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Regularity for a log-concave to log-concave mass transfer problem with near Euclidean cost

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If the cost function is not too far from the Euclidean cost, then the optimal map transporting Gaussians restricted to a ball will be regular. Similarly, given any cost function which is smooth in a neighborhood of two points on a manifold, there are small neighborhoods near each such that a Gaussian restricted to one is transported smoothly to a Gaussian on the other.

1. Introduction

This note deals with the regularity of the optimal transportation map, when the distributions under consideration are close to restricted Gaussians. From the work of Trudinger and Wang [10], regularity holds for arbitrary smooth distributions on c-convex domains when the cost satisfies the weak Ma–Trudinger–Wang condition (see [9]). It is established by Loeper [7] that without a weak MTW (A3w) condition on the cost function, there exist smooth distributions such that the optimal map between them is not continuous. However, for the well-mannered distributions that often greet each other in the streets, the question of when the map is regular is wide open. Here, we show that there are some restrictive situations in which we can expect smooth optimal transportation.

We show two results. The first is that when the transportation problem involves distributions somewhat like the standard Gaussian restricted to the unit ball, then if the cost function is close enough to the Euclidean distance squared cost, the map must be regular. As a corollary, given two points and any cost which is smooth near these points, we can find very focused Gaussians, restricted to very small balls near the points, so that the optimal transport is regular.

Our method yields a way to compute precisely how close the cost function need be to Euclidean, or relatedly, how small the balls must be around the given points. Recently other perturbative results for regularity of optimal transport have appeared: Delanoë and Ge [5] show regularity for certain densities on metrics near constant curvature. Caffarelli-Gonzalez-Nguyen [3] obtain perturbation results for cost Euclidean distance raised to powers other than 2.

The optimal transportation problem is the following. Given probability volume forms ρ and $\bar{\rho}$ on manifolds M and $\bar{M}$, and a cost function $c : M \times \bar{M} \to \mathbb{R}$, find a map $T : M \to \bar{M}$ which minimizes a cost integral,

$$\int_M c(x, T(x))\, d\rho$$

among all maps T which preserve the volume, i.e.

$$T_* \rho = \bar{\rho}.$$

To be clear with our conventions, we set up the following Kantorovich problem: if

$$J(u, v) = \int_M (-u) d\rho + \int_{\bar{M}} v\, d\bar{\rho},$$

the problem is to maximize J over all $-u(x) + v(\bar{x}) \leq c(x, \bar{x})$. One also considers a dual problem: if

$$I(\pi) = \int_{M \times \bar{M}} c(x, \bar{x})\, d\pi$$

find the minimum of I over all measures π on the product space $M \times \bar{M}$ which have marginals ρ and $\bar{\rho}$. It is well known (cf [12]) that

$$\sup_{-u(x)+v(\bar{x}) \leq c(x,\bar{x})} J(u, v) = \inf_{\pi \epsilon \Pi(\rho, \bar{\rho})} I(\pi).$$

With this setup in mind we can derive the optimal map T from u as follows:

Suppose $(x_0, \bar{x}_0)$ is a point where the equality $-u(x_0) + v(\bar{x}_0) = c(x_0, \bar{x}_0)$ occurs. The function

$$z_{\bar{x}_0}(x) = c(x, \bar{x}_0) + u(x)$$

must have a minimum at x_0. Then define the cost exponential $T(x_0, Du) = \bar{x}_0$, provided this is unique (condition (A1)). If differentiable, from the fact that $z_{\bar{x}_0}(x)$ is at a minimum we have

$$u_i + c_i(x, T(x, Du)) = 0, \tag{1.1}$$

where c_i refers to differentiation in the first variable. (One can check that $T(x, Du) = Du$ when $c(x, \bar{x}) = -x \cdot \bar{x}$). This only depends locally on the function u, and clearly requires that Du stay inside the range of $Dc(x,\cdot)$. The elliptic optimal transportation equation can be derived by taking another derivative and then a determinant

$$\begin{aligned} &u_{ij} + c_{ij} + c_{i\bar{s}} T_j^{\bar{s}} = 0 \\ &\quad \det(u_{ij} + c_{ij}) = \det(-c_{i\bar{s}} T_j^{\bar{s}}) = \det(-c_{i\bar{s}}) \frac{\rho(x)}{\bar{\rho}(T(x))}, \end{aligned}$$

using the fact that T locally pushes ρ forward to $\bar{\rho}$, so satisfies

$$\bar{\rho}(T(x)) \det DT = \rho(x).$$

That $c_{i\bar{s}}$ is nondegenerate is referred to as the (A2) condition.

Specifically, in this note, let $f, \bar{f}$ be functions on regions $\Omega, \bar{\Omega} \subset \mathbb{R}^n$, satisfying on Ω

$$|Df| \leq 1, \tag{a1}$$

$$1 \leq \delta \leq D^2 f \leq 2, \tag{a2}$$

$$|D^3 f| \leq 1, \tag{a3}$$

and similarly for $\bar{f}$ on $\bar{\Omega}$.

We define the following mass distributions:

$$m = \mathrm{e}^{-f(x)} \chi_\Omega, \tag{1.2}$$

$$\bar{m} = \mathrm{e}^{-\bar{f}(\bar{x})} \chi_{\bar{\Omega}}, \tag{1.3}$$

where we may add a constant to f so that both distributions have the same total mass.

The region Ω is defined by a function h so that on $\Omega = \{h \leq 0\}$, h satisfies the same three conditions (a1) to (a3) as f, as well as, along the boundary $\partial\Omega$

$$|Dh| \geq 1/2, \tag{1.4}$$

which implies the second fundamental form of the set $\partial\Omega = \{h = 0\}$ is bounded by 4. Similarly define an $\bar{h}$, $\bar{\Omega}$.

A solution of the optimal transportation equation for the mass densities above and a given cost function $c(x,\bar{x})$ is a function $u(x)$ which satisfies

$$\det w_{ij} = e^{-f(x)} e^{\bar{f}(T(x,Du))} |\det c_{is}(x,T(x,Du))|, \tag{1.5}$$

$$T(x,Du)\,(\Omega) = \bar{\Omega}, \tag{1.6}$$

where

$$w_{ij} = u_{ij}(x) + c_{ij}(x,T(x,Du)) = -c_{si}T^s_j \tag{1.7}$$

and $T(x,Du) = (T^1, T^2, \dots, T^n) \subset \bar{\Omega}$ is determined by (1.1). The function u is locally c-convex, which is equivalent to the (degenerate) ellipticity condition

$$u_{ij}(x) + c_{ij}(x,T(x,Du)) \geq 0.$$

(Such a solution must also be globally c-convex. In our setting, the classical notion of convexity is very close, so we do not belabor this point here. See Lemma 2.3.) We use the following convention: The derivatives of the cost function in the first variable x will be i,j,k, etc. The second variable $\bar{x}$ will be denoted by indices p,s,t, etc. Also upper index denotes inverse i.e, $c^{si} = (c_{is})^{-1}$.

Actually, the solution u of the above equation determines a map T, which will be one-to-one (see Lemma 2.3.) This map is a unique solution of the optimal transportation problem, and the inverse of T is a solution to the symmetric optimal problem of transporting the target mass to the source mass. Note that our conditions are symmetric, so properties which hold for u also hold for some other function $\bar{u}$ which solves a similar equation. Barred quantities will refer to corresponding quantities for the barred equation.

The cost $c(x,\bar{x})$ will satisfy the standard conditions (A1) and (A2) (this will follow from closeness to Euclidean cost) but not (A3) (see for example [9] Section 2). We will require further that the derivatives of the cost satisfy the following assumptions:

$$\left\| \left(-c^{si} - I\right) \right\| \leq \epsilon_0 \leq 1/20, \tag{c-a1}$$

$$\|c_{ij}\| \leq \epsilon_0 \leq 1/20, \tag{c-a2}$$

$$C(n)\left(\left\|D^3 c\right\| + \left\|D^4 c\right\|\right) \leq \epsilon_0 \leq 1/20, \tag{c-a3}$$

where $C(n)$ is a dimensional constant, and the derivative norms are with respect to both barred and unbarred directions. Finally, we will require that the densities are somewhat close to uniform

$$\text{(cm-a3)} \qquad \mathrm{e}^{-f(x)}e^{\bar{f}(\bar{x})}|\det c_{is}| \in [\Lambda^{-1}, \Lambda],$$

for all $x, \bar{x} \in \Omega \times \bar{\Omega}$ with

$$\text{(cm-a3b)} \qquad \Lambda \leq \left(\frac{n}{3/2}\right)^n.$$

We are now ready to state our result.

Theorem 1.1. *Let $m, \bar{m}$ be the mass densities defined by (1.2) (1.3) with $f, \bar{f}$ satisfying assumptions (a1) to (a3) on regions $\Omega, \bar{\Omega}$ whose defining functions also satisfy (a1) to (a3). There exists an $\epsilon_0(n)$ such that if the cost function satisfies assumptions (c-a1) to (c-a3) and (cm-a3) holds, then the optimal map transporting m to $\bar{m}$ is regular.*

Remark 1.1. These conditions are nonvacuous. For example, take $f, h, \bar{f}, \bar{h}$ all to be

$$\tfrac{2}{3}|x|^2 - \tfrac{1}{4},$$

and the Euclidean cost

$$c(x, \bar{x}) = -x \cdot \bar{x}.$$

One can check that all the assumptions are satisfied with plenty of room to perturb any of the problem's components.

The following theorem will follow by a change of coordinates and rescaling.

Theorem 1.2. *Let $x_0, \bar{x}_0$ be two points in manifolds $X, \bar{X}$ such that near $(x_0, \bar{x}_0)$ the cost function is smooth and satisfies standard nondegeneracy conditions (A1) and (A2). Then there exists a λ large depending on the cost function, so that the optimal map from the Gaussian (after a choice of coordinates)*

$$e^{-\lambda^2|x-x_0|^2/2}\chi_{B_{1/\lambda}(x_0)}$$

to

$$e^{-\lambda^2|\bar{x}-\bar{x}_0|^2/2}\chi_{B_{1/\lambda}(\bar{x}_0)}$$

is smooth.

Remark 1.2. We do not attempt to obtain any sharp results, rather the convenient smallness assumptions are to minimize crunchiness of the proof. Inspection of the proof will show that our choice of assumptions are robust. There is a rather large gap between what is covered here and the counterexamples, and we have no reason to suspect that these results are near sharp.

Remark 1.3. We would like to obtain a similar result for complete Gaussians, as Caffarelli obtained in the Euclidean case in [2]. In fact, it was an attempt to generalize the calculation in [2] that led to this result. A limitation of our current method is that we cannot force (cm-a3) to hold on large regions.

1.1. Proof Heuristic

We will solve the problem by continuity, starting with Euclidean cost, obtaining second derivative estimates using the approach of Urbas [11] and Trudinger and Wang [10], making use of the Ma-Trudinger-Wang [9] calculation together with the calculation of Caffarelli [2]. Making these methods work in the absence of the MTW condition, we use the following observation: the bound M on the second derivatives will satisfy the following type of inequality:

$$\delta M^2 - tM^{n+1} - 1 \leq 0. \tag{1.8}$$

When t is zero, this bounds M, so M is initially bounded. If t is small it follows that $M(t)$ lies either on a relatively small compact interval containing $[-1/2, 1/2]$ or on a noncompact interval. The bound $M(t)$ is changing continuously with t, thus the interval it lies in must not change, thus from the initial bound we may conclude that for all t in some interval of fixed size, $M(t)$ is bounded.

The quadratic coefficient δ in (1.8) (same δ as in (a2)) arises when the target distribution is log-concave, as is the case with Gaussians. This fact is essential to the proof.

2. Calculations

Recall the symmetric tensor w (1.7). We use the quantities defined as follows:

$$\begin{aligned} W(x) &= \sum w_{ii} \sim \max_i w_{ii} \sim \left\| T_j^s \right\|, \\ \bar{W}(x) &= \sum w^{ii} \sim 1/\min_i w_{ii}, \\ C_3 &\geq \left\| D^3 c \right\| C(n), \\ C_4 &\geq \left\| D^4 c \right\| C(n), \\ \frac{1}{C_2} \left|\xi\right|^2 &\leq -c^{si}\xi_i\xi_s \leq C_2 \left|\xi\right|^2 . \end{aligned}$$

From (cm-a3) and Newton–McLaurin inequalities, it follows that:

$$\bar{W}, W \geq n\frac{1}{\Lambda^{1/n}}, \tag{2.1}$$

$$\bar{W} \leq \frac{1}{n^{n-2}}\Lambda W^{n-1}, \tag{2.2}$$

$$W \leq \frac{1}{n^{n-2}}\Lambda \bar{W}^{n-1}, \tag{2.3}$$

and plugging in (cm-a3b)

$$W, \bar{W} \geq 3/2. \tag{2.4}$$

Notice that (a1), (a2), (c-a1), and (c-a3) imply the following inequality for any vector ξ in $\mathbb{R}^n$:

$$\left(\bar{h}_{st} - c^{pk}c_{kst}\bar{h}_p\right)\xi_s\xi_t \geq \frac{9}{10}|\xi|^2. \tag{2.5}$$

Throughout this section we will be assuming we have a smooth solution u to the Equation (1.5) on Ω. Our goal is to prove second derivative estimates.

We make use of a modification of the linearized operator used in [10]. Define

$$Lv = w^{ij}v_{ij} - \left(w^{ij}c_{ijs}c^{sk} - \bar{f}_s(T(x,Du))c^{sk} - c^{si}c_{sip}c^{pk}\right)v_k.$$

The following has an immediate consequence when maximums occur on the interior, and is also crucial in the boundary estimates in Section 4. The proof is a moderately long calculation and follows by the arguments in [9].

Lemma 2.1. *Suppose $u(x)$ is a solution to (1.5). Then*

$$\begin{aligned} Lw_{11} = {} & w^{ij}\left(c_{11is}T_j^s + c_{11sj}T_i^s + c_{11st}T_i^sT_j^t - 2c_{ij1s}T_1^s - c_{ijst}T_1^sT_1^t\right) \\ & - f_{11} + \bar{f}_{st}T_1^sT_1^t - c^{sj}c^{pi}\left(c_{jp1} + c_{jpv}T_1^v\right)\left(c_{is1} + c_{isv}T_1^v\right) \\ & + c^{si}\left(c_{is11} + 2c_{isv1}T_1^v + c_{isvp}T_1^vT_1^p\right) - \partial_1 w^{ij}\partial_1 w_{ij} \\ & - \left(w^{ij}c_{ijs} - \bar{f}_s - c^{pi}c_{ips}\right)\left(-2c^{sk}c_{1kp}T_1^p + c^{sk}c_{11p}T_k^p - c^{sk}c_{kpv}T_1^vT_1^p\right) \\ & + c_{11s}c^{sm}\big\{c^{pi}c_{mpv}T_i^v - w^{ij}c_{ijv}T_m^v - f_m + \bar{f}_vT_m^v \\ & \qquad + c^{pi}\left(c_{ipm} + c_{ipv}T_m^v\right)\big\}. \end{aligned}$$

Applying the maximum principle,

Corollary 2.1. *If the largest eigenvalue W of w is attained on the interior, it must satisfy*

$$\frac{\bar{\delta}}{C_2}W^2 - \left(C_4 + C_3 + C_3|Df|\right)W^{n+1} - |D^2 f| - C(C_3, C_4) \leq 0. \tag{2.6}$$

The next computation is implicit throughout (see [10, Sections 2 to 4]). We state it for concreteness.

Lemma 2.2. *Let $v(x) = F(x, T(x, Du))$. Then*

$$\begin{aligned} Lv = {} & w^{ij}F_{ij} - 2F_{is}c^{si} + F_{st}c^{si}c^{tj}w_{ij} \\ & + F_pc^{pk}\left(f_k + c_{ksj}c^{sj} - c_{kst}c^{si}c^{tj}w_{ij}\right) \\ & - F_kc^{sk}\left(w^{ij}c_{ijs} - \bar{f}_s - c^{ti}c_{sit}\right). \end{aligned}$$

Corollary 2.2. *Given conditions (c-a1) to (c-a3) and (a1), (a2), on the functions $f, \bar{f}$, h, and $\bar{h}$, which imply (2.5), we have*

$$Lh \geq \tfrac{9}{10}\delta\bar{W} - \tfrac{11}{10},$$

$$L\bar{h}(T(x, Du)) \geq \tfrac{9}{10}\delta W - \tfrac{11}{10}.$$

2.1. Obliqueness

We follow the argument from [10, Section 2]. Defining

$$\begin{aligned} \gamma &= Dh, \\ \beta_i &= -\bar{h}_s c^{si}, \end{aligned}$$

we let

$$\chi = -h_k \bar{h}_s c^{sk} = \gamma \cdot \beta.$$

From Lemma 2.2 with our assumptions we have

$$L\chi \leq \bar{W}(\left|D^3 h\right| + C_3 + C_4) + W(\left|D^3 \bar{h}\right| + C_3 + C_4) + C_5(n).$$

Then Corollary 2.2 gives

$$L\left\{\chi - \lambda h - \lambda \bar{h} \circ T(x))\right\} \leq \bar{W}(\tfrac{11}{10} - \tfrac{9}{10}\lambda) + W(\tfrac{11}{10} - \tfrac{9}{10}\lambda) + 2\tfrac{11}{10}\lambda + C_5(n),$$

which is negative for λ reasonably chosen. (Throughout we are using bounds (2.1) etc, and our initial assumptions.) This function will then have a minimum at the boundary, precisely at the point where χ achieves a minimum on the boundary, and at this point we have

$$\left\{D\chi - \lambda D(\bar{h} \circ T) - \lambda Dh\right\} \cdot \frac{\gamma}{|\gamma|} \leq 0$$

or

$$D\left\{\chi - \lambda \bar{h} \circ T\right\} = \tau\gamma,$$

for some $\tau \leq \lambda$.

Now computing (following [10, 2.31–2.33]), using (2.5) and (1.7) with our other assumptions including (1.4) we conclude

$$\begin{aligned} D\chi \cdot \beta &= c^{ti}\bar{h}_t \left(h_{ki}c^{sk}\bar{h}_s + h_k\left(c_i^{sk} + c_p^{sk}T_i^p\right)\bar{h}_s + h_k c^{sk}\bar{h}_{sp}T_i^p\right) \\ &= h_{ki}\beta_k\beta_i + c^{ti}\bar{h}_t h_k \bar{h}_s c_i^{sk} + c^{ti}\bar{h}_t h_k T_i^p(\bar{h}_{sp}c^{sk} - \bar{h}_s c^{sm} c^{rk} c_{mrp}) \\ &= h_{ki}\beta_k\beta_i + c^{ti}\bar{h}_t h_k \bar{h}_s c_i^{sk} + h_k \bar{h}_t T_a^t c^{pa} c^{rk}(\bar{h}_{rp} - \bar{h}_s c^{sm} c_{mrp}) \\ &\geq |\beta|^2\delta - C_3 \geq \frac{1}{5}\delta. \end{aligned} \tag{2.7}$$

The third term in (2.7) can be expressed as an inner product g of the gradients of the functions $h(x)$ and $\bar{h} \circ T(x)$, which are both multiples of the outward normal, where

$$g(\xi, \nu) = (\bar{h}_{rp} - \bar{h}_s c^{sm} c_{mrp}) c^{rk} c^{pa} \xi_k \nu_a.$$

Thus

$$\begin{aligned}\tau\gamma\cdot\beta &= D\chi\cdot\beta-\lambda D(\bar{h}\circ T)\cdot\beta\\ &\geq \delta/5-\lambda\bar{h}_s T_i^s\left(-c^{it}\right)\bar{h}_t\\ &= \delta/5-\lambda w_{\beta\beta}.\end{aligned}$$

Thus from $\tau\leq\lambda$,

$$\lambda\chi\geq\delta/5-\lambda w_{\beta\beta}. \tag{2.8}$$

Using symmetry, we replace all quantities with barred quantities and see that conditions in our problem do not change, (again see [10] and Lemma 2.3). There is a solution $\bar{u}$, known as the c-transpose of u, which satisfies a barred form of (1.5). In particular,

$$\bar{w}_{st}(\bar{x})=\bar{u}_{st}(\bar{x})+c_{st}(\bar{T}(\bar{x}),\bar{x}).$$

Applying the same arguments, we find that also

$$\lambda\chi\geq\delta/5-\lambda\bar{w}_{\gamma\gamma}. \tag{2.9}$$

Then, using the Urbas formula [11],[10, 2.13]

$$(\beta\cdot\gamma)^2=w^{ij}\gamma_i\gamma_j w_{\beta\beta}$$

or

$$\chi^2=\bar{w}_{\gamma\gamma}w_{\beta\beta}, \tag{2.10}$$

we have combined (2.8) to (2.10)

$$\chi\geq\frac{\delta}{10\lambda}=\theta. \tag{2.11}$$

Corollary 2.3. *The following holds, regarding the angle between β and γ*

$$\angle(\beta,\gamma)\leq\Delta<\pi/2.$$

2.2. Cost-convexity

Lemma 2.3. *Suppose $u\left(x\right)$ is a solution to (1.5) on a domain in $\mathbb{R}^n$. If $D^2u\geq 2\epsilon_0 I$, and the cost function differs from the Euclidean cost function* *by less than ϵ_0 in C^2, then u is c-convex, and the mapping $T(x,u)$ is one to one.*

Proof. Write the cost as $c = -x \cdot y + \phi(x,y)$, where ϕ is small in $C^2(\Omega \times \bar{\Omega})$. At a point x_0, we have $Du(x_0) = -D_x c\,(x_0, T(x_0, Du)) = T(x_0, Du) - D_x\phi(x_0, T(x_0))$. At another point, x_1

$$\langle Du(x_1) - Du(x_0), x_1 - x_0\rangle \geq 2\epsilon_0 |x_1 - x_0|^2.$$

Now suppose that u is not strictly c-convex. Clearly the issue would have to be nonlocal, as locally,

$$D^2u + D^2c \geq (2\epsilon_0 - \epsilon_0)\, I > 0.$$

Thus we can assume that there is a point x_0 and a locally supporting cost function

$$-c_{y_0}(x) = x \cdot T(x_0) - \phi(x, T(x_0)),$$

which contacts u from below near x_0 but touches u (possibly transversely) at a point x_1. It follows that:

$$\langle -Dc_{y_0}(x_1) + Dc_{y_0}(x_0), x_1 - x_0\rangle \geq \langle Du(x_1) - Du(x_0), x_1 - x_0\rangle,$$

that is

$$\|D\phi\|_{C^{0,1}} |x_1 - x_0|^2 \geq 2\epsilon_0 |x_1 - x_0|^2,$$

a contradiction of the smallness of $\left\|D^2\phi\right\|$. It follows that u is c-convex.

We have shown that $T(x, Du)$ is a single-valued map. To show that T is one to one, we argue that the same property holds for a solution $\bar{u}$ to a barred version of (1.5). This shows that the solution of the Kantorivich problem can be seen as a map from barred variable to unbarred as well. It follows that T is one to one. □

2.3. Boundary estimate

Let

$$M = \max_{|e|=1, e\in\mathbb{R}^n, x\in\Omega} w_{ee}$$

be the maximum of all eigenvalues of w over all of Ω. Throughout this section we will assume that the maximum occurs on the boundary.

Recalling (2.3) and Lemma 2.2, we may choose a C_6 so that

$$L(C_6 M^{n-2/n-1} h - \bar{h}(T(x, Du)) \geq 0.$$

Since $h, \bar{h}$ both vanish on the boundary, the derivatives must satisfy

$$D_\beta \bar{h} \circ T(x, Du) \leq C_6 M^{n-2/n-1},$$

that is

$$\bar{h}_s T_i^s \beta_i = \bar{h}_s c^{sj} w_{ij} \bar{h}_t c^{ti} = w_{\beta\beta} \leq C_6 M^{n-2/n-1}.$$

Lemma 2.4. *At a point x_0 on the boundary $\partial\Omega$, suppose $w_{ee} \leq M$ for unit directions e which are tangential to the boundary. If z is any vector in $T_{x_0}\Omega$, then*

$$w_{zz} \leq M|\hat{z}|^2 + \frac{1}{\theta^2}\langle z, \nabla h\rangle^2 w_{\beta\beta},$$

where

$$\hat{z} = z - \frac{\gamma \cdot z}{\gamma \cdot \beta}\beta = z - y,$$

and θ is defined by (2.11).

Proof. Dotting with γ verifies $\hat{z}$ is tangential, thus

$$0 = \partial_{\hat{z}} \bar{h} \circ T(x, Du) = \bar{h}_s T_j^s \hat{z}_j = -\bar{h}_s c^{is} w_{ij} \hat{z}_j.$$

Now

$$w_{zz} = w_{\hat{z}\hat{z}} + 2w_{\hat{z}y} + w_{yy},$$

but

$$w_{\hat{z}y} = -\frac{\gamma \cdot z}{\gamma \cdot \beta} w_{ij} \hat{z}_j \bar{h}_s c^{is} = 0,$$

so

$$w_{zz} \leq M|\hat{z}|^2 + \left(\frac{\gamma \cdot z}{\gamma \cdot \beta}\right)^2 w_{\beta\beta}.$$

□

Now suppose that the maximum tangential derivative $w_{11} = M^{\mathrm{T}}$ happens at a point x_0, where e_1 is a tangential direction. Define the function

$$\begin{aligned}\eta = w_{11} - M^{\mathrm{T}}|\hat{e}_1(x)|^2 - \ C_6\frac{1}{\theta^2}\langle e_1, \nabla h(x)\rangle^2 M^{n-2/n-1}\\ + C_7(M+1)(h + \bar{h}\circ T),\end{aligned}$$

where

$$|\hat{e}_1(x)|^2 = \left| e_1 - \frac{h_1(x)}{\xi(\bar{h}_s(T)(-c^{sk})h_k(x,T))}\beta\right|^2,$$

with ξ a smooth function satisfying $\xi(t) = t$ for $t > \theta/2$, and $\xi(t) \geq \theta/4$. Now computing, using Lemma 2.1, (2.2), and Corollary 2.2

$$\begin{aligned}L\eta \geq \delta w_{11}^2 - (C_4 + C_3)\,\bar{W}W^2 - C(n) - M\left|L|\hat{e}_1(x)|^2\right|\\ - C_6\frac{1}{\theta^2}|L\ \langle e_1, \nabla h(x)\rangle^2|M^{n-2/n-1}\\ + C_7(M+1)\left\{\frac{9}{10}\delta\left(\bar{W} + W\right) - \left(\frac{11}{10}\right)\right\},\end{aligned}$$

and using (considering Lemma 2.2)

$$\begin{aligned}\left|L|\hat{e}_1(x)|^2\right| \leq C_8(\bar{W} + 1 + W)\\ \left|LC_6\langle e_1, \nabla h(x)\rangle^2\right| \leq C_8(\bar{W} + 1 + W),\end{aligned}$$

we may choose

$$C_7 = C_8 + (C_4 + C_3)\left(\bar{M} + M\right),$$

so that

$$L\eta \geq 0.$$

Next we show a lower bound on $D_\beta w_{11}(x_0)$. First, observe that due to Lemma 2.4, η has a maximum at x_0. It follows from the Hopf maximum principle that $D\eta\cdot\beta = \nu\gamma\cdot\beta\ \geq 0$. Thus (recalling $h_1(x_0) = 0$)

$$\begin{aligned}D_\beta w_{11}(x_0) \geq M^T D_\beta|\hat{e}_1|^2 + D_\beta C_6\langle e_1, \nabla h(x)\rangle^2 M^{n-2/n-1}\\ - \left\{C_8 + (C_4 + C_3)\,(\bar{M} + M)\right\} M(D_\beta h + D_\beta\bar{h}\circ T)\\ \geq - C(n)M^T - \left\{C_8 + (C_4 + C_3)\left(\bar{M} + M\right)\right\}\\ \times C_6(n)(1 + M^{2n-3/n-1}).\end{aligned} \tag{2.12}$$

Finally we will derive a relation between the maximum M of all eigenvalues of w and for tangential eigenvalues M^{T}. Go to the point where the maximum of all eigenvalues for w happens. (Again, in this section we assume this happens along the boundary.) We diagonalize $w = diag(M, \lambda_2, \ldots \lambda_n)$ with respect to some coordinates $e_{1}, \ldots, e_n$, choosing $e_1 \cdot \gamma \geq 0$. Now

$$w_{\beta\beta} = (\beta \cdot e_1)^2 M + (\beta \cdot e_2)^2 \lambda_2 + \ldots (\beta \cdot e_n)^2 \lambda_n \leq C_6(n) M^{n-2/n-1},$$

thus

$$(\beta \cdot e_1)^2 \leq C_6(n) M^{-1/n-1}. \tag{2.13}$$

It follows that there is a C_{10} depending on $C_6(n)$ and Δ, (recall Corollary 2.3) such that if $M \geq C_{10}$, then

$$|\angle(\beta, e_1) - \pi/2| < \frac{1}{2}(\pi/2 - \Delta),$$

in particular

$$\angle(\gamma, e_1) \geq \frac{1}{2}(\pi/2 - \Delta).$$

Thus the length of projection of the maximum eigenvector of w onto the tangent plane is at least some value σM depending on Δ. So we may assume that either $M \leq C_{10}$, or the maximum tangential value M^{T} satisfies $M^{\mathrm{T}} \geq \sigma M$.

Proposition 2.1. *Suppose that the global maximum for eigenvalues of w is attained along the boundary. Then if $M \geq C_{10}$, M must satisfy*

$$M^2 - (C_4 + C_3) M^{n+1} \leq C_{11}.$$

Proof. Differentiating $\bar{h} \circ T(x, Du)$ twice tangentially,

$$\begin{aligned} \partial_{11} \bar{h} \circ T(x, Du)) &= \bar{h}_s T^s_{11} + \bar{h}_{st} T^s_1 T^t_1 = -\langle \nabla \bar{h} \circ T, S(\partial_1, \partial_1) \rangle \\ &= \bar{h}_p \left(-c^{pk} w_{11,k} - c^{pk} c_{11s} c^{sm} w_{mk} + c^{pk} c_{k1s} c^{sm} w_{m1} \right) \\ &\quad + \bar{h}_p \left(c^{pa} c^{sk} c_{as1} - c^{pa} c^{sk} c_{asv} c^{vm} w_{m1} T^s_1 \right) w_{k1} \\ &\quad + \bar{h}_{st} c^{si} w_{i1} c^{ti} w_{j1} \end{aligned}$$

as in [9, 4.11], with S denoting the second fundamental form of the boundary of Ω. Now using $-\bar{h}_p c^{pk} w_{11,k} = w_{11,\beta}$, (2.12) and the discussion in the previous paragraph (where σ is defined) we conclude that if $M \geq C_{10}$,

$$\delta\sigma M^2 - C(n)M^T - \left\{C_8 + (C_4 + C_3)\left(\bar{M} + M\right)\right\} C_6(n)M^{2n-3/n-1} - C_3M^2 \\ \leq C_6M^{n-2/n-1}.$$

Using Young's inequality to clean up the expression, we have

$$M^2 - (C_4 + C_3)\, M^{n+1} \leq C_{11}. \tag{2.14}$$

□

3. Proof of Theorem 1.1

We now go through the alternatives and make our choice of constants, in order to bound w and consequently D^2u.

First, if the maximum happens in the interior, then (2.6)

$$M^2 - (C_4 + C_3)\, M^{n+1} \leq C_{12}. \tag{3.1}$$

If not, then either (2.14)

$$M^2 - (C_4 + C_3)\, M^{n+1} \leq C_{11} \tag{3.2}$$

or

$$M \leq C_{10}, \tag{3.3}$$

by the discussion surrounding (2.13).

So we simply must choose $(C_4 + C_3)$ small enough, say

$$(C_4 + C_3) \leq \varepsilon_0,$$

so that the noncompact region defined by (3.1) does not intersect the compact regions defined by (3.2) and (3.3), similarly for the noncompact region defined by (3.2). Further, in order to have c-convexity, we must assume that the conditions of Lemma 2.3 are satisfied. The upper bounds in the above alternatives provide lower bounds on the Hessian, so we choose C_3 small enough so that Lemma 2.3 is satisfied.

Now by the theory of Delanoe [4], Caffarelli [1] and Urbas [11], we have a classical solution to the problem for Euclidean cost

$$c^0(x, y) = -x \cdot y$$

in Euclidean space.

We use the method of continuity. Openness is provided by Theorem 17.6 in [6], where we set

$$G : C^{2,\alpha}(\Omega) \times [0,1] \to C^{0,\alpha}(\Omega) \times C^{1,\alpha}(\partial\Omega)$$

with

$$G(u,t) = \begin{pmatrix} \ln\det\left[u_{ij} + c^{(t)}_{ij}(x, T^{(t)}(x,Du)\right] + f(x) - \bar{f}(T^{(t)}(x,Du)) \\ -\ln\det\left[-c^{(t)}_{is}(x, T^{(t)}(x,Du))\right],\ \bar{h}(T^{(t)}(x,Du)) \end{pmatrix},$$

where the cost function is changing from Euclidean to c as

$$c^{(t)} = (1-t)c^0 + tc$$

and $T^{(t)}$ defined by

$$D_x c^{(t)}(x, T^{(t)}(x,Du)) + Du = 0.$$

Our initial solution u_0 is smooth , so it satisfies the above estimates (3.1, etc) with $C_{3,}C_4 = 0$. These bounds change continuously with t so D^2u must stay in the compact components of (3.1) to (3.3). As is standard for this problem, we cite [8] to obtain the $C^{2,\alpha}$ estimates. By [6, Theorem 17.6], we have openness in t, and the estimates give us closeness as long as $\left|D^4c^{(t)}\right|, \left|D^3c^{(t)}\right| \leq \varepsilon_0$. This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

First we employ a change of coordinates (here we use (A2)), so that

$$c_{is}(x_0, \bar{x}_0) = -I_n.$$

Proof. Then, on a product of very small balls $B_{1/\lambda}(x_0) \times B_{1/\lambda}(\bar{x}_0)$ we have

$$\frac{1}{C_2}\left|\xi\right|^2 \leq -c^{si}\xi_i\xi_s \leq C_2\left|\xi\right|^2,$$

for some C_2 near 1, and $\left|D^3c\right|, \left|D^4c\right| \leq C$ which may be large but finite.

We now rescale and consider the following problem on $B_1(0) \times B_1(\bar{0})$: let

$$c^{(\lambda)}(y,\bar{y}) = \lambda^2 c\left(\frac{y}{\lambda}, \frac{\bar{y}}{\lambda}\right)$$

be the cost function, and let the distributions to be transported be Gaussians, satisfying (a1-3) on $B_1(0), B_1(\bar{0})$.

This cost function $c^{(\lambda)}$ now satisfies the conditions in our first theorem, as we see that choosing λ large enough will make the third and fourth derivatives arbitrarily small. The first derivatives are perhaps quite large, but we are free to subtract a linear cost without changing properties of the solution.

It follows by Theorem 1.1 that the solution to this rescaled optimal transportation problem is smooth. However, the coordinate change and "change of currency" do not change the underlying optimal transportation problem. Thus, we also have smoothness for the solution of the problem sending

$$m = e^{-\lambda^2|x-x_0|^2/2}\chi_{B_{1/\lambda}(x_0)}$$

to

$$\bar{m} = e^{-\lambda^2|\bar{x}-\bar{x}_0|^2/2}\chi_{B_{1/\lambda}(\bar{x}_0)}.$$

This completes the proof. □

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