frac{1}{2}]$.

For subcase (c), $P_4(3) > 0$, so $(1, 1, 1, 1, 3)$ is the smallest positive solution on the level $k = 3$. Let α_2 be defined by $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq 1 - \frac{3}{e} \triangleq \alpha_2$, $\alpha_2 \in (0, \frac{1}{4}]$, since $e \in (3, 4]$, and let $A = a\alpha_2$, $B = b\alpha_2$, $C = c\alpha_2$ and $D = d\alpha_2$. Also, notice that $A \geq 4$, $B \geq 3$, $C \geq 2$, $D \geq 1$, for the same reasons listed in Case V. Then, from the estimate provided in [11], we know that

$$\begin{aligned} 5!P_5 &= 5!(P_4(1) + P_4(2) + P_4(3)) \\ &\leq 5 \left[\left(A \frac{2 + \alpha_2}{3\alpha_2} - 1 \right) \left(B \frac{2 + \alpha_2}{3\alpha_2} - 1 \right) \left(C \frac{2 + \alpha_2}{3\alpha_2} - 1 \right) \right. \\ &\quad \times \left(D \frac{2 + \alpha_2}{3\alpha_2} - 1 \right) - \left(D \frac{2 + \alpha_2}{3\alpha_2} - 1 \right)^4 + \left(D \frac{2 + \alpha_2}{3\alpha_2} \right) \\ &\quad \times \left(D \frac{2 + \alpha_2}{3\alpha_2} - 1 \right) \left(D \frac{2 + \alpha_2}{3\alpha_2} - 2 \right) \left(D \frac{2 + \alpha_2}{3\alpha_2} - 3 \right) \\ &\quad + \left(A \frac{1 + 2\alpha_2}{3\alpha_2} - 1 \right) \left(B \frac{1 + 2\alpha_2}{3\alpha_2} - 1 \right) \left(C \frac{1 + 2\alpha_2}{3\alpha_2} - 1 \right) \\ &\quad \times \left(D \frac{1 + 2\alpha_2}{3\alpha_2} - 1 \right) - \left(D \frac{1 + 2\alpha_2}{3\alpha_2} - 1 \right)^4 + \left(D \frac{1 + 2\alpha_2}{3\alpha_2} \right) \\ &\quad \times \left(D \frac{1 + 2\alpha_2}{3\alpha_2} - 1 \right) \left(D \frac{1 + 2\alpha_2}{3\alpha_2} - 2 \right) \left(D \frac{1 + 2\alpha_2}{3\alpha_2} - 3 \right) \\ &\quad + (A - 1)(B - 1)(C - 1)(D - 1) - (D - 1)^4 \\ &\quad \left. + D(D - 1)(D - 2)(D - 3) \right]. \end{aligned}$$

It is sufficient to show that the RHS of the inequality above is strictly less than the RHS of Theorem 2.4. Let Δ_8 be the difference by subtracting the RHS of the inequality above from the RHS of Theorem 2.4, substituting

$a = \frac{A}{\alpha_2}, b = \frac{B}{\alpha_2}, c = \frac{C}{\alpha_2}, d = \frac{D}{\alpha_2}, e = \frac{3}{1-\alpha_2}, \nu = 4$, and the former.

$$\begin{aligned} \Delta_8 &= \frac{1}{81(1-\alpha_2)\alpha_2^4} (77ABCD - \alpha_2(27ABC + 27ABD + 27ACD \\ &\quad + 27BCD + 34ABCD - 270D^3) - \alpha_2^2(63AB + 63AC + 63BC \\ &\quad - 54ABC + 63AD + 63BD - 54ABD + 63CD - 54ACD \\ &\quad - 54BCD + 40ABCD + 1125D^2 - 270D^3) + \alpha_2^3(243A + 243B \\ &\quad - 54AB + 243C - 54AC - 54BC + 1053D - 54AD - 54BD \\ &\quad - 54CD + 40ABCD - 675D^2) + \alpha_2^4(324A - 19521 + 324B \\ &\quad - 270AB + 324C - 270AC - 270BC + 270ABC + 1134D \\ &\quad - 270AD - 270BD + 270ABD - 270CD + 270ACD \\ &\quad + 270BCD - 290ABCD - 1350D^2 + 540D^3) + \alpha_2^5(19764 \\ &\quad - 810A - 810B + 630AB - 810C + 630AC + 630BC - 540ABC \\ &\quad - 2430D + 630AD + 630BD - 540ABD + 630CD - 540ACD \\ &\quad - 540BCD + 490ABCD + 3150D^2 - 1080D^3) \\ &= \frac{1}{81(1-\alpha_2)\alpha_2^4} \Delta_9. \end{aligned}$$

Now, we apply the partial differentiation test on Δ_9 for $A \geq 4, B \geq 3, C \geq 2, D \geq 1$ and $\alpha_2 \in (0, \frac{1}{4}]$.

$$\begin{aligned} \frac{\partial^4 \Delta_9}{\partial A \partial B \partial C \partial D} &= 77 - 34\alpha_2 - 40\alpha_2^2 + 40\alpha_2^3 - 290\alpha_2^4 + 490\alpha_2^5 \\ &> 0 \end{aligned}$$

for $\alpha_2 \in (0, \frac{2}{5})$. It follows that $\frac{\partial^3 \Delta_9}{\partial A \partial B \partial C}$ is an increasing function of D for $D \geq 1$ and $\alpha_2 \in (0, \frac{2}{5})$, with a minimum at $D = 1$:

$$\begin{aligned} \left. \frac{\partial^3 \Delta_9}{\partial A \partial B \partial C} \right|_{D=1} &= 77 - 61\alpha_2 + 14\alpha_2^2 + 40\alpha_2^3 - 20\alpha_2^4 - 50\alpha_2^5 \\ &> 0 \end{aligned}$$

for $\alpha_2 \in (0, \frac{2}{5})$, so $\frac{\partial^3 \Delta_9}{\partial A \partial B \partial C} > 0$ for $D \geq 1$ and $\alpha_2 \in (0, \frac{2}{5})$. Because $\frac{\partial^2 \Delta_9}{\partial A \partial B}$ is symmetric with respect to C and D , we also know that $\frac{\partial^3 \Delta_9}{\partial A \partial B \partial D} > 0$ for $C \geq 1$ and $\alpha_2 \in (0, \frac{2}{5})$, and thus, $\frac{\partial^2 \Delta_9}{\partial A \partial B}$ is an increasing function of C and D

for $C \geq D \geq 1$ and $\alpha_2 \in (0, \frac{2}{5})$ with a minimum at $C = D = 1$.

$$\frac{\partial^2 \Delta_9}{\partial A \partial B} \Big|_{C=D=1} = 77 - 88\alpha_2 + 5\alpha_2^2 - 14\alpha_2^3 - 20\alpha_2^4 + 40\alpha_2^5 > 0$$

for $\alpha_2 \in (0, \frac{2}{5})$, so $\frac{\partial^2 \Delta_9}{\partial A \partial B} > 0$ for $C \geq D \geq 1$ and $\alpha_2 \in (0, \frac{2}{5})$. Due to the fact that $\frac{\partial \Delta_9}{\partial A}$ is symmetric with respect to B, C and D , we also know that $\frac{\partial^2 \Delta_9}{\partial A \partial C} > 0$ for $B \geq D \geq 1$, $\alpha_2 \in (0, \frac{2}{5})$ and $\frac{\partial^2 \Delta_9}{\partial A \partial D} > 0$ for $B \geq C \geq 1$, $\alpha_2 \in (0, \frac{2}{5})$. Thus, $\frac{\partial \Delta_9}{\partial A}$ is an increasing function of B, C and D for $B \geq C \geq D \geq 1$ and $\alpha_2 \in (0, \frac{2}{5})$ with a minimum at $B = C = D = 1$.

$$\frac{\partial \Delta_9}{\partial A} \Big|_{B=C=D=1} = (1 - \alpha_2)^3(77 + 116\alpha_2 + 50\alpha_2^2) > 0$$

for $\alpha_2 \in (0, \frac{2}{5})$, so $\frac{\partial \Delta_9}{\partial A} > 0$ for $B \geq C \geq D \geq 1$ and $\alpha_2 \in (0, \frac{2}{5})$. Because Δ_9 is symmetric with respect to A, B and C , we also know that $\frac{\partial \Delta_9}{\partial B} > 0$ for $A \geq C \geq D \geq 1$, $\alpha_2 \in (0, \frac{2}{5})$ and $\frac{\partial \Delta_9}{\partial C} > 0$ for $A \geq B \geq D \geq 1$, $\alpha_2 \in (0, \frac{2}{5})$. In the meantime, we also have

$$\frac{\partial^3 \Delta_9}{\partial D^3} = 1620(1 - \alpha_2)(\alpha_2)(1 + 2\alpha_2)(1 + 2\alpha_2^2) > 0$$

for $\alpha_2 \in (0, \frac{2}{5})$, so $\frac{\partial^2 \Delta_9}{\partial D^2}$ is an increasing function of D for $D \geq 1$ and $\alpha_2 \in (0, \frac{2}{5})$, with a minimum at $D = 1$.

$$\frac{\partial^2 \Delta_9}{\partial D^2} \Big|_{D=1} = 90(1 - \alpha_2)(\alpha_2)(18 + 11\alpha_2 - 4\alpha_2^2 + 2\alpha_2^3) > 0$$

for $\alpha_2 \in (0, \frac{2}{5})$, so $\frac{\partial \Delta_9}{\partial D}$ is an increasing function of D for $D \geq 1$ and $\alpha_2 \in (0, \frac{2}{5})$; furthermore, because $\frac{\partial \Delta_9}{\partial D}$ is symmetric with respect to A, B, C and D , it is also an increasing function of A, B and C for $A \geq B \geq C \geq D \geq 1$, $\alpha_2 \in (0, \frac{2}{5})$, with a minimum at $A = B = C = D = 1$.

$$\frac{\partial \Delta_9}{\partial D} \Big|_{A=B=C=D=1} = (1 - \alpha_2)(77 + 772\alpha_2 - 735\alpha_2^2 - 1154\alpha_2^3 - 1390\alpha_2^5) > 0$$

for $\alpha_2 \in (0, \frac{2}{5})$, so it follows that Δ_9 is an increasing function of D for $D \geq 1$ and $\alpha_2 \in (0, \frac{2}{5})$. We also know that Δ_9 is an increasing function of A, B and C for $A \geq B \geq C \geq 1$ and $\alpha_2 \in (0, \frac{2}{5})$, and therefore we only need to check if $\Delta_9 > 0$ at the minimum $A = 4, B = 3, C = 2, D = 1$.

$$\Delta_9|_{A=4, B=3, C=2, D=1} = 3(1 - \alpha_2)(616 - 16\alpha_2 - 456\alpha_2^2 + 89\alpha_2^3 - 6308\alpha_2^4) > 0$$

for $\alpha_2 \in (0, \frac{1}{4}]$, so $\Delta_9 > 0$ for $A \geq 4, B \geq 3, C \geq 2, D \geq 1$ and $\alpha_2 \in (0, \frac{1}{4}]$. □

2.5. Case II

We will now analyze the case that occurs when $e > 4, e \notin \mathbb{Z}_+$. We must also keep in mind the hypothesis that the non-integral portion of e, β has to be either $\frac{e}{a}, \frac{e}{b}, \frac{e}{c}$, or $\frac{e}{d}$, which is shown by Theorem 2.5 below.

Theorem 2.5. *A polynomial $f(z_1, \dots, z_n)$ is weighted homogeneous of type (w_1, \dots, w_n) and has an isolated singularity at the origin. Suppose $w_{i_1} \geq w_{i_2} \geq \dots \geq w_{i_n}$ and w_{i_n} is not an integer, where $\{i_1, i_2, \dots, i_n\} = \{1, 2, \dots, n\}$. Let $w_{i_n} = [w_{i_n}] + \beta$ with $0 < \beta < 1$. Then, there exists a $j \in \{1, \dots, n - 1\}$ such that $\beta = \frac{w_{i_n}}{w_{i_j}}$.*

Proof. Case 1: $w_{i_1} = w_{i_2} = \dots = w_{i_n}$. This means f is homogeneous, which implies that $w_{i_1} = \dots = w_{i_n}$ are integers, and that contradicts our initial assumption that w_{i_n} is not an integer.

Case 2: $w_{i_1} \geq w_{i_2} \geq \dots \geq w_{i_{n-1}} > w_{i_n}$. Write $f = \sum x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_n}^{\alpha_n}$, where $wt(x_{i_j}) = w_{i_j}$ and $1 \leq j \leq n$. Then, $\frac{\alpha_1}{w_{i_1}} + \dots + \frac{\alpha_{n-1}}{w_{i_{n-1}}} + \frac{\alpha_n}{w_{i_n}} = 1$. Since w_{i_n} is not an integer, $x_{i_n}^{\alpha_n} \notin \text{supp}(f)$. Because f has an isolated singularity at the origin and by the lemma in [17], there exists a $j \in \{1, 2, \dots, n - 1\}$ such that $x_{i_j} \cdot x_{i_n}^{\alpha_n} \in \text{supp}(f)$. Therefore,

$$\begin{aligned} \frac{1}{w_{i_j}} + \frac{\alpha_n}{w_{i_n}} &= 1 \\ \Rightarrow \frac{w_{i_n}}{w_{i_j}} &= w_{i_n} - \alpha_n \\ \Rightarrow \beta &= \frac{w_{i_n}}{w_{i_j}}. \end{aligned}$$

Case 3: $w_{i_1} \geq \dots \geq w_{i_j} > w_{i_{j+1}} = \dots = w_{i_n}$, where $1 \geq j \geq n - 2$.

Write $f = \sum x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_n}^{\alpha_n}$, where $wt(x_{i_j}) = w_{i_j}$ and $1 \leq j \leq n$. Then, $\frac{\alpha_1}{w_{i_1}} + \dots + \frac{\alpha_j}{w_{i_j}} + \frac{\alpha_{j+1} + \dots + \alpha_n}{w_{i_n}} = 1$. Since w_{i_n} is not an integer, $x_{i_{j+1}}^{\alpha_{j+1}} x_{i_{j+2}}^{\alpha_{j+2}} \dots x_{i_n}^{\alpha_n} \notin \text{supp}(f)$, and in particular, $x_{i_n}^{\alpha_n} \notin \text{supp}(f)$. Because f has an isolated singularity at the origin and by the lemma in [17], there exists a t such that $x_{i_t} x_{i_n}^{\alpha_n} \in \text{supp}(f)$, where $1 \leq t \leq j$. Therefore,

$$\begin{aligned} & \frac{1}{w_{i_t}} + \frac{\alpha_n}{w_{i_n}} = 1 \\ \Rightarrow & \frac{w_{i_n}}{w_{i_t}} = w_{i_n} - \alpha_n \\ \Rightarrow & \beta = \frac{w_{i_n}}{w_{i_t}}, \end{aligned}$$

and therefore the theorem is proved. □

Theorem 2.6. *Let $a \geq b \geq c \geq d \geq e > 4$ be real numbers, where e is not an integer. Consider $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1$. Let P_5 be the number of positive integral solutions of the above inequality; i.e., $P_5 = \#\{(x, y, z, v, w) \in \mathbb{Z}_+^5 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{v}{d} + \frac{w}{e} \leq 1\}$. Suppose e is not an integer and $e = [e] + \beta$ where β is either $\frac{e}{a}, \frac{e}{b}, \frac{e}{c}$ or $\frac{e}{d}$. Define $\mu = (a - 1)(b - 1)(c - 1)(d - 1)(e - 1)$. Then,*

$$\begin{aligned} (2.1) \quad 120P_5 &< \mu - (5v^4 - 25v^3 + 40v^2 - 19v - 1)|_{v=e-\beta+1} \\ &= abcde - (abcd + abce + abde + acde + bcde) \\ &\quad + (abc + abd + abe + acd + ace + ade + bcd + bce + bde + cde) \\ &\quad - (ab + ac + ad + ae + bc + bd + be + cd + ce + de) \\ &\quad + (a + b + c + d + e) - 1 - 5e^4 + 5e^3 + 5e^2 - 6e - 1 \\ &\quad + \beta(20e^3 - 15e^2 - 10e + 6) - \beta^2(30e^2 + 15e + 5) \\ &\quad + \beta^3(20e - 5) - 5\beta^4 + 1. \end{aligned}$$

Proof. By level $w = k$, we shall mean the intersection of the tetrahedron in Theorem 2.6 with the hyperplane $w = k$. The points within this intersection for level $w = k$ are in following four-dimensional tetrahedron

$$\frac{x}{a(1 - \frac{k}{e})} + \frac{y}{b(1 - \frac{k}{e})} + \frac{z}{c(1 - \frac{k}{e})} + \frac{v}{d(1 - \frac{k}{e})} \leq 1,$$

we can use GLY sharp upper estimate to bound the number of positive integral solutions at level $w = k$ if $d(1 - \frac{k}{e}) \geq 3$ or $d(1 - \frac{k}{e}) \geq 2$ with the

condition that there exist positive integral solutions at this level. By summing these upper bounds in various levels, this leads us to consider the following function:

$$\begin{aligned}
 &g(a, b, c, d, e, n) \\
 &= 5 \sum_{k=1}^n \left[abcd \left(1 - \frac{k}{e}\right)^4 - \frac{3}{2}(abc + abd + acd + bcd) \left(1 - \frac{k}{e}\right)^3 \right. \\
 &\quad \left. + \frac{11}{3}(ab + bc + ac) \left(1 - \frac{k}{e}\right)^2 - 2(a + b + c) \left(1 - \frac{k}{e}\right) \right].
 \end{aligned}$$

Define $S_5 = abcde$, $S_4 = abcd$, $S_3 = abc + abd + acd + bcd$, $S_2 = ab + ac + bc$ and $S_1 = a + b + c$. Then we have

$$\begin{aligned}
 &g(a, b, c, d, e, n) \\
 &= S_5 \left(\frac{n}{e}\right)^5 + \left[S_4 \left(\frac{1}{2e^4} - \frac{1}{e^3}\right) + \frac{3}{8} \frac{S_3}{e^3} \right] 5n^4 \\
 &\quad + \left[S_4 \left(\frac{1}{3e^4} - \frac{2}{e^3} + \frac{2}{e^2}\right) + 3S_3 \left(\frac{1}{4e^3} - \frac{1}{2e^2}\right) + \frac{11}{9} \frac{S_2}{e^2} \right] 5n^3 \\
 &\quad + \left[S_4 \left(\frac{-1}{e^3} + \frac{3}{e^2} - \frac{2}{e}\right) + 3S_3 \left(\frac{1}{8e^3} - \frac{3}{4e^2} + \frac{3}{4e}\right) \right. \\
 &\quad \left. + 11S_2 \left(\frac{1}{6e^2} - \frac{1}{3e}\right) + \frac{S_1}{e} \right] 5n^2 \\
 &\quad + \left[S_4 \left(\frac{-1}{30e^4} + \frac{1}{e^2} - \frac{2}{e}\right) + \frac{3}{4} S_3 \left(\frac{-1}{e^2} + \frac{3}{e}\right) + \frac{11}{3} S_2 \left(\frac{1}{6e^2} - \frac{1}{e}\right) + \frac{S_1}{e} \right] 5n \\
 &\quad + \left(5S_4 - \frac{15}{2} S_3 + \frac{55}{3} S_2 - 10S_1 \right) n.
 \end{aligned}$$

For a fixed a, b, c, d and e . $g(a, b, c, d, e, n)$ is a function of n and we denote it by $g(n)$.

We slice the five-dimensional simplex into the hyperplanes, $w = 1, w = 2, \dots, w = e - \beta - 1$ and $w = e - \beta$. It is obvious there are no positive integral solutions on level $w = e - \beta$. At level $w = e - \beta - 2$, the defining inequality of the simplex becomes $\frac{x}{e(\beta+2)} + \frac{y}{e(\beta+2)} + \frac{z}{e(\beta+2)} + \frac{v}{e(\beta+2)} \leq 1$ — note that $\frac{d}{e}(\beta + 2) > 3$ if $\beta = \frac{e}{d}$, and hence we can use GLY sharp upper estimate to bound the number of positive integral points at this level and for the levels lower than this level. At level $w = e - \beta - 1$, the defining inequality of the simplex becomes $\frac{x}{e(\beta+1)} + \frac{y}{e(\beta+1)} + \frac{z}{e(\beta+1)} + \frac{v}{e(\beta+1)} \leq 1$, We divide our

proof in two cases depending on whether the level $w = e - \beta - 1$ tetrahedron

$$(2.2) \quad \frac{x}{\frac{a}{e}(\beta + 1)} + \frac{y}{\frac{b}{e}(\beta + 1)} + \frac{z}{\frac{c}{e}(\beta + 1)} + \frac{v}{\frac{d}{e}(\beta + 1)} \leq 1$$

has positive integral solutions.

Case (A). $\frac{a}{e}(1 + \beta) < 4$ or $\frac{b}{e}(1 + \beta) \leq 3$ or $\frac{c}{e}(1 + \beta) \leq 2$ or $\frac{d}{e}(1 + \beta) \leq 1$.

In this case, there are no positive integral solutions at level $w = e - \beta - 1$.

Case (B). $\frac{a}{e}(1 + \beta) \geq 4$, $\frac{b}{e}(1 + \beta) > 3$, $\frac{c}{e}(1 + \beta) > 2$, and $\frac{d}{e}(1 + \beta) > 1$.

In this case, (2.2) may have positive integral solutions.

Case (A).

Case (A1). $\beta = \frac{e}{d}$.

$\beta = \frac{e}{d}$ implies $\frac{d}{e}(\beta + 2) > 3$ — hence we can use the GLY sharp upper estimate to bound the number of positive integral points at this level and for the levels lower than this level. Without loss of generality, we can assume there exist positive integral solutions at level $w = e - \beta - 2$ since the GLY upper estimate is positive at this level. Let $\Delta_{10} = \text{RHS of (2.1)} - g(e - \beta - 2)$, and $A = \frac{a}{e}, B = \frac{b}{e}, C = \frac{c}{e}, D = \frac{d}{e}$ and $\beta = \frac{1}{D}$ and substitute into Δ_{10} , then we have $A \geq B \geq C \geq D \geq 1$, and $e > 4$. Let $\Delta_{11} = 72D^4\Delta_{10}$. Now all we have to do is partial differentiate to ensure that Δ_{11} is positive throughout the domain considered. If the level $w = e - \beta - 2$ has positive integral solutions then $\frac{a}{e}(2 + \beta) \geq 4$, $\frac{b}{e}(2 + \beta) > 3$, $\frac{c}{e}(2 + \beta) > 2$ and $\frac{d}{e}(2 + \beta) > 1$. We get $A \geq \frac{4D}{2D+1}, B > \frac{3D}{2D+1}, C \geq 1$, and $D \geq 1$. We find that Δ_{11} is an increasing function with respect to A, B, C and D for $A \geq \frac{4D}{2D+1}, B > \frac{3D}{2D+1}, C \geq 1$ and $D \geq 1$, and $e > 4$. Evaluated at the minimum $A = \frac{4D}{2D+1}, B = \frac{3D}{2D+1}, C = 1$ and $D = 1$, $\Delta_{11} > 0$, so we can conclude that $\Delta_{10} > 0$ for this subcase.

$$\begin{aligned} \Delta_{10} = & \frac{1}{72e^4}(-180e^3a\beta^2 - 72ade^4 + 588bce^4 - 368ace^5 \\ & + 135abce^3 + 135abde^3 + 135acde^3 + 135bcde^3 \\ & - 360e^3a - 360e^3b - 360e^3c - 220e^3ab \\ & - 220e^3ac - 220e^3bc - 72e^4 + 360abcd - 120abcde^3 - 72bde^4 \\ & + 1320e^2ab + 1320e^2ac + 1320e^2bc - 540abce - 540abde \\ & - 540acde - 540bcde + 12abcde - 540e^3a\beta - 180e^3b\beta^2 - 540e^3b\beta \\ & - 180e^3c\beta^2 - 540e^3c\beta - 198abce^4 - 198abde^4 - 198acde^4 \\ & - 198bcde^4 + 63abce^5 + 63abde^5 + 63acde^5 + 63bcde^5 + 1980e^2ac\beta^2 \\ & + 2860e^2ac\beta + 1980e^2bc\beta^2 + 2860e^2bc\beta + 440e^2ab\beta^3 + 1980e^2ab\beta^2 \end{aligned}$$

$$\begin{aligned}
 &+ 2860e^2ab\beta + 72ade^5 - 368bce^5 - 368abe^5 - 72cde^4 - 1080e^6\beta \\
 &- 360\beta^4e^4 - 360\beta^3e^4 + 1440e^5\beta^3 - 2160e^6\beta^2 + 1440e^7\beta \\
 &+ 72cde^5 + 1080e^5\beta^2 + 360\beta^2e^4 - 720e^5\beta - 72de^5 \\
 &+ 108ce^5 + 108be^5 + 108ae^5 + 432\beta e^4 + 72de^4 - 108ce^4 \\
 &- 108be^4 - 360e^5 + 360e^7 + 360e^6 - 360e^8 - 108ae^4 + 72bde^5 \\
 &+ 588ace^4 + 588abe^4 + 108abcde^4 + 2160abcd\beta^2 + 1560abcd\beta^3 \\
 &+ 540abcd\beta^4 + 1428abcd\beta + 72abcd\beta^5 - 810abce\beta^3 - 1755abce\beta^2 \\
 &- 1620abce\beta + 440e^2ac\beta^3 + 440e^2bc\beta^3 - 810abde\beta^3 - 1755abde\beta^2 \\
 &- 1620abde\beta - 810acde\beta^3 - 1755acde\beta^2 - 1620acde\beta - 810bcde\beta^3 \\
 &- 1755bcde\beta^2 - 1620bcde\beta - 135abce\beta^4 - 135abde\beta^4 \\
 &- 135acde\beta^4 - 135bcde\beta^4),
 \end{aligned}$$

$$\begin{aligned}
 \Delta_{11} = &-360 - 360D - 72Ae^2D^5 - 108CeD^4 - 540CD^3 + 72Ce^3D^5 \\
 &- 180BD^2 - 72Ce^2D^5 - 72Be^2D^5 - 540BD^3 - 360CD^4 - 180CD^2 \\
 &+ 108Ae^2D^4 - 360BD^4 - 360AD^4 - 540AD^3 - 108BeD^4 \\
 &+ 108Ce^2D^4 + 72Ae^3D^5 + 108Be^2D^4 + 72Be^3D^5 - 360D^4e^4 \\
 &- 180AD^2 - 108AeD^4 + 1440De + 135Be^2CD^5 - 63ABC \\
 &- 300ACD^4 + 63Ae^4BD^5 - 198Ae^3BD^5 - 368Ae^3BD^4 \\
 &+ 135Ae^2BD^5 + 588Ae^2BD^4 - 220AeBD^4 - 540ABD^5 \\
 &- 300ABD^4 + 1105ABD^3 + 1170ABD^2 + 305ABD + 63Ae^4CD^5 \\
 &- 198Ae^3CD^5 - 368Ae^3CD^4 + 135Ae^2CD^5 + 588Ae^2CD^4 \\
 &- 220AeCD^4 - 540ACD^5 + 1105ACD^3 + 1170ACD^2 + 305ACD \\
 &+ 1170BCD^2 + 1105BCD^3 + 305BCD \\
 &+ 588Be^2CD^4 - 220BeCD^4 - 300BCD^4 - 540BCD^5 \\
 &- 198Be^3CD^5 + 63Be^4CD^5 - 368Be^3CD^4 + 360D^4e^3 - 72D^5e^2 \\
 &+ 360D^4e^2 + 1440D^3e^3 - 720D^3e - 360D^4e + 1080D^2e - 1080D^3e^2 \\
 &+ 432D^3 + 360D^2 - 72D^4 - 2160D^2e^2 + 72D^5e + 108Ae^4BCD^5 \\
 &+ 63Ae^4BCD^4 - 120Ae^3BCD^5 - 198Ae^3BCD^4 + 135Ae^2BCD^4 \\
 &+ 12AeBCD^5 + 360ABCD^5 + 888ABCD^4 + 540ABCD^3 \\
 &- 195ABCD^2 - 270ABCD.
 \end{aligned}$$

$$\begin{aligned}
 \Delta_{11}|_{A=\frac{4D}{2D+1}, B=\frac{3D}{2D+1}, C=1, D=1} \\
 = 4460 + \frac{1084}{3}e - \frac{1378}{3}e^3 - e^2 + 99e^4 \geq 0 \text{ for } e \geq 4
 \end{aligned}$$

Therefore we have $\Delta_{11} \geq 0$ on the region considered.

Case (A2). $\beta = \frac{e}{e}$. If $\frac{d}{e}(\beta + 2) \geq 3$, then the GLY sharp upper estimate or GLY rough upper estimate can be used; if $1 < \frac{d}{e}(\beta + 2) < 3$, we must use GLY rough upper estimate. Let

$$\begin{aligned} \Delta_{12} = & \text{RHS of (2.1)} - g(e - \beta - 3) \\ & - 5(A(\beta + 2) - 1)(B(\beta + 2) - 1)(C(\beta + 2) - 1)(D(\beta + 2) - 1). \end{aligned}$$

We use GLY rough upper estimate at level $w = e - \beta - 2$, and without loss of generality, we can assume there exist positive integral solutions at level $w = e - \beta - 2$ since the GLY rough upper estimate is positive at this level. From level $w = 1$ to level $w = e - \beta - 3$, $d(1 - \frac{w}{e}) \geq 3$ and so the GLY sharp upper estimate is used. If the level $w = e - \beta - 2$ has positive integral solutions then we have $\frac{a}{e}(2 + \beta) \geq 4$, $\frac{b}{e}(2 + \beta) > 3$, $\frac{c}{e}(2 + \beta) > 2$ and $\frac{d}{e}(2 + \beta) > 1$. Hence $A \geq \frac{4C}{2C+1}$, $B > \frac{3C}{2C+1}$, $C \geq 1$ and $D \geq 1$. Let $A = \frac{a}{e}$, $B = \frac{b}{e}$, $C = \frac{c}{e}$, $D = \frac{d}{e}$ and $\beta = \frac{1}{C}$ and substitute into Δ_{12} .

Let $\Delta_{13} = 72C^4\Delta_{12}$. We find that Δ_{13} is an increasing function with respect to A , B , C and D for $A \geq \frac{4C}{2C+1}$, $B > \frac{3C}{2C+1}$, $C \geq 1$ and $D \geq 1$ and $e > 4$. Evaluated at the minimum $A = \frac{4C}{2C+1}$, $B = \frac{3C}{2C+1}$, $C = 1$, and $D = 1$, $\Delta_{13} > 0$, so we can conclude that $\Delta_{12} > 0$ for $e \geq 4.194$, and we consider $4 < e < 4.194$ in Case (A5).

$$\begin{aligned} \Delta_{12} = & \frac{1}{72e^4}(6120abcd - 120abcde^3 + 135abce^3 + 135abde^3 + 135acde^3 \\ & + 135bcde^3 - 360e^4A\beta^2B - 1440e^4AB + 6600e^2ab + 6600e^2ac \\ & + 6600e^2bc + 12abcde - 4860abce - 4860abde - 4860acde - 4860bcde \\ & - 180e^3a\beta^2 + 72de^4 - 108ce^4 + 72bde^5 - 1080e^6\beta + 1440e^5\beta^3 \\ & - 2160e^6\beta^2 + 1440e^7\beta + 1080e^5\beta^2 + 360\beta^2e^4 - 720e^5\beta - 72de^5 \\ & + 108ce^5 + 108be^5 + 108ae^5 - 360\beta^4e^4 - 360\beta^3e^4 + 432\beta e^4 \\ & - 108be^4 - 108ae^4 - 198abce^4 - 198abde^4 - 198acde^4 - 198bcde^4 \\ & + 63abce^5 + 63abde^5 + 63acde^5 + 63bcde^5 - 360e^4B\beta^2C \\ & - 1440e^4A\beta D - 1440e^4B\beta C - 360e^4B\beta^2D - 360e^5 \\ & + 360e^7 + 360e^6 - 360e^8 + 108abcde^4 + 12948abcd\beta + 4440abcd\beta^3 \\ & + 10800abcd\beta^2 + 900abcd\beta^4 + 72abcd\beta^5 + 720e^4D + 720e^4A \\ & + 720e^4B + 720e^4C - 1440e^4A\beta B - 1440e^4A\beta C - 360e^4A\beta^2D \\ & + 2880e^4ACD - 1440e^4B\beta D + 2880e^4ABD - 360e^4C\beta^2D \\ & - 1440e^4C\beta D + 2880e^4BCD + 2880e^4ABC - 360e^4A\beta^2C \end{aligned}$$

$$\begin{aligned}
 &+ 360e^4 A\beta^3 BC + 2160e^4 A\beta^2 BC + 4320e^4 A\beta BC + 360e^4 A\beta^3 BD \\
 &+ 360e^4 A\beta^3 CD + 360e^4 B\beta^3 CD + 2160e^4 A\beta^2 BD + 4320e^4 A\beta BD \\
 &+ 2160e^4 A\beta^2 CD + 4320e^4 A\beta CD - 5760e^4 ABCD + 2160e^4 B\beta^2 CD \\
 &+ 4320e^4 B\beta CD - 360e^4 A\beta^4 BCD - 2880e^4 A\beta^3 BCD \\
 &- 8640e^4 A\beta^2 BCD - 11520e^4 A\beta BCD - 1350abce\beta^3 - 4995abce\beta^2 \\
 &- 8100abce\beta + 3300e^2 ab\beta^2 + 8140e^2 ab\beta - 1350abde\beta^3 - 4995abde\beta^2 \\
 &- 8100abde\beta - 1350acde\beta^3 - 4995acde\beta^2 - 8100acde\beta - 1350bcde\beta^3 \\
 &- 4995bcde\beta^2 - 8100bcde\beta - 135abce\beta^4 - 135abde\beta^4 - 135acde\beta^4 \\
 &- 135bcde\beta^4 + 3300e^2 ac\beta^2 + 8140e^2 ac\beta + 3300e^2 bc\beta^2 + 8140e^2 bc\beta \\
 &+ 440e^2 ab\beta^3 + 440e^2 ac\beta^3 + 440e^2 bc\beta^3 + 588bce^4 - 220e^3 ab \\
 &- 220e^3 ac - 220e^3 bc - 180e^3 b\beta^2 + 588abe^4 - 1080e^3 a - 1080e^3 b \\
 &- 1080e^3 c - 900e^3 c\beta - 432e^4 - 368ace^5 - 368bce^5 - 72bde^4 - 72cde^4 \\
 &+ 72cde^5 - 368abe^5 - 1440e^4 BC - 1440e^4 AD - 1440e^4 BD \\
 &- 1440e^4 CD + 360e^4 A\beta + 360e^4 B\beta + 360e^4 C\beta + 360e^4 D\beta - 72ade^4 \\
 &- 180e^3 c\beta^2 + 72ade^5 - 900e^3 a\beta - 900e^3 b\beta + 588ace^4 - 1440e^4 AC), \\
 \Delta_{13} = &-360 + 72DeC^4 + 588Ae^2C^5 + 5160AC^5 + 72C^5e^3D \\
 &- 368Ae^3C^5 - 720DC^4 + 5160BC^5 - 108BeC^4 + 6340BC^4 \\
 &- 1080e^2C^3 - 360C - 450ABCD + 1440Ce - 220eAC^5 \\
 &- 5220AC^4D + 1380ABC^4 - 5220BDC^4 - 1980AC^5D \\
 &- 1980BC^5D - 1980ABC^5 - 4275ADC^3 - 1350BDC^2 \\
 &+ 3865ABC^3 - 1350ADC^2 - 4275BDC^3 + 1950ABC^2 \\
 &+ 135Ae^2BC^5 - 198Ae^3BC^5 + 63Ae^4BC^5 + 360e^2C^4 - 360e^4C^4 \\
 &+ 360e^3C^4 - 108C^5e + 108C^5e^2 - 2160e^2C^2 + 1440e^3C^3 - 360eC^4 \\
 &+ 1080eC^2 - 720eC^3 + 252C^3 + 360C^2 - 360C^5 - 972C^4 \\
 &- 552ABDC^4 + 360ABC^5D - 1275ABDC^2 - 1620ABDC^3 \\
 &- 220eABC^4 + 588Ae^2BC^4 - 368Ae^3BC^4 \\
 &+ 72Ae^3DC^4 + 135Ae^2C^5D - 198Ae^3C^5D + 63Ae^4C^5D \\
 &- 72Ae^2DC^4 + 135Be^2C^5D + 72Be^3DC^4 - 198Be^3C^5D \\
 &+ 63Be^4C^5D - 72Be^2DC^4 - 120Ae^3BC^5D + 135Ae^2BDC^4 \\
 &+ 12AeBC^5D - 198Ae^3BDC^4 + 63Ae^4BDC^4 + 108Ae^4BC^5D \\
 &- 368Be^3C^5 + 2400AC^3 + 260AC^2 + 108Ae^2C^4 - 135ACD \\
 &- 63ABD - 108AeC^4 + 260BC^2 + 108Be^2C^4 + 588Be^2C^5
 \end{aligned}$$

$$\begin{aligned}
& -72C^5e^2D - 72De^2C^4 + 2400BC^3 - 135BCD - 1440C^5D \\
& + 6340AC^4 + 305ABC - 220eBC^5,
\end{aligned}$$

$$\begin{aligned}
\Delta_{13}|_{A=\frac{4C}{2C+1}, B=\frac{3C}{2C+1}, C=1, D=1} \\
= 1760 + \frac{1084}{3}e - \frac{1378}{3}e^3 - e^2 + 99e^4 > 0 \quad \text{for } e \geq 4.194
\end{aligned}$$

Therefore, we have $\Delta_{12} \geq 0$ for $e \geq 4.194$.

Case (A3). $\beta = \frac{e}{b}$. Similar to Case(A2). Substitute $\beta = \frac{e}{b}$ into Δ_{12} . Let $\Delta_{14} = 72B^4\Delta_{12}$. We find that Δ_{14} is an increasing function with respect to A, B, C and D for $A \geq \frac{4B}{2B+1}$, $B \geq 1$, $C \geq 1$ and $D \geq 1$ and $e > 4$. Evaluated at the minimum $A = \frac{4B}{2B+1}$, $B = 1$, $C = 1$ and $D = 1$, $\Delta_{14} > 0$ for $e \geq 4.194$, so we can conclude that $\Delta_{12} > 0$ for $e \geq 4.194$, and we consider $4 < e < 4.194$ in Case (A5).

$$\begin{aligned}
\Delta_{14} = & -360 + 72DeB^4 + 6340CB^4 + 5160B^5C + 260AB^2 - 108AeB^4 \\
& - 1440B^5D - 220eAB^5 + 260CB^2 - 368Ae^3B^5 + 2400AB^3 \\
& + 6340AB^4 - 450ABCD + 1440Be - 368B^5e^3C + 108Ce^2B^4 \\
& - 220eB^5C + 108Ae^2B^4 - 5220ADB^4 - 5220CDB^4 \\
& + 1380ACB^4 - 1980AB^5D - 1980B^5CD - 1980AB^5C \\
& + 3865ACB^3 - 1350ADB^2 - 1350CDB^2 - 4275CDB^3 \\
& + 1950ACB^2 - 4275AB^3D + 135Ae^2B^5C - 198Ae^3B^5C \\
& + 63Ae^4B^5C - 220eACB^4 - 368Ae^3CB^4 + 588Ae^2CB^4 \\
& - 198B^5e^3CD + 72Ce^3DB^4 + 135Ae^2B^5D + 63Ae^4B^5D \\
& - 72Ae^2DB^4 + 72Ae^3DB^4 + 135B^5e^2CD + 63B^5e^4CD \\
& + 360e^2B^4 - 1080e^2B^3 - 360eB^4 - 720eB^3 + 360B^2 + 1440e^3B^3 \\
& - 2160e^2B^2 + 1080eB^2 + 108B^5e^2 + 360e^3B^4 - 360B^5 - 972B^4 \\
& + 252B^3 - 1275ACDB^2 - 1620ACDB^3 - 72Ce^2DB^4 \\
& - 198Ae^3B^5D - 120Ae^3B^5CD + 135Ae^2CDB^4 + 12AeB^5CD \\
& - 198Ae^3CDB^4 + 63Ae^4CDB^4 + 108Ae^4B^5CD - 63ACD \\
& - 72De^2B^4 - 720DB^4 - 135ABD - 108CeB^4 + 2400CB^3 \\
& + 360AB^5CD - 360e^4B^4 + 72B^5e^3D + 588B^5e^2C - 135BCD \\
& - 72B^5e^2D + 5160AB^5 + 588Ae^2B^5 - 360B + 305ABC \\
& - 108B^5e - 552ACDB^4,
\end{aligned}$$

$$\begin{aligned} \Delta_{14} \Big|_{A=\frac{4B}{2B+1}, B=1, C=1, D=1} \\ = 1760 + \frac{1084}{3}e - \frac{1378}{3}e^3 - e^2 + 99e^4 > 0 \quad \text{for } e \geq 4.194 \end{aligned}$$

Therefore, we have $\Delta_{12} \geq 0$ for $e \geq 4.194$.

Case (A4). $\beta = \frac{e}{a}$. Similar to Case (A2). Substitute $\beta = \frac{e}{a}$ into Δ_{12} . Let $\Delta_{15} = 72A^4\Delta_{12}$. We find that Δ_{15} is an increasing function with respect to A, B, C and D for $A \geq 1.5, B \geq \frac{3A}{2A+1}, C \geq 1$ and $D \geq 1$ and $e > 4$. Evaluated at the minimum $A = 1.5, B = \frac{3A}{2A+1}, C = 1$ and $D = 1, \Delta_{15} > 0$, for $e \geq 4.896$, so we can conclude that $\Delta_{12} > 0$ for $e \geq 4.896$, and we consider $4 < e < 4.896$ at Case (A5).

$$\begin{aligned} \Delta_{15} = & -360 - 108BeA^4 - 220eA^5B - 450ABCD + 1440Ae + 1080eA^2 \\ & + 5160A^5B - 1440A^5D + 5160A^5C + 588A^5e^2B + 260BA^2 \\ & - 368A^5e^3B - 108CeA^4 + 108Ce^2A^4 - 220eA^5C + 260CA^2 \\ & - 368A^5e^3C + 588A^5e^2C + 72DeA^4 - 72De^2A^4 - 72A^5e^2D \\ & + 72A^5e^3D + 2400A^3C + 63A^5e^4BC - 972A^4 - 1080e^2A^3 \\ & - 2160e^2A^2 - 360e^4A^4 + 360e^2A^4 + 360e^3A^4 - 360eA^4 \\ & + 108A^5e^2 - 108A^5e - 720eA^3 + 252A^3 + 360A^2 - 360A^5 \\ & + 1440e^3A^3 + 1950BCA^2 + 1380BCA^4 - 5220BDA^4 - 5220CDA^4 \\ & - 1980A^5CD - 1980A^5BD - 1980A^5BC + 3865BCA^3 - 552BCDA^4 \\ & + 360A^5BCD - 4275BDA^3 - 1350CDA^2 - 4275CDA^3 - 1350A^2BD \\ & + 135A^5e^2BC - 198A^5e^3BC + 588Be^2CA^4 - 220eBCA^4 \\ & - 368Be^3CA^4 + 72Be^3DA^4 - 198A^5e^3BD + 63A^5e^4BD \\ & - 72Be^2DA^4 + 135A^5e^2BD - 198A^5e^3CD + 63A^5e^4CD - 72Ce^2DA^4 \\ & + 72Ce^3DA^4 + 135A^5e^2CD - 1275BCDA^2 - 1620A^3BCD \\ & - 120A^5e^3BCD + 135Be^2CDA^4 + 12A^5eBCD - 198Be^3CDA^4 \\ & + 63Be^4CDA^4 + 108A^5e^4BCD - 720DA^4 - 135ACD - 135ABD \\ & + 6340CA^4 - 360A + 6340BA^4 - 63BCD + 108Be^2A^4 + 305ABC \\ & + 2400A^3B, \end{aligned}$$

$$\begin{aligned} \Delta_{15} \Big|_{A=1.5, B=\frac{3A}{2A+1}, C=1, D=1} = & -7069.781e^3 - 5980.5e + 23593.5 \\ & + 10551.516e^2 + 1013.766e^4 > 0 \quad \text{for } e \geq 4.896. \end{aligned}$$

Case (A5). $e \in (4, 4.9)$. For $e \in (4, 4.9)$, consider the following two subcases:

- (a) The level $w = e - \beta - 2 = 2$ has no positive integral solutions:
 $P_4(2) = 0$.
- (b) The level $w = e - \beta - 2 = 2$ has positive integral solutions: $P_4(2) > 0$.

In Case (A5) the RHS of (2.1) becomes

$$(2.3) \quad \begin{aligned} \mu - (5v^4 - 25v^3 + 40v^2 - 19v - 1)|_{v=5} \\ = (a - 1)(b - 1)(c - 1)(d - 1)(e - 1) - 904. \end{aligned}$$

For subcase (a), the proof is almost exactly the same as the proof presented in Case IV. Because $P_5 > 0$, the set of positive integral solutions includes the point $(1, 1, 1, 1, 1)$, and let $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq 1 - \frac{1}{e} \triangleq \alpha, \alpha \in (\frac{3}{4}, \frac{39}{49})$, since $e \in (4, 4.9)$. Let $A = a\alpha, B = b\alpha, C = c\alpha$ and $D = d\alpha$; then, the following restrictions apply:

$$A \geq 4, B \geq 3, C \geq 2, D \geq \frac{\alpha}{1 - \alpha},$$

because $D = d\alpha \geq e\alpha = \frac{\alpha}{1 - \alpha}$. With $\alpha \in (\frac{3}{4}, \frac{39}{49}), \frac{\alpha}{1 - \alpha} \in (3, 3.9)$.

$$(2.4) \quad \begin{aligned} 5!P_5 \leq 5[(A - 1)(B - 1)(C - 1)(D - 1) \\ - (D - 1)^4 + D(D - 1)(D - 2)(D - 3)]. \end{aligned}$$

Let $\Delta_{16} = \text{RHS of (2.3)} - \text{RHS of (2.4)}$, and substitute $A = \frac{a}{e}, B = \frac{b}{e}, C = \frac{c}{e}$ and $D = \frac{d}{e}$ into Δ_{16}

$$\begin{aligned} \Delta_{16} = & -905 + 5A + 5B + 5C + 15D + e + Ae^3BC + Ae^3BD + Ae^3CD \\ & + Be^3CD - Ae^4BC - Ae^4BD - Ae^4CD - Be^4CD - 5ABCD \\ & + Ae + Be + Ce + De - Ae^2 - Be^2 - Ce^2 - De^2 - 5AB - 5AC \\ & - 5BC - 5AD - 5BD - 5CD - 25D^2 - Ae^2B - Ae^2C - Be^2C \\ & - Ae^2D - Be^2D - Ce^2D + Ae^3B + Ae^3C + Ae^3D + Be^3C + Be^3D \\ & + Ce^3D + 5ABC + 5ABD + 5ACD + 5BCD - Ae^4BCD \\ & + Ae^5BCD + 10D^3, \end{aligned}$$

$$\begin{aligned} \Delta_{16}|_{A=4, B=3, C=3, D=3} = & -945 + 14e - 76e^2 + 198e^3 - 243e^4 + 108e^5 \\ & > 0 \quad \text{for } e > 4. \end{aligned}$$

For subcase (b), $P_4(2) > 0$, which implies that $(1, 1, 1, 1, 2)$ is the smallest positive integer solution to the level $w = 2$. Then, we have $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq$

$1 - \frac{2}{e} \triangleq \alpha$, $\alpha \in (\frac{1}{2}, \frac{29}{49})$, since $e \in (4, 4.9)$. Let $A = a\alpha$, $B = b\alpha$, $C = c\alpha$ and $D = d\alpha$, and then we know $A \geq 4$, $B \geq 3$, $C \geq 2$, $D > 1$, $D = d\alpha \geq e\alpha = \frac{2\alpha}{1-\alpha}$. With $\alpha \in (\frac{1}{2}, \frac{29}{49})$, $\frac{2\alpha}{1-\alpha} \in (2, 2.9)$, we obtain the following:

$$\begin{aligned}
 (2.5) \quad 5!P_5 &= 5!(P_4(1) + P_4(2)) \\
 &\leq 5 \left[\left(A \left(\frac{1-\alpha}{2\alpha} + 1 \right) - 1 \right) \left(B \left(\frac{1-\alpha}{2\alpha} + 1 \right) - 1 \right) \right. \\
 &\quad \cdot \left(C \left(\frac{1-\alpha}{2\alpha} + 1 \right) - 1 \right) \left(D \left(\frac{1-\alpha}{2\alpha} + 1 \right) - 1 \right) \\
 &\quad - \left(D \left(\frac{1-\alpha}{2\alpha} + 1 \right) - 1 \right)^4 + D \left(\frac{1-\alpha}{2\alpha} + 1 \right) \\
 &\quad + \left(D \left(\frac{1-\alpha}{2\alpha} + 1 \right) - 1 \right) \left(D \left(\frac{1-\alpha}{2\alpha} + 1 \right) - 2 \right) \\
 &\quad \cdot \left(D \left(\frac{1-\alpha}{2\alpha} + 1 \right) - 3 \right) \\
 &\quad + (A-1)(B-1)(C-1)(D-1) - (D-1)^4 \\
 &\quad \left. + D(D-1)(D-2)(D-3) \right].
 \end{aligned}$$

Let $\Delta_{17} = \text{RHS of (2.3)} - \text{RHS of (2.5)}$ and by substituting $a = \frac{A}{\alpha}$, $b = \frac{B}{\alpha}$, $c = \frac{C}{\alpha}$, $d = \frac{D}{\alpha}$, $e = \frac{2}{1-\alpha}$ into Δ_{17} ,

$$\begin{aligned}
 16\alpha^4\Delta_{17} &= \alpha^4(-500D^2 - 16Be^2 - 85ABCD + 16Be^3D - 16Ce^2 \\
 &\quad - 16De^2 - 16Ae^2B - 16Ae^2C - 16Be^2C - 16Ae^2D - 16Be^2D \\
 &\quad - 16Ce^2D + 16Ae^3B + 16Ae^3C + 16Ae^3D + 16Be^3C - 100AC \\
 &\quad + 16Ce^3D + 90ABC + 90ABD + 90ACD + 90BCD \\
 &\quad + 120A + 120B + 120C + 360D + 180D^3 - 16Ae^4BCD \\
 &\quad + 16Ae^5BCD + 16Ae^3BD - 100AD - 100BC + 16Ae^3BC \\
 &\quad + 16Be + 16e - 14480 + 16Ae - 100AB + 16Ce + 16De - 16Ae^2 \\
 &\quad - 100BD - 100CD - 16Ae^4BD + 16Ae^3CD + 16Be^3CD \\
 &\quad - 16Ae^4BC - 16Ae^4CD - 16Be^4CD) \\
 &\quad + \alpha^3(-40CD + 40B + 60D^3 + 30BCD + 40A - 40BD + 40C \\
 &\quad - 20ABCD + 30ABC - 200D^2 - 40AC + 30ABD - 40AD \\
 &\quad - 40BC + 120D + 30ACD - 40AB)
 \end{aligned}$$

$$\begin{aligned}
 & + \alpha^2(30BCD - 20AB - 20AC - 20BD + 60D^3 - 20AD \\
 & + 30ABC + 30ACD - 100D^2 - 20CD - 30ABCD \\
 & - 20BC + 30ABD) + \alpha(10BCD + 20D^3 + 10ABC \\
 & + 10ABD + 10ACD - 20ABCD) - 5ABCD, \\
 16\alpha^4 \Delta_{17}|_{A=4, B=3, C=2, D=2} \\
 & = -240 + 192e\alpha^4 - 880e^2\alpha^4 - 40\alpha + 40\alpha^2 + 768e^5\alpha^4 \\
 & \quad - 14880\alpha^4 - 1984e^4\alpha^4 + 1920e^3\alpha^4 - 160\alpha^3 \\
 & > 0 \quad \text{for } \alpha \in \left(\frac{1}{2}, \frac{29}{49}\right) \quad e > 4.
 \end{aligned}$$

Case (B). $\frac{a}{e}(1 + \beta) \geq 4$, $\frac{b}{e}(1 + \beta) > 3$, $\frac{c}{e}(1 + \beta) > 2$ and $\frac{d}{e}(1 + \beta) > 1$, where $0 < \beta < 1$.

Case (B1). $\beta = \frac{e}{d}$. If there are no positive lattice points on level $w = e - \beta - 1$, the proof is the same as Case (A1), and so we can assume there exist positive integral solutions at level $w = e - \beta - 1$. $\beta = \frac{e}{d}$ implies $\frac{d}{e}(\beta + 1) > 2$; hence we can use GLY sharp upper estimate to bound the number of positive integral points at this level and for the levels lower than this level. Let $\Delta_{18} = \text{RHS of (2.1)} - g(e - \beta - 1)$. It suffices to prove that $\Delta_{18} > 0$. Let $A = \frac{a}{e}$, $B = \frac{b}{e}$, $C = \frac{c}{e}$, $D = \frac{d}{e}$ and $\beta = \frac{1}{D}$ — then substitute into Δ_{18} . Now all we have to do is partial differentiate to ensure that Δ_{18} is positive throughout the domain considered. From $A(1 + \beta) \geq 4$, $B(1 + \beta) > 3$, $C(1 + \beta) > 2$ and $D(1 + \beta) > 1$ and $\beta = \frac{1}{D}$, we obtain $A \geq \frac{4D}{D+1}$, $B > \frac{3D}{D+1}$, $C > \frac{2D}{D+1}$ and $D > 1$. Let $\Delta_{19} = 72D^4 \Delta_{18}$. We find that Δ_{19} is an increasing function with respect to A , B , C and D for $A \geq \frac{4D}{D+1}$, $B > \frac{3D}{D+1}$, $C > \frac{2D}{D+1}$ and $D > 1$, and $e > 4$. Evaluated at the minimum $A = \frac{4D}{D+1}$, $B = \frac{3D}{D+1}$, $C = 1$ and $D = 1$, $\Delta_{19} > 0$, so we can conclude that $\Delta_{18} > 0$ for this subcase.

$$\begin{aligned}
 \Delta_{18} = & \frac{1}{72e^4}(-72cde^4 + 135abce^3 + 135abde^3 + 135acde^3 + 135bcde^3 \\
 & - 180e^3b\beta^2 - 180e^3c\beta - 180e^3c\beta^2 - 72e^4 - 220e^3ac - 220e^3bc \\
 & + 72ade^5 - 198abde^4 - 198acde^4 - 198bcde^4 + 63abce^5 + 63abde^5 \\
 & + 63acde^5 + 63bcde^5 + 440e^2ac\beta^3 + 440e^2bc\beta^3 + 220e^2ab\beta \\
 & + 220e^2ac\beta + 220e^2bc\beta + 588bce^4 + 588abe^4 - 72bde^4 \\
 & + 108be^5 + 108ae^5 - 360\beta^4e^4 - 1080e^6\beta + 1440e^5\beta^3 \\
 & - 2160e^6\beta^2 + 1440e^7\beta + 1080e^5\beta^2 + 360\beta^2e^4
 \end{aligned}$$

$$\begin{aligned}
 & -720e^5\beta - 72de^5 - 360\beta^3e^4 + 108ce^5 + 432\beta e^4 \\
 & - 108be^4 - 108ae^4 + 72de^4 - 108ce^4 - 360e^5 + 360e^7 + 360e^6 \\
 & - 360e^8 + 108abcde^4 - 368abe^5 - 368ace^5 + 72bde^5 - 180e^3a\beta \\
 & + 120abcd\beta^3 + 180abcd\beta^4 + 72abcd\beta^5 + 660e^2ab\beta^2 \\
 & + 660e^2ac\beta^2 + 660e^2bc\beta^2 + 440e^2ab\beta^3 - 135abce\beta^4 \\
 & - 135abce\beta^2 - 135abde\beta^4 - 135abde\beta^2 - 135acde\beta^4 \\
 & - 135acde\beta^2 - 135bcde\beta^4 - 135bcde\beta^2 - 180e^3b\beta \\
 & - 270abce\beta^3 - 270abde\beta^3 - 270acde\beta^3 - 270bcde\beta^3 \\
 & - 12abcd\beta + 12abcde - 72ade^4 + 588ace^4), \\
 \Delta_{19} = & -360 - 360D - 108CeD^4 - 180BD^3 - 180CD^3 - 180AD^2 \\
 & + 72Ae^3D^5 + 360e^2D^4 - 120Ae^3BCD^5 - 72D^5e^2 - 72Be^2D^5 \\
 & - 72D^4 + 432D^3 + 305ACD + 305BCD + 390ABD^2 + 85ABD^3 \\
 & - 220eABD^4 + 360e^3D^4 + 360D^2 + 72D^5e - 360eD^4 + 1440e^3D^3 \\
 & - 2160e^2D^2 - 720eD^3 + 1080eD^2 - 12ABCD^4 - 198Ae^3CD^5 \\
 & + 588e^2ACD^4 + 63Ae^4CD^5 - 368Ae^3CD^4 + 135Ae^2CD^5 \\
 & - 220eACD^4 + 85ACD^3 + 390ACD^2 + 85BCD^3 \\
 & - 90ABCD + 12AeBCD^5 - 15ABCD^2 - 198Ae^3BCD^4 \\
 & + 108Ae^4BCD^5 + 63Ae^4BCD^4 + 135Ae^2BCD^4 + 1440De \\
 & - 1080e^2D^3 - 360e^4D^4 - 198Be^3CD^5 + 588e^2BCD^4 + 63Be^4CD^5 \\
 & - 368Be^3CD^4 + 135Be^2CD^5 - 220eBCD^4 + 390BCD^2 - 63ABC \\
 & + 305ABD - 198Ae^3BD^5 + 588e^2ABD^4 + 63Ae^4BD^5 \\
 & - 368Ae^3BD^4 + 135Ae^2BD^5 - 180CD^2 + 108Ae^2D^4 - 72Ce^2D^5 \\
 & - 72Ae^2D^5 - 108AeD^4 + 72Ce^3D^5 - 180AD^3 - 108BeD^4 \\
 & + 108Ce^2D^4 - 180BD^2 + 72Be^3D^5 + 108Be^2D^4, \\
 \Delta_{19}|_{A=\frac{4D}{D+1}, B=\frac{3D}{D+1}, C=1, D=1} & \\
 = 2910 - 368e - 2509e^3 + \frac{4629}{2}e^2 + \frac{1125}{2}e^4 \geq 0 & \text{ for } e \geq 4.
 \end{aligned}$$

Therefore, we have $\Delta_{18} \geq 0$ on the region considered.
Case (B2). $\beta = \frac{e}{c}$. If there are no positive lattice points at level $w = e - \beta - 1$, then the proof is the same as Case (A2), and so we can assume there exist positive integral solutions at level $w = e - \beta - 1$. $\frac{d}{c}(\beta + 2) > 2$ — hence we can use the GLY sharp upper estimate to bound the number of positive integral points at level $w = e - \beta - 2$ and for the levels lower than level

$w = e - \beta - 2$. We use Theorem 1.3 to bound the number of positive integral points at level $w = e - \beta - 1$. Let

$$\begin{aligned} \Delta_{20} = & \text{RHS of (2.1)} - g(e - \beta - 2) \\ & - \left[5 \left(\frac{a(\beta + 1)}{e} - 1 \right) \left(\frac{b(\beta + 1)}{e} - 1 \right) \left(\frac{c(\beta + 1)}{e} - 1 \right) \right. \\ & \times \left(\frac{d(\beta + 1)}{e} - 1 \right) - 5 \left(\frac{d(\beta + 1)}{e} - 1 \right)^4 + 5 \left(\frac{d}{e}(\beta + 1) \right) \\ & \left. \times \left(\frac{d(\beta + 1)}{e} - 1 \right) \left(\frac{d(\beta + 1)}{e} - 2 \right) \left(\frac{d(\beta + 1)}{e} - 3 \right) \right]. \end{aligned}$$

Let $A = \frac{a}{e}$, $B = \frac{b}{e}$, $C = \frac{c}{e}$, $D = \frac{d}{e}$ and $\beta = \frac{1}{C}$ and substitute into Δ_{20} . Let $\Delta_{21} = 72C^4\Delta_{20}$. From $A(1 + \beta) \geq 4$, $B(1 + \beta) > 3$, $C(1 + \beta) > 2$ and $D(1 + \beta) > 1$, and $\beta = \frac{1}{D}$, we obtain $A \geq \frac{4C}{C+1}$, $B > \frac{3C}{C+1}$, $C > 1$ and $D > 1$. We find that Δ_{21} is an increasing function with respect to A , B , C and D for $A \geq \frac{4C}{C+1}$, $B > \frac{3C}{C+1}$, $C > 1$ and $D > 1$. Evaluated at the minimum $A = \frac{4C}{C+1}$, $B = \frac{3C}{C+1}$, $C = 1$ and $D = 1$, $\Delta_{21} > 0$, so we can conclude that $\Delta_{20} > 0$ for this subcase.

$$\begin{aligned} \Delta_{20} = & \frac{1}{72e^4} (-180abc - 180abd - 180eacd - 180ebcd + 12abcde + 1080de^3 \\ & - 72cde^4 + 63e^5abc + 63e^5abd + 63e^5acd + 63e^5bcd \\ & - 198abc^4 - 198abde^4 - 198acde^4 - 198bcde^4 \\ & + 1620a\beta^2be^2 + 2140a\beta be^2 + 1620a\beta^2e^2c + 2140a\beta e^2c \\ & - 360a\beta^2e^2d + 1620e^2b\beta^2c + 2140e^2b\beta c - 360e^2b\beta^2d \\ & - 360e^2c\beta^2d - 720a\beta e^2d - 720e^2b\beta d + 720d^3e - 1800d^2e^2 \\ & - 1080e^6\beta - 360\beta^4e^4 - 360\beta^3e^4 + 1440e^5\beta^3 \\ & - 2160e^6\beta^2 + 1440e^7\beta + 1080e^5\beta^2 + 360\beta^2e^4 \\ & - 720e^5\beta - 72de^5 + 108ce^5 + 108be^5 + 108ae^5 + 432\beta e^4 \\ & + 72de^4 - 108ce^4 - 108be^4 - 108ae^4 - 180e^3a\beta - 135eabc\beta^4 \\ & - 135eabd\beta^4 - 135ebcd\beta^4 - 135eacd\beta^4 - 72e^4 - 360e^2bd - 72ade^4 \\ & - 180e^3c\beta^2 + 720d^3\beta^3e - 180e^3b\beta^2 - 180e^3b\beta \\ & + 135abce^3 + 135abde^3 + 135acde^3 + 135bcde^3 - 220e^3ab - 220e^3ac \\ & - 220e^3bc - 120abcde^3 - 180e^3c\beta - 72bde^4 - 360ae^2d - 360e^5 \\ & - 360e^8 + 360e^7 + 360e^6 - 368ace^5 - 368abe^5 - 368bce^5 + 72bde^5 \\ & + 72ade^5 + 588abe^4 + 588ace^4 + 588bce^4 + 72cde^5 - 180e^3a\beta^2 \end{aligned}$$

$$\begin{aligned}
 &+ 2160d^3\beta^2e - 1800d^2\beta^2e^2 + 2160d^3\beta e - 3600d^2\beta e^2 + 1080d\beta e^3 \\
 &+ 108abcde^4 + 120abcd\beta^3 - 12abcd\beta + 180abcd\beta^4 + 72abcd\beta^5 \\
 &- 360cde^2 + 960e^2bc + 960e^2ac + 960e^2ab - 450a\beta^3bed - 450a\beta^3ecd \\
 &- 450eb\beta^3cd - 450a\beta^3bce - 675a\beta^2bce - 675a\beta^2bed - 540a\beta bce \\
 &- 540a\beta bed - 675a\beta^2ecd - 540a\beta ecd - 675eb\beta^2cd - 540eb\beta cd \\
 &- 720e^2c\beta d + 440e^2ab\beta^3 + 440e^2ac\beta^3 + 440e^2bc\beta^3e^4), \\
 \Delta_{21} = &-360 - 1080C^3e^2 + 960BC^5 - 135BCD - 810BC^2D - 1395BC^3D \\
 &- 900BC^4D + 135Be^2C^5D - 72Be^2C^4D + 63Be^4C^5D \\
 &- 198Be^3C^5D + 72Be^3C^4D - 900AC^4D + 108Ae^4BC^5D \\
 &+ 63Ae^4BC^4D - 120Ae^3BC^5D - 198Ae^3BC^4D + 135Ae^2BC^4D \\
 &+ 12AeBC^5D - 192ABC^4D - 540ABC^3D - 555ABC^2D \\
 &- 270ABCD - 198Ae^3BC^5 + 588Ae^2BC^4 - 220AeBC^4 + 420ABC^4 \\
 &+ 1465ABC^3 + 1170ABC^2 + 305ABC - 180ABC^5 + 135Ae^2BC^5 \\
 &- 368Ae^3BC^4 - 63ABD + 63Ae^4C^5D - 198Ae^3C^5D + 72Ae^3C^4D \\
 &+ 135Ae^2C^5D - 72Ae^2C^4D - 180AC^5D - 180BC^5D - 1395AC^3D \\
 &- 810AC^2D - 135ACD + 720C^3D - 360C^4e^4 + 1080C^2e + 108C^5e^2 \\
 &- 108C^5e - 360C^4e + 1440C^3e^3 - 720C^3e + 360C^2 + 252C^3 - 252C^4 \\
 &+ 360C^4e^3 - 2160C^2e^2 + 63Ae^4BC^5 - 1800C^2D^2 + 2160C^2D^3 \\
 &- 360C^5D + 2160C^3D^3 - 3600C^3D^2 + 720C^4D^3 + 588Be^2C^5 \\
 &+ 2140BC^4 + 1440BC^3 + 108Ae^2C^4 + 72C^5e^3D + 108Be^2C^4 \\
 &- 72C^4e^2D - 108BeC^4 - 368Ae^3C^5 - 220BeC^5 + 260BC^2 \\
 &- 368Be^3C^5 + 72C^4eD - 108AeC^4 + 720CD^3 + 2140AC^4 - 72C^5e^2D \\
 &+ 360C^4D + 260AC^2 + 1440AC^3 - 1800C^4D^2 - 220AeC^5 \\
 &+ 360C^4e^2 + 1440Ce + 588Ae^2C^5 + 960AC^5 - 360C,
 \end{aligned}$$

$$\begin{aligned}
 \Delta_{21}|_{A=\frac{4C}{C+1}, B=\frac{3C}{C+1}, C=1, D=1} \\
 = 8430 - 368e + \frac{1125}{2}e^4 - 2509e^3 + \frac{4629}{2}e^2 \geq 0 \quad \text{for } e \geq 4.
 \end{aligned}$$

Case (B3). $\beta = \frac{e}{b}$. Similar to Case (B2). Let $A = \frac{a}{e}$, $B = \frac{b}{e}$, $C = \frac{c}{e}$, $D = \frac{d}{e}$ and $\beta = \frac{1}{B}$ and substitute into Δ_{20} . Let $\Delta_{22} = 72B^4\Delta_{20}$. From $\frac{a}{e}(1 + \beta) \geq 4$, $\frac{b}{e}(1 + \beta) > 3$, $\frac{c}{e}(1 + \beta) \geq 2$ and $\frac{d}{e}(1 + \beta) > 1$. We get $A \geq \frac{4B}{B+1}$, $B \geq 2$, $C \geq \frac{2B}{B+1}$ and $D \geq 1$. We find that Δ_{21} is an increasing function with respect to A , B , C and D for $A \geq \frac{4B}{B+1}$, $B \geq 2$, $C \geq \frac{2B}{B+1} \geq 1$ and $D \geq 1$. Evaluated at the minimum $A = \frac{4B}{B+1}$, $B = 2$, $C = 1$ and $D = 1$, $\Delta_{22} > 0$, so we can

conclude that $\Delta_{20} > 0$ for this subcase.

$$\begin{aligned} \Delta_{22} = & -360 - 720eB^3 + 108Ce^2B^4 + 1440AB^3 - 108eAB^4 + 260CB^2 \\ & - 2160e^2B^2 - 1080e^2B^3 + 360e^2B^4 + 360e^3B^4 - 360e^4B^4 \\ & + 108B^5e^2 - 360eB^4 - 108B^5e + 1440e^3B^3 + 1080eB^2 - 252B^4 \\ & + 252B^3 + 360B^2 + 260AB^2 + 108Ae^2B^4 + 588e^2B^5C \\ & - 220eAB^5 + 720D^3B^4 - 63ACD + 305ABC - 368B^5e^3C \\ & + 960AB^5 + 588e^2AB^5 - 198B^5e^3CD - 1395CDB^3 \\ & - 810CDB^2 + 135B^5e^2CD - 900CDB^4 - 72Ce^2DB^4 \\ & + 72Ce^3DB^4 - 198Ae^3B^5D + 63Ae^4B^5C - 180AB^5C \\ & + 1170ACB^2 + 1465ACB^3 + 420ACB^4 + 135Ae^2B^5C \\ & - 220eACB^4 + 588e^2ACB^4 - 198Ae^3B^5C - 368Ae^3CB^4 \\ & + 108Ae^4B^5CD - 198Ae^3CDB^4 + 63Ae^4CDB^4 \\ & - 120Ae^3B^5CD + 135Ae^2CDB^4 - 192ACDB^4 \\ & + 12AeB^5CD - 555ACDB^2 - 540ACDB^3 + 1440Be - 220eB^5C \\ & + 72B^5e^3D + 960B^5C - 72B^5e^2D - 135BCD - 135ABD \\ & + 1440CB^3 + 2140CB^4 + 2140AB^4 - 368Ae^3B^5 + 63Ae^4B^5D \\ & - 810ADB^2 - 180AB^5D - 1395ADB^3 - 900ADB^4 - 72Ae^2DB^4 \\ & + 135Ae^2B^5D + 72Ae^3DB^4 - 180B^5CD + 63B^5e^4CD + 72DeB^4 \\ & - 72De^2B^4 + 360DB^4 - 1800D^2B^4 - 3600D^2B^3 \\ & - 1800D^2B^2 + 2160D^3B^2 + 2160D^3B^3 + 720D^3B + 720DB^3 \\ & - 360B^5D - 108eCB^4 - 270ABCD - 360B, \end{aligned}$$

$$\begin{aligned} \Delta_{22}|_{A=\frac{4B}{B+1}, B=2, C=1, D=1} \\ = 149634 - 47136e + 117792e^2 - 93888e^3 + 18912e^4 \geq 0 \quad \text{for } e \geq 4. \end{aligned}$$

Case (B4). $\beta = \frac{e}{a}$. Similar to Case (B2). Let $A = \frac{a}{e}$, $B = \frac{b}{e}$, $C = \frac{c}{e}$, $D = \frac{d}{e}$ and $\beta = \frac{1}{A}$ and substitute into Δ_{20} . Let $\Delta_{23} = 72A^4\Delta_{20}$. From $\frac{a}{e}(1 + \beta) \geq 4$, $\frac{b}{e}(1 + \beta) > 3$, $\frac{c}{e}(1 + \beta) > 2$ and $\frac{d}{e}(1 + \beta) > 1$, we obtain $A \geq 3$, $B \geq \frac{3A}{A+1}$, $C \geq \frac{2A}{A+1}$ and $D \geq 1$. We find that Δ_{22} is an increasing function with respect to A , B , C and D for $A \geq 3$, $B \geq \frac{3A}{A+1}$, $C \geq \frac{2A}{A+1}$ and $D \geq 1$, and $e > 4$. Evaluated at the minimum $A = 3$, $B = \frac{3A}{A+1}$, $C = \frac{2A}{A+1}$ and $D = 1$, $\Delta_{23} > 0$, so we can conclude that $\Delta_{20} > 0$ for this subcase.

$$\begin{aligned} \Delta_{23} = & -360 - 2160e^2A^2 - 220eA^5C + 108Be^2A^4 + 12A^5eBCD \\ & + 360e^2A^4 + 260CA^2 + 1440BA^3 - 108eBA^4 + 260BA^2 - 368A^5e^3C \end{aligned}$$

$$\begin{aligned}
 &+ 2140BA^4 + 108Ce^2A^4 + 2140CA^4 + 1440Ae + 135Be^2CDA^4 \\
 &- 192BCDA^4 - 120A^5e^3BCD - 555BCDA^2 - 540BCDA^3 \\
 &+ 63Be^4CDA^4 + 108A^5e^4BCD - 198Be^3CDA^4 - 180A^5BD \\
 &- 900BDA^4 - 810BDA^2 - 72Be^2DA^4 - 1395BDA^3 - 220eBCA^4 \\
 &- 198A^5e^3BC - 368Be^3CA^4 + 420BCA^4 + 135A^5e^2BC - 180A^5BC \\
 &+ 1465BCA^3 + 1170BCA^2 + 588e^2BCA^4 + 63A^5e^4BC + 63A^5e^4BD \\
 &+ 135A^5e^2BD - 220eA^5B + 588e^2A^5C - 63BCD - 135ABD \\
 &- 135ACD + 305ABC + 960A^5B - 270ABCD + 588e^2A^5B \\
 &+ 63A^5e^4CD - 198A^5e^3CD - 180A^5CD - 810CDA^2 - 72Ce^2DA^4 \\
 &+ 135A^5e^2CD - 900CDA^4 - 1395CDA^3 + 72Ce^3DA^4 \\
 &- 198A^5e^3BD + 72Be^3DA^4 - 1080e^2A^3 + 1440e^3A^3 \\
 &- 720eA^3 + 360e^3A^4 - 360eA^4 - 360e^4A^4 - 108A^5e + 252A^3 \\
 &+ 360A^2 + 1080eA^2 + 960A^5C + 1440CA^3 + 720DA^3 + 720D^3A \\
 &+ 72A^5e^3D + 360DA^4 - 1800D^2A^4 + 2160D^3A^3 - 1800D^2A^2 \\
 &+ 2160D^3A^2 + 72DeA^4 - 72De^2A^4 - 72A^5e^2D \\
 &- 360A^5D - 3600D^2A^3 + 720D^3A^4 + 108A^5e^2 - 108eCA^4 \\
 &- 368A^5e^3B - 360A - 252A^4,
 \end{aligned}$$

$$\begin{aligned}
 \Delta_{23} \Big|_{A=3, B=\frac{3A}{A+1}, C=\frac{2A}{A+1}, D=1} \\
 &= -338553e + \frac{742851}{4}e^4 + 918072 + \frac{3846447}{4}e^2 - \frac{1647783}{2}e^3 \\
 &> 0 \quad \text{for } e > 4.
 \end{aligned}$$

□

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