

Local Palais–Smale sequences for the Willmore functional

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Using the reformulation in divergence form of the Euler–Lagrange equation for the Willmore functional as it was developed in the second author’s paper [24], we study the limit of a local Palais–Smale sequence of weak Willmore immersions with locally square-integrable second fundamental form. We show that the limit immersion is smooth and that it satisfies the *conformal Willmore equation*: it is a critical point of the Willmore functional restricted to infinitesimal conformal variations.

1. Introduction

Let Σ be a closed oriented surface without boundary immersed in $\mathbb{R}^{m \geq 3}$ through the action of a smooth positive immersion $\vec{\Phi}$. Its *Willmore energy* is the functional

$$(1.1) \quad W(\vec{\Phi}(\Sigma)) := \int_{\Sigma} |\vec{H}|^2 d\mu_g,$$

where $\vec{H} \in \mathbb{R}^m$ is the mean curvature vector, and $d\mu_g$ is the area form of the metric g induced on $\vec{\Phi}(\Sigma)$ via the canonical metric on \mathbb{R}^m .

The critical points of (1.1) for perturbations of the form $\vec{\Phi} + t\vec{\xi}$, where $\vec{\xi}$ is an arbitrary compactly supported smooth map on Σ into \mathbb{R}^m are known as *Willmore surfaces*. Examples of Willmore surfaces include, among many others, minimal surfaces (which are absolute minimizers), and round spheres. Further examples may be found in [20, 36], and the references therein.

The Willmore functional naturally arises in various areas of science [11, 13, 15, 16, 21]. Its importance is largely due to its invariance under conformal transformations [9, 34]. As the second author of the present paper showed in [23], the Euler–Lagrange equation arising from a two-dimensional conformally invariant Lagrangian with quadratic growth can be written in divergence form. These “conservation laws” fostered within variational problems involving conformally invariant Lagrangians offer a significant analytical help. The general ideas introduced in [23] are led to fruition in [24], where

conservation laws relative to the Willmore functional are developed and successfully applied to produce a variety of interesting results. Our present work stems from this alternative formulation of the Willmore equation.

Owing to the Gauß–Bonnet theorem, we note that the Willmore energy (1.1) may be equivalently expressed as

$$W(\vec{\Phi}(\Sigma)) = \int_{\Sigma} |\vec{\mathbb{I}}_g|^2 d\mu_g + \pi\chi(\Sigma),$$

where $\vec{\mathbb{I}}$ is the second fundamental form, and $\chi(\Sigma)$ is the Euler characteristic of Σ . Since the latter is a topological invariant, from the variational point of view, Willmore surfaces are the critical points of the energy $\int_{\Sigma} |\vec{\mathbb{I}}_g|^2 d\mu_g$. It thus appears natural to restrict our attention on a weak¹ immersion $\vec{\Phi} : \Sigma \rightarrow \mathbb{R}^m$ whose second fundamental forms are locally square-integrable. Such immersions were characterized in [30, 31], and in [19]. Frédéric Hélein (see Theorem 5.1.1 in [14]) showed that, locally about every point on Σ , there exist an open disk D and a homeomorphism ζ of D such that $\vec{\Phi} \circ \zeta$ is a conformal bilipschitz immersion. In this parametrization, the induced metric g on D^2 is continuous; and the Gauß map \vec{n} lies in $W^{1,2}(D^2, Gr_{m-2}(\mathbb{R}^m))$, relative to the induced metric g . Slightly abusing notation, we continue to denote $\vec{\Phi} \circ \zeta$ by $\vec{\Phi}$. Owing to the invariance of the Willmore energy under reparametrization, we shall henceforth assume that $\vec{\Phi}$ is a conformal weak immersion of the unit disc $D^2 = \{x = (x_1, x_2) \in \mathbb{R}^2; x_1^2 + x_2^2 < 1\}$:

$$|\partial_{x_1} \vec{\Phi}| = e^\lambda = |\partial_{x_2} \vec{\Phi}| \quad \text{and} \quad \partial_{x_1} \vec{\Phi} \cdot \partial_{x_2} \vec{\Phi} = 0,$$

where λ is the conformal parameter. An elementary computation shows that

$$d\mu_g = e^{2\lambda} dx \quad \text{and} \quad |\nabla \vec{n}|^2 dx = e^{2\lambda} |\vec{\mathbb{I}}_g|^2 dx = |\vec{\mathbb{I}}_g|^2 d\mu_g.$$

Hence, by hypothesis, we see that $\vec{n} \in W^{1,2}(D^2)$. Rescaling if necessary, we shall always assume that

$$\int_{D^2} |\nabla \vec{n}|^2 dx \leq \varepsilon,$$

where the adjustable parameter ε is chosen to fit our various needs (we will need it to be “small enough” in Theorem 2.2).

¹cf. [24] for details.

We set

$$\vec{e}_j := e^{-\lambda} \partial_{x_j} \vec{\Phi} \quad \text{for } j \in \{1, 2\}.$$

As $\vec{\Phi}$ is conformal, $\{\vec{e}_1(x), \vec{e}_2(x)\}$ forms an orthonormal basis of the tangent space $T_{\vec{\Phi}(x)} \vec{\Phi}(D^2)$. Owing to the topology of D^2 , there exists for almost every $x \in D^2$ a positively oriented orthonormal basis $\{\vec{n}_1, \dots, \vec{n}_{m-2}\}$ of the normal space $N_{\vec{\Phi}(x)} \vec{\Phi}(D^2)$, such that $\{\vec{e}_1, \vec{e}_2, \vec{n}_1, \dots, \vec{n}_{m-2}\}$ forms a basis of $T_{\vec{\Phi}(x)} \mathbb{R}^m$. From the Plücker embedding, realizing the Grassmannian $\text{Gr}_{m-2}(\mathbb{R}^m)$ as a submanifold of the projective space of the $(m-2)$ th exterior power $\mathbb{P}(\bigwedge^{m-2} \mathbb{R}^m)$, we can represent the Gauß map as the $(m-2)$ -vector $\vec{n} = \bigwedge_{\alpha=1}^{m-2} \vec{n}_\alpha$. Via the Hodge operator \star , we identify vectors and $(m-1)$ -vectors in \mathbb{R}^m , namely:

$$\star(\vec{n} \wedge \vec{e}_1) = \vec{e}_2, \quad \star(\vec{n} \wedge \vec{e}_2) = -\vec{e}_1, \quad \star(\vec{e}_1 \wedge \vec{e}_2) = \vec{n}.$$

In this notation, the second fundamental form is expressed as

$$\vec{\mathbb{I}} = \sum_{\alpha, i, j} h_{ij}^\alpha \vec{n}_\alpha (\vec{e}_i)^* \otimes (\vec{e}_j)^*, \quad \text{where } h_{ij}^\alpha := -e^{-\lambda} \vec{e}_i \cdot \partial_{x_j} \vec{n}_\alpha.$$

The mean curvature vector is

$$\vec{H} = \sum_{\alpha=1}^{m-2} H^\alpha \vec{n}_\alpha = \frac{1}{2} \sum_{\alpha=1}^{m-2} (h_{11}^\alpha + h_{22}^\alpha) \vec{n}_\alpha.$$

The Willmore equation satisfied by Willmore surfaces [32] becomes

$$(1.2) \quad \Delta_\perp \vec{H} + \sum_{\alpha, \beta, i, j} h_{ij}^\alpha h_{ij}^\beta H^\beta \vec{n}_\alpha - 2 |\vec{H}|^2 \vec{H} = 0,$$

with

$$\Delta_\perp \vec{H} := e^{-2\lambda} \pi_{\vec{n}} \operatorname{div}(\pi_{\vec{n}}(\nabla \vec{H})),$$

and $\pi_{\vec{n}}$ is the projection onto the normal space spanned by $\{\vec{n}_\alpha\}_{\alpha=1}^{m-2}$.

The Willmore equation (1.2) is a fourth-order nonlinear equation (in the coefficients of the induced metric, which depends on $\vec{\Phi}$). With respect to the coefficients H^α of the mean curvature vector, it is actually a strongly coupled system whose study is particularly challenging. In codimension 1, there is one equation for the scalar curvature; in higher codimension, the situation becomes significantly more complicated, and one must seek different techniques to approach (1.2). Fortunately, in a conformal parametrization,

the second authors showed that it is possible² to recast the system (1.2) in an equivalent, but analytically more suitable form [24]. Namely, there holds

$$(1.3) \quad \operatorname{div}(\nabla \vec{H} - 3\pi_{\vec{n}}(\nabla \vec{H}) + \star(\nabla^\perp \vec{n} \wedge \vec{H})) = 0,$$

where $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$. This reformulation in divergence form of the Willmore equation is the starting point of our analysis of local Palais–Smale sequences.

2. Main results

2.1. Local Palais–Smale sequences

Tom Willmore conjectured [35] that the Clifford torus minimizes, up to Möbius transformations, the Willmore energy in the class of smooth immersed tori in \mathbb{R}^3 . A satisfactory demonstration of this assertion has yet to be found. Amid the works aimed at solving this problem, one fundamental property of the Willmore functional was brought into light by Leon Simon [26]: for each $m \geq 3$, there exists a compact embedded real analytic torus in \mathbb{R}^m which minimizes the Willmore energy in the class of compact, genus 1, embedded surfaces without boundary. Unfortunately, it remains unknown whether this minimizer is the Willmore torus. Simon’s proof is variational and devises a minimizing sequence for the Willmore functional. This is one instance in which studying and understanding minimizing sequences of the Willmore functional are essential. Yet more generally, it is interesting to investigate Palais–Smale sequences.

We open our considerations with an “empirical” observation which will hopefully convince the reader that the results derived in [24] offer a suitable framework to acquire information on Palais–Smale sequences of Willmore surfaces. Let us consider a conformal weak Willmore immersion $\vec{\Phi}$ from the flat disk D^2 into \mathbb{R}^m with square-integrable second fundamental form. The “divergence-free” form given in (1.3) implies that, up to an additive constant, it is possible to define a map \vec{L} satisfying

$$(2.1) \quad \nabla^\perp \vec{L} := \nabla \vec{H} - 3\pi_{\vec{n}}(\nabla \vec{H}) + \star(\nabla^\perp \vec{n} \wedge \vec{H}).$$

²this procedure requires to choose the normal frame $\{\vec{n}_\alpha\}$ astutely. See [24] for details.

In [24], it is shown that the following system holds:³

$$(2.2) \quad \begin{cases} \nabla \vec{\Phi} \cdot \nabla^\perp \vec{L} = 0, \\ \nabla \vec{\Phi} \wedge \nabla^\perp \vec{L} = 2(-1)^m \nabla(\star(\vec{n} \lrcorner \vec{H})) \lrcorner \nabla^\perp \vec{\Phi}. \end{cases}$$

The Hodge operator \star and the contraction operator \lrcorner commute with all differential operators. Accordingly, the terms appearing in the system (2.2) enjoy a peculiar property: more than mere “products of derivatives,” they can be written in divergence form. This structural feature has the analytical advantage of being robust under weak limiting process. Hence it is legitimate to hope that a local Palais–Smale sequence of conformal weak Willmore immersions with, say, uniformly bounded square-integrable second fundamental forms, converges to an element $\vec{\Phi}$ satisfying the system (2.2) for some function \vec{L} (which may or may not be related to $\vec{\Phi}$ via (2.1)).

This is essentially the result that we shall establish. Prior to stating it precisely, it is necessary to specifically define the notion of *local Palais–Smale sequence* for the Willmore functional.

Definition 2.1. Let $(\vec{\Phi}_k)$ be a sequence of conformal immersions from the unit disc D^2 into \mathbb{R}^m such that

$$(2.3) \quad \|\vec{\Phi}_k\|_{W^{2,2} \cap W^{1,\infty}} + |\log |\nabla \vec{\Phi}_k|| \leq C$$

holds uniformly in k for some positive constant C . Denoting, respectively, by \vec{n}_k and \vec{H}_k the Gauß map and the mean curvature vector associated with the immersion $\vec{\Phi}_k$, we set

$$Q_k := \nabla \vec{H}_k - 3 \pi_{\vec{n}_k}(\nabla \vec{H}_k) + \star(\nabla^\perp \vec{n}_k \wedge \vec{H}_k).$$

The sequence $(\vec{\Phi}_k)$ is locally Palais–Smale if, in addition to (2.3), it satisfies

$$(2.4) \quad \operatorname{div} Q_k \longrightarrow 0 \quad \text{strongly in } (W^{2,2} \cap W^{1,\infty})'(D^2).$$

Remark 2.1. The first summand in (2.3) guarantees that the sequence $(\vec{\Phi}_k)$ is uniformly bounded in the appropriate space. Recall that the conformal parameter satisfies

$$\lambda_k = \frac{1}{2} \log \left(\frac{1}{2} |\nabla \vec{\Phi}_k|^2 \right).$$

³refer to the Introduction or Appendix A for the notation.

Accordingly, the second summand in (2.3) insures that (λ_k) remains uniformly bounded, so that the sequence of immersions does not degenerate and collapse. This condition is added here only for technical simplicity. As explained in [25], it is satisfied away from finitely many points, as one considers sequences of conformal immersions with uniformly bounded Willmore energy modulo some composition with elements from the Möbius group in \mathbb{R}^m .

The following result provides a first description of the limit of a Palais–Smale sequence of the Willmore functional.

Theorem 2.1. *Let $(\vec{\Phi}_k)$ be a local Palais–Smale sequence of conformal immersions from the unit disc D^2 into \mathbb{R}^m . There exist a conformal weak immersion $\vec{\Phi} \in (W^{2,2} \cap W^{1,\infty})(D^2)$ and an element⁴ $\vec{L} \in L^{2,\infty}(D^2)$ such that, up to extraction of a subsequence,*

$$\vec{\Phi}_k \longrightarrow \vec{\Phi} \quad \text{in } W^{2,2}(D^2)$$

and the system

$$(2.5) \quad \begin{cases} \nabla \vec{\Phi} \cdot \nabla^\perp \vec{L} = 0, \\ \nabla \vec{\Phi} \wedge \nabla^\perp \vec{L} = -2 \nabla \vec{\Phi} \wedge \nabla \vec{H} \end{cases}$$

holds in the sense of distributions, where \vec{H} denotes the mean curvature vector associated with $\vec{\Phi}$.

Remark 2.2. This result is one of the fundamental tools used in [25] to construct minimizers (and critical points) of the Willmore functional under various prescribed constraints (fixed conformal class, fixed surface area and fixed volume, etc.) and to apply the principles of the calculus of variations (e.g. minimizing principles, Ekeland variational principle, mountain-pass lemma, etc.).

The apparent difference between the systems (2.2) and (2.5) is fictitious only. Indeed, one verifies⁵ the identity

$$(2.6) \quad \nabla \vec{\Phi} \wedge \nabla \vec{H} = (-1)^{m-1} \nabla(\star(\vec{n} \lrcorner \vec{H})) \lrcorner \nabla^\perp \vec{\Phi}.$$

⁴The space $L^{2,\infty}$ is defined in Appendix A.2.

⁵cf. Equation (2.49) in the paper [24].

2.2. The conformal Willmore equation

The convergence result stated in Theorem 2.1 begs the following question: if an immersion $\vec{\Phi}$ satisfies, for some function \vec{L} , the system (2.5), is it true that $\vec{\Phi}$ is Willmore? Unfortunately, and perhaps surprisingly, the answer is negative (or rather, “not quite”). Nevertheless, the following description can be obtained.

Theorem 2.2. *Let $\vec{\Phi} \in W^{2,2} \cap W^{1,\infty}$ be a conformal weak immersion from the unit disc D^2 into \mathbb{R}^m , and such that*

$$\int_{D^2} |\nabla \vec{n}|^2 \leq \varepsilon,$$

for some $\varepsilon > 0$ small enough. Then there exists $\vec{L} \in L^{2,\infty}(D^2)$ satisfying (2.5) if and only if $\vec{\Phi}$ is smooth and there holds

$$(2.7) \quad e^{2\lambda} \left[\Delta_{\perp} \vec{H} + \sum_{\alpha,\beta,i,j} h_{ij}^{\alpha} h_{ij}^{\beta} H^{\beta} \vec{n}_{\alpha} - 2|\vec{H}|^2 \vec{H} \right] = \mathfrak{S}(\vec{H}_0 f),$$

for some holomorphic function f . The Weingarten operator is defined via

$$\vec{H}_0 := \frac{1}{2}(h_{11}^{\alpha} - h_{22}^{\alpha} + 2i h_{12}^{\alpha}) \vec{n}_{\alpha}.$$

Although the amount of information on the holomorphic function f is rather limited, we note that the way it appears on the right-hand side of (2.7) makes it play the rôle of a Lagrange multiplier in the Willmore equation (1.2). This observation enables one to give a natural geometric interpretation of f in the context of Teichmüller theory (cf. [6]).

The Willmore functional being conformally invariant, the left-hand side of (2.7) remains unchanged under a holomorphic change of coordinates. An elementary computation reveals that $\frac{1}{2}\vec{H}_0$ is the coefficient⁶ of $dz \otimes dz$ in the second fundamental form. Looking at the right-hand side of (2.7), we deduce that f must be the coordinate of a section $f(z) \partial_z \otimes \partial_z$. There is thus a one-to-one correspondence between the vector space to which f belongs and the vector space $\mathcal{H}^0(\Sigma)$ of holomorphic quadratic differentials. Let \mathfrak{g} be

⁶the variable z is defined as $x_1 + ix_2$, where (x_1, x_2) are the usual coordinates on the unit disc for the flat metric.

the genus of the immersed surface Σ . From the Riemann–Roch theorem (cf. Corollary 5.4.2 in [18]), the space $\mathcal{H}^0(\Sigma)$ satisfies

$$\dim_{\mathbb{C}} \mathcal{H}^0(\Sigma) = \begin{cases} 0, & \mathfrak{g} = 0, \\ 1, & \mathfrak{g} = 1, \\ 3(\mathfrak{g} - 1), & \mathfrak{g} \geq 2. \end{cases}$$

Equation (2.7) is not novelty; it has been studied in various contexts. Immersions that satisfy (2.7) are sometimes known as *constrained Willmore immersions*, such as in [6], where it is shown that (2.7) is the Euler–Lagrange equation deriving from the Willmore functional (1.1) under smooth compactly supported infinitesimal *conformal* variations. The corresponding critical points are the conformal-constrained Willmore surfaces. This notion clearly generalizes that of a Willmore surface, obtained via all smooth compactly supported infinitesimal variations. Conformal-constrained Willmore surfaces form a Möbius invariant class of surfaces fostering remarkable properties, some of which are studied in [6, 8, 22]. In the latter, it is in particular established that every constant mean curvature surface in a three-dimensional space form is conformal-constrained Willmore. We have chosen to refer to (2.7) as the *conformal Willmore equation*, rather than as the somewhat vaguer term “constrained Willmore equation.” There are indeed many ways to constrain the variations of the Willmore functional (e.g. restrictions on the volume and surface area in the Helfrich model). The adjective “conformal” appears more descriptive in our situation.

3. Proofs of the theorems

3.1. Proof of Theorem 2.1

Let us set

$$(3.1) \quad Q_k := \nabla \vec{H}_k - 3\pi_{\vec{n}}(\nabla \vec{H}_k) + \star(\nabla^\perp \vec{n}_k \wedge \vec{H}_k).$$

We suppose that

$$(3.2) \quad \operatorname{div} Q_k \longrightarrow 0 \quad \text{strongly in } (W^{2,2} \cap W^{1,\infty})'(D^2)$$

and

$$(3.3) \quad \|\vec{\Phi}_k\|_{W^{2,2} \cap W^{1,\infty}} \leq C \quad \text{uniformly, for some constant } C > 0.$$

By definition and from (3.3), it is not hard to see that the sequences (\vec{n}_k) , (\vec{H}_k) and (Q_k) are uniformly bounded in $W^{1,2}$, L^2 and $\mathbb{R}^2 \otimes (W^{-1,2} + L^1)$, respectively. These facts shall be recurrently used in the sequel.

We begin our study with an elementary result.

Lemma 3.1. *There holds*

$$\begin{cases} \vec{\Phi}_k \cdot \operatorname{div} Q_k \longrightarrow 0 \\ \vec{\Phi}_k \wedge \operatorname{div} Q_k \longrightarrow 0 \end{cases} \quad \text{in} \quad \mathcal{D}'(D^2).$$

Proof. Both convergences are obtained in analogous fashions (interchanging the dot and wedge products), so we concentrate only on the first one. For notational convenience, we set

$$X := W^{2,2} \cap W^{1,\infty},$$

and X_0 is the space of elements of X with vanishing trace on ∂D^2 . Let u be an arbitrary element of $\mathcal{D} = C_c^\infty(D^2)$. The Sobolev embedding theorem guarantees that the elements $\vec{\Phi}_k$ of X are Hölder continuous on $\overline{D^2}$. Since in addition X is an intersection of Sobolev spaces, it is clear that $\vec{\Phi}_k u$ is an element of X_0 . More precisely, from (3.3),

$$(3.4) \quad \|\vec{\Phi}_k u\|_{X_0} \lesssim \|\vec{\Phi}_k\|_X \|u\|_{\mathcal{D}} \leq C \|u\|_{\mathcal{D}}.$$

We can make sense of $\vec{\Phi}_k \cdot T_k$ as a distribution via

$$\langle \vec{\Phi}_k \cdot \operatorname{div} Q_k, u \rangle_{\mathcal{D}', \mathcal{D}} := \langle \operatorname{div} Q_k, \vec{\Phi}_k u \rangle_{X_0, X_0}.$$

It then follows immediately from (3.2) and (3.4) that $\vec{\Phi}_k \cdot \operatorname{div} Q_k$ converges to zero in the sense of distributions. □

By hypothesis, $\operatorname{div} Q_k$ is a bounded linear functional on X (with X as in the proof of the previous lemma). According to (A.3), the space $W_0^{2,(2,1)}$ is continuously embedded into X . Hence $\operatorname{div} Q_k$ belongs to $W^{-2,(2,\infty)}$, dual of $W_0^{2,(2,1)}$. Proposition A.1 grants therefore the existence of an element P_k in $\mathbb{R}^2 \otimes W^{-1,(2,\infty)}$ satisfying

$$(3.5) \quad \operatorname{div} Q_k = \operatorname{div} P_k$$

and

$$(3.6) \quad \|P_k\|_{W^{-1,(2,\infty)}} \leq \|\operatorname{div} Q_k\|_{W^{-2,(2,\infty)}} + \delta_k,$$

for some positive constant δ_k arbitrarily chosen.

We next establish a useful result.

Lemma 3.2. *There holds*

$$\begin{cases} \vec{\Phi}_k \cdot P_k \longrightarrow 0 \\ \vec{\Phi}_k \wedge P_k \longrightarrow 0 \end{cases} \quad \text{in} \quad \mathcal{D}'(D^2).$$

Proof. Owing to (3.2) and (3.6), the sequence (P_k) converges strongly to zero in $W^{-1,(2,\infty)} \equiv (W_0^{1,(2,1)})'$. Moreover, from (3.3) and (A.4), the sequence $(\vec{\Phi}_k)$ is uniformly bounded in $W^{1,(2,1)}$. The desired statements may now be reached by repeating *mutatis mutandis* the proof of Lemma 3.1, and letting δ_k tend to zero. □

Identity (3.5) says that

$$(3.7) \quad \operatorname{div} (Q_k - P_k) = 0.$$

We have seen that Q_k belongs to $\mathbb{R}^2 \otimes (W^{-1,2} + L^1)$ and that P_k belongs to $\mathbb{R}^2 \otimes W^{-1,(2,\infty)}$. Hence, altogether, the difference $(Q_k - P_k)$ lies in $\mathbb{R}^2 \otimes (W^{-1,2} + W^{-1,(2,\infty)} + L^1)$. As proved in Lemma A.1, Equation (3.7) implies the existence of an element \vec{L}_k in $L^{2,\infty}$ satisfying

$$(3.8) \quad Q_k - P_k = \nabla^\perp \vec{L}_k.$$

Because all sequences involved are uniformly bounded, one notices that likewise the sequence $(\nabla^\perp \vec{L}_k)$ is uniformly bounded in $\mathbb{R}^2 \otimes W^{-1,(2,\infty)}$.

Lemma 3.3. *There holds*

$$\begin{cases} \nabla \vec{\Phi}_k \cdot \nabla^\perp \vec{L}_k \longrightarrow 0 \\ \nabla \vec{\Phi}_k \wedge (\nabla^\perp \vec{L}_k + 2 \nabla \vec{H}_k) \longrightarrow 0 \end{cases} \quad \text{in} \quad \mathcal{D}'(D^2).$$

Proof. Observe that

$$\nabla \vec{\Phi}_k \cdot P_k = \operatorname{div} (\vec{\Phi}_k \cdot P_k) - \vec{\Phi}_k \operatorname{div} P_k = \operatorname{div} (\vec{\Phi}_k \cdot P_k) - \vec{\Phi}_k \operatorname{div} Q_k.$$

Whence, the results of Lemmas 3.1 and 3.2 show that

$$(3.9) \quad \nabla \vec{\Phi}_k \cdot P_k \longrightarrow 0 \quad \text{in} \quad \mathcal{D}'(D^2).$$

Similarly, one obtains

$$(3.10) \quad \nabla \vec{\Phi}_k \wedge P_k \longrightarrow 0 \quad \text{in} \quad \mathcal{D}'(D^2).$$

From (A.13) in Appendix A, we have the identities

$$\nabla \vec{\Phi}_k \cdot Q_k = 0 \quad \text{and} \quad \nabla \vec{\Phi}_k \wedge Q_k = -2 \nabla \vec{\Phi}_k \wedge \nabla \vec{H}_k.$$

The desired statement now ensues by combining (3.8) to (3.10). □

From the characterization provided in Proposition A.1, we can always arrange for the $L^{2,\infty}$ -norm of \vec{L}_k to come as close as we please to the $W^{-1,(2,\infty)}$ -norm of $\nabla^\perp \vec{L}_k$. In particular, the sequence (\vec{L}_k) is uniformly bounded in $L^{2,\infty}$. We may thus extract a weak* convergent subsequence (indexed for notational convenience as the original sequence) with

$$(3.11) \quad \begin{cases} \vec{L}_k \xrightarrow{*} \vec{L} & \text{in } L^{2,\infty}, \\ \nabla^\perp \vec{L}_k \xrightarrow{*} g & \text{in } \mathbb{R}^2 \otimes W^{-1,(2,\infty)}, \end{cases}$$

and $g = \nabla^\perp \vec{L}$ in the sense of distributions.

As the sequence $(\vec{\Phi}_k)$ is uniformly bounded in $W^{2,2} \cap W^{1,\infty}$ by hypothesis, the Banach–Alaoglu theorem yields a subsequence $(\vec{\Phi}_k)$ converging weak* in $W^{2,2} \cap W^{1,\infty}$ to some element $\vec{\Phi}$. In turn, the compact embeddings (A.4) provided by the Rellich–Kondrachov theorem enable us to further extract a subsequence, still denoted $(\vec{\Phi}_k)$, satisfying the strong convergences

$$(3.12) \quad \begin{cases} \vec{\Phi}_k \longrightarrow \vec{\Phi} & \text{in } \bigcap_{p<\infty} W^{1,p}, \\ \nabla \vec{\Phi}_k \longrightarrow \nabla \vec{\Phi} & \text{in } \mathbb{R}^2 \otimes \bigcap_{p<\infty} L^p. \end{cases}$$

Convergences (3.11) and (3.12) imply the next result.

Lemma 3.4.

$$\begin{cases} \nabla \vec{\Phi}_k \cdot \nabla^\perp \vec{L}_k \longrightarrow \nabla \vec{\Phi} \cdot \nabla^\perp \vec{L} \\ \nabla \vec{\Phi}_k \wedge \nabla^\perp \vec{L}_k \longrightarrow \nabla \vec{\Phi} \wedge \nabla^\perp \vec{L} \end{cases} \quad \text{in } \mathcal{D}'(D^2).$$

Proof. We shall only establish the first convergence, the second one being obtained *mutatis mutandis*. Owing to the general identity

$$\operatorname{div}(a \nabla^\perp b) = \nabla a \cdot \nabla^\perp b,$$

it suffices to show that

$$\vec{\Phi}_k \nabla^\perp \vec{L}_k \longrightarrow \vec{\Phi} \nabla^\perp \vec{L} \quad \text{in } \mathcal{D}'(D^2).$$

This is what we shall do. For convenience, we set $Y := \mathbb{R}^2 \otimes W^{1,(2,1)}$ and $Y_0 := \mathbb{R}^2 \otimes W_0^{1,(2,1)}$. Let g be as in (3.11), and u be an arbitrary test function in $\mathbb{R}^2 \otimes \mathcal{D}$. Clearly, $\vec{\Phi} u$ and $\vec{\Phi}_k u$ are elements of Y_0 with an estimate analogous to (3.4). Note that

$$\langle \vec{\Phi}_k \nabla^\perp \vec{L}_k - \vec{\Phi} g, u \rangle_{\mathcal{D}', \mathcal{D}} \equiv \langle \nabla^\perp \vec{L}_k, u(\vec{\Phi}_k - \vec{\Phi}) \rangle_{Y'_0, Y_0} + \langle \nabla^\perp \vec{L}_k - g, u\vec{\Phi} \rangle_{Y'_0, Y_0}.$$

Whence, from the uniform boundedness of $(\nabla^\perp \vec{L}_k)$ in $\mathbb{R}^2 \otimes W^{-1,(2,\infty)}$ and the convergences (3.11) to (3.12), the announced result follows. \square

As explained in Theorem 3.3.8 from [14], the fact that⁷

$$\Delta \lambda_k = -e^{2\lambda_k} K_k$$

is an element of the Hardy space \mathcal{H}^1 implies that (λ_k) is a sequence of elements in $W^{1,(2,1)}$, uniformly bounded in norm by a constant depending only upon the uniform bound on $(\|\vec{\Phi}_k\|_{W^{2,2} \cap W^{1,\infty}})$. Owing to the Rellich–Kondrachov theorem, we may extract a subsequence (λ_k) satisfying

$$(3.13) \quad \lambda_k \longrightarrow \lambda \quad \text{in} \quad \bigcap_{p < \infty} L^p,$$

for some λ . Furthermore, since $\lambda_k \simeq \log |\nabla \vec{\Phi}_k|$, we note from hypothesis (2.3) that the sequence (λ_k) is uniformly bounded in L^∞ , from which it follows that λ itself must be bounded.

Recall that for $j \in \{1, 2\}$, there holds

$$e^{2\lambda_k} = |\partial_{x_j} \vec{\Phi}_k|^2 \quad \text{and thus} \quad \partial_{x_i}(e^{2\lambda_k}) = 2 \partial_{x_j} \vec{\Phi}_k \cdot \partial_{x_j x_i} \vec{\Phi}_k.$$

Accordingly, from (2.3), the sequence $(e^{2\lambda_k})$ is uniformly bounded in $W^{1,2} \cap L^\infty$. Paired to the uniform boundedness of λ_k , this implies the same is true about the sequences $(e^{\pm\lambda_k})$. Combining this to the convergences (3.12) and (3.13), we find

$$(3.14) \quad \left\{ \begin{array}{l} e^{\pm\lambda_k} \longrightarrow e^{\pm\lambda} \\ (\vec{e}_j)_k := e^{-\lambda_k} \partial_{x_j} \vec{\Phi}_k \longrightarrow e^{-\lambda} \partial_{x_j} \vec{\Phi} =: \vec{e}_j \end{array} \right. \quad \text{in} \quad \bigcap_{p < \infty} L^p.$$

This argument sheds light onto the imposed assumption that (λ_k) be a uniformly bounded sequence. It enables us to conclude that λ lies in L^∞ ,

⁷where K_k denotes the Gaussian curvature of the immersion $\vec{\Phi}_k$.

and thus in particular that e^λ is bounded from below; thereby making the limit $\vec{\Phi}$ into a non-degenerate immersion without collapse.

Because the sequence (\vec{H}_k) is uniformly bounded in L^2 , we can extract a weakly convergent subsequence:

$$(3.15) \quad \vec{H}_k \rightharpoonup \vec{H} \quad \text{in} \quad L^2.$$

Altogether, (3.14) and (3.15) show that

$$(3.16) \quad (\vec{e}_1)_k \wedge (\vec{e}_2)_k \wedge \vec{H}_k \rightharpoonup \vec{e}_1 \wedge \vec{e}_2 \wedge \vec{H} \quad \text{in} \quad \bigcap_{1 \leq p < 2} L^p.$$

Equation (2.42) established in [24] states that

$$\star(\vec{n}_k \lrcorner \vec{H}_k) = (-1)^{m-1} (\vec{e}_1)_k \wedge (\vec{e}_2)_k \wedge \vec{H}_k.$$

Consequently, (3.16) is tantamount to

$$\star(\vec{n}_k \lrcorner \vec{H}_k) \rightharpoonup \star(\vec{n} \lrcorner \vec{H}) \quad \text{in} \quad \bigcap_{1 \leq p < 2} L^p.$$

Combining this to (3.12) shows that

$$(3.17) \quad \star(\vec{n}_k \lrcorner \vec{H}_k) \lrcorner \nabla^\perp \vec{\Phi}_k \rightharpoonup \star(\vec{n} \lrcorner \vec{H}) \lrcorner \nabla^\perp \vec{\Phi} \quad \text{in} \quad \bigcap_{1 \leq p < 2} L^p.$$

It takes little effort to verify that

$$\operatorname{div}(u \lrcorner \nabla^\perp v) = \nabla u \lrcorner \nabla^\perp v$$

holds in general. In particular, (3.17) yields

$$\nabla(\star(\vec{n}_k \lrcorner \vec{H}_k)) \lrcorner \nabla^\perp \vec{\Phi}_k \rightharpoonup \nabla(\star(\vec{n} \lrcorner \vec{H})) \lrcorner \nabla^\perp \vec{\Phi} \quad \text{in} \quad \mathcal{D}'(D^2).$$

Per identity (2.6), the latter is equivalent to

$$(3.18) \quad \nabla \vec{\Phi}_k \wedge \nabla \vec{H}_k \rightharpoonup \nabla \vec{\Phi} \wedge \nabla \vec{H} \quad \text{in} \quad \mathcal{D}'(D^2).$$

Finally, bringing altogether Lemmas 3.3, 3.4 and (3.18), we conclude as desired that, in the sense of distributions,

$$(3.19) \quad \begin{cases} \nabla \vec{\Phi} \cdot \nabla^\perp \vec{L} = 0, \\ \nabla \vec{\Phi} \wedge \nabla^\perp \vec{L} = -2 \nabla \vec{\Phi} \wedge \nabla \vec{H}. \end{cases}$$

3.2. Proof of Theorem 2.2

Since $\Delta\vec{\Phi} \wedge H = 2e^{2\lambda}\vec{H} \wedge \vec{H} = \vec{0}$, Equations (3.19) are in divergence form. In particular, we may introduce a scalar function S and an \mathbb{R}^m -valued two-form \vec{R} satisfying

$$(3.20) \quad \begin{cases} \nabla S := \nabla\vec{\Phi} \cdot \vec{L}, \\ \nabla\vec{R} := \nabla\vec{\Phi} \wedge \vec{L} + 2\nabla^\perp\vec{\Phi} \wedge \vec{H}. \end{cases}$$

Clearly, S and \vec{R} are defined on D^2 up to unimportant additive constants.

By hypothesis, \vec{L} is an element of $L^{2,\infty}$, while $\vec{\Phi}$ is Lipschitz. Accordingly, ∇S and $\nabla\vec{R}$ belong to $L^{2,\infty}(D^2)$ and to $\mathbb{R}^2 \otimes L^{2,\infty}(D^2)$, respectively. This observation and the particular structure of the right-hand side of the system (3.20) will enable us to deduce the regularity result we are seeking to obtain.

3.2.1. Regularity It is shown in [24]⁸ that S and \vec{R} satisfy the system

$$(3.21) \quad \begin{cases} \Delta S = \nabla(\star\vec{n}) \cdot \nabla^\perp\vec{R}, \\ \Delta\vec{R} = -\nabla(\star\vec{n}) \bullet \nabla^\perp\vec{R} - \nabla(\star\vec{n}) \cdot \nabla^\perp S. \end{cases}$$

The advantage of these equations lies essentially in their right-hand sides comprising Jacobians. This peculiar feature will enable us to apply techniques of integration by compensation, in particular Wente-type estimates, which we now recall. They are due in parts to contributions by Wente [33], Tartar [28], Coifman *et al.* [10], Bethuel [5] and Hélein [14].

Lemma 3.5. *Let Ω be an open subset of \mathbb{R}^2 with C^2 -boundary. Suppose that a and b are elements of $W^{1,2}(\Omega)$ and of $W^{1,(2,\infty)}(\Omega)$, respectively. If u satisfies*

$$\begin{cases} \Delta u = \nabla a \cdot \nabla^\perp b & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

then ∇u belongs to the space $L^2(\Omega)$ with the estimate

$$\|\nabla u\|_{L^2(\Omega)} \lesssim \|\nabla a\|_{L^2(\Omega)} \|\nabla b\|_{L^{2,\infty}(\Omega)},$$

up to a multiplicative constant depending only on Ω .

⁸the equations appearing in [24] are slightly different (although equivalent) to those given here. The system (3.21) is derived in [4].

The interested reader will find the proof of this result and further variations on the same theme in [12]. A second result (Theorem 3.4.1 in [14]) which will be useful to us is the following.

Lemma 3.6. *Let Ω be an open subset of \mathbb{R}^2 with C^1 -boundary. Suppose that a and b are elements of $W^{1,2}(\Omega)$. If u satisfies*

$$\begin{cases} \Delta u = \nabla a \cdot \nabla^\perp b & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

then u belongs to the space $W^{1,(2,1)}(\Omega) \subset C^0(\Omega)$ with the estimate

$$\|\nabla u\|_{L^{2,1}(\Omega)} \lesssim \|\nabla a\|_{L^2(\Omega)} \|\nabla b\|_{L^2(\Omega)},$$

up to a multiplicative constant depending only on Ω .

Gearred with these results, we are prepared to start our proof. Let us fix once and for all some point $x_0 \in D_{1/2}(0)$ and some radius $0 < r < 1/2$, so that the flat disk $D_r(x_0)$ of radius r and centered on the point x_0 is properly contained in the unit disc D^2 . We use an argument developed by Bethuel [5]. With the help of the theorem of Fubini, we may always choose $r_0 \in (r/4, r/2)$ such that

$$\begin{aligned} (3.22) \quad \int_{\partial D(x_0, r_0)} |\nabla S|^{\frac{3}{2}} &\lesssim \frac{1}{r} \int_{\partial D(x_0, r)} |\nabla S|^{\frac{3}{2}} \lesssim r^{-\frac{1}{2}} \|\nabla S\|_{L^{2,\infty}(D(x_0, r))}^{\frac{3}{2}} \\ &\leq r^{-\frac{1}{2}} \|\nabla S\|_{L^{2,\infty}(D^2)}^{\frac{3}{2}}. \end{aligned}$$

We define

$$(3.23) \quad S = S_0 + S_1 \quad \text{and} \quad \vec{R} = \vec{R}_0 + \vec{R}_1,$$

where the new variables, in accordance with (3.21), satisfy

$$(3.24) \quad \begin{cases} \Delta S_0 = 0, & \Delta \vec{R}_0 = \vec{0} & \text{in } D_{r_0}(x_0), \\ S_0 = S, & \vec{R}_0 = \vec{R} & \text{on } \partial D_{r_0}(x_0), \end{cases}$$

and

$$(3.25) \quad \begin{cases} \Delta S_1 = \Delta S, & \Delta \vec{R}_1 = \Delta \vec{R} & \text{in } D_{r_0}(x_0), \\ S_1 = 0, & \vec{R}_1 = \vec{0} & \text{on } \partial D_{r_0}(x_0). \end{cases}$$

We set

$$s := \frac{1}{2\pi r_0} \int_{\partial D_{r_0}(x_0)} S.$$

Classical Sobolev embedding theorems then give

$$(3.26) \quad \begin{aligned} \|\nabla S_0\|_{L^2(D_{r_0}(x_0))} &\leq \|S - s\|_{H^{1/2}(\partial D_{r_0}(x_0))} \lesssim r_0^{\frac{1}{3}} \|\nabla S\|_{L^{3/2}(\partial D_{r_0}(x_0))} \\ &\lesssim \|\nabla S\|_{L^{2,\infty}(D^2)} < \infty, \end{aligned}$$

where (3.22) was used. In exactly the same manner, there holds

$$(3.27) \quad \|\nabla \vec{R}_0\|_{L^2(D_{r_0}(x_0))} \lesssim \|\nabla \vec{R}\|_{L^{2,\infty}(D^2)} < \infty.$$

Owing to (3.26) and (3.27), we may invoke classical growth estimates (cf., e.g., Theorem 3.3.12 in [14]) for the harmonic functions S_0 and \vec{R}_0 , and so deduce that for $0 < k < 1$ one has:

$$(3.28) \quad \|\nabla S_0\|_{L^2(D_{kr_0}(x_0))} \lesssim k \|\nabla S_0\|_{L^2(D_{r_0}(x_0))},$$

and

$$(3.29) \quad \|\nabla \vec{R}_0\|_{L^2(D_{kr_0}(x_0))} \lesssim k \|\nabla \vec{R}_0\|_{L^2(D_{r_0}(x_0))}.$$

We next move on to obtaining information for the functions S_1 and \vec{R}_1 . Since the right-hand sides of (3.21), and thus of (3.25), comprise only Jacobians, Lemma 3.5 may be called upon so as to produce the estimates

$$\begin{cases} \|\nabla S_1\|_{L^2(D_{r_0}(x_0))} \lesssim \varepsilon \|\nabla \vec{R}\|_{L^{2,\infty}(D_{r_0}(x_0))}, \\ \|\nabla \vec{R}_1\|_{L^2(D_{r_0}(x_0))} \lesssim \varepsilon (\|\nabla S\|_{L^{2,\infty}(D_{r_0}(x_0))} + \|\nabla \vec{R}\|_{L^{2,\infty}(D_{r_0}(x_0))}), \end{cases}$$

up to some unimportant multiplicative constants. Here ε denotes the (adjustable) upper bound on the energy:

$$(3.30) \quad \|\nabla \vec{n}\|_{L^2(D^2)} \leq \varepsilon.$$

As by hypothesis S and \vec{R} are elements of $W^{1,(2,\infty)}$, the above estimates combined to (3.26) and (3.27) yield

$$\|\nabla S\|_{L^2(D_{r_0}(x_0))} + \|\nabla \vec{R}\|_{L^2(D_{r_0}(x_0))} < \infty.$$

We are now in position of applying Lemma 3.6 to the system (3.25), thereby obtaining (since $k < 1$),

$$\begin{cases} \|\nabla S_1\|_{L^{2,1}(D_{kr_0}(x_0))} \lesssim \varepsilon \|\nabla \vec{R}\|_{L^2(D_{r_0}(x_0))}, \\ \|\nabla \vec{R}_1\|_{L^{2,1}(D_{kr_0}(x_0))} \lesssim \varepsilon (\|\nabla S\|_{L^2(D_{r_0}(x_0))} + \|\nabla \vec{R}\|_{L^2(D_{r_0}(x_0))}). \end{cases}$$

In particular, since $L^{2,1} \subset L^2$ and $k < 1$, the latter paired to (3.28) and (3.29) shows that for some positive constants C_0 and C_1 , there holds

$$(3.31) \quad E_{D_{kr_0}(x_0)}(S, \vec{R}) \leq (C_0 k + C_1 \varepsilon) E_{D_{r_0}(x_0)}(S, \vec{R}),$$

where, for notational convenience, we have set the Dirichlet energy

$$E_{D_\rho(x_0)}(u, \vec{v}) := \|\nabla u\|_{L^2(D_\rho(x_0))} + \|\nabla \vec{v}\|_{L^2(D_\rho(x_0))}.$$

We have the freedom to adjust the positive parameter ε as we please. Because $k \in (0, 1)$, we may in particular arrange for the constant $(C_0 k + C_1 \varepsilon)$ to be smaller than 1. Then, iterating (3.31), we infer the existence of some $\gamma \in (0, 1)$ such that

$$(3.32) \quad E_{D_\rho(x)}(S, \vec{R}) \leq C \rho^\gamma$$

holds for all $0 < \rho < 1/2$, all points $x \in D_{1/2}(0)$, and some constant $C > 0$. With the help of the Poincaré inequality, the estimate (3.32) may be used to show that S and \vec{R} are locally Hölder continuous. We are however interested in another one of its corollaries. Consider the maximal function

$$M_{2-\gamma} F(x) := \sup_{\rho>0} \rho^{-\gamma} \int_{D_\rho(x)} |F(y)| dy.$$

Going back to (3.21), owing to (3.30) and (3.32), it follows that

$$M_{2-\gamma}(\chi_{D_{1/2}(0)} \Delta S)(x) \leq \varepsilon \sup_{0<\rho<\frac{1}{2}} \rho^{-\gamma} E_{D_\rho(x)} \leq C \varepsilon$$

is a uniformly bounded function on $D_{1/2}(0)$, and similarly with \vec{R} in place of S . Here $\chi_{D_{1/2}(0)}$ denotes the characteristic function of the disc $D_{1/2}(0)$. It is clear that ΔS and $\Delta \vec{R}$ belong to $L^1(D_{1/2}(0))$, since S , \vec{R} , and $\star \vec{n}$ are elements of $W^{1,2}(D_{1/2}(0))$. We may thus call upon Proposition 3.2 in [2] to

deduce that

$$\frac{1}{|x|} * \chi_{D_{1/2}(0)} \Delta S \quad \text{and} \quad \frac{1}{|x|} * \chi_{D_{1/2}(0)} \Delta \vec{R}$$

belong to $L^{q,\infty}(D_{1/2}(0))$, with $q = \frac{2-\gamma}{1-\gamma}$.

A classical estimate about Riesz kernels states that in general, there holds

$$|\nabla u|(y) \leq C_1 \frac{1}{|x|} * \chi_{D_{1/2}(0)} \Delta u + C_2, \quad \forall y \in D_{1/4}(0),$$

for two constants C_1 and C_2 . We then infer that ∇S and $\nabla \vec{R}$ are elements of $L^{q,\infty}(D_{1/4}(0))$, with q as above. In particular, because $\gamma \in (0, 1)$, it follows that $q > 2$, and thus

$$(3.33) \quad \nabla S, \nabla \vec{R} \in L^{2+\delta}(D_{1/4}(0)),$$

for some $\delta > 0$.

Now that we have this result at our disposal, we are ready to implement our regularity proof. It involves a bootstrapping argument, which will be performed on the following identity, established in Lemma A.4:

$$(3.34) \quad -2 \Delta \vec{\Phi} = \nabla \vec{R} \bullet \nabla^\perp \vec{\Phi} + \nabla S \cdot \nabla^\perp \vec{\Phi}.$$

By hypothesis, $\vec{\Phi}$ is a Lipschitz function. Combining (3.33) and the latter thus shows that

$$(3.35) \quad \nabla \vec{\Phi} \in W^{1,2+\delta}(D_{1/4}(0)).$$

Recall that

$$\star \vec{n} = \vec{e}_1 \wedge \vec{e}_2 \quad \text{with} \quad \vec{e}_j := \frac{\partial_{x_j} \vec{\Phi}}{|\partial_{x_j} \vec{\Phi}|}.$$

Hence there holds

$$\nabla \star \vec{n} = \frac{\sqrt{2}}{|\nabla \vec{\Phi}|} [\pi_{\vec{n}}(\partial_{x_1} \nabla \vec{\Phi}) \wedge \vec{e}_2 + \vec{e}_1 \wedge \pi_{\vec{n}}(\partial_{x_2} \nabla \vec{\Phi})],$$

and consequently

$$(3.36) \quad |\nabla \vec{n}| \simeq |\nabla \vec{\Phi}|^{-1} |\pi_{\vec{n}} \nabla^2 \vec{\Phi}|.$$

Accordingly, (3.35) and the fact that $|\nabla\vec{\Phi}|$ is bounded from above and below yield that

$$(3.37) \quad \nabla\vec{n} \in L^{2+\delta}(D_{1/4}(0)).$$

For the reader’s convenience, we recall

$$\begin{cases} \Delta S = \nabla(\star\vec{n}) \cdot \nabla^\perp \vec{R}, \\ \Delta \vec{R} = -\nabla(\star\vec{n}) \bullet \nabla^\perp \vec{R} - \nabla(\star\vec{n}) \cdot \nabla^\perp S. \end{cases}$$

Accounting for (3.33) and (3.37) in this system gives

$$(3.38) \quad \nabla S, \nabla \vec{R} \in W^{1,1+\delta/2} \subset L^\beta \quad \text{with} \quad \beta := 2 \frac{2+\delta}{2-\delta}.$$

Comparing the latter to (3.33) reveals that the regularity has been improved.

We are dealing with a non-degenerate immersion $\vec{\Phi}$, so in particular the modulus $|\nabla\vec{\Phi}|$ is bounded from both above and below. Then (3.38) and (3.35) brought into (3.34) show that

$$\nabla^2 \vec{\Phi} \in W^{1,\beta}(D_{1/4}(0)),$$

and hence

$$|\nabla\vec{\Phi}|, |\nabla\vec{\Phi}|^{-1} \in W^{2,\beta}(D_{1/4}(0)).$$

Put into (3.36), the latter gives that $\nabla\vec{n} \in W^{1,\beta}$. The above procedure may then be repeated until we reach that all functions involved are continuous (from belonging to a Sobolev space of high enough order). Finally, a standard “strong solution analysis” of the equations eventually yields that $\vec{\Phi}$ is smooth, thereby concluding the proof.

3.2.2. Conformal Willmore equation We open our derivations by introducing some notation. Let $z = x_1 + ix_2$ and \bar{z} be its complex conjugate. We set

$$\vec{e}_z := e^{-\lambda} \partial_z \vec{\Phi} = \frac{1}{2} (\vec{e}_1 - i \vec{e}_2) \quad \text{and} \quad \vec{e}_{\bar{z}} := e^{-\lambda} \partial_{\bar{z}} \vec{\Phi} = \frac{1}{2} (\vec{e}_1 + i \vec{e}_2).$$

Since $\vec{\Phi}$ is conformal, there holds

$$\partial_{x_j} \vec{\Phi} \cdot \partial_{x_k} \vec{\Phi} = e^{2\lambda} \delta_{jk} \quad \text{for} \quad (j, k) \in \{1, 2\},$$

and thus

$$\partial_a \vec{\Phi} \cdot \partial_b \vec{\Phi} = \frac{1}{2} e^{2\lambda} \delta_{a\bar{b}} \quad \text{for} \quad (a, b) \in \{z, \bar{z}\}.$$

From this, it follows easily that for any triple $(a, b, c) \in \{z, \bar{z}\}$ there holds

$$\begin{aligned} \partial_a \vec{\Phi} \cdot \partial_{bc} \vec{\Phi} &\equiv \delta_{ca} (\partial_a \vec{\Phi} \cdot \partial_{ab} \vec{\Phi}) + \delta_{cb} (\partial_a \vec{\Phi} \cdot \partial_{bb} \vec{\Phi}) \\ &= \frac{1}{2} (\delta_{ca} - \delta_{cb}) \partial_b |\partial_a \vec{\Phi}|^2 + \delta_{cb} \partial_b (\partial_a \vec{\Phi} \cdot \partial_b \vec{\Phi}) = \delta_{cb} \delta_{a\bar{b}} e^{2\lambda} \partial_b \lambda. \end{aligned}$$

Hence,

$$\begin{aligned} (3.39) \quad \vec{e}_a \cdot \partial_b \vec{e}_c &\equiv e^{-2\lambda} (\partial_a \vec{\Phi} \cdot \partial_{bc} \vec{\Phi} - (\partial_b \lambda) \partial_a \vec{\Phi} \cdot \partial_c \vec{\Phi}) \\ &= \frac{1}{2} (2 \delta_{cb} \delta_{a\bar{b}} - \delta_{a\bar{c}}) \partial_b \lambda. \end{aligned}$$

Observe furthermore that

$$(3.40) \quad \vec{e}_a \cdot \vec{e}_b = \frac{1}{2} \delta_{a\bar{b}}.$$

Thus, combining (3.39) and (3.40) gives

$$(3.41) \quad \partial_b \vec{e}_c = \partial_b \lambda \sum_{a \in \{z, \bar{z}\}} (2 \delta_{cb} \delta_{ab} - \delta_{ac}) \vec{e}_a + \pi_{\vec{n}} (\partial_b \vec{e}_c).$$

Next, we have

$$\begin{aligned} \vec{n}_\alpha \cdot \partial_{\bar{z}} \vec{e}_z &= -\vec{e}_z \cdot \partial_{\bar{z}} \vec{n}_\alpha = -\frac{1}{4} (\vec{e}_1 - i \vec{e}_2) (\partial_{x_1} + i \partial_{x_2}) \vec{n}_\alpha \\ &= \frac{e^\lambda}{4} (h_{11}^\alpha + h_{22}^\alpha) = \frac{e^\lambda}{2} H^\alpha, \end{aligned}$$

and similarly

$$\vec{n}_\alpha \cdot \partial_z \vec{e}_{\bar{z}} = \frac{e^\lambda}{2} H_0^\alpha.$$

Accordingly, we may now deduce from (3.41) that

$$(3.42) \quad \begin{cases} \partial_{\bar{z}} \vec{e}_z = -(\partial_{\bar{z}} \lambda) \vec{e}_z + \frac{e^\lambda}{2} \vec{H}, \\ \partial_z \vec{e}_{\bar{z}} = (\partial_z \lambda) \vec{e}_{\bar{z}} + \frac{e^\lambda}{2} \vec{H}_0. \end{cases}$$

Note that we can also easily obtain from the above computations the identity

$$(3.43) \quad \partial_{\bar{z}} \vec{n}_\alpha = -e^\lambda (H_0^\alpha \vec{e}_z + H^\alpha \vec{e}_{\bar{z}}) + \pi_{\vec{n}} (\partial_{\bar{z}} \vec{n}_\alpha).$$

These expressions shall come helpful in the sequel.

Suppose now that $\vec{\Phi}$ is smooth and satisfies for some \vec{L} the system

$$(3.44) \quad \begin{cases} \nabla \vec{\Phi} \cdot \nabla^\perp \vec{L} = 0, \\ \nabla \vec{\Phi} \wedge \nabla^\perp \vec{L} = -2 \nabla \vec{\Phi} \wedge \nabla \vec{H}. \end{cases}$$

Since $\vec{e}_j = e^{-\lambda} \partial_{x_j} \vec{\Phi}$, the first equation in the system is equivalent to

$$\vec{e}_1 \cdot \partial_{x_2} \vec{L} = \vec{e}_2 \cdot \partial_{x_1} \vec{L}.$$

Whence, we deduce that \vec{L} satisfies

$$(3.45) \quad \nabla^\perp \vec{L} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \end{pmatrix} + \begin{pmatrix} p_1^\alpha \\ p_2^\alpha \end{pmatrix} \vec{n}_\alpha,$$

for some suitable coefficients a, b, c and p_j^α .

Substituting this form in the second equation from (3.44) gives

$$\begin{aligned} & \vec{e}_1 \wedge [2(\vec{e}_1 \cdot \partial_{x_1} \vec{H}) \vec{e}_1 + 2(\vec{e}_2 \cdot \partial_{x_1} \vec{H}) \vec{e}_2 + 2(\vec{n}_\alpha \cdot \partial_{x_1} \vec{H}) \vec{n}_\alpha \\ & \quad + a \vec{e}_1 + b \vec{e}_2 + p_1^\alpha \vec{n}_\alpha] \\ & = -\vec{e}_2 \wedge [2(\vec{e}_1 \cdot \partial_{x_2} \vec{H}) \vec{e}_1 + 2(\vec{e}_2 \cdot \partial_{x_2} \vec{H}) \vec{e}_2 + 2(\vec{n}_\alpha \cdot \partial_{x_2} \vec{H}) \vec{n}_\alpha \\ & \quad + c \vec{e}_1 - a \vec{e}_2 + p_2^\alpha \vec{n}_\alpha], \end{aligned}$$

thereby yielding

$$c - b = 2(\vec{e}_2 \cdot \partial_{x_1} - \vec{e}_1 \cdot \partial_{x_2}) \vec{H} \quad \text{and} \quad p_j^\alpha = -2 \vec{n}_\alpha \cdot \partial_{x_j} \vec{H}.$$

However, because $\vec{H} = H^\alpha \vec{n}_\alpha$ and the second fundamental form is symmetric, there holds

$$\begin{aligned} c - b &= 2(\vec{e}_2 \cdot \partial_{x_1} - \vec{e}_1 \cdot \partial_{x_2}) \vec{H} = 2 H^\alpha (\vec{e}_1 \cdot \partial_{x_2} - \vec{e}_2 \cdot \partial_{x_1}) \vec{n}_\alpha \\ &= 2 H^\alpha e^\lambda (h_{21}^\alpha - h_{12}^\alpha) = 0. \end{aligned}$$

Whence, (3.45) may be recast in the form

$$\nabla^\perp \vec{L} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \end{pmatrix} - 2 \pi_{\vec{n}}(\nabla \vec{H}).$$

Equivalently, in the $\{\vec{e}_z, \vec{e}_{\bar{z}}\}$ -frame, this expression reads

$$(3.46) \quad \partial_z \vec{L} = A \vec{e}_{\bar{z}} - 2i \pi_{\vec{n}}(\partial_z \vec{H}),$$

where $A := b + ia$.

Owing to the identities (3.42), there holds

$$\begin{aligned}
 (3.47) \quad & \partial_{\bar{z}}\pi_{\bar{n}}\partial_z\vec{H} - \pi_{\bar{n}}\partial_{\bar{z}}\pi_{\bar{n}}\partial_z\vec{H} \\
 & \equiv 2(\vec{e}_z \cdot \partial_{\bar{z}}\pi_{\bar{n}}\partial_z\vec{H})\vec{e}_{\bar{z}} + 2(\vec{e}_{\bar{z}} \cdot \partial_{\bar{z}}\pi_{\bar{n}}\partial_z\vec{H})\vec{e}_z \\
 & = -2(\partial_{\bar{z}}\vec{e}_z \cdot \pi_{\bar{n}}\partial_z\vec{H})\vec{e}_{\bar{z}} - 2(\partial_{\bar{z}}\vec{e}_{\bar{z}} \cdot \pi_{\bar{n}}\partial_z\vec{H})\vec{e}_z \\
 & = -(e^\lambda\vec{H} \cdot \pi_{\bar{n}}\partial_z\vec{H})\vec{e}_{\bar{z}} - (e^\lambda\vec{H}_0 \cdot \pi_{\bar{n}}\partial_z\vec{H})\vec{e}_z \\
 & = -e^\lambda[(\vec{H} \cdot \partial_z\vec{H})\vec{e}_{\bar{z}} + (\vec{H}_0 \cdot \partial_z\vec{H})\vec{e}_z].
 \end{aligned}$$

From (3.46), the latter, and (3.43), we see that

$$\begin{aligned}
 (3.48) \quad & \partial_{\bar{z}z}\vec{L} = \partial_{\bar{z}}(A\vec{e}_{\bar{z}}) - 2i\partial_{\bar{z}}\pi_{\bar{n}}\partial_z\vec{H} \\
 & = \partial_{\bar{z}}(A\vec{e}_{\bar{z}}) + 2ie^\lambda[(\vec{H} \cdot \partial_z\vec{H})\vec{e}_{\bar{z}} + (\vec{H}_0 \cdot \partial_z\vec{H})\vec{e}_z] \\
 & \quad - 2i\pi_{\bar{n}}\partial_{\bar{z}}\pi_{\bar{n}}\partial_z\vec{H} \\
 & = e^{-\lambda}\partial_{\bar{z}}(e^\lambda A)\vec{e}_{\bar{z}} + 2ie^\lambda[(\vec{H} \cdot \partial_z\vec{H})\vec{e}_{\bar{z}} + (\vec{H}_0 \cdot \partial_z\vec{H})\vec{e}_z] \\
 & \quad + \frac{e^\lambda}{2}A\vec{H}_0 - 2i\pi_{\bar{n}}\partial_{\bar{z}}\pi_{\bar{n}}\partial_z\vec{H}.
 \end{aligned}$$

Because the imaginary part $\Im(\partial_{\bar{z}z}\vec{L}) = 0$, we separate this last identity into its normal and tangential components to discover

$$(3.49) \quad \partial_{\bar{z}}(e^\lambda A) = -2ie^{2\lambda}(\vec{H} \cdot \partial_z\vec{H} + \overline{\vec{H}_0} \cdot \partial_z\vec{H})$$

and

$$(3.50) \quad 4\Im[i\pi_{\bar{n}}\partial_{\bar{z}}\pi_{\bar{n}}\partial_z\vec{H}] = e^\lambda\Im(A\vec{H}_0).$$

Comparing (3.49) to the Codazzi–Mainardi equation (A.21) shows that

$$\partial_{\bar{z}}[e^\lambda(A + 2ie^\lambda\overline{\vec{H}_0} \cdot \vec{H})] = 0.$$

Accordingly, there exists some holomorphic function $f(z)$ such that

$$(3.51) \quad A = -e^{-\lambda}f(z) - 2ie^\lambda\overline{\vec{H}_0} \cdot \vec{H}.$$

In conformal coordinates, there holds

$$e^{2\lambda}\Delta_\perp\vec{H} = \pi_{\bar{n}}(\operatorname{div}\pi_{\bar{n}}(\nabla\vec{H})) \equiv 4\Im[i\pi_{\bar{n}}\partial_{\bar{z}}\pi_{\bar{n}}\partial_z\vec{H}],$$

so that (3.50) becomes

$$e^\lambda\Delta_\perp\vec{H} = \Im(A\vec{H}_0).$$

Introducing (3.51) in the latter gives thus

$$(3.52) \quad \Delta_{\perp} \vec{H} + 2 \Re(\overline{\vec{H}_0} \cdot \vec{H}) \vec{H}_0 = e^{-2\lambda} \Im(\vec{H}_0 f).$$

To complete our derivation, there remains to observe that

$$\begin{aligned} \Re[H_0^{\alpha} \overline{H_0^{\beta}}] &= \frac{1}{4} [(h_{11}^{\alpha} - h_{22}^{\alpha})(h_{11}^{\beta} - h_{22}^{\beta}) + 4h_{12}^{\alpha} h_{12}^{\beta}] \\ &= \frac{1}{2} \sum_{i,j=1}^2 h_{ij}^{\alpha} h_{ij}^{\beta} - \frac{1}{4} (h_{11}^{\alpha} + h_{22}^{\alpha})(h_{11}^{\beta} + h_{22}^{\beta}) \\ &= \frac{1}{2} \sum_{i,j=1}^2 h_{ij}^{\alpha} h_{ij}^{\beta} - H^{\alpha} H^{\beta}. \end{aligned}$$

Accordingly, (3.52) yields

$$(3.53) \quad \Delta_{\perp} \vec{H} + \sum_{\alpha,\beta=1}^{m-2} \sum_{i,j=1}^2 h_{ij}^{\alpha} h_{ij}^{\beta} H^{\beta} \vec{n}_{\alpha} - 2 |\vec{H}|^2 \vec{H} = e^{-2\lambda} \Im(\vec{H}_0 f).$$

This identity is the desired conformal Willmore equation announced in Theorem 2.2.

The converse of Theorem 2.2 is easily deduced from reversing the above computations. Assuming that (3.53) holds for some holomorphic function f , we may define A as in (3.51). The fact that f is holomorphic, along with our previous derivations, notably (3.47) and (3.48), yield

$$\begin{aligned} \partial_{\bar{z}}(A \vec{e}_{\bar{z}} - 2i \pi_{\vec{n}} \partial_z \vec{H}) &= e^{-\lambda} \partial_{\bar{z}}(e^{\lambda} A) \vec{e}_{\bar{z}} + \frac{e^{\lambda}}{2} A \vec{H}_0 + 2i \pi_{\vec{n}} \partial_{\bar{z}} \pi_{\vec{n}} \partial_z \vec{H} \\ &\quad - 2ie^{\lambda} ((\vec{H} \cdot \partial_z \vec{H}) \vec{e}_{\bar{z}} + (\vec{H}_0 \cdot \partial_z \vec{H}) \vec{e}_z) \\ &= -2ie^{-\lambda} \partial_{\bar{z}}(e^{2\lambda} \overline{\vec{H}_0} \cdot \vec{H}) \vec{e}_{\bar{z}} + \frac{1}{2} f \vec{H}_0 \\ &\quad - ie^{2\lambda} (\overline{\vec{H}_0} \cdot \vec{H}) \vec{H}_0 + 2i \pi_{\vec{n}} \partial_{\bar{z}} \pi_{\vec{n}} \partial_z \vec{H} \\ &\quad - 2ie^{\lambda} ((\vec{H} \cdot \partial_z \vec{H}) \vec{e}_{\bar{z}} + (\vec{H}_0 \cdot \partial_z \vec{H}) \vec{e}_z). \end{aligned}$$

Taking the imaginary part on both sides of the latter, and introducing the conformal Willmore equation (3.53) and the Codazzi–Mainardi equation (A.21), we find

$$\Im [\partial_{\bar{z}}(A \vec{e}_{\bar{z}} - 2i \pi_{\vec{n}} \partial_z \vec{H})] = 0.$$

On the contractible unit disc, we deduce that there exists some \vec{L} such that

$$\partial_z \vec{L} = A \vec{e}_z - 2i \pi_{\vec{n}} \partial_z \vec{H},$$

or equivalently

$$\nabla^\perp \vec{L} = e^{-\lambda} \begin{pmatrix} \Im(A) & \Re(A) \\ \Re(A) & -\Im(A) \end{pmatrix} \nabla \vec{\Phi} - 2 \pi_{\vec{n}} (\nabla \vec{H}).$$

It takes little effort to verify⁹ that

$$\nabla \vec{\Phi} \cdot \nabla^\perp \vec{L} = 0,$$

which is the first part of the system (2.5). Similarly, calling upon the identity (A.17), one finally finds the desired

$$\nabla \vec{\Phi} \wedge \nabla^\perp \vec{L} = -2 \nabla \vec{\Phi} \wedge \pi_{\vec{n}} (\nabla \vec{H}) = -2 \nabla \vec{\Phi} \wedge \nabla \vec{H}.$$

A. Appendix

A.1. Notational Conventions

We append an arrow to all the elements belonging to \mathbb{R}^m . To simplify the notation, by $\vec{\Phi} \in X(D^2)$ is meant $\vec{\Phi} \in X(D^2, \mathbb{R}^m)$ whenever X is a function space. Similarly, we write $\nabla \vec{\Phi} \in \mathbb{R}^2 \otimes X(D^2)$ for $\nabla \vec{\Phi} \in X(D^2, \mathbb{R}^{2m})$.

Although this custom may seem at first odd, we allow the differential operators classically acting on scalars to act on elements of \mathbb{R}^m . Thus, for example, $\nabla \vec{\Phi}$ is the element of $(\partial_{x_1} \vec{\Phi}, \partial_{x_2} \vec{\Phi}) \in \mathbb{R}^2 \otimes \mathbb{R}^m$. If S is a scalar and \vec{R} an element of \mathbb{R}^m , then we let

$$\begin{aligned} \vec{R} \cdot \nabla \vec{\Phi} &:= (\vec{R} \cdot \partial_{x_1} \vec{\Phi}, \vec{R} \cdot \partial_{x_2} \vec{\Phi}), \\ \nabla^\perp S \cdot \nabla \vec{\Phi} &:= \partial_{x_1} S \partial_{x_2} \vec{\Phi} - \partial_{x_2} S \partial_{x_1} \vec{\Phi}, \\ \nabla^\perp \vec{R} \cdot \nabla \vec{\Phi} &:= \partial_{x_1} \vec{R} \cdot \partial_{x_2} \vec{\Phi} - \partial_{x_2} \vec{R} \cdot \partial_{x_1} \vec{\Phi}, \\ \nabla^\perp \vec{R} \wedge \nabla \vec{\Phi} &:= \partial_{x_1} \vec{R} \wedge \partial_{x_2} \vec{\Phi} - \partial_{x_2} \vec{R} \wedge \partial_{x_1} \vec{\Phi}. \end{aligned}$$

Similar quantities are defined following the same logic.

⁹recall that the operator $\nabla \vec{\Phi} \cdot \pi_{\vec{n}}$ is trivial.

Two operations between multivectors are useful. The interior multiplication \lrcorner maps a pair comprising a q -vector γ and a p -vector β to a $(q-p)$ -vector. It is defined via

$$\langle \gamma \lrcorner \beta, \alpha \rangle = \langle \gamma, \beta \wedge \alpha \rangle \quad \text{for each } (q-p)\text{-vector } \alpha.$$

Let α be a k -vector. The first-order contraction operation \bullet is defined inductively through

$$\alpha \bullet \beta = \alpha \lrcorner \beta \quad \text{when } \beta \text{ is a 1-vector,}$$

and

$$\alpha \bullet (\beta \wedge \gamma) = (\alpha \bullet \beta) \wedge \gamma + (-1)^{pq} (\alpha \bullet \gamma) \wedge \beta,$$

when β and γ are, respectively, a p -vector and a q -vector.

A.2. Lorentz and Sobolev–Lorentz Spaces

For the reader's convenience, we recall in this section the fundamentals of Lorentz spaces. Detailed accounts may be found in [3, 17, 27].

For a real-valued measurable function f on an open subset of $U \subset \mathbb{R}^n$, its belonging to a Lorentz space is determined by a condition involving the non-decreasing rearrangement of $|f|$ on the interval $(0, |U|)$, where $|U|$ denotes the Lebesgue measure of U . The non-increasing rearrangement f^* of $|f|$ is the unique positive continuous function from $(0, |U|)$ into \mathbb{R} which is non-increasing and satisfies

$$|\{x \in U \mid |f(x)| \geq s\}| = |\{t \in (0, |U|) \mid f^*(t) \geq s\}|.$$

If $p \in (1, \infty)$ and $q \in [1, \infty]$, the Lorentz space $L^{p,q}(U)$ is the set of measurable functions $f : U \rightarrow \mathbb{R}$ for which

$$\int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t} < \infty \quad \text{if } q < \infty,$$

or

$$\sup_{t>0} t^{1/p} f^*(t) < \infty \quad \text{if } q = \infty.$$

A complete norm on $L^{p,q}(U)$ is given by

$$\|f\|_{L^{p,q}(U)} = \|t^{1/p} f^{**}\|_{L^q([0,\infty), dt/t)} \quad \text{where} \quad f^{**}(t) := \frac{1}{t} \int_0^t f^*(\tau) d\tau.$$

One verifies that

$$L^p = L^{p,p},$$

and that $L^{p,\infty}$ is the weak- L^p Marcinkiewicz space.

Moreover, we have the inclusions

$$L^{p,1} \subset L^{p,q'} \subset L^{p,q''} \subset L^{p,\infty} \quad \text{for} \quad 1 < q' < q'' < \infty;$$

and if U has finite measure, there holds for all q and q'

$$L^{p',q'}(U) \subset L^{p,q}(U) \quad \text{whenever} \quad p < p'.$$

More precisely,

$$(A.1) \quad \|f\|_{L^{p,q}(U)} \lesssim |U|^{\frac{p'-p}{p'p}} \|f\|_{L^{p',q'}(U)}.$$

Finally, if $q < \infty$, the space $L^{\frac{p}{p-1}, \frac{q}{q-1}}$ is the dual of $L^{p,q}$.

Similarly to Lebesgue spaces, Lorentz spaces obey a pointwise multiplication rule and a convolution product rule. More precisely, for $1 < p_1, p_2 < \infty$ and $1 \leq q_1, q_2 \leq \infty$, there holds

$$L^{p_1, q_1} \times L^{p_2, q_2} \subset L^{p, q} \quad \text{with} \quad \begin{cases} p^{-1} = p_1^{-1} + p_2^{-1}, \\ q^{-1} = q_1^{-1} + q_2^{-1}, \end{cases}$$

and

$$L^{p_1, q_1} * L^{p_2, q_2} \subset L^{p, q} \quad \text{with} \quad \begin{cases} p^{-1} = p_1^{-1} + p_2^{-1} - 1, \\ q^{-1} = q_1^{-1} + q_2^{-1}. \end{cases}$$

An interesting feature of Lorentz spaces is that they are interpolation spaces between Lebesgue spaces.

Lorentz spaces offer the possibility to sharpen the classical Sobolev embedding theorem. More precisely, it can be shown (see [3, 29]) that

$$W^{k,q}(\mathbb{R}^m) \subset L^{p,q}(\mathbb{R}^m)$$

is a continuous embedding so long as

$$1 \leq q \leq p < \infty \quad \text{and} \quad \frac{k}{m} = \frac{1}{q} - \frac{1}{p}.$$

Our study requires that we introduce Sobolev–Lorentz spaces. These spaces are defined analogously to the “standard” Sobolev spaces, but with

the Lebesgue norms replaced by Lorentz norms. In an effort to simplify the presentation, we shall focus only on the two-dimensional unit disc D^2 . Let $m \in \mathbb{N}$, $p \in (1, \infty)$, and $q \in [1, \infty]$. The (homogeneous) Sobolev–Lorentz space $W^{m,(p,q)}(D^2)$ consists of all locally summable functions u on D^2 such that $D^\alpha u$ exists in the weak sense and belongs to the Lorentz space $L^{p,q}(D^2)$ for each multiindex α with $|\alpha| = m$. The norm

$$\|u\|_{W^{m,(p,q)}} := \sum_{|\alpha|=m} \|D^\alpha u\|_{L^{p,q}}$$

clearly makes $W^{m,(p,q)}(D^2)$ into a Banach space. The space $W^{0,(p,q)}$ is understood to be $L^{p,q}$. For notational convenience, we shall from now on omit to precise that we work on the unit disc D^2 . This shall arise no confusion. We also focus on a case of particular interest to us, namely $(p, q) = (2, 1)$. Because $L^{2,1}$ is a subspace of L^2 , it follows immediately that for all $m \in \mathbb{N}$ there holds

$$(A.2) \quad W^{m,(2,1)} \subset W^{m,2},$$

where $W^{m,2}$ is the usual (homogeneous) Sobolev space. The embedding (A.2) shows that each element of $W^{m,(2,1)}$ has a well-defined trace on the boundary of the unit disc, for $m \geq 1$. If that trace is null, the said element belongs to the space $W_0^{m,(2,1)}$.

In [7], the authors prove¹⁰ that $W_0^{1,(2,1)}$ is a subspace of $L^\infty \cap C^0$, so that $W_0^{m,(2,1)}$ is a subspace of $W^{m-1,\infty} \cap C^{m-1}$. Altogether, there holds

$$(A.3) \quad W_0^{m,(2,1)} \subset W^{m,2} \cap W^{m-1,\infty} \cap C^{m-1}.$$

The Rellich–Kondrachov theorem states that $W^{1,2}$ is compact in L^r for all finite $r \geq 1$. By standard interpolation techniques, it follows that

$$(A.4) \quad W^{m,2} \subset\subset W^{m-1,(p,q)} \quad \text{for } 1 < p < \infty, \quad 1 \leq q \leq \infty$$

is a compact inclusion for every $m \in \mathbb{N}^*$.

It is instructive to characterize the dual space of $W_0^{m,(p,q)}$, denoted $W^{-m,(p',q')}$, where (p', q') is the conjugate pair of (p, q) :

$$(A.5) \quad p' = (1 - p^{-1})^{-1} \quad \text{and} \quad q' = (1 - q^{-1})^{-1}.$$

¹⁰see also Theorem 3.3.4 in [14].

Proposition A.1. *Let T be an element of $W^{-m,(p',q')}(D^2, \mathbb{R})$. For every $\delta > 0$, there exists an element P of $W^{1-m,(p',q')}(D^2, \mathbb{R}^2)$ such that*

$$(A.6) \quad T = \operatorname{div} P,$$

and

$$(A.7) \quad \|P\|_{W^{1-m,(p',q')}} \leq \|T\|_{W^{-m,(p',q')}} + \delta.$$

Proof. This is a mere adaption of the analogous statement for the Lebesgue–Sobolev space $W^{m,p}$, ultimately following from the Hahn–Banach theorem. The reader is referred to Theorem 3.10 in [1] for details. \square

We next bring into light an interesting Hodge decomposition result which follows from standard elliptic theory and the interpolation nature of Lorentz spaces (cf. Proposition 3.3.9 in [14]).

Proposition A.2. *To every vector field $\vec{g} \in L^1(D^2, \mathbb{R}^2)$ we associate the functions α , β , and h on D^2 which are solutions of*

$$\begin{cases} \Delta\alpha = \operatorname{div} \vec{g}, & \Delta\beta = \operatorname{curl} \vec{g} & \text{in } D^2, \\ \alpha = 0, & \beta = 0 & \text{on } \partial D^2, \end{cases}$$

with

$$\vec{g} = \nabla\alpha + \nabla^\perp\beta + \nabla h$$

so that h is harmonic on D^2 .

Then the operators

$$\vec{g} \longmapsto \nabla\alpha, \quad \vec{g} \longmapsto \nabla\beta, \quad \text{and} \quad \vec{g} \longmapsto \nabla h$$

map continuously $L^{p,q}(D^2, \mathbb{R}^2)$ into itself for every choice of $p \in (1, \infty)$ and $q \in [1, \infty]$.

This lemma is the central ingredient of the following result.

Lemma A.1. *Suppose that $G = G_1 + G_2 + G_3$ satisfies*

$$\operatorname{div} G = 0 \quad \text{in } D^2,$$

where

$$G_1 \in W^{-1,(2,\infty)}(D^2, \mathbb{R}^2), \quad G_2 \in W^{-1,2}(D^2, \mathbb{R}^2), \quad G_3 \in L^1(D^2, \mathbb{R}^2).$$

Then there exists an element L in the space $L^{2,\infty}(D^2, \mathbb{R})$ such that

$$(A.8) \quad G = \nabla^\perp L.$$

Proof. A classical result of Laurent Schwartz guarantees the existence of L_0 in L^1 such that $G = \nabla^\perp L_0$ holds in the sense of distributions. Owing to the density of C_c^∞ in $W_0^{1,(2,1)}$, it follows that

$$(A.9) \quad \int_{D^2} G \cdot F = - \int_{D^2} L_0 \operatorname{curl} F, \quad \forall F \in W_0^{1,(2,1)}(D^2, \mathbb{R}^2).$$

Next, let f be an arbitrary element of $L^{2,1}(D^2, \mathbb{R})$. Choosing

$$\vec{g} = \left(0, f - \int_{D^2} f \right)$$

in Proposition A.2, we infer the existence of an element $F := (-\alpha, \beta)$ in $W_0^{1,(2,1)}(D^2, \mathbb{R}^2)$ satisfying

$$(A.10) \quad \operatorname{curl} F = f - \int_{D^2} f$$

as well as

$$(A.11) \quad \|F\|_{W^{1,(2,1)}} \lesssim \|f\|_{L^{2,1}}.$$

Putting (A.10) into (A.9) then yields

$$(A.12) \quad - \int_{D^2} L f = \int_{D^2} G \cdot F, \quad \text{where } L := L_0 - \int_{D^2} L_0.$$

Clearly, the function L is a solution to (A.8). Recall that

$$W_0^{1,(2,1)} \subset W_0^{1,2} \cap L^\infty \quad \text{and} \quad W^{-1,(2,\infty)} = (W_0^{1,(2,1)})',$$

Accordingly, using (A.12) and (A.11), we find

$$\begin{aligned} \left| \int_{D^2} Lf \right| &\leq \left| \int_{D^2} G_1 \cdot F \right| + \left| \int_{D^2} G_2 \cdot F \right| + \left| \int_{D^2} G_3 \cdot F \right| \\ &\leq \|G_1\|_{W^{-1,(2,\infty)}} \|F\|_{W^{1,(2,1)}} + \|G_2\|_{W^{-1,2}} \|F\|_{W^{1,2}} + \|G_3\|_{L^1} \|F\|_{L^\infty} \\ &\lesssim (\|G_1\|_{W^{-1,(2,\infty)}} + \|G_2\|_{W^{-1,2}} + \|G_3\|_{L^1}) \|f\|_{L^{2,1}}. \end{aligned}$$

Because $L^{2,\infty}$ is the dual space of $L^{2,1}$, the announced statement ensues. \square

A.3. Miscellaneous identities

We use in this section the same notation as that defined in the Introduction.

Lemma A.2. *Let $\vec{\Phi}$, \vec{n} , and \vec{H} be as in Theorem 2.1. We set*

$$Q := \nabla \vec{H} - 3\pi_{\vec{n}}(\nabla \vec{H}) + \star(\nabla^\perp \vec{n} \wedge \vec{H}).$$

Then the following identities hold:

$$(A.13) \quad \nabla \vec{\Phi} \cdot Q = 0 \quad \text{and} \quad \nabla \vec{\Phi} \wedge Q = -2 \nabla \vec{\Phi} \wedge \nabla \vec{H}.$$

Proof. We first note that¹¹

$$\nabla^\perp \vec{n}_\alpha = \langle \vec{e}_1, \nabla^\perp \vec{n}_\alpha \rangle \vec{e}_1 + \langle \vec{e}_2, \nabla^\perp \vec{n}_\alpha \rangle \vec{e}_2 + \langle \vec{n}_\beta, \nabla^\perp \vec{n}_\alpha \rangle \vec{n}_\beta,$$

so that

$$\star(\vec{n} \wedge \nabla^\perp \vec{n}_\alpha) = \langle \vec{e}_1, \nabla^\perp \vec{n}_\alpha \rangle \vec{e}_2 - \langle \vec{e}_2, \nabla^\perp \vec{n}_\alpha \rangle \vec{e}_1.$$

Whence, since $\vec{n}_\alpha \wedge \vec{n} = 0$,

$$(A.14) \quad \star(\nabla^\perp \vec{n} \wedge \vec{n}_\alpha) = \begin{pmatrix} h_{22}^\alpha \\ -h_{12}^\alpha \end{pmatrix} \partial_{x_1} \vec{\Phi} + \begin{pmatrix} -h_{12}^\alpha \\ h_{11}^\alpha \end{pmatrix} \partial_{x_2} \vec{\Phi}.$$

Accordingly, we find

$$\nabla \vec{\Phi} \wedge \star(\nabla^\perp \vec{n} \wedge \vec{n}_\alpha) = (h_{12}^\alpha - h_{12}^\alpha) \partial_{x_1} \vec{\Phi} \wedge \partial_{x_2} \vec{\Phi} = 0,$$

and

$$\nabla \vec{\Phi} \cdot \star(\nabla^\perp \vec{n} \wedge \vec{n}_\alpha) = e^{2\lambda} (h_{11}^\alpha + h_{22}^\alpha) = 2e^{2\lambda} H^\alpha.$$

¹¹implicit summations over repeated indices are understood wherever appropriate.

The last two identities yield

$$(A.15) \quad \nabla \vec{\Phi} \wedge \star(\nabla^\perp \vec{n} \wedge \vec{H}) = 0$$

and

$$(A.16) \quad \nabla \vec{\Phi} \cdot \star(\nabla^\perp \vec{n} \wedge \vec{H}) = 2e^{2\lambda} |\vec{H}|^2.$$

For three indices $(a, b, c) \in \{x_1, x_2\}$, let

$$F(a, b, c) := \partial_a \vec{\Phi} \wedge \langle \vec{e}_b, \partial_c \vec{H} \rangle \vec{e}_b.$$

We have

$$\begin{aligned} F(a, b, c) &= e^\lambda \langle \vec{e}_b, \partial_c \vec{H} \rangle \vec{e}_a \wedge \vec{e}_b \\ &= -e^\lambda H^\alpha \langle \vec{n}_\alpha, \partial_c \vec{e}_b \rangle \vec{e}_a \wedge \vec{e}_b \quad \text{since } \vec{H} = H^\alpha \vec{n}_\alpha \\ &= -e^\lambda H^\alpha h_{cb}^\alpha \vec{e}_a \wedge \vec{e}_b. \end{aligned}$$

Using this, we obtain

$$\begin{aligned} (A.17) \quad \nabla \vec{\Phi} \wedge (\nabla \vec{H} - \pi_{\vec{n}}(\nabla \vec{H})) &\equiv \nabla \vec{\Phi} \wedge (\langle \vec{e}_1, \nabla \vec{H} \rangle \vec{e}_1 + \langle \vec{e}_2, \nabla \vec{H} \rangle \vec{e}_2) \\ &= F(x_1, x_1, x_1) + F(x_1, x_2, x_1) \\ &\quad + F(x_2, x_1, x_2) + F(x_2, x_2, x_2) \\ &= -e^\lambda H^\alpha h_{21}^\alpha \vec{e}_1 \wedge \vec{e}_2 - e^\lambda H^\alpha h_{12}^\alpha \vec{e}_2 \wedge \vec{e}_1 \\ &= 0. \end{aligned}$$

Hence

$$(A.18) \quad \nabla \vec{\Phi} \wedge (\nabla \vec{H} - 3\pi_{\vec{n}}(\nabla \vec{H})) = -2\nabla \vec{\Phi} \wedge \nabla \vec{H}.$$

An evident yet nonetheless useful identity is

$$\nabla \vec{\Phi} \cdot \pi_{\vec{n}}(\nabla \vec{H}) = 0,$$

thereby giving

$$(A.19) \quad \nabla \vec{\Phi} \cdot (\nabla \vec{H} - 3\pi_{\vec{n}}(\nabla \vec{H})) = \nabla \vec{\Phi} \cdot \nabla \vec{H}.$$

Bringing altogether (A.15) and (A.18) yields the second part of (A.13). Deriving the first part of (A.13) requires a bit more work. Combining (A.16)

and (A.19) shows that

$$(A.20) \quad \nabla \vec{\Phi} \cdot Q = \nabla \vec{\Phi} \cdot \nabla \vec{H} + 2e^{2\lambda} |\vec{H}|^2.$$

For three indices $(a, b, c) \in \{x_1, x_2\}$, let

$$G(a, b, c) := \partial_a \vec{\Phi} \cdot \langle \vec{e}_b, \partial_c \vec{H} \rangle \vec{e}_b.$$

Just as above, we verify easily that

$$G(a, b, c) = -e^{2\lambda} H^\alpha h_{cb}^\alpha \delta_{ab}.$$

Consequently, there holds

$$\begin{aligned} \nabla \vec{\Phi} \cdot \nabla \vec{H} &= G(x_1, x_1, x_1) + G(x_1, x_2, x_1) + G(x_2, x_1, x_2) + G(x_2, x_2, x_2) \\ &= -e^{2\lambda} H^\alpha h_{11}^\alpha - e^{2\lambda} H^\alpha h_{22}^\alpha \\ &= -2e^{2\lambda} |\vec{H}|^2. \end{aligned}$$

Putting this into (A.20) finally gives the first sought after identity in (A.13). □

Lemma A.3. *Using the notation introduced in Section 3.2.2, the Codazzi–Mainardi identity may be recast in the form*

$$(A.21) \quad e^{-2\lambda} \partial_{\bar{z}}(e^{2\lambda} \overrightarrow{H}_0 \cdot \vec{H}) = \vec{H} \cdot \partial_z \vec{H} + \overrightarrow{H}_0 \cdot \partial_{\bar{z}} \vec{H}.$$

Proof. Consider the 1-form

$$\eta_\alpha := \langle \vec{e}_{\bar{z}}, d\vec{n}_\alpha \rangle.$$

On one hand, from (3.43), there holds

$$\eta_\alpha = (\vec{e}_{\bar{z}} \cdot \partial_z \vec{n}_\alpha) dz + (\vec{e}_{\bar{z}} \cdot \partial_{\bar{z}} \vec{n}_\alpha) d\bar{z} = -\frac{e^\lambda}{2} (H^\alpha dz + H_0^\alpha d\bar{z}),$$

so that

$$(A.22) \quad \star d\eta_\alpha = -\frac{e^\lambda}{2} (H_0^\alpha \partial_z \lambda - H^\alpha \partial_{\bar{z}} \lambda + \partial_z H_0^\alpha - \partial_{\bar{z}} H^\alpha).$$

On the other hand, with (3.42) and (3.43), we find

$$\begin{aligned} \star d\eta_\alpha &= \star d\langle \vec{e}_z, d\vec{n}_\alpha \rangle = \partial_z \vec{e}_z \cdot \partial_{\bar{z}} \vec{n}_\alpha - \partial_{\bar{z}} \vec{e}_z \cdot \partial_z \vec{n}_\alpha \\ &= \frac{e^\lambda}{2} (H_0^\alpha \partial_z \lambda + H^\alpha \partial_{\bar{z}} \lambda + \vec{H} \cdot \partial_{\bar{z}} \vec{n}_\alpha - \vec{H}_0 \cdot \partial_z \vec{n}_\alpha). \end{aligned}$$

Comparing the latter to (A.22) yields

$$(A.23) \quad 2H_0^\alpha \partial_z \lambda = \partial_{\bar{z}} H^\alpha - \partial_z H_0^\alpha - \vec{H} \cdot \partial_{\bar{z}} \vec{n}_\alpha + \vec{H}_0 \cdot \partial_z \vec{n}_\alpha.$$

By antisymmetry, there holds

$$H^\alpha \vec{H} \cdot \partial_{\bar{z}} \vec{n}_\alpha = H^\alpha H^\beta \vec{n}_\beta \cdot \partial_{\bar{z}} \vec{n}_\alpha = 0,$$

and whence

$$(A.24) \quad H^\alpha (\partial_{\bar{z}} H^\alpha - \vec{H} \cdot \partial_{\bar{z}} \vec{n}_\alpha) = H^\alpha (\partial_{\bar{z}} H^\alpha + \vec{H} \cdot \partial_{\bar{z}} \vec{n}_\alpha) \equiv \vec{H} \cdot \partial_{\bar{z}} \vec{H}.$$

Similarly, we find

$$(A.25) \quad \begin{aligned} H^\alpha (\partial_z H_0^\alpha - \vec{H}_0 \cdot \partial_z \vec{n}_\alpha) &= \partial_z (\vec{H}_0 \cdot \vec{H}) - H_0^\alpha \partial_z H^\alpha - H^\alpha \vec{H}_0 \cdot \partial_z \vec{n}_\alpha \\ &= \partial_z (\vec{H}_0 \cdot \vec{H}) - \vec{H}_0 \cdot \partial_z \vec{H}. \end{aligned}$$

Multiplying (A.23) throughout by H^α , summing over α , and using (A.24) and (A.25) gives

$$2\vec{H} \cdot \vec{H}_0 \partial_z \lambda = -\partial_z (\vec{H} \cdot \vec{H}_0) + \vec{H} \cdot \partial_{\bar{z}} \vec{H} + \vec{H}_0 \cdot \partial_z \vec{H},$$

from which the (complex conjugate of the) desired (A.21) ensues. □

Lemma A.4. *Using the notation of Section 3.2, there holds*

$$(A.26) \quad -2\Delta\vec{\Phi} = \nabla\vec{R} \bullet \nabla^\perp\vec{\Phi} + \nabla S \cdot \nabla^\perp\vec{\Phi}.$$

Proof. Note that for any 1-vector \vec{a} , we have

$$(\vec{a} \wedge \vec{e}_j) \bullet \vec{e}_i = (\vec{e}_i \lrcorner \vec{a}) \wedge \vec{e}_j + \vec{a} \wedge (\vec{e}_i \lrcorner \vec{e}_j) = (\vec{e}_i \cdot \vec{a}) \vec{e}_j + \delta_{ij} \vec{a}.$$

From this, and $\vec{e}_i := e^{-\lambda} \partial_{x_i} \vec{\Phi}$, it follows that whenever

$$\vec{V} := V^i \vec{e}_i + V^\alpha \vec{n}_\alpha,$$

then

$$\begin{cases} (\vec{V} \wedge \nabla^\perp \vec{\Phi}) \bullet \nabla^\perp \vec{\Phi} = e^{2\lambda} (3 V^i \vec{e}_i + 2 V^\alpha \vec{n}_\alpha) \\ (\vec{V} \wedge \nabla \vec{\Phi}) \bullet \nabla^\perp \vec{\Phi} = e^{2\lambda} (V^2 \vec{e}_1 - V^1 \vec{e}_2) \equiv (\vec{V} \cdot \nabla \vec{\Phi}) \cdot \nabla^\perp \vec{\Phi}. \end{cases}$$

In particular, since $\vec{H} = H^\alpha \vec{n}_\alpha$, we find that

$$\begin{aligned} \nabla \vec{R} \bullet \nabla^\perp \vec{\Phi} &\equiv -(\vec{L} \wedge \nabla \vec{\Phi} + 2 \vec{H} \wedge \nabla^\perp \vec{\Phi}) \bullet \nabla^\perp \vec{\Phi} \\ &= -(\vec{L} \cdot \nabla \vec{\Phi}) \cdot \nabla^\perp \vec{\Phi} - 4 e^{2\lambda} \vec{H} \\ &= -\nabla S \cdot \nabla^\perp \vec{\Phi} - 4 e^{2\lambda} \vec{H}. \end{aligned}$$

Hence,

$$\nabla \vec{R} \bullet \nabla^\perp \vec{\Phi} + \nabla S \cdot \nabla^\perp \vec{\Phi} = -4 e^{2\lambda} \vec{H}.$$

Finally, there remains to recall that

$$\Delta \vec{\Phi} = 2 e^{2\lambda} \vec{H}$$

to reach the desired identity (A.26). □

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