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The odd Chern character and index localization formulae

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We describe geometric representatives for the generators of the cohomology ring of a model of the classifying space for the functor K^{-1}. The class corresponding to the degree one generator is closely related to the spectral flow of a one-parameter family of self-adjoint, Fredholm operators. We use intersection theory to derive localization formulae that express the cohomological index of a higher dimensional family of such operators as the Poincare dual of an explicit 0-cycle in the parameter space. We derive, under certain conditions, an equality that relates the cohomological index to the variation of the family of kernels.

1. Introduction

The motivation behind this article was three-fold. On one hand we had in mind to generalize to infinite dimensions the results of [18] in which the author builds certain geometrical representatives for the "fundamental" cohomology classes of the finite unitary group. On the other hand, we wanted to explain the new results in the context of "classical" index theory. The emphasis is on families of self-adjoint, Fredholm operators, hence complex, odd K-theory. We learned from [7] that this problem was posed by I. Singer [27] in the mid-1980s. Lastly, we sought to derive certain index theoretic consequences coming from the explicit description of the geometrical objects used to define the cohomology classes. These consequences have as their guiding example the related interpretations of the spectral flow as an isomorphism between the set of integers and $K^{-1}(S^1)$ on one hand, and as a count with sign of the 0-eigenvalues of a generic S^1-family of self-adjoint, Fredholm operators, on the other. The case of even K-theory has been treated in various forms in several works [16, 24], while the odd case, which requires, if not a different paradigm, at least different objects and approach has not appeared in the literature, in our knowledge.

The technique of analytic currents of Hardt, used by Nicolaescu in [18] to build *homology* classes is not available in the infinite-dimensional context.

Instead, inspired by [14], we were led to use sheaf cohomology which provides an ideal framework for dealing with stratified objects. The constraints imposed by working in infinite dimensions had the beneficial effect of forcing us to develop a better understanding of certain processes that pertain to symplectic topology (see Section 4).

There are many models for the classifying space of K^{-1}, the most well known being certain subgroups of the unitary group of operators acting on a complex, separable Hilbert space H, subgroups introduced by Palais [23]. It turns out that a more convenient class of classifying spaces is one that has a symplectic flavor. Concretely, we fix a Hilbert space H and we investigate the grassmannian $\mathcal{L}\mathrm{ag}$ of hermitian lagrangian subspaces of $\hat{H} := H \oplus H$. These are closed subspaces $L \subset \hat{H}$ that satisfy $JL = L^{\perp}$, where J is the obvious extra complex structure on $\hat{H}$.

The space $\mathcal{L}\mathrm{ag}$ is a real, Banach manifold and we extend to infinite dimensions a result of Arnold (see Th. 3.1) which gives a canonical diffeomorphism between the full unitary group and the full Lagrangian Grassmannian.

The Lagrangian Grassmannian $\mathcal{L}\mathrm{ag}$ contains a distinguished open set $\mathcal{L}\mathrm{ag}^-$, consisting of Lagrangian subspaces which are Fredholm pairs with the vertical space $H^- := 0 \oplus H$. We use symplectic techniques to prove that it has the homotopy type of $U(\infty)$ (see Th. 4.2). In fact, the same proof can be used to show that the Lagrangian Grassmannian $\mathcal{L}\mathrm{ag}^-_{\mathcal{K}}$ whose projection on the vertical axis is compact (or trace class, etc.), is classifying for K^{-1}. This Lagrangian Grassmannian corresponds via Arnold's isomorphisms to the Palais group of unitary operators of type $1 +$ compact (or $1 +$ trace class, etc.) and hence we get a new proof for the results of Palais [23].

Working with Lagrangians has the added bonus that it provides an elegant way for dealing with closed, self-adjoint, unbounded[1] operators simply because the graph of such an operator is Lagrangian. This observation leads to a natural connection with a different classifying space, a certain connected component, BFred_*, of the space of *bounded*, self-adjoint, Fredholm operators, introduced by Atiyah and Singer in [3]. This opens the gates for dealing with the second problem: the connection with index theory. In fact, one can use Nicolaescu's results to prove that the graph inclusion $\mathrm{BFred}_* \hookrightarrow \mathcal{L}\mathrm{ag}^-$, which takes an operator to its graph, is a weak homotopy equivalence. The advantage that $\mathcal{L}\mathrm{ag}^-$ has over the Atiyah–Singer classifying space is that it allows one to work freely with unbounded operators. Moreover, it enlarges

[1]Unbounded means, as usual, possibly unbounded.

the class of families of operators one can take into account, because the continuity of the graphs is a weaker requirement than the notion of continuity in the Atiyah–Singer classifying space, namely the one requiring that the family of zeroth order operators naturally associated to the initial family be continuous — the so-called Riesz continuity (see Section 10).

The differential topology of $\mathcal{L}\mathrm{ag}^-$ is very rich and can be understood in great detail. By fixing a complete, decreasing flag on the vertical Lagrangian,

$$H^- \supset W_1 \supset W_2 \supset \cdots$$

we get a decreasing filtration of $\mathcal{L}\mathrm{ag}^-$ with open subsets $\mathcal{L}\mathrm{ag}^{W_i}$ consisting of clean Lagrangians.

$$\mathcal{L}\mathrm{ag}^{W_i} := \{L \in \mathcal{L}\mathrm{ag}^- \mid L \cap W_i = \{0\} \text{ and } L + W_i \text{ closed}\}.$$

We prove that each of these open sets is the total space of an infinite-dimensional vector bundle over a finite dimensional Lagrangian Grassmannian, hence by Arnold's theorem, over a finite dimensional unitary group. The natural projection of this bundle has a symplectic description: it is given by a symplectic reduction process. In fact, this result holds more generally for the finite codimensional submanifolds of $\mathcal{L}\mathrm{ag}^-$

$$\mathcal{L}\mathrm{ag}_k^{W_i} := \{L \in \mathcal{L}\mathrm{ag}^- \mid \dim L \cap W_i = k, \text{ and } L + W_i \text{ closed}\}$$

with the only difference that the base space of the fibration is a product of finite unitary group with an infinite complex Grassmannian (see Proposition 4.3, Th. 4.1 and Cor. 4.1).

Things get even better. The complement $\overline{Z}_{i+1}$ of each $\mathcal{L}\mathrm{ag}^{W_i}$ have a simple description in terms of incidence relations which brings to mind the Schubert varieties of the complex Grassmannian,

$$\overline{Z}_{i+1} := \{L \mid \dim L \cap W_i \geq 1\}.$$

In fact, $\overline{Z}_{i+1} \subset \mathcal{L}\mathrm{ag}^-$ admits a stratification $\overline{Z}_{i+1} = F_0 \supset F_2$ such that the top stratum $Z_{i+1} = F_0 \setminus F_2$ is a finite codimensional, naturally cooriented submanifold of $\mathcal{L}\mathrm{ag}^-$, and $\overline{Z}_{i+1}$ does not have singularities in codimension one, i.e., $F_1 = F_2$. It turns out that these conditions are enough to induce a cohomology class $[\overline{Z}_{i+1}] \in H^{2i+1}(\mathcal{L}\mathrm{ag}^-, \mathbb{Z})$. On the other hand, every cohomology class in $\mathcal{L}\mathrm{ag}^-$ is uniquely determined by its restrictions to the finite unitary groups $U(n)$ which come with natural inclusions (induced by the flag and Arnold's theorem) to $\mathcal{L}\mathrm{ag}^-$. It turns out that the classes $[\overline{Z}_i]$ pull-back to some canonical generators $x_i \in H^{2i-1}(U(n),\mathbb{Z})$ of the cohomology ring of $U(n)$, generators apriori defined as transgression classes. Concretely,

$$x_i := \int_{S^1} c_i(E_n),$$

where $E_n \to S^1 \times U(n)$ is a certain canonical rank n complex vector bundle (see Th. 6.1).

We exploited this fact to relate $\overline{Z}_i$ to the cohomological index of a family of self-adjoint, Fredholm operators. We arrived at the following result.

Theorem 1.1. *The $2k-1$-th component of the cohomological index of a continuous family of unbounded, self-adjoint, Fredholm operators $F : M \to \mathcal{S}\mathrm{Fred}$ is a rational multiple of the geometric class $(\tilde{\Gamma}\circ F)^*[\overline{Z}_k]$ where $\tilde{\Gamma} : \mathcal{S}\mathrm{Fred} \to \mathcal{L}\mathrm{ag}^-$ is the (switched) graph map. In fact, the following relation holds:*

$$\mathrm{ch}_{2k-1}[F] = \frac{(-1)^{k-1}}{(k-1)!}(\tilde{\Gamma}\circ F)^*[\overline{Z}_k].$$

As an application to the previous result, we consider generic families of self-adjoint, Fredholm operators parametrized by a closed, odd-dimensional manifold and we give an explicit expression for the Poincaré dual of the top cohomology class obtained by pulling back the relevant geometric class via the classifying map. These are the localization formulae for which the spectral flow corresponds to the degree one cohomology class. The typical result is the following

Theorem 1.2. *Let M be a closed, oriented manifold of dimension $2k-1$, let $F : M \to \mathcal{S}\mathrm{Fred}$ be a smooth family of self-adjoint, Fredholm operators and let $W \subset H$ be a codimension $k-1$ subspace such that F is in general position with respect to W. Denote by M_W the set $M_W := \{p \in M \mid \dim \mathrm{Ker}\,(F(p)) \cap W = 1\}$. Then*

$$\mathrm{PD}\,\mathrm{ch}_{2k-1}([F]) = \frac{(-1)^{k-1}}{(k-1)!}\sum_{p\in M_W}\epsilon_p p,$$

where PD *means Poincaré dual and ϵ_p is the sign of a certain $(2p-1)\times(2p-1)$ determinant.*

In order to find ϵ_p we needed a concrete description of the normal bundle of the Schubert cell Z_k, description from which the coorientability of Z_k can be easily deduced.

A different type of localization formula is possible. If the parameter manifold has dimension three, then in generic conditions, the top cohomology class is related to the chern class of the line bundle of kernels, which lives over a closed, oriented two-dimensional submanifold.

Proposition 1.1. *Let M be a closed, oriented manifold of dimension* 3 *and let $T : M \to \mathcal{S}\mathrm{Fred}$ be a generic family. Then $M^1 := \{m \in M \,|\, \dim \mathrm{Ker}\, T_m = 1\}$ is a closed, cooriented surface. Let $\gamma \subset M^1 \times H$ be the tautological line bundle over M^1 with fiber $\gamma_m = \mathrm{Ker}\, T_m$. Then*

$$\int_M T^*[\overline{Z}_2] = \int_{M^1} c_1(\gamma^*).$$

For what generic means check Definition 10.5 and Lemma 10.2.

The results presented here are based on the author's dissertation [8] to which we refer several times in this paper, especially when dealing with certain results whose proofs would have clouded our main line of presentation.

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2. The Lagrangian Grassmannian

We start by introducing the main object of study, the space of lagrangian subspaces of a fixed complex Hilbert space endowed with an additional complex structure. In this section, we summarize basic definitions and properties of this space, most of them fairly standard. A fair body of work has been carried out dealing with spaces that carry the name of Lagrangian Grassmannian, ever since Arnold introduced the notion in 1967 in [1]. However, in the infinite-dimensional, complex context, there are not so many places where one can find a detailed study of these spaces. We mention here the work of Booss–Bavnbek and Zhu [6] and of Kirk and Lesch [15]. While there is a certain overlap with each of these papers, we preferred, for the convenience of both the reader and the writer to present the results that we needed later on with their proofs included. A standard reference for the basic definitions concerning self-adjoint operators on a Hilbert space is [25].

Definition 2.1. Let H be a separable, complex Hilbert space and let $\hat{H} = H \oplus H$. We denote by H^+ (resp. H^-) the space $H \oplus 0$ (resp. $0 \oplus H$) and call it the *horizontal subspace* (resp. the *vertical subspace*).

Notation: *We will use the letters* $\mathcal{B}$, $\mathcal{U}$, *resp.* $\mathcal{S}\mathrm{ym}$ *for bounded, resp. unitary, resp. bounded, self-adjoint operators. We will refer in brackets to the Hilbert space on which these operators act.*

Let $J\hat{H} \to \hat{H}$ *be the unitary operator which has the block decomposition relative* $\hat{H} = H \oplus H$

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Notice that J *satisfies* $J = -J^* = -J^{-1}$.

Definition 2.2. A complex subspace $L \subset \hat{H}$ is called *Lagrangian* if $JL = L^{\perp}$. The (hermitian) *Lagrangian Grassmannian*, $\mathcal{L}\mathrm{ag}(\hat{H}, J)$ or simply $\mathcal{L}\mathrm{ag}$ is the set of all lagrangian subspaces of $\hat{H}$.

Remark 2.1. Notice that $JL = L^{\perp}$ implies that L is closed since $L^{\perp}$ is always closed and J is unitary.

Remark 2.2. The alternative use of the adjective “Hermitian” is related to the fact that the notion of a Lagrangian Grassmannian of a complex vector space already appears in the literature, e.g. in [10]. The underlying structure in [10] is a non-degenerate, skew symmetric, *bilinear* form ω. The symplectic structure in our case, $\omega := \langle J(\cdot), \cdot \rangle$ is sesquilinear and hence skew symmetric in the hermitian sense, i.e., $\langle J(x), y \rangle = -\overline{\langle J(y), x \rangle}$. As a consequence, our $\mathcal{L}\mathrm{ag}$ is only a real manifold.

Example 2.1.
(i) Each one of the two subspaces $H^{\pm}$ is a lagrangian subspace and $JH^{\pm} = H^{\mp}$.

(ii) Let $T : D(T) \subset H \to H$ be a closed self-adjoint operator, bounded or unbounded $T : D(T) \subset H \to H$. We will carefully distinguish between the *graph* of T

$$\Gamma_T := \{(v, Tv) \mid v \in D(T)\}$$

and the *switched graph* of T:

$$\tilde{\Gamma}_T := \{(Tw, w) \mid w \in D(T)\}.$$

They are both Lagrangian subspaces in $\hat{H}$.

The Lagrangian Grassmannian is naturally endowed with a topology as follows. To each Lagrangian L one associates the orthogonal projection $P_L \in \mathcal{B}(\hat{H})$ such that $\operatorname{Ran} P_L = L$. The condition that L is a Lagrangian translates into the obvious relation

$$JP_L = P_{JL}J = P_L^{\perp}J = (1 - P_L)J.$$

If $R_L := 2P_L - 1$ denotes the reflection in L then L is Lagrangian if and only if

$$JR_L = -R_L J.$$

Conversely, if R is an orthogonal reflection that anticommutes with J then $\operatorname{Ker}(I - R)$ is a lagrangian subspace. We therefore get a bijection

$$\mathcal{L}\mathrm{ag} \leftrightarrow \{R \in \mathcal{B}(\hat{H}) \mid R^2 = 1,\ R = R^*,\ RJ = -JR\}$$

and so $\mathcal{L}\mathrm{ag}$ inherits a topology as a subset of $\mathcal{B}(\hat{H})$.

The following result is straightforward and therefore we omit the proof.

Lemma 2.1. (a) *If L is a Lagrangian and $S \in \mathcal{S}\mathrm{ym}\,(L)$ is a self-adjoint operator then the graph of $JS : L \to L^{\perp}$ is a Lagrangian as well.*

(b) *For a fixed Lagrangian L, if L_1 is both Lagrangian and the graph of an operator $T : L \to L^{\perp}$ then T has to be of the type JS with $S \in \mathcal{S}\mathrm{ym}(L)$ self-adjoint.*

It is a well-known fact that, in the finite dimensional case the sets $\mathcal{S}\mathrm{ym}$ (L) are mapped to open subsets of $\mathcal{L}\mathrm{ag}$ around L, turning the Lagrangian Grassmannian into a manifold. The situation in the infinite-dimensional case is identical. The following will make our life easier.

Proposition 2.1. *Let L be an Lagrangian space, $L \in \mathcal{L}\mathrm{ag}$. The following are equivalent:*

- (a) *L is the graph of an operator $JA : L_0 \to L_0^{\perp}$ where $A \in \mathcal{S}\mathrm{ym}(L_0)$.*
- (b) *$L \cap L_0^{\perp} = \{0\}$ and $L + L_0^{\perp}$ is closed.*
- (b') *$\hat{H} = L \oplus L_0^{\perp}$.*
- (b'') *$\hat{H} = L + L_0^{\perp}$.*
- (c) *$R_L + R_{L_0}$ is invertible.*

Proof. $(a) \Rightarrow (b)$ Clearly if L is the graph of an operator $L_0 \to L_0^\perp$ then L is a linear complement of $L_0^\perp$.
$(b) \Rightarrow (b') \Rightarrow (b'') \Rightarrow (b)$. We have the following equality:

$$(L + L_0^\perp)^\perp = L^\perp \cap L_0 = J(L \cap L_0^\perp) = \{0\}.$$

and this proves that $(b) \Rightarrow (b')$. Clearly (b') implies (b'') and (b'') implies (b) because if $z \in L \cap L_0^\perp$ then $Jz \perp L$ and $Jz \perp L_0$ and so $z = 0$.
$(b) \Rightarrow (c)$ It is easy to check that

$$\mathrm{Ker}\,(P_L - P_{L_0^\perp}) = L \cap L_0^\perp \oplus L^\perp \cap L_0 = \{0\}.$$

The injectivity now follows from $R_L + R_{L_0} = R_L - R_{L_0^\perp} = 2(P_L - P_{L_0^\perp})$. Part (b') gives also $\hat{H} = L^\perp \oplus L_0$. This implies that

$$\text{Range } (R_L - R_{L_0^\perp}) = \text{Range } (P_L - P_{L_0^\perp}) = L + L_0^\perp = \hat{H},$$

which proves surjectivity.
$(c) \Rightarrow (a)$ We show that the restriction to L of the projection onto L_0, $P_{L_0}|_L$ is an isomorphism. First $\mathrm{Ker}\, P_{L_0}|_L = L \cap L_0^\perp$ and since $\mathrm{Ker}\,(R_L + R_{L_0}) = L \cap L_0^\perp \oplus L^\perp \cap L_0$ one concludes that $L \cap L_0^\perp = \{0\}$.

Surjectivity boils down to showing that the adjoint $(P_{L_0}|_L)^*$ is bounded below [5]. But $(P_{L_0}|_L)^*$ is nothing else but $P_L|_{L_0}$.

For $x \in L_0$ one has the following string of equalities

$$\begin{aligned}
\|P_L|_{L_0}(x)\| &= \|P_L P_{L_0}(x)\| = 1/4\|(R_L + 1)(R_{L_0} + 1)(x)\| \\
&= 1/4\|(R_L + R_{L_0} - R_{L_0} + 1)(R_{L_0} + 1)(x)\| \\
&= 1/4\|(R_L + R_{L_0})(R_{L_0} + 1)(x)\| \\
&= 1/2\|(R_L + R_{L_0})P_{L_0}(x)\| = 1/2\|(R_L + R_{L_0})(x)\|.
\end{aligned}$$

The lower bound follows from the invertibility of $R_L + R_{L_0}$.

It is clear that L is the graph of an operator $T : L_0 \to L_0^\perp$, $T = P_{L_0^\perp}\big|_L \circ (P_{L_0}|_L)^{-1}$. This operator has to be of the type JA with $A \in \mathcal{S}\mathrm{ym}(L_0)$ by part $b)$ in the previous lemma. □

Corollary 2.1. *The set* $\{L \in \mathcal{L}\mathrm{ag} \mid L$ *is the graph of an operator* $L_0 \to L_0^\perp\}$ *is an open neighborhood around* L_0.

Proof. The invertibility of $R_L + R_{L_0}$ is an open condition. □

Definition 2.3. For a fixed lagrangian L, the map $\mathcal{A}_L : \mathcal{S}\mathrm{ym}(L) \to \mathcal{L}\mathrm{ag}$ which associates to an operator S the graph of JS is called the *Arnold chart* around L. We will sometimes use the same notation, $\mathcal{A}_L$ to denote the image of this map in $\mathcal{L}\mathrm{ag}$.

The following is now obvious.

Proposition 2.2. *The Arnold charts $\mathcal{A}_L$ turn the Lagrangian Grassmannian $\mathcal{L}\mathrm{ag}$ into a Banach manifold modeled on the space of bounded, self-adjoint operators $\mathcal{S}\mathrm{ym}\,(H)$.*

Let us a give an application of what we did so far. We will need this computation later.

Lemma 2.2. *Let $P : \mathcal{L}\mathrm{ag} \to \mathcal{S}\mathrm{ym}\,(\hat{H})$ be the map that associates to the lagrangian L the orthogonal projection onto L, P_L . The differential $d_L P : \mathcal{S}\mathrm{ym}\,(L) \to \mathcal{S}\mathrm{ym}\,(\hat{H})$ is given by the following expression relative $\hat{H} = L \oplus L^{\perp}$*

$$d_L P(\dot{S}) = \begin{pmatrix} 0 & \dot{S} J_L^{-1} \\ J_L \dot{S} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\dot{S} J \\ J\dot{S} & 0 \end{pmatrix},$$

where $J_L : L \to L^{\perp}$ is the restriction of J to L.

Proof. We need an expression for the projection $P_{\Gamma_{JS}}$ onto the graph $JS : L \to L^{\perp}$. That comes down to finding v in the equations

$$\begin{cases} a = v - SJ^{-1}w, \\ b = JSv + w, \end{cases}$$

where $a, v \in L$ and $b, w \in L^{\perp}$. We get

$$v = (1 + S^2)^{-1}(a + SJ^{-1}b),$$

so that the projection has the block decomposition relative $L \oplus L^{\perp}$

$$P_{\Gamma_{JS}} = \begin{pmatrix} (1+S^2)^{-1} & (1+S^2)^{-1}SJ^{-1} \\ J(1+S^2)^{-1}S & J(1+S^2)^{-1}S^2J^{-1} \end{pmatrix}. \tag{2.1}$$

Differentiating this at $S = 0$ we notice that the diagonal blocks vanish since we deal with even functions of S and so the product rule delivers the result. □

The space $\mathcal{L}\mathrm{ag}$ is not very interesting from a homotopy point of view and in the next section we will see that it is contractible. To get something non-trivial we restrict our attention to the subspace of vertical, Fredholm lagrangians which we now define.

Definition 2.4. A pair of Lagrangians (L_1, L_2) is called a *Fredholm pair* if the following two conditions hold

$$\dim(L_1 \cap L_2) < \infty \quad \text{and} \quad L_1 + L_2 \text{ is closed.}$$

The Grassmannian of *vertical, Fredholm Lagrangians* is

$$\mathcal{L}\mathrm{ag}^- := \{L \in \mathcal{L}\mathrm{ag} \mid (L, H^-) \text{ is a Fredholm pair}\}.$$

Remark 2.3. If $T : D(T) \to H$ is a a closed, self-adjoint, Fredholm operator then only its switched graph is a vertical, Fredholm lagrangian.

Our objects are primarily lagrangian subspaces. There is a related notion of a Fredholm pair when talking about orthogonal projections. We summarize here the main relations between the two notions. One can find the detailed proofs in the work of Avron *et al.*[4].

Definition 2.5. (a) A pair of orthogonal projections P and Q in a separable Hilbert space H is said to be a *Fredholm pair* if the linear operator

$$QP : \operatorname{Ran} P \to \operatorname{Ran} Q$$

is Fredholm.

(b) A pair of closed subspaces U and V of H is said to be a Fredholm pair if

$$\dim U \cap V < \infty, \quad \dim U^\perp \cap V^\perp < \infty \quad \text{and} \quad U + V \text{ closed.}$$

(c) Two subspaces U and V are said to form a *commensurate* Fredholm pair if $P_U - P_V$ is a compact operator.

When U and V are Lagrangian subspaces the middle condition in the definition of a Fredholm pair is superfluous.

Proposition 2.3. *Let (P, Q) be a pair of projections. Then the following statements are equivalent:*

(a) *The pair (P, Q) is a Fredholm pair.*

(b) *The pair (Q, P) is a Fredholm pair.*

(c) *The operators $P - Q \pm 1$ are Fredholm.*

(d) *The pairs of subspaces* $(\operatorname{Ran} P, \operatorname{Ker} Q) = (\operatorname{Ran} P, \operatorname{Ran} Q^{\perp})$ *and* $(\operatorname{Ran} Q, \operatorname{Ker} P) = (\operatorname{Ran} Q, \operatorname{Ran} P^{\perp})$ *are Fredholm pairs.*

Proof. See Proposition 3.1 and Theory 3.4 (a) in [4]. □

Proposition 2.4. *Suppose (U, V) is a Fredholm pair of closed subspaces and that W is another subspace commensurate with V. Then the pairs $(V^{\perp}, W)$ and (U, W) are Fredholm pairs.*

Proof. This follows from the previous proposition and Theorem 3.4 (c) in [4]. □

Let $P^{\pm}|_L$ be the orthogonal projections on $H^{\pm}$ restricted to the Lagrangian L and $P_L|_{H^{\pm}}$ be the projection on L restricted to $H^{\pm}$. The following is just a corollary of the previous propositions.

Corollary 2.2. *The set of vertical, Fredholm Lagrangians, $\mathfrak{Lag}^{-}$, coincides with the set*

$$\{L \in \mathfrak{Lag} \mid P^{+}|_L \text{ is Fredholm of index } 0\}.$$

3. Arnold's theorem

In this section, we generalize a result by Arnold ([2], see also [18]) to infinite dimensions. In his article, Arnold showed that the finite Lagrangian Grassmannian $\mathfrak{Lag}(N) \subset \operatorname{Gr}(N, 2N)$ is diffeomorphic to the unitary group $U(N)$. A similar result appears in [15].

We introduce now the main suspects. Consider the $\mp i$ eigenspaces of J, $\operatorname{Ker}(J \pm i)$ and let

$$\operatorname{Iso}_{-+} := \operatorname{Iso}(\operatorname{Ker}(J + i), \operatorname{Ker}(J - i)).$$

To each Lagrangian L we associate the restriction to $\operatorname{Ker}(J + i)$ of the reflection R_L. Since R_L anticommutes with J we get a well-defined *reflection map*

$$R_{-+} : \mathfrak{Lag} \to \operatorname{Iso}_{-+}, \qquad R_{-+}(L) = R_L\big|_{\operatorname{Ker}(J+i)}.$$

On the other hand, to each isomorphism $\alpha \in \mathrm{Iso}_{-+}$ we can associate its graph Γ_α which is a subspace of $\hat{H}$. It is, in fact, a Lagrangian. Indeed

$$w \in J\Gamma_\alpha \Leftrightarrow w = J(\alpha v + v) = -i\alpha v + iv = z - \alpha^* z \in \tilde{\Gamma}_{-\alpha^*}$$

for some $v \in \mathrm{Ker}\,(J+i)$, with $z = -iuv \in \mathrm{Ker}\,(J-i)$. It is standard that the switched graph $\tilde{\Gamma}_{-\alpha^*}$ is the orthogonal complement of Γ_α. Therefore, we get a second well-defined *graph map*

$$\Gamma : \mathrm{Iso}_{-+} \to \mathcal{L}\mathrm{ag}, \qquad \Gamma(\alpha) = \Gamma_\alpha.$$

Lemma 3.1. *Let $L \subset \hat{H}$ be a Lagrangian. Then the following maps are canonical isomorphisms of Hilbert spaces:*

$$\phi_\pm(L) : L \to \mathrm{Ker}\,(J \pm i), \qquad \phi_\pm(L)(v) = 1/\sqrt{2}(v \pm iJv)$$

Proof. The injectivity is straightforward. For surjectivity, let $w \in \mathrm{Ker}\,(J+i)$ be written as

$$w = v + v^\perp, \qquad v \in L \text{ and } v^\perp \in L^\perp.$$

Then $Jw = -iw$ implies that $Jv = -iv^\perp$ and so $v^\perp = iJv$. □

Notation: *When there is no possibility for confusion we will use $\phi_\pm := \phi_\pm(L)$.*

The previous lemma identifies in a canonical way Iso_{-+} *with the set of unitary operators* $\mathcal{U}(L)$ *via*

$$\alpha \to \phi_-^{-1}\alpha\phi_+ =: U_\alpha.$$

The graph, Γ_α, is expressed in terms of the unitary operator U_α as the set

$$\Gamma_\alpha = \{(1+U_T)v + iJ(1-U_T)v \mid v \in L\}.$$

Definition 3.1. Let L be a Lagrangian. The associated *Cayley graph map* is the following application:

$$\mathcal{C}_L : \mathcal{U}(L) \to \mathcal{L}\mathrm{ag}$$
$$\mathcal{C}_L(U) := \Gamma(\phi_- \circ U \circ \phi_+^{-1}) = \mathrm{Ran}\{L \ni v \mapsto (1+U)v + iJ(1-U)v \in \hat{H}\}.$$

We have the following generalization of a result of Arnold [2].

Theorem 3.1. (a) *The reflection map,* $R_{-+} : \mathcal{L}\mathrm{ag} \to \mathrm{Iso}_{-+}$ *and the graph map* $\Gamma : \mathrm{Iso}_{-+} \to \mathcal{L}\mathrm{ag}$ *are inverse to each other.*

(b) *For every Lagrangian* L*, the Cayley graph map,* $\mathcal{C}_L : \mathcal{U}(L) \to \mathcal{L}\mathrm{ag}$ *is a diffeomorphism of real Banach manifolds.*

(c) *The restriction of the Cayley graph map induces a diffeomorphism of the following open subsets of* $\mathcal{U}(L)$*, resp.* $\mathcal{L}\mathrm{ag}$*.*

$$\mathcal{U}_{-1}(L) := \{U \in \mathcal{U}(L) \mid 1 + U \quad \textit{is Fredholm}\ \}$$

and

$$\mathcal{L}\mathrm{ag}^{-}(L) := \{L_1 \in \mathcal{L}\mathrm{ag} \mid (L_1, L) \quad \textit{is a Fredholm pair}\}.$$

Proof. (a) Notice that the operator from $\hat{H}$ which relative to the decomposition $\hat{H} = \mathrm{Ker}\,(J + i) \oplus \mathrm{Ker}\,(J - i)$, has the expression

$$\begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}$$

is an orthogonal reflection whose eigenvalue 1-eigenspace is Γ_α. Hence $R_{-+}\Gamma(\alpha) = \alpha$.

In order to prove that $\Gamma \circ R_{-+} = \mathrm{id}$ it suffices to show only that $\Gamma \circ R_{-+}(L) \subset L$. Take $v \in \mathrm{Ker}\,(J + i)$. Then

$$\Gamma(R_{-}(L)) \ni v + R_L(v) = 2P_L(v) \in L.$$

(b) By part (a) the map is a bijection. We prove differentiability. Fix a Lagrangian L and let $\mathcal{S}\mathrm{ym}(L) \ni S \to \Gamma_{JS} \in \mathcal{L}\mathrm{ag}$ be the Arnold chart centered at L. Then using (2.1) we get the following expression for the reflection $R_{\Gamma_{JS}}$ relative to $\hat{H} = L \oplus L^\perp$

$$R_{\Gamma_{JS}} = 2P_{\Gamma_{JS}} - 1 = \begin{pmatrix} (1 - S^2)(1 + S^2)^{-1} & 2S(1 + S^2)^{-1}J^{-1} \\ 2JS(1 + S^2)^{-1} & -J(1 - S^2)(1 + S^2)^{-1}J^{-1} \end{pmatrix}.$$

which is differentiable function of S. Since $R_{-+} = P^i \circ R\big|_{\mathrm{Ker}\,(J+i)}$, where P^i is the projection on the i-eigenspace of J, we conclude that R_{-+} is smooth and therefore its inverse Γ is smooth and so is $\mathcal{C}_L$.

(c) Fix the Lagrangian $L := H^+$. Let $\mathcal{C}_+ = \mathcal{C}_{H^+}$ Since the Fredholm property is an open condition it follows that $\mathcal{U}_{-1}$ is open in $\mathcal{U}(H)$.

By standard spectral theory, the Fredholm property of $1 + U$ implies that $\mathrm{Ker}\,(1 + U) = \mathcal{C}(U) \cap H^-$ is finite dimensional and also that $-1 \notin \sigma(U|_{\mathrm{Ker}\,(1+U)^\perp})$.

We can now factor out $\mathrm{Ker}(1+U)$. To that end, let $\breve{H} = \mathrm{Ker}\,(1+U)^{\perp}$, and $\breve{H}^{\pm}$ be the horizontal/vertical copy of $\breve{H}$ in $\breve{H} \oplus \breve{H}$ and $\breve{U} = U|_{\breve{H}}$. Since $1+\breve{U}$ is invertible, the Cayley graph of $\breve{U}$ is in the Arnold chart of $\breve{H}^{+}$ and so $\mathcal{C}(\breve{U}) + \breve{H}^{-}$ is closed by Proposition 2.1. On the other hand

$$\mathcal{C}(U) + H^{-} = \mathcal{C}(\breve{U}) + \breve{H}^{-} + \mathrm{Ker}\,(1+U),$$

where $\mathrm{Ker}\,(1+U) \subset H^{-}$ is finite dimensional and this proves that $\mathcal{C}(U) + H^{-}$ is closed.

Conversely, let $L \cap H^{-}$ be finite dimensional and $L + H^{-}$ be closed. If we let $\breve{L}$ and $\breve{H}^{-}$ be the orthogonal complements of $L \cap H^{-}$ in L and H^{-}, respectively, these two spaces are Lagrangians in $J\breve{H}^{-} \oplus \breve{H}^{-}$ whose intersection is empty, and whose sum is closed. Their sum is closed because of the relation

$$\breve{L} + \breve{H} = (L \cap H^{-})^{\perp},$$

where the orthogonal complement is taken in $L + H^{-}$.

By Lemma 2.1, $\breve{L}$ is in the Arnold chart of $J\breve{H}^{-}$, hence $\breve{L} = \Gamma_S$ with $S : J\breve{H}^{-} \to J\breve{H}^{-}$ self-adjoint. We get $L = \mathcal{C}(U)$ where U is the extension by -1 on $J(L \cap H^{-})$ of the Cayley transform of S. It is clear that $1+U$ is Fredholm. □

Notation: *Whenever we have a map or object depending on a Lagrangian, a sub/superscript $\pm$ indicates that the lagrangian is $H^{\pm}$.*

Our main interest is in $\mathcal{L}\mathrm{ag}^{-}$ so we will state the theorem in this case separately.

Corollary 3.1. *The Cayley graph map $\mathcal{C}_{+} : \mathcal{U}(H) \to \mathcal{L}\mathrm{ag}$*

$$\mathcal{C}_{+}(U) := \mathrm{Ran}\{H \ni v \to ((1+U)v, -i(1-U)v) \in H \oplus H\}$$

induces a diffeomorphism between $\mathcal{U}_{-1}$ and $\mathcal{L}\mathrm{ag}^{-}$.

Proof. This is just the case $L = H^{+}$ in Theory 3.1. The reason for $-i$ in the second component is that under the canonical identifications $H = H^{+}$ and $H = H^{-}$, $J\big|_{H^{+}}$ acts as minus the identity. □

Remark 3.1. We chose the name Cayley graph map because the composition

$$\mathcal{C}_{+}^{-1} \circ \tilde{\Gamma} : \mathcal{S}\mathrm{ym}(H) \to \mathcal{U}(H), \qquad A \to (A-i)(A+i)^{-1}$$

is the well-known Cayley transform. Notice that $A = 0$ corresponds to $-Id$.

Remark 3.2. Our choice of the reflection map, R_{-+} to go from $\mathrm{Ker}\,(J+i)$ to $\mathrm{Ker}\,(J-i)$ rather then the other way was dictated by orientation considerations.

In the case $H = \mathbb{C}$ we wanted the composition

$$\mathcal{C}_+^{-1} \circ \tilde{\Gamma} : \mathcal{S}\mathrm{ym}(\mathbb{C}) \to S^1 \setminus \{1\}, \qquad A \to (A-i)(A+i)^{-1}$$

to be an orientation preserving diffeomorphism, where S^1 is given the counterclockwise orientation.

For each Lagrangian L, we introduce the change of basis isomorphism

$$(*) \qquad \Phi_L : L \oplus L^\perp \to \mathrm{Ker}\,(J+i) \oplus \mathrm{Ker}\,(J-i), \quad \Phi_L = \begin{pmatrix} \phi_+ & 0 \\ 0 & \phi_- \circ J^{-1} \end{pmatrix}.$$

where $J^{-1} : L^\perp \to L$ is the inverse of the restriction $J : L \to L^\perp$. Notice that Φ_L diagonalizes J relative to the decomposition $\hat{H} = L \oplus L^\perp$, i.e.,

$$\Phi_L^{-1} J \Phi_L = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

Lemma 3.2. *Let $U \in \mathcal{U}(L)$ be a unitary map. Then the reflection in the Lagrangian $L_U := \mathcal{C}_L(U)$ has the following expression relative $\hat{H} = L \oplus L^\perp$*

$$R_{L_U} = \Phi_L \begin{pmatrix} 0 & U^* J^{-1} \\ JU & 0 \end{pmatrix} \Phi_L^{-1}.$$

Proof. Let $\alpha_U : \mathrm{Ker}\,(J+i) \to \mathrm{Ker}\,(J-i)$ be the isomorphism that corresponds to $\mathcal{C}_L(U)$, in other words $\alpha_U = \phi_- U \phi_+^{-1}$. Then

$$R_{L_U} = \begin{pmatrix} 0 & \alpha_U^* \\ \alpha_U & 0 \end{pmatrix} = \begin{pmatrix} \phi_+ & 0 \\ 0 & \phi_- \end{pmatrix} \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix} \begin{pmatrix} \phi_+^{-1} & 0 \\ 0 & \phi_-^{-1} \end{pmatrix}.$$

The claim follows from the expression for Φ_L. □

Using the previous lemma we get a different characterization of $\mathcal{L}\mathrm{ag}^-$.

Corollary 3.2. *The space of vertical, Fredholm Lagrangians has the following characterization $\mathcal{L}\mathrm{ag}^- = \{L \in \mathcal{L}\mathrm{ag} \mid R_L + R_{H^+}$ is Fredholm$\}$.*

Proof. If U is the operator L is coming from via the Cayley graph map then

$$R_L + R_+ = \Phi_+ \begin{pmatrix} 0 & (1+U)^* J^{-1} \\ J(1+U) & 0 \end{pmatrix} \Phi_+^{-1}.$$

Clearly, $R_L + R_+$ is Fredholm if and only if $1+U$ is Fredholm. □

Corollary 3.3. *The Cayley graph map, $\mathcal{C}$ takes the Arnold chart around U_0 bijectively onto the Arnold chart around $L_0 := \mathcal{C}(U_0)$.*

Proof. Let $T := U + U_0$, $L_U := \mathcal{C}(U)$. Then

$$R_{L_U} + R_{L_0} = \Phi_+ \begin{pmatrix} 0 & -T^* \\ -T & 0 \end{pmatrix} \Phi_+^{-1}.$$

is invertible if and only if T is invertible and hence by Proposition 2.1 we get $\mathcal{C}_+(U) \in \mathcal{A}_{L_0}$ if and only if T is invertible. □

Corollary 3.4. *Let M be a differentiable manifold and $F : M \to \mathcal{L}\mathrm{ag}$ be a map. Then F is differentiable if and only if $R \circ F$ is differentiable where $R : \mathcal{L}\mathrm{ag} \to \mathcal{B}(\hat{H})$ associates to every Lagrangian L its reflection.*

Proof. Clearly F is differentiable if and only if $F_1 := \mathcal{C}_+^{-1} \circ F : M \to \mathcal{U}(H^+)$ is differentiable. Now $R \circ \mathcal{C}_+ : \mathcal{U}(H) \to \mathcal{B}(\hat{H})$ has the following expression

$$R \circ \mathcal{C}_+(U) = \Phi_+ \begin{pmatrix} 0 & -U^* \\ -U & 0 \end{pmatrix} \Phi_+^{-1}.$$

So F_1 is differentiable if and only if $R \circ \mathcal{C}_+ \circ F_1$ is differentiable. □

A closer look at the Cayley graph map suggests a useful reformulation of Theorem 3.1. Notice that the group of unitary operators

$$\mathcal{U}_J(\hat{H}) := \{U \in \mathcal{U}(\hat{H}) \mid UJ = JU\}$$

acts on $\mathcal{L}\mathrm{ag}$. Theorem 3.1 says that given two Lagrangians L_1 and L_2 there exists a *canonical* $U^\sharp \in \mathcal{U}_J(\hat{H})$ such that $U^\sharp L_1 = L_2$. Indeed, let $U := \mathcal{C}_{L_1}^{-1}(L_2)$. Then the operator

$$\tilde{U} : L_1 \to L_2, \qquad \tilde{U}(v) = 1/2((1+U)v + iJ(1-U)v)) \tag{3.1}$$

is a Hilbert space isomorphism. Define $U^\sharp \in \mathcal{U}(\hat{H})$, by $U^\sharp := \tilde{U} \oplus J\tilde{U}J^{-1}$. Relative to the decomposition $\hat{H} = L \oplus L^\perp$ this automorphism has the expression:

$$U^\sharp = \frac{1}{2}\begin{pmatrix} 1+U & -i(1-U)J^{-1} \\ iJ(1-U) & J(1+U)J^{-1} \end{pmatrix} = \Phi_L \begin{pmatrix} 1 & 0 \\ 0 & JUJ^{-1} \end{pmatrix} \Phi_L^{-1}.$$

Theorem 3.2. (a) *For a fixed Lagrangian L, the following map is a diffeomorphism of real Banach manifolds*

$$\mathcal{C}_L : \mathcal{U}(L) \mapsto \mathcal{L}\mathrm{ag}, \qquad \mathcal{C}_L(U) = \mathcal{O}_L(U)L,$$

where

$$\mathcal{O}_L : \mathcal{U}(L) \mapsto \mathcal{U}(\hat{H}), \qquad U \mapsto \Phi_L \begin{pmatrix} 1 & 0 \\ 0 & JUJ^{-1} \end{pmatrix} \Phi_L^{-1}.$$

The block decomposition is relative to $\hat{H} = L \oplus L^\perp$.

(b) *The map $\mathcal{O}_L$ has the expression:*

$$\mathcal{O}_L(U) = \begin{pmatrix} 1 & 0 \\ 0 & \phi_- U \phi_-^{-1} \end{pmatrix}$$

relative to the decomposition $\hat{H} = \mathrm{Ker}\,(J+i) \oplus \mathrm{Ker}\,(J-i)$. Hence the image of $\mathcal{O}_L$ depends only on J.

(c) *The bundle $\mathcal{U}_\tau \subset \mathcal{L}\mathrm{ag} \times \mathcal{B}(\hat{H})$ over $\mathcal{L}\mathrm{ag}$ whose fiber over a Lagrangian L consists of unitary operators $U \in \mathcal{U}(L)$ is canonically trivializable and the map:*

$$\mathcal{O} : \mathcal{U}_\tau \to \mathcal{U}(\hat{H}), \qquad \mathcal{O}(L,U) := \mathcal{O}_L(U)$$

is differentiable.

Proof. (a) The job is done by Theorem 3.1.

(b) Straightforward.

(c) Let us notice that the tautological bundle $\tau \subset \mathcal{L}\mathrm{ag} \times \hat{H}$, $\tau := \{(L,v) \mid v \in L\}$ over $\mathcal{L}\mathrm{ag}$ is naturally trivializable. A natural trivialization is given as follows. For every Lagrangian L, let $U_L := \mathcal{C}_+^{-1}(L)$ be the unitary operator on H^+ corresponding to L via the Cayley graph map. Then the following map is a trivialization of the tautological bundle:

$$\alpha : \tau \to \mathcal{L}\mathrm{ag} \times H^+, \qquad \alpha(L,v) = (L, (\mathcal{O}_+(U_L))^{-1}(v))$$

since both $\mathcal{C}_+^{-1}$ and $\mathcal{O}_+$ are differentiable. It is straightforward that $\mathcal{U}_\tau$ is naturally trivializable.

In order to show that $\mathcal{O}$ is differentiable it is enough to prove that the map

$$\Phi : \mathcal{L}\mathrm{ag} \to \mathcal{U}(\hat{H}), \qquad L \to \Phi_L$$

is differentiable. Since $\mathcal{O}_+$ and $\mathcal{C}_+$ are differentiable, the following identity proves this claim:

$$\Phi(\mathcal{C}_+(U)) = \mathcal{O}_+(U)\Phi_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -U \\ -i & -iU \end{pmatrix}, \qquad \forall U \in \mathcal{U}(H^+),$$

where the decomposition is relative $\hat{H} = H \oplus H$. □

Remark 3.3. Part (b) of the Theorem is saying that the image of $\mathcal{O}$ is only "half" of $\mathcal{U}_J(\hat{H})$ which consists of unitary operators with diagonal block decomposition relative $\hat{H} = \mathrm{Ker}\,(J+i) \oplus \mathrm{Ker}\,(J-i)$. One consequence is that $\mathcal{U}_J(\hat{H})$ acts transitively on $\mathcal{L}\mathrm{ag}$. Let us notice that $U_J(\hat{H})$ is a subgroup of the symplectic group, i.e., the invertible operators $X \in \mathrm{GL}(\hat{H})$ satisfying $X^*JX = J$. This bigger group also acts transitively and differentiably on $\mathcal{L}\mathrm{ag}$ (see Prop 4.5 in [8]).

Remark 3.4. It is not hard to see that the canonical unitary operators $\tilde{U}(L_1, L_2)$ defined at (3.1) satisfy the relation

$$U(L_2, L_3) \circ U(L_1, L_2) = U(L_1, L_3).$$

In fact, if $\tau \subset \mathcal{L}\mathrm{ag} \times \hat{H}$ is the tautological bundle over $\mathcal{L}\mathrm{ag}$, which, as we saw before, is trivializable, then the collection of unitary operators $\tilde{U}$ represents the parallel transport for the pull-back to τ of the trivial connection on the trivial bundle $\mathcal{L}\mathrm{ag} \times \hat{H}$.

The main object of study is $\mathcal{L}\mathrm{ag}^-$. However, all the results stated here hold for other infinite Lagrangian Grassmannians. Let $\mathcal{I} \subset \mathcal{K} \subset \mathcal{B}$ be a non-trivial, two-sided ideal of bounded(compact) operators, ideal endowed with a topology at least as strong as the norm topology [26] (examples are trace class operators with the trace norm or Hilbert–Schmidt operators, etc.). The Palais unitary group, $\mathcal{U}_{\mathcal{I}}$, introduced in [23], is $\mathcal{U}_{\mathcal{I}} := \mathcal{U} \cap (1 + \mathcal{I})$. Notice that $\mathcal{U}_{\mathcal{I}} \subset \mathcal{U}_{-1}$ since the only essential spectral value of $U \in \mathcal{U}_{\mathcal{I}}$ is 1. If we let

$$\mathcal{L}\mathrm{ag}_{\mathcal{I}} := \mathcal{C}_+(\mathcal{U}_{\mathcal{I}})$$

be the corresponding space of Lagrangians, it is not hard to see that (compare with Corollary 2.2)

$$\mathcal{L}\mathrm{ag}_{\mathcal{I}} = \{L \in \mathcal{L}\mathrm{ag}(H) \mid P^+|_L \text{ is Fredholm of index 0 and } P^-|_L \in \mathcal{I}\}$$

where $P^-|_L \in \mathcal{I}$ means that $P^-|_L$ is a compact operator and its singular values satisfy the same boundedness condition as the one that describes $\mathcal{I}$.

Lemma 3.3. *Let $T : D(T) \subset H \to H$ be a self-adjoint, Fredholm operator. Then the switched graph $\tilde{\Gamma}_T$ belongs to $\mathcal{L}\mathrm{ag}_{\mathcal{I}}$ if and only if the resolvent $R_T(\lambda)$ belongs to $\mathcal{I}$ for some $\lambda \notin \sigma(T)$.*

Proof. Let $U := \mathcal{C}_+^{-1}(\tilde{\Gamma}_T)$ be the unitary operator that corresponds to T in Theorem 3.1 Then $X := 1 - U$ is bounded and induces a bijection $X : H \to D(T)$. If we endow $D(T)$ with the norm

$$\langle v, w\rangle_{D(T)} := \langle X^{-1}v, X^{-1}w\rangle_H, \quad \forall v, w \in D(T)$$

(which is nothing else but the graph norm) then $T : D(T) \to H$ is a bounded operator and

$$T = iX^{-1}(2 - X).$$

Hence $1 - U = X = 2i(T + i)^{-1} = 2iR_T(-i)$, which proves the claim. □

The space $\mathcal{L}\mathrm{ag}_{\mathcal{I}}$ is also a real Banach manifold modeled on the space of bounded, self-adjoint operators $S \in \mathcal{I}$.

4. Symplectic reduction

In this section, we describe the symplectic reduction process that we will need further on. While other people have written about and used (linear) symplectic reduction (see for example [15, 19, 21]) we will need in section 8 a generalized version (see definition 4.6) of this process that we develop here. As a byproduct, we get in 4.1 a description of symplectic reduction as the projection map of a natural vector bundle.

We conclude this section with an application of symplectic reduction. We show in Theorem 4.2 that $\mathcal{L}\mathrm{ag}^-$ is classifying for K^{-1}-theory. This result, with a different proof, first appears in [15].

Definition 4.1. An *isotropic* subspace of the pair $(\hat{H}, J)$ is a *closed* subspace $W \subseteq \hat{H}$ such that $W \subset \Lambda$ for some Lagrangian Λ.

The space $W^\omega := (JW)^\perp$ is called the *annihilator* of W. Notice that

$$W \subset \Lambda = (J\Lambda)^\perp \subset W^\omega.$$

The orthogonal complement of W in W^ω, denoted H_W is called the (symplectically) *reduced* space of $\hat{H}$.

Remark 4.1. The Hilbert space H_W is the orthogonal complement of $W \oplus JW$ in $\hat{H}$ and hence it is J-invariant. Notice also that

$$\dim \operatorname{Ker}(i \pm J|_{H_W}) = \frac{1}{2} \dim H_W = \dim W_\Lambda^\perp,$$

where $W_\Lambda^\perp$ is the orthogonal complement of W in Λ. The dimension could be infinite.

Definition 4.2. The isotropic space W is called *cofinite* if H_W is finite dimensional. The codimension of W is half of the dimension of H_W.

Definition 4.3. Let W be a fixed, cofinite, isotropic space and k a non-negative integer. A lagrangian L is called *k-clean* with respect to W (or just clean if $k = 0$) if it belongs to the set:

$$\mathfrak{Lag}_k^W := \{L \in \mathfrak{Lag} \mid \dim L \cap W = k \text{ and } L + W \text{ is closed}\}.$$

Let $\mathfrak{Lag}^W := \bigcup_k \mathfrak{Lag}_k^W$ be the space of Lagrangians which are Fredholm pairs with W.

If $W = L_0$ is a Lagrangian itself, then $\mathfrak{Lag}^W = \mathcal{A}_{L_0^\perp}$.

Proposition 4.1. *Let W be a cofinite, isotropic space.*

(a) *Let $L \in \mathfrak{Lag}_k^W$ for some integer $k \geq 0$. Then $P_{H_W}(L \cap W^\omega)$ is a Lagrangian subspace of H_W where P_{H_W} is the orthogonal projection onto H_W.*

(b) *Let $L \in \mathfrak{Lag}_0^W$. Then there exists a Lagrangian $L_0 \supset W$ such that $L + L_0 = \hat{H}$. Hence $L \in \mathcal{A}_{L_0^\perp} \subset \mathfrak{Lag}_0^W$ and therefore $\mathfrak{Lag}_0^W$ is open in $\mathfrak{Lag}$.*

Proof. (a) Note first that $\dim L \cap W^\omega < \infty$ since $\operatorname{Ker} P_{H_W}\big|_{L \cap W^\omega} = L \cap W$. Let $W^\perp$ be the orthogonal complement of W in $\hat{H}$. We have $L^\perp \cap W^\perp =$

$J(L\cap W^\omega)$ so that $L^\perp\cap W^\perp$ is finite dimensional as well. This implies that $L^\perp\cap W^\perp+H_W^\perp$ is closed.

Let $\ell := P_{H_W}(L\cap W^\omega)$ and let ℓ' be the orthogonal complement of ℓ in H_W. Notice that

$$\ell = (L+W)\cap H_W$$

and so

$$\begin{aligned}\ell' &= H_W\cap\big(L^\perp\cap W^\perp + H_W^\perp\big) = H_W\cap J\big(L\cap W^\omega + H_W^\perp\big)\\ &= J\big(H_W\cap(L\cap W^\omega+H_W^\perp)\big) = JP_{H_W}(L\cap W^\omega) = J\ell\end{aligned}$$

which proves that ℓ is a lagrangian in H_W.

(b) Let $L_0 := J\ell + W$ where, as before, $\ell := P_{H_W}(L\cap W^\omega)$. Then L_0 is a Lagrangian since

$$\begin{aligned}L_0^\perp &= (J\ell+W)^\perp = J\ell^\perp\cap W^\perp = (\ell+W+JW)\cap W^\perp\\ &= (\ell+JW)\cap W^\perp = \ell+JW = JL_0.\end{aligned}$$

On the other hand, $L+L_0$ is closed and $L\cap L_0=\{0\}$. We will prove this last claim. Take $z=x+y\in(J\ell+W)\cap L$, with $x\in J\ell\subset H_W$ and $y\in W$. Then $z\in L\cap W^\omega = L\cap(H_W\oplus W)$ which also means that $x=P_{H_W}(z)\in\ell$ and since $x\in J\ell$ we conclude that $x=0$ and therefore $z=y\in L\cap W=\{0\}$.

In order to show that $\mathcal{A}_{L_0^\perp}\subset \mathcal{L}\mathrm{ag}_0^W$ notice that if $\tilde L\in\mathcal{A}_{L_0^\perp}$ then $L\cap W\subset \tilde L\cap L_0=\{0\}$. On the other hand $(\tilde L, L_0)$ is a Fredholm pair and L_0 and W are commensurate; hence by Proposition 2.4 $\tilde L+W$ is closed. □

Definition 4.4. Let W be a cofinite, isotropic space. The map

$$\mathcal{R}_W : \mathcal{L}\mathrm{ag}^W \to \mathcal{L}\mathrm{ag}(H_W), \qquad L\mapsto P_{H_W}(L\cap W^\omega)$$

is called *the symplectic reduction* with W.

The symplectic reduction map so defined is not continuous. However, its restriction to each $\mathcal{L}\mathrm{ag}_k^W$ is. In order not to overload the notations we will use the same symbol $\mathcal{R}_W$ also for the restriction maps. We have

Proposition 4.2. *The symplectic reduction is a differentiable map $\mathcal{R}_W : \mathcal{L}\mathrm{ag}_0^W \to \mathcal{L}\mathrm{ag}(H_W)$. In fact, if $L_0\supset W$ is a Lagrangian and $W^\perp$ is the orthogonal complement of W in L_0, then $\mathcal{R}(\mathcal{A}_{L_0^\perp}) = \mathcal{A}_{JW^\perp}$ and in these Arnold coordinates, symplectic reduction is the linear map $\mathcal{R}(T)=P_{W^\perp}\, TP_{JW^\perp}$, i.e., projection onto the $JW^\perp\times W^\perp$ block, for every $T: L_0^\perp\to L_0$.*

Proof. Let $T : L_0^{\perp} \to L_0$ be an operator such that as a map $JW \oplus JW^{\perp} \to W \oplus W^{\perp}$ has the block decomposition:

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}.$$

It is an easy exercise to show that $\mathcal{R}_W(\Gamma_T) = \Gamma_{T_4} \subset JW^{\perp} \oplus W^{\perp}$. □

We want to show that $\mathcal{R}$ is a differentiable map when restricted to $\mathcal{L}\mathrm{ag}_k^W$ as well. But first we need to prove that $\mathcal{L}\mathrm{ag}_k^W$ is a differentiable manifold. From now on we will consider that $W \subset H^-$ is a subspace of the vertical lagrangian of codimension c, therefore cofinite, isotropic. Notice that $\mathcal{L}\mathrm{ag}_k^W \subset \mathcal{L}\mathrm{ag}^-$ for all non-negative integers k and, in fact, Proposition 2.4 implies that

$$\mathcal{L}\mathrm{ag}^- = \bigcup_{k \geq 0} \mathcal{L}\mathrm{ag}_k^W = \mathcal{L}\mathrm{ag}^W .$$

Lemma 4.1. *For every Lagrangian $L \in \mathcal{L}\mathrm{ag}_k^W$ the intersection $L \cap W^{\omega}$ has dimension equal to $k + c = \dim L \cap W + 1/2 \dim H_W$. Moreover $L \cap W^{\omega}$ decomposes orthogonally as*

$$L \cap W^{\omega} = L \cap W \oplus \big(L \cap (V^{\perp} \oplus H_W)\big),$$

where $V^{\perp}$ is the orthogonal complement of $V := L \cap W$ in W.

Proof. The image of the projection $P_{H_W} : L \cap W^{\omega} \to H_W$ has dimension equal to $1/2 \dim H_W$ and the kernel is just $L \cap W$. The two spaces that appear in the sum are orthogonal and subsets of $L \cap W^{\omega}$. It is enough to prove that $L \cap (V^{\perp} \oplus H_W)$ has dimension p. It is not hard to see that

$$L \cap (V^{\perp} \oplus H_W) = L \cap \big(J(L \cap W) \oplus V^{\perp} \oplus H_W\big) =: L \cap W_L^{\omega},$$

where $W_L := J(L \cap W) \oplus V^{\perp}$ is an isotropic space, clean with L and such that $H_{W_L} = H_W$. Hence $P_{H_W} : L \cap W_L^{\omega} \to H_W$ is injective and the image is a lagrangian in H_W, which has dimension c. □

Definition 4.5. For every Lagrangian $L \in \mathcal{L}\mathrm{ag}_k^W$, let $V := L \cap W$, $V^{\perp}$ be the orthogonal complement of V in W and let ℓ be the symplectic reduction of L with W. The space $L_W := \ell \oplus V \oplus JV^{\perp}$ is called the *associated Lagrangian* or simply the *associate.*

Proposition 4.3. (a) *For any $L \in \mathcal{L}\mathrm{ag}_k^W$, the associated Lagrangian, L_W, is in $\mathcal{L}\mathrm{ag}_k^W$. Moreover, every $L \in \mathcal{L}\mathrm{ag}_k^W$ is in the Arnold chart of its associate and it is given by the graph of JS where $S \in \mathcal{S}\mathrm{ym}(L_W)$ has the block decomposition:*

$$\begin{pmatrix} 0 & 0 & S_2^* \\ 0 & 0 & 0 \\ S_2 & 0 & S_4 \end{pmatrix}.$$

(b) *Let $W = V \oplus V^\perp$ be an orthogonal decomposition of W such that V is k-dimensional and let $\ell \subset H_W$ be a Lagrangian. Then $\ell \oplus V \oplus JV^\perp \in \mathcal{L}\mathrm{ag}_k^W$ and the set $\mathcal{L}\mathrm{ag}_k^W \cap \mathcal{A}_{\ell\oplus V\oplus JV^\perp}$ is described in this Arnold chart by linear equations. More precisely, given $S \in \mathcal{S}\mathrm{ym}(\ell \oplus V \oplus JV^\perp)$ then $\Gamma_{JS} \in \mathcal{L}\mathrm{ag}_k^W$ if and only if its $V \times V$ and $V \times \ell$ blocks of S are zero.*

(c) *The space $\mathcal{L}\mathrm{ag}_k^W$ is a submanifold of $\mathcal{L}\mathrm{ag}^-$ of real codimension $k^2 + 2ck$ and the symplectic reduction map:*

$$\mathcal{R} : \mathcal{L}\mathrm{ag}_k^W \to \mathcal{L}\mathrm{ag}(H_W), \qquad L \to \ \mathrm{Range}\, P_{H_W}|_{L\cap W^\omega}$$

is differentiable.

Proof. (a) The fact that the associated Lagrangian is indeed a Lagrangian is straightforward. Now $L_W \cap W = V$, hence clearly L_W is in $\mathcal{L}\mathrm{ag}_k^W$.

For the second claim, notice that $(L, V^\perp)$ is a Fredholm pair and $V^\perp$ and $L_W^\perp$ ar commensurable, so $(L, L_W^\perp)$ is a Fredholm pair. Moreover, the intersection $L \cap L_W^\perp$ is trivial. Indeed let

$$\underset{L}{x} = \underset{J\ell}{a} + \underset{JV}{b} + \underset{V^\perp}{c}.$$

Then $b \in L^\perp$ and so $b = 0$. From $x = a + c$ it follows that $x \in L \cap W^\omega$ and $a = P_{H_W}(x) \in \ell$ so $a = 0$. This implies $x = c \in L \cap V^\perp = \{0\}$.

For the last part notice that if L is the graph of an operator $JS : L_W \to L_W^\perp$ then $JS|_{L\cap L_W} = 0$. Indeed

$$JSv = P_{L_W^\perp} \circ (P_{L_W}|_L)^{-1} v = P_{L_W^\perp} v = 0, \qquad \forall v \in L \cap L_W.$$

On the other hand, $V \subset L \cap L_W$. This and the self-adjointness imply that the middle row and column of S are zero. The vanishing of the top, left block follows from the following considerations. The symplectic reduction of any lagrangian $\tilde{L}$ with $W_L := JV \oplus V^\perp$ in the Arnold chart of L_W is just the graph of the $\ell \times \ell$ block of the self-adjoint operator on L_W that gives $\tilde{L}$. But the operator $\ell \to \ell$ whose graph ℓ is of course the zero operator. We implicitly used the fact that $\ell = \mathcal{R}_{W_L}(L) = \mathcal{R}_W(L)$ which can be easily checked.

(b) Clearly $\ell \oplus V \oplus JV^\perp \in \mathcal{L}ag_k^W$.

Now, every lagrangian in the Arnold chart $\mathcal{A}_{\ell \oplus V \oplus JV^\perp}$ is just the graph of an operator JS where $S \in \mathcal{S}ym\,(\ell \oplus V \oplus JV^\perp)$. So S has a block decomposition

$$S = \begin{pmatrix} S_{\ell,\ell} & S_{V,\ell} & S_{JV^\perp,\ell} \\ S_{\ell,V} & S_{V,V} & S_{JV^\perp,V} \\ S_{\ell,JV^\perp} & S_{V,JV^\perp} & S_{JV^\perp,JV^\perp} \end{pmatrix}.$$

The condition $v + JSv \in W$ where $v = (v_1, v_2, v_3) \in \ell \oplus V \oplus JV^\perp$ implies that the sum

$$\begin{array}{ccc} \underset{\ell}{v_1} + \underset{V}{v_2} + \underset{JV^\perp}{v_3} & + & \underset{J\ell}{(JS_{\ell,\ell}v_1 + JS_{V,\ell}v_2 + JS_{JV^\perp,\ell}v_3)} \\ + \underset{JV}{(JS_{\ell,V}v_1 + JS_{V,V}v_2 + JS_{JV^\perp,V}v_3)} & + & \underset{V^\perp}{(JS_{\ell,JV^\perp}v_1 + JS_{V,JV^\perp}v_2 + JS_{JV^\perp,JV^\perp}v_3)} \end{array}$$

is in $V \oplus V^\perp$. Since $\ell \oplus J\ell \oplus JV \oplus JV^\perp \perp V \oplus V^\perp$ we get

$$\begin{aligned} v_1 = v_3 = (JS_{\ell,\ell}v_1 + JS_{V,\ell}v_2 + JS_{JV^\perp,\ell}v_3) = (JS_{\ell,V}v_1 + JS_{V,V}v_2 \\ + JS_{JV^\perp,V}v_3) = 0 \end{aligned}$$

and

$$v_2 + (JS_{\ell,JV^\perp}v_1 + JS_{V,JV^\perp}v_2 + JS_{JV^\perp,JV^\perp}v_3) \in V \oplus V^\perp.$$

We conclude that in order for $v + JSv$ to be in W one must have $v_2 \in \operatorname{Ker} T$ where $T := (S_{V,\ell}, S_{V,V}) : V \to \ell \oplus V$. Also $\Gamma_{JS} \cap W$ is the graph of the restriction $(JS_{V,JV^\perp}|_{\operatorname{Ker} T})$. The only way the graph of $JS_{V,JV^\perp}|_{\operatorname{Ker} T}$ can have dimension equal to the dimension of V is if $\operatorname{Ker} T = V$, i.e.,

$$(S_{V,\ell}, S_{V,V}) = 0.$$

Hence the intersection $\mathcal{A}_{\ell \oplus V \oplus JV^\perp} \cap \mathcal{L}ag_k^W$ consists of graphs of operators whose $V \times V$, $V \times \ell$ and $\ell \times V$ blocks are zero.

(c) Every Lagrangian $L \in \mathcal{L}ag_k^W$ is in the Arnold chart of its associate which is of the type required by part (b). In these charts $\mathcal{L}ag_k^W$ is described by linear equations and one can very fast see that the codimension is the one indicated.

In the Arnold chart of $\ell \oplus V \oplus JV^{\perp}$ the symplectic reduction with W of any lagrangian $\tilde{L} \in \mathcal{L}\mathrm{ag}_k^W$ is the graph of the projection onto the $\ell \times \ell$ block of the operator that gives $\tilde{L}$ and the differentiability follows. □

We will use the symplectic reduction process to shed some light on the diffeomorphism type of $\mathcal{L}\mathrm{ag}_k^W$.

Definition 4.6. Let $\mathrm{Gr}(k, W)$ denote the Grassmannian of k-dimensional subspaces of W. The *generalized reduction* is the map:

$$\begin{gathered} \mathcal{G} : \mathcal{L}\mathrm{ag}_k^W \to \mathcal{L}\mathrm{ag}(H_W) \times \mathrm{Gr}(k, W), \\ L \to (\mathcal{R}_W(L), L \cap W). \end{gathered}$$

Notice that for $k = 0$ the generalized reduction coincides with the symplectic reduction. We will see that the generalized reduction is a vector bundle projection. First let us notice that $\mathcal{G}$ comes with a natural section called *the associate section* namely

$$\mathcal{S} : \mathcal{L}\mathrm{ag}(H_W) \times \mathrm{Gr}(k, W) \to \mathcal{L}\mathrm{ag}_k^W, \quad (\ell, V) \to \ell \oplus V \oplus JV^{\perp}.$$

Every associate Lagrangian lies on this section.

Theorem 4.1. (a) *The restriction to the associate section of the tangent space of $\mathcal{L}\mathrm{ag}_k^W$, $T\,\mathcal{L}\mathrm{ag}_k^W\big|_{\mathcal{S}}$, can be naturally identified with the vector subbundle of $T\,\mathcal{L}\mathrm{ag}^-\big|_{\mathcal{S}}$ whose fiber at $\ell \oplus V \oplus JV^{\perp}$ consists of self-adjoint operators $S \in \mathcal{S}\mathrm{ym}(\ell \oplus V \oplus JV^{\perp})$ with block decomposition:*

$$S = \begin{pmatrix} S_1 & 0 & S_2^* \\ 0 & 0 & S_3^* \\ S_2 & S_3 & S_4 \end{pmatrix}.$$

(b) *The generalized reduction map is differentiable. Moreover* $(\mathrm{Ker}\ d\mathcal{G})\big|_{\mathcal{S}}$ *can be identified with the vector subbundle of $T\,\mathcal{L}\mathrm{ag}_k^W\big|_{\mathcal{S}}$ whose fiber at $\ell \oplus V \oplus JV^{\perp}$ consists of self-adjoint operators $S \in \mathcal{S}\mathrm{ym}(\ell \oplus V \oplus JV^{\perp})$ with block decomposition.*

$$S = \begin{pmatrix} 0 & 0 & S_2^* \\ 0 & 0 & 0 \\ S_2 & 0 & S_4 \end{pmatrix}. \tag{4.1}$$

(c) *The natural map*

$$\mathcal{N} : (\mathrm{Ker}\ \ d\mathcal{G})\big|_{\mathcal{S}} \to \mathcal{L}\mathrm{ag}_k^W, \qquad \mathcal{N}(\ell \oplus V \oplus JV^\perp, S) = \Gamma_{JS}$$

is a diffeomorphism that makes the diagram commutative

$$\begin{array}{ccc} (\mathrm{Ker}\, d\mathcal{G})\big|_{\mathcal{S}} & \xrightarrow{\quad\mathcal{N}\quad} & \mathcal{L}\mathrm{ag}_k^W \\ & \searrow \qquad \swarrow_{\mathcal{G}} & \\ & \mathcal{L}\mathrm{ag}(H_W) \times \mathrm{Gr}(k, W) & \end{array}.$$

(d) *The space* $\mathcal{L}\mathrm{ag}_k^W$ *is diffeomorphic to* $\mathcal{L}\mathrm{ag}(H_W) \times (\tau^\perp)^p \oplus \mathcal{S}\mathrm{ym}(\tau^\perp)$ *where* $\tau^\perp$ *is the tautological quotient bundle over* $\mathrm{Gr}(k, W)$*.*

Proof. (a) This is obvious since, as in the proof of Proposition 4.3, in the charts centered at $L = \ell \oplus V \oplus JV^\perp$, the manifold $\mathcal{L}\mathrm{ag}_k^W$ can be described exactly as the set of those self-adjoint operators with the claimed block decomposition.

(b) In what concerns differentiability, we only have to prove that the second component, $\mathcal{G}^2$, is differentiable. For that we look again at the proof of Lemma 4.3 where we saw that in the Arnold chart $\mathcal{A}_{\ell \oplus V \oplus JV^\perp}$ the intersection $\Gamma_{JS} \cap W$ is just $\Gamma_{JS_{V,JV^\perp}}$ for every $S \in \mathcal{L}\mathrm{ag}_k^W$.

The second claim is obvious when one works in the Arnold charts centered at $L = \ell \oplus V \oplus JV^\perp$ since then $d_L\mathcal{G}$ is just the projection on the $\ell \times \ell$ and $V \times JV^\perp$ blocks.

(c) We construct an inverse for $\mathcal{N}$. To every $L \in \mathcal{L}\mathrm{ag}_k^W$ we associate the Lagrangian $L_W := \ell \oplus V \oplus JV^\perp$ where $V = L \cap W$ and ℓ is the symplectic reduction with W. By part (a) of the previous proposition, in the Arnold chart centered at L_W, the lagrangian L is the graph Γ_{JS} of an operator S of type:

$$S = \begin{pmatrix} 0 & 0 & S_2^* \\ 0 & 0 & 0 \\ S_2 & 0 & S_4 \end{pmatrix}.$$

The inverse of $\mathcal{N}$ takes $L \in \mathcal{L}\mathrm{ag}_k^W$ to the Lagrangian L_W and the operator S that gives L in the Arnold chart of L_W. By part (b) this is well-defined.

(d) At (b) and (c) we have identified the fiber of $\mathcal{L}\mathrm{ag}_k^W$ over $\mathcal{L}\mathrm{ag}_{H_W} \times \mathrm{Gr}(k, W)$ at (ℓ, V) with the vector space $\mathrm{Hom}(\ell, JV^\perp) \oplus \mathcal{S}\mathrm{ym}\,(JV^\perp)$. We know that the tautological bundle over $\mathcal{L}\mathrm{ag}_k^W$ is naturally trivializable so the bundle with fiber $\mathrm{Hom}(\ell, JV^{\perp})$ over $\mathcal{L}\mathrm{ag}_{H_W} \times \mathrm{Gr}(k, W)$ is naturally isomorphic with the bundle $\mathrm{Hom}(W^{\perp}, JV^{\perp})$ where the lagrangian $W^{\perp} \in \mathcal{L}\mathrm{ag}(H_W)$ is just the orthogonal complement of W in H^-. A choice of a basis on $W^{\perp}$ identifies $\mathrm{Hom}(W^{\perp}, JV^{\perp})$ with $(\tau^{\perp})^p$. □

It is worth having the case $k = 0$ of the previous results stated explicitly

Corollary 4.1. *The symplectic reduction map* $\mathcal{R} : \mathcal{L}\mathrm{ag}_0^W \to \mathcal{L}\mathrm{ag}(H_W)$ *together with the associated* 0*-section*

$$\mathcal{S} : \mathcal{L}\mathrm{ag}(H_W) \to \mathcal{L}\mathrm{ag}_0^W, \qquad \ell \mapsto \ell \oplus JW$$

is diffeomorphic with the vector bundle $\mathrm{Ker}\, d\mathcal{R}\big|_{\mathcal{S}} \to \mathcal{L}\mathrm{ag}(H_W)$.

As an application to symplectic reduction we give a geometric proof of the following important result which first appears in [15].

Theorem 4.2 Kirk–Lesch. *The spaces* $\mathcal{L}\mathrm{ag}^-$ *and* $\mathcal{L}\mathrm{ag}_{\mathcal{J}}^-$ *are weak homotopy equivalent to* $U(\infty)$.

Proof. The proof is no different for $\mathcal{L}\mathrm{ag}^-$ than for $\mathcal{L}\mathrm{ag}_{\mathcal{J}}^-$ and is based on the fact that the symplectic reduction map is a vector bundle. In the case of $\mathcal{L}\mathrm{ag}_{\mathcal{J}}^-$ the operators 4.1 are from the ideal $\mathcal{J}$.

We start by fixing a complete decreasing flag of finite codimensional subspaces of H^-.

$$H^- = W_0 \supset W_1 \supset W_2 \supset \dots .$$

Notice that $\mathcal{L}\mathrm{ag}^{W_i} \subset \mathcal{L}\mathrm{ag}^{W_{i+1}}$ is an inclusion of open subsets of $\mathcal{L}\mathrm{ag}$ for all i and that

$$\bigcup_{i \geq 0} \mathcal{L}\mathrm{ag}^{W_i} = \mathcal{L}\mathrm{ag}^- .$$

Indeed the "$\subset$" inclusion follows by noticing that dim $L \cap H^- \leq$ codim $W_i = i$ and $L + H^- = L + W_i + W_i^{\perp}$ is closed since the orthogonal complement $W_i^{\perp}$ finite dimensional. The "$\supset$" inclusion is a consequence of the observation that the intersection of the decreasing sequence $L \cap W_i$ is zero and of Proposition 2.4 applied to the commensurate pair (H^-, W_i) and the Fredholm pair (L, H^-).

If we can prove that for a fixed k there exists an n big enough such that the pair $(\mathcal{L}\mathrm{ag}^-, \mathcal{L}\mathrm{ag}^{W_n})$ is k-connected than we are done because $\mathcal{L}\mathrm{ag}^{W_n}$ is homotopy equivalent with $\mathcal{L}\mathrm{ag}(H_{W_n})$ which is diffeomorphic by Theorem 3.1 with $U(n)$. This will imply that the induced map

$$U(\infty) \to \mathcal{L}\mathrm{ag}(\infty) := \lim_n \mathcal{L}\mathrm{ag}(H_{W_n}) \longmapsto \lim_n \mathcal{L}\mathrm{ag}^{W_n} = \mathcal{L}\mathrm{ag}^-$$

is a weak homotopy equivalence where the first map is induced by the Cayley graph map $\mathcal{C} : U(n) \to \mathcal{L}\mathrm{ag}(H_{W_n})$.

We have of course that $\mathcal{L}\mathrm{ag}^{W_n} = \mathcal{L}\mathrm{ag}^- \setminus \{L \mid \dim L \cap W_n \geq 1\}$ and the set $\overline{Z}_{n+1} := \{L \mid \dim L \cap W_n \geq 1\}$ is a finite codimensional stratified subset of $\mathcal{L}\mathrm{ag}^-$ whose top stratum, $\mathcal{L}\mathrm{ag}_1^{W_n}$, has codimension $2n+1$. We therefore fix $n > 1/2(k-1)$ and show the induced map on homotopy groups

$$\pi_i(\mathcal{L}\mathrm{ag}^{W_n}) \mapsto \pi_i(\mathcal{L}\mathrm{ag}^-), \qquad \forall i \leq k$$

is an isomorphism.

Every continuous map $\sigma : S^k \to \mathcal{L}\mathrm{ag}^-$ is contained in an open set $\mathcal{L}\mathrm{ag}^{W_N}$ for some $N > n$ big enough so one can deform it to a map $S^k \to \mathcal{L}\mathrm{ag}(H_{W_N}) \hookrightarrow \mathcal{L}\mathrm{ag}^-$ simply by composing with the symplectic reduction which is a deformation retract. The new map $\sigma_1 : S^k \to \mathcal{L}\mathrm{ag}(H_{W_N})$ can be deformed into a smooth map and can also be put into transversal position with $\overline{Z}_{n+1} \cap \mathcal{L}\mathrm{ag}(H_{W_N})$ which is a Whitney stratified set (see Remark 5.5) of codimension $2n+1$ in the finite-dimensional manifold $\mathcal{L}\mathrm{ag}(H_{W_N})$. But for $k < 2n+1$ this means that there is no intersection and hence the resulting map σ_2 has its image in $\mathcal{L}\mathrm{ag}^{W_n}$. This proves the surjectivity of the map on homotopy groups.

The injectivity follows by noting that every map $I \times S^k \to \mathcal{L}\mathrm{ag}^-$ can be deformed to a map $I \times S^k \to \mathcal{L}\mathrm{ag}^{W_n}$ by the same type of argument as before for $2n > k$.

The same proof works for $\mathcal{L}\mathrm{ag}_{\mathcal{J}}^-$. $\square$

Remark 4.2. In the case of the unitary group, symplectic reduction has a nice explicit description as an algebraic map (see [8], Section 2.5).

5. Schubert cells and varieties

The topological structure of the vertical Lagrangian Grassmanian, $\mathcal{L}\mathrm{ag}^-$ is intimately connected with the structure of the finite Lagrangian Grassmanians which are nothing else but the classical unitary groups. A detailed topological study of these spaces has been undertaken by Nicolaescu in [18].

In that paper, the author shows that the Poincaré duals of the generators of the cohomology group of $U(n)$ can be represented by integral currents supported by semialgebraic varieties. That approach is not available in this infinite-dimensional context. However, we have on our side symplectic reduction which translates most of the problems into their finite-dimensional counterpart.

In the proof of Th. 4.2 we introduced a complete, decreasing flag of finite-codimensional subspaces

$$\mathcal{W}: \qquad H^- =: W_0 \supset W_1 \supset W_2 \supset \dots.$$

We now fix an orthonormal basis $\{f_1, f_2, \dots\}$ of H^- such that $W_n^\perp = \langle f_1, f_2, \dots, f_n\rangle$ and we set $e_i := Jf_i$.

To every k-tuple of positive integers $I = \{i_1 < i_2 < \dots < i_k\}$ we associate the following vector subspaces of $\hat{H}$.

$$F_I = \langle f_i \mid i \in I\rangle, F_{I^c} = \overline{\langle f_i \mid i \in I^c\rangle} \text{ and } H_I^+ = F_I \oplus JF_{I^c}$$

Definition 5.1. Let $I = \{i_1 < i_2 < \dots < i_k\}$ be a k-tuple of positive integers. Set $i_0 := 0$ and $i_{k+1} := \infty$. The weight of the k-tuple is the integer:

$$N_I := \sum_{i \in I} (2i - 1)$$

The *Schubert cell* of type I denoted Z_I is a subset of $\mathcal{L}\mathrm{ag}^-$ defined by the following incidence relations with respect to the fixed flag

$$Z_I = \{L \in \mathcal{L}\mathrm{ag}^- \mid \dim L \cap W_j = k - p,\ \forall\, 0 \le p \le k,\ \forall\, j \text{ such that } i_p \le j < i_{p+1}\}$$

Remark 5.1. One way to look at the incidence relations is by thinking that the k-tuple $(i_1, i_2, \dots, i_k)$ records the "nodes" in the flag where the dimension of the intersection with the lagrangian L drops by one. □

Remark 5.2. Notice that the orthogonal complement $W_n^\perp$ of W_n in H^- is naturally a lagrangian in $H_{W_n} := W_n^\perp \oplus JW_n^\perp$ and $W_n^\perp \subset H_{W_n}$ will play the role of the vertical subspace. The flag $W_0 = H^- \supset W_1 \supset W_2$ induces a complete, decreasing flag of $W_n^\perp$:

$$\tilde{W_0} := W_n^\perp \supset \tilde{W_1} := W_1/W_n \supset \ldots \supset \tilde{W_n} := W_n/W_n = \{0\}$$

We let $Z_I(n)$ be the Schubert cell in $\mathcal{L}\mathrm{ag}(H_{W_n})$ described by the same incidence relations as the sets Z_I above with $\tilde{W}_i$ replacing W_i and $H_I^+(n)$ be the space corresponding to H_I^+. □

The following description of Schubert cells proves that they are actually Banach spaces when regarded in the right charts.

Proposition 5.1. *The Schubert cell Z_I is a vector subspace of codimension N_I in the Arnold chart $\mathcal{A}_{H_I^+}$. More precisely, $\Gamma_{JA} \in \mathcal{A}_{H_I^+} \cap Z_I$ if and only if the bounded self-adjoint operator satisfies the linear equations*

$$\begin{aligned} \langle Af_i, f_j \rangle &= 0, \quad \forall\, j \leq i,\ i,j \in I, \\ \langle Af_i, e_j \rangle &= 0, \quad \forall\, j \leq i,\ i \in I,\ j \in I^c. \end{aligned}$$

Proof. We will show first that $Z_I \subset \mathcal{A}_{H_I^+}$. Let $L \in Z_I$. Notice that (L, H_I^-) is a Fredholm pair by Proposition 2.4 since H_I^- is commensurate with H^-.

We will show that $L \cap H_I^- = \{0\}$ thus proving that $L = \Gamma_{JA} \in \mathcal{A}_{H_I^+}$ with $A \in \mathcal{S}\mathrm{ym}\, H_I^+$.

Let us remark that $L \cap F_{I^c} = \{0\}$ because otherwise the dimension of $L \cap W_j$ would drop at "nodes" other than $i_1, i_2, \ldots i_k$, (take $v = \sum_{j \in I^c} a_j f_j \in L \cap F_{I^c}$ with $p = \min\{j \in I^c \mid a_j \neq 0\}$ then $v \in L \cap W_{p-1} \setminus L \cap W_p$). This is saying that $L \cap H^-$ is the graph of an operator $T : F_I \to F_{I^c}$.

To see that $L \cap H_I^- = \{0\}$, let $x = v_1 + v_2 \in L \cap JF_I \oplus F_{I^c}$. Then $Jx \in L^\perp$ and so $\langle Jx, w + Tw \rangle = 0$, for all $w \in F_I$. This implies

$$\langle Jv_1, w \rangle = 0, \quad \forall\, w \in F_I.$$

We get $v_1 = 0$ and so $x = v_2 \in L \cap F_{I^c} = 0$, thus finishing the proof that $L = \Gamma_{JA} \in \mathcal{A}_{H_I^+}$.

Let now $A \in \mathcal{S}\mathrm{ym}\,(H_I^+)$ such that $\Gamma_{JA} \in Z_I$ and let

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

be the block decomposition of A relative to $H_I^+ = F_I \oplus JF_{I^c}$.

One checks immediately that the intersection $\Gamma_{JA} \cap H^-$ is just the graph of the restriction $JA_3|_{\mathrm{Ker}\, A_1}$ which has the same dimension as $\mathrm{Ker}\, A_1 \subset F_I$.

Since F_I has dimension k, one concludes that

$$\dim \Gamma_{JA} \cap H^- = k \iff A_1 = 0 \iff \langle Af_i, f_j\rangle = 0,\ \forall\, j \leq i,\ i,j \in I.$$

To prove the rest of the relations, i.e., $\langle Af_i, e_j\rangle = 0$, $\forall i \in I,\ j \in I^c,\ j \leq i$ we observe first that

$$\Gamma_{JA} \cap H^- = \Gamma_{JA_3} \text{ and } \langle Af_i, e_j\rangle = -\langle JA_3 f_i, f_j\rangle.$$

The graph of $JA_3 = T : F_I \to F_{I^c}$ satisfies the incidence relations if and only if the required coefficients vanish, otherwise we would have dimension drops at the wrong places again. □

Remark 5.3. For every two-sided symmetrically normed ideal $\mathcal{J}$ we can define $Z_I(\mathcal{J}) = Z_I \cap \mathcal{L}\mathrm{ag}^-_{\mathcal{J}}$. Since the next results are true for Z_I, as well as for $Z_I(\mathcal{J})$ making only the minimal changes, we choose to work with Z_I to keep the indices to a minimum.

Notice that $Z_I \subset \mathcal{L}\mathrm{ag}^{W_n}$ for all $n \geq \max\{i \mid i \in I\}$ so we could look at the symplectic reduction of Z_I. We record the obvious:

Lemma 5.1. *For $n \geq \max\{i \,|\, i \in I\}$ the symplectic reduction $\mathcal{R} : \mathcal{L}\mathrm{ag}^{W_n} \to \mathcal{L}\mathrm{ag}(H_W)$ takes $Z_I \subset \mathcal{L}\mathrm{ag}^{W_n}$ to $Z_I(n)$. The stronger $\mathcal{R}^{-1}(Z_I(n)) = Z_I$ is also true.*

Proof. For $n \geq \max\{i \mid i \in I\}$ we have that $F_I \subset W_n^{\perp}$ and so $\mathcal{A}_{H_I^+} \subset Lag^{W_n}$. The reduction with W_n of the Arnold chart centered at H_I^+ is the Arnold chart centered at $H_I^+(n)$. The reduction in the Arnold chart being just the projection, the lemma easily follows. □

Definition 5.2. For every k-tuple $I = \{i_1, i_2, \ldots, i_k\}$ the *Schubert variety* is the closure of Z_I in $\mathcal{L}\mathrm{ag}^-$, denoted $\overline{Z}_I$.

Lemma 5.2. *The Schubert variety $\overline{Z}_I$ can be described by the following incidence relations:*

$$\overline{Z}_I = \{L \in \mathcal{L}\mathrm{ag}^- \mid \mathit{dim}\ L \cap W_j \geq k - p,\ \forall\ 0 \leq p \leq k,\ \forall\ j \text{ such that } i_p \leq j < i_{p+1} \text{ where } i_0 = 0, i_{k+1} = \infty \text{ and } i_p \in I,\ \forall\ 1 \leq p \leq k\}.$$

Proof. The fact that the closure is included in the right-hand side is a consequence of the upper semi-continuity of the functions:

$$L \to \dim L \cap W_j, \qquad \forall j \geq 0.$$

Conversely, let us notice that for n big enough we have the following obvious equalities $\mathcal{L}\mathrm{ag}^{W_n} \cap \overline{Z}_I = \mathbf{cl}_n(Z_I) = \mathcal{R}^{-1}(\overline{Z_I(n)})$ where $\mathbf{cl}_n(Z_I)$ is the closure of Z_I in $\mathcal{L}\mathrm{ag}^{W_n}$. Now, a lagrangian that satisfies the incidence relations in the lemma is in some $\mathcal{L}\mathrm{ag}^{W_n}$ and its reduced space will satisfy the same incidence relations with respect to the flag $\tilde{W} \supset \tilde{W}_1 \supset \ldots \supset \tilde{W}_n$. But this means it is in $\overline{Z_I(n)}$, since the finite version of the lemma is true by results from [18], namely Prop. 4.3, 4.4 and 4.6. □

Remark 5.4. The Schubert variety $\overline{Z}_I$ is not included in any of the clean sets $\mathcal{L}\mathrm{ag}^{W_n}$. However, the intersection has a very simple description: $\overline{Z}_I \cap Lag^{W_n} = \mathcal{R}^{-1}(\overline{Z}_I(n))$.

We can now describe the strata in the Schubert variety $\overline{Z}_I$. Notice first that if $Z_J \subset \overline{Z}_I$ then $|J| \geq |I|$ since $|J| = \dim L \cap H^-$ for every $L \in Z_J$. Say $J = \{j_1 < j_2 < \cdots < j_l\}$ and $I = \{i_1 < i_2 < \cdots < i_k\}$ with $l \geq k$. We deduce that $i_1 \leq j_{l-k+1}$ since j_{l-k+1} records the node where the dimension of the intersection of $L \in Z_J$ with the flag drops to $k-1$ and similarly $i_s \leq j_{l-k+s}$ for all $1 \leq s \leq k$. We record this:

Lemma 5.3. (a) *If $Z_J \subset \overline{Z}_I$ then $|J| = l \geq k = |I|$ and $i_s \leq j_{l-k+s}$ for all $1 \leq s \leq k$.*

(b) *If $Z_J \subset \overline{Z}_I$ has codimension $N_I + 1$ in $\mathcal{L}\mathrm{ag}^-$ where $N_I = \sum_{i \in I}(2i-1)$ is the codimension of Z_I in $\mathcal{L}\mathrm{ag}^-$ then $|J| = k+1$, $j_1 = 1$ and $j_{s+1} = i_s$ for all $1 \leq s \leq k$.*

Corollary 5.1. *The fundamental Schubert variety $\overline{Z}_n$ can be described by the simple incidence relation:*

$$\overline{Z}_n = \{L \mid \dim(L \cap W_{n-1}) \geq 1\}.$$

Proof. Let $J = \{j_1, \ldots, j_l\}$, $L \in Z_J$ and $Z_J \subset \overline{Z}_n$. The previous lemma tells us that $j_l \geq n$ and so the node where the dimension of the intersection of L with the flag drops to 0 is bigger than $n-1$. This proves the "$\subset$" inclusion. The other inclusion is obvious. □

The previous corollary is key to dealing with the singularities in codimension 1 of $\overline{Z}_n$. It says that the following inclusions hold:

$$Z_n \subset \mathcal{L}\mathrm{ag}_1^{W_{n-1}} \subset \overline{Z}_n = \overline{\mathcal{L}\mathrm{ag}_1^{W_{n-1}}}.$$

In other words, we can find a new stratification of $\overline{Z}_n$ in which the top stratum is $\mathcal{L}\mathrm{ag}_1^{W_{n-1}}$ and there aren't any singularities in codimension 1. We will do this in more detail in Section 8. One can play the same game with the Schubert varieties $\overline{Z}_I$. Indeed, let $I := \{i_1 < i_2 < \cdots < i_k\}$ and let $J^I := I - i_1 + 1$ be two k-tuples of positive integers. For every set J with $|J| = k$ (not necessarily of the previous type) and every complete, decreasing flag $W =: V_0 \supset V_1 \supset \cdots$ of a finite codimensional, isotropic space W define

$$\begin{aligned}\mathrm{Sch}_J^W := \{V \in \mathrm{Gr}(k, W) \mid \dim L \cap V_s = k - p,\ \forall 0 \leq p \leq k,\ \text{such that}\\ j_p \leq s < j_{p+1}\}\end{aligned}$$

to be the "standard" Schubert cell in $\mathrm{Gr}(k, W)$ of complex codimension $\sum_p j_p - p$. Let

$$\mathcal{L}\mathrm{ag}_I^W := (\mathcal{G}_2)^{-1}(\mathrm{Sch}_{J^I}^W)$$

be a subset of $\mathcal{L}\mathrm{ag}^-$, where $\mathcal{G}_2$ is the second component of the symplectic reduction. Clearly $\mathcal{L}\mathrm{ag}_I^W$ is a manifold. Moreover, the following inclusions hold:

$$Z_I \subset \mathcal{L}\mathrm{ag}_I^{W_{i_1-1}} \subset \overline{Z}_I,$$

where the flag in W_{i_1-1} is, obviously, $W_{i_1} \supset W_{i_1+1} \supset \ldots$. The manifolds $\mathcal{L}\mathrm{ag}_I^W$ are the top strata in a stratification of Z_I without singularities in codimension 1.

Remark 5.5. In the proof of Theorem 4.2 we claimed that $\overline{Z}_{n+1}(N)$ is a Whitney stratified subset of $\mathcal{L}\mathrm{ag}(H_{W_N})$ of codimension $2n+1$, where $N > n$. The codimension assertion follows from noting that $\overline{Z}_{n+1}(N)$ is the closure of $\mathcal{L}\mathrm{ag}_1^{W_n}$ which has codimension $2n+1$. The fact that the stratification is Whitney is because $\overline{Z}_{n+1}(N)$ is a semi-algebraic orbit of a certain subgroup of the symplectic group (see Section 3 in [18]).

6. The cohomology ring and geometrical representatives

Our plan is to define geometrical representatives for certain cohomology classes of $\mathcal{L}\mathrm{ag}^-$. For the infinite-dimensional framework, we are dealing with the sheaf theoretic methods are very general and efficient. In the case of $\mathcal{L}\mathrm{ag}^-$

one could, in principle, use some ad-hoc method to define these cohomology classes without appeal to sheaf theory. However, we prefer to summarize in an appendix the general principles which can be of use in other situations.

Notice that by fixing a decreasing flag:

$$H^- \supset W_1 \supset W_2 \cdots ,$$

we have canonical inclusions

$$\mathfrak{L}\mathrm{ag}(H_{W_n}) \hookrightarrow \mathfrak{L}\mathrm{ag}(H_{W_{n+1}}) \hookrightarrow \mathfrak{L}\mathrm{ag}^- .$$

Proposition 6.1. (a) *The inclusion map*

$$i : \mathfrak{L}\mathrm{ag}(H_{W_n}) \hookrightarrow \mathfrak{L}\mathrm{ag}^-, \qquad L \to L + JW_n$$

induces an isomorphism of cohomology groups

$$H^q(\mathfrak{L}\mathrm{ag}^-, \mathbb{Z}) \simeq H^q(\mathfrak{L}\mathrm{ag}(H_{W_n}), \mathbb{Z})$$

for $q \leq 2n - 1$.

(b) *Let* I *be an indexing set. Suppose* $z_i \in H^*(\mathfrak{L}\mathrm{ag}^-, \mathbb{Z})$ *with* $i \in I$ *are cohomology classes such that for each* n *the set of pull-backs* $z_i(n) \in H^*(\mathfrak{L}\mathrm{ag}(H_{W_n}), \mathbb{Z})$ *generate the cohomology ring of* $\mathfrak{L}\mathrm{ag}(H_{W_n})$*. Then* $(z_i)_{i \in I}$ *generate the cohomology ring of* $\mathfrak{L}\mathrm{ag}^-$*.*

Proof. (a) By Corollary 4.1 we have that the inclusion induces a homotopy equivalence of $\mathfrak{L}\mathrm{ag}(H_{W_n})$ with $\mathfrak{L}\mathrm{ag}^{W_n} = \mathfrak{L}\mathrm{ag}^- \setminus \overline{Z}_{n+1}$.

On the other hand, $\overline{Z}_{n+1}$ is a stratified subset whose top stratum has codimension $2n + 1$ hence by the extension property, see Propositions A.2 and A.4, the natural map

$$H^q(\mathfrak{L}\mathrm{ag}^-) \to H^q(\mathfrak{L}\mathrm{ag}^- \setminus \overline{Z}_{n+1})$$

is an isomorphism.

(b) It follows from (a). □

Remark 6.1. From now on we will identify $\mathfrak{L}\mathrm{ag}(H_{W_n})$ with the unitary group $U(n)$ via the Cayley graph map. Technically speaking, we only have a *canonical* isomorphism between $\mathfrak{L}\mathrm{ag}(H_{W_n})$ and $U(JW_n^{\perp})$ (where $JW_n^{\perp}$ is the horizontal subspace of H_{W_n}) given by the Arnold theorem and an identification of $U(JW_n^{\perp})$ with $U(n)$ via a *non-canonical* unitary map $JW_n^{\perp} \simeq \mathbb{C}^n$.

From a cohomological point of view it does not matter what this non-canonical unitary map is since any other choice will induce the same isomorphism

$$H^*(U(JW_n^{\perp})) \simeq H^*(U(n)),$$

simply because every unitary map on a vector space is homotopic to the identity. □

Following [18], the groups $U(n)$ have canonically defined cohomology classes $x_i \in H^{2i-1}(U(n), \mathbb{Z})$. On the product $S^1 \times U(n)$ there is a rank n (universal) complex vector bundle E_n. The bundle is obtained by modding out a $\mathbb{Z}$-action on the $\mathbb{Z}$-equivariant bundle

$$\mathbb{R} \times U(n) \times \mathbb{C}^n \to \mathbb{R} \times U(n).$$

The action on the total space is given by

$$k(t, U, v) := (t + k, U, U^k v), \quad \forall (t, U, v) \in \mathbb{R} \times U(n) \times \mathbb{C}^n, \quad k \in \mathbb{Z},$$

whereas on the base space, $\mathbb{Z}$ acts in the obvious way on the $\mathbb{R}$ component. The classes x_i are transgressions of the Chern classes of E_n, i.e.,

$$x_i(n) := \int_{S^1} c_i(E_n)$$

The classes $x_i(n) \in H^{2i-1}(U(n))$, $1 \leq i \leq n$, generate the cohomology ring of $U(n)$, i.e.,

$$H^*(U(n), \mathbb{Z}) \simeq \Lambda(x_1, \ldots, x_n).$$

Notice that via the canonical inclusion

$$S^1 \times U(n) \hookrightarrow S^1 \times U(n+1)$$

the bundle E_{n+1} pulls-back to give a bundle isomorphic to $E_n \oplus \underline{\mathbb{C}}$. Therefore, the class $x_i(n+1)$ pulls back to $x_i(n)$.

This compatibility with the natural inclusions of the classes $x_i(n)$ prompts the following definition.

Definition 6.1. The fundamental *transgression classes* on $\mathfrak{L}\mathrm{ag}^-$ are the unique cohomology classes $z_i \in H^{2i-1}(\mathfrak{L}\mathrm{ag}^-)$ that pull-back to the classes

$x_i(n) \in H^{2i-1}(U(n))$ via the induced map

$$U(n) \xhookrightarrow{i} \mathcal{L}\mathrm{ag}^- , \qquad x_i(n) = i^*(z_i),$$

where i is the composition of the natural inclusion $\mathcal{L}\mathrm{ag}(n) \hookrightarrow \mathcal{L}\mathrm{ag}^-$ with the Cayley graph diffeomorphism.

For every set of positive integers $I = \{i_1, \ldots, i_k\}$, define the product class $z_I \in H^{N_I}(\mathcal{L}\mathrm{ag}^-)$ to be the cup product of fundamental transgression classes:

$$z_I = z_{i_1} \wedge z_{i_2} \wedge \ldots \wedge z_{i_k}.$$

We now turn to the Schubert varieties $\overline{Z}_I$. In Appendix A, we describe how one can define a cohomology class when dealing with a cooriented, stratified space without singularities in codimension 1. We summarize the main definitions and procedures.

Definition 6.2. Let X be a Banach manifold. A *quasi-submanifold* of X of codimension c is a closed subset $F \subset X$ together with a decreasing filtration by closed subsets

$$\boldsymbol{F} : \qquad F = F^0 \supset F^1 \supset F^2 \supset F^3 \subset \cdots$$

such that the following hold.

(i) $F^1 = F^2$.

(ii) The strata $S^k = F^k \setminus F^{k+1}$, are submanifolds of X of codimension $k + c$.

The quasi-submanifold is called *coorientable* if S^0 is coorientable. A coorientation of a quasi-submanifold is then a coorientation of its top stratum.

The main ingredients to define a cohomology class out of a coorientable quasi-submanifold are:

- A Thom isomorphism of the top dimensional stratum S^0, which is a submanifold and closed subset of $X \setminus F^2$.

$$H^0(S^0) \simeq H^c(X \setminus F^2, X \setminus F^0).$$

This depends on the choice of a coorientation.

- An extension isomorphism in cohomology, over the singular stratum F^2, which exists because F^2 has codimension at least two bigger than S^0.

$$H^c(X) \to H^c(X \setminus F^2).$$

The cohomology class determined by the triple $(F, \boldsymbol{F}, \omega)$ is the image of $1 \in H^0(S^0)$ via the composition:

$$H^0(S^0) \simeq H^c(X \setminus F^2, X \setminus F^0) \to H^c(X \setminus F^2) \simeq H^c(X)$$

and is denoted $[F, \omega]$. Although this class depends on the stratification $\boldsymbol{F}$ (see Remark A.4) we will not make this fact explicit in the notation.

Notation: $$Z_I^\circ := Z_I \cup Z_{I\cup 1}, \qquad \partial Z_I := \overline{Z}_I \setminus Z_I^\circ.$$

Definition 6.3. The *standard filtration* on $\overline{Z}_I$ is the following filtration.

$$F^0 := \overline{Z}_I \; ; \qquad F^1 = F^2 := \partial Z_I \; ; \qquad F^k := \bigcup_{\substack{Z_J \subset \overline{Z}_I \\ N_J \geq N_I + k}} Z_J.$$

Theorem 6.1. *The standard stratification on the Schubert variety $\overline{Z}_I$ turns it into a coorientable quasi-submanifold of $\mathcal{L}\mathrm{ag}^-$ of codimension N_I. There exists a canonical choice of a coorientation ω_I on the top stratum such that the following equality of cohomology classes holds*

$$[\overline{Z}_I, \omega_I] = z_I.$$

Before we go into the proof, a short digression on the results of [18] is necessary. In that article, Nicolaescu uses the theory of analytic currents to build out of the finite-dimensional Schubert variety, $\overline{Z}_I(n) \subset \mathcal{L}\mathrm{ag}(n)$, endowed with an orientation, a homology class. He shows that this class is Poincare dual to the class $x_I(n) \in H^{N_I}(\mathcal{L}\mathrm{ag}(n))$.

We summarize the main results:

Proposition 6.2. *The sets $Z_I(n)^\circ$ are orientable, smooth, subanalytic manifolds of codimension N_I in $\mathcal{L}\mathrm{ag}(n)$.*

Proof. See [18], Lemma 5.7. Alternatively, $Z_I(n)^\circ$ is open in the manifold $\mathcal{L}\mathrm{ag}_I^{W_{i_1-1}}(n)$ (see the end of the previous section). In the next section we prove, via a different method, the fact that $\mathcal{L}\mathrm{ag}_1^{W_k} \supset Z_k^\circ$ is naturally cooriented. □

Proposition 6.3. *The closed set $\overline{Z}_I(n)$ with the canonical orientation ω_I is an analytical cycle and so it defines a homology class in $H_{n^2-N_I}(U(n))$ which is Poincare dual to $x_I(n)$.*

Proof. See [18], Theorem 6.1. □

Proof of Theorem 6.1: We consider $n \geq 1/2(N_I+1)$ and we look at the symplectic reduction $\mathcal{R}_n : \mathcal{L}\mathrm{ag}^{W_n} \to \mathcal{L}\mathrm{ag}(H_{W_n})$. We have $n \geq \max\{i \in I\}$ and so $Z_I^\circ \subset \mathcal{L}\mathrm{ag}^{W_n}$. By Lemma 5.1 we have that $\mathcal{R}_n^{-1}(Z_I^\circ(n)) = Z_I^\circ$. Since $\mathcal{R}_n$ is a vector bundle, the normal bundle to Z_I° is canonically isomorphic with the pull-back via $\mathcal{R}_n$ of the normal bundle of $Z_I(n)^\circ$. Hence it induces an orientation. With this coorientation on $\overline{Z}_I$ we get a cohomology class $[\overline{Z}_I, \omega_I] \in H^{N_I}(\mathcal{L}\mathrm{ag}^-)$. By Proposition 6.1 this class is uniquely determined by its restriction to $\mathcal{L}\mathrm{ag}(H_{W_n})$.

Now, the inclusion map $i : \mathcal{L}\mathrm{ag}(H_{W_n}) \hookrightarrow \mathcal{L}\mathrm{ag}^-$ is transversal to $\overline{Z}_I$ and the following set equality holds:

$$i^{-1}(\overline{Z}_I) = \overline{Z}_I(n).$$

To see why it is transversal notice that the image of i is the zero section, i.e., $\mathcal{S}(\mathcal{L}\mathrm{ag}(H_{W_n}))$, of the symplectic reduction. It is therefore enough to prove the transversality of $\mathcal{S}(\mathcal{L}\mathrm{ag}(H_{W_n}))$ with $\overline{Z}_I \cap \mathcal{L}\mathrm{ag}^{W_n}$ (which is a stratified set with a finite number of strata). This is true because of Lemma 5.1.

By Proposition A.5 we have

$$i^*([\overline{Z}_I, \omega_I]) = [\overline{Z}_I(n), \omega_I(n)] \in H^{N_I}(\mathcal{L}\mathrm{ag}(H_{W_n})).$$

By Nicolaescu's results the class on the right equals $x_I(N)$. Since in $\mathcal{L}\mathrm{ag}^-$ there is another class that restricts to $x_I(N)$, namely z_I we get the desired equality. □

Remark 6.2. In the proof of the previous theorem we used Nicolaescu's results in a form slightly different than stated in Proposition 6.3. We used the *cohomology* class represented by a stratified set together with an induced coorientation. It is clear that an coorientation on $\mathcal{Z}_I(n)^\circ$ induces a orientation on the same space by the normal-bundle-first convention. An oriented quasi-submanifold C determines a homology class which is Poincaré dual to the cohomology class induced by the same quasi-submanifold. Moreover, as shown by Hardt [11, 12], there exists an isomorphism of the Borel–Moore homology groups of X with the analytic currents homology that takes the geometric class determined by the quasi-submanifold C to the current C induces.

Definition 6.4. The triple composed of the Schubert variety $\overline{Z}_I$ with the standard filtration and the coorientation ω_I is called the Schubert cocycle or the geometric representative of z_I. The cohomology class it represents is denoted $[\overline{Z}_I, \omega_I]$.

We consider now families of vertical, Fredholm Lagrangians.

Definition 6.5. A map $F : M \to \mathfrak{L}\mathrm{ag}^-$ is said to be (standard) transversal to $\overline{Z}_I$ if it is transversal to every stratum in the standard stratification.

Lemma 6.1. *Any smooth family $F : M \to \mathfrak{L}\mathrm{ag}^-$ can be deformed by a smooth homotopy to a family transversal to $\overline{Z}_k$.*

Proof. Since M is compact, transversality with $\overline{Z}_I$ means actually transversality of the reduced family with $\overline{Z}_I$ for n big enough. Transversality with Whitney stratified spaces is an open, dense condition in the space of all smooth maps $G : M \to \mathfrak{L}\mathrm{ag}(n)$. □

Proposition 6.4. *Let M be a closed oriented manifold and let $F : M \to \mathfrak{L}\mathrm{ag}^-$ be a family transversal to $\overline{Z}_I$. Then $F^{-1}(\overline{Z}_I)$ is quasi-submanifold of M with a naturally induced coorientation $F^*\omega_I$ and*

$$[F^{-1}(\overline{Z}_I), F^*\omega_I] = F^*[\overline{Z}_I, \omega_I].$$

Proof. This is just Proposition A.5. The pull-back of the normal bundle to Z_I° is naturally isomorphic with the normal bundle to $F^{-1}(Z_I^\circ)$ and the coorientation $F^*\omega_I$ is the one induced via this isomorphism. □

In the infinite-dimensional context, Poincaré Duality does not make sense. Instead we aim for an expression of Poincaré Duality for families of lagrangians parametrized by a closed, oriented manifold M. In Appendix A, we show that when the manifold M is compact then any oriented quasi-submanifold $F \subset M$ of dimension d determines a homology class $\lfloor F \rfloor \in H_d(M)$ which is Poincaré dual to the cohomology class $[F] \in H^{n-d}(M)$ induced by the obvious coorientation. Combining this with the previous proposition we get.

Theorem 6.2. *Let $F : M \to \mathfrak{L}\mathrm{ag}^-$ be a smooth map from an oriented, closed manifold M of dimension n to $\mathfrak{L}\mathrm{ag}^-$. Suppose F is transversal to $\overline{Z}_I$. Then the preimage $F^{-1}(\overline{Z}_I)$ has a naturally induced orientation and so it defines a homology class $\lfloor F^{-1}(\overline{Z}_I) \rfloor \in H_{n-N_I}(M)$ which is Poincaré dual to the class $F^*[\overline{Z}_I]$.*

7. The odd Chern character

Let M be a finite, CW-complex, hence compact. The Chern character is a ring homomorphism:

$$\mathrm{ch} : K^0(M) \to H^{\mathrm{even}}(M, \mathbb{Q}).$$

The suspension isomorphism, which is actually taken to be the definition of K^{-1}, helps us extend the Chern character to the odd component:

$$\begin{array}{ccc} \tilde{K}^{-1}(M) & \overset{\Sigma}{=\!=\!=} & \tilde{K}^0(\Sigma M) \\ \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} \\ \tilde{H}^{\mathrm{odd}}(M, \mathbb{Q}) & \xrightarrow{\Sigma} & \tilde{H}^{\mathrm{even}}(\Sigma M, \mathbb{Q}) \end{array}$$

It is well known that $U(\infty)$ is a classifying space for K^{-1}. Hence, every element in $\tilde{K}^{-1}(M)$ can be represented by the homotopy class of a (pointed) map $f : M \to U(\infty)$. Let $[f] \in K^{-1}(M)$ be the element this map represents. Then $\Sigma f : \Sigma M \to \Sigma U(\infty)$ represents an element in $\tilde{K}^0(\Sigma M)$ which corresponds to f via the suspension isomorphism. The previous commutative diagram can be written as

$$\Sigma \circ \mathrm{ch}\,[f] = \mathrm{ch}\,([\Sigma f]). \tag{7.1}$$

A short digression is necessary at this point. The space $\Sigma U(\infty)$ comes with a principal $U(\infty)$-bundle $\breve{U}$, namely the one obtained via the clutching map given by the identity. More precisely, one starts with the trivial $U(\infty)$ bundle over $[0,1] \times U(\infty)$ and identifies $(0, U, g)$ with $(1, U, Ug)$ for all $(U, g) \in U(\infty) \times U(\infty)$. This is an old acquaintance of ours. Indeed the pull-back of this bundle to $\Sigma U(n)$ is nothing else but the frame bundle associated to the vector bundle E_n, which we considered in Section 6.

Another way of looking at these bundles is via the periodicity map (see [22], pp. 224–225)

$$\Sigma U(n) \to \mathrm{Gr}(n, 2n) \hookrightarrow \mathrm{Gr}(n, \infty) \simeq BU(n),$$

where the first map is given explicitly as follows:

$$[0, \pi] \times U(n) \to \mathrm{Gr}(n, 2n), \quad (t, U) \to \cos t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \sin t \begin{pmatrix} 0 & U^{-1} \\ U & 0 \end{pmatrix}.$$

The right-hand side is an involution of $\mathbb{C}^n \oplus \mathbb{C}^n$. The bundle E_n is the pull-back of the universal $U(n)$vector bundle $EU(n)$. In the same way $\breve{U}$ comes from the universal $U(\infty)$-bundle over $BU(\infty)$.

Now every continuous map $f : M \to U(\infty)$ defined on a compact set M is homotopy equivalent with a map (which we denote by the same paper) $f : M \to U(n)$. The class $[f] \in K^{-1}(M)$ or the class $[\Sigma f] \in \tilde{K}^0(\Sigma M)$ can be represented by the bundle $(\Sigma f)^* E_n$ (which determines a stable isomorphism class). Using equation (7.1) we get that

$$\mathrm{ch}\,[f] = \Sigma^{-1}\,\mathrm{ch}((\Sigma f)^* E_n)) = \Sigma^{-1}((\Sigma f)^*\,\mathrm{ch}\,E_n). \tag{7.2}$$

The inverse of the suspension isomorphism Σ is easy to describe. It is the composition

$$\tilde{H}^{\mathrm{even}}(\Sigma M, \mathbb{Z}) \xrightarrow{\pi^*} \tilde{H}^{\mathrm{even}}(S^1 \times M, \mathbb{Z}) \xrightarrow{/dt} H^{\mathrm{even}-1}(M, \mathbb{Z}),$$

where $\pi : S^1 \times M \to \Sigma M$ stands for the projection and $/dt$ stands for the slant product with the fundamental class of S^1. So

$$\begin{aligned}\Sigma^{-1}((\Sigma f)^*\,\mathrm{ch}\,E_n) &= (\pi^*(\Sigma f)^*\,\mathrm{ch}\,E_n)/dt = ((\Sigma f \circ \pi)^*\,\mathrm{ch}\,E_n)/dt \\ &= ((\mathrm{id}_{S^1} \times f)^*\,\mathrm{ch}\,E_n)/dt = f^*(\mathrm{ch}\,E_n/dt).\end{aligned} \tag{7.3}$$

The class $\mathrm{ch}\,E_n/dt \in H^{\mathrm{odd}}(U(n), \mathbb{Q})$ is called the *transgression* class of the Chern character. Of course, one can do slant product componentwise and get, for each positive integer k a class:

$$\mathrm{ch}^{\tau}_{2k-1} := \mathrm{ch}_{2k}\,(E_n)/dt \in H^{2k-1}(U(n), \mathbb{Q}).$$

There is nothing special about the Chern character. The same transgression process can be applied to any characteristic class of E_n, in particular to the Chern classes and we have already done this in Section 6 where we denoted those classes by x_i. We use a different notation now which is more appropriate to this context,

$$c^{\tau}_{2k-1} := c_k(E_n)/dt \in H^{2k-1}(U(n), \mathbb{Z}).$$

There is a very simple relation between $\mathrm{ch}^{\tau}_{2k-1}$ and c^{τ}_{2k-1}:

Lemma 7.1.

$$\mathrm{ch}^{\tau}_{2k-1} = \frac{(-1)^{k-1}}{(k-1)!} c^{\tau}_{2k-1}.$$

Proof. First of all, $\operatorname{ch}_{2k}(E_n) \in \tilde{H}^{2k}(S^1 \times U(n), \mathbb{Q})$ is a polynomial in the variables $c_1(E_n), \ldots, c_k(E_n)$ and the coefficient of $c_k(E_n)$ is $(-1)^{k-1}/(k-1)!$. On the other hand, every element in $H^{2k}(S^1 \times U(n), \mathbb{Z})$ is a sum:

$$z = x + Dt \wedge y,$$

where $x \in H^{2k}(U(n), \mathbb{Z})$, $y \in H^{2k}(U(n), \mathbb{Z})$ and $Dt \in H^1(S^1, \mathbb{Z})$ satisfies $Dt(dt) = 1$. We claim that for every characteristic class of E_n its $H^{2k}(U(n), \mathbb{Z})$ component vanishes. Indeed the class x is the pull-back of z via the inclusion $\{1\} \times U(n) \to S^1 \times U(n)$ and the claim follows by noticing that the pull-back of the bundle E is trivial over $U(n)$.

We conclude that the cup product of any two characteristic classes of E_n is zero and so we have

$$\operatorname{ch}_{2k}(E_n) = \frac{(-1)^{k-1}}{(k-1)!} c_k(E_n),$$

which after taking the slant product gives the identity we were after. □

Suppose now that M is a closed, oriented manifold and $f : M \to \mathcal{L}\mathrm{ag}^-$. Theorem 6.1 is saying that the pull-back $f^* x_k = f^*[\overline{Z}_k, \omega_k]$. On the other hand, by the previous lemma, relations (7.2), (7.3) and Proposition 6.4 we have the following result:

Proposition 7.1. *Let M be a closed manifold and let $f : M \to \mathcal{L}\mathrm{ag}^-$ be a smooth map transversal to $\overline{Z}_k$. The following holds:*

$$\operatorname{ch}_{2k-1}([f]) = \frac{(-1)^{k-1}}{(k-1)!} f^*[\overline{Z}_k, \omega_k] = \frac{(-1)^{k-1}}{(k-1)!} [f^{-1}(\overline{Z}_k), f^*\omega_k]. \qquad \square$$

We will give now a first application to what we said so far. Let us take $M := S^{2N-1}$ in the previous proposition. Notice that we have a map

$$\Pi_N : \pi_{2N-1}(\mathcal{L}\mathrm{ag}^-) \to \mathbb{Z}, \quad \Pi_N([f : S^{2N-1} \to \mathcal{L}\mathrm{ag}^-]) = \int_{S^{2N-1}} f^*[\overline{Z}_N, \omega_N].$$

Using Bott divisibility (see Th. IV.1.4 in [17]) and Bott periodicity arguments (such as Th. 24.5.3 in [13]) one can show that the following result holds

Theorem 7.1. *The map Π_N is injective and the image is the subgroup $(n-1)!\mathbb{Z}$.*

Corollary 7.1. *The homotopy type of a map $f : S^{2N-1} \to \mathcal{L}\mathrm{ag}^-$ is determined by the integer*

$$\int_{S^{2N-1}} f^*[\overline{Z}_N, \omega_N],$$

which is always divisible by $(N-1)!$. If f is transversal to $\overline{Z}_N$ then this integer is the total intersection number of f and $\overline{Z}_N$.

Remark 7.1. Any map $f : S^{2N-1} \to \mathcal{L}\mathrm{ag}^-$ can be deformed to a map $S^{2N-1} \to \mathcal{L}\mathrm{ag}(N)$. After identifying $\mathcal{L}\mathrm{ag}(N)$ with $U(N)$ one gets a map $\mathcal{L}\mathrm{ag}(N) \to S^{2N-1}$ coming from the fibration $p : U(N) \to S^{2N-1}$. The degree of the composition $p \circ f : S^{2N-1} \to S^{2N-1}$ is exactly the integer from the corollary.

8. The normal bundle

The main goal of this paper is to give concrete *local* intersection formulae for Theorem 6.2 in the particular case of the fundamental Schubert classes $[Z_k, \omega_k]$. To that end we need a good description of the normal bundle of the top stratum of $\overline{Z}_k$, i.e., Z_k°, in $\mathcal{L}\mathrm{ag}^-$. It turns out that it is more convenient to work with the submanifold $\mathcal{L}\mathrm{ag}_1^{W_{k-1}} \supset Z_k^\circ$. Hence, we will use a different stratification in which the top stratum is $\mathcal{L}\mathrm{ag}_1^{W_{k-1}}$. A legitimate question is then what role does the filtration play in the definition of the cohomology class determined by a quasi-submanifold? In the appendix A of [8] we showed that if a quasi-submanifold W comes with two different filtrations (W, F) and (W, G), which have common refinement (W, H), where by refinement we understand that $\mathcal{H}^2 \subset F^2 \cup G^2$ and the coorientation on $W \setminus H^2$ restricts to the coorientations of $W \setminus F^2$ and $W \setminus G^2$ then they define the same cohomology class. It is possible that any two filtrations of a quasi-submanifold have a common refinement. However, we could not prove that.

Definition 8.1. The *non-standard stratification* on $\overline{Z}_k$:

$$\overline{Z}_k := F_0 \supset F_2 \supset F_3 \supset$$

has as its top stratum the manifold $F_0 \setminus F_2 := \mathcal{L}\mathrm{ag}_1^{W_{k-1}}$, while the other strata, $F_i \setminus F_{i+1}$ are unions of $Z_J \subset \overline{Z}_k \setminus \mathcal{L}\mathrm{ag}_1^{W_{k-1}}$ each of which has codimension $(2k-1)+i$ in $\mathcal{L}\mathrm{ag}^-$.

A function $F : M \to \mathcal{L}\mathrm{ag}^-$ is (non-standard) transversal to $\overline{Z}_k$ if it is transversal to every stratum in the non-standard stratification.

In this section, we aim to give a concrete description for a splitting of the differential of the inclusion $\mathcal{L}\mathrm{ag}_k^W$ where $W \subset H^-$ is finite codimensional. We have a canonical choice of this splitting in the charts along the zero section as Theorem 4.1 shows. We prove that a similar results holds everywhere.

Definition 8.2. Let $j : E \to F$ be an injective morphism of vector bundles over a smooth Banach manifold X. An *algebraic complement* G of E is a vector bundle over X that splits j. This means that there exists an injective morphism $k : G \to F$ such that

$$F = E \oplus G.$$

Notation: *Let $F : X_1 \to X_2$ be a smooth immersion of Banach manifolds. An algebraic complement of the tangent bundle TX_1 is denoted by NX_1.*

Lemma 8.1. *Let $F : X_1 \to X_2$ be a smooth immersion of Banach manifolds. Then every algebraic complement of TX_1 is naturally isomorphic with the normal bundle νX_1.*

Proof. The natural projection $NX_1 \to \nu X_1$ is an isomorphism. □

The following two results help us generalize the results of Theorem 4.1.

Lemma 8.2. *Let L_0 and $L \in \mathcal{L}\mathrm{ag}^-$ be two lagrangians such that $L \in \mathcal{A}_{L_0}$ (2.3). Then the differential at L of the transition map between the Arnold chart centered at L_0 and the Arnold chart centered at L is the map:*

$$d_L : \mathcal{S}\mathrm{ym}\,(L_0) \to \mathcal{S}\mathrm{ym}\,(L) \quad d_L(\dot{S}) = P_L|_{L_0} \circ \dot{S} \circ P_{L_0}|_L.$$

Proof. Let $L_1 \in \mathcal{A}_{L_0} \cap \mathcal{A}_L$. This means that L_1 can be described both as Γ_{JX} where $X \in \mathcal{S}\mathrm{ym}\,(L_0)$ and Γ_{JS} where $S \in \mathcal{S}\mathrm{ym}\,(L)$. It is not hard to see what S should be.

$$JS = P_{L^\perp} \circ (I, JX) \circ [P_L \circ (I, JX)]^{-1}.$$

The image of the map $(I, JX) : L_0 \to \hat{H}$ gives the Lagrangian L_1 and the inverse of $P_L \circ (I, JX)$ is a well-defined operator $L \to L_1$ since L_1 is in $\mathcal{A}_L$. We consider the function:

$$F : \mathcal{S}\mathrm{ym}\,(L_0) \to \mathcal{S}\mathrm{ym}\,(L), \quad F(X) = -JP_{L^\perp} \circ (I, JX) \circ [P_L \circ (I, JX)]^{-1}.$$

Notice that for $X_0 = -JP_{L_0^\perp} \circ (P_{L_0}|_L)^{-1}$ we have $F(X_0) = 0$ since $X_0 \in \mathcal{S}\mathrm{ym}\,(L_0)$ is the self-adjoint operator such that $L = \Gamma_{JX_0}$. The differential of

F at X_0 is

$$\begin{aligned} d_{X_0}F(\dot S) &= -JP_{L^\perp} \circ (0, J\dot S) \circ [P_L \circ (I, JX_0)]^{-1} - JP_{L^\perp} \circ (I, JX_0) \circ [\ldots] \\ &= -JP_{L^\perp} \circ (0, J\dot S) \circ [P_L \circ (I, JX_0)]^{-1}. \end{aligned}$$

The reason for the cancelation of the second term is that the image of (I, JX_0) is in L. It is easy to see that $[P_L \circ (I, JX_0)]^{-1} = P_{L_0}|_L$, the restriction to L of the projection onto L_0. Also since $P_{JL}(Jv) = JP_L(v)$ for any lagrangian L and for any $v \in \hat H$, we get that $-JP_{L^\perp} \circ (0, J\dot S) = P_L \circ \dot S$. So

$$d_{X_0}F(\dot S) = P_L|_{L_0} \circ \dot S \circ P_{L_0}|_L$$

and this is our d_L. □

It is convenient to have another description of the differential of the transition map. To this end, let us recall that Theorem 3.1 provides a canonical unitary isomorphism:

$$\tilde U : L_0 \to L, \qquad \tilde U(v) := 1/2[(1+U)v + \mathrm{i}J(1-U)v], \quad \forall\, v \in L_0,$$

where $U \in \mathcal{U}(L_0)$ is the Cayley transform of the self-adjoint operator $X_0 \in \mathcal{S}\mathrm{ym}(L_0)$ that gives L as a graph of $JX_0 : L_0 \to L_0^\perp$. Notice first that the projection $P_L|_{L_0}$ has a description in terms of the same self-adjoint operator X_0. The orthogonal $L^\perp$ is the switched graph of $-(JX_0)^* = X_0J$. So in order to find the projection $P_L|_{L_0}$ in terms of X_0 one needs to solve the system

$$\begin{cases} a = v + X_0Jw, \\ 0 = JX_0v + w, \end{cases}$$

where $a, v \in L_0$ and $w \in L_0^\perp$. This is easy and one gets

$$v = (1 + X_0^2)^{-1}(a),$$

which yields the expression for the projection:

$$P_L|_{L_0}(a) = (1 + X_0^2)^{-1}(a) + JX_0(1 + X_0^2)^{-1}(a).$$

We now plug in

$$X_0 = \mathrm{i}(1+U)^{-1}(1-U)$$

to conclude that

$$P_L|_{L_0}(a) = 1/2\; \tilde U((1 + U^*)(a)).$$

Since $P_{L_0}|_L = (P_L|_{L_0})^*$ we have just proved the following result:

Lemma 8.3. *Let L_0 and L be two Lagrangians such that $L \in \mathcal{A}_{L_0}$, let $U = \mathcal{C}_{L_0}^{-1}(L)$ and let $\tilde{U}$ be the canonical unitary isomorphism $\tilde{U} : L_0 \to L$ as in Theorem 3.2. Then the differential d_L in Lemma 8.2 can be written as:*

$$d_L(\dot{S}) = 1/4\ \tilde{U}(1+U^*)\dot{S}(1+U)\tilde{U}^*.$$

Proposition 8.1. (a) *Every Lagrangian $L \in \mathcal{L}\mathrm{ag}_k^W$ has an orthogonal decomposition $L = \ell \oplus L \cap W \oplus \Lambda$ where ℓ is the orthogonal complement of $L \cap W$ in $L \cap W^\omega$ and Λ is the orthogonal complement of $L \cap W^\omega$ in L. Then the space of operators $S \in \mathcal{S}\mathrm{ym}\,(L$ with block decomposition*

$$S = \begin{pmatrix} 0 & S_1^* & 0 \\ S_1 & S_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{8.1}$$

is an algebraic complement of $T_L\,\mathcal{L}\mathrm{ag}_k^W$.

(b) *The algebraic complement of $T\,\mathcal{L}\mathrm{ag}^-|_{\mathcal{L}\mathrm{ag}_k^W}$ described above is a finite dimensional, orientable bundle. If $k = 1$, it has a natural orientation.*

Proof. (a) The claim is clearly true for any associate Lagrangian L_W by Lemma 4.3. We want to use the transition maps between two different Arnold charts at L, namely the one given by L_W and the one centered at L to show that the claim is true in general.

Let $\ell_0 := \mathcal{R}_W\,L$ be the symplectic reduction of L with W. So $L_W = \ell_0 \oplus L \cap W \oplus JV^\perp$ where $V^\perp$ is the orthogonal complement of $V := L \cap W$ in W.

By definition $\ell_0 = P_{H_W}(\ell)$. We are looking for a relation between ℓ and ℓ_0 in terms of the unitary isomorphism U where $U \in \mathcal{U}(L_W)$ is the Cayley transform of the self-adjoint operator $X \in \mathcal{S}\mathrm{ym}(L_W)$ whose graph is L, i.e.,

$$L = \Gamma_{JX}, \quad X = \mathrm{i}\frac{1-U}{1+U} \quad \text{and} \quad U = \frac{\mathrm{i}-X}{\mathrm{i}+X}.$$

Let $\tilde{\ell} := \tilde{U}^{-1}(\ell) \subset L_W$. We claim that

$$(\mathrm{i}+X)\ell_0 = \tilde{\ell}.$$

Indeed, we use first Lemma 4.3 to conclude that ℓ is the graph of the restriction $JX\big|_{\ell_0}$. Now $\tilde{U} : L_W \to L$ has the following expression:

$$\tilde{U}v = 1/2[(1+U)v + \mathrm{i}J(1-U)v].$$

which implies that

$$2\tilde{U}(1+U)^{-1}w = w + JXw, \qquad \forall w \in L_W.$$

We deduce that

$$2\tilde{U}(1+U)^{-1}\ell_0 = \ell,$$

which can be rewritten as

$$2(1+U)^{-1}\ell_0 = \tilde{\ell} \quad \text{or} \quad -\mathrm{i}(\mathrm{i}+X)\ell_0 = \tilde{\ell} \tag{8.2}$$

and that proves the claim.

Let $\tilde{\Lambda} := \tilde{U}^*\Lambda$. Since $\tilde{U}\big|_{L\cap W} = \mathrm{i}d$ we deduce that $\tilde{U}^*$ takes the decomposition $L = \ell \oplus L\cap W \oplus \Lambda$ to an orthogonal decomposition $L_W = \tilde{\ell} \oplus L \cap W \oplus \tilde{\Lambda}$, where $\tilde{\Lambda} := \tilde{U}\Lambda$. The operators $S \in \mathcal{S}\mathrm{ym}\,(L)$ with block decomposition (8.1) go via conjugation by $\tilde{U}$ to operators $\tilde{S} \in \mathcal{S}\mathrm{ym}\,(L_W)$ with the same type of block decomposition relative $L_W = \tilde{\ell} \oplus L \cap W \oplus \tilde{\Lambda}$.

In order to finish the proof we notice that Lemma 8.3 implies, due to dimension constraints, that the only thing one needs to prove is that the equation in $B \in \mathcal{S}\mathrm{ym}\,(L_W)$ and $S \in \mathcal{S}\mathrm{ym}\,(L_W)$

$$1/4(1+U^*)B(1+U) = S \tag{8.3}$$

has only the trivial solution $B = 0$, $S = 0$, where

$$B = \begin{pmatrix} B_1 & 0 & B_2^* \\ 0 & 0 & B_3^* \\ B_2 & B_3 & B_4 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & S_1^* & 0 \\ S_1 & S_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Notice that the block decomposition of B is relative to $L_W = \ell_0 \oplus L \cap W \oplus JV^\perp$ and the decomposition of S is relative to $L_W = \tilde{\ell} \oplus L \cap W \oplus \tilde{\Lambda}$. Notice that (8.3) can be written as

$$B = -(\mathrm{i} - X)S(\mathrm{i} + X).$$

This is the same thing as

$$\langle Bv, w\rangle = -\langle(\mathrm{i} - X)S(\mathrm{i} + X)v, w\rangle = \langle S(\mathrm{i}+X)v, (\mathrm{i}+X)w\rangle, \quad \forall v, w \in L_W.$$

We take first $v \in \ell_0$ and $w \in L \cap W$. Relation (8.2) and $X = 0$ on $L \cap W$ imply

$$0 = -\mathrm{i}\langle S(\mathrm{i}+X)v, w\rangle = -\mathrm{i}\langle S_1(\mathrm{i}+X)v, w\rangle.$$

We conclude that $S_1 \equiv 0$. Similarly taking $v, w \in L \cap W$, we get $S_2 \equiv 0$, which finishes the proof.

(b) Let us notice that we have two tautological bundles over $\mathcal{L}\mathrm{ag}_k^W$ namely ϑ, resp. ϑ^ω whose fiber at L consists of $L \cap W$, resp. $L \cap W^\omega$.

$$\begin{array}{ccc} \vartheta & \hookrightarrow & \mathcal{L}\mathrm{ag}_k^W \times W \\ \downarrow & & \downarrow \\ \vartheta^\omega & \hookrightarrow & \mathcal{L}\mathrm{ag}_k^W \times W^\omega \end{array}.$$

We have of course that ϑ is a subbundle of ϑ^ω and if we let θ be the orthogonal complement of ϑ in ϑ^ω, then the bundle described in the statement is $\mathcal{S}\mathrm{ym}\,(\vartheta) \oplus \mathrm{Hom}\,(\theta, \vartheta)$. Hence it is the direct sum of a complex bundle, always naturally oriented and the bundle of self-adjoint endomorphisms associated to a complex bundle. But this last one is up to isomorphism the bundle associated to the principal bundle of unitary frames via the adjoint action of the unitary group on its Lie algebra. This is clearly orientable.

In the case $k = 1$, ϑ is a line bundle and $\mathcal{S}\mathrm{ym}\,(\vartheta)$ is oriented by the identity operator. □

9. Local intersection numbers

We are now ready for doing intersection theory on $\mathcal{L}\mathrm{ag}^-$. In this section, M will be a closed, oriented manifold of fixed dimension $2k-1$. This is the codimension of the Schubert variety $\overline{Z}_k$. We consider on $\overline{Z}_k$ the non-standard stratification. Let $F : M \to \mathcal{L}\mathrm{ag}^-$ be a smooth map. We will call such a map a (smooth, compact) family of lagrangians.

By Proposition 8.1, the inclusion of tangent bundles $T\,\mathcal{L}\mathrm{ag}_1^{W_{k-1}} \hookrightarrow T\,\mathcal{L}\mathrm{ag}^-$ has an algebraic complement $N\,\mathcal{L}\mathrm{ag}_1^{W_{k-1}}$ which is naturally oriented as follows.

Let $L \in \mathcal{L}\mathrm{ag}_1^{W_{k-1}}$, $V := L \cap W_{k-1}$ and ℓ be the orthogonal complement of $L \cap W_{k-1}$ in $L \cap W_{k-1}^\omega$. The algebraic complement to $T_L\,\mathcal{L}\mathrm{ag}_1^{W_{k-1}}$ is the vector subspace of $\mathcal{S}\mathrm{ym}\,(L)$ of operators coming from

$$\mathcal{S}\mathrm{ym}(V) \oplus \mathrm{Hom}(\ell, V).$$

The space V is one-dimensional and so $\mathcal{S}\mathrm{ym}(V)$ is a one-dimensional *real* vector space, naturally oriented by the identity map. Concretely a nonzero operator $A \in \mathcal{S}\mathrm{ym}(V)$ is positively oriented if the following number is positive:

$$\langle Av, v\rangle \text{ for some } v \in L \cap W_{k-1}.$$

The canonical orientation on $\mathrm{Hom}(\ell, V)$ is given by the following data. Let v be a unit vector in V and $\{\lambda_1, \lambda_2, \dots, \lambda_{k-1}\}$ be a complex orthonormal basis for ℓ. We say that a basis $T_1, \dots T_{2k-2}$ is positively oriented for $\mathrm{Hom}(\ell, V)$ if the following determinant is positive:

$$\begin{vmatrix} \mathrm{Re}\langle T_1\lambda_1, v\rangle & \mathrm{Im}\langle T_1\lambda_1, v\rangle & \dots & \mathrm{Re}\langle T_1\lambda_{k-1}, v\rangle & \mathrm{Im}\langle T_1\lambda_{k-1}, v\rangle \\ \mathrm{Re}\langle T_2\lambda_1, v\rangle & \mathrm{Im}\langle T_2\lambda_1, v\rangle & \dots & \mathrm{Re}\langle T_2\lambda_{k-1}, v\rangle & \mathrm{Im}\langle T_2\lambda_{k-1}, v\rangle \\ \dots & \dots & \dots & \dots & \dots \\ \mathrm{Re}\langle T_{2k-2}\lambda_1, v\rangle & \mathrm{Im}\langle T_{2k-2}\lambda_1, v\rangle & \dots & \mathrm{Re}\langle T_{2k-2}\lambda_{k-1}, v\rangle & \mathrm{Im}\langle T_{2k-2}\lambda_{k-1}, v\rangle \end{vmatrix}.$$

One can check that the orientation does not depend on the choice of v or of the basis $\{\lambda_1, \lambda_2, \dots, \lambda_{k-1}\}$.

The following is straightforward:

Lemma 9.1. *Let $T \in \mathcal{S}\mathrm{ym}(L)$ be a self-adjoint operator. Let $v \in V$ be a unit complex number. Then the $\mathcal{S}\mathrm{ym}(V) \oplus \mathrm{Hom}(\ell, V)$ block of S is the operator*

$$x \mapsto \langle Tx, v\rangle v, \qquad \forall x \in \ell \oplus V.$$

This lemma and the previous observations prompt the following definition:

Definition 9.1. Let $F : M \to \mathcal{L}\mathrm{ag}^-$ be an oriented family of lagrangians of dimension $2k-1$ transversal to $\overline{Z_k}$ with the non-standard stratification and let $p \in F^{-1}(\overline{Z_k}) = F^{-1}(\mathcal{L}\mathrm{ag}^{W_{k-1}}(1))$ be a point in M.

Let $\{\epsilon_1, \dots \epsilon_{2k-1}\}$ be an oriented basis for M at p, v be a unit vector in $F(p) \cap W_{k-1}$ and $\{\lambda_1, \lambda_2, \dots, \lambda_{k-1}\}$ be a unitary basis of $\ell(p)$, the orthogonal complement of $F(p) \cap W_{k-1}$ in $F(p) \cap W_{k-1}^{\omega}$.

The *intersection number* at p of F and $\overline{Z_k}$, denoted $\sharp(M \cap \overline{Z_k})_p$ is the sign of the determinant

(9.1)
$$\begin{vmatrix} \langle d_pF(\epsilon_1)v, v\rangle & \mathrm{Re}\langle d_pF(\epsilon_1)\lambda_1, v\rangle & \dots & \mathrm{Im}\langle d_pF(\epsilon_1)\lambda_{k-1}, v\rangle \\ \langle d_pF(\epsilon_2)v, v\rangle & \mathrm{Re}\langle d_pF(\epsilon_2)\lambda_1, v\rangle & \dots & \mathrm{Im}\langle d_pF(\epsilon_2)\lambda_{k-1}, v\rangle \\ \dots & \dots & \dots & \dots \\ \langle d_pF(\epsilon_{2k-1})v, v\rangle & \mathrm{Re}\langle d_pF(\epsilon_{2k-1})\lambda_1, v\rangle & \dots & \mathrm{Im}\langle d_pF(\epsilon_{2k-1})\lambda_{k-1}, v\rangle \end{vmatrix}.$$

It is remarkable that a similar formula holds when one replaces the differential of F by the differential of the associated projections.

Lemma 9.2. *Let $F : M \to \mathcal{L}\mathrm{ag}^-$ be a smooth family of Lagrangians transversal to Z_k and let $P : \mathcal{L}\mathrm{ag}^- \to \mathcal{B}(\hat{H})$ be the smooth map that takes a*

Lagrangian to its orthogonal projection. Denote by $P_F : M \to \mathcal{B}(\hat{H})$ the composition $-JP \circ F$. The intersection number $\sharp(M \cap \overline{Z_k})_p$ is equal to the sign of the determinant 9.1 where one replaces the differential of F with the differential of P_F everywhere.

Proof. By Lemma 2.2 we have

$$d_P(\dot{S}) = \begin{pmatrix} 0 & \dot{S}J_L^{-1} \\ J_L\dot{S} & 0 \end{pmatrix}$$

and so $(-Jd_LP)|_{\mathcal{S}\mathrm{ym}(L)} = \mathrm{id}_{\mathcal{S}\mathrm{ym}(L)}$. Since all the vectors $v, \lambda_1, \ldots, \lambda_{k-1}$ belong to $L := F(p)$ the proposition follows. □

The intersection numbers for $k = 1$ have received a particular attention, especially in the real case.

Definition 9.2. For every family $F : S^1 \to \mathcal{L}\mathrm{ag}^-$, transversal to $\overline{Z}_1$ the intersection number

$$\sum_{p \in F^{-1}(\overline{Z}_1)} \sharp(M \cap \overline{Z}_1)_p$$

is called the *Maslov index.*

Proposition 9.1. *The Maslov index is a homotopy invariant that provides an isomorphism:*

$$\pi_1(\mathcal{L}\mathrm{ag}^-) \simeq \mathbb{Z}$$

Proof. This is obvious in light of the fact that the Maslov index is the evaluation over S^1 of the pull-back of the cohomology class $[\overline{Z}_1, \omega_1]$. □

The top stratum of $\overline{Z}_1$ in the non-standard stratification, $\mathcal{L}\mathrm{ag}_1^{H^-}$ has a particular nice structure that allows one to derive a different type of formula.

Notice first that we have the following set equalities:

$$Z_{\geq k} := \bigcup_{i \geq k} Z_i = \mathcal{G}^{-1}(\mathbb{P}(W_{k-1})) = \mathcal{L}\mathrm{ag}_1^{H^-} \cap \mathcal{L}\mathrm{ag}_1^{W_{k-1}},$$

where $\mathcal{G}$ is the generalized reduction $\mathcal{G} : \mathcal{L}\mathrm{ag}_1^{H^-} \to \mathbb{P}(H^-)$. This implies in particular that $Z_{\geq k}$ is a smooth manifold of codimension $2k - 2$ in $\mathcal{L}\mathrm{ag}_1^{H^-}$.

Definition 9.3. A smooth $2k-1$ dimensional family $F : M \to \mathfrak{L}\mathrm{ag}^-$ is *strongly transversal* to $\overline{Z}_k$ if the following conditions hold:

- F is transversal to $\overline{Z}_k$.
- F is transversal to $\overline{Z}_1$.
- $F^{-1}(\overline{Z}_1) = F^{-1}(\mathfrak{L}\mathrm{ag}_1^{H^-})$, i.e., $\dim F(m) \cap H^- \leq 1$ for all $m \in M$.

Remark 9.1. The first and the third conditions of strong transversality imply that $F^{-1}(\overline{Z}_k) = F^{-1}(Z_{\geq k})$. Indeed, the first condition implies that $F^{-1}(\overline{Z}_k) = F^{-1}(\mathfrak{L}\mathrm{ag}_1^{W_{k-1}})$, whereas the third implies that $F^{-1}(\overline{Z}_k) \subset F^{-1}(\mathfrak{L}\mathrm{ag}_1^{H^-})$.

Remark 9.2. Every smooth family can be deformed to a family that satisfies the first two transversality conditions. However, the third condition of strong transversality is not amenable to perturbations, since there are topological obstructions to achieving that. An example is a family for which the cohomology class $F^*[\overline{Z}_{\{1,2\}}, \omega_{\{1,2\}}]$ is non-trivial.

Things are good when $k = 2$.

Lemma 9.3. *Let* $\dim M = 3$. *Any family* $F : M^3 \to \mathfrak{L}\mathrm{ag}^-$ *can be deformed to a strongly transversal family to* $\overline{Z}_2$.

Proof. First deform the family to a map transversal to $\overline{Z}_2$ and then move it off $\bigcup_{k\geq 2} Z_{1,k}$ which has codimension 4 and has the property that $\bigcup_{k\geq 2} Z_{1,k} = \mathfrak{L}\mathrm{ag}_1^{W_1} \setminus \mathfrak{L}\mathrm{ag}_1^{H^-}$. □

Proposition 9.2. *Let M be an oriented, closed manifold of dimension $2k-1$ and let $F : M \to \mathfrak{L}\mathrm{ag}^-$ be a family, strongly transversal to $\overline{Z}_k$. Then $M^1 := \{m \in M \mid \dim F(m) \cap H^- = 1\}$ is a closed, cooriented submanifold of M of dimension $2k-2$. Let $\gamma \subset M^1 \times \mathbb{P}(H^-)$ be the tautological bundle over M^1 with fiber $\gamma_m := L \cap H^-$. Then*

$$\int_M F^*[\overline{Z}_k, \omega_k] = \int_{M^1} c_1(\gamma^*)^{k-1}.$$

Proof. Notice that $M^1 = F^{-1}(\mathfrak{L}\mathrm{ag}_1^{H^-})$. The fact that M^1 is a cooriented submanifold of M of codimension 1 follows from the second condition of strong transversality and the fact that $\mathfrak{L}\mathrm{ag}_1^{H^-}$ is a cooriented submanifold of $\mathfrak{L}\mathrm{ag}^-$ of codimension 1. It therefore inherits the coorientation of $\mathfrak{L}\mathrm{ag}_1^{H^-}$

in $\mathcal{L}\mathrm{ag}^-$. It is closed because it is the preimage of $\overline{Z}_1$ by the third condition of strong transversality.

We have the following commutative diagram

$$\begin{array}{ccccc}
F^{-1}(\overline{Z}_k) & \xrightarrow{F} & Z_{\geq k} & \xrightarrow{\mathcal{G}} & \mathbb{P}(W_{k-1}) . \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
M^1 & \xrightarrow{F|_{M^1}} & \mathcal{L}\mathrm{ag}_1^{H^-} & \xrightarrow{\mathcal{G}} & \mathbb{P}(H^-) \\
\big\downarrow & & \big\downarrow & & \\
M & \xrightarrow{F} & \mathcal{L}\mathrm{ag}^- & &
\end{array}$$

The local intersection number of M and $Z_{\geq k}$ in $\mathcal{L}\mathrm{ag}^-$ at a point $m \in M$ is the local intersection number of M^1 and $\mathcal{L}\mathrm{ag}_1^{H^-}$ at $m \in M$, which is the local intersection number of $\mathcal{G} \circ F|_{M^1}$ with $\mathbb{P}(W_{k-1})$ at $m \in M$.

Let τ^* be the dual to the tautological bundle of $\mathbb{P}(H^-)$. Then the Poincaré dual of $\mathbb{P}(W_{k-1})$ in $\mathbb{P}(H^-)$ is $c_1(\tau^*)^{k-1}$ where c_1 is the first Chern class of τ^*. The total intersection number of M^1 and $\mathbb{P}(W_{k-1})$ is the evaluation of the pull-back of the Poincaré dual to $\mathbb{P}(W_{k-1})$ on M^1. The next equality finishes the proof

$$\gamma = (\mathcal{G} \circ F|_{M^1})^*(\tau).$$

□

10. Intersection formulae for families of operators

The motivating example for this paper was Nicolaescu's Theorem 3.14 from [21] which is saying, roughly, that the spectral flow of a generic path of Dirac operators equals the Maslov index of an appropriate path of lagrangians. The spectral flow can be defined more generally for a family F of self-adjoint, Fredholm operators parametrized by a compact manifold M to be the first component of the odd Chern character applied to the $K^{-1}(M)$ class determined by F. For every path $\alpha : S^1 \to M$ in M this class can be integrated over the path to give an integer which is what is regularly called the spectral flow of a path of operators. When the path is generic this integer is a count with sign of the 0-eigenvalues (assumed simple) of the operators on the path. Now, it is not true that every family of self-adjoint, Fredholm operators gives rise to a K^{-1} class. In fact, Atiyah and Singer [3] showed that a classifying space for K^{-1} is the space of *bounded*, self-adjoint, Fredholm operators having both positive and negative essential spectra. The important word here is *bounded*. The typical way of passing from unbounded operators to bounded operators is via functional calculus as we now describe:

Definition 10.1. Let $\mathcal{S}$Fred denote the set of all closed, densely defined, self-adjoint, Fredholm operators on H, let BFred be the space of bounded, self-adjoint, Fredholm operators on H and let $\text{BFred}_* \subset \text{BFred}$ be the Atiyah–Singer classifying space. The map

$$\text{Ri} : \mathcal{S}\text{Fred} \to \text{BFred}, \qquad \text{Ri}(T) := T\left(\sqrt{1+T^2}\right)^{-1}$$

called the Riesz map is an injection. The topology induced by Ri on $\mathcal{S}$Fred is called the Riesz topology.

On the other hand, there is another injective map, the switched graph

$$\tilde{\Gamma} : \mathcal{S}\text{Fred} \to \mathcal{L}\text{ag}^-,$$

through which $\mathcal{S}$Fred is endowed with another topology called the gap topology. Nicolaescu proved the following in [20, Lemma 1.2, Prop. 3.1 and Th. 3.3].

Proposition 10.1. *Every Riesz continuous family of operators* $F : M \to \mathcal{S}\text{Fred}$ *is also gap continuous.*

Theorem 10.1. *The graph map* $\tilde{\Gamma} : \text{BFred}_* \to \mathcal{L}\text{ag}^-$ *is a weak homotopy equivalence and for every Riesz continuous family* $F : M \to \mathcal{S}\text{Fred}$ *the families of Lagrangians* $\tilde{\Gamma} \circ F$ *and* $\tilde{\Gamma} \circ \text{Ri} \circ F$ *are homotopic.*

These results are saying that the "old" analytic index a la Atiyah–Singer is the same as the "new" index, defined using $\mathcal{L}\text{ag}^-$. Therefore we make the following definition.

Definition 10.2. Let $F : M \to \mathcal{S}\text{Fred}$ be a family of self-adjoint, Fredholm operators. Then F is continuous if $\tilde{\Gamma} \circ F$ is gap continuous. The *analytic index* of a continuous family of self-adjoint, Fredholom operators is the homotopy class of the map $\tilde{\Gamma} \circ F : M \to \mathcal{L}\text{ag}^-$ and is denoted by $[F]$. The *cohomological index* of F is the class $\text{ch}[F] \in H^{\text{odd}}(M, \mathbb{Q})$.

Theorem 10.2. *The* $2k-1$ *component of the cohomological index of a continuous family of operators* $F : M \to \mathcal{S}\text{Fred}$ *is a rational multiple of the* *geometric class* $(\tilde{\Gamma} \circ F)^*[\overline{Z}_k, \omega_k]$. *In fact, the following relation holds:*

$$\mathrm{ch}_{2k-1}[F] = \frac{(-1)^{k-1}}{(k-1)!}(\tilde{\Gamma} \circ F)^*[\overline{Z}_k, \omega_k].$$

Proof. This is a rephrasing of Proposition 7.1. □

Before we proceed let us mention the following useful criterion for deciding gap continuity and, more generally, gap differentiability. The proof can be found in [8, Theorem 4.6].

Theorem 10.3. *Let* $(T_x)_{x\in\mathbb{R}^n} : D(T_x) \subset H \to H$ *be a family of densely-defined, closed, self-adjoint, Fredholm operators. Let* $H_0 := D(T_0)$ *and suppose* H_0 *comes equipped with an inner product such that:*

(1) *the inclusion* $H_0 \to H$ *is bounded and*

(2) *the operator* $T_0 : H_0 \to H$ *is bounded.*

Suppose there exists a differentiable family of bounded, invertible operators $U : \mathbb{R}^n \to GL(H)$ *such that*

(a) $U_x^*(H_0) = D(T_x)$;

(b) *the new family of operators* $\tilde{T}_x := U_x T_x U_x^*$ *is a differentiable family of bounded operators in* $\mathcal{B}(H_0, H)$.

Then the family of switched graphs associated to $(T_x)_{x\in\mathbb{R}^n}$ *is differentiable at* T_0 *in* $\mathcal{L}\mathrm{ag}^-$.

The localization part in the title was motivated by the interpretation of the spectral flow as $\mathrm{ch}_1[F] \in H^1(M, \mathbb{Z})$. Classically, the spectral flow associates to a generic loop (or more generally to a path) of self-adjoint operators an integer, which is a count with sign of the 0-eigenvalues of the family of operators. To be more precise, if $\lambda_1, \ldots \lambda_k : S^1 \to \mathbb{R}$ are the eigencurves that take the value 0 eventually[2], then the spectral flow is the sum

$$\sum_{t_0 \mid \lambda_i(t_0)=0} \mathrm{sgn}\, \dot{\lambda}_i(t_0).$$

[2]We are also assuming that these eigencurves exist and 0 is a simple eigenvalue for the uninvertible operators, conditions that justify the word generic.

If the family F of operators is bounded and differentiable then one can check that

$$\dot{\lambda}(t_0) = \langle \dot{F}_{t_0} v_0, v_0 \rangle,$$

where v_0 is a unit eigenvector for the 0-eigenvalue of F_{t_0}. It is easy to see that the spectral flow is the degree of the 0-cycle whose Poincaré dual is $\mathrm{ch}_1[F]$. Indeed this degree is, in transversal conditions the total intersection number of $\tilde{\Gamma} \circ F$ with Z_1. We can ask what is a similar interpretation in terms of eigenvalues and eigenspaces for $\mathrm{ch}_{2k-1}[F]$ where k is a positive integer. Based on Proposition 7.1 we will rephrase this problem as follows:

> "Give localization formulae for the Poincaré dual of $F^*[\overline{Z}_{k-1}, \omega_{k-1}] \in H^{2k-1}(M, \mathbb{Z})$, where $F : M \to \mathcal{S}\mathrm{Fred}$ is a gap differentiable family of operators parametrized by an oriented manifold of dimension $2k-1$."

We are aiming, of course, for intersection formulae in generic conditions, which in our case means transversality.

Let W be a codimension $k-1$ subspace of H^-. We consider the associated $2k-1$ codimensional cocycle whose underlying space is the following Schubert variety:

$$\overline{Z}_W = \{L \in \mathcal{L}\mathrm{ag}^- \mid \dim L \cap W \geq 1\}.$$

Definition 10.3. A smooth family $F : M \to \mathcal{S}\mathrm{Fred}$ is said to be in general position with respect to W if $\tilde{\Gamma} \circ F$ is transversal to the Schubert variety $\overline{Z}_W$ with the non-standard stratification.

If M has complementary dimension to $\overline{Z}_W$, i.e., $\dim M = 2k-1$, the condition to be in general position with respect to W implies that there are only a finite number of points $p \in M$ such that

$$\dim \tilde{\Gamma}_{F(p)} \cap W = 1. \tag{10.1}$$

This means that

$$\dim \mathrm{Ker}\,(F(p)) \cap W = 1.$$

Notation: *Let* $F : M \to \mathcal{S}\mathrm{Fred}$ *be a smooth family in general position with respect to* W. *For every* $p \in M$ *such that* $\dim \mathrm{Ker}\,(F(p)) \cap W = 1$ *denote by* $\epsilon_p \in \{\pm 1\}$ *the intersection number at* p *of* $\tilde{\Gamma} \circ F$ *with* $\hat{Z}_W := \mathcal{L}\mathrm{ag}_1^W = \{L \mid \dim L \cap W = 1\} \subset \overline{Z}_W$.

Theorem 10.4. *Let M be a closed oriented manifold of dimension $2k-1$, let $F : M \to \mathcal{S}\mathrm{Fred}$ be a smooth family of self-adjoint, Fredholm operators and let $W \subset H$ be a codimension $k-1$ subspace such that F is in general position with respect to W. Denote by M_W the set $M_W := \{p \in M \mid \dim \operatorname{Ker}(F(p)) \cap W = 1\}$. Then*

$$\mathrm{PD}\, \mathrm{ch}_{2k-1}([F]) = \frac{(-1)^{k-1}}{(k-1)!} \sum_{p \in M_W} \epsilon_p p,$$

where PD *means Poincaré dual.*

Proof. This is a restatement of Theorem 6.2 using Proposition 7.1. □

Our main goal in this section is to give a formula for the intersection numbers ϵ_p. This is a local problem. We restrict our attention to bounded operators although similar formulae hold for nice families of unbounded operators, such as the families that we called affine in [8] (see Section 4.1). The reason for which these formulas are available is that we can make sense of what the differential of a family of operators means.

Let B be the unit ball in $\mathbb{R}^{2k-1}$ and let $T : B \to \mathcal{S}\mathrm{ym}(H)$ be a family of bounded, self-adjoint, Fredholm operators, differentiable at zero. Let us notice that if $T_0 \in \mathcal{S}\mathrm{ym}(H)$ is a bounded self-adjoint, Fredholm operator, the projection $P_0^- : \tilde{\Gamma}_{T_0} \to H^-$ is a *Banach* space isomorphism.

The next result relates the operator differential to the graph differential.

Lemma 10.1. *The family of switched graphs $(\tilde{\Gamma}_{T_x})_{x\in B}$ is differentiable at zero. Moreover, for every unit vector $v \in \mathbb{R}^n$, the following equality holds between the graph and the operator partial derivatives of the family at* 0

$$P_0^- \circ \left.\frac{\partial \tilde{\Gamma}}{\partial v}\right|_0 \circ (P_0^-)^{-1} = (1+T_0^2)^{-1} \circ \left.\frac{\partial T}{\partial v}\right|_0 \quad \in \mathcal{S}\mathrm{ym}(H).$$

Here P_0^- is the projection of the switched graph of T_0 onto H^-.

Proof. For differentiability one can suppose without loss of generality that the operators are invertible. Then the family of inverses is differentiable, which implies that the family of switched graphs is differentiable.

For $\|x\|$ small the switched graph of T_x is in the Arnold chart of $\tilde{\Gamma}_{T_0}$. Therefore, it is the graph of an operator $JS_x : \tilde{\Gamma}_{T_0} \to J\tilde{\Gamma}_{T_0}$, where $S_x \in \mathcal{S}\mathrm{ym}\,(\tilde{\Gamma}_{T_0})$. We fix such an x. We are looking for an expression for $P_0^- S_x$ $(P_0^-)^{-1}$ as an operator on H.

For every vector $v \in H$, we have a decomposition:

$$(T_x v, v) = (T_0 z, z) + J(T_0 y, y) = (T_0 z + y, z - T_0 y).$$

It is not hard to see that

$$y = (1 + T_0^2)^{-1}(T_x - T_0)v,$$
$$v = (1 + T_0 T_x)^{-1}(1 + T_0^2)z.$$

The last relation makes sense, since $1 + T_0 T_x$ approaches the invertible operator $1 + T_0^2$. The operator $P_0^- S_x (P_0^-)^{-1} : H \to H$ is nothing else but the correspondence $z \to y$ hence the expression:

$$P_0^- S_x (P_0^-)^{-1} = (1 + T_0^2)^{-1}(T_x - T_0)(1 + T_0 T_x)^{-1}(1 + T_0^2).$$

Differentiating this expression with respect to x finishes the proof. $\square$

In order not always to repeat ourselves, we make the following:

Definition 10.4. A smooth family of bounded, self-adjoint, Fredholm operators $F : B \to \mathcal{S}\mathrm{ym}(H)$ is called *localized* at 0 with respect to W if the following two conditions hold:

- F is in general position with respect to W;
- $(\tilde{\Gamma} \circ F)^{-1}(\hat{Z}_W) = \{0\}$.

The fact that the switched graph of $F(0)$ is in Z_W implies that $1 \leq \dim \operatorname{Ker} F(0) \leq k$ by Corollary 4.1.

We treat first a particular, non-generic case, because the formula looks quite simple in this case.

Proposition 10.2. *Let $F : B \to \mathrm{BFred}$ be a family of bounded, self-adjoint, Fredholm operators localized at 0 with respect to W. Suppose that $\dim \operatorname{Ker} F(0) = k$. Let $\phi \in \operatorname{Ker} F(0) \cap W$ be a unit vector, let $\phi^\perp$ be the orthogonal complement of $\langle \phi \rangle$ in $\operatorname{Ker} F(0)$ and let $\{\psi_1, \dots \psi_{k-1}\}$ be an orthonormal basis of $\phi^\perp$.*

The intersection number, ϵ_0, is given by the sign of the determinant:

$$\begin{vmatrix} \langle \partial_1 F\phi, \phi\rangle & \mathrm{Re}\langle \partial_1 F\psi_1, \phi\rangle & \dots & \mathrm{Im}\langle \partial_1 F\psi_{k-1}, \phi\rangle \\ \langle \partial_2 F\phi, \phi\rangle & \mathrm{Re}\langle \partial_2 F\psi_1, \phi\rangle & \dots & \mathrm{Im}\langle \partial_2 F\psi_{k-1}, \phi\rangle \\ \dots & \dots & \dots & \dots \\ \langle \partial_{2k-1} F\phi, \phi\rangle & \mathrm{Re}\langle \partial_{2k-1} F\psi_1, \phi\rangle & \dots & \mathrm{Im}\langle \partial_{2k-1} F\psi_{k-1}, \phi\rangle \end{vmatrix},$$

where $\partial_i F$ is the partial derivative of F at zero in the ith coordinate direction of $\mathbb{R}^{2k-1}$.

Proof. Since $\dim \operatorname{Ker} F(0) = k$ we get that

$$\tilde{\Gamma}_{JF(0)} \cap W^{\omega} = \tilde{\Gamma}_{JF(0)} \cap H^{-} = \operatorname{Ker} F(0)$$

and so the vectors $\lambda_1 = \psi_1, \dots, \lambda_{k-1} = \psi_{k-1}$ in the definition of the intersection number 9.1 can all be taken to be from $\operatorname{Ker} F(0)$.

Let $\tilde{F} := \tilde{\Gamma} \circ F$. The claim that proves the lemma is

$$\langle d_0 \tilde{F}(x)\psi, \phi\rangle = \langle d_0 F(x)(0)\psi, \phi\rangle,$$

for every unit vector x and every $\psi \in \langle\{\phi, \psi_1, \dots, \psi_{k-1}\}\rangle$. In order to prove the claim let P_0^- be the projection of $\tilde{\Gamma}_{F(0)}$ onto H^- and let $w := (1 + F_0^2)^{-1} \circ d_0 F(x) g$. Then

$$(P_0^-)^{-1} \circ (1 + F_0^2)^{-1} \circ d_0 F(x) \circ P_0^-(0, g) = (F_0 w, w).$$

Therefore, by using Lemma 10.1 we get

$$\langle d_0 \tilde{F}(x) g, \phi\rangle = \langle (F_0 w, w), (0, \phi)\rangle = \langle w, \phi\rangle.$$

Then

$$\langle w, \phi\rangle = \langle d_0 F(x) g, (1 + F_0^2)^{-1}\phi\rangle = \langle d_0 F(x) g, \phi\rangle.$$

The last equality holds because $\phi \in \operatorname{Ker} F_0$. □

In the case $k = 2$, the intersection numbers still have a quite simple description. Suppose for now that B is the three dimensional ball.

Proposition 10.3. *Let $T : B \to \mathrm{BFred}$ be a family of bounded, self-adjoint, Fredholm operators. Let $e \in H$ be a vector and suppose that T is localized at 0 with respect to $\langle e\rangle^{\perp}$. Let $0 \neq \phi$ be a generator of $\operatorname{Ker} T_0 \cap \langle e\rangle^{\perp}$. Then only one of the two situations is possible*

I $\dim \operatorname{Ker} T_0 = 1$, *in which case we let* ψ *be a non-zero vector satisfying the following two relations*:

$$\begin{cases} \langle \phi, \psi \rangle = 0, \\ T_0 \psi = e. \end{cases} \tag{10.2}$$

II $\dim \operatorname{Ker} T_0 = 2$, *in which case we let* $\psi \in \operatorname{Ker} T_0$ *be a non-zero vector such that* $\psi \perp \phi$.

The intersection number of T *with* $\overline{Z}_{e^\perp}$ *is given by the sign of the determinant*

$$\begin{vmatrix} \langle \partial_1 T\phi, \phi \rangle & \operatorname{Re}\langle \partial_1 T\psi, \phi \rangle & \operatorname{Im}\langle \partial_1 T\psi, \phi \rangle \\ \langle \partial_2 T\phi, \phi \rangle & \operatorname{Re}\langle \partial_2 T\psi, \phi \rangle & \operatorname{Im}\langle \partial_2 T\psi, \phi \rangle \\ \langle \partial_3 T\phi, \phi \rangle & \operatorname{Re}\langle \partial_3 T\psi, \phi \rangle & \operatorname{Im}\langle \partial_3 T\psi, \phi \rangle \end{vmatrix}_{t=0},$$

where $\partial_i T$ *is the directional derivative of* T *in the* i*th coordinate direction of* $\mathbb{R}^3$.

Proof. Let $W = \langle e \rangle^\perp$. The intersection of the switched graph of T_0 with W^ω is two-dimensional. Hence the kernel of T is either one or two-dimensional. One vector in the intersection $\tilde{\Gamma}_T \cap W^\omega$ is $(0, \phi)$. If the kernel of T_0 is two-dimensional, then $\tilde{\Gamma}_{T_0} \cap W^\omega = \operatorname{Ker} T_0$ and so the second vector in the intersection formulae is a generator of the orthogonal complement in $\operatorname{Ker} T_0$ of $\langle \phi \rangle$.

If $\dim \operatorname{Ker} T_0 = 1$, the condition $(T_0\alpha, \alpha) \in W^\omega$ imposes that $T_0 \alpha = ae$ for some constant a. We are looking for a solution when $a \neq 0$, otherwise α is a multiple of ϕ. The projection of $\frac{1}{a}\alpha$ onto $\operatorname{Ker} {T_0}^\perp$ is an element of $W^\perp \subset W^\omega$ and a generator of the orthogonal complement of $\operatorname{Ker} T_0$ in $\tilde{\Gamma}_{T_0} \cap W^\omega$. It satisfies the two conditions required from ψ.

The fact that one can replace the partial derivatives of the switched graphs in the $g_1 = (T\psi, \psi)$ direction by the partial derivatives of T in the ψ direction is a computation exactly as in Proposition 10.2. □

We state now the general case.

Proposition 10.4. *Let* $W \subset H$ *be a* $k-1$ *codimensional subspace and let* $T : B \to \mathrm{BFred}$ *be a family of bounded, self-adjoint, Fredholm operators localized at* 0 *with respect to* W. *Suppose that* $\dim \operatorname{Ker} T_0 = p \leq k$. *Let* ϕ *be a generator of* $\operatorname{Ker} T_0 \cap W$ *and let* $\phi_1, \dots, \phi_{p-1}$ *be a basis of the orthogonal complement of* ϕ *in* $\operatorname{Ker} T_0$.

The space $W_T := W \cap \operatorname{Ran} T_0$ *has dimension* $k - p$. *Let* $\psi_1, \dots, \psi_{k-p}$ *be an orthonormal basis of* $P\Big|_{(\operatorname{Ker} T)^\perp} {T_0}^{-1}(W_T)$.

Then the intersection number of T with $\overline{Z}_W$ is the sign of the determinant

$$\begin{vmatrix} \langle\partial_1 T\phi,\phi\rangle & \langle\partial_2 T\phi,\phi\rangle & \dots & \langle\partial_{2k-1} T\phi,\phi\rangle \\ \mathrm{Re}\langle\partial_1 T\phi_1,\phi\rangle & \mathrm{Re}\langle\partial_2 T\phi_1,\phi\rangle & \dots & \mathrm{Re}\langle\partial_{2k-1} T\phi_1,\phi\rangle \\ \dots & \dots & \dots & \dots \\ \mathrm{Im}\langle\partial_1 T\phi_{p-1},\phi\rangle & \mathrm{Im}\langle\partial_2 T\phi_{p-1},\phi\rangle & \dots & \mathrm{Im}\langle\partial_{2k-1} T\phi_{p-1},\phi\rangle \\ \mathrm{Re}\langle\partial_1 T\psi_1,\phi\rangle & \mathrm{Re}\langle\partial_2 T\psi_1,\phi\rangle & \dots & \mathrm{Re}\langle\partial_{2k-1} T\psi_1,\phi\rangle \\ \dots & \dots & \dots & \dots \\ \mathrm{Im}\langle\partial_1 T\psi_{k-p},\phi\rangle & \mathrm{Im}\langle\partial_2 T\psi_{k-p},\phi\rangle & \dots & \mathrm{Im}\langle\partial_{2k-1} T\psi_{k-p},\phi\rangle \end{vmatrix}_{t=0}.$$

Proof. One only needs to make sense of what the orthogonal complement of $\mathrm{Ker}\, T_0 \cap W$ in $\tilde{\Gamma}_T \cap W^\omega$ is. □

We close this section with a reformulation of Proposition 9.2 in terms of operators. We make the following definition based on Definition 9.3:

Definition 10.5. A smooth family $T : M \to \mathcal{S}\mathrm{Fred}$ of bounded self-adjoint, Fredholm operators is called strongly transversal to $\overline{Z}_k$, if $\tilde{\Gamma} \circ T$ is strongly transversal to $\overline{Z}_k$

Lemma 10.2. *Let* $\dim M = 3$. *Any smooth family* $T : M \to \mathrm{BFred}$ *can be deformed to a strongly transversal family to* $\overline{Z}_2$.

Proof. This is just proof of Lemma 9.3 with the addition that one has to make sure that in the course of the deformation one stays inside BFred. This is true because the map $\tilde{\Gamma} : \mathrm{BFred} \to \mathcal{L}\mathrm{ag}^-$ is open. □

Proposition 10.5. *Let M be a closed, oriented manifold of dimension $2k - 1$ and let $T : M \to \mathcal{S}\mathrm{Fred}$ be a strongly transversal family to $\overline{Z}_k$. Then $M^1 := \{m \in M \mid \dim \mathrm{Ker}\, T_m = 1\}$ is a closed, cooriented manifold. Let $\gamma \subset M^1 \times H$ be the tautological line bundle over M^1 with fiber $\gamma_m = \mathrm{Ker}\, T_m$. Then*

$$\int_M T^*[\overline{Z}_k, \omega_k] = \int_{M^1} c_1(\gamma^*)^k.$$

Proof. This is just Proposition 9.2 formulated in terms of operators. □

We describe the coorientation of M^1 in concrete terms. Let $m \in M^1$ and $v \in T_m M \setminus T_m M^1$ be a vector. The vector v is said to be positively oriented if given a curve $\alpha : (-\epsilon, \epsilon) \to M$ such that $\alpha \cap M^1 = m = \alpha(0)$ and $\alpha'(0) = v$, the curve of operators $T \circ \alpha$ has local spectral flow equal to $+1$, i.e., $\mathrm{sgn}\, \langle \frac{d}{dt}(T \circ \alpha)\big|_{t=0} v, v\rangle = +$.

Appendix A Representatives of cohomology classes in Banach manifolds

We describe in this appendix how certain stratified spaces in a Banach manifold define cohomology classes. We will work with sheaf cohomology groups and our presentation is inspired from the work of Iversen [14].

Let X be a metric Banach manifold. Because of paracompactness the sheaf cohomology groups are isomorphic with the Cech cohomology groups. Also, because X is locally contractible, one has an isomorphism between the sheaf cohomology groups with coefficients in the constant sheaf $\underline{\mathbb{Z}}$ and singular cohomology with $\mathbb{Z}$ coefficients.

$$H^*(X,\underline{\mathbb{Z}}) \simeq H^*_{\text{sing}}(X,\mathbb{Z}). \tag{A.1}$$

Let us recall now the definition of the local cohomology groups.

Definition A.1. Let $C \subset X$ be a closed subset and let $\mathcal{F}$ be a sheaf on X. Then to $\mathcal{F}$ one associates a new sheaf $\mathcal{F}_C$ on X with supports in C defined as follows:

$$\mathcal{F}_C(U) := \{s \in \mathcal{F}(U) \mid \operatorname{supp} s \in U \cap C\}, \qquad \forall\, U \subset X \text{ open.}$$

The pth local cohomology group of C in X with values in the sheaf $\mathcal{F}$ is the pth right derived functor

$$H^p_C(X,\mathcal{F}) := R^p(\Gamma(\mathcal{F}_C)),$$

where Γ is the global section functor.

Remark A.1. Notice that the local cohomology groups $H^p_C(X,\mathcal{F})$ fit into a long exact sequence:

$$\to H^p_C(X,\mathcal{F}) \to H^p(X,\mathcal{F}) \to H^p(X\setminus C, j^*\mathcal{F}) \to H^{p+1}_C(X,\mathcal{F}) \to, \tag{A.2}$$

where $j : X\setminus C \to X$ is the natural inclusion. In the case of the constant sheaf, $\mathcal{F} = \underline{\mathbb{Z}}$ and when the spaces involved, X and $X\setminus C$ are locally contractible then

$$H^*_C(X,\underline{\mathbb{Z}}) \cong H^*_{\text{sing}}(X, X\setminus C,\mathbb{Z}),$$

because of the naturality of the isomorphism A.1.

From now on we will work only with the constant sheaf $\underline{\mathbb{Z}}$, which we will not include in the notation.

One important property of local cohomology is *the excision exact sequence* [14, Prop. II.9.5]:

Proposition A.1. *Let $C_1 \supset C_2$ be two closed subsets of the topological space X. Then one has the following long exact sequence:*

$$\to H^k_{C_2}(X) \to H^k_{C_1}(X) \to H^k_{C_1\setminus C_2}(X \setminus C_2) \to H^{k+1}_{C_2}(X) \to \tag{A.3}$$

Remark A.2. When C_1 and C_2 are locally contractible then the previous sequence corresponds to the long exact sequence in singular cohomology associated to the triple $(X, X \setminus C_1, X \setminus C_2)$.

For any closed subset C, we denote by $\mathcal{H}^k_C$ the sheaf on X associated to the presheaf $\widetilde{\mathcal{H}}^k_C$ such that for any open set $U \subset X$ we have

$$\Gamma(U \cap C, \widetilde{\mathcal{H}}^k_C) = H^k_{C\cap U}(U).$$

Remark A.3. An equivalent way to define the sheaf $\mathcal{H}^k_C$ is as the kth right-derived functor of the left exact functor

$$-_C : \text{ sheaves on } X \to \text{ sheaves on } X, \qquad\qquad \mathcal{F} \to \mathcal{F}_C$$

evaluated on the constant sheaf $\underline{\mathbb{Z}}$, i.e., $\mathcal{H}^k_C = R^k(-_C)\big|_{\mathcal{F}=\underline{\mathbb{Z}}}$.

Notice that the sheaf $\mathcal{H}^k_C$ has support on C and because of that we have:

$$H^*(X, \mathcal{H}^k_C) \simeq H^*(C, i^{-1}\mathcal{H}^k_C),$$

where $i : C \hookrightarrow X$ is the natural inclusion.

Definition A.2. The closed space C is said to have *homological codimension in X at least c* if and only if $\mathcal{H}^k_C = 0$, $\forall k < c$. We write this as

$$\operatorname{codim}^h_X C \geq c.$$

The Grothendieck spectral sequence for local cohomology (see [9, Remark 2.3.16]) whose E_2 term is

$$E_2^{p,k} = H^p(X, \mathcal{H}^k_C) = R^p(\Gamma) \circ R^k(-_C)\big|_{\mathcal{F}=\underline{\mathbb{Z}}}$$

converges to $H^{p+k}_C(X) = R^{p+k}(\Gamma \circ -_C)$. An immediate consequence is

Proposition A.2 (extension property). *If* $\operatorname{codim}^h_X(S) \geq c$*, then for any* $q < c-1$ *the restriction map*

$$H^q(X) \to H^q(X \setminus S)$$

is an isomorphism. More generally, if $C \supset S$ *is a closed subset of* X *then*

$$H^q_C(X) \to H^q_{C\setminus S}(X \setminus S)$$

is an isomorphism. We refer to any of this isomorphisms as extension across S*.*

Proof. The E_2 term in the Grothendieck spectral sequence vanishes on the first $c-1$ rows, which implies the vanishing of $H^q_S(X)$ for $q \leq c-1$. The long exact sequences A.2 and A.1 complete the proof. □

Definition A.3. A (topological) submanifold of codimension c of X is a subset Y such that for every point $y \in Y$ there exist an open neighborhood $N \subset X$ with the property that the pair $(N, N \cap Y)$ is homeomorphic with a pair $(\mathbb{R}^c \times B, 0 \times B)$ where B is a Banach vector space.

We clearly have that a submanifold C of codimension c, which is also closed as a subset has homological codimension at least c, since in this case the sheaf local cohomology is isomorphic with singular local cohomology. Moreover, the sheaf $\mathcal{H}^c_C$ is locally isomorphic to the constant sheaf $\underline{\mathbb{Z}}$. We say that $\mathcal{H}^c_C$ is the *coorientation sheaf* of $C \hookrightarrow X$ and we will denote it by Ω_C. The submanifold is called coorientable if the sheaf Ω_C is isomorphic to the constant sheaf $\underline{\mathbb{Z}}$. A coorientation is a choice of an isomorphism $\underline{\mathbb{Z}} \to \Omega_C$ and is uniquely determined by an element $\omega_C \in H^0(C, \Omega_C)$ which, viewed as a section of Ω_C, has the property that $\omega_C(w)$ generates the stalk $\Omega_C(w)$ for any $w \in C$.

Proposition A.3 (Thom isomorphism). *Let* $C \hookrightarrow X$ *be a cooriented submanifold of codimension c. There exists an isomorphism*

$$\mathcal{T}_C : H^*(C, \Omega_C) \to H^*_C(X)[c] := H^{*+c}_C(X).$$

Proof. This is an immediate consequence of the Grothendieck spectral sequence since for a submanifold C

$$E_2^{p,k} = H^p(X, \mathcal{H}^k_C) = \begin{cases} 0, & k \neq c \\ H^p(C, \Omega_C), & k = c. \end{cases}$$

converges to $H^{p+k}_C(X)$ and therefore $H^p(C, \Omega_C) \simeq H^{p+k}_C(X)$. □

Definition A.4. A stratified subspace of codimension c of a Banach manifold X is a pair $(C, \boldsymbol{F})$, where C is a closed subset endowed with a filtration

$$\boldsymbol{F}: \qquad C = F_0 \supset F_1 \supset F_2 \supset \cdots$$

with closed sets $F_i \subset X$ such that the stratum $F_i \setminus F_{i+1}$ is a submanifold of codimension $c+i$ or is the empty set. The set C is called coorientable (cooriented) if the top stratum $F_0 \setminus F_1$ is coorientable (cooriented).

Proposition A.4. *Every stratified set $(C, \boldsymbol{F})$ of codimension c has homological codimension* $\operatorname{codim}_X^h(C) \geq c$.

Proof. It is enough to show that given $w \in F^0$

$$H^k_{F_0}(U) = 0, \qquad \forall\ k \leq c,$$

for all small open neighborhoods U of w. But for U open small enough, $U \cap F^0$ is a stratified space with a finite stratification because there exists an n such that $w \in F_n \setminus F_{n+1}$ and $U \cap F_{n+1} = \emptyset$. So without restriction of the generality we can suppose that the stratification is finite, i.e., $F^{n+1} = \emptyset$. There exists a maximal $N < n+1$ such that F_N is a nonempty, closed submanifold of codimension $c+N$ in X. Therefore, F_N is normally non-singular and so it has homological codimension at least $c+N$. We use induction on the number of strata to prove $\operatorname{codim}_X^h C = 0$. Suppose we have proved that $\operatorname{codim}_X^h F^1 \geq c+1$. Then in the excision exact sequence:

$$H^k_{F_1}(U) \to H^k_{F_0}(U) \to H^k_{F_0\setminus F_1}(U \setminus F_1) \to H^{k+1}_{F_1}(U)$$

the first and the last group are zero for all $k < c$. On the other hand, $F_0 \setminus F_1$ is a submanifold of codimension c in $U \setminus F_1$ and so $H^k_{F_0\setminus F_1}(U \setminus F_1) = H^k_{F_0}(U) = 0$, for all $k < c$ and this finishes the proof. □

Definition A.5. A stratified set $(C, \boldsymbol{F})$ is called quasi-submanifold if $F_1 = F_2$.

Proposition A.5. *A cooriented quasi-submanifold of codimension c, $(C, \boldsymbol{F}, \omega_C)$ naturally determines a cohomology class*

$$[C, \boldsymbol{F}, \omega_C] \in H^c(X).$$

If Y is another Banach manifold and $g : Y \to X$ is a differentiable map transversal to C, meaning that g is transversal to every stratum $F_i \setminus F_{i+1}$

then $(g^{-1}(C), g^{-1}(\boldsymbol{F}), g^*\omega_C)$ *is a cooriented quasi-submanifold of codimension* c *and*

$$[g^{-1}(C), g^{-1}\boldsymbol{F}, g^*\omega_C)] = g^*[(C, \boldsymbol{F}, \omega_C)].$$

Proof. Let us denote by S the top stratum $S := F_0 \setminus F_2$. It is a closed submanifold of $X \setminus F_2$. The class $[C, \boldsymbol{F}, \omega_C] \in H^c(X, \mathbb{Z})$ is the image of the coorientation class $\omega_S \in H^0(S, \Omega_S)$ via the following sequence of maps:

$$H^0(S, \Omega_S) \simeq H^c_S(X \setminus F_2) \to H^c(X \setminus F_2) \simeq H^c(X),$$

where the first map is the Thom isomorphism and the last is the extension across F_2.

The second part of the proposition follows from the naturality of the Grothendieck spectral sequence. □

Remark A.4. It is important to keep in mind that, the class $[C, \boldsymbol{F}, \omega]$ depends on the stratification $\boldsymbol{F}$. If C has three quasi-submanifold stratifications $\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{H}$ such that $\boldsymbol{F}$ is a "refinement" of the two, i.e., $G_2, H_2 \subset F_2$, then the classes determined by the three stratifications coincide. However, it is not clear that any two stratifications have a common refinement.

When the ambient manifold is finite dimensional then one can speak about duality and the "correct" dual space for local cohomology is Borel–Moore homology. For the general definition of Borel–Moore homology see Chapter IX in [14].

Remark A.5. The Borel–Moore homology group of a compact space M is isomorphic with its singular homology. When the locally compact space M has a "nice" compactification $\overline{M}$, e.g., when $(\overline{M}, \overline{M} \setminus M)$ is a CW-pair then

$$H^{\mathrm{BM}}_i(M) := H_i(\overline{M}, \overline{M} \setminus M).$$

Let M be an oriented manifold of dimension n. Then any oriented quasi-submanifold F of dimension d defines a Borel–Moore homology $\lfloor F \rfloor \in H^{\mathrm{BM}}_d(M)$ class as follows. The top stratum $S := F \setminus F^2$ determines a fundamental class $\lfloor S \rfloor \in H^{\mathrm{BM}}_d(S)$. Then the long exact sequence (see IX.2 in [14])

$$\to H^{\mathrm{BM}}_d(F_2) \to H^{\mathrm{BM}}_d(F) \to H^{\mathrm{BM}}_d(S) \to H^{\mathrm{BM}}_{d-1}(F_2) \to \qquad \forall\, p,$$

where the first and the last groups are zero (because a similar exact sequence and a quick induction proves that the Borel–Moore homology of a stratified space vanishes in degree bigger than the dimension, since that is true for a manifold), implies that the middle map

$$j : H_d^{\mathrm{BM}}(F) \to H_d^{\mathrm{BM}}(S)$$

is an isomorphism. Now $j^{-1}\lfloor S \rfloor \in H_d^{\mathrm{BM}}(S)$ can be pushed forward via

$$H_d^{\mathrm{BM}}(F) \to H_d^{\mathrm{BM}}(M)$$

to a class $\lfloor F \rfloor \in H_d^{\mathrm{BM}}(M)$.

One of the most important features of Borel–Moore homology is that it appears in the Poincaré–Verdier Duality theorem (see Th. 4.7, Ch. IX.4 in [14]).

$$H^p(M) \simeq H_{n-p}^{\mathrm{BM}}(M), \qquad \forall\ 0 \le p \le n.$$

The isomorphism is given by capping with the fundamental class $\lfloor M \rfloor$. Moreover via this isomorphism the cohomology class $[F] \in H^{n-d}(M)$ determined by a cooriented quasi-submanifold F goes to the homology class $\lfloor F \rfloor \in H_d^{\mathrm{BM}}(M)$ determined by the same quasi-submanifold with the induced orientation (see Ch. X.4 in [14]).

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