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Polynomial Bridgeland stability conditions for the derived category of sheaves on surfaces

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For the derived category of bounded complexes of coherent sheaves on a smooth projective surface, we study the standard polynomial Bridgeland stability conditions introduced by Bayer [2] (see also [19, 20]). Assuming certain conditions on the stability vector, we prove that the standard polynomial Bridgeland stability remains to be the same when the polarization varies in a chamber in the usual sense of [16, 17]. Furthermore, when the polarization is contained in a chamber, we show that the polynomial Bridgeland stability and Gieseker stability can be identified.

1. Introduction

Derived categories and triangulated categories have been studied extensively in recent years. Bridgeland [4] introduced the concept of stability conditions on triangulated categories, which can be viewed as a mathematical approach to understand Douglas' work [6] on Π-stability for D-branes in string theory. These stability conditions on the derived categories of sheaves have been constructed and classified for certain varieties (see for example [1, 5, 14, 19] and the references given there). However, the existence of a Bridgeland stability condition on the derived categories of sheaves over a general variety is still unknown at the present. On the other hand, Bayer [2] defined polynomial Bridgeland stability conditions which generalize Bridgeland stability conditions on triangulated categories (see [20] for related work), and proved the existence of the standard polynomial Bridgeland stability conditions on the derived categories of sheaves over any normal projective variety. By the Proposition 4.1 in [2], the standard polynomial Bridgeland stability conditions are related to the large volume limits of Bridgeland stability conditions.

In this paper, we study the standard polynomial Bridgeland stability conditions for surfaces. Fix a smooth projective surface X. A stability data $\Omega_L = (L, \rho, p, U)$ consists of an ample divisor L on X, a stability vector

$\rho = (\rho_0, \rho_1, \rho_2) \in (\mathbb{C}^*)^3$, a perversity function $p : \{0, 1, 2\} \to \mathbb{Z}$ associated to ρ, and a unipotent element $U \in A^*(X)_{\mathbb{C}}$ (see Definition 2.5). Throughout the paper, using the $\widetilde{\mathrm{GL}}_+(2, \mathbb{R})$-action and derived duals, we will assume that p and ρ satisfy either Case 1 or Case 2 listed in the second paragraph of Section 4. By [2], there exists a standard polynomial Bridgeland stability condition $(Z_{\Omega_L}, \mathcal{P}_{\Omega_L})$ on the derived categories $\mathcal{D}^b(X)$ of bounded complexes of coherent sheaves over X. An element $E \in \mathcal{D}^b(X)$ is *of type* (r, c_1, c_2) if $\mathrm{rk}(E) = r, c_1(E) = c_1$ and $c_2(E) = c_2$. In the context of torsion free sheaves (i.e., when $r > 0$), the notions of walls and chambers of type (r, c_1, c_2) were introduced in [16, 17]. To handle objects E in the abelian category $\mathcal{A}^p$ which will be defined in (2.5), we generalize these notions to cover the case $r < 0$ in Definition 3.1. Let $\mathrm{Num}(X)$ be the group of divisors in X modulo numerical equivalence. For $\xi \in \mathrm{Num}(X) \otimes \mathbb{R}$, define

$$W^\xi = \mathbb{C}_X \cap \{\alpha \in \mathrm{Num}(X)_{\mathbb{R}} \mid \alpha \cdot \xi = 0\},$$

where $\mathbb{C}_X$ is the ample cone. Let $\mathbb{H} \subset \mathbb{C}$ be the strict upper half plane

$$\mathbb{H} = \{z \in \mathbb{C} \mid z \in \mathbb{R}_+ \cdot e^{i\pi\phi(z)}, \quad 0 < \phi(z) \le 1\},$$

and let $\phi(z)$ be the phase of $z \in \mathbb{H}$. Our first result asserts that the Bogomolov inequality holds for Z_{Ω_L}-semistable objects, and the Z_{Ω_L}-stability remains to be the same when L varies within a chamber.

Theorem 1.1. *Let X be a smooth projective surface, and let Ω_L denote the data (L, ρ, p, U) where $p(0) = p(1) = 0$ and $U = 1 + u_1 + u_2$ with $u_i \in A^i(X)_{\mathbb{R}}$. Let $0 \ne E \in \mathcal{D}^b(X)$ be of type (r, c_1, c_2) with $r \ne 0$.*

(i) *If E is Z_{Ω_L}-semistable, then the Bogomolov inequality holds*

$$2rc_2 \ge (r - 1)c_1^2.$$

(ii) *Let L and H be contained in the same chamber of type (r, c_1, c_2). When $r < 0$, we further assume that $(c_1 + ru_1)$ does not satisfy*

$$(c_1 + ru_1) \cdot L = 0 < (c_1 + ru_1) \cdot H.$$

Then E is Z_{Ω_L}-semistable if and only if it is Z_{Ω_H}-semistable.

Let $\overline{\mathfrak{M}}_L^G(r, c_1, c_2)$ be the moduli space of torsion free sheaves which are of type (r, c_1, c_2) and are Gieseker-semistable with respect to L. Similarly, let

$\overline{\mathfrak{M}}_{\Omega_L}(r, c_1, c_2)$ be the *set* of all objects $E \in \mathcal{A}^p$ which are of type (r, c_1, c_2) and are Z_{Ω_L}-semistable. We remark that it is unknown whether $\overline{\mathfrak{M}}_{\Omega_L}(r, c_1, c_2)$ exists as a scheme. Notice that there are several works on moduli stacks of stable objects [9, 12, 13, 18]. Our second result identifies $\overline{\mathfrak{M}}_{\Omega_L}(r, c_1, c_2)$ with certain Gieseker moduli space as sets, and provides strong evidence that $\overline{\mathfrak{M}}_{\Omega_L}(r, c_1, c_2)$ should exist as a scheme.

Proposition 1.1. *Let X be a smooth projective surface, and fix a numerical type (r, c_1, c_2) with $r \neq 0$. Let $\Omega_L = (L, \rho, p, U)$ where $p = 0$ is the constant perversity function, $U \in A^*(X)_{\mathbb{R}}$, and $L \in \mathbb{C}_X$ does not lie on any wall of type (r, c_1, c_2). Then $A \in \overline{\mathfrak{M}}_{\Omega_L}(r, c_1, c_2)$ if and only if $A \in \overline{\mathfrak{M}}_L^G(r, c_1, c_2)$.*

Proposition 1.1 implies that for a generic stability data $\Omega_L = (L, \rho, p, U)$ on a surface X, the Z_{Ω_L}-stability can be identified with the Gieseker stability. This has been observed and studied by Bridgeland [5], Kawatani [11], Ohkawa [15], and Toda [18] in the context of Bridgeland stability. One can also see important results on moduli spaces of stable objects on surfaces in Toda's paper [18] and the paper [1] by Arcara, Bertram and Lieblich where they studied the moduli stacks and spaces of Bridgeland semi-stable objects on surfaces with special attentions to $K3$ surfaces.

It would be interesting to see to what extent results analogous to Proposition 1.1 hold for a higher-dimensional variety X. For instance, when the perversity function p is a constant function and U is the Todd class $\mathrm{td}(X)$ of X, it has been proved by Bayer [2] that polynomial Bridgeland stability and Gieseker stability coincide. On the other hand, the same statement definitely does not hold for 3-folds, as seen by the PT/DT wall-crossing studied in [22, 23]. We also note that the wall-crossing inside a wall can be extremely interesting (see e.g., [21]).

The paper is organized as follows. In Section 2, we recall the definitions and results from [2]. In Section 3, we generalize the definitions and results in [16, 17] regarding walls and chambers. In Section 4, we prove Theorem 1.1 and Proposition 1.1.

Conventions. The ith cohomology of a sheaf E on a variety X is denoted by $H^i(X, E)$, and its usual dual sheaf $\mathcal{H}om(E, \mathcal{O}_X)$ is denoted by E^*. The derived category of bounded complexes of coherent sheaves on X is denoted by $\mathcal{D}^b(X)$. The ith cohomology sheaf of an object $E \in \mathcal{D}^b(X)$ is denoted by $\mathcal{H}^i(E)$, and the derived dual of E is denoted by $E^\vee = \mathbb{R}\,\mathcal{H}om(E, \mathcal{O}_X) \in \mathcal{D}^b(X)$.

2. Polynomial Bridgeland stability

In this section, we recall the polynomial Bridgeland stability defined in [2]. All the definitions in this section are from Sections 2 and 3 of [2].

Definition 2.1. Let $(S, \succeq)$ be a linearly ordered set equipped with an order-preserving bijection $S \to S, \phi \mapsto \phi + 1$ satisfying $\phi + 1 \succeq \phi$. An *S-valued slicing* of a triangulated category $\mathcal{D}$ is given by full additive extension-closed subcategories $\mathcal{P}(\phi)$ for all $\phi \in S$ such that the following properties are satisfied:

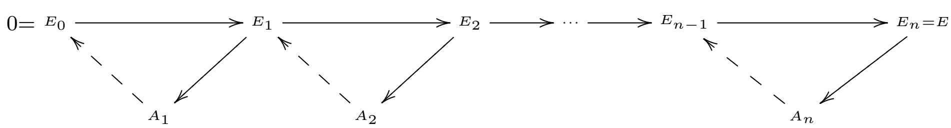
(i) for all $\phi \in S$, we have $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$;

(ii) if $\phi \succ \psi$ for $\phi, \psi \in S$ and $A \in \mathcal{P}(\phi)$, $B \in \mathcal{P}(\psi)$, then $\mathrm{Hom}_{\mathcal{D}}(A, B) = 0$;

(iii) for every nonzero object $E \in \mathcal{D}$, there exist a finite sequence

$$\phi_1 \succ \phi_2 \succ \ldots \succ \phi_n$$

of elements in S and a sequence of exact triangles with $A_i \in \mathcal{P}(\phi_i)$:



The sequence of exact triangles in Definition 2.1 (iii) is known as the *Harder–Narasimhan filtration* of the nonzero object $E \in \mathcal{D}$.

Definition 2.2. The set S of *polynomial phase functions* is the set of continuous function germs $\phi : \mathbb{R} \cup \{+\infty\} \to \mathbb{R}$ such that there exists a polynomial $Z(m) \in \mathbb{C}[m]$ with $Z(m) \in \mathbb{R}_+ \cdot \mathrm{e}^{i\pi\phi(m)}$ for $m \gg 0$. It is linearly ordered by setting

$$\phi \prec \psi \quad \Longleftrightarrow \quad \phi(m) < \psi(m) \quad \text{for } 0 \ll m < +\infty,$$

and its shift $\phi \mapsto \phi + 1$ is given by point-wise addition.

In the rest of the paper, S denotes the set of polynomial phase functions. For a triangulated category $\mathcal{D}$, denote its Grothendieck group by $K(\mathcal{D})$.

Definition 2.3. A *polynomial Bridgeland stability condition* on a triangulated category $\mathcal{D}$ is given by a pair $(Z, \mathcal{P})$, where $\mathcal{P}$ is an S-valued slicing of $\mathcal{D}$ and $Z : K(\mathcal{D}) \to \mathbb{C}[m]$ is a group homomorphism with the following property: if $0 \neq E \in \mathcal{P}(\phi)$, then $Z(E)(m) \in \mathbb{R}_+ \cdot \mathrm{e}^{\mathrm{i}\pi\phi(m)}$ for $m \gg 0$.

Definition 2.4. A *polynomial Bridgeland stability function* on an Abelian category $\mathcal{A}$ is a group homomorphism $Z : K(\mathcal{A}) \to \mathbb{C}[m]$ such that there exists a polynomial phase function $\phi_0 \in S$ with the following property: for every $0 \neq E \in \mathcal{A}$, there exists $\phi_E \in S$ such that $\phi_0 \prec \phi_E \preceq \phi_0 + 1$ and $Z(E)(m) \in \mathbb{R}_+ \cdot \mathrm{e}^{\mathrm{i}\pi\phi_E(m)}$ for $m \gg 0$.

It is known [2, 4] that giving a polynomial Bridgeland stability condition on $\mathcal{D}$ is equivalent to giving a bounded t-structure on $\mathcal{D}$ and a polynomial Bridgeland stability function on its heart with the Harder–Narasimhan property.

Next, let X be a smooth projective complex variety of dimension n. Let $\mathcal{A} = \mathrm{Coh}(X)$ be the category of coherent sheaves on X, and $\mathcal{D}^b(X)$ be the derived category of bounded complexes of coherent sheaves on X. A function $p : \{0, 1, \dots, n\} \to \mathbb{Z}$ is called *a perversity function* if

$$p(d) \geq p(d+1) \geq p(d) - 1. \tag{2.1}$$

For a perversity function p, define the abelian subcategory $\mathcal{A}^{p,\leq k}$ of $\mathcal{A}$ by

$$\mathcal{A}^{p,\leq k} = \{0\} \cup \{A \in \mathcal{A} \mid p(\dim \mathrm{Supp} A) \geq -k\}. \tag{2.2}$$

By Bezrukavnikov [3] and Kashiwara [10], the following pair defines a bounded t-structure on $\mathcal{D}^b(X)$

$$\mathcal{D}^{p,\leq 0} = \{E \in \mathcal{D}^b(X) \mid \mathcal{H}^{-k}(E) \in \mathcal{A}^{p,\leq k} \text{ for all } k \in \mathbb{Z}\}, \tag{2.3}$$

$$\mathcal{D}^{p,\geq 0} = \{E \in \mathcal{D}^b(X) \mid \mathrm{Hom}(A, E) = 0 \text{ for all } A \in \mathcal{A}^{p,\leq k}[k+1] \text{ and } k \in \mathbb{Z}\}. \tag{2.4}$$

Denote the heart (or core) of this t-structure on $\mathcal{D}^b(X)$ by $\mathcal{A}^p$, i.e.,

$$\mathcal{A}^p = \mathcal{D}^{p,\leq 0} \cap \mathcal{D}^{p,\geq 0}. \tag{2.5}$$

Lemma 2.1. *Let $n = \dim(X)$ and p be a perversity function with $p(0) = 0$.*

(i) *Let $0 \neq E \in \mathcal{A}^p$. Let k be the largest integer with $\mathcal{H}^{-k}(E) \neq 0$, and let d be the dimension of the support of $\mathcal{H}^{-k}(E)$. Then $p(d) = -k$, the sheaf $\mathcal{H}^{-k}(E)$ has no torsion in dimension d' whenever $p(d') > -k$, and all other cohomology sheaves of E are supported in smaller dimension.*

(ii) *Let* $0 \ne E \in \mathcal{A}^p$. *Then,* $\mathcal{H}^i(E) = 0$ *whenever* $i > 0$ *or* $i < p(n)$.

(iii) *If* $F \in \mathcal{A} = \mathrm{Coh}(X)$ *and* $k_0 \ge 0$, *then* $F[-k_0] \in D^{p,\ge 0}$.

(iv) *If* $F \in \mathcal{A} = \mathrm{Coh}(X)$ *and* $p(\dim \mathrm{Supp}(F)) = 0$, *then* $F \in \mathcal{A}^p$.

(v) *If* $p = 0$ *is the constant perversity function, then* $\mathcal{A}^p = \mathcal{A} = \mathrm{Coh}(X)$.

Proof. (i) This is the Lemma 3.2.3 in [2].

(ii) Suppose first that $\mathcal{H}^i(E) \ne 0$ for some $i > 0$. Since $E \in D^{p,\le 0}$, $\mathcal{H}^i(E) \in \mathcal{A}^{p,\le -i}$ by (2.3). Thus by (2.2), $p\big(\dim \mathrm{Supp}\, \mathcal{H}^i(E)\big) \ge i$. This is impossible since p cannot be positive. So $\mathcal{H}^i(E) = 0$ if $i > 0$. Next, we see from (i) that $\mathcal{H}^i(E) = 0$ if $-i > k = -p(d)$. It follows that $\mathcal{H}^i(E) = 0$ if $i < p(n)$ since $p(n) \le p(d)$.

(iii) Let $0 \ne A \in \mathcal{A}^{p,\le k}[k+1]$ where $k \in \mathbb{Z}$. We want to prove that

$$\mathrm{Hom}(A, F[-k_0]) = 0. \tag{2.6}$$

We have $A = B[k+1]$ for some $B \in \mathcal{A}^{p,\le k}$. By the definition of $\mathcal{A}^{p,\le k}$,

$$p\big(\dim \mathrm{Supp}\, B\big) \ge -k. \tag{2.7}$$

Since $p(0) \ge p(\dim \mathrm{Supp}\, B)$, we have $0 \ge -k$. It follows that:

$$\begin{aligned} \mathrm{Hom}(A, F[-k_0]) &= \mathrm{Hom}(B[k+1], F[-k_0]) \\ &= \mathrm{Hom}(B, F[-k_0 - k - 1]) \\ &\cong \mathrm{Ext}_{\mathcal{A}}^{-k_0-k-1}(B, F) \\ &= 0, \end{aligned}$$

since $-k_0 - k - 1 \le -k_0 - 1 \le -1$. This proves (2.6).

(iv) Recall that $\mathcal{A}^p = D^{p,\le 0} \cap D^{p,\ge 0}$. By (iii), it remains to prove that $F \in D^{p,\le 0}$. When $k \ne 0$, $\mathcal{H}^{-k}(F) = 0 \in \mathcal{A}^{p,\le k}$ since $0 \in \mathcal{A}^{p,\le k}$ by convention. When $k = 0$,

$$p(\dim \mathrm{Supp}\, \mathcal{H}^{-k}(F)) = p(\dim \mathrm{Supp}\, F) = p(0) = -k,$$

i.e., $\mathcal{H}^{-k}(F) \in \mathcal{A}^{p,\le k}$ as well. Therefore, we have $F \in D^{p,\le 0}$.

(v) The conclusion follows directly from (ii) and (iv). $\square$

Definition 2.5. A *stability vector* ρ is a sequence $(\rho_0, \rho_1, \dots, \rho_n) \in (\mathbb{C}^*)^{n+1}$ such that for every $0 \le d \le (n-1)$, ρ_d/ρ_{d+1} is contained in the interior of

$$\mathbb{H} = \{z \in \mathbb{C} \mid z \in \mathbb{R}_+ \cdot \mathrm{e}^{i\pi\phi(z)},\ 0 < \phi(z) \le 1\}.$$

A perversity function $p : \{0, 1, \dots, n\} \to \mathbb{Z}$ is *associated to* ρ if we have $(-1)^{p(d)}\rho_d \in \mathbb{H}$ for all $0 \le d \le n$. An element $\omega \in \mathrm{Num}(X)_\mathbb{R}$ is *ample* if $\omega^d \cdot \alpha > 0$ for all $1 \le d \le n$ and for every nonzero effective class $\alpha \in A_d(X)$. An element $U \in A^*(X)_\mathbb{C}$ is *unipotent* if $U = 1 + N$, where N is concentrated in positive degree.

The following is the main theorem in [2], noting that we have replaced the condition $\omega \in A^1(X)_\mathbb{R}$ there by $\omega \in \mathrm{Num}(X)_\mathbb{R}$.

Theorem 2.1. *Let the data $\Omega = (\omega, \rho, p, U)$ be given, consisting of*

- *an ample class $\omega \in \mathrm{Num}(X)_\mathbb{R}$,*
- *a stability vector $\rho = (\rho_0, \rho_1, \dots, \rho_n)$,*
- *a perversity function p associated to ρ,*
- *a uniponent class $U \in A^*(X)_\mathbb{C}$.*

Let $Z_\Omega : K(\mathcal{D}^b(X)) = K(X) \to \mathbb{C}[m]$ be the central charge defined by

$$Z_\Omega(E)(m) = \int_X \sum_{d=0}^{n} \rho_d \omega^d m^d \cdot \mathrm{ch}(E) \cdot U. \tag{2.8}$$

Then $Z_\Omega(E)(m)$ is a polynomial Bridgeland stability function for $\mathcal{A}^p$ with the Harder–Narasimhan property, and thus induces a polynomial Bridgeland stability condition $(Z_\Omega, \mathcal{P}_\Omega)$ on $\mathcal{D}^b(X)$.

Remark 2.1. Note that the polynomial Bridgeland stability condition associated to the data $(a\omega, \rho, p, U), a \in \mathbb{R}_+$ is independent of the parameter $a \in \mathbb{R}_+$.

Finally, for the smooth variety X, let $\mathbb{D} : \mathcal{D}^b(X) \to \mathcal{D}^b(X)$ be the dual functor defined by the derived dual $\mathbb{D}(E) = \mathbb{R}\mathcal{H}om(E, \mathcal{O}_X) = E^\vee$, and let

$$P : A^*(X) \to A^*(X) \tag{2.9}$$

be the parity operator acting by multiplication by $(-1)^i$ on $A^i(X)$. Then the *dual* polynomial Bridgeland stability condition $(Z_{\Omega^*}, \mathcal{P}_{\Omega^*})$ of $(Z_\Omega, \mathcal{P}_\Omega)$ is defined by the data $\Omega^* = (\omega, \rho^*, p^*, U^*)$, where $\rho_d^* = (-1)^d \overline{\rho_d}$, p^* is the dual perversity function defined by $p^*(d) = -d - p(d)$, and $U^* = P(\overline{U})$.

3. Walls and chambers for surfaces

In this section, X denotes a smooth projective surface. Our goal is to recall and generalize the basic definitions and results in [8, 16, 17] regarding walls, chambers, and variations of μ-stability as the polarization changes.

When $r > 0$, the following definition can be found in [16, 17] (see also [8]).

Definition 3.1. Let $\mathbb{C}_X \subset \text{Num}(X)_{\mathbb{R}}$ be the ample cone of the smooth projective surface X. Fix two integers $r, c_2 \in \mathbb{Z}$ and a divisor c_1 on X.

(i) For a class $\xi \in \text{Num}(X) \otimes \mathbb{R}$, we define

$$W^\xi = \mathbb{C}_X \cap \{\alpha \in \text{Num}(X)_{\mathbb{R}} \mid \alpha \cdot \xi = 0\}. \tag{3.1}$$

We say that $L, H \in \text{Num}(X)_{\mathbb{R}}$ are *separated by* W^ξ if $L \cdot \xi \le 0 < H \cdot \xi$, or $L \cdot \xi < 0 \le H \cdot \xi$, or $L \cdot \xi \ge 0 > H \cdot \xi$, or $L \cdot \xi > 0 \ge H \cdot \xi$.

(ii) Let $\mathcal{W}(r, c_1, c_2)$ be the set whose elements are of the form W^ξ, where ξ is the numerical equivalence class $(rF - sc_1)$ for some divisor F and some integer s with $0 < s < |r|$ satisfying the inequalities

$$-\frac{r^2}{4}(2rc_2 - (r-1)c_1^2) \le \xi^2 < 0. \tag{3.2}$$

(iii) A *wall of type* (r, c_1, c_2) is an element in $\mathcal{W}(r, c_1, c_2)$, while a *chamber of type* (r, c_1, c_2) is a connected component in the complement $\mathbb{C}_X - \mathcal{W}(r, c_1, c_2)$.

(iv) A sheaf E (or in general, a bounded complex E of sheaves) on X is *of type* (r, c_1, c_2) if $\text{rk}(E) = r, c_1(E) = c_1$ and $c_2(E) = c_2$.

Remark 3.1. For polynomial Bridgeland stability conditions, sometimes a "wall" may also refer to a situation where some of the stability vectors ρ_i overlap. We refer to [2, 23] for more details.

Fix a triple (r, c_1, c_2). By the results in [7], the set $\mathcal{W}(r, c_1, c_2)$ of wall of type (r, c_1, c_2) is locally finite, i.e., given a compact subset K of the ample cone $\mathbb{C}_X$, there are only finitely many walls W of type (r, c_1, c_2) such that $W \cap K \neq \emptyset$.

Lemma 3.1. *Let* $\tilde{r} = -r \neq 0$ *and* $1 + \tilde{c}_1 + \tilde{c}_2 = (1 + c_1 + c_2)^{-1} \in A^*(X)$. *Then,* ξ *defines a wall of type* (r, c_1, c_2) *if and only if it defines a wall of type* $(\tilde{r}, \tilde{c}_1, \tilde{c}_2)$.

Proof. We have $\tilde{c}_1 = -c_1$ and $\tilde{c}_2 = c_1^2 - c_2$. Our result follows from the observations that $(rF - sc_1) = (\tilde{r}\widetilde{F} - \tilde{s}\tilde{c}_1)$ where $\widetilde{F} = (r/|r|)c_1 - F$ and $\tilde{s} = |r| - s$, and that

$$-\frac{r^2}{4}(2rc_2 - (r-1)c_1^2) = -\frac{\tilde{r}^2}{4}(2\tilde{r}\tilde{c}_2 - (\tilde{r}-1)\tilde{c}_1^2). \qquad \square$$

The following lemma is well-known, and its proof is omitted.

Lemma 3.2. *Let* L *be an ample divisor, and assume a filtration of torsion free sheaves* $0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_{n-1} \subset E_n = E$ *such that all the quotients* E_i/E_{i-1} *with* $i > 0$ *are torsion free and* μ*-semistable with respect to* L. *If* $\mu_L(E_i) = \mu_L(E)$ *for all* $i > 0$, *then* E *is* μ*-semistable with respect to* L. $\square$

Lemma 3.3. *Let* A *be a torsion free sheaf of type* (r, c_1, c_2). *Let* B *be a proper subsheaf of* A *with torsion free quotient* $C = A/B$. *Assume that* $\mu_L(A) = \mu_L(B)$, *and* A *(hence* B*) is* μ*-semistable with respect to* L.

(i) *Either the class* $\xi \stackrel{\text{def}}{=} rc_1(B) - \text{rk}(B)c_1 \equiv 0$ *(numerically equivalent to zero) or* ξ *defines a wall of type* (r, c_1, c_2) *satisfying* $\xi \cdot L = 0$.

(ii) *The integer* $2rc_2 - (r-1)c_1^2$ *is bounded below by*

$$\max(2\text{rk}(B)c_2(B) - (\text{rk}(B)-1)c_1(B)^2,\ 2\text{rk}(C)c_2(C) - (\text{rk}(C)-1)c_1(C)^2).$$

Proof. (i) This is the Theorem 4.C.3 in [8]. Note that we also have

$$\xi = \text{rk}(C)c_1(B) - \text{rk}(B)c_1(C).$$

(ii) From our setup, we have the exact sequence

$$0 \to B \to A \to C \to 0. \tag{3.3}$$

A straight–forward computation shows that $2rc_2 - (r-1)c_1^2$ is equal to

$$\begin{aligned} &\frac{r}{\text{rk}(B)}(2\text{rk}(B)c_2(B) - (\text{rk}(B)-1)c_1(B)^2) \\ &+ \frac{r}{\text{rk}(C)}(2\text{rk}(C)c_2(C) - (\text{rk}(C)-1)c_1(C)^2) - \frac{\xi^2}{\text{rk}(B)\text{rk}(C)}. \end{aligned} \tag{3.4}$$

Since B and C are μ-semistable, they satisfy the Bogomolov inequality:

$$2\,\mathrm{rk}(B)c_2(B) \geq (\mathrm{rk}(B)-1)c_1(B)^2,$$
$$2\,\mathrm{rk}(C)c_2(C) \geq (\mathrm{rk}(C)-1)c_1(C)^2.$$

Since $\xi^2 \leq 0$ by the Hodge Index Theorem, our conclusion follows from (3.4). □

A weaker version of the following lemma can be found in [17].

Lemma 3.4. *Let A be a torsion free sheaf of type (r, c_1, c_2), and let L, $H \in \mathbb{C}_X$.*

(i) *When A is strictly μ-semistable with respect to H, there is an exact sequence*

$$0 \to B \to A \to C \to 0, \tag{3.5}$$

such that C is torsion free, and that either $\xi \stackrel{\text{def}}{=} rc_1(B) - \mathrm{rk}(B)c_1 \equiv 0$, or ξ defines a wall of type (r, c_1, c_2) with $\xi \cdot H = 0$.

(ii) *If A is not μ-semistable with respect to H but is μ-semistable with respect to L, then there exists an exact sequence (3.5) such that C is torsion free, and that ξ defines a wall of type (r, c_1, c_2) with $\xi \cdot L < 0 < \xi \cdot H$ or $\xi \cdot L = 0$.*

Proof. (i) Since A is strictly μ-semistable with respect to H, there exists a proper subsheaf B of A with $\mu_H(B) = \mu_H(A)$. Moreover, we may assume that the quotient $C = A/B$ is torsion free. Then our result follows from Lemma 3.3 (i).

(ii) First of all, assume that A is μ-stable with respect to L. Let $I_{\mathbb{Q}}$ consist of all the rational points on the line segment $\overline{LH}$ connecting L and H. Recall that the conditions of being μ-stable with respect to M and of being not μ-semistable with respect to M are both open conditions for $M \in I_{\mathbb{Q}}$. It follows that there exists $M \in I_{\mathbb{Q}}$ such that A is strictly μ-semistable with respect to M. By (i), there exists an exact sequence (3.5) such that either $\xi = rc_1(B) - \mathrm{rk}(B)c_1 \equiv 0$ or ξ defines a wall of type (r, c_1, c_2) with $\xi \cdot M = 0$. The case $\xi \equiv 0$ cannot happen: indeed, $\mu_L(B) < \mu_L(A)$ since A is μ-stable with respect to L; so $\xi \cdot L < 0$ and $\xi \not\equiv 0$. Also, $\xi \cdot H > 0$ since $\xi \cdot L < 0$, $\xi \cdot M = 0$, and M lies in the interior of $\overline{LH}$.

Next, let A be strictly μ-semistable with respect to L. Then there is a filtration

$$0 = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_{n-1} \subset A_n = A, \tag{3.6}$$

such that all the quotients A_i/A_{i-1} are torsion free and μ-stable with respect to L, and $\mu_L(A_i) = \mu_L(A)$ for all i. If $rc_1(A_i) - \mathrm{rk}(A_i)c_1 \not\equiv 0$ for some $1 \le i \le (n-1)$, then setting $B = A_i$ and $\xi = rc_1(A_i) - \mathrm{rk}(A_i)c_1$, we see from Lemma 3.3 (i) that ξ defines a wall of type (r, c_1, c_2) with $\xi \cdot L = 0$. In the following, assume that

$$rc_1(A_i) - \mathrm{rk}(A_i)c_1 \equiv 0$$

i.e., $c_1(A_i)/\mathrm{rk}(A_i) \equiv c_1/r$ for all $1 \le i \le (n-1)$. Then $\mu_H(A_i) = \mu_H(A)$ for all $1 \le i \le (n-1)$. Since A is not μ-semistable with respect to H, we conclude from Lemma 3.2 that A_i/A_{i-1} is not μ-semistable with respect to H for some $1 \le i \le n$. Since $\widetilde{A} \overset{\text{def}}{=} A_i/A_{i-1}$ is μ-stable with respect to L, we see from the preceding paragraph that there exists an exact sequence

$$0 \to \widetilde{B} \to \widetilde{A} \to \widetilde{C} \to 0, \tag{3.7}$$

such that $\widetilde{C}$ is torsion free, and that $\tilde{\xi} \overset{\text{def}}{=} \mathrm{rk}(\widetilde{A})c_1(\widetilde{B}) - \mathrm{rk}(\widetilde{B})c_1(\widetilde{A})$ defines a wall of type $\big(\mathrm{rk}(\widetilde{A}), c_1(\widetilde{A}), c_2(\widetilde{A})\big)$ with $\tilde{\xi} \cdot L < 0 < \tilde{\xi} \cdot H$. Let B be the kernel of the composition $A_i \to \widetilde{A} \to \widetilde{C}$. Then $\widetilde{B} = B/A_{i-1}$. Letting $C = A/B$, we obtain (3.5). Note that C is torsion free since it sits in the exact sequence

$$0 \to A_i/B \to C \to A/A_i \to 0 \tag{3.8}$$

with both A/A_i and $A_i/B \cong \widetilde{C}$ being torsion free. Setting $\xi = rc_1(B) - \mathrm{rk}(B)c_1$, then $\xi \equiv r\tilde{\xi}/\mathrm{rk}(\widetilde{A})$ since $c_1(A_j)/\mathrm{rk}(A_j) \equiv c_1/r$ for every j. So $\xi^2 < 0$ and $\xi \cdot L < 0 < \xi \cdot H$. Applying Lemma 3.3 (ii) to the inclusions $A_{i-1} \subset A_i \subset A$ and noting $\mu_L(A) = \mu_L(A_i) = \mu_L(A_{i-1})$, we obtain

$$\begin{aligned} 2rc_2 - (r-1)c_1^2 &\ge 2\mathrm{rk}(A_i)c_2(A_i) - (\mathrm{rk}(A_i) - 1)c_1(A_i)^2 \\ &\ge 2\mathrm{rk}(\widetilde{A})c_2(\widetilde{A}) - (\mathrm{rk}(\widetilde{A}) - 1)c_1(\widetilde{A})^2. \end{aligned}$$

Since $\tilde{\xi}$ defines a wall of type $\big(\mathrm{rk}(A_i), c_1(A_i), c_2(A_i)\big)$, we obtain

$$\begin{aligned}\xi^2 &= \frac{r^2}{\mathrm{rk}(\widetilde{A})^2}\,\tilde{\xi}^2 \\ &\ge \frac{r^2}{\mathrm{rk}(\widetilde{A})^2}\cdot\left\{-\frac{\mathrm{rk}(\widetilde{A})^2}{4}\Big(2\mathrm{rk}(\widetilde{A})c_2(\widetilde{A}) - (\mathrm{rk}(\widetilde{A})-1)c_1(\widetilde{A})^2\Big)\right\} \\ &= -\frac{r^2}{4}\Big(2\mathrm{rk}(\widetilde{A})c_2(\widetilde{A}) - (\mathrm{rk}(\widetilde{A})-1)c_1(\widetilde{A})^2\Big) \\ &\ge -\frac{r^2}{4}(2rc_2 - (r-1)c_1^2).\end{aligned}$$

This shows that ξ defines a wall of type (r, c_1, c_2) with $\xi \cdot L < 0 < \xi \cdot H$. □

Our next result generalizes the Theorem 1.2.3 in [16].

Theorem 3.1. *Let A be a torsion free sheaf of type (r, c_1, c_2) on a surface X. Let $\mathcal{C}_-$ and $\mathcal{C}_+$ be two adjacent chambers of type (r, c_1, c_2) sharing a common wall W. Let $L \in \mathcal{C}_-$, $H \in \mathcal{C}_+$, and $\{M\} = W \cap \overline{LH}$. Then, A is μ-stable with respect to L but μ-unstable with respect to H if and only if all the following conditions hold*:

(i) *A is strictly μ-semistable with respect to M,*

(ii) *there exists an exact sequence $0 \to B \to A \to C \to 0$ such that C is torsion free and $\xi \overset{\mathrm{def}}{=} rc_1(B) - \mathrm{rk}(B)c_1$ represents W with $\xi \cdot L < 0 < \xi \cdot H$,*

(iii) *$\mu_L(\widetilde{B}) < \mu_L(A)$ for every proper subsheaf $\widetilde{B}$ of A with $\mu_M(\widetilde{B}) = \mu_M(A)$.*

Proof. First of all, assume that A is μ-stable with respect to L but μ-unstable with respect to H. Since A is μ-stable with respect to L, $\mu_L(\widetilde{B}) < \mu_L(A)$ for every proper subsheaf $\widetilde{B}$ of A. This proves (iii). By Lemma 3.4 (ii), we obtain (ii). Moreover, from the first paragraph in the proof of Lemma 3.4 (ii), we see that A is strictly μ-semistable with respect to M. This proves (i).

Conversely, assume (i) to (iii). Since $\xi \cdot H > 0$, we have $\mu_H(B) > \mu_H(A)$. So A is μ-unstable with respect to H. Assume that A is not μ-stable with respect to L. Since L does not lie on any wall of type (r, c_1, c_2), A cannot be strictly μ-semistable with respect to L. Thus A is μ-unstable with respect to L. Let $\widetilde{B}$ be the first nonzero term in the Harder–Narasimhan filtration of A with respect to L. By the Proposition 4.3.6 in [17], $\widetilde{B}$ is also the first nonzero term in the Harder–Narasimhan filtration of A with respect to every $\widetilde{L} \in \mathcal{C}_-$. So $\mu_{\widetilde{L}}(\widetilde{B}) > \mu_{\widetilde{L}}(A)$ for all $\widetilde{L} \in \mathcal{C}_-$. Since M is on the boundary of

$\mathcal{C}_-$, $\mu_M(\widetilde{B}) \geq \mu_M(A)$. Since A is μ-semistable with respect to M, $\mu_M(\widetilde{B}) \leq \mu_M(A)$. Hence $\mu_M(\widetilde{B}) = \mu_M(A)$. Therefore, $\mu_L(\widetilde{B}) < \mu_L(A)$ by (iii). This contradicts to the choice of $\widetilde{B}$. □

4. Polynomial Bridgeland stability under change of polarizations

In this section, we will assume that X is a smooth projective complex surface and the class $L \in A^1(X)_{\mathbb{R}}$ in a stability data $\Omega_L = (L, \rho, p, U)$ is represented by an $\mathbb{R}$-ample divisor on X. Our goal is to study the variation of the polynomial Bridgeland stability condition $(Z_{\Omega_L}, \mathcal{P}_{\Omega_L})$ on $\mathcal{D}^b(X)$ when we change L.

We begin with some simplifications of the perversity function $p : \{0, 1, 2\} \to \mathbb{Z}$ and the stability vector $\rho = (\rho_0, \rho_1, \rho_2)$. Let $\widetilde{\mathrm{GL}}_+(2, \mathbb{R})$ be the universal covering space of $\mathrm{GL}_+(2, \mathbb{R})$ which denotes the group of 2×2-matrices with positive determinants. As in [4], the group $\widetilde{\mathrm{GL}}_+(2, \mathbb{R})$ acts on the space of polynomial Bridgeland stability conditions. Using this action, we may assume that $\rho_0 = -1$ and $\rho_1 = i$. It follows that $p(0) = p(1) = 0$ (up to a swift of p by $-p(0)$). Using the derived duality mentioned in the paragraph containing (2.9), we may further assume that ρ_2 is contained in the closure $\overline{\mathbb{H}}$. This leaves two cases,

Case 1. $p(0) = p(1) = p(2) = 0$ $(\rho_0 = -1, \rho_1 = i, \rho_2 \in \mathbb{H})$.

Case 2. $p(0) = p(1) = 0$ and $p(2) = -1$ $(\rho_0 = -1, \rho_1 = i, \rho_2 > 0)$.

In the rest of the paper, we will work with these two cases. Note that when X is a $K3$ surface, the second case has been treated in [5]. In addition, in the second case, the stability condition is essentially self-dual (up to modifying U), and so we have the convenience of applying the derived duality.

Next, let $\Omega_L = (L, \rho, p, U)$ where $U = 1 + u_1 + u_2$ with $u_i \in A^i(X)_{\mathbb{C}}$. Let $E \in \mathcal{D}^b(X)$ be of type (r, c_1, c_2). We calculate from (2.8) that

(4.1)
$$Z_{\Omega_L}(E)(m) = \rho_2\, rL^2\, m^2 + \rho_1\big(ru_1L + c_1L\big)m + \rho_0(ru_2 + c_1\, u_1 + c_1^2/2 - c_2).$$

If the rank r is nonzero, let $\mu_L(E) = (c_1 \cdot L)/r$ be the μ-slope function. Then,

(4.2)
$$\frac{Z_{\Omega_L}(E)(m)}{r} = \rho_2 L^2 m^2 + \rho_1\big(u_1 L + \mu_L(E)\big)m + \rho_0 \frac{ru_2 + c_1\, u_1 + c_1^2/2 - c_2}{r}.$$

In the next two lemmas, we handle Case 1 when $p(0) = p(1) = p(2) = 0$.

Lemma 4.1. *Let $\Omega_L = (L, \rho, p, U)$ where $U \in A^*(X)_{\mathbb{R}}$, L is an ample divisor, and $p = 0$ is the constant perversity function. Let $0 \neq A \in \mathcal{A}^p$ be of type (r, c_1, c_2).*

(i) *If $r \neq 0$, then $r > 0$.*

(ii) *If A is Z_{Ω_L}-semistable, then A is a pure dimensional sheaf.*

(iii) *Let $r > 0$. If A is Z_{Ω_L}-semistable, then A is μ-semistable with respect to L; moreover, if A is strictly μ-semistable with respect to L, then either L lies on some wall of type (r, c_1, c_2), or A is semistable with respect to the stability data $(L, \rho, p, \widetilde{U})$ for every $\widetilde{U} \in A^*(X)_{\mathbb{R}}$.*

Proof. (i) Note from Lemma 2.1 (v) that $\mathcal{A}^p = \mathcal{A} = \mathrm{Coh}(X)$. So A is a nonzero sheaf on X. Since $r \neq 0$, we must have $r > 0$.

(ii) Since $p(0) = p(1) = p(2) = 0$, we see that $\rho_0, \rho_1, \rho_2 \in \mathbb{H}$ and $\phi(\rho_0) > \phi(\rho_1) > \phi(\rho_2)$. Now let d be the dimension of the support of A. If A is not pure dimensional, then A contains a d'-dimensional torsion T with $d' < d$. By (4.1),

$$\phi\big(Z_{\Omega_L}(A)(+\infty)\big) = \phi(\rho_d),$$
$$\phi\big(Z_{\Omega_L}(T)(+\infty)\big) = \phi(\rho_{d'}).$$

Since $\phi(\rho_d) < \phi(\rho_{d'})$, we would obtain $\phi\big(Z_{\Omega_L}(A)(m)\big) < \phi\big(Z_{\Omega_L}(T)(m)\big)$ for $m \gg 0$. This is impossible since A is Z_{Ω_L}-semistable and there is an inclusion $T \hookrightarrow A$.

(iii) By (ii), A is torsion free. It follows from (4.2) that the polynomial semistability implies the μ-semistability. Thus, A is μ-semistable with respect to L.

Assume that A is strictly μ-semistable with respect to L, and that L does not lie on any wall of type (r, c_1, c_2). Let $\Omega_{L,\widetilde{U}} = (L, \rho, p, \widetilde{U})$. Let B be any proper subsheaf of A. If $\mu_L(B) < \mu_L(A)$, then $\phi\big(Z_{\Omega_{L,\widetilde{U}}}(B)(m)\big) < \phi\big(Z_{\Omega_{L,\widetilde{U}}}(A)(m)\big)$ for $m \gg 0$. If $\mu_L(B) = \mu_L(A)$, then $rc_1(B) - \mathrm{rk}(B)c_1 \equiv 0$ by Lemma 3.3 (i). By (4.2),

$$\frac{Z_{\Omega_L}(A)(m)}{r} = \rho_2 L^2 m^2 + \rho_1\big(u_1 L + \mu_L(A)\big)m + \rho_0 \left(u_2 + \frac{c_1}{r}u_1 + \frac{c_1^2 - 2c_2}{2r}\right).$$

Since A is Z_{Ω_L}-semistable, $\phi\big(Z_{\Omega_L}(B)(m)\big) \le \phi\big(Z_{\Omega_L}(A)(m)\big)$ for $m \gg 0$. So

$$u_2 + \frac{c_1(B)}{\mathrm{rk}(B)}u_1 + \frac{c_1(B)^2 - 2c_2(B)}{2\mathrm{rk}(B)} \le u_2 + \frac{c_1}{r}u_1 + \frac{c_1^2 - 2c_2}{2r}.$$

Since $c_1(B)/\mathrm{rk}(B) \equiv c_1/r$, we conclude immediately that

$$\frac{c_1(B)^2 - 2c_2(B)}{2\mathrm{rk}(B)} \leq \frac{c_1^2 - 2c_2}{2r}.$$

This in turn implies $\phi\big(Z_{\Omega_{L,\widetilde{U}}}(B)(m)\big) \leq \phi\big(Z_{\Omega_{L,\widetilde{U}}}(A)(m)\big)$ for $m \gg 0$. Therefore, A is semistable with respect to the stability data $(L, \rho, p, \widetilde{U})$ for all $\widetilde{U} \in A^*(X)_{\mathbb{R}}$. □

Lemma 4.2. *Let $\Omega_L = (L, \rho, p, U)$ and $\Omega_H = (H, \rho, p, U)$, where $U \in A^*(X)_{\mathbb{R}}$, and L and H are two $\mathbb{R}$-ample divisors on X. Let $p(0) = p(1) = p(2) = 0$, and let $0 \neq A \in \mathcal{A}^p$ be of type (r, c_1, c_2) with $r > 0$. If A is Z_{Ω_L}-semistable but not Z_{Ω_H}-semistable, then there exists an exact sequence of sheaves*

$$0 \to B \to A \to C \to 0, \tag{4.3}$$

such that C is torsion free and that $\xi \stackrel{\mathrm{def}}{=} rc_1(B) - \mathrm{rk}(B)c_1$ defines a wall of type (r, c_1, c_2) satisfying $\xi \cdot H = 0$, or $\xi \cdot L < 0 < \xi \cdot H$, or $\xi \cdot L = 0$.

Proof. By Lemma 4.1 (iii), the sheaf A is μ-semistable with respect to L. Since A is not Z_{Ω_H}-semistable with respect to H, there exists a proper subsheaf $\widetilde{B} \subset A$ such that $\phi\big(Z_{\Omega_H}(\widetilde{B})(m)\big) > \phi\big(Z_{\Omega_H}(A)(m)\big)$ for $m \gg 0$. By (4.2),

$$\frac{Z_{\Omega_H}(A)(m)}{r} = \rho_2 H^2 m^2 + \rho_1\big(u_1 H + \mu_H(A)\big)m + \rho_0 a,$$
$$\frac{Z_{\Omega_H}(\widetilde{B})(m)}{\mathrm{rk}(\widetilde{B})} = \rho_2 H^2 m^2 + \rho_1\big(u_1 H + \mu_H(\widetilde{B})\big)m + \rho_0 b,$$

where $a, b \in \mathbb{R}$ are independent of H. So $\mu_H(\widetilde{B}) \geq \mu_H(A)$, and A is not μ-stable with respect to H. Note that since extending $\widetilde{B}$ by a torsion sheaf increases either $\mu_H(\widetilde{B})$ or b, we may assume that $A/\widetilde{B}$ is torsion free.

If A is strictly μ-semistable with respect to H, then let $B = \widetilde{B}$. We have $\mu_H(B) = \mu_H(A)$. By Lemma 3.4 (i), either $\xi = \big(rc_1(B) - \mathrm{rk}(B)c_1\big) \equiv 0$ or ξ defines a wall of type (r, c_1, c_2) with $\xi \cdot H = 0$. We claim that $\xi \equiv 0$ can not happen. Indeed, if $\xi \equiv 0$, then $\mu_L(B) = \mu_L(A)$. So $\phi\big(Z_{\Omega_L}(B)(m)\big) \leq \phi\big(Z_{\Omega_L}(A)(m)\big)$ for $m \gg 0$ implies $b \leq a$, which in turn implies that $\phi\big(Z_{\Omega_H}(B)(m)\big) \leq \phi\big(Z_{\Omega_H}(A)(m)\big)$ for $m \gg 0$. This contradicts to our assumption about $B = \widetilde{B}$.

Assume now that A is not μ-semistable with respect to H. By Lemma 3.4 (ii), there exists an exact sequence (4.3) such that $\xi = rc_1(B) - \mathrm{rk}(B)c_1$ defines a wall of type (r, c_1, c_2) satisfying either $\xi \cdot L < 0 < \xi \cdot H$ or $\xi \cdot L = 0$. □

Next, we handle Case 2 when $p(0) = p(1)$ and $p(2) = -1$.

Lemma 4.3. *Let $\Omega_L = (L, \rho, p, U)$, where $U \in A^*(X)_{\mathbb{R}}$, L is an ample divisor, and $p(0) = p(1) = 0$ and $p(2) = -1$. Let $0 \ne E \in \mathcal{A}^p$ be of type (r, c_1, c_2).*

(i) *The category $\mathcal{A}^p$ consists of two-term complexes F with $\mathcal{H}^{-1}(F)$ being torsion free and $\mathcal{H}^0(F)$ being torsion. In particular, if $r \ne 0$, then $r < 0$.*

(ii) *Let $r < 0$, and let E be Z_{Ω_L}-semistable. Then, the torsion sheaf $\mathcal{H}^0(E)$ is not one-dimensional, the sheaf $\mathcal{H}^{-1}(E)$ is μ-semistable with respect to L, and the Bogomolov inequality holds for E:*

$$2rc_2 \ge (r-1)c_1^2. \tag{4.4}$$

Proof. (i) This follows from Lemma 2.1. Since the conclusion has been stated in Sect. 4 of [2], we omit the detailed proof here.

(ii) Since $p(0) = p(1) = 0$ and $p(2) = -1$, we have $\rho_0, \rho_1, -\rho_2 \in \mathbb{H}$ with $\phi(\rho_0), \phi(-\rho_2) > \phi(\rho_1)$. Note that in $\mathcal{A}^p$, we have a short exact sequence

$$0 \to \mathcal{H}^{-1}(E)[1] \to E \to \mathcal{H}^0(E) \to 0. \tag{4.5}$$

Since E is Z_{Ω_L}-semistable, $\phi\big(Z_{\Omega_L}\big(\mathcal{H}^0(E)\big)(m)\big) \ge \phi\big(Z_{\Omega_L}(E)(m)\big)$ for $m \gg 0$. Since

$$Z_{\Omega_L}(E)(m) = -Z_{\Omega_L}\big(\mathcal{H}^{-1}(E)\big)(m) + Z_{\Omega_L}\big(\mathcal{H}^0(E)\big)(m)$$

and $\mathcal{H}^0(E)$ is torsion, we have $\phi\big(Z_{\Omega_L}(E)(+\infty)\big) = \phi(-\rho_2)$. So

$$\phi\big(Z_{\Omega_L}\big(\mathcal{H}^0(E)\big)(+\infty)\big) \ge \phi(-\rho_2).$$

If $\mathcal{H}^0(E)$ is one-dimensional, then $\phi\big(Z_{\Omega_L}(E)(+\infty)\big) = \phi(\rho_1)$. Thus $\phi(\rho_1) \ge \phi(-\rho_2)$, but this contradicts to $\phi(-\rho_2) > \phi(\rho_1)$. Hence $\mathcal{H}^0(E)$ is not one-dimensional.

Let B be any proper subsheaf of $\mathcal{H}^{-1}(E)$ with torsion free quotient. Then, $B[1]$ is a sub-object of E in $\mathcal{A}^p$. Since E is Z_{Ω_L}-semistable,

$\phi\big(Z_{\Omega_L}(B[1])(m)\big) \leq \phi\big(Z_{\Omega_L}(E)(m)\big)$ for $m \gg 0$. By (4.2), we obtain

$$\begin{aligned}\frac{Z_{\Omega_L}(E)(m)}{-r} &= -\rho_2 L^2 m^2 + \rho_1\big(- u_1 L - \mu_L\big(\mathcal{H}^{-1}(E)\big)\big)m - \rho_0 a,\\ \frac{Z_{\Omega_L}(B[1])(m)}{\mathrm{rk}(B)} &= -\rho_2 L^2 m^2 + \rho_1\big(- u_1 L - \mu_L(B)\big)m - \rho_0 b,\end{aligned}$$

where $a, b \in \mathbb{R}$ are independent of m. Since $\phi(-\rho_2) > \phi(\rho_1)$, we must have

$$-u_1 L - \mu_L(B) \geq -u_1 L - \mu_L\big(\mathcal{H}^{-1}(E)\big).$$

Thus $\mu_L(B) \leq \mu_L\big(\mathcal{H}^{-1}(E)\big)$. Hence $\mathcal{H}^{-1}(E)$ is μ-semistable with respect to L.

Let $A = \mathcal{H}^{-1}(E)$, and let $\ell = \ell\big(\mathcal{H}^0(E)\big)$ be the length of the zero-dimensional torsion sheaf $\mathcal{H}^0(E)$. By (4.5), $-\mathrm{ch}(A) = \mathrm{ch}(E) - \ell$. Thus,

$$\mathrm{rk}(A) = -r, \quad c_1(A) = -c_1, \quad c_2(A) = c_1^2 - c_2 - \ell.$$

Applying the usual Bogomolov inequality to the μ-semistable sheaf A, we see that $2(-r)c_2(A) \geq \big((-r) - 1\big)c_1^2$. Therefore, we obtain

$$2rc_2 = 2r(-c_2(A) + c_1^2 - \ell) \geq 2r(-c_2(A) + c_1^2) \geq (r-1)c_1^2. \qquad \square$$

Lemma 4.4. *Let $\Omega_L = (L, \rho, p, U)$, where $U = 1 + u_1 + u_2 \in A^*(X)_{\mathbb{R}}$, and L and H are two $\mathbb{R}$-ample divisors on X. Let p be the perversity function $p(0) = p(1) = 0$ and $p(2) = -1$. Let $0 \neq E \in \mathcal{A}^p$ be of type (r, c_1, c_2) with $r < 0$. If E is Z_{Ω_L}-semistable but not Z_{Ω_H}-semistable, then either $(c_1 + ru_1) \cdot L = 0 < (c_1 + ru_1) \cdot H$, or there is an exact sequence in $\mathcal{A}^p$:*

$$0 \to B[1] \to E \to C \to 0, \tag{4.6}$$

such that B is a torsion free sheaf, and that $\xi \stackrel{\text{def}}{=} rc_1(B[1]) - \mathrm{rk}(B[1])c_1$ defines a wall of type (r, c_1, c_2) satisfying $\xi \cdot H = 0$, or $\xi \cdot L < 0 < \xi \cdot H$, or $\xi \cdot L = 0$.

Proof. Since E is not Z_{Ω_H}-semistable, there exists a Z_{Ω_H}-destablizing subobject $F \in \mathcal{A}^p$. Moreover, we may assume that F is Z_{Ω_H}-semistable. Let $G = E/F \in \mathcal{A}^p$. Then we have a long exact sequence of cohomologies

(4.7)

$$0 \to \mathcal{H}^{-1}(F) \to \mathcal{H}^{-1}(E) \to \mathcal{H}^{-1}(G) \to \mathcal{H}^0(F) \to \mathcal{H}^0(E) \to \mathcal{H}^0(G) \to 0.$$

First of all, assume that $\mathcal{H}^{-1}(F) = 0$. Then F is a torsion sheaf by Lemma 4.3 (i). Let d be the dimension of the support of F. Then $d = 0$ or 1.

If $d = 1$, then $\phi\big(Z_{\Omega_H}(F)(+\infty)\big) = \phi(\rho_1)$; since $\phi\big(Z_{\Omega_H}(E)(+\infty)\big) = \phi(-\rho_2)$ and $\phi(-\rho_2) > \phi(\rho_1)$, we see that $\phi\big(Z_{\Omega_H}(E)(m)\big) > \phi\big(Z_{\Omega_H}(F)(m)\big)$ for $m \gg 0$; this contradicts to the assumption that F is Z_{Ω_H}-destablizing. So $d = 0$. Since $\rho_0 = -1$,

$$Z_{\Omega_L}(F)(m) = \rho_0 \cdot \ell(F) = -\ell(F)$$

is a constant polynomial. Since

$$1 = \phi\big(Z_{\Omega_L}(F)(m)\big) \le \phi\big(Z_{\Omega_L}(E)(m)\big)$$

for $m \gg 0$, we have $\phi\big(Z_{\Omega_L}(E)(m)\big) = 1$ for $m \gg 0$. By (4.1), $(c_1 + ru_1) \cdot L = 0$. Similarly, since

$$1 = \phi\big(Z_{\Omega_H}(F)(m)\big) > \phi\big(Z_{\Omega_H}(E)(m)\big)$$

for $m \gg 0$, we obtain $(c_1 + ru_1) \cdot H > 0$. Thus $(c_1 + ru_1) \cdot L = 0 < (c_1 + ru_1) \cdot H$.

Next, assume that $\mathrm{rk}\mathcal{H}^{-1}(F) = \mathrm{rk}\mathcal{H}^{-1}(E)$. Thus G is a zero-dimensional torsion sheaf. So $\phi\big(Z_{\Omega_H}(G)(m)\big) = 1 \ge \phi\big(Z_{\Omega_H}(E)(m)\big)$ for $m \gg 0$, contradicting to the assumption that G is Z_{Ω_H}-destablizing for E. Hence this case cannot happen.

Finally, assume $0 < \mathrm{rk}\mathcal{H}^{-1}(F) < \mathrm{rk}\mathcal{H}^{-1}(E)$. Let $\widetilde{B} = \mathcal{H}^{-1}(F)$ and $A = \mathcal{H}^{-1}(E)$. Then A is μ-semistable with respect to L by Lemma 4.3 (ii). Since F is Z_{Ω_H}-semistable, $\mathcal{H}^0(F)$ is a zero-dimensional torsion by Lemma 4.3 (ii) as well. So

$$\begin{aligned}
\frac{Z_{\Omega_H}(E)(m)}{-r} &= -\rho_2 H^2 m^2 + \rho_1\big(-u_1 H - \mu_H(A)\big)m - \rho_0 \tilde{a}, \\
\frac{Z_{\Omega_H}(F)(m)}{-\mathrm{rk}(F)} &= -\rho_2 H^2 m^2 + \rho_1\big(-u_1 H - \mu_H(\widetilde{B})\big)m - \rho_0 \tilde{b},
\end{aligned}$$

where $-\mathrm{rk}(F) = \mathrm{rk}(\widetilde{B}) > 0$, and $\tilde{a}, \tilde{b} \in \mathbb{R}$ are independent of m. Since $\phi(-\rho_2) > \phi(\rho_1)$ and $\phi\big(Z_{\Omega_H}(F)(m)\big) > \phi\big(Z_{\Omega_H}(E)(m)\big)$ for $m \gg 0$, we must have

$$(-u_1 H - \mu_H(\widetilde{B})) \le (-u_1 H - \mu_H(A)).$$

So $\mu_H(\widetilde{B}) \ge \mu_H(A)$. Now there are two cases. In the first case, A is strictly μ-semistable with respect to H. Let $B = \widetilde{B}$. We have $\mu_H(B) = \mu_H(A)$. Note from (4.7) that A/B is torsion free. By Lemma 3.4 (i), either

$$\xi = (-r)c_1(B) - \mathrm{rk}(B)(-c_1) \equiv 0$$

or ξ defines a wall of type (r, c_1, c_2) with $\xi \cdot H = 0$. As in the second paragraph in the proof of Lemma 4.2, $\xi \equiv 0$ cannot happen. In the second case, A is not μ-semistable with respect to H. By Lemma 3.4 (ii), there exists an exact sequence

$$0 \to B \to A \to \widetilde{C} \to 0,$$

such that $\widetilde{C}$ is torsion free and $\xi = (-r)c_1(B) - \mathrm{rk}(B)(-c_1)$ defines a wall of type (r, c_1, c_2) satisfying $\xi \cdot L < 0 < \xi \cdot H$ or $\xi \cdot L = 0$. Note that in both cases,

$$\xi = rc_1(B[1]) - \mathrm{rk}(B[1])\, c_1.$$

In addition, since A/B is torsion free, we have the inclusions $B[1] \subset A[1] \subset E$ in $\mathcal{A}^p$. Letting $C \in \mathcal{A}^p$ be the quotient $E/B[1]$ gives rise to (4.6). □

Theorem 4.1. *Let X be a smooth projective surface, and let Ω_L denote the data (L, ρ, p, U), where $p(0) = p(1) = 0$ and $U = 1 + u_1 + u_2$ with $u_i \in A^i(X)_{\mathbb{R}}$. Let $0 \neq E \in \mathcal{D}^b(X)$ be of type (r, c_1, c_2) with $r \neq 0$.*

(i) *If E is Z_{Ω_L}-semistable, then the Bogomolov inequality holds*

$$2rc_2 \geq (r-1)c_1^2.$$

(ii) *Let L and H be contained in the same chamber of type (r, c_1, c_2). When $r < 0$, we further assume that $(c_1 + ru_1)$ does not satisfy*

$$(c_1 + ru_1) \cdot L = 0 < (c_1 + ru_1) \cdot H.$$

Then E is Z_{Ω_L}-semistable if and only if it is Z_{Ω_H}-semistable.

Proof. (i) By definition, the semistable objects in $\mathcal{D}^b(X)$ are precisely the shifts of the semistable objects in the heart $\mathcal{A}^p$. So $E = \widetilde{E}[k]$ for some Z_{Ω_L}-semistable object $\widetilde{E} \in \mathcal{A}^p$. When $p(2) = 0$, we apply the usual Bogomolov inequality to the μ-semistable sheaf $\widetilde{E}$; when $p(2) = -1$, we apply (4.4) to $\widetilde{E}$. So we obtain

$$2\,\mathrm{rk}(\widetilde{E})\, c_2(\widetilde{E}) \geq \big(\mathrm{rk}(\widetilde{E}) - 1\big) c_1(\widetilde{E})^2.$$

When k is even, we have $\mathrm{rk}(\widetilde{E}) = r$ and $c(\widetilde{E}) = c(E)$; when k is odd, $\mathrm{rk}(\widetilde{E}) = -r$ and $1 + \tilde{c}_1 + \tilde{c}_2 \stackrel{\mathrm{def}}{=} c(\widetilde{E}) = c(E)^{-1} = 1 - c_1 + (c_1^2 - c_2)$. In either case, we conclude that E satisfies the Bogomolov inequality (4.4) as well.

(ii) Again, the object E is Z_{Ω_L}-semistable if and only if $E = \widetilde{E}[k]$ for some Z_{Ω_L}-semistable object $\widetilde{E} \in \mathcal{A}^p$. By Lemma 3.1, the chambers of type

(r, c_1, c_2) coincide with the chambers of type $(-r, \tilde{c}_1, \tilde{c}_2)$. By Lemmas 4.2 and 4.4, $\widetilde{E} \in \mathcal{A}^p$ is Z_{Ω_L}-semistable if and only if it is Z_{Ω_H}-semistable. Hence, we see that E is Z_{Ω_L}-semistable if and only if it is Z_{Ω_H}-semistable. □

Definition 4.1. Fix a type (r, c_1, c_2) on the surface X and an ample divisor L.

(i) Define $\overline{\mathfrak{M}}_L^G(r, c_1, c_2)$ to be the moduli space of torsion free sheaves which are of type (r, c_1, c_2) and are Gieseker-semistable with respect to L.

(ii) For $\Omega_L = (L, \rho, p, U)$, define $\overline{\mathfrak{M}}_{\Omega_L}(r, c_1, c_2)$ to be the set of all objects $E \in \mathcal{A}^p$ which are of type (r, c_1, c_2) and are Z_{Ω_L}-semistable.

We remark that it is unknown whether $\overline{\mathfrak{M}}_{\Omega_L}(r, c_1, c_2)$ exists as a scheme.

Lemma 4.5. *Let X be a smooth projective surface, and fix a numerical type (r, c_1, c_2) with $r \neq 0$. Let $\Omega_{L,U} = (L, \rho, p, U)$, where $p = 0$ is the constant perversity function, $U \in A^*(X)_\mathbb{R}$, and $L \in \mathbb{C}_X$ does not lie on any wall of type (r, c_1, c_2). Then, $\overline{\mathfrak{M}}_{\Omega_{L,U}}(r, c_1, c_2)$ is independent of U.*

Proof. Note from Lemma 4.1 (i) that we have $r > 0$. Let $A \in \overline{\mathfrak{M}}_{\Omega_{L,U_0}}(r, c_1, c_2)$ for some $U_0 \in A^*(X)_\mathbb{R}$. By Lemma 4.1 (iii), A is a torsion free sheaf μ-semistable with respect to L. If A is μ-stable with respect to L, then $A \in \overline{\mathfrak{M}}_{\Omega_{L,\widetilde{U}}}(r, c_1, c_2)$ for every $\widetilde{U} \in A^*(X)_\mathbb{R}$. If A is strictly μ-semistable with respect to L, then since L does not lie on any wall of type (r, c_1, c_2), we see from Lemma 4.1 (iii) again that $A \in \overline{\mathfrak{M}}_{\Omega_{L,\widetilde{U}}}(r, c_1, c_2)$ for every $\widetilde{U} \in A^*(X)_\mathbb{R}$. □

Proposition 4.1. *Let X be a smooth projective surface, and fix a numerical type (r, c_1, c_2), with $r \neq 0$. Let $\Omega_L = (L, \rho, p, U)$, where $p = 0$ is the constant perversity function, $U \in A^*(X)_\mathbb{R}$, and $L \in \mathbb{C}_X$ does not lie on any wall of type (r, c_1, c_2). Then $A \in \overline{\mathfrak{M}}_{\Omega_L}(r, c_1, c_2)$ if and only if $A \in \overline{\mathfrak{M}}_L^G(r, c_1, c_2)$.*

Proof. It has been proved in [2] that if $U = \mathrm{td}(X)$, then $A \in \overline{\mathfrak{M}}_{\Omega_L}(r, c_1, c_2)$ if and only if $A \in \overline{\mathfrak{M}}_L^G(r, c_1, c_2)$. So our result follows from Lemma 4.5. □

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