Smooth metric measure spaces with non-negative curvature

Ovidiu Munteanu and Jiaping Wang

In this paper, we study both function theoretic and spectral properties on complete non-compact smooth metric measure space $(M, g, \mathrm{e}^{-f}dv)$ with non-negative Bakry–Émery Ricci curvature. Among other things, we derive a gradient estimate for positive f-harmonic functions and obtain as a consequence the strong Liouville property under the optimal sublinear growth assumption on f. We also establish a sharp upper bound of the bottom spectrum of the f-Laplacian in terms of the linear growth rate of f. Moreover, we show that if equality holds and M is not connected at infinity, then M must be a cylinder. As an application, we conclude steady Ricci solitons must be connected at infinity.

1. Introduction

On a Riemannian manifold (M, g), the consideration of weighted measure of the form $\mathrm{e}^{-f}dv$, where f is a smooth function and dv is the volume element induced by the metric g, arises naturally in various situations. It can be viewed as the volume form of a suitable conformal change of the metric g. Perhaps a more notable example is in the work of Perelman [27], where he introduces a functional involving the integral of the scalar curvature with respect to such a weighted measure and formulates the Ricci flow as the gradient flow of the functional. The triple $(M, g, \mathrm{e}^{-f}dv)$ is customarily called a smooth metric measure space. The differential operator Δ_f, which is called f-Laplacian and given by

$$\Delta_f := \Delta - \nabla f \cdot \nabla$$

is more naturally associated with such a smooth metric measure space than the classical Laplacian as it is symmetric with respect to the measure $\mathrm{e}^{-f}dv$.

That is,

$$\int_M \langle \nabla\varphi, \nabla\psi \rangle \mathrm{e}^{-f} = -\int_M (\Delta_f\varphi)\psi \mathrm{e}^{-f},$$

for any $\varphi, \psi \in C_0^\infty(M)$.

Again, we point out that the operator f-Laplacian is very much related to the Laplacian of a suitable conformal change of the background Riemannian metric. It also appears as the generator of a class of stochastic diffusion processes, the Brownian motion with drifts.

The Bakry–Émery Ricci tensor [1] of the metric measure space $(M, g, \mathrm{e}^{-f} dv)$ is defined by

$$\mathrm{Ric}_f := \mathrm{Ric} + \mathrm{Hess}(f),$$

where Ric denotes the Ricci curvature of M and $\mathrm{Hess}(f)$ the Hessian of f. This curvature relates to f-Laplacian via the following Bochner formula:

$$\Delta_f |\nabla u|^2 = 2|\mathrm{Hess}(u)|^2 + 2\langle \nabla u, \nabla \Delta_f u \rangle + 2\mathrm{Ric}_f(\nabla u, \nabla u).$$

This of course suggests the important role of Ric_f in the analysis of f-Laplacian. Perhaps as a more prominent example, Ric_f also appears in the study of the Ricci flow. The gradient solitons of the Ricci flow, which arise from the singularity analysis of the Ricci flow, are defined to be complete manifolds (M, g) that the following equation

$$\mathrm{Ric}_f = \lambda g$$

holds for some function f and constant λ. Obviously, the Einstein manifolds are gradient Ricci solitons. The gradient Ricci solitons are called shrinking, steady and expanding accordingly when $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$, [15].

The classification of gradient Ricci solitons is an important problem from the point of view of both the Ricci flow singularity analysis and purely as a class of geometric partial differential equations. The problem has received much attention recently. The book [7] is a good source for some of the important results. But it seems fair to say that the whole picture is far from clear for now.

Partially motivated by interest in the study of gradient Ricci solitons, various attempts have been made recently to study the geometry and analysis on general metric measure spaces. We will refer the readers to [22, 31] for some of the results. It should be noted however a while back, Lichnerowicz [23] has already done some pioneering work in this direction. In

particular, he has extended the classical Cheeger–Gromoll splitting theorem to the metric measure spaces with $\text{Ric}_f \geq 0$ and f bounded.

In this paper, we will investigate some function theoretic and spectral properties of metric measure space $(M, g, e^{-f}dv)$. While the results are of independent interest, applications to the steady gradient Ricci solitons are no doubt of our main focus. Throughout, we will assume $\text{Ric}_f \geq 0$ and f is of linear growth unless otherwise noted. Recall that f is of linear growth if for all x in M,

$$|f|(x) \leq \alpha\, r(x) + \beta$$

for some constants α and β, where $r(x) := d(p, x)$ is the geodesic distance function to a fixed point p in M. The linear growth rate a of f is then defined as the infimum of all such values α.

Clearly, a steady gradient Ricci soliton (M, g) satisfies $\text{Ric}_f = 0$ for some f. It is also well-known that the potential function f in this case is of linear growth. So all of our results are applicable to the steady gradient solitons. On the other hand, it should also be noted that many, if not all, of our results will fail without the growth assumption on function f.

Our first result gives a gradient estimate for positive f-harmonic functions on $(M, g, e^{-f}dv)$.

Theorem 1.1. *Let $(M, g, e^{-f}dv)$ be a complete non-compact smooth metric measure space with $\text{Ric}_f \geq 0$. Assume that f has linear growth rate a and let $u > 0$ be f-harmonic on M, i.e., $\Delta_f u = 0$ on M. Then the following gradient estimate holds true on M:*

$$|\nabla \log u| \leq C(n)a,$$

where the constant $C(n)$ depends only on the dimension n. In particular, if f is of sublinear growth, then any positive f-harmonic function u on M must be constant.

This estimate is sharp as demonstrated by the following simple example.

Example 1.1 [26]. Let $M = \mathbb{R} \times \mathbb{S}^{n-1}$ and $f(t, \theta) = at$ for $t \in \mathbb{R}$ and $\theta \in \mathbb{S}^{n-1}$. Then $u(t, \theta) := e^{at}$ is positive f-harmonic on M. Clearly, the linear growth rate of f is a and $|\nabla \log u| = a$.

Although the statement in the theorem takes the form of Yau's classical result on the positive harmonic functions, we would like to point out that our proof is quite different. In the classical case of the Ricci curvature,

Yau [34] directly works with $\log u$ and obtains an estimate on $|\nabla \log u|$ via the Bochner formula. This approach works also with the N-Bakry–Émery Ricci curvature given by

$$\mathrm{Ric}_f^N := \mathrm{Ric} + \mathrm{Hess}_f - \frac{1}{N} df \otimes df$$

as demonstrated by Li [22]. For the case of curvature Ric_f, if one imposes suitable assumption on $|\nabla f|$, say, it is bounded, then it is still possible to utilize Yau's argument as shown by Wu [32]. However, with only the growth assumption on f, this direct approach seems to run into essential obstacles. Our argument relies on both Yau's idea and the well-known De Giorgi–Nash–Moser theory. In a recent paper [2], Brighton, by applying Yau's idea to function u^ϵ instead of $\log u$, proved bounded f-harmonic functions must be constant without any assumption on f so long as $\mathrm{Ric}_f \geq 0$. In our proof, we first refined Brighton's argument and derived the gradient estimate under the assumption that u is of exponential growth. Here, no growth assumption on f is necessary. In particular, this implies that any sub-exponential growth positive f-harmonic on M with $\mathrm{Ric}_f \geq 0$ is constant. The growth assumption on f was then used to get the desired growth control on u. For that, we utilize a different set of techniques including a mean value inequality obtained through the Moser's iteration argument.

We also deal with the polynomial growth f-harmonic functions. Here, the result is very much parallel to the case of harmonic functions on a manifold with non-negative Ricci curvature, obtained by Cheng and Yau [10], Li–Tam [18], Li [17] and Colding–Minicozzi [11], respectively. We define the space

$$\mathcal{H}^d(M) := \{u : \ \Delta_f u = 0 \quad \text{and} \quad |u|(x) \leq C(r(x)+1)^d\}.$$

Theorem 1.2. *Let $(M, g, e^{-f}dv)$ be a complete non-compact smooth metric measure space with $\mathrm{Ric}_f \geq 0$ and f bounded. Then we have*

$$\dim \mathcal{H}^d(M) = 1, \quad \text{if } d < 1$$
$$\dim \mathcal{H}^d(M) \leq n + 1, \quad \text{if } d = 1$$

and

$$\dim \mathcal{H}^d(M) \leq C d^{n-1}, \quad \text{if } d \geq 1.$$

Our second objective is to study the spectrum of the f-Laplacian on $(M, g, \mathrm{e}^{-f}dv)$. Define $\lambda_1(\Delta_f) := \min \mathrm{Spec}(-\Delta_f)$ to be the bottom spectrum

of Δ_f. By the variational characterization, we have

$$\lambda_1(\Delta_f) = \inf_{\phi \in C_0^\infty(M)} \frac{\int_M |\nabla\phi|^2 \mathrm{e}^{-f}}{\int_M \phi^2 \mathrm{e}^{-f}}.$$

The following result summarizes what we proved concerning the bottom spectrum.

Theorem 1.3. *Let $(M, g, \mathrm{e}^{-f} dv)$ be a complete non-compact smooth metric measure space with $Ric_f \geq 0$. Then*

$$\lambda_1(\Delta_f) \leq \frac{1}{4}a^2,$$

where a is the linear growth rate of f. Moreover, if $\lambda_1(\Delta_f) = \frac{1}{4}a^2$, then M is connected at infinity; or M is isometric to $\mathbb{R} \times N$ for some compact manifold N.

Note that the splitting case in the theorem does occur. Indeed, for $M = \mathbb{R} \times N$, if we take $f(t, y) = at$ for $(t, y) \in \mathbb{R} \times N$, then $|\nabla f| = a$, $\Delta_f \mathrm{e}^{\frac{1}{2}at} = -\frac{1}{4}a^2 \mathrm{e}^{\frac{1}{2}at}$ and $\lambda_1(\Delta_f) = \frac{1}{4}a^2$.

Applying the preceding result to the gradient steady Ricci solitons, we obtain the following.

Corollary 1.1. *A non-trivial gradient steady Ricci soliton must be connected at infinity.*

This is because for a steady gradient Ricci soliton (M, g, f) such that $Ric_f = 0$ on M and $\sup |\nabla f| = a$, one can show $\lambda_1(\Delta_f) = \frac{1}{4}a^2$. Note now that the splitting case cannot arise as otherwise $Hess(f) = 0$ and M would be Ricci flat. We remark that previous attempts on the issue of connectedness at infinity for steady solitons can be found in [12, 24].

Historically, Cheng [9] proved a sharp upper bound of the bottom spectrum of the Laplacian on a complete manifold with Ricci curvature bounded below. Later on, in [19, 20], Li and the second author studied the rigidity issue when the sharp upper bound is achieved. Our results here are very much in the same spirit. Our arguments, however, follow [21] more closely.

The paper is organized as follows. In Section 2, we discuss Laplacian and volume comparison estimates and establish the upper bound estimate for the bottom spectrum $\lambda_1(\Delta_f)$. In Section 3, we prove the gradient estimate for positive f-harmonic functions and related Liouville type results concerning

f-harmonic functions of polynomial growth. Finally, in Section 4 we discuss the structure of manifolds with maximal bottom spectrum and the resulting application to the steady Ricci solitons.

In a sequel to this paper [25], we will address some similar issues on smooth metric measure spaces with Ric_f bounded below. Among other things, we obtain a gradient estimate as Theorem 1.1 of the form $|\nabla \log u| \leq c(n, a)$ if $\mathrm{Ric}_f \geq -(n-1)$, where a now is the maximum oscillation of f over all unit balls in M. A parallel result to Theorem 1.3 also holds true. More precisely, the bottom spectrum satisfies $\lambda_1(\Delta_f) \leq \frac{1}{4}(n-1+a)^2$ if $\mathrm{Ric}_f \geq -(n-1)$, where a again is the linear growth rate of f. Under slightly stronger assumption on f that its gradient is bounded by a, then that the equality holds implies the manifold M must be connected at infinity unless it is a product $\mathbb{R} \times N$ with N being compact.

2. Volume comparison theorems

In this section, we discuss Laplacian and volume comparison estimates for smooth metric measure spaces with non-negative Bakry–Émery Ricci curvature. The estimates in this section are instrumental in proving other results of this paper. Also, as an immediate application, we obtain upper bound estimates for the bottom spectrum of Δ_f.

Let $(M, g, \mathrm{e}^{-f} dv)$ be a smooth metric measure space. Take any point $x \in M$ and denote the volume form in geodesic polar coordinates centered at x with

$$dV|_{\exp_x(r\xi)} = J(x, r, \xi) dr d\xi,$$

where $r > 0$ and $\xi \in S_x M$, a unit tangent vector at x. It is well-known that if $y \in M$ is any point such that $y = \exp_x (r\xi)$, then

$$\Delta d(x, y) = \frac{J'(x, r, \xi)}{J(x, r, \xi)} \quad \text{and} \quad \Delta_f d(x, y) = \frac{J'_f(x, r, \xi)}{J_f(x, r, \xi)},$$

where $J_f(x, r, \xi) := \mathrm{e}^{-f} J(x, r, \xi)$ is the f-volume form in geodesic polar coordinates. For a fixed point $p \in M$ and $R > 0$, define

$$(2.1) \qquad\qquad A(R) := \sup_{x \in B_p(3R)} |f|(x).$$

For a set Ω, we will denote by $V(\Omega)$ the volume of Ω with respect to the usual volume form dv, and $V_f(\Omega)$ the f-volume of Ω. The following result has been established in [33].

Lemma 2.1. *Let $(M, g, e^{-f} dv)$ be a smooth metric measure space with $Ric_f \geq 0$. Then along any minimizing geodesic starting from $x \in B_p(R)$ we have*

$$\frac{J_f(x, r_2, \xi)}{J_f(x, r_1, \xi)} \leq e^{4A} \left(\frac{r_2}{r_1}\right)^{n-1} \quad and \quad \frac{J(x, r_2, \xi)}{J(x, r_1, \xi)} \leq e^{6A} \left(\frac{r_2}{r_1}\right)^{n-1}$$

for any $0 < r_1 < r_2 < R$. In particular, for any $0 < r_1 < r_2 < R$,

$$\frac{V_f(B_x(r_2))}{V_f(B_x(r_1))} \leq e^{4A} \left(\frac{r_2}{r_1}\right)^{n} \quad and \quad \frac{V(B_x(r_2))}{V(B_x(r_1))} \leq e^{6A} \left(\frac{r_2}{r_1}\right)^{n}.$$

Here, $A = A(R)$ as defined by (2.1).

Proof of Lemma 2.1. We include a proof here for the reader's convenience. Some of the ingredients of this proof will also be used later.

Let γ be the minimizing geodesic from x to y such that $\gamma(0) = x$ and $\gamma(r) = y$. Recall the following Laplace comparison theorem [12, 31],

$$(2.2) \qquad \Delta_f d(x, y) \leq \frac{n-1}{r} - \frac{2}{r^2} \int_0^r t f'(t) dt,$$

where $f(t) := f(\gamma(t))$. Integrating by parts, we get

$$(2.3) \qquad \Delta_f d(x, y) \leq \frac{n-1}{r} - \frac{2}{r} f(r) + \frac{2}{r^2} \int_0^r f(t) dt.$$

For $0 < r_1 < r_2 < R$, integrating (2.3) from r_1 to r_2 yields

$$(2.4) \quad \log\left(\frac{J_f(x, r_2, \xi)}{J_f(x, r_1, \xi)}\right) \leq (n-1) \log\left(\frac{r_2}{r_1}\right) + 2 \int_{r_1}^{r_2} \frac{1}{r^2} \left(\int_0^r f(t) dt\right) dr$$
$$- 2 \int_{r_1}^{r_2} \frac{1}{r} f(r) dr.$$

However,

$$\int_{r_1}^{r_2} \frac{1}{r^2} \left(\int_0^r f(t) dt\right) dr = -\frac{1}{r} \left(\int_0^r f(t) dt\right) \Big|_{r_1}^{r_2} + \int_{r_1}^{r_2} \frac{1}{r} f(r) dr.$$

Plugging into (2.4), we conclude

$$(2.5) \qquad \frac{J_f(x, r_2, \xi)}{J_f(x, r_1, \xi)} \leq \left(\frac{r_2}{r_1}\right)^{n-1} \exp\left(\frac{2}{r_1} \int_0^{r_1} f(t) dt - \frac{2}{r_2} \int_0^{r_2} f(t) dt\right).$$

Therefore,

$$\frac{J_f(x, r_2, \xi)}{J_f(x, r_1, \xi)} \leq e^{4A} \left(\frac{r_2}{r_1}\right)^{n-1}$$

for all $x \in B_p(R)$ and $0 < r_1 < r_2 < R$. Clearly, the corresponding result for $J(x, r, \xi)$ follows by using again the definition (2.1).

To establish the volume comparison, we use that

$$\frac{J_f(x, t, \xi)}{J_f(x, s, \xi)} \leq e^{4A} \left(\frac{t}{s}\right)^{n-1}$$

for any $0 < s < r_1 < t < r_2 < R$. Integrating in t from r_1 to r_2 and s from 0 to r_1, we get

$$\frac{V_f(B_x(r_2)) - V_f(B_x(r_1))}{V_f(B_x(r_1))} \leq e^{4A} \frac{(r_2)^n - (r_1)^n}{(r_1)^n}.$$

This implies that

$$\frac{V_f(B_x(r_2))}{V_f(B_x(r_1))} \leq e^{4A} \left(\frac{r_2}{r_1}\right)^n + 1 - e^{4A} \leq e^{4A} \left(\frac{r_2}{r_1}\right)^n.$$

Now the volume comparison for $V(B_x(r))$ follows directly from here. The lemma is proved. $\qquad\square$

Using the proof of Lemma 2.1 we obtain a sharp upper bound for $\lambda_1(\Delta_f)$, the bottom spectrum of Δ_f, by assuming f is of linear growth. Recall

$$\lambda_1(\Delta_f) := \inf_{\phi \in C_0^\infty(M)} \frac{\int_M |\nabla \phi|^2 e^{-f}}{\int_M \phi^2 e^{-f}}.$$

Theorem 2.1. *Let $(M, g, e^{-f} dv)$ be a complete non-compact smooth metric measure space with $Ric_f \geq 0$. If there exist positive constants $a, b > 0$ such that*

$$|f|(x) \leq ar(x) + b \quad \text{for all } x \in M,$$

then we have the upper bound estimate

$$\lambda_1(\Delta_f) \leq \frac{1}{4}a^2.$$

In particular, if f has sublinear growth, then $\lambda_1(\Delta_f) = 0$.

Proof of Theorem 2.1. By setting $x = p$, $r_1 = 1$ and $r_2 = R > 1$ and noting $|f|(x) \leq ar(x) + b$ on M, (2.5) implies

$$J_f(p, R, \xi) \leq CR^{n-1}e^{-\frac{2}{R}\int_0^R f(t)dt} \leq CR^{n-1}e^{aR}$$

for all $R > 1$. Therefore,

(2.6) $$V_f(B_p(R)) \leq CR^n e^{aR}.$$

We now claim $\lambda_1(\Delta_f) \leq \frac{1}{4}a^2$. Indeed, take a cut-off ψ on $B_p(R)$ such that $\psi = 1$ on $B_p(R-1)$, $\psi = 0$ on $M \backslash B_p(R)$ and $|\nabla\psi| \leq c$. Consider $\phi(y) := e^{-\frac{1}{2}(a+\varepsilon)r(y)}\psi(y)$ as a test function in the variational principle for $\lambda_1(\Delta_f)$. Using the volume growth (2.6) we get immediately that $\lambda_1(\Delta_f) \leq \frac{1}{4}(a+\varepsilon)^2$. Since $\varepsilon > 0$ is arbitrary this implies the desired estimate. The theorem is proved. $\qquad\square$

As indicated in the Introduction, the estimate in Theorem 2.1 is sharp. We now show that the bottom spectrum of steady Ricci solitons also attains this upper bound. Recall a gradient steady Ricci soliton is a manifold (M, g, f) satisfying $R_{ij} + f_{ij} = 0$. It is known that there exists a positive constant $a > 0$ such that $|\nabla f|^2 + S = a^2$, where S is the scalar curvature of M. It is also known that $S \geq 0$ for any gradient steady Ricci soliton [4, 8]. Another useful relation is $\Delta f + S = 0$, which is obtained directly from $\mathrm{Ric}_f = 0$ by taking trace. In summary, a steady Ricci soliton satisfies the following.

(2.7) $$|\nabla f|^2 + S = a^2, \quad \text{for some constant } a > 0$$
$$\Delta f + S = 0$$
$$S \geq 0.$$

We first recall a well-known result.

Lemma 2.2. *Let $(M, g, e^{-f}dv)$ be a smooth metric measure space. If there exists a positive function $v > 0$ such that $\Delta_f v \leq -\lambda v$ for some constant $\lambda > 0$, then $\lambda_1(\Delta_f) \geq \lambda$.*

Proof of Lemma 2.2. For completeness, we include a proof of this result. For an exhaustion of M by compact domains $\Omega_i \subset\subset M$, consider the first

Dirichlet eigenfunction u_i.

$$\Delta_f u_i = -\lambda_1(\Omega_i)u_i \quad \text{in } \Omega_i$$
$$u_i = 0 \quad \text{on } \partial\Omega_i.$$

It is known that we may assume $u_i > 0$. By the strong maximum principle, $\frac{\partial u_i}{\partial \eta} < 0$ on $\partial\Omega_i$. Now,

$$(\lambda_1(\Omega_i) - \lambda)\int_{\Omega_i} vu_i e^{-f} \geq \int_{\Omega_i}(u_i\Delta_f v - v\Delta_f u_i)e^{-f}$$
$$= \int_{\partial\Omega_i}\left(u_i\frac{\partial v}{\partial \eta} - v\frac{\partial u_i}{\partial \eta}\right)e^{-f} = -\int_{\partial\Omega_i} v\frac{\partial u_i}{\partial \eta}e^{-f} > 0.$$

Since both u_i and v are positive, this shows $\lambda_1(\Omega_i) \geq \lambda$. The Lemma follows from

$$\lim_{i\to\infty}\lambda_1(\Omega_i) = \lambda_1(\Delta_f).$$

$\square$

Proposition 2.1. *Let (M, g, f) be a gradient steady Ricci soliton, normalized as in (2.7). Then $\lambda_1(\Delta_f) = \frac{a^2}{4}$.*

Proof of Proposition 2.1. Since $S \geq 0$, it follows that $|\nabla f| \leq a$. Therefore, $|f|(x) \leq ar(x) + b$ for any $x \in M$. So by Theorem 2.1, $\lambda_1(\Delta_f) \leq \frac{1}{4}a^2$. To show the equality, we proceed as follows. Observe that

$$\Delta_f e^{\frac{1}{2}f} = \left(\frac{1}{2}\Delta_f(f) + \frac{1}{4}|\nabla f|^2\right)e^{\frac{1}{2}f}.$$

But for a steady soliton,

$$\Delta_f(f) = \Delta f - |\nabla f|^2 = -(S + |\nabla f|^2) = -a^2.$$

Since $|\nabla f| \leq a$, we conclude that

$$\Delta_f e^{\frac{1}{2}f} = \left(-\frac{1}{2}a^2 + \frac{1}{4}|\nabla f|^2\right)e^{\frac{1}{2}f}$$
$$\leq \left(-\frac{1}{2}a^2 + \frac{1}{4}a^2\right)e^{\frac{1}{2}f} = -\frac{a^2}{4}e^{\frac{1}{2}f}.$$

By Lemma 2.2, we have $\lambda_1(\Delta_f) \geq \frac{a^2}{4}$. This proves Proposition 2.1. $\square$

3. f-harmonic functions

In this section, we continue to assume $\mathrm{Ric}_f \geq 0$ on $(M, g, \mathrm{e}^{-f} dv)$. The main objective is to derive the following global gradient estimate for positive f-harmonic functions defined on M. This in particular leads to a strong Liouville theorem under optimal growth assumption on f.

For a fixed point $p \in M$, we let $r(x) := d(p, x)$.

Theorem 3.1. *Let $(M, g, e^{-f} dv)$ be a complete non-compact smooth metric measure space with $\mathrm{Ric}_f \geq 0$. Assume there exist positive constants a and b such that*

$$|f|(x) \leq a\, r(x) + b \quad on\ M.$$

Let $u > 0$ be f-harmonic on M. Then the following gradient estimate holds true on M:

$$|\nabla \log u| \leq C(n)a,$$

where constant $C(n)$ depending only on n, the dimension of M.

An immediate consequence is the following strong Liouville property. As noted by the explicit example in first section, the growth assumption on f is optimal.

Corollary 3.1. *Let $(M, g, e^{-f} dv)$ be a complete non-compact smooth metric measure space with $\mathrm{Ric}_f \geq 0$. If f is of sublinear growth, then any positive f-harmonic function u on M is constant.*

We prove Theorem 3.1 in several steps. First, we show that $|\nabla \log u|$ can be controlled from above by $\frac{1}{R} \sup_{B_p(R)} \log(u + 1)$. This follows by adapting Yau's [34] argument. We then verify that u must be of exponential growth with the exponent controlled by the constant a. To achieve this, we use the Moser iteration technique, following the ideas in [13, 28].

Proposition 3.1. *Let $(M, g, e^{-f} dv)$ be a complete non-compact smooth metric measure space with $\mathrm{Ric}_f \geq 0$. Assume that there exist constants $a > 0$ and $b > 0$ such that*

$$|f|(x) \leq ar(x) + b \quad on\ M.$$

Then for any positive f-harmonic function u on M, we have

$$\sup_M |\nabla \log u|^2 \leq C(n)(\Omega(u)^2 + a\Omega(u)),$$

where $C(n) > 0$ is a constant only depending on the dimension n of M and

$$\Omega(u) := \limsup_{R \to \infty} \left\{ \frac{1}{R} \sup_{B_p(R)} \log(u+1) \right\}.$$

Proof of Proposition 3.1. The proof is based on Yau's argument using the Bochner technique, with a modification similar to that in [2]. Let $u > 0$ be f-harmonic. For $0 < \epsilon < \frac{1}{2}$, define

$$h := \frac{1}{\epsilon} u^\epsilon.$$

Then a direct computation gives $\Delta_f h = (\epsilon - 1) u^{\epsilon-2} |\nabla u|^2$.

Let us denote $\sigma := |\nabla h|^2 = u^{2\epsilon-2} |\nabla u|^2$. The Bochner formula asserts that

$$\begin{aligned}
\frac{1}{2} \Delta_f \sigma &= |h_{ij}|^2 + \langle \nabla h, \nabla(\Delta_f h) \rangle + \mathrm{Ric}_f(\nabla h, \nabla h) \\
&\geq \langle \nabla h, \nabla(\Delta_f h) \rangle = (\epsilon - 1) \langle \nabla h, \nabla(u^{\epsilon-2} |\nabla u|^2) \rangle \\
&= (\epsilon - 1) \langle \nabla h, \nabla(u^{-\epsilon} \sigma) \rangle.
\end{aligned}$$

Notice that

$$\begin{aligned}
\langle \nabla h, \nabla(u^{-\epsilon} \sigma) \rangle &= -\epsilon \langle \nabla h, \nabla u \rangle u^{-\epsilon-1} \sigma + u^{-\epsilon} \langle \nabla h, \nabla \sigma \rangle \\
&= -\epsilon |\nabla h|^2 u^{-2\epsilon} \sigma + u^{-\epsilon} \langle \nabla h, \nabla \sigma \rangle.
\end{aligned}$$

Consequently,

(3.1) $$\frac{1}{2} \Delta_f \sigma \geq \epsilon(1 - \epsilon) u^{-2\epsilon} \sigma^2 + (\epsilon - 1) u^{-\epsilon} \langle \nabla h, \nabla \sigma \rangle.$$

We now take a function $\phi : [0, 2R] \to [0, 1]$ with the following properties:

$$\begin{aligned}
\phi &= 1 \quad \text{on} \quad [0, R] \\
\mathrm{supp}(\phi) &\subseteq [0, 2R] \\
-\frac{c}{R} &\leq \frac{\phi'}{\sqrt{\phi}} \leq 0 \\
|\phi''| &\leq \frac{c}{R^2},
\end{aligned}$$

where $c > 0$ is a universal constant. We continue to denote by $c > 0$ some universal constant, which may differ from line to line. We use this function

to define a cut-off on M by taking $\phi(x) := \phi(r(x))$. Define $G := \phi\sigma$. Then G is non-negative on M and has compact support in $B_p(2R)$. Therefore, it achieves its maximum at some point $y \in B_p(2R)$.

Without loss of generality we can assume that y is not in the cut-locus of p. So ϕ is smooth at y. Now at point y, we have

$$(3.2) \qquad\qquad \Delta G(y) \le 0 \quad \text{and} \quad \nabla G(y) = 0.$$

By (2.3) together with $|f|(x) \le ar(x) + b$, we get

$$\Delta_f r(x) \le \frac{n-1+4b}{r} + 3a.$$

So there exists $r_0 > 0$ such that $\Delta_f r(x) \le 4a$ for $x \in M \backslash B_p(r_0)$. Therefore, for $R > r_0$, we have

$$\Delta_f \phi = \phi' \Delta_f r + \phi'' \ge -c\left(\frac{a}{R} + \frac{1}{R^2}\right)$$

and

$$\phi^{-1}|\nabla\phi|^2 \le \frac{c}{R^2}.$$

Combining with inequality (3.1), we find that

$$\begin{aligned}
\frac{1}{2}\Delta_f G &= \frac{1}{2}\phi\Delta_f\sigma + \frac{1}{2}\sigma\Delta_f\phi + \langle\nabla\phi, \nabla\sigma\rangle \\
&\ge \epsilon(1-\epsilon)u^{-2\epsilon}\phi^{-1}G^2 + (\epsilon-1)u^{-\epsilon}\langle\nabla h, \nabla(\phi^{-1}G)\rangle\phi \\
&\quad - c\left(\frac{a}{R} + \frac{1}{R^2}\right)\phi^{-1}G + \langle\nabla\phi, \nabla(\phi^{-1}G)\rangle \\
&= \epsilon(1-\epsilon)u^{-2\epsilon}\phi^{-1}G^2 + (\epsilon-1)u^{-\epsilon}\langle\nabla h, \nabla\phi^{-1}\rangle G\phi \\
&\quad + (\epsilon-1)u^{-\epsilon}\langle\nabla h, \nabla G\rangle - c\left(\frac{a}{R} + \frac{1}{R^2}\right)\phi^{-1}G \\
&\quad - |\nabla\phi|^2\phi^{-2}G + \phi^{-1}\langle\nabla\phi, \nabla G\rangle.
\end{aligned}$$

After multiplying both sides by ϕ and invoking (3.2), we conclude that at y,

$$0 \ge \epsilon(1-\epsilon)u^{-2\epsilon}G^2 - (1-\epsilon)u^{-\epsilon}|\nabla h||\nabla\phi|G - c\left(\frac{a}{R} + \frac{1}{R^2}\right)G.$$

Note that

$$u^{-\epsilon}|\nabla h||\nabla\phi| \le \frac{c}{R}u^{-\epsilon}\sigma^{\frac{1}{2}}\phi^{\frac{1}{2}} = \frac{c}{R}(u^{-2\epsilon}G)^{\frac{1}{2}}.$$

This shows at point y,

$$\epsilon(u^{-2\epsilon}G) - \frac{c}{R}(u^{-2\epsilon}G)^{\frac{1}{2}} - c\left(\frac{a}{R} + \frac{1}{R^2}\right) \le 0.$$

Solving this as a quadratic inequality in $(u^{-2\epsilon}G)^{\frac{1}{2}}$ we get

$$u^{-2\epsilon}(y)G(y) \le \frac{c}{(\epsilon R)^2} + \frac{ca}{\epsilon R}.$$

This proves that

$$\sup_{B_p(R)} (u^{2\epsilon}|\nabla \log u|^2) = \sup_{B_p(R)} \sigma \le \sup_{B_p(2R)} G$$
$$\le \left(\frac{c}{(\epsilon R)^2} + \frac{ca}{\epsilon R}\right) \sup_{B_p(2R)} (u^{2\epsilon}).$$

We now observe that if u is globally bounded on M, then the estimate implies u is constant by letting $R \to \infty$. That means the gradient estimate claimed in the Proposition is automatically true. For unbounded u, we let

$$\epsilon := \left(2 + \sup_{B_p(2R)} \log(u+1)\right)^{-1} > 0.$$

Then,

$$\sup_{B_p(2R)} (u^{2\epsilon}) \le \mathrm{e}^2.$$

So we obtain, for any $R > 0$ and $r < R$, that

$$\sup_{B_p(r)} (u^{2\epsilon}|\nabla \log u|^2) \le \left(\frac{c}{R} \sup_{B_p(2R)} \log(u+1)\right)^2$$
$$+ \frac{ca}{R} \sup_{B_p(2R)} \log(u+1) + \frac{c(a+1)}{R}.$$

Since u is unbounded, it is clear that $\epsilon \to 0$ as $R \to \infty$. Therefore, after letting $R \to \infty$ with r fixed, we arrive at

$$\sup_{B_p(r)} |\nabla \log u|^2 \le C(n)(\Omega(u)^2 + a\Omega(u)).$$

Since r is arbitrary, this proves the proposition. $\qquad\square$

Let us point out that it is possible to prove another version of Proposition 3.1 without any growth assumption on f. It takes the form of $\sup_M |\nabla \log u|^2 \leq C(\Omega(u)^2 + \Omega(u))$ for some constant C depending on the Ricci curvature lower bound and $\sup |\nabla f|$ on $B_p(1)$. This is because we can use the Laplace comparison Theorem 3.1 of [31] instead, which does not require any assumption on f. It in particular says that a positive f-harmonic function of sub-exponential growth on a complete manifold with $\mathrm{Ric}_f \geq 0$ must be a constant.

In the next step, we will establish an upper bound estimate for $\Omega(u)$ defined in Proposition 3.1 by using Moser iteration argument. First, we will establish a local Sobolev inequality on M, following the arguments in [3, 13, 14, 28]. Since it will be crucial to have explicit and accurate dependency of the constants appearing in the inequality in terms of the growth of f, we provide details of the proof here.

We now use Lemma 2.1 to prove a Neumann Poincaré inequality. For this, we follow Buser's proof, see [3] (also cf.[5, p. 354]). There is an alternate proof, see [29, p. 176], which first establishes a weaker version of Neumann Poincaré and then uses a covering argument to prove the strong version. For the proof of Theorem 3.1, the weaker version of Neumann Poincaré inequality is in fact sufficient. Recall

$$(3.3) \qquad A(R) := \sup_{x \in B_p(3R)} |f|(x).$$

In the following, we will suppress R in $A(R)$ and simply call it A.

Lemma 3.1. *Let $(M, g, e^{-f}dv)$ be a smooth metric measure space with $\mathrm{Ric}_f \geq 0$. Then for any $x \in B_p(R)$ we have*

$$\int_{B_x(r)} |\varphi - \varphi_{B_x(r)}|^2 \leq c_1 e^{c_2 A} \cdot r^2 \int_{B_x(r)} |\nabla \varphi|^2$$

for all $0 < r < R$ and $\varphi \in C^\infty(B_x(r))$, where $\varphi_{B_x(r)} := V^{-1}(B_x(r)) \int_{B_x(r)} \varphi$. The constants c_1 and c_2 depend only on the dimension n.

Proof of Lemma 3.1. First, we show that Lemma 2.1 and the argument in [3] imply a lower bound on the isoperimetric constant. Let Γ be a smooth hypersurface in $B_x(r)$ with $\bar{\Gamma}$ imbedded in $\overline{B_x(r)}$. Let D_1 and D_2 be disjoint open subsets in $B_x(r)$ such that $D_1 \cup D_2 = B_x(r) \backslash \Gamma$. We will show that

$$(3.4) \qquad \min\{V(D_1), V(D_2)\} \leq rc_1 e^{c_2 A} A(\Gamma).$$

Then, by Cheeger's theorem, the inequality (3.4) gives the claimed Neumann Poincaré inequality.

We fix D_1 so that $V\left(D_1 \cap B_x\left(\frac{r}{2}\right)\right) \leq \frac{1}{2}V\left(B_x\left(\frac{r}{2}\right)\right)$. For a fixed $\alpha \in (0,1)$ to be chosen later, consider first the case when $V\left(D_1 \cap B_x\left(\frac{r}{2}\right)\right) \leq \alpha V(D_1)$. We denote by $C(x)$ the cut-locus of x. For $y \in D_1 \backslash C(x)$, let y^* be the first intersection point of yx, the unique minimizing geodesic from y to x, with Γ. In the case yx does not intersect Γ, we set $y^* = x$.

Define

$$\mathcal{A}_1 := \left\{ y \in D_1 \backslash \left(C(x) \cup B_x\left(\frac{r}{2}\right) \right) : \quad y^* \notin B_x\left(\frac{r}{4}\right) \right\}$$

$$\mathcal{A}_2 := \left\{ y \in D_1 \backslash \left(C(x) \cup B_x\left(\frac{r}{2}\right) \right) : \quad y^* \in B_x\left(\frac{r}{4}\right) \right\}$$

$$\mathcal{A}_3 := \left(B_x\left(\frac{r}{2}\right) \backslash B_x\left(\frac{r}{4}\right) \right) \cap \left(\cup_{y \in \mathcal{A}_2} \mathrm{rod}(y) \right),$$

where $\mathrm{rod}(y) := \left\{ \exp_x(\tau\xi) : \quad \frac{r}{4} < \tau < s \right\}$ for $y = \exp_x(s\xi)$ with $\xi \in S_x M$ and $0 < s < r$.

From now on, we will use c_1 and c_2 to denote constants depending only on dimension n. By Lemma 2.1 we have $\frac{V(\mathcal{A}_2)}{V(\mathcal{A}_3)} \leq c_1 \mathrm{e}^{c_2 A}$. Note that $V(\mathcal{A}_1) + V(\mathcal{A}_2) = V\left(D_1 \backslash B_x\left(\frac{r}{2}\right)\right) \geq (1-\alpha)V(D_1)$ and $V(\mathcal{A}_3) \leq V\left(D_1 \cap B_x\left(\frac{r}{2}\right)\right) \leq \alpha V(D_1)$. Thus,

$$(1-\alpha)V(D_1) \leq V(\mathcal{A}_1) + V(\mathcal{A}_2) \leq V(\mathcal{A}_1) + c_1 \mathrm{e}^{c_2 A} V(\mathcal{A}_3)$$
$$\leq V(\mathcal{A}_1) + \alpha c_1 \mathrm{e}^{c_2 A} V(D_1),$$

or

$$(3.5) \qquad V(\mathcal{A}_1) \geq (1 - \alpha c_1 \mathrm{e}^{c_2 A})V(D_1).$$

By setting

$$(3.6) \qquad \alpha := \frac{1}{2}(c_1)^{-1}\mathrm{e}^{-c_2 A},$$

we conclude $V(\mathcal{A}_1) \geq \frac{1}{2}V(D_1)$. Moreover, Lemma 2.1 implies that

$$\frac{A(\Gamma)}{V(\mathcal{A}_1)} \geq (c_3)^{-1}\mathrm{e}^{-c_4 A}r^{-1},$$

where constants c_3 and c_4 depending only on n.

This, together with (3.5), proves the lemma in the case when $V\left(D_1 \cap B_x\left(\frac{r}{2}\right)\right) \leq \alpha V(D_1)$, where α is given by (3.6).

Now we assume that $V\left(D_1 \cap B_x\left(\frac{r}{2}\right)\right) \geq \alpha V(D_1)$ and finish the proof of the lemma. We need the following general fact, see [3]. Set $W_0 := D_1 \cap B_x\left(\frac{r}{2}\right)$ and $W_1 := D_2 \cap B_x\left(\frac{r}{2}\right)$ or vice versa $W_0 := D_2 \cap B_x\left(\frac{r}{2}\right)$ and $W_1 := D_1 \cap B_x\left(\frac{r}{2}\right)$. Then, for at least one of the two choices of $\{W_0, W_1\}$, there exist a point $w_0 \in W_0$ and a measurable set $\mathcal{W}_1 \subset W_1$ so that $V(\mathcal{W}_1) \geq \frac{1}{2}V(W_1)$. Moreover, for each $y \in \mathcal{W}_1$, the minimizing geodesic yw_0 from y to w_0 intersects Γ and the first intersection point y^* satisfies $d(y, y^*) \leq d(y^*, w_0)$.

Observe that $\alpha V(D_1) \leq V\left(D_1 \cap B_x\left(\frac{r}{2}\right)\right) \leq \frac{1}{2}V\left(B_x\left(\frac{r}{2}\right)\right)$. Therefore, $\alpha V(D_1) \leq V\left(D_2 \cap B_x\left(\frac{r}{2}\right)\right)$, too. So, regardless of how $\mathcal{W}_1$ is picked, we have $\alpha V(D_1) \leq 2V(\mathcal{W}_1)$. Now, to establish the lemma, we need only to show $\frac{A(\Gamma)}{V(\mathcal{W}_1)} \geq (c_3)^{-1}\mathrm{e}^{-c_4 A}r^{-1}$ for some c_3 and c_4 depending only on n. For this, we use polar coordinates at w_0. For $y \in \mathcal{W}_1$, write $y = \exp_{w_0}(t_0\xi)$. Let y^* be the first intersecting point of yw_0 with Γ. Define t_1 by $y^* = \exp_{w_0}(t_1\xi)$. From the choice of w_0 and $\mathcal{W}_1$, we know $t_1 \geq \frac{1}{2}t_0$. Let t_2 be the maximal t so that $\exp_{w_0}(t\xi) \in \mathcal{W}_1 \backslash C(w_0)$. Clearly, $t_2 \leq 2t_1$. From Lemma 2.1 we conclude that

$$\int_{t_1}^{t_2} J(w_0, t, \xi)dt \leq (t_2 - t_1)J(w_0, t_1, \xi)c_1\mathrm{e}^{c_2 A}$$

$$\leq rJ(w_0, t_1, \xi)c_1\mathrm{e}^{c_2 A}.$$

The desired result then follows after integrating in ξ. This proves the lemma. $\qquad\square$

Combining Lemma 2.1, Lemma 3.1 and the argument in [14], we get a local Neumann Sobolev inequality of the following form.

Lemma 3.2. *Let* $(M, g, e^{-f}dv)$ *be a smooth metric measure space with* $Ric_f \geq 0$. *Then there exist constants* $\nu > 2$, c_1 *and* c_2, *all depending only on* n *such that*

$$\left(\int_{B_p(R)} |\varphi - \varphi_{B_p(R)}|^{\frac{2\nu}{\nu-2}}\right)^{\frac{\nu-2}{\nu}} \leq c_1 e^{c_2 A}\frac{R^2}{V(B_p(R))^{\frac{2}{\nu}}} \int_{B_p(R)} |\nabla\varphi|^2$$

for $\varphi \in C^\infty(B_p(R))$, *where* $\varphi_{B_p(R)} := V^{-1}(B_p(R)) \int_{B_p(R)} \varphi$.

Proof of Lemma 3.2. For $y \in B_p(R)$, let $\gamma(t)$ be a minimizing geodesic from p to y such that $\gamma(0) = p$ and $\gamma(L) = y$. Define $y_0 := p$ and $y_i := \gamma\left(\sum_{j=1}^i \frac{R}{2^j}\right)$ for $1 \leq i \leq i_0$, where i_0 is the largest integer i so that $\sum_{j=1}^i \frac{R}{2^j} < L$. Define also $B_i := B_{y_i}\left(\frac{R}{2^{i+1}}\right)$ for $i < i_0$ and $B_i := B_y\left(\frac{R}{2^{i+1}}\right)$ for $i \geq i_0$.

Let $\varphi_{B_i} := V^{-1}(B_i) \int_{B_i} \varphi$. Then $\lim_{i \to \infty} \varphi_{B_i} = \varphi(y)$. Thus, we have

$$|\varphi_{B_0} - \varphi(y)| \le \sum_{i \ge 0} |\varphi_{B_i} - \varphi_{B_{i+1}}| \le \sum_{i \ge 0} (|\varphi_{B_i} - \varphi_{D_i}| + |\varphi_{D_i} - \varphi_{B_{i+1}}|).$$

Here, $D_i := B_{z_i}\left(\frac{R}{2^{i+3}}\right) \subset B_i \cap B_{i+1}$, and $z_i := \gamma\left(\sum_{j=1}^i \frac{R}{2^j} + \frac{3R}{2^{i+3}}\right)$.

By Lemma 2.1 it is easy to see that

$$|\varphi_{B_i} - \varphi_{D_i}| \le V(D_i)^{-1} \int_{B_i} |\varphi - \varphi_{B_i}| \le c_1 e^{c_2 A} V(B_i)^{-1} \int_{B_i} |\varphi - \varphi_{B_i}|.$$

Note that a similar bound for $|\varphi_{D_i} - \varphi_{B_{i+1}}|$ also holds. So we conclude that

$$\begin{aligned}
|\varphi_{B_0} - \varphi(y)| &\le c_1 e^{c_2 A} \sum_{i \ge 0} V(B_i)^{-1} \int_{B_i} |\varphi - \varphi_{B_i}| \\
&\le c_1 e^{c_2 A} \sum_{i \ge 0} (V(B_i)^{-1} \int_{B_i} |\varphi - \varphi_{B_i}|^2)^{\frac{1}{2}} \\
&\le c_1 e^{c_2 A} \sum_{i \ge 0} \frac{R}{2^{i+1}} (V(B_i)^{-1} \int_{B_i} |\nabla\varphi|^2)^{\frac{1}{2}},
\end{aligned}$$

where in the second line we have use the Cauchy–Schwarz inequality and in the last line we have used Lemma 3.1. On the other hand,

$$|\varphi_{B_0} - \varphi(y)| = c R^{-\frac{1}{2}} \sum_{i \ge 0} \left(\frac{R}{2^{i+1}}\right)^{\frac{1}{2}} |\varphi_{B_0} - \varphi(y)|,$$

where c is a universal constant. So for $R_i := \frac{R}{2^{i+1}}$, we have

$$\sum_{i \ge 0} (R_i)^{\frac{1}{2}} |\varphi_{B_0} - \varphi(y)| \le c_1 e^{c_2 A} R^{\frac{1}{2}} \sum_{i \ge 0} R_i \left(V(B_i)^{-1} \int_{B_i} |\nabla\varphi|^2\right)^{\frac{1}{2}}.$$

Hence, there exists an i (depending on y) so that

$$|\varphi_{B_0} - \varphi(y)|^2 \le c_1 e^{c_2 A} (R R_i) \frac{1}{V(B_i)} \int_{B_i} |\nabla\varphi|^2.$$

Since $B_i \subset B_y(3R_i)$, it follows that for each $y \in B_p(R)$ there exists $r_y > 0$ so that

$$(3.7) \qquad |\varphi_{B_0} - \varphi(y)|^2 \le c_1 e^{c_2 A} (R r_y) V(B_y(r_y))^{-1} \int_{B_y(r_y) \cap B_p(R)} |\nabla\varphi|^2.$$

According to Lemma 2.1,

$$(3.8) \qquad \frac{V(B_p(R))}{V(B_y(r_y))} \leq \frac{V(B_y(2R))}{V(B_y(r_y))} \leq c_1 e^{c_2 A} \left(\frac{R}{r_y}\right)^n,$$

Solving r_y from (3.8) and plugging into (3.7) then gives

$$(3.9) \qquad |\varphi_{B_0} - \varphi(y)|^2 \leq c_1 e^{c_2 A} R^2 V(B_p(R))^{-\frac{1}{n}} V(B_y(r_y))^{\frac{1}{n}-1}$$
$$\times \int_{B_y(r_y) \cap B_p(R)} |\nabla \varphi|^2.$$

We now define $A_t := \{y \in B_p(R) : |\varphi_{B_0} - \varphi(y)| \geq t\}$. Applying the Vitali covering lemma, we find a countable disjoint collection $\{B_i(r_i)\}_{i \in I}$ of balls from $\{B_y(r_y) : y \in A_t\}$ such that for any $y \in A_t$, there exists $i \in I$ such that $B_i(r_i) \cap B_y(r_y) \neq \phi$ and $B_y(r_y) \subset B_i(3r_i)$. Then it follows, by Lemma 2.1 and (3.9), that

$$V(A_t)^{1-\frac{1}{n}} \leq c_1 e^{c_2 A} \sum_{i \in I} V(B_i)^{1-\frac{1}{n}}$$
$$\leq c_1 e^{c_2 A} \frac{R^2}{t^2} V(B_p(R))^{-\frac{1}{n}} \sum_{i \in I} \int_{B_i(r_i) \cap B_p(R)} |\nabla \varphi|^2$$
$$= c_1 e^{c_2 A} \frac{R^2}{t^2} V(B_p(R))^{-\frac{1}{n}} \int_{B_p(R)} |\nabla \varphi|^2.$$

This may be rewritten as

$$(3.10) \qquad V(A_t) \leq t^{-\frac{2n}{n-1}} B$$

with $B := c_1 e^{c_2 A} R^{\frac{2n}{n-1}} V(B_p(R))^{-\frac{1}{n-1}} \left(\int_{B_p(R)} |\nabla \varphi|^2\right)^{\frac{n}{n-1}}$. Now, for any $\frac{2n}{n-1} > q > 2$, we have

$$\int_{B_p(R)} |\varphi - \varphi_{B_0}|^q = q \int_0^\infty t^{q-1} V(A_t) dt$$
$$= q \int_0^T t^{q-1} V(A_t) dt + q \int_T^\infty t^{q-1} V(A_t) dt$$
$$\leq T^q V(B_p(R)) + \frac{q}{\frac{2n}{n-1} - q} T^{q-\frac{2n}{n-1}} B,$$

where we have used (3.10) to bound the second integral in the second line. Choosing $q = \frac{2n-1}{n-1}$ and $T := \left(R^2 \frac{1}{V(B_p(R))} \int_{B_p(R)} |\nabla \varphi|^2 \right)^{\frac{1}{2}}$, we get

$$\left(\int_{B_p(R)} |\varphi - \varphi_{B_0}|^{\frac{2\nu}{\nu-2}} \right)^{\frac{\nu-2}{\nu}} \leq c_1 e^{c_2 A} \frac{R^2}{V(B_p(R))^{\frac{2}{\nu}}} \int_{B_p(R)} |\nabla \varphi|^2,$$

where $\nu := \frac{2q}{q-2} = 4n - 2$. This proves the theorem. $\qquad\square$

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let u be a positive solution to $\Delta_f u = 0$. Applying the Moser iteration scheme to the equation $\Delta_f u = 0$ with the help of Lemma 2.1 and Lemma 3.2, we obtain

$$(3.11) \qquad \sup_{B_p\left(\frac{1}{2}R\right)} u \leq \frac{c_1 e^{c_2 A}}{V(B_p(R))} \int_{B_p(R)} u,$$

where c_1 and c_2 are constants depending only on n. For this, notice that the Neumann Sobolev inequality, Lemma 3.2, also holds true when integrals are with respect to the measure $e^{-f} dv$, without changing the nature of the dependency of the constants on A.

Now we start using the assumption on f that $|f|(x) \leq ar(x) + b$ on M. The following argument follows closely the proof of Proposition 2.1 in [30]. Applying (2.3) we obtain, for any $r > 0$, that $\Delta_f r(x) \leq \frac{n-1+4b}{r} + 3a$. Therefore, we can find $r_0 > 0$ so that

$$(3.12) \qquad \Delta_f r(x) \leq 4a \quad \text{for any } x \in M \backslash B_p(r_0).$$

So for $r > r_0$,

$$\begin{aligned}
4a \int_{B_p(r)\backslash B_p(r_0)} u e^{-f} &\geq \int_{B_p(r)\backslash B_p(r_0)} u (\Delta_f r) e^{-f} \\
&= r_0 \int_{\partial B_p(r_0)} \langle \nabla u, \nabla r \rangle e^{-f} - r \int_{\partial B_p(r)} \langle \nabla u, \nabla r \rangle e^{-f} \\
&\quad + \int_{\partial B_p(r)} u e^{-f} - \int_{\partial B_p(r_0)} u e^{-f} \\
&= \int_{\partial B_p(r)} u e^{-f} - \int_{\partial B_p(r_0)} u e^{-f},
\end{aligned}$$

where we have used the fact that $\Delta_f u = 0$ and

$$\int_{\partial B_p(r)} \langle \nabla u, \nabla r \rangle \mathrm{e}^{-f} = \int_{B_p(r)} (\Delta_f u) \mathrm{e}^{-f} = 0.$$

Denote by

$$U(r) := \int_{B_p(r) \backslash B_p(r_0)} u \mathrm{e}^{-f} \quad \text{and}$$

$$C_0 := \int_{\partial B_p(r_0)} u \mathrm{e}^{-f}.$$

Then the preceding inequality implies that for $r > r_0$,

$$U'(r) \leq 4aU(r) + C_0.$$

After integrating from r_0 to $R > r_0$, we obtain

$$U(R) \leq C_1 \mathrm{e}^{4aR}$$

with $C_1 := U(r_0) + \frac{1}{a} C_0$. Consequently, we have

$$\int_{B_p(R)} u \leq C_2 \mathrm{e}^{5aR}.$$

Plugging this into (3.11), we conclude that

$$\sup_{B_p\left(\frac{1}{2}R\right)} u \leq C_3 \mathrm{e}^{c(n)aR},$$

where $c(n)$ depends only on dimension n and C_3 is independent of R. This shows that

$$\Omega(u) := \limsup_{R \to \infty} \left\{ \frac{1}{R} \sup_{B_p(R)} \log(u+1) \right\} \leq c(n)a,$$

where $c(n)$ depends only on n. By Proposition 3.1, we conclude

$$\sup_M |\nabla \log u| \leq C(n)a$$

with $C(n)$ being a constant depending only on n, as claimed in Theorem 3.1. It is evident that if f has sublinear growth, then a can be taken as close to zero as we wish, thus proving that u must be constant. $\quad\square$

We now turn to polynomial growth f-harmonic functions.

Theorem 3.2. *Let $(M, g, e^{-f}dv)$ be a complete non-compact smooth metric measure space with $Ric_f \geq 0$ and f bounded. Then a sublinear growth f-harmonic function on M must be a constant.*

Proof of Theorem 3.2. Let w be an f-harmonic function so that

$$(3.13) \qquad \lim_{x \to \infty} \frac{|w|(x)}{r(x)} = 0.$$

The Bochner formula asserts that

$$\frac{1}{2}\Delta_f |\nabla w|^2 = |w_{ij}|^2 + \text{Ric}_f(\nabla w, \nabla w) \geq 0.$$

So $|\nabla w|^2$ is f-subharmonic. Applying the Moser iteration scheme, we obtain a mean value inequality of the form

$$(3.14) \qquad \sup_{B_p(\frac{1}{2}R)} |\nabla w|^2 \leq \frac{C}{V(B_p(R))} \int_{B_p(R)} |\nabla w|^2 e^{-f}$$

for some constant C depending on n and $\sup |f|$. Note that now the constant A in Lemma 2.1 and Lemma 3.2 is independent of R as f is assumed to be bounded.

We now choose a cut-off ϕ such that $\phi = 1$ on $B_p(R)$, $\phi = 0$ on $M \backslash B_p(2R)$ and $|\nabla \phi| \leq \frac{C}{R}$. Integrating by parts and using $\Delta_f w = 0$ we get

$$\begin{aligned}
\int_M |\nabla w|^2 \phi^2 e^{-f} &= -2 \int_M w\phi \langle \nabla w, \nabla \phi \rangle e^{-f} \\
&\leq 2 \int_M |w|\phi |\langle \nabla w, \nabla \phi \rangle| e^{-f} \\
&\leq \frac{1}{2} \int_M |\nabla w|^2 \phi^2 e^{-f} + 2 \int_M w^2 |\nabla \phi|^2 e^{-f}.
\end{aligned}$$

This shows that

$$\int_{B_p(R)} |\nabla w|^2 \mathrm{e}^{-f} \leq 4 \int_M w^2 |\nabla \phi|^2 \mathrm{e}^{-f} \leq \frac{C}{R^2} \int_{B_p(2R) \setminus B_p(R)} w^2 \mathrm{e}^{-f}$$

$$\leq \frac{C}{R^2} \left(\sup_{B_p(2R)} w^2 \right) V(B_p(2R))$$

$$\leq \frac{C}{R^2} \left(\sup_{B_p(2R)} w^2 \right) V(B_p(R)),$$

where in the last line we have used Lemma 2.1.

Together with (3.13) we obtain

$$\lim_{R \to \infty} \frac{1}{V(B_p(R))} \int_{B_p(R)} |\nabla w|^2 \mathrm{e}^{-f} = 0.$$

So by (3.14), $|\nabla w| = 0$ on M. The theorem is proved. $\qquad\square$

Our next result is a dimension estimate for the space of polynomial growth f-harmonic functions. For this, we use again Moser iteration and Sobolev inequality. However, the situation here is considerably easier as f is assumed to be bounded.

Theorem 3.3. *Let $(M, g, \mathrm{e}^{-f} dv)$ be a complete non-compact smooth metric measure space with $Ric_f \geq 0$ and f bounded. We have*

$$\dim \mathcal{H}^d(M) \leq C d^{n-1} \quad \text{for any } d \geq 1.$$

Moreover, we have the sharp estimate

$$\dim \mathcal{H}^1(M) \leq n + 1.$$

Proof of Theorem 3.3. We first establish the second result about the dimension of the space of linear growth f-harmonic functions. Consider u an f-harmonic function with linear growth, i.e., $\Delta_f u = 0$ and $|u|(x) \leq C$ $(r(x) + 1)$ on M. Using the argument in the proof of Theorem 3.2 we want to claim that $|\nabla u|$ is bounded on M. Indeed, since f is bounded, (3.14) is true here, too. Now the reverse Poincaré inequality for f-harmonic function yields

$$\frac{1}{V(B_p(R))} \int_{B_p(R)} |\nabla u|^2 \mathrm{e}^{-f} \leq \frac{C}{R^2} \sup_{B_p(2R)} u^2 \leq C.$$

By (3.14) this shows that $|\nabla u|$ is bounded on M.

We now prove a mean value theorem at infinity of the following form. For any bounded positive f-subharmonic function v we have

$$(3.15) \qquad \lim_{R\to\infty} \frac{1}{V_f(B_p(R))} \int_{B_p(R)} v\mathrm{e}^{-f} = \sup_M v.$$

When f is constant this was first established by P. Li [36] by a heat equation method. Here, we follow the argument in [6] which uses a monotonicity formula.

Let $w = \sup_M v - v$, which is a positive function that satisfies

$$\Delta_f w \le 0 \quad \text{and} \quad \inf_M w = 0.$$

To prove (3.15) we show instead

$$(3.16) \qquad \lim_{R\to\infty} \frac{1}{V_f(B_p(R))} \int_{B_p(R)} w\mathrm{e}^{-f} = 0.$$

Let h_R solve $\Delta_f h_R = 0$ in $B_p(R)$ with $h_R = w$ on $\partial B_p(R)$. By the maximum principle, h_R is positive and uniformly bounded. Moreover, since $\inf_M w = 0$, for any $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that

$$\inf_{B_p(R)} w < \varepsilon \quad \text{for any } R > R_\varepsilon.$$

Again, by the maximum principle, it follows that

$$\inf_{B_p(R)} h_R < \varepsilon \quad \text{for any } R > R_\varepsilon.$$

Notice the following Harnack inequality holds.

$$\sup_{B_p\left(\frac{1}{2}R\right)} h_R \le C \inf_{B_p\left(\frac{1}{2}R\right)} h_R,$$

where C depends only on n and $\sup_M |f|$. Indeed, this follows from Lemma 2.1, Lemma 3.1 and Lemma 3.2 by the Moser iteration argument as in [29, Chapter II].

We therefore conclude that

(3.17)
$$\sup_{B_p\left(\frac{1}{2}R\right)} h_R < C\varepsilon \quad \text{for any } R > 2R_\varepsilon.$$

Furthermore, for $R > r > 0$, we have

$$
\begin{aligned}
0 &= \int_{B_p(r)} (\Delta_f h_R) \mathrm{e}^{-f} = \int_{\partial B_p(r)} \frac{\partial h_R}{\partial r} \mathrm{e}^{-f} \\
&= \frac{\partial}{\partial r} \int_{\partial B_p(r)} h_R \mathrm{e}^{-f} - \int_{\partial B_p(r)} h_R \Delta_f(r) \mathrm{e}^{-f} \\
&\geq \frac{\partial}{\partial r} \int_{\partial B_p(r)} h_R \mathrm{e}^{-f} - \frac{C}{r} \int_{\partial B_p(r)} h_R \mathrm{e}^{-f},
\end{aligned}
$$

where in the last line we have used (2.3) and the fact that f is bounded. This shows that $\log\left(\frac{1}{r^C} \int_{\partial B_p(r)} h_R \mathrm{e}^{-f}\right)$ is decreasing as a function of r for $0 < r < R$. In particular, it shows that

$$\int_{\partial B_p(R)} h_R \mathrm{e}^{-f} \leq C \int_{\partial B_p\left(\frac{1}{2}R\right)} h_R \mathrm{e}^{-f}$$

for a constant C depending only on n and $\sup_M |f|$.

So for $R > 2R_\varepsilon$,

$$
\begin{aligned}
\int_{\partial B_p(R)} w \mathrm{e}^{-f} &= \int_{\partial B_p(R)} h_R \mathrm{e}^{-f} \leq C \int_{\partial B_p\left(\frac{1}{2}R\right)} h_R \mathrm{e}^{-f} \\
&\leq C\varepsilon A_f\left(\partial B_p\left(\frac{1}{2}R\right)\right),
\end{aligned}
$$

where in the second line we have used (3.17). Since this inequality is true for all $R > 2R_\varepsilon$, integrating this from $2R_\varepsilon$ to R gives

$$
\begin{aligned}
\int_{B_p(R)\backslash B_p(2R_\varepsilon)} w \mathrm{e}^{-f} &\leq C\varepsilon \int_{2R_\varepsilon}^R A_f\left(\partial B_p\left(\frac{1}{2}t\right)\right) dt \\
&\leq C\varepsilon V_f\left(B_p\left(\frac{1}{2}R\right)\right).
\end{aligned}
$$

Hence, for R sufficiently large,

$$
\begin{aligned}
\frac{1}{V_f(B_p(R))} \int_{B_p(R)} w \mathrm{e}^{-f} &= \frac{1}{V_f(B_p(R))} \int_{B_p(R) \setminus B_p(2R_\varepsilon)} w \mathrm{e}^{-f} \\
&+ \frac{1}{V_f(B_p(R))} \int_{B_p(2R_\varepsilon)} w \mathrm{e}^{-f} \\
&\leq C\varepsilon + \frac{1}{V_f(B_p(R))} \int_{B_p(2R_\varepsilon)} w \mathrm{e}^{-f} \leq 2C\varepsilon.
\end{aligned}
$$

This proves (3.16).

The rest of the argument now follows verbatim [18]. For completeness we sketch it below.

For $u, v \in \mathcal{H}^1(M)$ define

$$
\langle\langle u, v \rangle\rangle := \lim_{R \to \infty} \frac{1}{V_f(B_p(R))} \int_{B_p(R)} \langle \nabla u, \nabla v \rangle \mathrm{e}^{-f}.
$$

This is well defined in view of (3.16). Also, $\langle\langle , \rangle\rangle$ defines an inner product on

$$
\mathcal{H}' := \{ u \in \mathcal{H}^1(M) : \ u(p) = 0 \}.
$$

Consider any finite dimensional subspace $\mathcal{H}''$ of $\mathcal{H}'$, of dimension l. Let $\{u_1, \ldots, u_l\}$ be an orthonormal basis of $(\mathcal{H}'', \langle\langle , \rangle\rangle)$ and define

$$
F^2(x) := \sum_{i=1}^{l} u_i^2(x).
$$

Note F is independent of the choice of $\{u_i\}$. For a fixed point $x \in M$, we may choose $\{u_i\}$ so that $u_i(x) = 0$ for all $i \neq 1$. Then it follows that

$$
\begin{aligned}
F^2(x) &= u_1^2(x) \quad \text{and} \\
F(x)\nabla F(x) &= u_1(x)\nabla u_1(x).
\end{aligned}
$$

Since $\langle\langle u_1, u_1 \rangle\rangle = 1$, we have

$$
\sup_M |\nabla u_1| = 1.
$$

This shows that $|\nabla F|(x) \leq 1$, too. Integrating along minimizing geodesics and using $F(p) = 0$, we get that $F(x) \leq r(x)$. On the other hand,

$$(3.18) \quad 2\sum_{i=1}^{l} \int_{B_p(R)} |\nabla u_i|^2 \mathrm{e}^{-f} = \int_{B_p(R)} (\Delta_f F^2)\mathrm{e}^{-f}$$

$$\leq 2\int_{\partial B_p(R)} F|\nabla F|\mathrm{e}^{-f} \leq 2RA_f(\partial B_p(R)).$$

Since $\{u_i\}$ is orthonormal with respect to $\langle\langle,\rangle\rangle$, for any $\varepsilon > 0$ there exists R_ε such that for $R > R_\varepsilon$,

$$\sum_{i=1}^{l} \frac{1}{V_f(B_p(R))} \int_{B_p(R)} |\nabla u_i|^2 \mathrm{e}^{-f} \geq l - \varepsilon.$$

So, according to (3.18), for any $R \geq R_\varepsilon$,

$$\frac{l-\varepsilon}{R} \leq \frac{(V_f(B_p(R)))'}{V_f(B_p(R))}.$$

Integrating the inequality from R_ε to R, we then conclude that

$$(3.19) \qquad \left(\frac{R}{R_\varepsilon}\right)^{l-\varepsilon} \leq \frac{V_f(B_p(R))}{V_f(B_p(R_\varepsilon))}$$

On the other hand, according to Lemma 2.1, we have

$$V_f(B_p(R)) \leq CR^n.$$

Plugging into (3.19), we conclude that

$$\dim \mathcal{H}^1(M) \leq n + 1.$$

This proves the second claim of Theorem 3.3.

To prove that $\dim \mathcal{H}^d(M) \leq Cd^{n-1}$ for $d > 1$, we observe first that since f is bounded we have for any $x \in M$,

$$\frac{V(B_x(R))}{V(B_x(r))} \leq C_v \left(\frac{R}{r}\right)^n \quad \text{for } r < R,$$

where C_v depends on n and $\sup_M |f|$. Moreover, the Sobolev inequality Lemma 3.2 and the Moser iteration imply that we have a mean value inequality of the form

$$u^2(x) \leq C_\mathcal{M} \frac{1}{V(B_x(R))} \int_{B_x(R)} u^2$$

for any non-negative f-subharmonic function u on M. The result in [17] then implies $\dim \mathcal{H}^d(M) \leq Cd^{n-1}$. The theorem is proved. $\qquad\square$

4. Rigidity

In this section, we investigate the structure at infinity of smooth metric measure spaces whose $\lambda_1 (\Delta_f)$ attains its upper bound in Theorem 2.1. We prove our result by using a Busemann function argument, which is similar to the one in [21]. A manifold M is called connected at infinity if it has only one end.

Theorem 4.1. *Let $(M, g, e^{-f} dv)$ be a smooth metric measure space such that $Ric_f \geq 0$. Assume that $\lambda_1 (\Delta_f) = \frac{1}{4} a^2$, where a is the linear growth rate of f. Then, either M is connected at infinity or M is isometric to $\mathbb{R} \times N$ for some compact manifold N.*

Before proving the theorem, we recall some terminology. First, a manifold is f-non-parabolic if Δ_f admits a positive Green's function. Otherwise, it is called f-parabolic. For an end of the manifold, the same definition applies, where the Green's function now refers to the one satisfying the Neumann boundary conditions. We will divide our proof into two cases according to the type of ends of M. In fact, the result for f-non-parabolic ends does not require any assumption on f or $\lambda_1(\Delta_f)$. We state it in the following lemma.

Lemma 4.1. *Let $(M, g, e^{-f} dv)$ be a smooth metric measure space with $Ric_f \geq 0$. Then M has at most one f-non-parabolic end.*

Proof of Lemma 4.1. Suppose M has two f-non-parabolic ends. Then M admits a positive non-constant bounded f-harmonic function v with $\int_M |\nabla v|^2 e^{-f} < \infty$. This kind of result was first discovered by Li and Tam (see [35]). Now according to a result of Brighton [2], since v is bounded, it must be a constant function. This is a contradiction.

Alternatively, the lemma may be proved as follows. Using the Bochner formula

$$\frac{1}{2}\Delta_f|\nabla v|^2 = |v_{ij}|^2 + \langle\nabla\Delta_f v, \nabla v\rangle + \mathrm{Ric}_f(\nabla v, \nabla v) \geq |v_{ij}|^2$$

and a cut-off argument, we get

$$2\int_M |v_{ij}|^2 \mathrm{e}^{-f}\phi^2 \leq \int_M (\Delta_f|\nabla v|^2)\phi^2\mathrm{e}^{-f}$$
$$= -\int_M \langle\nabla|\nabla v|^2, \nabla\phi^2\rangle\mathrm{e}^{-f}$$
$$\leq \int_M |v_{ij}|^2\phi^2\mathrm{e}^{-f} + 4\int_M |\nabla v|^2|\nabla\phi|^2\mathrm{e}^{-f}.$$

Thus,

$$\int_M |v_{ij}|^2\mathrm{e}^{-f}\phi^2 \leq 4\int_M |\nabla v|^2|\nabla\phi|^2\mathrm{e}^{-f}.$$

Since $\int_M |\nabla v|^2\mathrm{e}^{-f} < \infty$, the right hand side goes to 0 after taking the limit $R \to \infty$ by choosing ϕ to be the standard cut-off function so that $\phi = 1$ on $B_p(R)$ and $\phi = 0$ outside $B_p(2R)$. This forces $v_{ij} = 0$. Therefore, $|\nabla v|$ must be a constant on M. Now the weighted volume $V_f(M) = \int_M \mathrm{e}^{-f}dv = \infty$ as M is f-non-parabolic. Using $\int_M |\nabla v|^2\mathrm{e}^{-f} < \infty$ again, we conclude $|\nabla v| = 0$ and v is a constant. This finishes our proof. $\square$

We now prove Theorem 4.1.

Proof of Theorem 4.1. First, since $\lambda_1(\Delta_f) > 0$, we know that M is f-non-parabolic. This follows exactly as in the case $f = $ constant. We refer to [16] for details.

Let us assume that M has at least two ends. By Lemma 4.1 exactly one end is f-non-parabolic, and all other are f-parabolic. We denote E to be the f-non-parabolic end and let $F := M\backslash E$. Then F is an f-parabolic end.

First, we claim the fact that $\lambda_1(\Delta_f) = \frac{1}{4}a^2$ implies

(4.1) $$V_f(F\backslash B_p(R)) \leq C\mathrm{e}^{-aR}.$$

This claim can be verified by following the argument of Li and Wang in [20]. Since they only use the variational principle for the bottom spectrum and integration by parts, it is easy to check that their estimates can be carried over to our setting. We shall omit the details here.

For a geodesic ray γ contained in the end F, define the associated Busemann function by

$$\beta(x) := \lim_{t \to \infty} (t - d(x, \gamma(t))).$$

Denote by $\tau_t(s)$ the normal minimizing geodesic from $\gamma(t)$ to x. According to (2.3) we have

$$\Delta_f(d(\gamma(t), x)) \le \frac{n-1}{r} - \frac{2}{r}f(x) + \frac{2}{r^2}\int_0^r f(s)ds,$$

where $r := d(\gamma(t), x)$ and $f(s) := f(\tau_t(s))$. Let us assume that there exist $\alpha > 0$ and $C > 0$ such that $|f(z)| \le \alpha d(p, z) + C$, for all $z \in M$. Then,

$$
\begin{aligned}
|f(s)| = |f(\tau_t(s))| &\le \alpha d(p, \tau_t(s)) + C \\
&\le \alpha(d(p, x) + d(x, \tau_t(s))) + C \\
&= \alpha(r - s) + (\alpha d(p, x) + C),
\end{aligned}
$$

for any $0 < s < r$. Therefore, since $|f(x)| \le \alpha d(p, x) + C$, we conclude

$$
\begin{aligned}
\Delta_f(d(\gamma(t), x)) &\le \frac{n - 1 + 4(\alpha d(p, x) + C)}{r} + \frac{2}{r^2}\int_0^r \alpha(r - s)ds \\
&= \frac{n - 1 + 4(\alpha d(p, x) + C)}{r} + \alpha.
\end{aligned}
$$

Now, by using this and the definition of the Busemann function, it is standard to see that the following inequality holds in the sense of distributions:

$$\Delta_f \beta(x) \ge -\alpha.$$

Now, if a denotes the linear growth rate of f i.e., it is the infimum of such α, it follows that

(4.2) $$\Delta_f \beta(x) \ge -a.$$

To estimate the volume growth of the f-non-parabolic end E we will use (4.2) and the fact that $|\nabla \beta| = 1$. Note

$$\Delta_f e^{a\beta} \ge 0.$$

We also know that on the end E the Busemann function is equivalent to the distance function, that is, there exists a constant C such that

$$-r(x) - C \le \beta(x) \le -r(x) + C$$

for $x \in E$, see [21]. Integrating $\Delta_f e^{a\beta}$ on $\{-t \leq \beta \leq -r\} \cap E$ and using Stokes theorem we get

$$
\begin{aligned}
0 &\leq \frac{1}{a} \int_{\{-t \leq \beta \leq -r\} \cap E} (\Delta_f e^{a\beta}) e^{-f} \\
&= \int_{\{\beta = -r\} \cap E} e^{a\beta} e^{-f} - \int_{\{\beta = -t\} \cap E} e^{a\beta} e^{-f} \\
&= e^{-ar} A_f(\{\beta = -r\} \cap E) - e^{-at} A_f(\{\beta = -t\} \cap E),
\end{aligned}
$$

where we have used $|\nabla\beta| = 1$. Here, $A_f(\Omega)$ denotes the area of the set Ω with respect to the weighted area form. This shows for a fixed r and all $t > r$,

$$
A_f(\{\beta = -t\} \cap E) \leq C(r) e^{at}.
$$

Integrating with respect to t from r to R we obtain an upper bound for the volume of the sublevel sets of the Busemann function. Since the Busemann function is equivalent with the distance function on E, it follows that for $R > 0$,

$$
(4.3) \qquad\qquad V_f(B_p(R) \cap E) \leq C e^{aR}.
$$

Consider the function

$$
B := e^{\frac{1}{2} a\beta}.
$$

By (4.2) and $|\nabla\beta| = 1$, we have

$$
(4.4) \qquad\qquad \Delta_f B \geq -\frac{1}{4} a^2 B.
$$

We now use B as a test function for the variational formula of $\lambda_1(\Delta_f) = \frac{1}{4} a^2$. For this sake, we define a cut-off function ϕ with support in $B_p(2R)$ such that $\phi = 1$ on $B_p(R)$ and $|\nabla\phi| \leq \frac{C}{R}$. Then, according to the variational

characterization of $\lambda_1(\Delta_f)$, we have

$$
\begin{aligned}
\frac{1}{4}a^2 \int_M (B\phi)^2 \mathrm{e}^{-f} &\le \int_M |\nabla(B\phi)|^2 \mathrm{e}^{-f} \\
&= \int_M |\nabla B|^2 \phi^2 \mathrm{e}^{-f} + \frac{1}{2}\int_M \langle \nabla B^2, \nabla \phi^2 \rangle \mathrm{e}^{-f} + \int_M |\nabla \phi|^2 B^2 \mathrm{e}^{-f} \\
&= \int_M |\nabla B|^2 \phi^2 \mathrm{e}^{-f} - \frac{1}{2}\int_M (\Delta_f B^2)\phi^2 \mathrm{e}^{-f} + \int_M |\nabla \phi|^2 B^2 \mathrm{e}^{-f} \\
&= -\int_M B(\Delta_f B)\phi^2 \mathrm{e}^{-f} + \int_M |\nabla \phi|^2 B^2 \mathrm{e}^{-f} \\
&= \frac{1}{4}a^2 \int_M B^2 \phi^2 \mathrm{e}^{-f} + \int_M |\nabla \phi|^2 B^2 \mathrm{e}^{-f} - \int_M B\left(\Delta_f B + \frac{1}{4}a^2 B\right)\phi^2 \mathrm{e}^{-f}.
\end{aligned}
$$

Therefore,

$$
\int_M B\left(\Delta_f B + \frac{1}{4}a^2 B\right)\phi^2 \mathrm{e}^{-f} \le \int_M |\nabla \phi|^2 B^2 \mathrm{e}^{-f}.
$$

From (4.1) and (4.3) and the construction of B, arguing as in [21], we have

$$
\int_M |\nabla \phi|^2 B^2 \mathrm{e}^{-f} \le \frac{C}{R} \to 0.
$$

In view of (4.4), we conclude $\Delta_f B + \frac{1}{4}a^2 B = 0$. In particular, B is smooth. Therefore,

$$
\Delta_f \beta = -a \quad \text{and} \quad |\nabla \beta| = 1
$$

hold everywhere on M. Now, by the Bochner formula,

$$
0 = \frac{1}{2}\Delta_f |\nabla \beta|^2 = |\beta_{ij}|^2 + \langle \nabla \Delta_f \beta, \nabla \beta \rangle + Ric_f(\nabla \beta, \nabla \beta) \ge |\beta_{ij}|^2.
$$

This proves $\beta_{ij} = 0$, which implies M is a direct product given by $M = \mathbb{R} \times N$. That N is compact follows from the assumption of M having two ends. This proves the theorem. $\qquad\square$

We now discuss an important application of this result to the gradient Ricci solitons. Recall that a gradient steady Ricci soliton is a manifold (M, g, f) such that $R_{ij} + f_{ij} = 0$. Here we prove that gradient steady Ricci solitons are either connected at infinity or they are trivial, that is, isometric to a Ricci flat cylinder.

Theorem 4.2. *Let (M, g, f) be a gradient steady Ricci soliton. Then either M is connected at infinity or M is isometric to $\mathbb{R} \times N$ for a compact Ricci flat manifold N.*

Proof of Theorem 4.2. Let us assume that M has at least two ends. It is known that for any gradient steady Ricci soliton there exists $a > 0$ so that $|\nabla f|^2 + S = a^2$, see (2.7). By Proposition 2.1 we know that $\lambda_1(\Delta_f) = \frac{1}{4}a^2$. Then, by Theorem 4.1 we conclude $M = \mathbb{R} \times N$ for some compact manifold N. It is easy to see N has to be a steady Ricci soliton also. However, any compact steady gradient Ricci soliton must be Ricci flat. This concludes the proof. $\qquad\square$

Acknowledgments

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DEPARTMENT OF MATHEMATICS
COLUMBIA UNIVERSITY
NEW YORK, NY 10027
USA
E-mail address: omuntean@math.columbia.edu

School of Mathematics
University of Minnesota
Minneapolis, MN 55455
USA
E-mail address: `jiaping@math.umn.edu`