

On existence of the prescribing k -curvature problem on manifolds with boundary

YAN HE AND WEIMIN SHENG

In this paper, we study the problem of conformally deforming a metric to a prescribed k th order symmetric function of the eigenvalues of the Schouten tensor on compact Riemannian manifolds with totally geodesic boundary. We prove the solvability of the problem and the compactness of the solution set for the case $k \geq n/2$, provided the conformal class admits a k -admissible metric. These results have been proved by Gursky and Viaclovsky, Trudinger and Wang for the manifolds without boundary, and by Jin et al. and S. Chen for the locally conformally flat manifolds with boundary.

1. Introduction

In this paper, we study the existence of the solution to a prescribing k -curvature problem on manifolds with boundary and the compactness of the solution set.

Let (M^n, g) be a smooth, compact Riemannian manifold of dimension $n \geq 3$. The Schouten tensor of g is defined by

$$A_g = \frac{1}{n-2} \left(\text{Ric}_g - \frac{R_g}{2(n-1)}g \right),$$

where Ric and R are the Ricci and scalar curvatures of g , respectively. Let $[g]$ be the set of metrics conformal to g . For $\tilde{g} = e^{-2u}g \in [g]$, we consider the equation

$$(1.1) \quad \sigma_k^{1/k}(\lambda(\tilde{g}^{-1}A_{\tilde{g}})) = f(x),$$

where $\sigma_k : R^n \rightarrow R$ denotes the k th elementary symmetric function ($1 \leq k \leq n$), and $\lambda(g^{-1}A_g)$ the eigenvalues of $g^{-1}A_g$. $\sigma_k(\lambda(g^{-1}A_g))$ is called k -curvature. The Schouten tensor transforms according to the formula

$$A_{\tilde{g}} = \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + A_g,$$

where ∇u and $\nabla^2 u$ denote the gradient and Hessian of u with respect to g . Consequently, (1.1) is equivalent to

$$(1.2) \quad \sigma_k^{1/k} \left(\lambda \left(g^{-1} \left[\nabla^2 u + du \otimes du - \frac{1}{2} |\nabla u|^2 g + A_g \right] \right) \right) = f(x) e^{-2u}.$$

Let $\Gamma_k \subset R^n$ denote the component of $\{x \in R^n \mid \sigma_k(x) > 0\}$ containing the positive cone $\{x \in R^n \mid x_1 > 0, \dots, x_n > 0\}$ and $[g]_k = \{\tilde{g} \in [g] \mid \lambda(\tilde{g}^{-1} A_{\tilde{g}}) \in \Gamma_k\}$. We call a metric in $[g]_k$ k -admissible, or simply admissible. The function u is called k -admissible, if $e^{-2u} g \in [g]_k$. The k -Yamabe problem is to find a solution of (1.2) with $f(x) = \text{constant}$ for some given metric g on M^n with $g \in \Gamma_k$. When $k = 1$, it reduces to the classical Yamabe problem. For compact manifolds without boundary, the classical Yamabe problem (i.e., $k = 1$) has been solved by Yamabe [1], Trudinger [2], Aubin [3] and Schoen [4]. For $k \geq 2$, the existence of the solutions to the k -Yamabe equation ((1.2) for $f(x) = \text{constant}$) has been solved for the cases $k = 2, n = 4$ [5, 6], $k = 2, n \geq 4$ [12], $k = n/2$ [7], $k > n/2$ [8, 9] or for locally conformally flat manifolds [10–12, 33]. The compactness of the solution set has also been proved when $k \geq n/2$ in the above papers, and for locally conformally flat case, by [13].

For compact Riemannian manifold (M^n, g) with non-empty smooth boundary ∂M , there are two classes of boundary conditions for the existence problem of equation (1.2). One is the Dirichlet boundary condition, which has been studied by Bo Guan in [14]. Another is the Neumann problem, has been studied by S. Chen, Jin–Li–Li and Li–Li [15–18], etc. For $k = 1$, there are also several results (e.g. [19–22], etc.). Under various conditions, they derive local estimates for solutions and establish some existence results.

In this paper, we are interested in the case $k \geq n/2$ with the Neumann boundary condition. Under the assumption that the boundary is totally geodesic, we obtain the existence of the solutions to the Neumann problem and the compactness of the solution set. In fact, we have the following

Theorem 1.1. *Let (M^n, g) be compact n -dimensional Riemannian manifold with totally geodesic boundary, $n \geq 3$. Assume g is k -admissible with $k > n/2$ and M is not conformally equivalent to the standard hemisphere. Then for any given smooth positive function $f \in C^\infty(M)$, there exists a smooth function $u \in C^\infty(M)$ such that the conformal metric $\tilde{g} = e^{-2u} g$ satisfies*

$$\sigma_k^{1/k} (\lambda(\tilde{g}^{-1} A_{\tilde{g}})) = f(x)$$

and M has totally geodesic boundary under \tilde{g} . In addition, the set of all such solutions is compact in the C^m -topology for any $m \geq 0$.

Theorem 1.2. *Let (M^n, g) be a compact n -dimensional Riemannian manifold with totally geodesic boundary, $n \geq 4$. Assume that g is k -admissible with $k = n/2$ and (M, g) is not conformally equivalent to (S_n^+, g_c) , where g_c is the standard metric on the hemisphere. Then for any given smooth positive function $f \in C^\infty(M)$ there exists a smooth function $u \in C^\infty(M)$ such that the conformal metric $\tilde{g} = e^{-2u}g$ satisfies*

$$\sigma_k^{\frac{1}{k}}(\lambda(\tilde{g}^{-1}A_{\tilde{g}})) = f(x)$$

and M has totally geodesic boundary under \tilde{g} . In addition, the set of all such solutions is compact in the C^m -topology for any $m \geq 0$.

Here Theorem 1.1 generalizes a result in [17] where it assumes that (M, g) is locally conformally flat near ∂M . Theorem 1.2 improves a corresponding result in [15] and [17] for the case $k = n/2$, where $f(x) \equiv \text{constant}$, and (M, g) is locally conformally flat.

Recall that the second fundamental form L of ∂M with respect to g is defined as

$$L(X, Y) = -g(\nabla_X \nu, Y), \quad X, Y \in T_x(\partial M),$$

where $T_x(\partial M)$ denotes the tangent space of ∂M at x , ν is the unit inward normal vector field to ∂M in (M, g) , ∇ is the Levi-Civita connection of g . A point $x \in \partial M$ is umbilic if $L(X, Y) = \tau_g(x)g(X, Y)$ for all $X, Y \in T_x(\partial M)$. The boundary is called umbilic if every point of ∂M is umbilic. A totally geodesic boundary is umbilic with $\tau_g \equiv 0$. Note that the umbilicity is conformally invariant. In fact, we have

$$\tilde{L}(X, Y) e^u = \frac{\partial u}{\partial \nu} g(X, Y) + L(X, Y), \quad \text{for any } X, Y \in T_x(\partial M),$$

where \tilde{L} denotes the second fundamental form of ∂M with respect to $\tilde{g} = e^{-2u}g$. When the boundary is umbilic, the above formula becomes

$$\tau_{\tilde{g}} e^{-u} = \frac{\partial u}{\partial \nu} + \tau_g.$$

Especially, if (M, g) has totally geodesic boundary and the conformal metric \tilde{g} has totally geodesic boundary as well, then the k -Yamabe problem with totally geodesic boundary becomes to consider the following equation:

$$(1.3) \quad \begin{cases} \sigma_k^{1/k} \left(\lambda \left(g^{-1} \left[\nabla^2 u + du \otimes du - \frac{1}{2} |\nabla u|^2 g + A_g \right] \right) \right) = f(x) e^{-2u}, & \text{in } M, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial M. \end{cases}$$

For $k > \frac{n}{2}$, we study (1.3) along the line of [8]. By use of the deformation (2.1), which is defined in [8], we may get (1.3) when $t = 1$ and the equation when $t = 0$ is easier to analyze. The Leray–Schauder degree, defined in [23]([15] for the boundary case) is non-zero. By homotopy-invariance of the degree, the existence of the solution reduces to establish a priori estimates for $t \in [0, 1]$. To prove this, we argue it by contradiction. Assuming that there exists a sequence of solutions $\{u_i\}$ which fails on C^0 -bound, we study its blow-up. In Section 3, we prove that there are only finite blow-up points. Then we show that the super limit of solution u_i is $+\infty$ at regular point in Section 4. Hence, in Section 5, we can obtain a better rescaled function w_i . Then by gluing two copies of M along the boundary, we derive a $C_{\text{loc}}^{1,1}$ function \tilde{w} on a closed $C^{2,1}$ manifold \widetilde{M} . Therefore, by the argument in sections 6 and 7 of [8], we know $(M^n, e^{-2w}g)$ is in fact the half-plane in Euclidean space, which contradicts with the condition that the manifold is not conformally equivalent to standard hemisphere.

However, when $k = \frac{n}{2}$, the Ricci tensor is only positive semi-definite, it is not enough to prove the existence as the case $k > n/2$. So, we need to employ another method. In [7], Trudinger and X.-J. Wang provided an another approach. By analyzing the asymptotic behavior of the solution at singular points, they prove the existence of the solutions to equation (1.2) for manifolds without boundary. By the similar argument as [7], we may glue two copies of M along the boundary as above, and prove Theorem 1.2 in Section 6.

2. Deformation and C^1 and C^2 estimates

2.1. Deformation

To prove the existence of solution to the equation (1.3), we employ the following deformation which is defined in [24]:

$$(2.1) \quad \begin{cases} \sigma_k^{1/k} \left(\lambda \left(g^{-1} \left[\lambda_k (1 - \psi(t))g + \psi(t)A_g + \nabla^2 u + du \otimes du \right. \right. \right. \\ \left. \left. \left. - \frac{1}{2} |\nabla u|^2 g \right] \right) \right) = \psi(t)f(x)e^{-2u} + (1-t)(\int e^{-(n+1)u})^{\frac{2}{n+1}}, & \text{in } M, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial M. \end{cases}$$

where $\psi \in C^1[0, 1]$ satisfies $0 \leq \psi(t) \leq 1$, $\psi(0) = 0$, $\psi(t) = 1$ for $t \geq \frac{1}{2}$; and $\lambda_k = \binom{n}{k}^{-\frac{1}{k}} \text{vol}(M, g)^{\frac{2}{n+1}}$.

As [24], at $t = 1$ (2.1) becomes (1.3). While at $t = 0$, it becomes

$$\begin{cases} \sigma_k^{1/k} \left(\lambda \left(g^{-1} \left[\lambda_k g + \nabla^2 u + du \otimes du - \frac{1}{2} |\nabla u|^2 g \right] \right) \right) \\ = \left(\int e^{-(n+1)u} \right)^{\frac{2}{n+1}}, & \text{in } M, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial M. \end{cases}$$

It has been pointed out in [24] that the above equation has a unique solution $u(x) \equiv 0$, if $\partial M = \emptyset$. We can show it is also true in our case.

In fact, it is obvious that $u = 0$ is a solution. Now we can show its uniqueness.

At the maximum point x_0 of u , no matter x_0 is an interior or a boundary point, it is true that $\nabla u|_{x_0} = 0$, and $\nabla^2 u|_{x_0}$ is negative semi-definite. In fact, if x_0 is interior point, it is easy to be understood. Now we consider the case that x_0 is boundary point. Since $\frac{\partial u}{\partial \nu}|_{\partial M} = 0$, we have $\frac{\partial u}{\partial \nu}|_{x_0} = 0$ and $\frac{\partial^2 u}{\partial \nu \partial x^\alpha}|_{x_0} = 0$. It is also true that $\frac{\partial u}{\partial x^\alpha}|_{x_0} = 0$ and $\frac{\partial^2 u}{\partial x^\alpha \partial x^\beta}|_{x_0}$ is negative semi-definite, where $\{x^\alpha\}_{1 \leq \alpha \leq n-1}$ is a local coordinates on the boundary ∂M around x_0 . For $\frac{\partial^2 u}{\partial \nu^2}|_{x_0}$, we denote $u(x) = u(x', x^n)$, $x_0 = (x'_0, 0)$. For fixed x'_0 , we consider $u = u(x', x^n)$ as the function of x^n . Since $u(x_0) = u(x'_0, 0)$ is the maximum of $u(x', x^n)$ for $0 \leq x^n < \epsilon$, $\frac{\partial^2 u}{(\partial x^n)^2}|_{x_0} \leq 0$. This means that $\frac{\partial^2 u}{\partial \nu^2}|_{x_0}$ is nonpositive. Therefore, $\nabla^2 u|_{x_0}$ is negative semi-definite. Now at x_0 we have

$$\begin{aligned} \lambda_k \binom{n}{k}^{1/k} &= \lambda_k \sigma_k^{\frac{1}{k}} (\lambda (g^{-1} \cdot g)) \\ &\geq \sigma_k^{1/k} \left(\lambda \left(g^{-1} \left[\nabla^2 u + du \otimes du - \frac{1}{2} |\nabla u|^2 g + \lambda_k g \right] \right) \right) \\ &= \left(\int e^{-(n+1)u} \right)^{\frac{2}{n+1}}. \end{aligned}$$

Similarly, at the minimum point of u , we have $\lambda_k \binom{n}{k}^{1/k} \leq \left(\int e^{-(n+1)u} \right)^{\frac{2}{n+1}}$. Therefore, $\lambda_k \binom{n}{k}^{1/k} = \left(\int e^{-(n+1)u} \right)^{\frac{2}{n+1}}$.

By Newton-MacLaurin inequality, we can immediately get $\sigma_k^{1/k} \leq \frac{1}{n} \binom{n}{k}^{1/k} \sigma_1$. Hence,

$$\begin{aligned} \lambda_k \binom{n}{k}^{1/k} &= \sigma_k^{1/k} \left(\lambda \left(g^{-1} \left[\lambda_k g + \nabla^2 u + du \otimes du - \frac{1}{2} |\nabla u|^2 g \right] \right) \right) \\ &\leq \frac{1}{n} \binom{n}{k}^{1/k} \sigma_1 \left(\lambda \left(g^{-1} \left[\lambda_k g + \nabla^2 u + du \otimes du - \frac{1}{2} |\nabla u|^2 g \right] \right) \right) \\ &= \frac{1}{n} \binom{n}{k}^{1/k} \left(\Delta u + \left(1 - \frac{n}{2} \right) |\nabla u|^2 + n \lambda_k \right). \end{aligned}$$

Then

$$\left(\frac{n}{2} - 1\right) \int_M |\nabla u|^2 \leq \int_M \Delta u = \int_{\partial M} \frac{\partial u}{\partial \nu} = 0,$$

and $u \equiv \text{constant} = 0$.

Thus the operator

$$\begin{aligned} \Psi_t[u] = & \sigma_k^{1/k} \left(\lambda \left(g^{-1} \left[\lambda_k (1 - \psi(t)) g + \psi(t) A_g + \nabla^2 u + du \otimes du - \frac{1}{2} |\nabla u|^2 g \right] \right) \right) \\ & - \psi(t) f(x) e^{-2u} - (1 - t) \left(\int e^{-(n+1)u} \right)^{\frac{2}{n+1}} \end{aligned}$$

satisfies Leray–Schauder degree $\deg(\Psi_0, \mathcal{O}_0, 0) \neq 0$ at $t = 0$, where the Leray–Schauder degree is defined by [23] ([15] for the boundary case) and \mathcal{O}_0 is a neighborhood of the zero solution in $\{u \in C^{4,\alpha}(M) : u \text{ is } k\text{-admissible, } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial M\}$. When we obtain the homotopy-invariance of degree, we can derive that the Leray–Schauder degree is non-zero at $t = 1$ which implies equation (1.3) is solvable.

2.2. C^1 and C^2 estimates

We use Fermi coordinates in a boundary neighborhood at first. In this local coordinates, we take the geodesic in the inner normal direction $\nu = \frac{\partial}{\partial x^n}$ parameterized by arc length, and (x^1, \dots, x^{n-1}) forms a local chart on the boundary. The metric can be expressed as $g = g_{\alpha\beta} dx^\alpha dx^\beta + (dx^n)^2$. The Greek letters $\alpha, \beta, \gamma, \dots$ stand for the tangential direction indices, $1 \leq \alpha, \beta, \gamma, \dots \leq n - 1$, while the Latin letters i, j, k, \dots stand for the full indices, $1 \leq i, j, k, \dots \leq n$. For Fermi coordinates, see [25] for details. In Fermi coordinates, the half ball is defined by $\bar{E}_r^+ = \{x_n \geq 0, \sum_i x_i^2 \leq r^2\}$ and the segment on the boundary by $\Sigma_r = \{x_n = 0, \sum_i x_i^2 \leq r^2\}$. We denote the RHS of the first equation of (2.1) by $h(x, u)$ and define

$$\begin{aligned} \overline{c_{\text{sup}}}(r) = & \sup_{\bar{E}_r^+} \left(|h| + |\nabla_x h(x, u)| + |h_z(x, u)| + |\nabla_x^2 h(x, u)| \right. \\ & \left. + |\nabla_x h_z(x, u)| + |h_{zz}(x, u)| + \left| \frac{|\nabla_x h(x, u)|}{\inf_M h} \right| + \left| \frac{|h_z(x, u)|}{\inf_M h} \right| \right). \end{aligned}$$

Note that the constant $(1-t)(f e^{-(n+1)u})^{\frac{2}{n+1}}$ is less than $\sigma_k^{1/k}(\lambda(g^{-1}[\lambda_k(1-\psi(t))g + \psi(t)A_g]))$. According to Theorem 1 in [28], we have

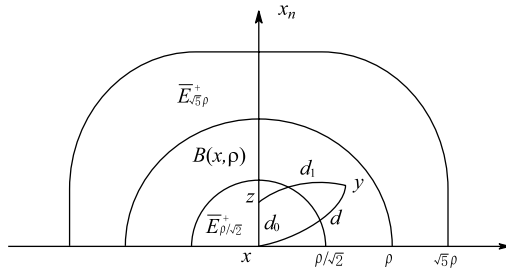
$$(2.2) \quad \sup_{\overline{E}_{r/2}^+} (|\nabla u|^2 + |\nabla^2 u|) \leq C_1 \cdot \overline{c}_{\text{sup}}(r) = C_2 \cdot (1 + e^{-2 \inf_{\overline{E}_r^+} u}),$$

where $C_2 = C_2(n, g, r, f)$.

Now we can immediately get a boundary estimate on the geodesic half ball $B(x, r) = \{y \in M \mid \text{dist}(x, y) < r\}$, since there is a following relationship between the half balls in Fermi coordinates and the half geodesic balls:

$$\overline{E}_{\rho/\sqrt{2}}^+ \subset B(x, \rho) \subset \overline{E}_{\sqrt{5}\rho}^+.$$

In fact, we may assume the fermi coordinate of y is (y_1, \dots, y_n) , z is on the x_n -axis satisfying $\text{dist}(z, y) = \text{dist}(x_n\text{-axis}, y)$ and let $d = \text{dist}(x, y)$, $d_0 = \text{dist}(x, z)$ and $d_1 = \text{dist}(z, y)$. Now for any $y \in \overline{E}_{\rho/\sqrt{2}}^+$, we have $d_0^2 + d_1^2 = \sum_{\alpha} (y_{\alpha})^2 + (y_n)^2 < \rho^2/2$. Thus the triangle inequality implies $d^2 = \text{dist}(x, y)^2 \leq (\text{dist}(x, z) + \text{dist}(z, y))^2 \leq 2(d_0^2 + d_1^2) < \rho^2$. Therefore, $y \in B(x, \rho)$. On the other hand, for any $y \in B(x, \rho)$, we have $d^2 < \rho^2$. Thus $d_1^2 = \text{dist}(z, y)^2 \leq \text{dist}(x, y)^2 = d^2 < \rho^2$ and $d_0 = \text{dist}(z, x) \leq 2\text{dist}(x, y) = 2d < 2\rho$, otherwise $d \geq d_0 - d_1 > 2d - d = d$. Hence, $d_0^2 + d_1^2 < 5\rho^2$ and $y \in \overline{E}_{\sqrt{5}\rho}^+$.



Then (2.2) implies

$$(2.3) \quad \begin{aligned} \sup_{B(x_0, r)} (|\nabla u|^2 + |\nabla^2 u|) &\leq \sup_{\overline{E}_{\sqrt{5}r}^+} (|\nabla u|^2 + |\nabla^2 u|) \\ &\leq C_3 \left(1 + e^{-2 \inf_{\overline{E}_{2\sqrt{5}r}^+} u} \right) \\ &\leq C_3 \left(1 + e^{-2 \inf_{B(x_0, 2\sqrt{10}r)} u} \right), \end{aligned}$$

where x_0 is a boundary point and $C_3 = C_3(n, g, r, f)$.

We can get interior estimate as well. Let

$$\begin{aligned} c_{\text{sup}}(r) = & \sup_{B(x_0, r)} (|h| + |\nabla_x h(x, u)| + |h_z(x, u)| + |\nabla_x^2 h(x, u)| \\ & + |\nabla_x h_z(x, u)| + |h_{zz}(x, u)| + \left| \frac{|\nabla_x h(x, u)|}{\inf_M h} \right| + \left| \frac{|h_z(x, u)|}{\inf_M h} \right|), \end{aligned}$$

where x_0 is an interior point. Then by the estimates in [26, 27] (or Theorem 2 in [28]) we have

$$\sup_{B(x_0, r/2)} (|\nabla u|^2 + |\nabla^2 u|) \leq C_4 \cdot \left(1 + e^{-2 \inf_{B(x_0, r)} u}\right),$$

where $C_4 = C_4(n, g, r, f)$.

Now we may assume that $\inf_M u_i \rightarrow -\infty$. Otherwise, by above estimate and Harnack inequality we can get the upper bound, therefore completes the proof.

There are following two possibilities.

(A) One is that the blow-up subsequence u_{t_i} happens at $t_i \leq 1 - \delta < 1$ for $\delta > 0$. We still denote it by u_i for simplicity. Then, at the maximum point of u_i , which is either an interior point or a boundary point, we have

$$\delta \left(\int e^{-(n+1)u_i} \right)^{\frac{2}{n+1}} \leq \sigma_k^{1/k} \left(\lambda \left(g^{-1} \left[\lambda_k (1 - \psi(t)) g + \psi(t) A_g \right] \right) \right) \leq C_0.$$

Then we can take $\epsilon_i = e^{\inf_M u_i} \triangleq e^{u_i(z_i^0)}$, where $z_i^0 \in M$ is u_i 's minimum point. Defining a map:

$$\begin{aligned} \mathcal{T}_i : \mathcal{B}(0, c_0) \subset T_{z_i^0} M &\rightarrow B(z_i^0, c_0 \cdot \epsilon_i) \subset M, \\ y &\rightarrow \exp_{z_i^0}(\epsilon_i y), \end{aligned}$$

where the metric on tangent space is $\tilde{g}_i = \epsilon_i^{-2} \mathcal{T}_i^* g$ and $\mathcal{B}(0, c_0)$ is a geodesic ball in $\exp_{z_i^0}^{-1}(M)$ with radius $c_0 > 0$. Then we can obtain the functions with lower bounds in $\mathcal{B}(0, c_0)$ on the tangent space $T_{z_i^0} M$: $w_i(y) = u_i(\mathcal{T}_i(y)) - \log \epsilon_i \geq 0$.

Furthermore, w_i satisfies

$$\begin{aligned} & \sigma_k^{1/k} \left(\lambda \left(\tilde{g}_i^{-1} (\epsilon_i^2 \lambda_k (1 - \psi(t_i)) \tilde{g}_i + \psi(t_i) A_{\tilde{g}_i} + \nabla^2 w_i + dw_i \otimes dw_i - \frac{1}{2} |\nabla w_i|_{\tilde{g}_i}^2 \tilde{g}_i) \right) \right) \\ & = \psi(t_i) f(\mathcal{T}_i(y)) e^{-2w_i} + \epsilon_i^2 (1 - t_i) \left(\int e^{-(n+1)u_i} \right)^{\frac{2}{n+1}}, \quad \text{in } \mathcal{B}(0, c_0). \end{aligned}$$

Then by the interior and boundary estimates on $\mathcal{B}(0, \frac{c_0}{2})$, we can obtain the upper bound of w_i immediately. We then obtain the following contradiction:

$$C(n, g, c_0, f) \leq \int_{\mathcal{B}(0, \frac{c_0}{4})} e^{-(n+1)w_i} \leq \epsilon_i \int_{B(z_i^0, c_0 \epsilon_i / 2)} e^{-(n+1)u_i} \leq \epsilon_i (C_0 / \delta)^{\frac{n+1}{2}} \rightarrow 0.$$

Therefore, we have the following boundary version of Theorem 2.1 in [24].

Lemma 2.1. *For any fixed $0 < \delta < 1$, there is a constant $C = C(\delta, n, g, f)$ such that any k -admissible solution of (2.1) with $t \in [0, 1 - \delta]$ satisfies $\|u\|_{C^{4,\alpha}} \leq C$.*

(B) So without loss of generality, we may assume that u_{t_i} tends to $-\infty$ at the time $t_i \rightarrow 1$, where u_{t_i} is the solution of (2.1) at $t = t_i$, which will be denoted by u_i in follows. Thus equation (2.1) is changed as

$$(2.4) \quad \begin{cases} \sigma_k^{1/k} \left(\lambda \left(g^{-1} (A_g + \nabla^2 u + du \otimes du - \frac{1}{2} |\nabla u|^2 g) \right) \right) \\ = (1-t)o + f(x) e^{-2u}, & \text{in } M, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial M. \end{cases}$$

where u is assumed to be k -admissible, and $o \geq 0$ is a constant.

We will obtain more exact boundary and interior estimates which are both in geodesic coordinates.

Lemma 2.2. *Let $u \in C^4(M)$ be a k -admissible solution of (2.1) in $B(x, r)$ and $0 \leq r < 1$. Then there is a constant $C = C(n, g, f)$ such that*

$$(2.5) \quad (|\nabla^2 u| + |\nabla u|^2)(x') \leq C \left(r^{-2} + e^{-2 \inf_{B(x, 2\sqrt{10}r)} u} \right).$$

for all $x' \in B(x, r)$.

Proof. If x is on the boundary ∂M , we define a local diffeomorphism

$$\begin{aligned} \mathcal{O} : \mathcal{B}(0, 2\sqrt{10}) \subset T_x M &\rightarrow B(x, 2\sqrt{10}r), \\ y &\rightarrow \exp_x(ry), \end{aligned}$$

where $\mathcal{B}(0, 2\sqrt{10})$ is the geodesic ball in $\exp_x^{-1}(M)$ with the metric $\tilde{g} = r^{-2} \mathcal{O}^* g$.

Let $w(y) = u(\mathcal{O}(y)) - \log r$ in $\mathcal{B}(0, 2\sqrt{10})$, then w satisfies

$$\begin{aligned} \sigma_k^{1/k} \left(\lambda \left(\tilde{g}^{-1} \left[A_{\tilde{g}} + \nabla^2 w + dw \otimes dw - \frac{1}{2} |\nabla w|_{\tilde{g}}^2 \tilde{g} \right] \right) \right) \\ = f(\mathcal{O}(y)) e^{-2w} + r^2(1-t)o, \quad \text{in } \mathcal{B}(0, 2\sqrt{10}). \end{aligned}$$

By a similar argument as (2.3) we can obtain

$$\begin{aligned} \sup_{B(x,r)} r^2 (|\nabla u|^2 + |\nabla^2 u|) \\ \leq \sup_{\mathcal{B}(0,1)} (|\nabla w|^2 + |\nabla^2 w|) \\ \leq C \cdot \left(e^{-2 \inf_{\mathcal{B}(0, 2\sqrt{10})} w} + 1 \right) \\ \leq C \cdot \left(e^{-2 \inf_{B(x, 2\sqrt{10}r)} u} r^2 + 1 \right). \end{aligned}$$

Therefore,

$$\sup_{B(x,r)} |\nabla^2 u|(x) + |\nabla u|^2(x) \leq C \left(r^{-2} + e^{-2 \inf_{B(x, 2\sqrt{10}r)} u} \right),$$

where $C = C(n, g, f)$.

If x is an interior point, the interior estimate in [8] has implied the result. \square

3. Finite blow-up points

In this section, we prove that there are finite blow-up points for prescribing k -curvature problem, where $k > n/2$. By [29], we have $\text{Ric}_g \geq \frac{(2k-n)(n-1)}{n(k-1)} \sigma_1(A_g)g$ for k -admissible metric g .

Lemma 3.1 below has been proved by Guan and Wang in [26] for the interior point. If we only focus on the boundary point, their proof also can be applied, we just notice that the ball is half ball, i.e., $B(x, \rho)$ means $B(x, \rho) \cap M$. Here we omit the proof.

Lemma 3.1. *There exist ς and $C = C(\varsigma, n, g, f)$ such that any solution $u \in C^2(B(x, \rho))$ with $\int_{B(x, \rho)} e^{-nu} < \varsigma$ satisfies $\inf_{B(x, \rho/2)} u \geq \log \rho - C$.*

Similar as [24], for a given point $x \in M$, we define the mass of x by

$$m(\{u_i\}; x) = \lim_{r \rightarrow 0} \limsup_{i \rightarrow \infty} \int_{B(x,r)} e^{-nu_i}.$$

We also denote $\Sigma[\{u_i\}] = \{x \mid m(\{u_i\}; x) \neq 0\}$.

By use of the ϵ -regularity result (Lemma 3.1) and the volume comparison theorem, we may get the following propositions. See [30] for their proofs.

Proposition 3.1 (Lemma 2.3 of [30]). *Given a sequence of smooth solutions of (2.4) with $\inf_M u_i \rightarrow -\infty$. Then there exists a positive constant $\mu(n, g, f, \varsigma)$ such that either $m(\{u_i\}; x) = 0$ or $m(\{u_i\}; x) \geq \mu$.*

Proposition 3.2 (Section 2 of [30]). *Suppose u_i is a blow-up sequence of (2.4), then*

- (1) $\Sigma[\{u_i\}]$ is non-empty;
- (2) $\Sigma[\{u_i\}]$ is finite.

Corollary 3.1 (Corollary 4.7 of [8]). *Let u_i be a solution of (2.4), then*

- (1) when $u_i(x_i)$ converges to $-\infty$, any accumulated point of x_i must belong to $\Sigma[\{u_i\}]$.
- (2) for any $x_0 \in \Sigma[\{u_i\}]$, there exists a subsequence $\{u_{k_i}\}$ and a sequence points x_i such that $\lim_{i \rightarrow \infty} x_i = x_0$ and $\lim_{i \rightarrow \infty} u_{k_i}(x_i) = -\infty$.

Proof. (1) Let $u_i(x_i)$ be a blow-up sequence converging to $-\infty$. Since the manifold M is compact, then there is a convergent subsequence x_{i_k} . Suppose $\lim_{k \rightarrow \infty} x_{i_k} = x_0$.

We assert that x_0 must be in $\Sigma[\{u_i\}]$. Otherwise, by Lemma 3.1, there exist constants r_0 and J such that $\inf_{B(x_0, r_0)} u_i \geq -C$, where $i \geq J$. Furthermore, we can find K such that $x_{i_k} \in B(x_0, r_0)$ for any $k \geq K$. Therefore, $u_j(x_{i_k}) \geq -C$ when $j \geq J$ and $k \geq K$, which contradicts the fact that $\lim_{k \rightarrow \infty} u_{i_k}(x_{i_k}) = -\infty$.

(2) Otherwise, there exists a neighborhood of x_0 , for example $B(x_0, r_0)$, and a constant C such that $\inf_{B(x_0, r_0)} u_i \geq -C$. Then

$$m(\{u_i\}; x_0) = \lim_{r \rightarrow 0} \limsup_{i \rightarrow \infty} \int_{B(x_0, r)} e^{-nu_i} \leq \lim_{r \rightarrow 0} \int_{B(x_0, r)} e^{nC} = 0,$$

which contradicts the fact that $x_0 \in \Sigma[\{u_i\}]$. □

For simplicity, we still denote the subsequence by $\{u_i\}$.

4. Tend to $+\infty$ in regular set

In this section, we prove that at any regular point $x \in M \setminus \Sigma[\{u_i\}]$, $\limsup_{i \rightarrow \infty} u_i(x) = +\infty$, where $\text{Ric}_g \geq \frac{(2k-n)(n-1)}{n(k-1)}\sigma_1(A_g)g$, when g is k -admissible ($k > n/2$). We prove the result by contradiction.

Suppose there is a regular point x' such that $\limsup_{i \rightarrow \infty} u_i(x') < +\infty$.

Lemma 4.1. *We can find a subsequence $u_{i_k} \rightarrow u \in C^\infty(M \setminus \Sigma[\{u_i\}])$, where the convergence is C^m on compact sets away from $\Sigma[\{u_i\}]$.*

Proof. Suppose K is a compact subset in $M \setminus \Sigma[\{u_i\}]$. Then we can find compact sets K' and \tilde{K}' , such that $K \cup \{x'\} \subset K' \subset \tilde{K}'$. Since compact set \tilde{K}' is covered by finite open sets, then by Lemma 2.2, we may get $\sup_{K'} |\nabla u_i| \leq C_1$, and $\sup_K u_i \leq \sup_{K'} u_i \leq \inf_{K'} u_i + C_2 \leq \limsup_{i \rightarrow \infty} u_i(x') + C_2 \leq C_3$. Therefore, by regularity theory, there is a subsequence u_{i_k} which converges uniformly to u on the compact set K . \square

For simplicity we still denote the subsequence u_{i_k} in Lemma 4.1 by u_i .

Lemma 4.2. *Let $u \in C^2(M)$. Assume $g_u = e^{-2u}g$ has non-negative scalar curvature, where $\frac{\partial u}{\partial \nu}|_{\partial M} = 0$. Suppose there is a ball $B(x, \rho) \subset M$ and constants $\alpha_0 > 0$ and $B_0 > 0$ with*

$$(4.1) \quad \int_{B(x, \rho)} e^{\alpha_0 u} dV_g \leq B_0.$$

Then there is a constant $C = C(n, g, \rho, \alpha_0, B_0)$, such that

$$(4.2) \quad \max_M u \leq C.$$

Proof. This is the boundary version of Proposition 3.3 in [8]. The proof for compact manifolds without boundary is given in [8]. We just present a proof for manifolds with boundary. We denote R_u the scalar curvature of g_u , then

$$\frac{1}{2(n-1)} R_u e^{-2u} = \frac{1}{2(n-1)} R + \Delta u - \frac{n-2}{2} |\nabla u|^2.$$

By the k -admissible ($k > n/2$) condition, we know that both R_u and R are positive.

If we denote $v = e^{-\frac{n-2}{2}u}$, we have

$$\frac{n-2}{2} \frac{R_u}{2(n-1)} v^{\frac{4}{n-2}+1} = -\Delta v + \frac{R(n-2)}{4(n-1)} v,$$

then

$$(4.3) \quad \Delta v \leq C_0 v.$$

Let $\varepsilon < \frac{2\alpha_0}{n-2}$. Multiplying by $v^{-2\varepsilon-1}$ on both sides of (4.3), and integrating by parts, we have

$$\begin{aligned} C_0 \int_M v^{-2\varepsilon} &\geq \int_M v^{-2\varepsilon-1} \Delta v = (1+2\varepsilon) \int_M v^{-2\varepsilon-2} |\nabla v|^2 + \int_{\partial M} v^{-2\varepsilon-1} \frac{\partial v}{\partial \nu} \\ &= \frac{1+2\varepsilon}{\varepsilon^2} \int_M |\nabla(v^{-\varepsilon})|^2. \end{aligned}$$

Then, by the lower bound of the first eigenvalue η_1 for Neumann boundary condition (see [31]), we see that $\int_M v^{-2\varepsilon}$ can be controlled by a constant depending on $\text{vol}(B(x, \rho))$ and the bound of $\int_{B(x, \rho)} v^{-\varepsilon} < B_0^{\frac{n-2}{2\alpha_0}\varepsilon}$ $\text{vol}(B(x, \rho))^{1-\frac{(n-2)\varepsilon}{2\alpha_0}} \triangleq A_0$:

$$\begin{aligned} \int_M v^{-2\varepsilon} &\leq \frac{(\int_M v^{-\varepsilon})^2}{\text{vol}(M)} + \frac{1}{\eta_1} \int_M |\nabla(v^{-\varepsilon})|^2 \leq \frac{(\int_M v^{-\varepsilon})^2}{\text{vol}(M)} + \frac{C_0 \varepsilon^2}{\eta_1 (1+2\varepsilon)} \int_M v^{-2\varepsilon} \\ &\leq \frac{(\int_{B(x, \rho)} v^{-\varepsilon} + (\int_{B(x, \rho)^c} v^{-2\varepsilon})^{\frac{1}{2}} \text{vol}(B(x, \rho)^c)^{\frac{1}{2}})^2}{\text{vol}(M)} \\ &\quad + \frac{C_0 \varepsilon^2}{\eta_1 (1+2\varepsilon)} \int_M v^{-2\varepsilon} \\ &\leq \frac{A_0^2 + A_0 \theta \int_{B(x, \rho)^c} v^{-2\varepsilon} + \frac{A_0}{\theta} \text{vol}(B(x, \rho)^c) + \text{vol}(B(x, \rho)^c) \int_{B(x, \rho)^c} v^{-2\varepsilon}}{\text{vol}(M)} \\ &\quad + \frac{C_0 \varepsilon^2}{\eta_1 (1+2\varepsilon)} \int_M v^{-2\varepsilon} \\ &\leq C_1(A_0, \theta) + \left(\frac{A_0 \theta + \text{vol}(B(x, \rho)^c)}{\text{vol}(M)} + \frac{C_0 \varepsilon^2}{\eta_1 (1+2\varepsilon)} \right) \int_M v^{-2\varepsilon}. \end{aligned}$$

Then by choosing a suitable constants ε and θ such that

$$\frac{A_0 \theta + \text{vol}(B(x, \rho)^c)}{\text{vol}(M)} + \frac{C_0 \varepsilon^2}{\eta_1 (1+2\varepsilon)}$$

is strictly less than 1, we can obtain the upper bound of $\int_M v^{-2\varepsilon}$ which depends on g, A_0, ρ and α_0 . In other words, when ε is small enough, (4.1) implies a global integral upper bound of $v^{-2\varepsilon}$.

Now we complete the proof via Green representation theorem. Note when $s < \frac{n}{n-1}$, the Green function G and its gradient $|\nabla G|$ are L^s integrable. If we denote $\frac{1}{1-\frac{2\varepsilon}{s}}$ by s' , we wish to find a good enough $L^{s'}$ integrable function. Denote $w = v^{-\frac{2\varepsilon}{s'}}$, then $w \in L^{s'}(M)$, and

$$\begin{aligned} \Delta w &= -\frac{2\varepsilon}{s'} v^{-2\frac{\varepsilon}{s'}-1} \Delta v + \frac{2\varepsilon}{s'} \left(\frac{2\varepsilon}{s'} + 1 \right) v^{\frac{2\varepsilon}{s'}-2} |\nabla v|^2 \\ &\geq -C_2 \frac{2\varepsilon}{s'} w. \end{aligned}$$

Consequently, by Green representation theorem we have

$$\begin{aligned} w(a) &= -\int_M G(a, \cdot) \Delta w + \int_{\partial M} \frac{\partial G(a, \cdot)}{\partial \nu} w \\ &\leq C_3 \int_M G(a, \cdot) w + \int_{\partial M} \frac{\partial G(a, \cdot)}{\partial \nu} w \\ &\leq C_4 (\|G(a, \cdot)\|_{L^s} \|w\|_{L^{s'}} + \|\nabla G(a, \cdot)\|_{L^s} \|w\|_{L^{s'}}) \\ &\leq C_5(n, g, A_0, \rho, \alpha_0). \end{aligned}$$

This gives (4.2). □

Proposition 4.1 (Proposition 4.6. of [8]). *There is a neighborhood $B(x_0, \bar{\rho})$ of $x_0 \in \Sigma[\{u_i\}]$ and constant $C(n, g, f, \bar{\rho}, \varsigma)$, such that for any $x \in B(x_0, \bar{\rho}) \setminus \{x_0\}$,*

$$u(x) \geq \log d_g(x, x_0) - C.$$

Proof. Since u_{i_k} is bounded above in some neighborhood U of regular point, so

$$\int_U e^{\alpha u_{i_k}} \leq C_1,$$

where α is a constant. Then according to Lemma 4.2.

$$\max_M u_{i_k} \leq C_2.$$

Let $u(x) = \limsup_{k \rightarrow \infty} u_{i_k}$ in M , then the limit satisfies

$$\sup_{M \setminus \Sigma[\{u_i\}]} u \leq C_2.$$

By Volume Comparison Theorem, and Fatou Lemma,

$$\int_M e^{-nu} \leq \liminf_{k \rightarrow \infty} \int_M e^{-nu_{i_k}} = \text{vol}(g_{u_{i_k}}) \leq v_0.$$

Hence, there exists $\bar{\rho}$ small enough such that

$$\int_{B(x_0, 2\bar{\rho})} e^{-nu} \leq \varsigma/2,$$

where ς is the constant in Lemma 3.1. Then for any point $x \in B(x_0, \bar{\rho}) \setminus \{x_0\}$, $B(x, d_g(x, x_0)/2) \subset B(x_0, 2\bar{\rho})$ and

$$\int_{B(x, \frac{1}{2}d_g(x, x_0))} e^{-nu} \leq \varsigma/2.$$

Since u is a $C^2(B(x, \frac{1}{2}d(x, x_0)))$ solution of (2.4), by Lemma 3.1

$$u(x) \geq \inf_{B(x, \frac{1}{2}d(x, x_0))} u(x) \geq \log d_g(x, x_0) - C. \quad \square$$

Similar as the case of the manifolds without boundary in [8], for the manifolds with non-empty boundary, we can get the following propositions. Their proofs are also similar as in [8], we just notice the boundary condition $\frac{\partial u}{\partial \nu} = 0$, when we take integral on the boundary. We omit their proofs.

Proposition 4.2 (Theorem 3.5. of [8]). *Let $u \in C_{loc}^{1,1}(A(\frac{1}{2}r_1, 2r_2))$, where $x_0 \in M$ and $A(\frac{1}{2}r_1, 2r_2)$ denotes the annulus $B(x_0, 2r_2)/B(x_0, \frac{1}{2}r_1)$, with $0 < r_1 < r_2$. Assume $\frac{\partial u}{\partial \nu} = 0$ on the boundary ∂M and $g_u = e^{-2u}g$ satisfies $\text{Ric}(g_u) - 2\delta\sigma_1(A_u)g \geq 0$ almost everywhere in $A(\frac{1}{2}r_1, 2r_2)$ for some $0 \leq \delta < \frac{1}{2}$. Define $\alpha_\delta = \frac{n-2}{1-2\delta}\delta \leq 0$ and $p = n + 2\alpha_\delta \geq n$. Then for any given $\alpha \geq \alpha_\delta$, there is constant $C = C((\alpha - \alpha_\delta)^{-1}, n, g) > 0$ such that*

$$\begin{aligned} & \int_{A(r_1, r_2)} |\nabla u|^p e^{\alpha u} \\ & \leq C \left(\int_{A(\frac{1}{2}r_1, 2r_2)} e^{\alpha u} |\text{Ric}|^{\frac{p}{2}} + \frac{1}{r_1^p} \int_{A(\frac{r_1}{2}, r_2)} e^{\alpha u} + \frac{1}{r_2^p} \int_{A(r_1, 2r_2)} e^{\alpha u} \right). \end{aligned}$$

Corollary 4.1 (Corollary 3.9. of [8]). *Let $u \in C_{loc}^{1,1}(M)$ satisfies $\frac{\partial u}{\partial \nu} = 0$ on the boundary ∂M . Assume $g_u = e^{-2u}g$ is k -admissible with $(k > n/2)$. Suppose δ satisfies $0 < \delta \leq \min \left\{ \frac{1}{2}, \frac{(2k-n)(n-1)}{2n(k-1)} \right\}$. Define $\alpha_\delta = \frac{n-2}{1-2\delta}\delta$. Then*

for any $\alpha > \alpha_\delta$, there exists a constant $C = C(\delta, n, g, \alpha)$ such that

$$\|e^{(\alpha/p)u}\|_{C^{\gamma_0}} \leq C \|e^{(\alpha/p)u}\|_{L^p},$$

where $\gamma_0 = \frac{2\alpha_\delta}{n+2\alpha_\delta}$.

Proposition 4.3 (Proposition 4.5. of [8]). *Suppose $x_0 \in \Sigma[\{u_i\}]$ and u_i is a blow-up sequence near x_0 . Then for any given $\theta > 0$ there exists neighborhood U of x_0 and a constant $C = C(\theta, n, g, f)$ such that the function $u = \limsup_{i \rightarrow \infty} u_i$ satisfies*

$$u(x) \leq (2 - \theta) \log d_g(x, x_0) + C,$$

for all $x \neq x_0$ in U .

Now from Propositions 4.1 and 4.3 we obtain a contradiction. This implies that the assumption $\limsup_i u_i(x') < +\infty$ for some regular point x' is impossible. Thus we have the following:

Proposition 4.4. (1) $\limsup_i u_i = +\infty$ in $M/\Sigma[\{u_i\}]$.

(2) *There is a subsequence u_{i_k} converging uniformly to $+\infty$ on compact set $K \subset M/\Sigma[\{u_i\}]$.*

Proof of (2). Note that $\limsup_i u_i(x') = +\infty$, we may suppose $\lim_k u_{i_k}(x') = +\infty$.

There are compact sets K' and \tilde{K}' , such that $K \cup \{x'\} \subset K' \subset \tilde{K}'$. Applying Lemma 2.2 on \tilde{K}' , we have $\sup_{K'} |\nabla u_i| \leq C_1$ and $\inf_K u_i \geq \inf_{K'} u_i \geq \sup_{K'} u_i - C_2$.

Thus for any fixed $N \in \mathbb{N}$, since $\lim_k u_{i_k}(x') = +\infty$ we can find $J \in \mathbb{N}$, such that $u_{i_k}(x') > N + C_2$ when $k > J$. Hence, $\inf_K u_{i_k} \geq u_{i_k}(x') - C_2 > N$. \square

In the rest of the proof we consider the subsequence u_{i_k} chosen above (still denoted by u_i) and a non-empty set $\Sigma_0[\{u_{i_k}\}] \subset \Sigma[\{u_i\}]$, which will be denoted by Σ_0 for simplicity.

5. Complete the proof of Theorem 1.1.

Note that u_i satisfies $\lim_i u_i(x') = +\infty$, for some regular point $x' \in M \setminus \Sigma_0$. Let

$$w_i(x) = u_i(x) - u_i(x').$$

We will show that w_i converges to a $C_{\text{loc}}^{1,1}$ -limit in $M \setminus \Sigma_0$.

Proposition 5.1. *We have*

- (1) $S_{w_i} = Ric_{w_i} - 2\delta\sigma_1(A_{w_i})g$ is positive semi-definite.
- (2) Let $M_r = M \setminus \cup_{x_k \in \Sigma} B(x_k, r)$, where $r > 0$ small enough. Then we can find constants $J = J(r)$ and $C = C(n, g, f)$ such that

$$|\nabla^2 w_i| + |\nabla w_i|^2 \leq Cr^{-2}$$

for all $x \in M_r$ and $i > J = J(r)$.

- (3) The sequence $\{w_i\}$ has a global upper bound $\max_M w_i \leq C$ and a L^∞ bound in M_r .

Proof. (1) $S_{w_i} = Ric_{w_i} - 2\delta\sigma_1(A_{w_i})g = Ric_{u_i} - 2\delta\sigma_1(A_{u_i})g \geq 0$.

(2) By Proposition 4.4 (2), we can find $J(r) \in \mathbb{N}$ such that $e^{-2\inf_{M_r/2} u_i} \leq r^{-2}$ when $i > J$. Therefore, for any $x \in M_r$ and $i > J$, we have

$$\begin{aligned} & \sup_{B(x, r/4)} (|\nabla^2 w_i| + |\nabla w_i|^2) \\ &= \sup_{B(x, r/4)} (|\nabla^2 u_i| + |\nabla u_i|^2) \\ &\leq C_1 \left(r^{-2} + e^{-2\inf_{M_{\sqrt{10}r/2}} u_i} \right) \\ &\leq 2C_1 r^{-2}. \end{aligned}$$

Since M_r is compact, by the finite covering argument, we know that there is a constant $C = C(n, g, f)$ such that

$$|\nabla^2 w_i| + |\nabla w_i|^2 \leq Cr^{-2}$$

for all $x \in M_r$ and $i > J = J(r)$.

(3) We may assume that r is small enough and M_r contains x' . By (2) $|\nabla w_i|^2(x) \leq 2C_1 r^{-2}$ in M_r . Then

$$\sup_{M_r} w_i \leq \inf_{M_r} w_i + C_2 \leq w_i(x') + C_2 = C_2,$$

where C_2 depends on n, g, f and r . By Lemma 4.2., we obtain a global upper bound $\max_M w_i \leq C$. For the lower bound, we have

$$\inf_{M_r} w_i \geq \sup_{M_r} w_i - C_2 \geq w_i(x') - C_2 = -C_2. \quad \square$$

Then Arzela–Ascoli theorem implies that a subsequence of w_i (denoted by w_i again) converges on compact sets $K \subset M \setminus \Sigma_0$ in $C^{1,\alpha}(K)$ where

$\alpha \in (0, 1)$. Hence, from Rademacher theorem, $\nabla^2 w$ is well defined almost everywhere. We can obtain the following corollary immediately.

Corollary 5.1. (1) *The limit $w = \lim_i w_i$ is in $C_{loc}^{1,1}(M \setminus \Sigma_0)$.*

(2) *$S_w = Ric_w - 2\delta\sigma_1(A_w)g$ is positive semi-definite almost everywhere in M .*

Now we may consider a doubling manifold \widetilde{M} of M by gluing two copies of M along the boundary ∂M . With the given smooth Riemannian metric g on M , there is a standard metric \widetilde{g} on \widetilde{M} induced from g . When ∂M is totally geodesic in (M, g) , \widetilde{g} is $C^{2,1}$ on \widetilde{M} ; see [19].

If we denote the corresponding double of Σ_0 by $\widetilde{\Sigma}_0$, we can extend w to a $C_{loc}^{1,1}(\widetilde{M} \setminus \widetilde{\Sigma}_0)$ function \widetilde{w} as follows:

Near the boundary we take Fermi Coordinates, \widetilde{w} is then defined as

$$\widetilde{w}(x_1, \dots, x_n) = \begin{cases} w(x_1, \dots, x_n), & x_n \geq 0, \\ w(x_1, \dots, -x_n), & x_n \leq 0. \end{cases}$$

Since ∇w is locally Lipschitz, $\nabla \widetilde{w}$ is the same. In fact, taking a geodesic convex neighborhood $\widetilde{B}(x, \widetilde{r})$ centered at any $x \in \partial M$, we may assume p and q are two points with $x_n(p) \geq 0$ and $x_n(q) \leq 0$. Then the geodesic connecting p and q is contained in $\widetilde{B}(x, \widetilde{r})$ and pass across the boundary ∂M . Thus there exists a point z in $\widetilde{B}(x, \widetilde{r}) \cap \partial M$ such that $\widetilde{dist}(p, q) = \widetilde{dist}(p, z) + \widetilde{dist}(z, q)$ where the distance function under metric \widetilde{g} denote by $\widetilde{dist}(\cdot, \cdot)$. Since $\nabla_{\frac{\partial}{\partial x^i}} w$ ($1 \leq i \leq n$) is a locally Lipschitz function we know that there exist a constant L such that $|\nabla_{\frac{\partial}{\partial x^i}} w(p) - \nabla_{\frac{\partial}{\partial x^i}} w(z)| \leq L \cdot \widetilde{dist}(p, z)$ and $|\nabla_{\frac{\partial}{\partial x^i}} \widetilde{w}(z) - \nabla_{\frac{\partial}{\partial x^i}} \widetilde{w}(q)| \leq L \cdot \widetilde{dist}(z, q)$. Therefore,

$$\begin{aligned} & |\nabla_{\frac{\partial}{\partial x^i}} \widetilde{w}(p) - \nabla_{\frac{\partial}{\partial x^i}} \widetilde{w}(q)| \\ & \leq |\nabla_{\frac{\partial}{\partial x^i}} \widetilde{w}(p) - \nabla_{\frac{\partial}{\partial x^i}} \widetilde{w}(z)| + |\nabla_{\frac{\partial}{\partial x^i}} \widetilde{w}(z) - \nabla_{\frac{\partial}{\partial x^i}} \widetilde{w}(q)| \\ & \leq L \left(\widetilde{dist}(p, z) + \widetilde{dist}(z, q) \right) \\ & = L \cdot \widetilde{dist}(p, q). \end{aligned}$$

It is obviously that \widetilde{w} is a C^1 function. Now we may conclude that $\widetilde{w} \in C_{loc}^{1,1}(\widetilde{M} \setminus \widetilde{\Sigma}_0)$. Then the following corollary is immediately.

Corollary 5.2. (1) *The limit $\widetilde{w} = \lim_i \widetilde{w}_i$ is in $C_{loc}^{1,1}(\widetilde{M} \setminus \widetilde{\Sigma}_0)$.*

(2) *$S_{\widetilde{w}} = Ric_{\widetilde{w}} - 2\delta\sigma_1(A_{\widetilde{w}})\widetilde{g}$ is positive semi-definite almost every where in (M, \widetilde{g}) .*

By a similar proof as it in sections 6 and 7 of [8], we can get the following proposition.

Proposition 5.2. (1) *There exists an isometry*

$$\Phi : (\widetilde{M}_{reg}, e^{-2\widetilde{w}}\widetilde{g}) \rightarrow (\mathbb{R}^n, g_{Euc}),$$

where g_{Euc} is the Euclidean metric and $\widetilde{M}_{reg} = \widetilde{M} \setminus \widetilde{\Sigma}_0$.

(2) $(M_{reg}, e^{-2w}g)$ is isometric to the half-plane in Euclidean space, where $M_{reg} = M \setminus \Sigma_0$.

Thus, Theorem 1.1 follows immediately, since that (M, g) is assumed to be not conformally equivalent to a standard hemisphere.

6. Proof of Theorem 1.2.

Note that when $k = n/2$, we cannot find a positive δ such that $\text{Ric}_g \geq \delta\sigma_1(A_g)g$. Therefore, the method of the proof in Theorem 1.1 cannot be applied in the case $k = n/2$. Nevertheless, any k -admissible solution w ($k \geq n/2$) on (M, g) satisfies another crucial inequality (see [7])

$$(6.1) \quad W_{nn} + \frac{1}{n-2} \sum_{k=1}^n W_{kk} \geq 0,$$

where $W_{ij} = w_{ij} + w_i w_j - \frac{g_{ij}}{2} (\sum_{k=1}^n w_k)^2 + (A_g)_{ij}$.

Let u_i be a sequence of k -admissible solutions to equation (1.2). In [7], Trudinger and Wang consider the rescaled k -admissible solutions $w_j = u_j - \sup_M u_j$ and prove the rescaled sequence w_j converges in $W^{1,p}$ (for any $1 < p < \frac{n}{n-1}$) to an admissible function w . Roughly speaking, if \bar{x} is a blow-up point of w , inequality (6.1) implies one side of the estimate for the limit function w near \bar{x} :

$$(6.2) \quad w(x) \leq 2 \log d(x, \bar{x}) + C.$$

Furthermore, they prove

$$(6.3) \quad w(x) = 2 \log d(x, \bar{x}) + o(1),$$

where $o(1) \rightarrow 0$ when $d(x, \bar{x}) \rightarrow 0$. From (6.3), one can show that each blow-up point is isolated, which implies that the number of blow-up points is finite. Combining the fact that $\text{Ric}_g \geq 0$, by the volume comparison theorem, one

can show as in [8, 9] the ratio of the volume of the geodesic ball of radius r in the metric $e^{-2w}g$ with that of the Euclidean ball is non-increasing. Therefore w has exactly one blow-up point 0 and the manifold $(M \setminus \{0\}, e^{-2w}g)$ is isometric to the Euclidean space, which contradicts with the assumption. Therefore there is a uniform L^∞ bound for solutions and the set of solutions is compact.

Now, similarly, when dealing with manifold with boundary we expect to prove the conformal metric $e^{-2w}g$ is in fact Euclidean metric on half-plane and get a contradiction, where w is the limit function of the rescaled sequence $w_j = u_j - \sup_M u_j$ on manifold with boundary.

To this end, we double the manifold (M, g) . Given a smooth Riemannian metric g on M , there is a standard metric \widehat{g} on \widehat{M} induced from g , which is glued by two copies of M along the boundary ∂M . When ∂M is totally geodesic in (M, g) , then \widehat{g} is $C^{2,1}$ on \widehat{M} , see [19].

Then we extend the functions w_j to a function \widehat{w}_j on \widehat{M} as follows:

$$\widehat{w}_j(x_1, \dots, x_n) = \begin{cases} w_j(x_1, \dots, x_n), & x_n \geq 0, \\ w_j(x_1, \dots, -x_n), & x_n \leq 0, \end{cases}$$

where we take Fermi Coordinates near the boundary as before. We firstly verify that \widehat{w}_j satisfies the preliminary Lemmas in section 2 of [7]. However, from the boundary condition we can see that \widehat{w}_j are in fact C^2 k -admissible functions on $(\widehat{M}, \widehat{g})$. We calculate under Fermi coordinates:

$$\begin{aligned} & \lim_{x_n \rightarrow 0^+} \frac{\partial \widehat{w}_j}{\partial x^n}(x_1, \dots, x_n) = \frac{\partial w_j}{\partial x^n}(x_1, \dots, x_{n-1}, 0) \\ = & 0 = -\frac{\partial w_j}{\partial x^n}(x_1, \dots, x_{n-1}, 0) = \lim_{x_n \rightarrow 0^-} \frac{\partial \widehat{w}_j}{\partial x^n}(x_1, \dots, x_n), \end{aligned}$$

and

$$\lim_{x_n \rightarrow 0^+} \frac{\partial^2 \widehat{w}_j}{\partial (x^n)^2}(x_1, \dots, x_n) = \lim_{x_n \rightarrow 0^-} \frac{\partial^2 \widehat{w}_j}{\partial (x^n)^2}(x_1, \dots, x_n).$$

Thus from the k -admissible property of w_j we know that \widehat{w}_j are k -admissible and sub-harmonic with some elliptic operator.

As a matter of fact, we may extend the definition of k -admissible and sub-harmonic(super-harmonic) in the viscosity sense (see [7] for details).

We call a metric $\tilde{g} = \mathcal{X}g$ is k -admissible if \mathcal{X} is lower semi-continuous, does not equal to ∞ , and there exists a sequence of k -admissible functions $\mathcal{X}_m \in C^2(M)$ such that $\mathcal{X}_m \rightarrow \mathcal{X}$ almost everywhere in M .

We say a function v is super-harmonic with respect to a elliptic operator \mathcal{L} if (i) v is lower semi-continuous (l.s.c.); (ii) v does not equal to ∞ in

any open set; (iii) for any open subset $D \subset M$ and any function $h \in C^2(\bar{D})$ satisfying $\mathcal{L}(h) = 0$ in D and $h \leq v$ on ∂D , we have $h \leq v$ in D . Subharmonic functions are defined as the negative of super-harmonic ones (See p. 131 of [32]).

As a result, if \hat{w}_j is k -admissible, then the function $\hat{v}_j = e^{-\frac{n-2}{2}\hat{w}_j}$ is super-harmonic with respect to the conformal Laplace operator $\mathcal{L} \triangleq \Delta_g - \frac{n-2}{4(n-1)}R_g$. The corresponding maximal (minimal) radial functions are also super-harmonic (subharmonic), where the maximal and minimal radial functions in $B_R(x_0)$ are defined by

$$\tilde{v}(x) = \inf\{\hat{v}(y) : y \in \partial B_r(x_0), r = d(x, x_0)\}$$

and

$$\tilde{w}(x) = \sup\{\hat{w}(y) : y \in \partial B_r(x_0), r = d(x, x_0)\},$$

respectively, where $r \leq R$.

Now \hat{w}_j (\hat{v}_j) and \tilde{w}_j (\tilde{v}_j) satisfying the subharmonic (super-harmonic) property, thus Lemma 2.1, Lemma 2.2, Corollary 2.1 and hence Corollary 2.2 and Corollary 2.3 in [7] still hold for \hat{v} :

Lemma 6.1 (Corollary 2.2 of [7]). *Let \hat{v} be super-harmonic with respect to the conformal Laplacian operator \mathcal{L} . Then the maximal radial function $\tilde{v}(r)$ is locally uniformly Hölder continuous away from 0, with Hölder exponent $\alpha \in (0, 1/n)$.*

Lemma 6.2 (Corollary 2.3 of [7]). *Let \hat{v}_j be a sequence of super-harmonic functions which converges to \hat{v} in $L^1(B_r(0))$. Then the corresponding maximal radial functions $\tilde{v}_j(r)$ converges locally uniformly to $\tilde{v}(r)$.*

Proof of Theorem 1.2. By use of the argument in [7] and [8], we only sketch the proof here. Suppose $x_{0,j}$ is a blow up sequence of u_j and $\bar{x} = \lim_{j \rightarrow \infty} x_{0,j}$. Let x_0^j be the maximum point of u_j . Notice that $e^{-2\sup u_j} f(x_0^j) = e^{-2u_j(x_0^j)} f(x_0^j) \leq C(\Delta u_j + A_g)(x_0^j) \leq C$. Since $\sup_M u_j > -\infty$, $x_{0,j}$ is also a blow-up sequence of $w_j = u_j - \sup_M u_j$, and $\hat{w}_j(x_{0,j}) \rightarrow -\infty$, $\hat{v}_j(x_{0,j}) \rightarrow +\infty$.

Now we are going to prove the limit of $\hat{w} = \lim_{j \rightarrow \infty} \hat{w}_j$ satisfies (6.3). We begin with two observations:

- (1) If we denote $\hat{v}_j(x_{0,j})^{\frac{1}{n-2}}$ and $\frac{1}{1-2^{-\frac{1}{n-2}}}$ by R_j and A_0 respectively, when j is large enough in $B(x_{0,j}, A_0 R_j^{-1})$ there must be some local maximum points of \hat{v}_j , denoted by x_j . Furthermore, x_j is the local maximum points of \hat{v}_j in $B(x_j, \frac{1}{2}\hat{v}_j(x_j)^{-\frac{1}{n-2}})$ yet (see Lemma 3.2. in [7] for details).
- (2) Note that the maximal and the minimal radial functions depend only on distance to the center. Thus, we may denote that

$$\tilde{w}_j(r) = \sup\{\hat{w}_j(y) : y \in \partial B_r(x_j)\},$$

and

$$\tilde{w}(r) = \sup\{\hat{w}(y) : y \in \partial B_r(\bar{x})\}.$$

In virtue of Lemmas 6.1 and 6.2, we can obtain $\tilde{w}(r) = \lim_{j \rightarrow \infty} \tilde{w}_j(r)$.

Then by a similar argument in section 3 of [7], we can see that \hat{w} satisfies (6.3) in $(\widehat{M}, \widehat{g})$ and singular points are isolated. Furthermore, since the Ricci curvature of $(\widehat{M}, \widehat{g})$ is still positive semi-definite, by the volume comparison theorem, there is at most one end and it is away from the singular points; the metric $e^{-2\hat{w}}\widehat{g}$, a doubling of $e^{-2w}g$, is in fact a Euclidean one (see section 7 of [8] for details). Finally, restricting the argument to manifold M , we can see $(M \setminus \{\bar{x}\}, e^{-2w}g)$ is just the half-plane in Euclidean space, which contradicts with the assumption. Therefore, there is a uniform L^∞ bound for solutions and the set of solutions is compact. This completes the proof of Theorem 1.2. \square

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CENTRE FOR MATHEMATICAL SCIENCES, ZHEJIANG UNIVERSITY, HANGZHOU 310027, CHINA

E-mail address: helenai@hotmail.com

DEPARTMENT OF MATHEMATICS, ZHEJIANG UNIVERSITY, HANGZHOU 310027, CHINA

E-mail address: weimins@zju.edu.cn

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