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The classification of toroidal Dehn surgeries on Montesinos knots

Ying-Qing Wu

Exceptional Dehn surgeries have been classified for two-bridge knots and Montesinos knots of length at least 4. In this paper, we classify all toroidal Dehn surgeries on Montesinos knots of length 3.

1. Introduction

A Dehn surgery on a hyperbolic knot K along a non-trivial slope δ is said to be *exceptional* if the resulting manifold K_δ is either reducible, toroidal or a small Seifert fibered manifold. By the Geometrization Conjecture proved by Perelman [1], a non-trivial surgery is exceptional if and only if K_δ is non-hyperbolic. By Thurston's Hyperbolic Surgery Theorem, all but finitely many Dehn surgeries on a hyperbolic knot produce hyperbolic manifolds, hence there are only finitely many exceptional surgeries.

It is known that there are no exceptional surgeries on Montesinos knots of length at least four [5]. Exceptional surgeries for two-bridge knots have been classified in [2]. Thus length 3 knots are the only ones among the Montesinos knots, which have not been settled. In this paper, we will classify toroidal surgeries for such knots. See Theorems 1.1 and 1.2 below. By [3] there is no reducible surgery on a hyperbolic Montesinos knot because it is strongly invertible. It remains a challenging open problem to determine all small Seifert fibred surgeries on Montesinos knots of length 3.

Hatcher and Oertel [4] have an algorithm to determine all boundary slopes of a given Montesinos knot. We will therefore focus on finding all length 3 knots such that some of their boundary slopes are toroidal slopes. Each incompressible surface in the exterior of K corresponds to three "allowable edgepaths" $\gamma_1, \gamma_2, \gamma_3$. We will define an Euler number for allowable edge paths, and show that if $F(\gamma_1, \gamma_2, \gamma_3)$ is a punctured torus then one of the γ_i must have non-negative Euler number. We then analyze the graph of Hatcher–Oertel (figure 1), and show that the ending point of one of the above γ_i must lie in a subgraph consisting of seven edges. This breaks the

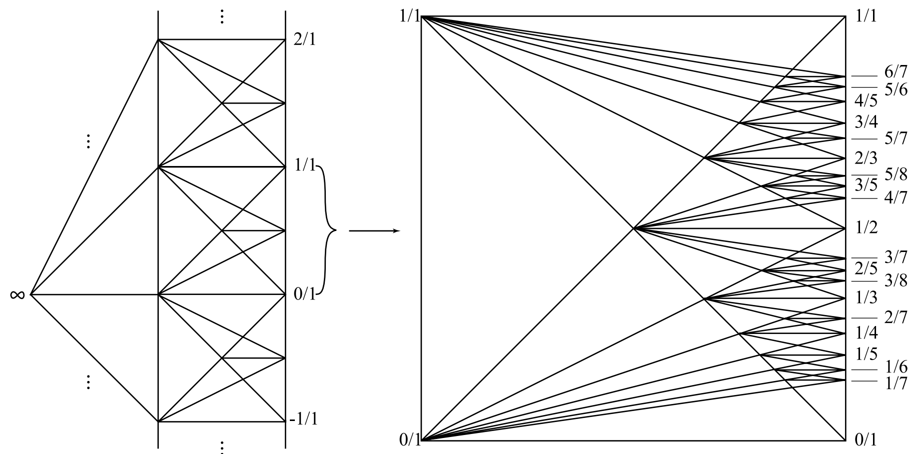

Figure 1:

problem down to several different cases. We will then use the properties of allowable edge paths to find all possible solutions in each case.

Two knots are considered *equivalent* if there is a (possibly orientation reversing) homeomorphism of S^3 sending one knot to the other. Thus K_1 is equivalent to K_2 if K_1 is isotopic to K_2 or its mirror image. Similarly, if $N(K_i)$ is a neighborhood of K_i and δ_i is a slope on $\partial N(K_i)$, then (K_1, δ_1) is equivalent to (K_2, δ_2) if there is a homeomorphism of S^3 sending $N(K_1)$ to $N(K_2)$ and δ_1 to δ_2. The following is the classification theorem for toroidal boundary slopes of Montesinos knots of length 3. Some knots are listed more than once, with different boundary slopes, which means that they admit more than one toroidal surgery. The variable $\bar{u}$ is the u coordinate of the ending points of the edge paths, which will be defined in Section 2. Note that some knots may have the same toroidal boundary slope at different $\bar{u}$ values, in which case we will only list one $\bar{u}$ value.

Theorem 1.1. *Let K be a hyperbolic Montesinos knot of length 3, let $E(K) = S^3 - \mathrm{Int}N(K)$, and let δ be a slope on $\partial E(K)$. Then $E(K)$ contains an essential surface F of genus one with boundary slope δ if and only if (K, δ) is equivalent to one of the pairs in the following list.*

(1) $K = K(1/q_1, 1/q_2, 1/q_3)$, *$q_i$ odd,* $|q_i| > 1$, $\delta = 0$; $\bar{u} = 1$.

(2) $K = K(1/q_1, 1/q_2, 1/q_3)$, *$q_1$ even, q_2, q_3 odd,* $|q_i| > 1$, $\delta = 2(q_2 + q_3)$; $\bar{u} = 1$.

(3) $K = K(-1/2, 1/3, 1/(6+1/n))$, $n \neq 0, -1$, $\delta = 16$ *if* n *is odd, and* 0 *if* n *is even;* $\bar{u} = 6$.

(4) $K = K(-1/3, -1/(3+1/n), 2/3)$, $n \neq 0, -1$, $\delta = -12$ *when* n *is odd, and* $\delta = 4$ *when* n *is even;* $\bar{u} = 3$.

(5) $K = K(-1/2, 1/5, 1/(3+1/n))$, n *even, and* $n \neq 0$, $\delta = 5 - 2n$*;* $\bar{u} = 3$.

(6) $K = K(-1/2, 1/3, 1/(5+1/n))$, n *even, and* $n \neq 0$, $\delta = 1 - 2n$*;* $\bar{u} = 3$.

(7) $K = K(-1/(2+1/n), 1/3, 1/3)$, n *odd,* $n \neq -1$, $\delta = 2n$*;* $\bar{u} = 2$.

(8) $K = K(-1/2, 1/3, 1/(3+1/n))$, n *even,* $n \neq 0$, $\delta = 2 - 2n$*;* $\bar{u} = 2$.

(9) $K = K(-1/2, 2/5, 1/9)$, $\delta = 15$*;* $\bar{u} = 5$.

(10) $K = K(-1/2, 2/5, 1/7)$, $\delta = 12$*;* $\bar{u} = 4$.

(11) $K = K(-1/2, 1/3, 1/7)$, $\delta = 37/2$*;* $\bar{u} = 2.5$.

(12) $K = K(-2/3, 1/3, 1/4)$, $\delta = 13$*;* $\bar{u} = 2.5$.

(13) $K = K(-1/3, 1/3, 1/7)$, $\delta = 1$*;* $\bar{u} = 2.5$.

For each case in Theorem 1.1, the candidate system $(\gamma_1, \gamma_2, \gamma_3)$ is given in the proofs of the lemmas, hence it is straightforward using the algorithm of Hatcher–Oertel to calculate the boundary slope of $F(\gamma_1, \gamma_2, \gamma_3)$ and show that it is an incompressible toroidal surface. For each individual knot this can also be verified using a computer program of Dunfield [16]. We will therefore concentrate on showing the "only if" part, that is, if the exterior of K has an incompressible toroidal surface with boundary slope δ then (K, δ) must be one of those in the list.

In general, the existence of a toroidal incompressible surface F with boundary slope δ in the exterior of a knot K does not guarantee that K_δ is toroidal, because the corresponding closed surface $\hat{F}$ may be compressible in K_δ. However, the following theorem shows that this does not happen for Montesinos knots of length 3; hence the above theorem actually gives a classification of all toroidal surgeries for Montesinos knots of length 3.

Theorem 1.2. *Let* K *be a hyperbolic Montesinos knot of length* 3*, and let* δ *be a slope on* $T = \partial N(K)$*. Then* K_δ *is toroidal if and only if* (K, δ) *is equivalent to one of those in the list of Theorem* 1.1*.*

Together with [2] and [5], this gives a complete classification of toroidal surgeries on all Montesinos knots. The following are some of the consequences.

(1) The only non-integral toroidal surgery on a Montesinos knot is the $37/2$ surgery on $K(-1/2, 1/3, 1/7)$.

(2) No Montesinos knot admits more than three toroidal surgeries, and the figure 8 knot and $K(-1/2, 1/3, 1/7)$ are the only ones admitting three toroidal surgeries.

(3) By [2], a 2-bridge knot admits exactly two toroidal surgeries if and only if it is associated to the rational number $1/(2 + 1/n)$ for some $|n| > 2$. By checking the list in Theorem 1.1 for knots which are listed more than once, we see that $K(t_1, t_2, t_3)$ admits exactly two toroidal surgeries if and only if it is equivalent to one of the following 5 knots.

$$\begin{aligned} &K(-1/2, 1/3, 2/11), && \delta = 0 \text{ and } -3; \\ &K(-1/3, 1/3, 1/3), && \delta = 0 \text{ and } 2; \\ &K(-1/3, 1/3, 1/7), && \delta = 0 \text{ and } 1; \\ &K(-2/3, 1/3, 1/4), && \delta = 12 \text{ and } 13; \\ &K(-1/3, -2/5, 2/3), && \delta = 4 \text{ and } 6. \end{aligned}$$

(4) A toroidal essential surface F in Theorem 1.1 has at most four boundary components. In case (1) of Theorem 1.1 F is a Seifert surface with a single boundary component. In all other cases F is a separating surface and the result follows from the proof of Theorem 1.2.

(5) Also follows from results of Gordon and Luecke [6] and Eudave–Munoz [7], which classified non-integral toroidal surgeries on all knots in S^3. There are many other interesting results about toroidal Dehn surgery, see for example [8–15].

The paper is organized as follows. In Section 2 we give a brief introduction to some definitions and results of Hatcher and Oertel in [4], then define and explore the properties of Euler numbers $e(\gamma)$ for any edge path in the Hatcher–Oertel graph $\mathcal{D}$ shown in figure 1. It will be shown that if $F(\gamma_1, \gamma_2, \gamma_3)$ is a punctured torus then up to equivalence the ending point v_1 of γ_1 must lie on the subgraph G in figure 4. Sections 3, 4 and 5 discuss the cases that v_1 lies on a horizontal edge in G, and Section 6 deals with the remaining cases. The proofs of Theorems 1.1 and 1.2 will be given in Section 7.

2. Preliminaries

In this section we first give a brief introduction to some results of Hatcher–Oertel in [4]. We will then define Euler numbers for points and edge paths on the Hatcher–Oertel diagram $\mathcal{D}$, and show how they are related to the Euler characteristic of the corresponding surfaces. The main result is Theorem 2.8, which will play a key role in finding Montesinos knots which admit toroidal surgeries.

2.1. The diagram $\mathcal{D}$

The diagram $\mathcal{D}$ of Hatcher–Oertel is a two-complex on the plane $\mathbb{R}^2$ consisting of vertices, edges and triangular faces described as follows. See figure 1, which is the same as [4, figure 1.3]. Unless otherwise stated, we will always write a rational number as p/q, where p, q are coprime integers, and $q > 0$.

(1) To each rational number $y = p/q$ is associated a vertex $\langle y \rangle$ in $\mathcal{D}$, which has Euclidean coordinates $(x, y) = ((q-1)/q, p/q)$.

(2) For each rational number $y = p/q$, there is also an "ideal" vertex $\langle y \rangle_0$ with Cartesian coordinates $(1, p/q)$.

(3) There is a vertex $\infty = 1/0$ located at $(-1, 0)$.

(4) There is an edge $E = \langle p/q, p'/q' \rangle$ connecting $\langle p/q \rangle$ to $\langle p'/q' \rangle$ if and only if $|pq' - p'q| = 1$. Thus for example, there is an edge connecting ∞ to each vertex $\langle p/1 \rangle$, and there is an edge connecting $\langle p/q \rangle$ to $\langle 0 \rangle$ if and only if $p = \pm 1$.

(5) For each rational number y there is a horizontal edge $L(y)$ connecting $\langle y \rangle$ to the ideal vertex $\langle y \rangle_0$.

(6) A face of $\mathcal{D}$ is a triangle bounded by three non-horizontal edges of $\mathcal{D}$.

Note that a non-horizontal line segment in the figure with one endpoint on an ideal vertex (i.e., a vertex on the vertical line $x = 1$) is not an edge of $\mathcal{D}$. It is a union of infinitely many edges of $\mathcal{D}$ and contains infinitely many vertices of $\mathcal{D}$. Similarly a triangle Δ in the figure is not a face of $\mathcal{D}$ if it contains a horizontal edge because there are edges in the interior of Δ. Actually in this case Δ is a union of infinitely many faces of $\mathcal{D}$. On the other hand, if all three vertices of a triangle Δ in the figure are non-ideal vertices of $\mathcal{D}$, and if all three boundary edges of Δ are edges of $\mathcal{D}$ as defined above, then Δ is a face of $\mathcal{D}$; in particular, its interior contains no other edges or vertices of $\mathcal{D}$.

2.2. Allowable edge paths, candidate systems and candidate surfaces

An *edge path* γ in $\mathcal{D}$ is a piecewise linear path in the one-skeleton of $\mathcal{D}$. Note that the endpoints of γ may not be vertices of $\mathcal{D}$. An edge path γ is a *constant path* if its image is a single point.

Let $K = K(t_1, t_2, t_3)$ be a Montesinos knot of length 3. Let $\gamma_1, \gamma_2, \gamma_3$ be three edge paths in $\mathcal{D}$. According to [4, p. 457], we say that the three edge paths form a *candidate system* for $K(t_1, t_2, t_3)$ if they satisfy the following conditions:

(1) The starting point of γ_i is on the horizontal edge $L(t_i)$, and if this starting point is not the vertex $\langle t_i \rangle$ then γ_i is a constant path.

(2) γ_i is minimal in the sense that it never stops and retraces itself, or goes along two sides of a triangle of $\mathcal{D}$ in succession.

(3) The ending points of γ_i are rational points $\mathcal{D}$ which all lie on one vertical line and whose vertical coordinates add up to zero.

(4) γ_i proceeds monotonically from right to left, "monotonically" in the weak sense that motion along vertical edges is permitted.

Each γ_i above is called an *allowable edge path.* By definition, an allowable edge path must be of one of the following three types:

(1) A constant path on a horizontal edge, possibly at a vertex of $\mathcal{D}$.

(2) An edge path with both endpoints on vertices of $\mathcal{D}$.

(3) An edge path starting from a vertex of $\mathcal{D}$ and ending in the interior of a non-horizontal edge.

For each candidate system, one can construct a surface $F = F(\gamma_1, \gamma_2, \gamma_3)$ in the exterior of K, called a *candidate surface.* We refer the reader to [4, p. 457] for the construction of the surface. Denote by $\hat{F} = \hat{F}(\gamma_1, \gamma_2, \gamma_3)$ the corresponding closed surface obtained by capping off each boundary component of F with a disk. Let $N(K)$ be a regular neighborhood of K. If δ is the boundary slope of F on $\partial N(K)$ then $\hat{F}$ is a closed surface in the manifold K_δ obtained by δ surgery on K.

When γ_i ends at $\langle \infty \rangle$ there may also be some "augmented" candidate surface, but fortunately this does not happen for Montesinos knots of length 3. The following is [4, Proposition 1.1].

Proposition 2.1. *Every incompressible, ∂-incompressible surface in $S^3 - K$ having non-empty boundary of finite slope is isotopic to one of the candidate surfaces.*

To find all toroidal surgeries on Montesinos knots of length 3, it suffices to find all candidate systems $(\gamma_1, \gamma_2, \gamma_3)$ such that $\hat{F} = \hat{F}(\gamma_1, \gamma_2, \gamma_3)$ is a torus. By Theorem 1.2 all toroidal $\hat{F}$ are incompressible.

2.3. The u-coordinate of a point and the length of an edge path

Any rational point (x, y) in the diagram represents some curve system (a, b, c) on a four-punctured sphere as shown in figure 2. (When c is negative, reverse the tangency of the train track, and relabel c by $-c$.) The parameters (a, b, c) and (x, y) are related as follows.

$$y = \frac{c}{a+b}, \quad x = \frac{b}{a+b}$$

See [4, p. 455].

Note that (a, b, c) is determined by (x, y) up to scalar multiplication, i.e., (a, b, c) and $k(a, b, c)$ correspond to the same rational point in $\mathcal{D}$, so for any rational point (x, y) one can choose a, b, c to be integers with $a > 0$.

A rational point in the interior of an edge $\langle p/q, r/s\rangle$ in $\mathcal{D}$ corresponds to a curve system $(1, b, c)$, which can be written as a linear combination

$$(1, b, c) = \alpha(1, s-1, r) + \beta(1, q-1, p),$$

where α, β are positive rational numbers, and $\alpha + \beta = 1$. We write

$$v = \alpha\langle r/s\rangle + \beta\langle p/q\rangle$$

to indicate that the point v is related to $\langle p/q\rangle$ and $\langle r/s\rangle$ as above. The number α (resp. β) is called the *length* of the edge segment from $\langle p/q\rangle$ (resp. $\langle r/s\rangle$) to v. It is important to note that this is not the euclidean length of the segments of the edge cut by v, even if the length of the edge is normalized to 1. From the construction of the candidate surface ([4, p. 457]), we see that traveling from the vertex $\langle r/s\rangle$ to the point v above corresponds to adding $m\beta$ saddles to the surface, where m is the number of times the surface intersects a meridian of K, which must be an integer. This fact will be useful in the calculation of the Euler number of the resulting surface.

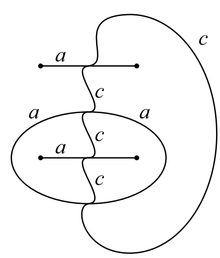

Figure 2:

To make calculation easier, we introduce the *u-coordinate* of a point v. Define

$$u = u(v) = \frac{1}{1-x},$$

where x is the x-coordinate of the point v in $\mathcal{D}$. Thus, we have $x = (u-1)/u$. The u-coordinate has two important properties.

(1) The u-coordinate of a vertex $\langle p/q \rangle$ is q.

(2) The length of an edge segment is equal to its length in u-coordinate when the length of the edge is normalized to 1, as shown in the following lemma.

Lemma 2.1. *Let $v = \alpha\langle r/s \rangle + \beta\langle p/q \rangle$. Let $u = u(v)$, $u_0 = q$ and $u_1 = s$ be the u-coordinates of v, $\langle p/q \rangle$ and $\langle r/s \rangle$ respectively. Then*

$$u = \alpha u_1 + \beta u_0.$$

In particular, α and β can be calculated by the following formulas:

$$\alpha = \frac{u-u_0}{u_1-u_0} = \frac{u-q}{s-q},$$
$$\beta = \frac{u_1-u}{u_1-u_0} = \frac{s-u}{s-q}.$$

Proof. Suppose $\langle p/q \rangle$ is represented by the curve system (a_0, b_0, c_0), and $\langle r/s \rangle$ by (a_1, b_1, c_1). By definition we may choose (a_1, b_1, c_1) so that $a_1 = a_0$. Then the x-coordinates of these points are $x_i = b_i/(a_i + b_i)$, hence $u_i = 1/(1-x) = (a_i + b_i)/a_i = 1 + (b_i/a_i)$ for $i = 0, 1$.

By definition $v = a\langle r/s \rangle + \beta\langle p/q \rangle$ is represented by $\langle \alpha a_1 + \beta a_0, \alpha b_1 + \beta b_0, \alpha c_1 + \beta c_0 \rangle$. Using the facts that $\alpha + \beta = 1$ and $a_0 = a_1$, we can calculate the u-coordinate of $\alpha\langle r/s \rangle + \beta\langle p/q \rangle$ as follows

$$\begin{aligned} u &= 1 + \frac{\alpha b_1 + \beta b_0}{\alpha a_1 + \beta a_0} = 1 + \frac{\alpha b_1 + \beta b_0}{a_0} \\ &= \alpha\left(1 + \frac{b_1}{a_1}\right) + \beta\left(1 + \frac{b_0}{a_0}\right) \\ &= \alpha u_1 + \beta u_0. \end{aligned}$$

□

Definition 2.1. The length $|\gamma|$ of an edge path γ in $\mathcal{D}$ is defined by counting the length of a full edge as 1, and the length of a partial edge from $\langle r/s \rangle$ to $\alpha\langle r/s\rangle + \beta\langle p/q\rangle$ as β.

2.4. Euler numbers of points, edge paths and surfaces

The knot $K = K(t_1, t_2, t_3)$ in S^3 can be constructed as follows. Let (B_i, T_i) be a rational tangle of slope $t_i = p_i/q_i$, where $B_i = D^2 \times I$, and T_i consists of two strings with endpoints on the vertical diameters of $D^2 \times \partial I$. Gluing the end disks $D^2 \times \partial I$ of the tangles in a cyclic way, we get a knot in a solid torus V, which can be trivially embedded in S^3 to produce the knot $K = K(t_1, t_2, t_3)$ in S^3.

Denote by M_i the tangle space $B_i - \text{Int}N(T_i)$. Let E_i be a disk in M_i separating the two arcs of T_i, and let $D_i = D_i^1 \cup D_i^2$ be a pair of disks properly embedded in M_i such that D_i^j intersects the meridian of the j-th string of T_i at a single point and is disjoint from the meridians of the other string.

Define a number m_i as follows. If γ_i is a not a constant path, let m_i be the minimal positive integer such that $m_i \times |\gamma_i|$ is an integer. If γ_i is a constant path on $L(p_i/q_i)$ at a point with u-coordinate $\bar{u}$, let m_i be the smallest positive integer such that $m_i\bar{u}/q_i$ is an integer. Let $n = \text{lcm}(m_1, m_2, m_3)$ be the least common multiple of m_1, m_2, m_3, and let m be a multiple of n. A candidate surface is said to have m *sheets* if it intersects the meridian of K (and hence the meridian of each strand of the tangles) at m points.

Lemma 2.2. *Let $F(\gamma_i)$ be an m-sheet surface in the tangle space M_i corresponding to the edge path γ_i constructed in [4, p. 457].*

(1) *If γ_i is not a constant path then $\chi(F(\gamma_i)) = m(2 - |\gamma_i|)$.*

(2) *If γ_i is a constant path on the horizontal edge $L(t_i)$ with u-coordinate $\bar{u}$, then $\chi(F(\gamma_i)) = m(1 + \bar{u}/q_i)$*

(3) *Let m_i be defined as above, and let $n = \text{lcm}(m_1, m_2, m_3)$. Then there exists an m-sheet orientable candidate surface $F = F(\gamma_1, \gamma_2, \gamma_3)$ with $m = n$ or $2n$.*

Proof. (1) If γ_i is not a constant path in the interior of $L(t_i)$ then according to [4, p. 457], $F(\gamma_i)$ is obtained from m copies of D_i by adding some saddles. For each full edge in γ_i one adds m saddles, and for a partial edge of length β_i one adds $m\beta_i$ saddles. (By the choice of m in the construction, $m\beta_i$ must be an integer.) Since $\chi(mD_i) = 2m$, and adding a saddle reduces the Euler characteristic by 1, we have $\chi(F(\gamma_i)) = 2m - m|\gamma_i|$.

(2) Suppose γ_i is a constant path in the interior of the horizontal edge $L(t_i)$. Then by [4, p. 457], $F(\gamma_i)$ consists of m copies of D_i and k copies of E_i for some k, hence $\chi(F(\gamma_i)) = 2m + k$. We need to determine the number k.

Let (a', b', c') be the parameters of ∂D_i on the four-punctured sphere ∂M_i. Since it is a vertex on $L(t_i)$, we have $y_i = p_i/q_i = c'/(a'+b')$, and $x_i = (q_i - 1)/q_i = b'/(a'+b')$. Also $a' = 1$ because a meridian intersects ∂D_i at a single point. Solving these equations gives $b' = q_i - 1$, and $c' = p_i$.

Let (a'', b'', c'') be the parameters of ∂E_i. Then $a'' = 0$ because ∂E_i is disjoint from the meridians of T_i. Examining the curve on ∂B_i explicitly we see that $b'' = q_i$ and $c'' = p_i$, hence it has parameters $(0, q_i, p_i)$.

Now the parameters of $mD_i + kE_i$ are given by

$$m(1, q_i - 1, p_i) + k(0, q_i, p_i) = (m, (m+k)q_i - m, (m+k)p_i).$$

Hence the x-coordinate and u-coordinate of the constant path γ_i satisfy

$$x_i = \frac{(m+k)q_i - m}{(m+k)q_i},$$
$$\bar{u} = \frac{1}{1 - x_i} = \frac{(m+k)q_i}{m} = q_i + \frac{kq_i}{m}.$$

Solving the last equation gives $k = m\bar{u}/q_i - m$, therefore $\chi(F(\gamma_i)) = 2m + k = m + m\bar{u}/q_i$.

(3) Let F' be an n-sheet candidate surface corresponding to the candidate system, constructed in [4, p. 457]. If F' is orientable then we are done. If not, let F be the boundary of a regular neighborhood of F'. Then F is an orientable surface. It can be constructed by doubling the intersection of F' with each tangle space and therefore is a candidate surface corresponding to the same candidate system, with $m = 2n$ sheets. □

Definition 2.2. Let v be a rational point in $\mathcal{D}$ with $u = u(v)$ as its u-coordinate, and let γ be an edge path with v as its ending point.

(1) If v is not on a horizontal line, define $e(v) = \frac{1}{3}(4 - u(v))$.

(2) If v is on a horizontal line $L(p/q)$, define $e(v) = \frac{1}{3} + u(v)(\frac{1}{q} - \frac{1}{3})$.

(3) For an allowable edge path γ with ending point v, define $e(\gamma) = e(v) - |\gamma|$.

(4) Given a candidate edge path system $(\gamma_1, \gamma_2, \gamma_3)$, define $\bar{e} = \bar{e}(\gamma_1, \gamma_2, \gamma_3) = \sum e(\gamma_i)$.

The numbers $e(v)$, $e(\gamma)$ and $\bar{e}$ are called the *Euler number* of a point, an edge path, and a candidate system, respectively.

Example 2.1. (a) If $v \in L(p/2)$ then $e(v) = \frac{1}{3} + \frac{1}{6}u > 0$.

(b) If $v \in L(p/3)$ then $e(v) = \frac{1}{3} > 0$.

(c) If $v \in L(p/q)$ and $q \geq 4$ then $e(v) = \frac{1}{3} + u(\frac{1}{q} - \frac{1}{3}) \leq \frac{1}{3} + q(\frac{1}{q} - \frac{1}{3}) = \frac{4}{3} - \frac{q}{3} \leq 0$, and $e(v) = 0$ if and only if $q = u = 4$.

(d) At a vertex $v = \langle p/q \rangle$, $u(v) = q$, so $e(v)$ can be rewritten as $e(v) = \frac{1}{3}(4 - q)$; in particular, $e(\langle p/q \rangle) < 0$ for all $q > 4$.

Lemma 2.3. *Let γ be an edge path with $|\gamma| < 1$. Let v be the ending point of γ, and let $u = u(v)$ be the u-coordinate of v. Then*

(1) *$e(\gamma) > 0$ if and only if (i) v is on the horizontal edge $L(p/q)$ with $q \leq 3$, (ii) v is on $\langle p/1, r/s \rangle$ for some $s \leq 3$, or (iii) v is on $\langle p/2, r/3 \rangle$ and $u > 2.5$.*

(2) *$e(\gamma) = 0$ if and only if (i) v is on $\langle p/1, r/4 \rangle$, or (ii) v is on $\langle p/2, r/3 \rangle$ and $u = 2.5$.*

Proof. Since $|\gamma| < 1$, γ cannot contain a full edge, hence if v is a vertex then γ is a constant path. By definition γ is also a constant path if v is in the interior of a horizontal edge. Thus if v is on a horizontal line $L(p/q)$ then the result follows from the calculations in Example 2.1(4) because $e(\gamma) = e(v)$.

We now assume that $v \in \langle p/q, r/s \rangle$, and $v \neq \langle p/q \rangle, \langle r/s \rangle$. Let $\bar{u}$ be the u-coordinate of v. Then by definition we have $e(\gamma) = \frac{1}{3}(4 - \bar{u}) - (s - \bar{u})/(s - q)$, which is a linear function of $\bar{u}$, $e(\gamma) = e(\langle p/q \rangle) - 1$ when $\bar{u} = q$, and $e(\gamma) = e(\langle r/s \rangle)$ when $\bar{u} = s$. Thus when $s \geq 5$ we have $e(\gamma) \leq 0$ at $\bar{u} = q$, and <0 at $\bar{u} = s$, hence $e(\gamma) < 0$ for all $q < \bar{u} \leq s$. The cases where $1 \leq q < s \leq 4$ can be done one by one. We omit the details. □

The set of v such that $e(\gamma) > 0$ for some γ ending at v is shown in figure 3 for the square $[0, 1] \times [0, 1]$. Those in the other squares are vertical translations of this graph.

Theorem 2.1. *Let $K = K(p_1/q_1, p_2/q_2, p_3/q_3)$ be a Montesinos knot of length 3, let $(\gamma_1, \gamma_2, \gamma_3)$ be a candidate system, let $F = F(\gamma_1, \gamma_2, \gamma_3)$ be the associated candidate surface, and let $\hat{F} = \hat{F}(\gamma_1, \gamma_2, \gamma_3)$ be the corresponding closed surface. Denote by $r = a/b$ the boundary slope of F, where a, b are coprime integers. Then $\hat{F}$ is a torus if and only if*

$$\bar{e} = \sum e(\gamma_i) = \frac{b - 1}{b}.$$

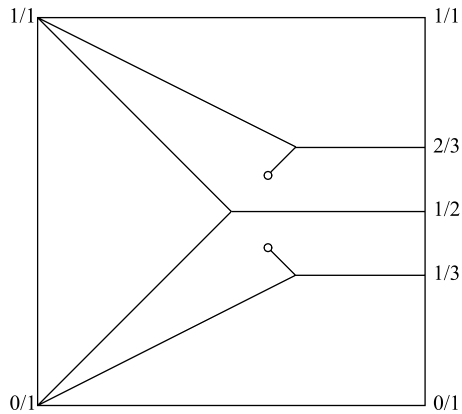

Figure 3:

In particular, if r is an integer slope then $\bar{e} = 0$*, and if r is a half integer slope then* $\bar{e} = \frac{1}{2}$.

Proof. Let (a_i, b_i, c_i) be the parameters of the ending point of γ_i, chosen so that $a_1 = a_2 = a_3$ for all i, which will be denoted by m. Since $x_i = b_i/(b_i + a_i)$ are the same for a candidate system, we have $b_1 = b_2 = b_3$, which we denote by b.

First consider the surface F' obtained by gluing $F(\gamma_i)$ along the three twice punctured disks P_j on the boundary of the tangle spaces. Each $F(\gamma_i)$ intersects P_j at $2m + b$ arcs, hence after gluing the $F(\gamma_i)$ to each other, we have

$$\chi(F') = \sum \chi(F(\gamma_i)) - 3(2m + b).$$

By construction $F = F(\gamma_1, \gamma_2, \gamma_3)$ is obtained from F' by adding $2m + 2b$ disjoint meridional disks in the solid torus $S^3 - \cup B_i$, hence

$$\chi(F) = \chi(F') + 2m + 2b = \sum \chi(F(\gamma_i)) - 4m - b.$$

From $x = b_i/(a_i + b_i) = b/(m + b)$ and $\bar{u} = 1/(1 - x)$, one can solve b to obtain

$$b = \frac{mx}{1 - x} = m(\bar{u} - 1).$$

When r is an integer slope, we need to attach m disks to F to obtain $\hat{F}$, hence

$$\begin{aligned}\chi(\hat{F}) &= \chi(F) + m = \sum \chi(F(\gamma_i)) - 3m - m(\bar{u}-1)\\ &= m \sum \left(\frac{1}{m}\chi(F(\gamma_i)) - \frac{2}{3} - \frac{1}{3}\bar{u} \right).\end{aligned}$$

If γ_i is a constant path, by Lemma 2.2(2) and Definition 2.2 we have

$$\begin{aligned}\frac{1}{m}\chi(F(\gamma_i)) - \frac{2}{3} - \frac{1}{3}\bar{u} &= \left(1 + \frac{\bar{u}}{q_i}\right) - \frac{2}{3} - \frac{1}{3}\bar{u}\\ &= \frac{1}{3} + \bar{u}\left(\frac{1}{q_i} - \frac{1}{3}\right) = e(\gamma_i).\end{aligned}$$

If γ_i is not a constant path, by Lemma 2.2(1) and Definition 2.2 we have

$$\begin{aligned}\frac{1}{m}\chi(F(\gamma_i)) - \frac{2}{3} - \frac{1}{3}\bar{u} &= (2 - |\gamma_i|) - \frac{2}{3} - \frac{1}{3}\bar{u}\\ &= \frac{4}{3} - \frac{1}{3}\bar{u} - |\gamma_i| = e(\gamma_i).\end{aligned}$$

Therefore, we always have $\chi(\hat{F}) = m\sum e(\gamma_i)$, hence $\hat{F}$ is a torus if and only if $\sum e(\gamma_i) = 0$.

The proof for $r = a/b$ and $b \neq 1$ is similar. In this case F has m/b boundary components, so $\hat{F}$ is obtained by attaching m/b disks to F. Hence a similar calculation shows that

$$\chi(\hat{F}) = \chi(F) + \frac{m}{b} = (\chi(F) + m) - \frac{b-1}{b}m = m\left(\sum e(\gamma_i) - \frac{b-1}{b}\right).$$

Therefore in this case $\hat{F}$ is a torus if and only if $\sum e(\gamma_i) = (b-1)/b$. □

Proposition 2.2. *Let $(\gamma_1, \gamma_2, \gamma_3)$ be a candidate system such that $\hat{F}(\gamma_1, \gamma_2, \gamma_3)$ is a torus. Suppose $\bar{u} \leq 1$. Then (i) $\bar{u} = 1$, and (ii) $K = K(1/q_1, 1/q_2, 1/q_3)$ for some (possibly negative) integers q_i, such that $|q_i| > 1$, and at most one q_i is even.*

The knots and the corresponding toroidal slopes are the same as those in Theorem 1.1(1) and (2).

Proof. Let γ'_i be the part of γ_i in the strip of $x \in [0, 1)$ (i.e., $u \geq 1$), and let y'_i be the ending points of γ'_i. Then $|\gamma'_i|$ are all nonzero integers, hence we have

$$0 \le \bar{e} = (4 - \bar{u}) - \sum |\gamma_i| \le 3 - \sum |\gamma'_i| \le 0.$$

Thus all the inequalities above are equalities, and we have $\bar{u} = |\gamma_i| = |\gamma'_i| = 1$ for all i, so $\gamma_i = \gamma'_i$ contains only one edge. Since $\bar{u} = 1$, $y_i = y'_i$ are integers. By definition of candidate system we have $\sum y_i = 0$, hence by choosing the parameters properly we may assume that $y_i = 0$ for all i. It is now easy to see that $K = K(1/q_1, 1/q_2, 1/q_3)$ for some q_i. Since K is of length 3, $|q_i| > 1$. Since K is a knot, at most one q_i is even.

The knots are the same as those in Theorem 1.1(1) and (2). The toroidal surface corresponding to a candidate system above is the pretzel surface S, or its double cover if S is nonorientable. One can draw the pretzel surface and show that the boundary slope of F is the same as that in Theorem 1.1(1) and (2). □

Up to equivalence we may change the parameters of $K = K(t_1, t_2, t_3)$ by the following moves.

(1) Replace all t_i by $-t_i$;

(2) Permute t_i;

(3) Replace (t_1, t_2, t_3) by $(t_1 + k_1, t_2 + k_2, t_3 + k_3)$, where k_i are integers, and $\sum k_i = 0$.

If $(\gamma_1, \gamma_2, \gamma_3)$ is a candidate system for $K(t_1, t_2, t_3)$, and (t'_1, t'_2, t'_3) is equivalent to (t_1, t_2, t_3) by the above relations, then we can obtain a candidate system $(\gamma'_1, \gamma'_2, \gamma'_3)$ for $K(t'_1, t'_2, t'_3)$ in the obvious way. For example, when (t_1, t_2, t_3) is replaced by $(t_1 + 1, t_2 - 1, t_3)$, the edge path γ'_1 is obtained by moving γ_1 upward by one unit, and γ'_2 downward by one unit. Clearly the surface $F(\gamma_1, \gamma_2, \gamma_3)$ is homeomorphic to $F(\gamma'_1, \gamma'_2, \gamma'_3)$.

Let $G = \langle 0, -\frac{1}{3}\rangle \cup \langle 0, -\frac{1}{2}\rangle \cup \langle -\frac{1}{2}, -\frac{1}{3}\rangle \cup \langle -\frac{1}{2}, -\frac{2}{3}\rangle \cup \langle -1, -\frac{1}{2}\rangle \cup L(-\frac{1}{2}) \cup L(-\frac{1}{3})$, as shown in figure 4.

Lemma 2.4. *Let $(\gamma_1, \gamma_2, \gamma_3)$ be a candidate edge path system for $K = K(t_1, t_2, t_3)$ such that the corresponding surface $\hat{F}(\gamma_1, \gamma_2, \gamma_3)$ is a torus. Let $v_i = (x_i, y_i)$ be the ending point of γ_i, and let $\bar{u}$ be the u-coordinate of v_i. Assume $\bar{u} > 1$. Then the following hold up to re-choosing the parameters of K.*

(1) $\sum y_i = 0$;

(2) $|y_i| + |y_j| \le 1$ *for any* $i \ne j$;

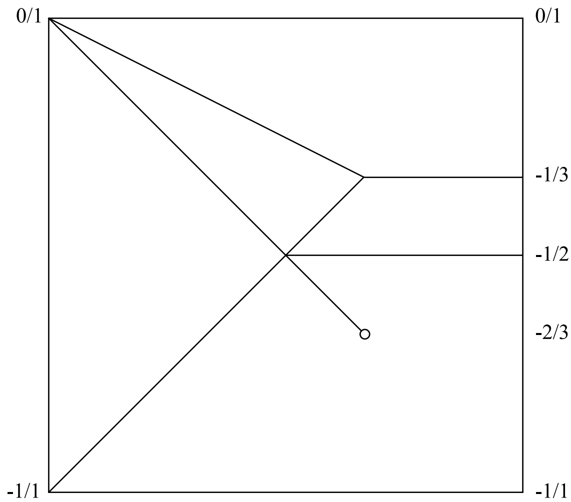

Figure 4:

(3) $0 < |y_i| \leq \frac{2}{3}$;

(4) v_1 *is on the subgraph* G *in figure 4.*

Proof. We may assume that the parameters of the knot has been chosen, among equivalent knots, so that $\sum |y_i|$ is minimal. The minimum can be reached because (i) $\sum |y_i|$ remains the same when permuting the parameters or replacing (t_1, t_2, t_3) by $(-t_1, -t_2, -t_3)$, and (ii) $\sum |y_i|$ goes to ∞ when (t_1, t_2, t_3) is replaced by $(t_1 + k_1, t_2 + k_2, t_3 - k_1 - k_2)$ and at least one k_i goes to ∞.

(1) This follows from the definition of candidate system.

(2) By permuting the t_i and simultaneously changing their signs if necessary, we may assume without loss of generality that $-y_1 \geq y_2 \geq y_3 \geq 0$. If the result is false then $-y_1 + y_2 > 1$. But then replacing (t_1, t_2, t_3) of K by $(t_1 + 1, t_2 - 1, t_3)$ will give a candidate system such that the y-coordinates of the ending points are $y_1' = -y_1 + 1$, $y_2' = y_2 - 1$, and $y_3' = y_3$, respectively. One can check that $\sum |y_i'| < \sum |y_i|$ if $|y_1| + |y_2| > 1$.

(3) Since $\bar{u} > 1$, v_i cannot be on $L(0)$ as otherwise γ_i would be a constant path on $L(0)$, so t_i would be 0, contradicting the assumption that the parameters of $K = K(t_1, t_2, t_3)$ are non-integers. Therefore, $y_i \neq 0$. If $|y_1| > \frac{2}{3}$, say, then since $y_1 = -y_2 - y_3$, we would have $|y_i| > \frac{1}{3}$ for $i = 2$ or 3, which implies $|y_1| + |y_i| > 1$, contradicting (2).

(4) Up to relabeling we may assume that $e(\gamma_1) \geq e(\gamma_i)$ for $i = 2, 3$, and by taking the mirror image of K if necessary we may assume that $y_1 < 0$.

By Theorem 2.1, $\bar{e} = \sum e(\gamma_i) \geq 0$, hence either $e(\gamma_1) > 0$, or $e(\gamma_i) = 0$ for all i.

First assume $e(\gamma_1) > 0$. Since $-1 < y_1 < 0$, by Lemma 2.3 v_1 is on one of the edges in G, except that it may also be on the edge $L(-\frac{2}{3})$. However, if $v_1 \in L(-\frac{2}{3})$ then since $-y_1 = y_2 + y_3$ and $-y_1 + y_i \leq 1$, we must have $y_2 = y_3 = \frac{1}{3}$, hence replacing (t_1, t_2, t_3) by $(-t_2, -t_3, -t_1)$ will give a new candidate system such that the ending point of the first edge path is on $L(-\frac{1}{3})$, as required.

Now assume $e(\gamma_i) = 0$ for all i. We may assume that no v_i is on G or its reflection along the line $y = 0$, as otherwise we may choose the parameters of K so that $v_1 \in G$. Thus by Lemmas 2.3 and 2.4(3), each v_i must be on $\langle 0, \pm\frac{1}{4}\rangle$. However, in this case one can show that $\sum y_i \neq 0$, contradicting Lemma 2.4(1). Therefore this case cannot happen. □

2.5. Calculation of boundary slopes

Denote by e_- (resp. e_+) the number of edges in all the γ_i on which a point moves downward (resp. upward) when traveling from right to left. Then the *twist number* of the edge path system $(\gamma_1, \gamma_2, \gamma_3)$ is defined as

$$\tau = \tau(\gamma_1, \gamma_2, \gamma_3) = 2(e_- - e_+).$$

Denote by $\delta = \delta(\gamma_1, \gamma_2, \gamma_3)$ the boundary slope of the surface $F(\gamma_1, \gamma_2, \gamma_3)$. The following lemma is due to Hatcher and Oertel [4], and can be used to calculate the boundary slope δ for a given edge path system.

Lemma 2.5. *Let t_i be rational numbers, and let $(\gamma_1, \gamma_2, \gamma_3)$ be a candidate system with γ_i starting at a point on $L(t_i)$. Then $\delta - \tau$ depends only on t_i and is independent of the paths γ_i.*

Thus, if $F' = F(\gamma_1', \gamma_2', \gamma_3')$ has boundary slope δ' and γ_i' has starting point on $L(t_i)$, then $\delta = \tau + \delta' - \tau'$, where $\delta' = \delta(\gamma_1', \gamma_2', \gamma_3')$ and $\tau' = \tau(\gamma_1', \gamma_2', \gamma_3')$. In particular if F' is a Seifert surface then $\delta = \tau - \tau'$.

Proof. This is in page 460 of [4], where it was shown that $\delta = \tau - \tau'$, where τ' is the twist number of the edge path system corresponding to the Seifert surface of K, starting from the vertices $\langle t_i \rangle$. Therefore $\delta - \tau$ depends only on (t_1, t_2, t_3). □

2.6. Notations and conventions

Throughout this paper we will denote by γ_i the edge path for the i-th tangle, by v_i the ending point of γ_i, by $\bar{u}$ the u-coordinate of v_i, which must be the same for all i, and by y_i the y-coordinate of v_i. Let L be the union of the two horizontal edges in G, i.e., $L = L(-1/2) \cup L(-1/3)$.

The case $\bar{u} \leq 1$ has been discussed in Proposition 2.2. Hence in Sections 3–6 we will assume that $\bar{u} > 1$. By Lemma 2.4, in this case we may choose the parameters t_i of $K = K(t_1, t_2, t_3)$ to satisfy the conclusions of that lemma; in particular, the ending point v_1 of γ_1 lies on the subgraph G of $\mathcal{D}$ in figure 4. In Sections 3–6 we will determine K case by case, according to the position of v_1 in G.

3. The case that $v_1 \in L$ and $\alpha_i = 0$ for $i = 2$ or 3

In this section, we will discuss the case that one of the vertices, say v_1, lies on the horizontal lines $L = L(-1/2) \cup L(-1/3)$, and $\alpha_2 = 0$. Note that the second condition is equivalent to that either γ_2 is a constant path, or v_2 is a vertex of $\mathcal{D}$.

Lemma 3.1. *γ_i cannot all be constant paths.*

Proof. Let $y_i = p_i/q_i$ be the y-coordinates of the ending points v_i of γ_i, where p_i, q_i are coprime integers. By Lemma 2.4(1) we have

$$\sum y_i = \sum \frac{p_i}{q_i} = 0. \tag{3.1}$$

If all γ_i are constant paths, $K = K(p_1/q_1, p_2/q_2, p_3/q_3)$ and since K is a knot, at most one of the q_i is even. If one of the q_i, say q_1, is even, then we have

$$q_2q_3p_1 + q_1(q_3p_2 + q_2p_3) = 0.$$

Since the first term is odd and the other two are even, this is impossible. If all q_i are odd, then equation (3.1) implies that

$$q_2q_3p_1 + q_1q_3p_2 + q_1q_2p_3 \equiv p_1 + p_2 + p_3 \equiv 0 \mod 2,$$

which implies that either one or three p_i are even. However, in this case K is a link of two components, which is again a contradiction. □

Lemma 3.2. *If v_1 is in the interior of L, and v_2 is in the interior of $L(p_2/q_2)$ for some $q_2 \le 3$, then $K = K(-1/2, 1/3, 1/(6+1/n))$ for some $n \ne 0, -1$, and $\bar{u} = 6$.*

Proof. Recall that we have assumed that y_i satisfy the conclusions of Lemma 2.4, hence $y_1 + y_2 + y_3 = 0$, $0 < |y_i| \le \frac{2}{3}$, and $|y_i| + |y_j| \le 1$ for $i \ne j$.

First assume $y_1 = -\frac{1}{2}$. Then the above and the assumption of $v_2 \in L(p_2/q_2)$ for $q_2 \le 3$ imply that $y_2 = \frac{1}{3}$, and $y_3 = -y_1 - y_2 = 1/6$. The horizontal line $y = 1/6$ intersects the graph $\mathcal{D}$ at the horizontal edge $L(1/6)$ and one point on each edge $\langle 0, 1/q \rangle$ with $q \le 6$. By Lemma 3.1, v_3 cannot be in the interior of $L(1/6)$ as otherwise we would have three constant paths. It follows that v_3 must be on some $\langle 0, 1/q \rangle$ with $q \le 6$. By calculating the intersection point of $y = 1/6$ with $\langle 0, 1/q \rangle$ we see that $u \le 3$ when $q \le 5$, which would be a contradiction because $v_2 \in \text{Int}L(\frac{1}{3})$ implies that $\bar{u} > 3$. Therefore, we must have $q = 6$, in which case v_3 is the vertex $\langle 1/6 \rangle$.

By Definition 2.2 we have

$$\sum e(v_i) = \left(\frac{1}{3} + 6\left(\frac{1}{2} - \frac{1}{3}\right)\right) + \frac{1}{3} + \frac{1}{3}(4-6) = 1,$$

therefore by Theorem 2.1 we must have $\bar{e} = \sum e(\gamma_i) = \sum e(v_i) - \sum |\gamma_i| = 0$, so there is exactly one edge in $\cup \gamma_i$, which must be in γ_3 because γ_1 and γ_2 are constant paths. Therefore

$$K = K\left(-\frac{1}{2}, \frac{1}{3}, \frac{1}{6 + \frac{1}{n}}\right).$$

Since γ_3 must be an allowable edge path, we have $n \ne 0, -1$, and the result follows.

Now assume $y_1 = -\frac{1}{3}$. The case of $y_2 = \frac{1}{2}$ is similar to the above, and we obtain the same knot up to equivalence. If $y_2 = -\frac{1}{2}$ then $|y_2| + |y_3| > 1$, contradicting Lemma 2.4. In all other cases we have $v_i \in \text{Int}L(p_i/3)$ for each i, which implies that γ_i are all constant paths, contradicting Lemma 3.1. □

Lemma 3.3. *Suppose $v_1 \in L(-1/3)$. Then v_2 cannot be in the interior of a horizontal edge $L(p_2/q_2)$ with $q_2 \ge 4$.*

Proof. If this is not true then γ_2 is a constant path, so by Lemma 3.1 γ_3 cannot be a constant path, hence $|\gamma_3| > 0$. By Definition 2.2 we have

$$e(\gamma_1) = \frac{1}{3},$$

$$\begin{aligned}
e(\gamma_2) &= \frac{1}{3} + \bar{u}\left(\frac{1}{q_2} - \frac{1}{3}\right), \\
e(\gamma_3) &= \frac{1}{3}(4-\bar{u}) - |\gamma_3| \\
0 \le \bar{e} &= \sum e(\gamma_i) = 2 - \frac{5}{12}\bar{u} - \bar{u}\left(\frac{1}{4} - \frac{1}{q_2}\right) - |\gamma_3| \\
&\le 2 - \frac{5}{12}\bar{u} - |\gamma_3| < 2 - \frac{5}{12}\bar{u}.
\end{aligned}$$

This gives $5 > \bar{u}$. Since $\bar{u} \ge q_3$ and $q_3 \ge 4$, we must have $q_2 = 4$. Assume v_3 is on the edge $\langle p_3/q_3, t_3/s_3\rangle$. Then $s_3 > \bar{u} \ge 4$, so $s_3 \ge 5$.

Define $\beta_3(u) = (s_3 - u)/(s_3 - q_3)$. Then $\beta_3(\bar{u})$ is the length of the last edge segment in γ_3, so $\beta_3(\bar{u}) \le |\gamma_3|$.

The function $e(u) = 2 - \frac{5}{12}u - \beta_3(u)$ is a linear function of u. We have $e(5) < 0$, and $e(\bar{u}) \ge \bar{e} \ge 0$, for some $4 < \bar{u} < 5$, so $e(4) > 0$, and hence $\beta_3(4) < \frac{1}{3}$. Since $\beta_3(4) = (s_3 - 4)/(s_3 - q_3)$, this is true if and only if $q_3 = 1$ and $s_3 = 5$. Hence v_3 is on an edge $E_3 = \langle p_3/1, t_3/5\rangle$ for some p_3, t_3. Since $0 < |y_3| \le \frac{2}{3}$, we must have $E_3 = \langle 0, \pm 1/5\rangle$.

By assumption we have $y_1 = -\frac{1}{3}$. Since $|y_2| = |p_2/q_2| = |p_2|/4$ and $|y_1| + |y_2| \le 1$, we must have $|y_2| = \frac{1}{4}$, hence $(y_2, y_3) = (-\frac{1}{4}, \frac{5}{12})$ or $(\frac{1}{4}, \frac{1}{12})$. It is easy to see that the horizontal line $y = \frac{5}{12}$ does not intersect the edge E_3 above. Therefore, we must have $(y_2, y_3) = (\frac{1}{4}, \frac{1}{12})$, hence $E_3 = \langle 0, \frac{1}{5}\rangle$.

The line equation of E_3 is given by $y = x/4$, hence the only solution for $y_3 = 1/12$ and $x > 0$ is at $x = 1/3$, which has u-coordinate $u = 1/(1-x) = 3/2 < 4$. Therefore, there is no solution in this case because $\bar{u} > 4$. □

Lemma 3.4. *Suppose $v_1 \in L(-\frac{1}{2})$. Then v_2 cannot be in the interior of a horizontal edge $L(p_2/q_2)$ with $q_2 \ge 4$.*

Proof. Similar to Lemma 3.3, we have

$$\begin{aligned}
e(\gamma_1) &= \frac{1}{3} + \bar{u}\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3} + \frac{1}{6}\bar{u}, \\
e(\gamma_2) &= \frac{1}{3} + \bar{u}\left(\frac{1}{q_2} - \frac{1}{3}\right), \\
e(\gamma_3) &= \frac{1}{3}(4-\bar{u}) - |\gamma_3|, \\
0 \le \bar{e} &= \sum e(\gamma_i) = 2 - \bar{u}\left(\frac{1}{2} - \frac{1}{q_2}\right) - |\gamma_3|.
\end{aligned}$$

Since K is a knot, q_2 must be odd, hence $q_2 \geq 5$. Hence from the above we have $2 - \bar{u}(\frac{1}{2} - \frac{1}{5}) \geq 0$, so $\bar{u} < 7$. Since $q_2 < \bar{u}$, we must have $q_2 = 5$. Therefore, $y_2 = p_2/q_2 = \pm k/5$ for some $k = 1, 2, 3, 4$. Since $|y_i| + |y_j| \leq 1$ and $|y_1| = \frac{1}{2}$, we must have $y_2 = \frac{1}{5}$ or $\frac{2}{5}$. (The cases of $y_2 = -\frac{1}{5}$ and $-\frac{2}{5}$ are impossible because then $y_3 > \frac{1}{2}$, so $|y_1| + |y_3| > 1$.)

When $y_2 = \frac{2}{5}$, we have $y_3 = -y_1 - y_2 = \frac{1}{10}$. The intersection of the line $y = \frac{1}{10}$ and $\mathcal{D}$ is the union of $L(\frac{1}{10})$ and one point in each edge $\langle 0, 1/q_3 \rangle$ for $q_3 \leq 10$. Since $\bar{u} < 7$, v_2 cannot be on $L(\frac{1}{10})$. By direct calculation we see that the u value of the intersection between $y = \frac{1}{10}$ and $\langle 0, 1/q_3 \rangle$ is

$$u = \frac{1}{1 - (q_3 - 1)y} = \frac{10}{11 - q_3},$$

which gives $u \leq 5$ for $q_3 \leq 9$, and $u = 10$ for $q_3 = 10$. Since $5 < \bar{u} < 7$, there is no solution in this case.

When $y_2 = \frac{1}{5}$, we have $y_3 = -y_1 - y_2 = \frac{3}{10}$. The horizontal line $y = \frac{3}{10}$ intersects $\mathcal{D}$ at $L(\frac{3}{10})$, and one point on each of $\langle 0, \frac{1}{2} \rangle$, $\langle 0, \frac{1}{3} \rangle$, $\langle \frac{1}{3}, \frac{1}{4} \rangle$, and $\langle \frac{1}{3}, \frac{2}{7} \rangle$. (These are all the edges $\langle t_1, t_2 \rangle$ with t_1 and t_2 on opposite sides of $y = \frac{3}{10}$.) As above, one can calculate the u-coordinate of the intersection to show that there is no intersection point on the interval $5 < u < 7$. Hence there is no solution in this case either. □

We now assume that $v_1 \in L$, and v_2, v_3 are not in the interior of horizontal edges. Since $\alpha_2 = 0$, the ending point v_2 of γ_2 must be a vertex of $\mathcal{D}$. The following two lemmas determine all knots with this property.

Lemma 3.5. *Suppose $v_1 \in L(-1/3)$, v_2 is a vertex of $\mathcal{D}$, and v_3 is not in the interior of a horizontal edge. Then $K = K(-1/3, -1/(3+1/n), 2/3)$ for some odd $n \neq -1$, and $\bar{u} = 3$.*

Proof. By Definition 2.2 we have

$$0 \leq \bar{e} = \sum e(\gamma_i) \leq \frac{1}{3} + 2 \times \frac{1}{3}(4 - \bar{u}) - \sum |\gamma_i|,$$

which gives $\bar{u} \leq 4.5$. Since v_2 is vertex and $v_1 \in L(-1/3)$, $\bar{u}$ is an integer, and $\bar{u} \geq 3$. Hence $\bar{u} = 3$ or 4. Thus $v_2 = \langle \pm 1/3 \rangle$, $\langle \pm 2/3 \rangle$, $\langle \pm 1/4 \rangle$ or $\langle \pm 3/4 \rangle$. Since $|y_i| \leq \frac{2}{3}$, we cannot have $|y_2| = 3/4$.

When $y_2 = -1/3$, we have $y_3 = 2/3$, so all the three v_i are vertices, and $\sum e(v_i) = 1$. Since $\bar{e} = \sum e(v_i) - \sum |\gamma_i|$, by Theorem 2.1 we must have $\sum |\gamma_i| = 1$, so there is one full edge in some γ_i. Because of symmetry K is equivalent to $K(-1/3, -1/(3+1/n), 2/3)$. Since γ_3 is allowable, $n \neq -1$, and since K is a knot, n must be odd. This gives the knots listed in the lemma.

When $y_2 = 1/3$ or $-2/3$, $y_3 = 0$ or 1, which is not a solution. When $y_2 = 2/3$ we have $y_3 = -1/3$, which gives the same solution as above.

When $y_2 = \frac{1}{4}$, we have $y_3 = -y_1 - y_2 = \frac{1}{12}$. Now v_3 lies on the intersection of $y = \frac{1}{12}$ and $u = 4$, which is a point on the edge $\langle 0, 1/10\rangle$. We have

$$0 \le \bar{3} \le \sum e(v_i) = \frac{1}{3} + 0 + \left(-\frac{2}{3}\right) < 0,$$

hence it is not a solution.

When $y_2 = -\frac{1}{4}$, $y_3 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$. One can check that the point of $(u, y) = (4, \frac{7}{12})$ lies on the edge $\langle \frac{1}{2}, \frac{3}{5}\rangle$. We have $\beta_3 = (\bar{u} - 2)/(5 - 2) = 1/3$. Since $\beta_1 = \beta_2 = 0$, by Lemma 2.5 the boundary slope of the surface is $\delta \equiv 2(e_- - e_+) \equiv \pm 2\beta_3 = \pm 2/3 \mod 1$, hence by Theorem 2.1 we have $\bar{e} = 2/3$. On the other hand, we have

$$\bar{e} = \sum e(\gamma_i) = \frac{1}{3} + 0 + \left(\frac{1}{3}(4-4) - \frac{1}{3}\right) = 0.$$

This contradiction completes the proof of the lemma. □

Lemma 3.6. *Suppose $v_1 \in L(-\frac{1}{2})$, v_2 is a vertex of $\mathcal{D}$, and v_3 is not in the interior of a horizontal edge. Then K and $\bar{u}$ are given by one of the following.*

(i) *$K(-1/2, 1/5, 2/7)$, $\bar{u} = 5$;*

(ii) *$K(-1/2, 2/5, 1/9)$, $\bar{u} = 5$;*

(iii) *$K(-1/2, 1/5, 1/(3 + 1/n))$, n even, $n \neq 0$, $\bar{u} = 3$;*

(iv) *$K(-1/2, 1/3, 1/(5 + 1/n))$, n even, $n \neq 0$, $\bar{u} = 3$.*

Proof. Let $v_2 = \langle y_2\rangle = \langle p_2/q_2\rangle$. By definition of $e(\gamma_i)$ and Theorem 2.1 we have

$$\begin{aligned} 0 \le \bar{e} = \sum e(\gamma_i) &= \left(\frac{1}{3} + \frac{1}{6}\bar{u}\right) + 2 \times \frac{1}{3}(4 - \bar{u}) - |\gamma_2| - |\gamma_3| \\ &= 3 - \frac{1}{2}\bar{u} - |\gamma_2| - |\gamma_3|, \end{aligned}$$

which gives $\bar{u} = q_2 \le 6$. We have $y_2 > 0$ since otherwise $y_3 = -y_1 - y_2 > \frac{1}{2}$, so $|y_1| + |y_3| > 1$, contradicting Lemma 2.4(2). Similarly we must have

$y_2 \le \frac{1}{2}$ as otherwise we would have $|y_1| + |y_2| > 1$. Moreover, $y_2 \neq \frac{1}{2}$ as otherwise we would have $y_3 = -y_1 - y_2 = 0$, contradicting Lemma 2.4(3). Therefore $y_2 = \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\ \frac{2}{5}$, or $\frac{1}{6}$. In each case v_3 is uniquely determined by the facts that $u(v_3) = \bar{u} = q_2$, and $y_3 = -y_1 - y_2$. We separate the cases.

Case 1. $y_2 = \frac{1}{3}$.

We have $\bar{u} = 3$, and $y_3 = \frac{1}{6}$. The point v_3 lies on the edge $\langle 0, \frac{1}{5}\rangle$, with $\beta_3 = (5-3)/(5-1) = 1/2$, which is the length of the last segment of γ_3. Hence

$$\sum e(v_i) - \beta_3 = \frac{5}{6} + \frac{1}{3} + \frac{1}{3} - \frac{1}{2} = 1,$$

so there is an extra edge, whose ending point is either $\langle 1/3\rangle$ or $\langle 1/5\rangle$. Since $K = K(-\frac{1}{2}, a_2/b_2, a_3/b_3)$ is a knot, the numbers b_2 and b_3 must be odd. Combining these, we see that K is equivalent to a knot of type (iii) or (iv) in the Lemma.

Case 2. $y_2 = \frac{1}{4}$.

We have $\bar{u} = 4$ and $y_3 = -y_1 - y_2 = \frac{1}{4}$, so v_3 is also at $\langle \frac{1}{4}\rangle$. Since $\sum e(v_i) = (\frac{1}{3} + \frac{1}{6}\bar{u}) + 0 + 0 = 1$, there is one extra edge. It follows that $K = K(-\frac{1}{2}, \frac{1}{4}, 1/(4+\frac{1}{n}))$, which is a link of at least two components. Therefore there is no solution in this case.

Case 3. $y_2 = \frac{1}{5}$.

We have $\bar{u} = 5$ and $y_3 = \frac{1}{2} - \frac{1}{5} = \frac{3}{10}$. The vertex v_3 lies on the edge $\langle \frac{1}{3}, \frac{2}{7}\rangle$, and $\beta_3 = (7-5)/(7-3) = 1/2$. We have $\sum e(v_i) - \beta_3 = (\frac{1}{3} + \frac{5}{6}) + \frac{1}{3}(4-5) + \frac{1}{3}(4-5) - \frac{1}{2} = 0$, so there is no extra edge. The knot is $K(-\frac{1}{2}, \frac{1}{5}, \frac{2}{7})$.

Case 4. $y_2 = \frac{2}{5}$.

Then $\bar{u} = 5$ and $y_3 = \frac{1}{10}$. The point v_3 lies on the edge $\langle 0, \frac{1}{9}\rangle$, and $\beta_3 = (9-5)/(9-1) = 1/2$. We have $\sum e(v_i) - \beta_3 = (\frac{1}{3} + \frac{5}{6}) + \frac{1}{3}(4-5) + \frac{1}{3}(4-5) - \frac{1}{2} = 0$, so there is no extra edge. The knot is $K(-\frac{1}{2}, \frac{2}{5}, \frac{1}{9})$.

Case 5. $y_2 = \frac{1}{6}$.

Then $\bar{u} = 6$ and $y_3 = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$. The point v_3 is in the interior of the horizontal line $L(\frac{1}{3})$, which contradicts the assumption. Therefore there is no solution in this case. □

Proposition 3.1. *Suppose $v_1 \in L$ and $\alpha_i = 0$ for $i = 2$ or 3. Then K is equivalent to one of the knots listed in Lemma 3.2, 3.5 or 3.6.*

Proof. By symmetry we may assume that $\alpha_2 = 0$, so v_2 is either a vertex or in the interior of a horizontal line $L(p_2/q_2)$. The first case is covered by Lemmas 3.5 and 3.6. In the second case by Lemmas 3.3 and 3.4 we must have $q_2 \le 3$. Since $y_2 \neq 0$, we have $q_2 \ge 2$. We may now apply Lemma 3.2 unless

v_1 is a vertex of L, which happens only if $q_2 = 2$ and $v_1 = \langle -\frac{1}{3} \rangle$. If that is the case, then we may consider the (equivalent) knot $K(-p_2/q_2, -p_1/q_1, -p_3/q_3)$ instead, which has the property that the ending points of the corresponding edge paths are (v_1', v_2', v_3'), with $v_1' \in L(-\frac{1}{2})$ and $v_2' = \langle \frac{1}{3} \rangle$, which has been covered by Lemma 3.6. $\square$

4. The case that $v_1 \in L(-\frac{1}{3})$ and $\alpha_i \neq 0$ for $i = 2, 3$

In this section we will assume that $v_1 \in L(-\frac{1}{3})$, and v_i is in the interior of a non-horizontal edge $E_i = \langle p_i/q_i, r_i/s_i \rangle$ and hence $0 < \alpha_i < 1$ for $i = 2, 3$. We have $\bar{u} = u(v_1) \geq 3$.

Define

$$\beta_i(u) = \frac{s_i - u}{s_i - q_i},$$
$$e(u) = 3 - \frac{2}{3}u - \beta_2(u) - \beta_3(u).$$

Then $e(u)$ is a linear function of u. Recall that $\beta_i = (s_i - \bar{u})/(s_i - q_i) = \beta_i(\bar{u})$. We have

Lemma 4.1. *Suppose $v_1 \in L(-\frac{1}{3})$, and $\alpha_i \neq 0$ for $i = 2, 3$. Then*

(1) *$0 \leq \bar{e} \leq e(\bar{u})$, and*

(2) *$3 \leq \bar{u} < 4.5$.*

Proof. (1) By definition we have

$$0 \leq \bar{e} = \sum e(\gamma_i) = \frac{1}{3} + 2 \times \left(\frac{1}{3}(4 - \bar{u}) \right) - |\gamma_2| - |\gamma_3|$$
$$\leq 3 - \frac{2}{3}\bar{u} - \beta_2 - \beta_3 = e(\bar{u}).$$

(2) Since v_i is in the interior of non-horizontal edge, $\beta_i > 0$ for $i = 2, 3$. Hence the above inequality implies that $\bar{u} < 4.5$. Since $v_1 \in L(-\frac{1}{3})$, we have $\bar{u} \geq 3$. $\square$

Lemma 4.2. *Suppose $v_1 \in L(-\frac{1}{3})$, and $\alpha_i \neq 0$ for $i = 2, 3$. Then $q_i \leq 3$ for $i = 2, 3$.*

Proof. Since $q_i \leq \bar{u}$, by Lemma 4.1(2) we have $q_i \leq 4$ for $i = 2, 3$. If $q_2 = 4$ then by Lemma 4.1(1) we have $4 < \bar{u} < 4.5$, so $\beta_2 = (s_2 - \bar{u})/(s_2 - 4) \geq 1/2$.

Therefore $e(\bar{u}) = 3 - \frac{2}{3}\bar{u} - \beta_2 - \beta_3 < 0$, which is a contradiction to Lemma 4.1(1). □

Lemma 4.3. *Suppose $v_1 \in L(-\frac{1}{3})$ and $\alpha_i \neq 0$ for $i = 2, 3$. Then $q_2 \leq 2$.*

Proof. Assume to the contrary that $q_2 > 2$. Then by Lemma 4.2 we must have $q_2 = 3$. In this case $\beta_2(3) = -1$, so $e(3) = 3 - \frac{2}{3}(3) - 1 - \beta_3 < 0$.

First assume $s_2 \geq 5$. If $s_3 \geq 5$ then $\beta_i(5) \geq 0$ for $i = 2, 3$, so $e(5) = 3 - \frac{2}{3} \times 5 - \beta_2(5) - \beta_3(5) < 0$, and by linearity we have $e(\bar{u}) < 0$, which is a contradiction. If $s_3 = 4$ then $3 < \bar{u} < 4$. Since $\beta_2(4) \geq \frac{1}{2}$ we have $e(4) < 3 - \frac{2}{3}4 - \beta_2(4) < 0$, which again contradicts the fact that $e(\bar{u}) \geq 0$.

We may now assume $s_2 = 4$. Then $3 < \bar{u} < 4$. We have $e(3) < 0$, and $e(\bar{u}) \geq 0$, hence $e(4) = \frac{1}{3} - \beta_3(4) > 0$, i.e., $\beta_3(4) = (s_3 - 4)/(s_3 - q_3) < \frac{1}{3}$.

If $q_3 = 3$ then $\beta_3(4) < \frac{1}{3}$ implies that $s_3 = 4$. Since $|y_i| \leq \frac{2}{3}$, $E_3 \neq \langle \pm\frac{2}{3}, \pm\frac{3}{4} \rangle$, so we must have $E_3 = \langle \pm\frac{1}{3}, \pm\frac{1}{4} \rangle$. Since $(q_2, s_2) = (3, 4)$, the same is true for E_2, hence $\frac{1}{4} < |y_i| < \frac{1}{3}$ for $i = 2, 3$. Therefore either $y_2 + y_3 = 0$ (if $E_2 \neq E_3$), or $|y_2 + y_3| > \frac{1}{2}$, either case contradicting the fact that $y_2 + y_3 = -y_1 = \frac{1}{3}$.

If $q_3 = 2$ then since s_3 is coprime with q_3, $s_3 \neq 4$, so we must have $s_3 \geq 5$. Thus $\beta_3(4) = (s_3 - 4)/(s_3 - 2) \geq 1/3$, which is a contradiction.

If $q_3 = 1$ then $\beta_3(4) = (s_3 - 4)/(s_3 - 1) < 1/3$ implies that $s_3 = 4$ or 5, so $E_3 = \langle 0, \pm\frac{1}{4} \rangle$ or $\langle 0, \pm\frac{1}{5} \rangle$. As above, $E_2 = \langle \pm\frac{1}{3}, \pm\frac{1}{4} \rangle$, and one can check that there is no solution to the equation $y_1 + y_2 + y_3 = 0$ in these cases. □

Proposition 4.1. *Suppose $v_1 \in L(-\frac{1}{3})$, and $\alpha_i \neq 0$ for $i = 2, 3$. Then K is equivalent to either $K(-1/3, 1/5, 1/5)$ or $K(-1/3, 1/4, 1/7)$, and $\bar{u} = 3$.*

Proof. First assume that $s_2, s_3 \geq 5$. Then $\beta_i(5) \geq 0$, hence $e(5) = 3 - \frac{2}{3} \times 5 - \beta_2(5) - \beta_3(5) < 0$. By Lemma 4.1 we have

$$e(3) = 3 - \frac{2}{3} \times 3 - \beta_2(3) - \beta_3(3) = 1 - \beta_2(3) - \beta_3(3).$$

Since $e(\bar{u}) \geq 0$ for some $3 \leq \bar{u} < 5$, by linearity we have $e(3) \geq 0$. Therefore, one of the β_i, say β_2, satisfies

$$\beta_2(3) = \frac{s_2 - 3}{s_2 - q_2} \leq \frac{1}{2}.$$

Since $s_2 \geq 5$, this is true if and only if $(q_2, s_2) = (1, 5)$, in which case $\beta_2(3) = \frac{1}{2}$, hence $\beta_3(3) \leq \frac{1}{2}$, and the same argument as above shows that

$(q_3, s_3) = (1, 5)$. It follows that $K = K(-1/3, 1/5, 1/5)$, and $\bar{u} = 3$ because $e(\bar{u}) < 0$ if $\bar{u} > 3$.

Now assume $s_2 = 4$. Then $3 \le \bar{u} < 4$. Since q_2 is coprime with s_2 and $q_2 < 3$ by Lemma 4.3, we must have $q_2 = 1$, so $(q_2, s_2) = (1, 4)$. Since $|y_2| \le \frac{2}{3}$, $E_2 \neq \langle \pm 1, \pm\frac{3}{4}\rangle$. Therefore we must have $E_2 = \langle 0, \pm\frac{1}{4}\rangle$.

Suppose $E_2 = \langle 0, -\frac{1}{4}\rangle$. Then $-E_1 - E_2$ coincides with the edge $\langle \frac{1}{2}, \frac{3}{5}\rangle$ on $3 \le u \le 4$, so $E_3 = \langle \frac{1}{2}, \frac{3}{5}\rangle$ if there is a solution. By Example 2.1 and Definition 2.2 we have

$$\bar{e} = \sum e(\gamma_i) \le \sum e(v_i) - \beta_3 = \frac{1}{3} + 0 + \frac{1}{3}(4 - \bar{u}) - \frac{5 - \bar{u}}{5 - 2} = 0.$$

On the other hand, since E_2 goes upward and E_3 downward when traveling from right to left, we have $e_- \equiv \beta_3$ and $e_+ \equiv \beta_2$ mod 1, hence by Lemma 2.5 the boundary slope of the surface δ satisfies

$$\delta \equiv 2(e_- - e_+) \equiv 2(\beta_3 - \beta_2) = 2\left(\frac{5 - \bar{u}}{5 - 2} - \frac{4 - \bar{u}}{4 - 1}\right) = \frac{2}{3}.$$

It follows from Theorem 2.1 that $\bar{e} = \frac{2}{3}$, which is a contradiction. Therefore, there is no solution in this case.

Now assume $E_2 = \langle 0, \frac{1}{4}\rangle$. Then $y_2 = x/3$, and $y_3 = -y_1 - y_2 = (1 - x)/3$. One can check that when $\bar{u} = 3$, v_3 is on the edge $\langle 0, \frac{1}{7}\rangle$, in which case we have $\bar{e} = 0$, and $K = K(-\frac{1}{3}, \frac{1}{4}, \frac{1}{7})$. We need to show that there is no solution when $\bar{u} > 3$.

The line segment $y = (1 - x)/3$ is below the line $y = 1 - x$, hence from figure 1 we see that y_3 is on an edge $E_3 = \langle 0, 1/s_3\rangle$ for some s_3. Since the line segment has negative slope, and since it intersects $\langle 0, \frac{1}{7}\rangle$ at $u = 3$, we must have $s_3 > 7$ when $u > 3$. By definition we have

$$\bar{e} = \sum e(\gamma_i) = \frac{1}{3} + 0 + \left(\frac{1}{3}(4 - \bar{u}) - \frac{s_3 - \bar{u}}{s_3 - 1}\right).$$

For $s_3 > 7$, the right hand side is negative for $\bar{u} = 3$ and 4, hence by linearity it is negative for all $3 \le \bar{u} \le 4$. By Theorem 2.1 there is no solution in this case. □

5. The case that $v_1 \in L(-\frac{1}{2})$ and $\alpha_i \neq 0$, for $i = 2, 3$

In this section, we will assume that $v_1 \in L(-\frac{1}{2})$, and v_i is in the interior of a non-horizontal edge $E_i = \langle p_i/q_i, r_i/s_i\rangle$ and hence $0 < \alpha_i < 1$ for $i = 2, 3$. By

Lemma 2.4 (2) and (3) we must have $0 < y_i < \frac{1}{2}$ for $i = 2, 3$, hence $0 \leq z \leq \frac{1}{2}$ when $z = p_i/q_i$ or r_i/s_i and $i = 2, 3$.

As before, define $\bar{e} = \sum e(\gamma_i)$, and $\beta_i(u) = (u - q_i)/(s_i - q_i)$. Let $\beta_i = \beta_i(\bar{u})$. Define a function

$$e(u) = 3 - \frac{1}{2}u - \beta_2(u) - \beta_3(u).$$

Note that this is different from the function $e(u)$ defined in Section 4.

Define $l_i = s_i - q_i$. Given an edge E in $\mathcal{D}$ and a number t, we use $t - E$ to denote the set of points $\{t - t' \mid t' \in E\}$.

Lemma 5.1. *Suppose $v_1 \in L(-\frac{1}{2})$ and $\alpha_i \neq 0$ for $i = 2, 3$. Then*

(1)

$$\begin{aligned} 0 \leq \bar{e} \leq e(\bar{u}) &= 3 - \frac{1}{2}\bar{u} - \beta_2 - \beta_3 = 1 - \frac{1}{2}\bar{u} + \alpha_2 + \alpha_3 \\ &= 1 - \frac{1}{2}\bar{u} + \frac{\bar{u} - q_2}{l_2} + \frac{\bar{u} - q_3}{l_3} \\ &= \left(1 - \frac{q_2}{l_2} - \frac{q_3}{l_3}\right) - \left(\frac{1}{2} - \frac{1}{l_2} - \frac{1}{l_3}\right)\bar{u} \end{aligned}$$

(2) $2 \leq \bar{u} < 6$.

Proof. (1) By Theorem 2.1 we have $\bar{e} = \sum e(\gamma_i) \geq 0$. Since $v_1 \in L(-\frac{1}{2})$, by Definition 2.2 and Example 2.1 we have

$$\bar{e} = \left(\frac{1}{3} + \frac{1}{6}\bar{u}\right) + 2 \times \frac{1}{3}(4 - \bar{u}) - |\gamma_2| - |\gamma_3| \leq e(\bar{u})$$

The other equalities are just different expressions of $e(\bar{u})$.

(2) Since $v_1 \in L(-\frac{1}{2})$, we have $\bar{u} \geq 2$. Since v_i is in the interior of non-horizontal edges, we have $\beta_i > 0$, so $0 \leq e(\bar{u}) < 3 - \frac{1}{2}\bar{u}$, hence $\bar{u} < 6$. □

Lemma 5.2. *Suppose $v_1 \in L(-\frac{1}{2})$, and $\alpha_i \neq 0$ for $i = 2, 3$. Then $q_i \leq 3$ for $i = 2, 3$.*

Proof. Note that $q_i \leq \bar{u} < 6$. If $q_2 = 5$ then $\bar{u} \in [5, 6)$. We have $e(6) \leq 3 - \frac{1}{2} \times 6 = 0$, and $e(5) = 3 - \frac{5}{2} - \beta_2(5) - \beta_3(5) < 0$ because $\beta_2(5) = 1$. Since $e(u)$ is linear, $e(\bar{u}) < 0$, a contradiction. Therefore we may assume that $q_2 = 4$.

First assume $s_2 > 5$. We have $e(4) = 1 - \beta_2(4) - \beta_3(4) = -\beta_3(4) < 0$ and $e(\bar{u}) \geq 0$. If $s_3 > 5$ then $\beta_i(6) \geq 0$, so $e(6) \leq 3 - \frac{1}{2} \times 6 = 0$, which contradicts $4 < \bar{u} < 6$ and the linearity of $e(u)$. If $s_3 = 5$ then $4 < \bar{u} < 5$, and $e(5) = 3 - \frac{1}{2} \times 5 - \beta_2(5) \leq 0$, which again is a contradiction.

Now assume $s_2 = 5$. Then $E_2 = \langle p_2/4, r_2/5\rangle$. Since $0 < y_2 < \frac{1}{2}$, we must have $E_2 = \langle \frac{1}{4}, \frac{1}{5}\rangle$. However, one can check that the interior of the line segment $\frac{1}{2} - E_2$ lies in the interior of the triangle with vertices $\langle\frac{1}{3}\rangle$, $\langle\frac{1}{4}\rangle$, $\langle\frac{2}{7}\rangle$, hence there is no solution to the equation $\sum y_i = 0$ for $v_2 \in \text{Int}E_2$. This contradiction completes the proof of the lemma. □

Lemma 5.3. *If $v_1 \in L(-\frac{1}{2})$, $\alpha_i > 0$ for $i = 2, 3$, and $q_3 \leq q_2 = 3$, then $K = K(-1/2, 2/5, 1/7)$ and $\bar{u} = 4$, or $K(-1/2, 1/5, 2/7)$ and $3 < \bar{u} < 5$.*

Proof. Since $y_2 \in (0, \frac{1}{2})$, we must have $E_2 = \langle 1/3, r_2/s_2\rangle$. From figure 1 we see that $r_2/s_2 \geq \frac{1}{4}$, hence $y_2 > \frac{1}{4}$, and $y_3 = -y_1 - y_2 < \frac{1}{4}$. Since $q_3 \leq 3$ and $y_3 < 1/4$, from figure 1, we see that $p_3/q_3 = 0$, so $E_3 = \langle 0, 1/s_3\rangle$ for some $s_3 \geq 4$. As before, put $l_i = s_i - q_i$.

Since $y_1 = -\frac{1}{2}$ and $y_3 = -y_1 - y_2 = \frac{1}{2} - y_2$, we see that v_3 lies on the intersection of E_3 and $\frac{1}{2} - E_2$. When $E_2 = \langle 1/3, 1/4\rangle$, one can check that the interior of the line segment $1/2 - E_2$ lies in the interior of the triangle $\langle 0, 1/4, 1/5\rangle$. Therefore there is no solution in this case.

When $E_2 = \langle 1/3, 2/5\rangle$, we have the following calculation.

$$
\begin{aligned}
y_2 &= \frac{1}{3} + \frac{1}{2}\left(x - \frac{2}{3}\right) = \frac{x}{2}, \\
y_3 &= \frac{1}{l_3}x \qquad \text{and} \\
y_3 &= -y_1 - y_2 = \frac{1}{2} - y_2 = \frac{1}{2} - \frac{x}{2}, \quad \text{hence} \\
x &= \frac{l_3}{l_3 + 2}, \\
\bar{u} &= \frac{1}{1 - x} = \frac{l_3 + 2}{2}.
\end{aligned}
$$

Since $v_2 \in \langle 1/3, 2/5\rangle$ and is not a vertex, we have $3 < \bar{u} < 5$, so the above gives $4 < l_3 < 8$. Since $K(-1/2, 2/5, 1/s_3)$ is a knot, s_3 must be odd. Therefore the only possibility is that $s_3 = 7$, in which case $l_3 = 6$, $\bar{u} = 4$, and the knot is $K(-1/2, 2/5, 1/7)$.

When $E_2 = \langle 1/3, 2/7\rangle$, $1/2 - E_2$ lies on the edges $\langle 0, 1/5\rangle$ and $\langle 1/5, 2/9\rangle$. Since $q_3 \leq 3$, we must have $E_3 = \langle 0, 1/5\rangle$. Note that $l_2 = l_3$ and the slopes of these edges add up to zero. Therefore by Lemma 2.5 all solution surfaces on these edges have the same boundary slope. We have $K = K(-1/2, 1/5, 2/7)$, and $3 < \bar{u} < 5$.

We have $s_2 \neq 6$ because s_2 is coprime with q_2. Thus it remains to consider the case that $E_2 = \langle 1/3, r_2/s_2 \rangle$ for some $s_2 \geq 8$. It is clear from figure 1 that $1/2 - E_2$ does not intersect the interior of $\langle 0, 1/s \rangle$ for $s \leq 4$. Therefore, we may assume that $E_3 = \langle 0, 1/s_3 \rangle$ for some $s_3 \geq 5$. We now have $l_2 = s_2 - q_2 \geq 5$ and $l_3 = s_3 - q_3 \geq 4$. Since $\bar{u} > q_2 = 3$, by Lemma 5.1 we have

$$\begin{aligned}
\bar{e} &\leq \left(1 - \frac{q_2}{l_2} - \frac{q_3}{l_3}\right) + \left(-\frac{1}{2} + \frac{1}{l_2} + \frac{1}{l_3}\right)\bar{u} \\
&= \left(1 - \frac{3}{l_2} - \frac{1}{l_3}\right) + 3\left(-\frac{1}{2} + \frac{1}{l_2} + \frac{1}{l_3}\right) + \left(-\frac{1}{2} + \frac{1}{l_2} + \frac{1}{l_3}\right)(\bar{u} - 3) \\
&= \left(-\frac{1}{2} + \frac{2}{l_3}\right) + \left(-\frac{1}{2} + \frac{1}{l_2} + \frac{1}{l_3}\right)(\bar{u} - 3) \\
&\leq \left(-\frac{1}{2} + \frac{2}{4}\right) + \left(-\frac{1}{2} + \frac{1}{5} + \frac{1}{4}\right)(\bar{u} - 3) < 0.
\end{aligned}$$

This contradicts Theorem 2.1. Therefore, there is no solution in this case. □

Lemma 5.4. *Suppose $v_1 \in L(-\frac{1}{2})$, $\alpha_i > 0$ for $i = 2, 3$, and $q_3 \leq q_2 = 2$. Then $K = K(-\frac{1}{2}, \frac{1}{3}, \frac{1}{7})$, and $\bar{u} = \frac{5}{2}$.*

Proof. The minimum value of y_2 on edges of type $\langle 1/2, r_2/s_2 \rangle$ is achieved at the vertex $\langle 1/3 \rangle$ on $\langle \frac{1}{2}, \frac{1}{3} \rangle$. If $q_3 = 2$ then $y_2 + y_3 \geq 2/3 > y_1$, so there is no solution to $\sum y_i = 0$. Therefore we must have $q_3 = 1$. By the remark at the beginning of the section we have $y_2 < 1/2$, so $E_2 = \langle 1/2, r_2/s_2 \rangle$ for some $r_2/s_2 < 1/2$, and $E_3 = \langle 0, 1/s_3 \rangle$.

We have the following calculation.

$$\begin{aligned}
y_2 &= \frac{1}{2} - \frac{1}{l_2}\left(x - \frac{1}{2}\right), \\
y_3 &= \frac{1}{l_3}x \quad \text{and} \\
y_3 &= \frac{1}{2} - y_2 = \frac{1}{l_2}\left(x - \frac{1}{2}\right), \quad \text{hence} \\
x &= \frac{l_3}{2(l_3 - l_2)}
\end{aligned}$$

$$\begin{aligned}
\bar{u} &= \frac{1}{1-x} = \frac{2(l_3 - l_2)}{l_3 - 2l_2} = 2 + \frac{2l_2}{l_3 - 2l_2}, \\
\alpha_2 &= \frac{\bar{u} - q_2}{l_2} = \frac{\bar{u} - 2}{l_2} = \frac{2}{l_3 - 2l_2}, \\
\alpha_3 &= \frac{\bar{u} - q_3}{l_3} = \frac{\bar{u} - 1}{l_3} = \frac{1}{l_3 - 2l_2}, \\
0 &\le \bar{e} \le e(\bar{u}) = 1 - \frac{1}{2}\bar{u} + \alpha_2 + \alpha_3 = \frac{3 - l_2}{l_3 - 2l_2}.
\end{aligned}$$

Since $v_2 \in \langle 1/2, r_2/s_2 \rangle$ and is not a vertex, we have $\bar{u} < s_2 = l_2 + 2$. By the above formula of $\bar{u}$, this gives $2 + 2l_2/(l_3 - 2l_2) < 2 + l_2$, hence

$$(*) \qquad l_3 - 2l_2 > 2.$$

Note that the slope of E_2 is negative and the slope of E_3 is positive, so by Lemma 2.5 the boundary slope δ of the surface satisfies

$$\delta \equiv 2(e_- - e_+) \equiv 2(\beta_3 - \beta_2) \equiv 2(\alpha_2 - \alpha_3) = \frac{2}{l_3 - 2l_2} \not\equiv 0 \mod 1.$$

By Theorem 2.1 this means that $\frac{1}{2} \le \bar{e} < 1$, hence

$$e(\bar{u}) = \frac{3 - l_2}{l_3 - 2l_2} = \frac{1}{2} \times \frac{3 - l_2}{(l_3/2) - l_2} \ge \bar{e} \ge \frac{1}{2}.$$

By (*) we have $(l_3/2) - l_2 > 0$, so the above inequality gives

$$l_3 \le 6.$$

Together with (*), this implies that $(l_2, l_3) = (1, 5)$ or $(1, 6)$. By definition $e(\bar{u}) - \bar{e}$ equals the number of full edges in $\cup e(\gamma_i)$. In both cases $e(\bar{u}) < 1$, so there are no full edges. In the first case we have $K = K(-1/2, 1/3, 1/6)$, which is not a knot. In the second case we have $\bar{u} = 2.5$, $s_2 = 3$, and $s_3 = 7$, hence the knot is $K = K(-1/2, 1/3, 1/7)$. This solution gives the well-known 37/2 toroidal surgery on $K(-1/2, 1/3, 1/7)$. □

Lemma 5.5. *Suppose $v_1 \in L(-\frac{1}{2})$, $\alpha_i > 0$, and $E_i = \langle 0, 1/s_i \rangle$ for $i = 2, 3$. Then*

$$\begin{aligned}
&K = K(-1/(2 + 1/n), 1/3, 1/3), \quad n \text{ odd}, \ n \ne -1, \bar{u} = 2; \text{ or} \\
&K = K(-1/2, 1/3, 1/(3 + 1/n)), \quad n \text{ even}, n \ne 0, \bar{u} = 2.
\end{aligned}$$

Proof. Since v_i is in the interior of the edge E_i above, we have $y_i < 1/s_i$, so $\sum y_i = 0$ has no solution if $s_i \geq 4$ for $i = 2, 3$. Also since $v_1 \in L(-\frac{1}{2})$, we have $\bar{u} \geq 2$, hence $s_i > 2$. Therefore we may assume that $s_2 = 3$ and $s_3 \geq 3$. We have

$$\begin{aligned}
& y_2 + y_3 = \frac{x}{l_2} + \frac{x}{l_3} = \left(\frac{1}{l_2} + \frac{1}{l_3}\right) x = \frac{1}{2} = -y_1, \\
& x = \frac{l_2 l_3}{2(l_2 + l_3)}, \\
& \bar{u} = \frac{1}{1 - x} = \frac{2(l_2 + l_3)}{2(l_2 + l_3) - l_2 l_3}, \\
& \alpha_2 = \frac{\bar{u} - q_2}{l_2} = \frac{l_3}{2(l_2 + l_3) - l_2 l_3}, \\
& \alpha_3 = \frac{\bar{u} - q_3}{l_3} = \frac{l_2}{2(l_2 + l_3) - l_2 l_3}, \\
& \alpha_2 + \alpha_3 = \frac{1}{2}\bar{u}, \\
& \bar{e} \equiv e(\bar{u}) = 1 - \frac{1}{2}\bar{u} + \alpha_2 + \alpha_3 = 1.
\end{aligned}$$

Since both E_2 and E_3 have positive slope, $e_- \equiv \beta_2 + \beta_3 \equiv -(\alpha_2 + \alpha_3)$ mod 1; by Lemmas 2.11 the boundary slope of the surface satisfies $\delta \equiv 2(e_- - e_+) \equiv -2(\alpha_2 + \alpha_3) = -\bar{u}$. Since $\bar{e} \equiv 0$ mod 1, by Theorem 2.1 $\delta = -\bar{u}$ must be an integer slope, and $\bar{e} = 0$. Since $v_1 \in L(-\frac{1}{2})$ and v_2 is in the interior of $\langle 0, 1/3 \rangle$, we have $2 \leq \bar{u} < 3$, hence $\bar{u} = 2$. From the formula of $\bar{u}$ above we have $l_2 = l_3 = 2$. Also

$$0 = \bar{e} = e(\bar{u}) - \sum(|\gamma_i| - \beta_i) = 1 - \sum(|\gamma_i| - \beta_i).$$

Hence there is an extra edge, which may end at either $\langle \frac{1}{2} \rangle$ or $\langle \frac{1}{3} \rangle$. Therefore K is one of the following knots.

$$\begin{aligned}
& K\left(-\frac{1}{2 + 1/n}, \frac{1}{3}, \frac{1}{3}\right), \quad n \text{ odd, and } n \neq -1; \\
& K\left(-\frac{1}{2}, \frac{1}{3}, \frac{1}{3 + 1/n}\right), \quad n \text{ even, } n \neq 0.
\end{aligned}$$

The extra conditions on n is to guarantee that γ_i are allowable, and K is a knot. □

Proposition 5.1. *If $v_1 \in L(-\frac{1}{2})$ and $\alpha_i > 0$ for $i = 2, 3$, then K is one of the knots in Lemma 5.3, 5.4 or 5.5.*

Proof. Because of symmetry we may assume $q_2 \geq q_3 \geq 1$. By Lemma 5.2 we have $q_2 \leq 3$, so $q_2 = 3$, 2 or 1, which are covered by Lemmas 5.3, 5.4 and 5.5, respectively. □

6. No horizontal edges

In this section, we study the case that no v_i is on a horizontal edge. As before, let $E_i = \langle p_i/q_i, r_i/s_i \rangle$. By Lemma 2.4(4) we may assume that v_1 is in the interior of a non-horizontal edge in the graph G shown in figure 4, hence $1 < \bar{u} < 3$ and $\bar{u} \neq 2$. Since $q_i \leq \bar{u}$, we have $q_i \leq 2$ for all i.

Similar to the previous sections, we define

$$\begin{aligned} \beta_i(u) &= \frac{s_i - u}{s_i - q_i}, \\ \alpha_i(u) &= \frac{u - q_i}{s_i - q_i}, \\ e(u) &= 4 - u - \sum \beta_i(u) = 1 - u + \sum \alpha_i(u). \end{aligned}$$

Since no v_i is in the interior of a horizontal edge, by Definition 2.2 we have

$$\begin{aligned} \bar{e} &= \sum e(\gamma_i) = (4 - \bar{u}) - \sum |\gamma_i| \\ &\leq 4 - \bar{u} - \sum \beta_i = 1 - \bar{u} + \sum \alpha_i = e(\bar{u}). \end{aligned} \tag{6.1}$$

Note also that $e(\bar{u}) - \bar{e}$ is a nonnegative integer, which equals the number of full edges in $\cup \gamma_i$.

Let δ be the boundary slope of $F(\gamma_1, \gamma_2, \gamma_3)$. Then

$$\delta \equiv 2(e_- - e_+) \equiv -2 \sum \operatorname{sign}(r_i/s_i - p_i/q_i)\alpha_i \mod 1.$$

Lemma 6.1. *Suppose $v_1 \in G - L$. If $q_i = 2$, then $s_i = 3$.*

Proof. Assume to the contrary that $q_2 = 2$ and $s_2 > 3$. We have $E_2 = \langle \pm 1/2, r_2/s_2 \rangle$. From figure 1 we see that any point (x, y_2) in the interior of E_2 satisfies $x/2 < |y_2| < x$.

First assume that $s_3 = 3$. Denote by $\pm Q = \langle 0, \pm\frac{1}{3} \rangle \cup \langle \pm\frac{1}{2}, \pm\frac{1}{3} \rangle \cup \langle \pm\frac{1}{2}, \pm\frac{2}{3} \rangle$. Note that any point (x, y) on $\pm Q$ satisfies $x/2 \leq |y| \leq x$.

Since $\bar{u} > q_2 = 2$, we have $s_1 = 3$, so $v_1 \in G - L$ implies that $v_1 \in -Q$. We assumed $s_3 = 3$ and by Lemma 2.4 we have $|y_3| \leq \frac{2}{3}$, hence $v_3 \in \pm Q$. Thus $x/2 \leq |y_i| \leq x$ for $i = 1, 3$. This implies that if $y_3 < 0$ then $-y_1 - y_3 \geq (x/2) + (x/2) = x$, and if $y_3 < 0$ then $|-y_1 - y_3| \leq x - x/2 = x/2$, either case contradicting the fact that $y_2 = -y_1 - y_3$ satisfies $x/2 < |y_2| < x$.

We may now assume that $s_3 > 3$. Consider the function $e(u) = 1 - u + \sum \alpha_i(u)$, where $\alpha_i(u) = (u - q_i)/(s_i - q_i)$. By (6.1) we have $e(\bar{u}) \geq \bar{e} \geq 0$.

When $u = 2$ we have $\alpha_1(2) \leq \frac{1}{2}$, $\alpha_2(2) = 0$ because $q_2 = 2$, and $\alpha_3(2) \leq \frac{1}{3}$ because $s_3 \geq 4$, hence $e(2) = (1-2) + \sum \alpha_i(u) < 0$.

Now calculate $e(3)$. Since s_2 is coprime with $q_2 = 2$ and $s_2 > 3$, we have $s_2 \geq 5$. Hence $\alpha_2(3) \leq \frac{1}{3}$. Also, $\alpha_3(3) \leq \frac{2}{3}$ because $s_3 > 3$, and $\alpha_1(3) \leq 1$. Hence $e(3) = (1-3) + \sum \alpha_i(3) \leq 0$. By the linearity of $e(u)$ we have $\bar{e} \leq e(\bar{u}) < 0$ because $2 = q_2 < \bar{u} < s_1 = 3$. This contradicts Theorem 2.1. □

Lemma 6.2. *Suppose* $v_1 \in \text{Int}\langle -\frac{1}{2}, -\frac{2}{3}\rangle$, *and* $|y_1| > y_2 \geq y_3 > 0$. *Then* $\bar{u} = 2.5$, *and* $K = K(-2/3, 1/3, 1/4)$.

Proof. The equation for E_1 is $y_1 = -x$. By Lemma 2.4(2) we have $|y_1| + |y_2| \leq 1$, hence $y_2 \leq 1 - x$. Since $y_2 + y_3 = -y_1$ and $y_2 \geq y_3$, we have $y_2 \geq \frac{1}{2}(-y_1) = \frac{1}{2}x$. From figure 1 we see that $E_2 = \langle \frac{1}{2}, \frac{1}{3}\rangle$ or $\langle 0, \frac{1}{3}\rangle$.

If $E_2 = \langle 0, \frac{1}{3}\rangle$ then $y_3 = -y_1 - y_2 = x - x/2 = x/2 = y_2$, so $E_3 = E_2$. In this case $E_1 = -E_2 - E_3$, so there are infinitely many solutions, all giving the same slope. We have $\alpha_1 = \bar{u} - 2$, $\alpha_2 = \alpha_3 = (\bar{u} - 1)/2$, hence

$$\bar{e} \leq e(\bar{u}) = 1 - \bar{u} + \sum \alpha_i = \bar{u} - 2 < 1.$$

Therefore there are no extra edges, hence $K = K(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$. Since this is a link of two components. there is no solution in this case.

Now assume $E_2 = \langle \frac{1}{2}, \frac{1}{3}\rangle$. Then $y_3 = -y_1 - y_2 = x - (1 - x) = 2x - 1 < \frac{1}{3}$ because $x < \frac{2}{3}$. Hence from figure 1 we see that $E_3 = \langle 0, 1/s_3\rangle$ for some $s_3 \geq 3$. Define $l_i = s_i - q_i$. We have the following calculation.

$$\begin{aligned}
&-y_1 - y_2 = 2x - 1 = \frac{x}{l_3} = y_3, \\
&x = \frac{1}{2 - 1/l_3} = \frac{l_3}{2l_3 - 1}, \\
&\bar{u} = \frac{1}{1 - x} = 2 + \frac{1}{l_3 - 1}, \\
&\alpha_1 = \frac{\bar{u} - 2}{l_1} = \frac{1}{l_3 - 1}, \\
&\alpha_2 = \frac{\bar{u} - 2}{l_2} = \frac{1}{l_3 - 1}, \\
&\alpha_3 = \frac{\bar{u} - 1}{l_3} = \frac{1}{l_3 - 1}, \\
&0 \leq \bar{e} \leq e(\bar{u}) = 1 - \bar{u} + \sum \alpha_i = -1 + \frac{2}{l_3 - 1}.
\end{aligned}$$

This gives $l_3 = 2$ or 3. When $l_3 = 2$, $\bar{u} = 3$, which is a contradiction because $\bar{u} < 3$. When $l_3 = 3$, we have $\bar{u} = 2.5$ and $\bar{e} = 0$, so there is no extra edge, hence $K = K(r_1/s_2, r_2/s_2, r_3/s_3) = K(-2/3, 1/3, 1/4)$. □

Lemma 6.3. *Suppose* $v_1 \in \text{Int}\langle -1/2, -1/3\rangle$, *and* $|y_1| > y_2 \geq y_3 > 0$. *Then* $K = K(-1/3, 1/3, 1/7)$ *and* $\bar{u} = 2.5$.

Proof. Since $-y_1 > y_2 \geq y_3 > 0$, E_2 and E_3 must be below the edge $E_1 = \langle \frac{1}{2}, \frac{1}{3}\rangle$, hence we must have $E_i = \langle 0, 1/s_i\rangle$ for $i = 2, 3$. We have the following calculation.

$$\begin{aligned}
&\sum y_i = (x-1) + x/l_2 + x/l_3 = 0,\\
&x = \frac{1}{(1/l_2) + (1/l_3) + 1},\\
&\bar{u} = \frac{1}{1-x} = 1 + \frac{l_2 l_3}{l_2 + l_3},\\
&\alpha_1 = \frac{\bar{u} - 2}{l_1} = \bar{u} - 2 = \frac{l_2 l_3}{l_2 + l_3} - 1,\\
&\alpha_2 = \frac{\bar{u} - 1}{l_2} = \frac{l_3}{l_2 + l_3},\\
&\alpha_3 = \frac{\bar{u} - 1}{l_3} = \frac{l_2}{l_2 + l_3},\\
&\delta \equiv 2(-\alpha_1 - \alpha_2 - \alpha_3) \equiv \frac{-2 l_2 l_3}{l_2 + l_3} \equiv -2\bar{u},\\
&0 \leq \bar{e} \leq 1 - \bar{u} + \sum \alpha_i = 0.
\end{aligned}$$

The last inequality gives $\bar{e} = 0$, so by Theorem 2.1 the boundary slope of the surface must be an integer, hence by Lemma 2.5 we have $\delta \equiv 2\bar{u} \equiv 0 \bmod 1$. Since $2 < \bar{u} < 3$, we have $\bar{u} = 2.5$. The only solutions for $\bar{u} = 1 + l_2 l_3/(l_2 + l_3) = 2.5$ are $(l_2, l_3) = (2, 6)$ or $(3, 3)$. Since $e = 0$, there is no extra edge, so in the second case we would have $K = K(-1/3, 1/4, 1/4)$, which is not a knot. Therefore the only solution in this case is $K = K(-1/3, 1/3, 1/7)$ and $\bar{u} = 5/2$. □

Lemma 6.4. *Suppose* $v_1 \in G$, *and* $|y_1| > y_2 \geq y_3 > 0$. *Then* $v_1 \notin \text{Int}\langle 0, -1/2\rangle$.

Proof. If $v_1 \in \text{Int}\langle 0, -1/2\rangle$, then $|y_1| > y_i > 0$ implies that $E_i = \langle 0, 1/s_i\rangle$ for some $s_i \geq 3$. We have $y_1 = -x$, and $y_i = x/l_i$ for $i = 2, 3$, hence from the equations $y_1 + y_2 + y_3 = 0$ and $x > 0$ we have

$$-1 + \frac{1}{l_2} + \frac{1}{l_3} = 0.$$

Since $s_i \geq 3$, this gives $l_2 = l_3 = 2$. Note that $\alpha_1 = \bar{u} - 1$, and $\alpha_2 = \alpha_3 = (\bar{u} - 1)/2 < 1/2$, hence

$$\bar{e} = 4 - \bar{u} - \sum |\gamma_i| \equiv 1 - \bar{u} + \sum \alpha_i = \alpha_2 + \alpha_3 = \bar{u} - 1 \not\equiv 0 \mod 1.$$

By Theorem 2.1 this implies that the boundary slope δ of the surface is not an integer slope. On the other hand, since E_1 has positive slope and E_2, E_3 have negative slope, by Lemma 2.5 we have

$$\delta \equiv 2(e_- - e_+) = 2(\beta_2 + \beta_3 - \beta_1) \equiv 2(-\alpha_2 - \alpha_3 + \alpha_1) = 0 \mod 1$$

so δ is an integer slope, which is a contradiction. □

Lemma 6.5. *Suppose* $v_1 \in \text{Int}\langle 0, -1/3 \rangle$, *and* $|y_1| > y_2 \geq y_3 > 0$. *Then* $K = K(-1/3, 1/4, 1/7)$ *or* $K(-1/3, 1/5, 1/5)$, *and the boundary slopes are the same as the pretzel slopes corresponding to the candidate systems in Proposition 2.2.*

Proof. Similar to the proof of Lemma 6.4, we have

$$\sum y_i = -\frac{1}{2} + \frac{1}{l_2} + \frac{1}{l_3} = 0,$$

which gives $(l_2, l_3) = (3, 6)$ or $(4, 4)$. We have

$$0 \leq \bar{e} \leq e(\bar{u}) = 1 - \bar{u} + \sum \alpha_i(\bar{u}) = 1 - \bar{u} + \frac{\bar{u} - 1}{2} + \frac{\bar{u} - 1}{l_2} + \frac{\bar{u} - 1}{l_3} = 0.$$

Therefore there are no extra edges. The knots are $K = K(-1/3, 1/4, 1/7)$ and $K(-1/3, 1/5, 1/5)$. The boundary slopes are the same for all $\bar{u}$, which is also the boundary slope of the candidate system at $\bar{u} = 1$, as given in Proposition 2.2. □

Lemma 6.6. *Suppose* $v_1 \in G$, *and* $|y_1| > y_2 \geq y_3 > 0$. *Then* $v_1 \notin \text{Int}\langle -1, -\frac{1}{2} \rangle$.

Proof. If $v_1 \in \text{Int}\langle -1, -\frac{1}{2} \rangle$ then $1 < \bar{u} < 2$, and we have $E_i = \langle 0, 1/s_i \rangle$ for $i = 2, 3$. If both $s_i > 2$ then it is easy to show that $|y_1| > |y_2| + |y_3|$ for

$0 < x < \frac{1}{2}$, so there is no solution to $\sum y_i = 0$ when $1 < \bar{u} < 2$. Hence we must have $s_2 = 2$, so $y_1 = -1 + x$, $y_2 = x$, and $y_3 = x/l_3$. We have the following calculations.

$$\begin{aligned}
&\sum y_i = (-1+x) + x + \frac{x}{l_3} = 0, \\
&x = \frac{l_3}{2l_3+1}, \\
&\bar{u} = \frac{1}{1-x} = 2 - \frac{1}{l_3+1} = 2 - \frac{1}{s_3}, \\
&\alpha_1 = \frac{\bar{u} - q_1}{l_1} = \frac{(2-1/s_3)-1}{1} = 1 - \frac{1}{s_3}, \\
&\alpha_2 = \frac{\bar{u} - q_2}{l_2} = \frac{(2-1/s_3)-1}{1} = 1 - \frac{1}{s_3}, \\
&\alpha_3 = \frac{\bar{u} - q_3}{l_3} = \frac{(2-1/s_3)-1}{s_3-1} = \frac{1}{s_3}, \\
&\delta \equiv 2(-\alpha_1 - \alpha_2 - \alpha_3) = \frac{2}{s_3}, \\
&\bar{e} \equiv e(\bar{u}) = 1 - \bar{u} + \sum \alpha_i = 1 \equiv 0 \mod 1.
\end{aligned}$$

By Theorem 2.1, $\bar{e} \equiv 0 \mod 1$ implies that δ is an integer slope, hence from $\delta \equiv 2/s_3 \mod 1$ and $s_3 \geq 2$ we see that $s_3 = 2$. Since $\bar{e} = 1$ there is one extra edge, but since $(r_1/s_1, r_2/s_2, r_3/s_3) = (-1/2, 1/2, 1/2)$ or $(-1/2, 1/2, 1/4)$, adding one extra edge will make a link of type $K(-1/2, 1/2, 1/(2+1/n))$ or $K(-1/2, 1/2, 1/(4+1/n))$, which has at least two components. Therefore there is no solution in this case. □

Proposition 6.1. *Suppose some $v_i \in G - L$, and $\bar{u} > 1$. Then K and $\bar{u}$ are equivalent to one of the pairs in Lemmas 6.2, 6.3 and 6.5.*

Proof. Since $\bar{u} > 1$, we must have $y_j \neq 0$. Up to permutation and changing of signs of the parameters of K we may assume that $-y_1 \geq |y_i|$ for $i = 2, 3$. Before this modification we have some $v_i \in G - L$. We need to show that $v_1 \in G - L$ after the modification.

The assumption of $v_i \in G - L$ implies that $\bar{u} < 3$, hence $p_1/q_1 \in \{0, -\frac{1}{2}, -1\}$. By Lemma 2.4(2) we have $|y_1| \leq \frac{2}{3}$, so from figure 1 we see that if $p_1/q_1 = -1$ then $E_1 = \langle -1, -\frac{1}{2} \rangle$, hence $v_1 \in G - L$. If $p_1/q_1 = -\frac{1}{2}$ then by Lemma 6.1 we have $r_1/s_1 = -\frac{1}{3}$ or $-\frac{2}{3}$, so again $v_1 \in G - L$. If $p_1/q_1 = 0$ and $v_1 \notin G - L$ then from figure 1 and the assumption of $|y_i| \leq |y_1|$ we see that each v_j is in $\langle 0, \pm\frac{1}{s_j} \rangle$ for some $s_j \geq 4$, which contradicts the assumption that some v_i is in $G - L$ before the modification of parameters of K.

We now have $v_1 \in G - L$, and $-y_1 \geq |y_i|$ for $i = 2, 3$. Since $\bar{u} > 1$, no v_i is on the horizontal line $L(0)$ as otherwise the corresponding parameter of K would be 0, contradicting the assumption that the length of K is 3. Hence from $\sum y_i = 0$ we have that $|y_1| > y_i > 0$. Permuting the second and third parameters of K if necessary we may assume $y_2 \geq y_3$. Therefore we have $|y_1| > y_2 \geq y_3 > 0$.

There are only 5 edges in $G - L$, so E_1 must be one of them, which have been discussed in Lemma 6.2–6.6 respectively. The above discussion shows that y_i satisfy the conditions of the lemmas, hence the result follows from these lemmas. □

7. The classification

Lemma 7.1. *Let K be a Montesinos knot of length 3. Then $E(K)$ contains a candidate surface F of genus one with boundary slope δ if and only if (K, δ) is equivalent to one of the pairs listed in Theorem 1.1.*

Proof. When $\bar{u} \leq 1$, the knots are given in Proposition 2.2. When $\bar{u} > 1$, by Lemma 2.4(4) we may assume that $v_1 \in G$. Propositions 3.7, 4.4 and 5.6 covered the case of $v_1 \in L$, and Proposition 6.1 covered the case of $v_1 \in G - L$. The list in Theorem 1.1 contains all the knots given in these propositions. Here are more details.

Parts (1) and (2) of Theorem 1.1 include the knots in Proposition 2.2, as well as $K(-1/3, 1/5, 1/5)$ and $K(-1/3, 1/4, 1/7)$ in Proposition 4.1 and Lemma 6.5. Note that the boundary slopes for the last two are the same as those in the list, but the $\bar{u}$ values are different, which is allowed by the remark before the statement of Theorem 1.1.

Part (3) is given by Lemma 3.2, and part (4) by Lemma 3.5. Parts (5), (6) and (9) are in Lemma 3.6. Parts (7) and (8) are from Lemma 5.5, Parts (10) from Lemma 5.3, Parts (11) from Lemma 5.4, Parts (12) from Lemma 6.2, and (13) from Lemma 6.3. Note that the knot $K(-1/2, 1/5, 2/7)$ in Lemmas 3.6 and 5.3 is included in Parts (5) (with $n = 2$) because they all have the same boundary slope. The boundary slopes can be calculated using the algorithm of Hatcher and Oertel in Lemma 2.5. □

Proof of Theorem 1.1. Most of the work has been done throughout the paper and summarized in Lemma 7.1, as it was shown that the list in Theorem 1.1 contains all possible (K, δ). It remains to show that the candidate surfaces in Lemma 7.1 are incompressible in $E(K)$ when K is hyperbolic. Since the candidate systems are already determined in the proofs of the lemmas and the propositions above, it is straightforward to follow the procedure of [4] to verify the incompressibility of the candidate surface F. For each individual knot the toroidal slopes can also be verified using the computer program of Nathan Dunfield [16].

Here are more details. If $\bar{u} \geq q_i$ for some i then the path γ_i is a constant path, so it follows from [4, Proposition 2.1] that F is incompressible. This takes care of all the cases except (1), (2), (7), (12) and (13) in Theorem 1.1. Recall from [4, Page 463] that the r-value of a path is determined by this rule: Extend the last segment of the path to meet the right-hand border of the the diagram in a point whose slope has denominator r. (For length 3 knot we do not have vertical line segments in γ_i and hence $r \neq 0$.) In this case by [4, Corollary 2.4] F is incompressible unless the r-value cycle is $(1, 1, r_3)$ or $(1, 2, r_3)$. In cases (1) and (2) the r-value cycle is $(|q_1| - 1, |q_2| - 1, |q_3| - 1)$, so F is incompressible unless $|q_1| = 2$ and $|q_2| = 3$. ($|q_2| \neq 2$ since K is a knot.) By [4, Proposition 2.7] F is incompressible unless the slopes of γ_1 is of opposite sign to those of γ_2 and γ_3 and $r_3 = |q_3| - 1 = 2$ or 4. It follows that F is incompressible unless K is the non-hyperbolic knot $K(-1/2,\ 1/3,\ 1/3)$ or $K(-1/2,\ 1/3,\ 1/5)$, which has been excluded since we assumed that K is hyperbolic. In case (7) $r_i \neq 1$ unless $K = K(-1/3,\ 1/3,\ 1/3)$, in which case the r-cycle is $(1, 2, 2)$. Since the slopes of the three edges are all positive (the first one is on $\langle -1/2, -1/3\rangle$ and the other two are on $(\langle 0/1, , 1/3\rangle)$, by [4, Proposition 2.7] F is incompressible. In case (12) the r-cycle is $(1, 1, 3)$ and the first two paths are edges of the same slope -1, so F is incompressible by [4, Proposition 2.6]. In case (13) the r-cycle is $(1, 2, 6)$ and the edges all have positive slopes, hence again the incompressibility of F follows from [4, Proposition 2.7]. □

If F is a surface in a three-manifold M, denote by $M|F$ the manifold obtained by cutting M along F. Similarly, if C is a set of curves on a surface F then $F|C$ denotes the surface obtained by cutting F along C. All surfaces in three-manifolds below are assumed compact, connected, orientable, and properly embedded. Recall that a surface in M is *essential* if it is incompressible, ∂-incompressible, and is not boundary parallel. Denote by $|\partial F|$ the number of boundary components of F.

Lemma 7.2. *Let M be a compact orientable 3-manifold with $\partial M = T$ a torus. Let F be a genus one separating essential surface in M such that $|\partial F| \leq 4$. Let M_1, M_2 be the components of $M|F$, and let A be a component of $\partial M_1 - F$. If $F \cup A$ is incompressible in M_1, then M contains a closed essential surface.*

Proof. This is due to Gordon, and is true for any number of components on ∂F. If $|\partial F| = 2$ then by assumption $F \cup A$ is incompressible and the result follows, so we assume $|\partial F| = 4$. Let A' be the annulus on T which contains A and such that $\partial A' = \partial F - \partial A$. Then we can push the part of $F \cup A$ near A into the interior of M to obtain a surface F' with $\partial F' = A'$, and then push $F' \cup A'$ into the interior of M to obtain a closed surface F''. Number the annuli $T|\partial F$ successively as $A_1, \ldots, A_4$, with $A = A_1$. One can show that $M|F''$ consists of two components M_1'', M_2'', such that M_1'' is obtained by gluing $T \times I$ to M_1 along the annulus A_3 and a non-trivial annulus on $T \times 0$, and M_2'' is obtained by gluing a $A' \times I$ to M_2, where A' is an annulus, $A' \times 0$ identified to A_2, and $A' \times 1$ to A_4. An innermost circle argument shows that if $F \cup A$ is incompressible then F'' is incompressible in both M_i'', hence is incompressible in M. □

Lemma 7.3. *Let M be a compact orientable irreducible three-manifold with $\partial M = T$ a torus. Let F be a genus one separating incompressible surface in M with boundary slope δ, and let $\hat{F}$ be the corresponding torus in the Dehn filling manifold $M(\delta)$. If (i) M contains no closed incompressible surface, and (ii) F has at most four boundary components, then $\hat{F}$ is incompressible in $M(\delta)$.*

Proof. Since F is separating, $|\partial F| = 2$ or 4. We assume the latter, as the proof for the former case is similar and simpler. Let M_1, M_2 be the components of $M|F$, and let $A_1, \ldots, A_4$ be the annuli $T|\partial F$, labeled so that $\partial M_1 = F \cup A_1 \cup A_3$.

Since M contains no closed essential surface, each M_i is a handlebody of genus 3, and by Lemma 7.2 the surface $F_1 = F \cup A_1$ is compressible in M_1. Let D be a compressing disk of F_1. If D is separating then since M_1 is a handlebody and $\partial M_1 - F_1 = A_3$ is connected, we can find a nonseparating compressing disk in a component of $M_1|D$ disjoint from A_3. Therefore, we may assume without loss of generality that D is a nonseparating compressing disk. It follows that after attaching a two-handle to M_1 along A_3 the resulting manifold M' has compressible boundary, because D remains a compressing disk of $\partial M'$ in M'.

We have $A_1 \subset \partial M'$. We want to show that $F_2 = \partial M' - A_1$ is incompressible in M'. Consider the surface $F_3 = F \cup A_3 = \partial M_1 - A_1$. For the same reason as above, we know that F_3 is compressible in M_1. By assumption $F_3 - A_3 = F$ is incompressible. Therefore, by the Handle Addition Lemma (see [17] or [18]), we know that after attaching a two-handle to M_1 along A_3, the resulting surface $F_2 = \partial M' - A_1$ is incompressible in M'.

We have shown that $\partial M'$ is compressible, and $\partial M' - A_1 = F_2$ is incompressible. Applying the Handle Addition Lemma again, we see that after attaching a two-handle to A_1 the boundary of the resulting manifold M'' is incompressible. Note that M'' is a component of $M(\delta)|\hat{F}$, and $\partial M'' = \hat{F}$. For the same reason, $\hat{F}$ is incompressible in the other component of $M(\delta)|\hat{F}$. It follows that $\hat{F}$ is incompressible in $M(\delta)$. □

Remark. The above result is similar to a special case of Proposition 2.2.1 of [19]. However, that proposition requires that the number of boundary components of the surface is minimal among all incompressible surfaces with the same boundary slope. In our case there is no guarantee that there is no higher genus surface with fewer boundary components of the same slope. Lemma 7.3 is probably false if there is no constraint about the number of components in ∂F.

Proof of Theorem 1.2. If K_δ is toroidal then clearly there is a toroidal incompressible surface in the exterior of K, so by Theorem 1.1 the pair (K, δ) must be one of those in the list.

To prove the other direction, we would like to show that if (K, δ) is in the list of Theorem 1.1 then the corresponding toroidal incompressible surface $F = F(\gamma_1, \gamma_2, \gamma_3)$ gives rise to an incompressible torus $\hat{F}$ in K_δ. By Oertel [20], the exterior of $K(t_1, t_2, t_3)$ contains no closed essential surface. Therefore by Lemma 7.3 it suffices to show that F has at most four boundary components.

Let m_i be the number defined before the statement of Lemma 2.2, and let $n = \text{lcm}(m_1, m_2, m_3)$. Then by Lemma 2.2(3) there is an orientable candidate surface $F = F(\gamma_1, \gamma_2, \gamma_3)$ with $|\partial F| \leq 2n$. Since F is toroidal, by choosing a component of F if necessary we may assume that F is connected, so it is of genus one. Therefore if $n \leq 2$ then by Lemma 7.3 the surface $\hat{F}$ is an incompressible torus in K_δ and we are done. By definition m_i can be easily calculated from $\bar{u}$ and $E_i = \langle p_i/q_i, r_i/s_i \rangle$, which can be found in the proof of the corresponding lemma for that (K, δ). We leave it to the reader to check that $m_i \leq 2$ for all i in all the cases listed in the theorem, except that $m_3 = 4$ in case (13). (For each individual knot, one may also use Dunfield's program [16] to calculate $n = \text{lcm}(m_1, m_2, m_3)$, which is shown as "number of sheets" in the program.) Therefore Theorem 1.2 follows from Lemma 7.3, except in Case (13) of Theorem 1.1.

For Case (13), let $F' = F'(\gamma_1, \gamma_2, \gamma_3)$ be the surface in the exterior of K constructed using the candidate system $(\gamma_1, \gamma_2, \gamma_3)$ given by Lemma 6.3, such that $|\partial F'| = n = 4$. By the proof of Lemma 2.2(3) we have $F = F'$, or its double cover if F' is nonorientable. In the first case we have $|\partial F| = 4$ and the result follows from Lemma 7.3. Hence we assume that $\hat{F}$ is a double cover of $\hat{F}'$, so $\hat{F}'$ is a Klein bottle in $M = K_\delta$. On the other hand, from Theorem 1.1 we see that in this case $\delta = 1$, hence $H_1(K_\delta, \mathbb{Z}_2) = 0$, which is a contradiction because by duality a $\mathbb{Z}_2$-homology sphere cannot contain a Klein bottle. □

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Department of Mathematics
University of Iowa
Iowa City
IA 52242
USA
E-mail address: wu@math.uiowa.edu

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