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A compactification of the space of algebraic maps from $\mathbb{P}^1$ to $\mathbb{P}^n$

Yi Hu, Jiayuan Lin and Yijun Shao

We provide a natural smooth projective compactification of the space of algebraic maps from $\mathbb{P}^1$ to $\mathbb{P}^n$ by adding a divisor with simple normal crossings.

1. Introduction

Fix a vector space V of dimension $n+1$. Let N_d be the Quot scheme parameterizing the exact sequences

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-d) \xrightarrow{f} V \otimes \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{Q} \longrightarrow 0,$$

where $\mathcal{Q}$ is a coherent sheaf over $\mathbb{P}^1$ of degree d and rank n. The locus of points of N_d where $\mathcal{Q}$ is locally free can be identified with the space $\mathring{N}_d$ of algebraic maps of degree d from $\mathbb{P}^1$ to $\mathbb{P}(V)$. The boundary $N_d \setminus \mathring{N}_d$, consisting of points parameterizing the exact sequences above where $\mathcal{Q}$ is not locally free, has rather complicated singularities. One of the main theme of the current paper is to resolve the singularities of $N_d \setminus \mathring{N}_d$. For this, we find that $N_d \setminus \mathring{N}_d$ comes equipped with a natural filtration by subschemes

$$Z_{d,0} \subset Z_{d,1} \subset \cdots \subset Z_{d,d-1} = N_d \setminus \mathring{N}_d,$$

where $Z_{d,k}$ is supported on the subset

$$\{[f] \in N_d \,|\, \text{the torsion part of } \mathcal{Q} \text{ has degree} \geq d-k\},$$

for all $0 \leq k \leq d-1$. As noted in pages 4738-39 of [8], it is expected that one can successively blow up N_d along the subschemes $Z_{d,0}, Z_{d,1}, \ldots, Z_{d,d-1}$ such that the resulting final scheme M_d is smooth and the boundary $M_d \setminus \mathring{N}_d$ is a divisor with normal crossings. The main purpose of the current paper is to execute this conjectural construction in details.

We now briefly outline the main constructions. For each integer $m \geq 0$, we let

$$\rho_{f,m} : \mathrm{Hom}(V, \mathcal{O}_{\mathbb{P}^1}(m)) \longrightarrow \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(-d), \mathcal{O}_{\mathbb{P}^1}(m))$$

be the homomorphism obtained by applying $\mathrm{Hom}(-, \mathcal{O}_{\mathbb{P}^1}(m))$ to f. Fix any $m \geq d-1$. We set

$$W_m := \mathrm{Hom}(V, \mathcal{O}_{\mathbb{P}^1}(m)) \quad \text{and} \quad V_{d+m} := H^0(\mathcal{O}_{\mathbb{P}^1}(d+m)).$$

Let $0 \leq k \leq d-1$. Then the exterior power $\bigwedge^{k+2+m} \rho_{d,m}$ is a section of

$$\mathrm{Hom}(\bigwedge^{k+2+m} W_m, \bigwedge^{k+2+m} V_{d+m}) \otimes \mathcal{O}_{N_d}(k+2+m).$$

We proved that the scheme of zeros of $\bigwedge^{k+2+m} \rho_{d,m}$ is independent of $m \geq d-1$ and it is by definition the subscheme $Z_{d,k}$.

For any two vector spaces E and F, we let $S(E,F)$ denote $\mathbb{P}(\mathrm{Hom}(E,F))$. Then our main theorem reads

Theorem 1.1. *The variety M_d is a compactification of $\mathring{N}_d$ such that the following hold.*

(1) *M_d is isomorphic to the closure of the graph of the rational map*

$$N_d \dashrightarrow \prod_{l=0}^{d-1} S(\bigwedge^{m+2+l} W_m, \bigwedge^{m+2+l} V_{d+m})$$
$$[f] \mapsto ([\bigwedge^{m+2} \rho_{f,m}], [\bigwedge^{m+3} \rho_{f,m}], \ldots, [\bigwedge^{m+d+1} \rho_{f,m}]),$$

for all $m \geq d-1$.

(2) *M_d is a nonsingular projective variety.*

(3) *The complement $M_d \setminus \mathring{N}_d$ is a divisor with simple normal crossings.*

In the course of proofs, we discover that our space M_d possesses structures strikingly similar to the classic and modern theories on complete collineations, complete correlations, and complete quadrics. Indeed, our proofs rely on Vainsencher's construction of the spaces of the complete collineations [20, 21]. The beautiful stories on these complete objects went all the way back to the works of Schubert in 19th century, to the works of[1] Severi, Van de Waerden, Semple, Tyrrell in the early and middle of the last century, and to the modern treatments, refinements and advances of Laskov [12, 13], Vainsencher [20, 21], Thorup–Kleiman [19], De Concini–Procesi [3], Demazure, and De Concini–Procesi–Goresky–MacPherson [4] in the 1980's. Needless to say, this way of producing good compactifications is nowadays very standard with the Fulton–MacPherson compactification [5] and Procesi–MacPherson compactification [14] being the prime examples. Related works in 1990's include the works of De Concini–Procesi schools on hyperplane arrangements and the works of Bifet–De Concini–Procesi on regular embeddings. Lately there are further works in this direction by Wenchuan Hu and Li Li. For the topological aspects of the stories, see [16, 18].

We believe that there are some lurking geometric objects, analogous to the above classic complete objects, for which our space M_d is a parameter space. This is being pursued in a forthcoming publication [9].

Using Quot schemes of coherent sheaves of higher co-ranks, similar spaces can be constructed to provide good compactifications of the spaces of maps from the smooth rational curve to Grassmannians. This has been carried out in the third author's PhD dissertation [17].

Two further problems to consider are to generalize to higher genus curves and to compare our compactification with the Kontsevich moduli space of stable maps (see, for example,[1, 2, 6, 11]).

Throughout the paper, we will work with a fixed algebraically closed base field of characteristic zero, unless otherwise stated.

2. Conventions and Terminology

2.1. From the introduction, N_d is the Quote scheme parameterizing the exact sequences

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-d) \xrightarrow{f} V \otimes \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{Q} \longrightarrow 0, \tag{2.1}$$

where $\mathcal{Q}$ is a coherent sheaf over $\mathbb{P}^1$ of degree d and rank n. We will denote the corresponding point of (2.1) by

$$[f] \in N_d.$$

[1]At the risk of inadvertently omitting many authors who made important contributions to these areas, we mention only a few.

Note that we have the following identification

$$N_d = \mathbb{P}(\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(-d), V \otimes \mathcal{O}_{\mathbb{P}^1})).$$

The Quote scheme N_d comes equipped with the universal family which is an exact sequence of coherent sheaves on $\mathbb{P}^1 \times N_d$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-d) \otimes \mathcal{O}_{N_d}(-1) \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}^1 \times N_d} \longrightarrow \mathcal{Q} \longrightarrow 0, \tag{2.2}$$

where $\mathcal{Q}$ is a coherent sheaf of rank n, of relative degree d and flat over N_d. The restriction of (2.2) to the fiber of the projection $\mathbb{P}^1 \times N_d \to N_d$ at the point $[f]$ is exactly the exact sequence (2.1).

2.2. Alternatively, we may realize N_d as

$$N_d = \{[f_0, \ldots, f_n] \mid f_0, \ldots, f_n \in H^0(\mathcal{O}_{\mathbb{P}^1}(d)), \text{ not all are zero }\}.$$

From this perspective, the boundary strata of $N_d \setminus \mathring{N}_d$ are

$$C_{d,k} = \{[f_0, \ldots, f_n] \mid f_0, \ldots, f_n \text{ have a common factor of degree} \geq d-k\},$$

for all $0 \leq k \leq d-1$. More details along this line is to be given in Section 3.2.

2.3. For any nonnegative integer m, we set

$$V_m := H^0(\mathcal{O}_{\mathbb{P}^1}(m)),$$
$$W_m := \mathrm{Hom}(V, \mathcal{O}_{\mathbb{P}^1}(m)).$$

In addition, we have the identifications

$$V_{d+k} = \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(-d), \mathcal{O}_{\mathbb{P}^1}(k)),$$
$$N_d = \mathbb{P}(V_d \otimes V).$$

3. Resultant homomorphisms

3.1. Resultant homomorphism and degree of torsion

3.1. Consider the exact sequence (2.1)

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-d) \xrightarrow{f} V \otimes \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

For each integer $k \geq 0$, recall from the introduction that

$$\rho_{f,k} : \mathrm{Hom}(V, \mathcal{O}_{\mathbb{P}^1}(k)) \longrightarrow \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(-d), \mathcal{O}_{\mathbb{P}^1}(k)) \tag{3.1}$$

is the map obtained by applying $\mathrm{Hom}(-, \mathcal{O}_{\mathbb{P}^1}(k))$ to f. We call $\rho_{f,k}$ the *kth resultant homomorphism* of f.

Proposition 3.1. *Let $\mathcal{T}$ be the torsion submodule of $\mathcal{Q}$. Then we have*

$$\operatorname{rank} \rho_{f,k} \begin{cases} = k+1+d-\deg \mathcal{T}, & \textit{if } \; k \geq d - \deg \mathcal{T} - 1, \\ \geq 2(k+1), & \textit{if } \; 0 \leq k \leq d - \deg \mathcal{T} - 1. \end{cases}$$

In particular, $\operatorname{rank} \rho_{f,k} = 2(k+1)$ *when* $k = d - \deg \mathcal{T} - 1$.

Proof. First, we write $\mathcal{Q} = \mathcal{F} \oplus \mathcal{T}$ such that $\mathcal{F}$ is a locally free sheaf of rank n and degree $(d - \deg \mathcal{T})$. By applying the functor $\mathrm{Hom}(-, \mathcal{O}_{\mathbb{P}^1}(k))$ to the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-d) \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{Q} \longrightarrow 0,$$

we get an exact sequence

$$0 \longrightarrow \mathrm{Hom}(\mathcal{Q}, \mathcal{O}_{\mathbb{P}^1}(k)) \longrightarrow \mathrm{Hom}(V, \mathcal{O}_{\mathbb{P}^1}(k)) \xrightarrow{\rho_{f,k}} \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(-d), \mathcal{O}_{\mathbb{P}^1}(k)).$$

Thus we have

$$\begin{aligned} \operatorname{rank} \rho_{f,k} &= \dim \mathrm{Hom}(V, \mathcal{O}_{\mathbb{P}^1}(k)) - \dim \mathrm{Hom}(\mathcal{Q}, \mathcal{O}_{\mathbb{P}^1}(k)) \\ &= \dim H^0(V^\vee(k)) - \dim \mathrm{Hom}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^1}(k)) \\ &= (n+1)(k+1) - \dim H^0(\mathcal{F}^\vee(k)). \end{aligned}$$

We write $\mathcal{F}$ as $\bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(d_i)$ with $d_i \geq 0$ and $\sum_{i=1}^n d_i = d - \deg \mathcal{T}$. Then

$$H^0(\mathcal{F}^\vee(k)) = H^0\left(\bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(k - d_i)\right) = \bigoplus_{i=1}^n H^0(\mathcal{O}_{\mathbb{P}^1}(k - d_i)).$$

Consequently,

$$h^0(\mathcal{F}^\vee(k)) = \sum_{i=1}^n \max\{k - d_i + 1, 0\}.$$

If $k \geq d - \deg \mathcal{T} - 1$, then $k - d_i + 1 \geq 0$ for all i. In this case

$$h^0(\mathcal{F}^\vee(k)) = \sum_{i=1}^{n}(k - d_i + 1) = n(k+1) - (d - \deg \mathcal{T})$$

and it implies

$$\operatorname{rank} \rho_{f,k} = k + 1 + d - \deg \mathcal{T}.$$

This proves the first case of the proposition.

On the other hand, if $0 \leq k \leq d - \deg \mathcal{T} - 1$, then we claim that

$$h^0(\mathcal{F}^\vee(k)) \leq (n-1)(k+1).$$

There are two cases to consider. First, $k - d_i + 1 \geq 0$ for all i. In this case,

$$h^0(\mathcal{F}^\vee(k)) = \sum_{i=1}^{n}(k - d_i + 1) = n(k+1) - (d - \deg \mathcal{T}) \leq (n-1)(k+1).$$

Second, $k - d_i + 1 < 0$ for some i, say for $i = 1$. In this case,

$$h^0(\mathcal{F}^\vee(k)) = \sum_{i=2}^{n} \max\{k - d_i + 1, 0\} \leq \sum_{i=2}^{n}(k+1) = (n-1)(k+1).$$

Thus, in either case, $\operatorname{rank} \rho_{f,k} \geq (n+1)(k+1) - (n-1)(k+1) = 2(k+1)$.

This completes the proof. □

Corollary 3.1. *Let the notations be as in above.*

(1) *Assume that* $\operatorname{rank} \rho_{f,k} \leq 2k+1$. *Then we have*

$$\deg \mathcal{T} \geq d - k \quad \textit{and} \quad \operatorname{rank} \rho_{f,k} = k + 1 + d - \deg \mathcal{T}.$$

Further, for all $m \geq k - 1$,

$$\operatorname{rank} \rho_{f,m} - \operatorname{rank} \rho_{f,k} = m - k.$$

(2) *For any fixed* $l \geq 0$,

$$\deg \mathcal{T} = d - k \textit{ if and only if } \operatorname{rank} \rho_{f,k+l} = 2k + 1 + l.$$

(3) *For any fixed* $l \geq 0$,

$$\deg \mathcal{T} \geq d-k \textit{ if and only if } \operatorname{rank} \rho_{f,k+l} \leq 2k+1+l.$$

Proof. (1) By Proposition 3.1, we must have $k > d - \deg \mathcal{T} - 1$, that is $\deg \mathcal{T} \geq d-k$. In this case, $\operatorname{rank} \rho_{f,k} = k+1+d-\deg T$. Further, when $m \geq k-1$, because $k+1+d-\deg \mathcal{T} = \operatorname{rank} \rho_{f,k} \leq 2k+1$, we must also have $m \geq d - \deg \mathcal{T} - 1$. Hence by Proposition 3.1 again,

$$\rho_{f,m} - \rho_{f,k} = (m+1+d-\deg \mathcal{T}) - (k+1+d-\deg \mathcal{T}) = m-k.$$

(2) $\deg \mathcal{T} = d-k$ if and only if (by Proposition 3.1) $\operatorname{rank} \rho_{f,k} = 2k+1$ if and only if (by (1)) $\operatorname{rank} \rho_{f,k+l} = 2k+1+l$.

(3) $\deg \mathcal{T} \geq d-k$ if and only if (by Proposition 3.1) $\operatorname{rank} \rho_{f,k} \leq 2k+1$ if and only if(by (1)) $\operatorname{rank} \rho_{f,k+l} \leq 2k+1+l$. □

3.2. Resultant homomorphism and degree of common factor

In this subsection, we reinterpret the results of Section 3.1 in terms of "common factors of polynomials". For this, we use the following identification:

$$\operatorname{Hom}(\mathcal{O}_{\mathbb{P}^1}(-d), V) = V \otimes \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^1}(-d), \mathcal{O}_{\mathbb{P}^1}) = V \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(d)).$$

Then we let $\{x, y\}$ be a basis for $H^0(\mathcal{O}_{\mathbb{P}^1}(1))$ and let $\{e_0, \ldots, e_n\}$ be a basis for V. This way, any $f \in \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^1}(-d), V)$ can be written as

$$f = e_0 \otimes f_0 + \cdots + e_n \otimes f_n,$$

where $f_i = \sum_{j=0}^{d} a_{ij} x^{d-j} y^j$ are homogeneous polynomials in x, y of degree d. Recall that under this view of points of the space N_d, $\deg \mathcal{T}$ is simply the degree of the greatest common factors of $f_0, \ldots, f_n$.

3.2. Consider the kth resultant homomorphism (3.1)

$$\rho_{f,k} : \operatorname{Hom}(V, \mathcal{O}_{\mathbb{P}^1}(k)) \longrightarrow \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^1}(-d), \mathcal{O}_{\mathbb{P}^1}(k)).$$

We abbreviate $\rho_{f,k}$ as $\rho_{f,k} : W_k \longrightarrow V_{d+k}$, where

$$W_k := \mathrm{Hom}(V, \mathcal{O}_{\mathbb{P}^1}(k)) = V^\vee \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(k)),$$
$$V_{d+k} := \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(-d), \mathcal{O}_{\mathbb{P}^1}(k)) = H^0(\mathcal{O}_{\mathbb{P}^1}(d+k)).$$

Using the basis $\{e_i^\vee \otimes x^k y^{k-j} : i = 0, \ldots, n;\ j = 0, \ldots, k\}$ for W_k and the basis $\{x^{d+k-j}y^j : j = 0, \ldots, d+k\}$ for V_{d+k}, the linear map $\rho_{f,k}$ is represented by the following $(k+1)(n+1) \times (d+k+1)$ matrix:

$$A_{f,k} = \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0d} & & & & \\ \vdots & \vdots & \cdots & \vdots & & & & \\ a_{n0} & a_{n1} & \cdots & a_{nd} & & & & \\ & a_{00} & a_{01} & \cdots & a_{0d} & & & \\ & \vdots & \vdots & \cdots & \vdots & & & \\ & a_{n0} & a_{n1} & \cdots & a_{nd} & & & \\ & & \cdots & \cdots & \cdots & \cdots & & \\ & & & \cdots & \cdots & \cdots & \cdots & \\ & & & & a_{00} & a_{01} & \cdots & a_{0d} \\ & & & & \vdots & \vdots & \cdots & \vdots \\ & & & & a_{n0} & a_{n1} & \cdots & a_{nd} \end{pmatrix}, \tag{3.2}$$

which acts on elements of W_k by multiplication from the right.

Corollary 3.1 takes the following form in this setting.

Corollary 3.2. *Let g be a greatest common factor of $f_0, \ldots, f_n$.*

(1) *Assume that* $\mathrm{rank}\, A_{f,k} \leq 2k+1$. *Then we have*

$$\deg g \geq d - k \quad \textit{and} \quad \mathrm{rank}\, A_{f,k} = k + 1 + d - \deg g.$$

Further, for all $m \geq k-1$,

$$\mathrm{rank}\, A_{f,m} - \mathrm{rank}\, A_{f,k} = m - k.$$

(2) *For any fixed $l \geq 0$,*

$$\deg g = d - k \ \textit{ if and only if } \ \mathrm{rank}\, A_{f,k+l} = 2k + 1 + l.$$

(3) *For any fixed* $l \geq 0$,

$$\deg g \geq d-k \quad \textit{if and only if} \quad \operatorname{rank} A_{f,k+l} \leq 2k+1+l.$$

We remark here that Proposition 3.1 implies Kakie's Proposition 3 in [10]. For more on resultants, see [7].

4. Determinantal subschemes

4.1. The universal resultant homomorphisms

4.1. Let $\pi : \mathbb{P}^1 \times N_d \to N_d$ denote the second projection. For each integer $m \geq 0$, by applying the functor $\pi_* \mathcal{H}om(-, \mathcal{O}_{\mathbb{P}^1}(m))$ to the homomorphism $\mathcal{O}_{\mathbb{P}^1}(-d) \otimes \mathcal{O}_{N_d}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}^1 \times N_d}$ (which comes from the universal family (2.2)), we obtain a nowhere zero $\mathcal{O}_{N_d}$-homomorphism

$$\rho_{d,m} : W_m \to V_{d+m} \otimes \mathcal{O}_{N_d}(1).$$

It is a nowhere zero section of $\operatorname{Hom}(W_m, V_{d+m}) \otimes \mathcal{O}_{N_d}(1)$ whose restriction to every point $[f] \in N_d$ is $\rho_{f,m}$. We call $\rho_{d,m}$ the mth *universal resultant homomorphism.*

4.2. For $m, k \geq 0$, the exterior power $\bigwedge^{k+2+m} \rho_{d,m}$ is a section of

$$\operatorname{Hom}\left(\bigwedge^{k+2+m} W_m, \bigwedge^{k+2+m} V_{d+m}\right) \otimes \mathcal{O}_{N_d}(k+2+m).$$

Fix any $1 \leq k \leq d-1$. Then using Corollary 3.1, one checks that the scheme $Z_{d,k;m}$ of zeros of $\bigwedge^{k+2+m} \rho_{d,m}$ is supported on $C_{d,k}$ whenever $m \geq k$. The ideal sheaf $I_{d,k;m}$ of $Z_{d,k;m}$ is the image of the induced homomorphism

$$\operatorname{Hom}\left(\bigwedge^{k+2+m} W_m, \bigwedge^{k+2+m} V_{d+m}\right)^{\vee} \otimes \mathcal{O}_{N_d}(-k-2-m) \twoheadrightarrow I_{d,k;m} \subset \mathcal{O}_{N_d}.$$

4.3. We suspect that for any fixed $1 \leq k \leq d-1$,

$$I_{d,k;m} = I_{d,k;k} \tag{4.1}$$

holds whenever $m \geq k$. This would imply that $I_{d,k;m}$ with $m \geq k$ all endow the same scheme structure on $C_{d,k}$. Rather than proving this, we will show the weaker Proposition 4.1 below, which already suffice for our purpose. To pave the way for its proof, we need some preparation.

4.4. Let R be a ring and A a $p \times q$ matrix over R. We let $I_l(A)$ be the ideal generated by all $l \times l$ minors of A with $1 \leq l \leq p, q$. Suppose B is an invertible $p \times p$ matrix and C is an invertible $q \times q$ matrix. Then one checks directly that

$$I_l(A) = I_l(BA) = I_l(AC), \tag{4.2}$$

for all $1 \leq l \leq p, q$. In more concrete terms, (4.2) means that the following three operations on the matrix A preserve the ideal $I_l(A)$:

(1) multiply a row or a column by units;

(2) interchanging two rows or two columns;

(3) multiply one row (column) by an element of R and add the result to another row (column).

Proposition 4.1. *Fix* $1 \leq k \leq d-1$. *Then*

(1) $I_{d,0;m} = I_{d,0;0}$ *for all* $m \geq 0$;

(2) $I_{d,k;m} = I_{d,k;d-1}$ *for all* $m \geq d-1$.

Proof. Consider any nonzero $f \in \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(-d), V)$. Recall that using the bases as chosen in 3.2, we express it as $f = e_0 \otimes f_0 + \cdots + e_n \otimes f_n$ where $f_i = \sum_{j=0}^{d} a_{ij} x^{d-j} y^j$. This way, the coefficients (a_{ij}) become the homogeneous coordinates of N_d. Observe in addition that for any fixed $m \geq 0$, (a_{ij}) are also the entries in the first block of the matrix $A_{f,m}$ of the m-version of (3.2). We regard $A_{f,m}$ as a matrix over the polynomial ring $\mathbf{k}[a_{ij}]$. Observe that the ideal sheaf $I_{d,k;m}$ coincides with the sheaf $(I_{k+2+m}(A_{f,m}))^{\sim}$ associated to the module $I_{k+2+m}(A_{f,m})$. Our strategy of the proof is to cover N_d by the standard affine open subsets and prove the statements over the open subsets.

We first localize to the affine open set $U_0 = (a_{00} \neq 0)$. By using the affine coordinates $b_{ij} = a_{ij}/a_{00}$, the matrix $A_{f,m}$ is reduced to $B_{f,m}$ such that all the entries a_{ij} are replaced by b_{ij} except that a_{00} is replaced by 1. Then the localization of each ideal $I_{k+2+m}(A_{f,m})$ to U_0 is $I_{k+2+m}(B_{f,m})$. Now using $b_{00} = 1$, we can eliminate the entries b_{i0} by some appropriate row operations and reduce the matrix $B_{f,m}$ to

$$C_{f,m} = \begin{pmatrix} 1 & c_{01} & \cdots & c_{0d} & & & & & \\ 0 & c_{11} & \cdots & c_{1d} & & & & & \\ \vdots & \vdots & \cdots & \vdots & & & & & \\ 0 & c_{n1} & \cdots & c_{nd} & & & & & \\ & 1 & c_{01} & \cdots & c_{0d} & & & & \\ & 0 & c_{11} & \cdots & c_{1d} & & & & \\ & \vdots & \vdots & \cdots & \vdots & & & & \\ & 0 & c_{n1} & \cdots & c_{nd} & & & & \\ & & \cdots & \cdots & \cdots & \cdots & & & \\ & & & \cdots & \cdots & \cdots & \cdots & & \\ & & & & 1 & c_{01} & \cdots & c_{0d} \\ & & & & 0 & c_{11} & \cdots & c_{1d} \\ & & & & \vdots & \vdots & \cdots & \vdots \\ & & & & 0 & c_{n1} & \cdots & c_{nd} \end{pmatrix},$$

where $c_{0j} = b_{0j}, 1 \leq i \leq d$ and $c_{ij} = b_{ij} - b_{i0}b_{0j}, 1 \leq i \leq n, 1 \leq j \leq d$.

We are now ready to prove (1) over the open subset U_0. One sees by direct calculations that

$$I_2(C_{f,0}) = \langle c_{ij} \,|\, 1 \leq i \leq n, 1 \leq j \leq d \rangle.$$

For the ideal $I_{2+m}(C_{f,m})$ with $m \geq 1$, it is trivial that

$$I_{2+m}(C_{f,m}) \subset I_2(C_{f,0}).$$

On the other hand, observe that $C_{f,m}$ has a $(n+1+m) \times (d+m+1)$ submatrix of the form

$$\begin{pmatrix} 1 & c_{01} & \cdots & c_{0d} & & & & \\ & 1 & c_{01} & \cdots & c_{0d} & & & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & & & 1 & c_{01} & \cdots & c_{0d} \\ & & & 0 & c_{11} & \cdots & c_{1d} \\ & & & \vdots & \vdots & \cdots & \vdots \\ & & & 0 & c_{n1} & \cdots & c_{nd} \end{pmatrix}.$$

Using the $(m+1)$ 1's on the diagonal, one easily finds $(m+2) \times (m+2)$ minors such that their determinants are c_{ij} for all $1 \leq i \leq n$ and $1 \leq j \leq d$.

This implies that $I_2(C_{f,0}) \subset I_{2+m}(C_{f,m})$. Thus,

$$I_{2+m}(C_m) = I_2(C_0)$$

for all $m \geq 1$. This proves (1) over the open subset U_0.

We now turn to the statement (2) over U_0. Consider the matrix $C_{f,m}$ with $m \geq d$. It is routine to check that the following holds:

$$\text{row}_i + \sum_{j=1}^{d} (c_{ij}\text{row}_{j(n+1)+i-1} + c_{0j}\text{row}_{j(n+1)+i}) = 0, \tag{4.3}$$

for all $2 \leq i \leq n+1$. This means that we can eliminate row_i for all $2 \leq i \leq n+1$. We can also easily eliminate the entries c_{0i} of the first row by using the first column. This implies that $I_{k+2+m}(C_{f,m}) = I_{k+1+m}(C_{f,m-1})$. This process can be repeated until we reach $I_{k+1+d}(C_{f,d-1})$. Thus we obtain

$$I_{k+2+m}(C_{f,m}) = I_{k+1+d}(C_{f,d-1}),$$

for all $m \geq d-1$. This completes the proof of (1) and (2) over the open subset U_0.

To investigate (1) and (2) over the rest of open charts of N_d, we use the symmetry of N_d. The group

$$\text{GL}(H^0(\mathcal{O}_{\mathbb{P}^1}(1))) \times \text{GL}(V) \cong \text{GL}_2 \times \text{GL}_{n+1}$$

acts on N_d. If $g \in \text{GL}(H^0(\mathcal{O}_{\mathbb{P}^1}(1)))$, it corresponds to a change of basis of $H^0(\mathcal{O}_{\mathbb{P}^1}(1))$; since it induces bases changes in both W_m and V_{d+m}, we see that g acts on the matrix $A_{f,m}$ by multiplying invertible matrices from both the left and the right. Likewise, an element $g \in \text{GL}(V)$ acts on $A_{f,m}$ by multiplying an invertible matrix from the left. By 4.4, $g^*(I_l(A_{f,m})) = I_l(A_{f,m})$ for any $g \in \text{GL}(H^0(\mathcal{O}_{\mathbb{P}^1}(1))) \times \text{GL}(V)$.

Now, for $[f] \in N_d \setminus U_0$, we have $a_{00} = 0$. If one of a_{0i} is not zero, say $a_{0j} \neq 0$. It is routine to find a basis change of $H^0(\mathcal{O}_{\mathbb{P}^1}(1))$ such that under the new basis $a'_{00} \neq 0$. This means that there is $g \in \text{GL}(H^0(\mathcal{O}_{\mathbb{P}^1}(1)))$ such that $g \cdot [f] \in U_0$. If all of a_{0i} are zero, then there is $i \geq 1$ and j such that $a_{ij} \neq 0$. Then let $g \in \text{GL}(V)$ correspond to interchanging e_0 and e_j, we see that $g \cdot [f]$ places us in the previous situation. In either case, by the invariance of the ideals $g^*(I_l(A_{f,m})) = I_l(A_{f,m})$, we conclude that the statements (1) and (2) hold everywhere in N_d.

This completes the proof. □

4.2. Determinantal subschemes and their basic properties

4.5. Let $1 \leq k \leq d-1$. We set

$$I_{d,0} = I_{d,0;m},\ m \geq 0 \quad \text{and} \quad I_{d,k} = I_{d,k;m},\ m \geq d-1.$$

By Proposition 4.1, these are well-defined.

Definition 4.1. *For any* $0 \leq k \leq d-1$*, we let* $Z_{d,k}$ *be the subscheme of* N_d *defined by the ideal* $I_{d,k}$*.*

By 4.2, $Z_{d,k}$ is supported on $C_{d,k}$.

4.6. Since

$$N_d = \mathbb{P}(\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(-d), V \otimes \mathcal{O}_{\mathbb{P}^1})) \quad \text{and} \quad N_k = \mathbb{P}(\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(-k), V \otimes \mathcal{O}_{\mathbb{P}^1})),$$

using the identification $V_{d-k} = \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(-d), \mathcal{O}_{\mathbb{P}^1}(-k))$, we obtain a natural morphism

$$\begin{aligned} \varphi_{d,k} \ : \ \mathbb{P}(V_{d-k}) \times N_k &\longrightarrow N_d \\ ([h],[g]) &\mapsto [g \circ h]. \end{aligned} \tag{4.4}$$

When $d = 0$, $N_0 = \mathbb{P}(V)$. Also we have the identification $N_d = \mathbb{P}(V_d \otimes V)$. A direct computation shows that

Proposition 4.2. *The morphism* $\varphi_{d,0} : \mathbb{P}(V_d) \times \mathbb{P}(V) \longrightarrow N_d = \mathbb{P}(V_d \otimes V)$ *is the Segre embedding and its image scheme is exactly* $Z_{d,0}$*.*

In particular, this implies that $Z_{d,0}$ is smooth. For general $\varphi_{d,k}$, we have

Proposition 4.3. *The restriction of* $\varphi_{d,k}$ *to* $\mathbb{P}(V_{d-k}) \times (N_k \setminus Z_{k,k-1})$ *gives rise to an isomorphism*

$$\varphi'_{d,k} : \mathbb{P}(V_{d-k}) \times (N_k \setminus Z_{k,k-1}) \xrightarrow{\cong} Z_{d,k} \setminus Z_{d,k-1}. \tag{4.5}$$

Proof. The idea of the proof is taken from the third author's thesis [17]. We give sufficient sketch here.

We just need to produce the inverse to $\varphi'_{d,k}$.

First, recall that the Quot scheme N_d comes equipped with a universal exact sequence of sheaves over $\mathbb{P}^1 \times N_d$

$$0 \to \mathcal{E} \to V \otimes \mathcal{O}_{\mathbb{P}^1 \times N_d} \to \mathcal{Q} \to 0, \tag{4.6}$$

where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(-d) \otimes \mathcal{O}_{N_d}(-1)$ and $\mathcal{Q}$ is a coherent sheaf of rank n, relative degree d and flat over N_d. Similarly, $\mathbb{P}(V_{d-k})$ comes equipped with a universal exact sequence of sheaves over $\mathbb{P}^1 \times \mathbb{P}(V_{d-k})$

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-d) \otimes \mathcal{O}_{\mathbb{P}(V_{d-k})}(-1) \to \mathcal{O}_{\mathbb{P}^1}(-k) \otimes \mathcal{O}_{\mathbb{P}(V_{d-k})} \to \mathcal{T} \to 0, \tag{4.7}$$

where $\mathcal{T}$ is a torsion sheaf of relative degree $d-k$ and is flat over $\mathbb{P}(V_{d-k})$.

Next, taking dual of the exact sequence (4.6), we obtain

$$V^{\vee}_{\mathbb{P}^1 \times N_d} \to \mathcal{O}_{\mathbb{P}^1}(d) \otimes \mathcal{O}_{N_d}(1) \to \mathcal{G} \to 0, \tag{4.8}$$

where $\mathcal{G} = \mathcal{E}xt^1(\mathcal{Q}, \mathcal{O}_{\mathbb{P}^1 \times N_d})$. Here and below, we use $\mathcal{E}xt^1$ for $\mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^1 \times N_d}}$. Tensoring (4.8) by $\mathcal{O}_{\mathbb{P}^1}(m)$ for $m \gg 0$ and applying π_* where $\pi : \mathbb{P}^1 \times N_d \to N_d$ is the projection map, we then obtain

$$\pi_*(V^{\vee}_{\mathbb{P}^1 \times N_d}(m)) \to \pi_*(\mathcal{O}_{\mathbb{P}^1}(d+m) \otimes \mathcal{O}_{N_d}(1)) \to \pi_*(\mathcal{G}(m)) \to 0, \tag{4.9}$$

where the first map is simply

$$\rho_{d,m} : W_m = \pi_*(V^{\vee}_{\mathbb{P}^1 \times N_d}(m)) \to \pi_*(\mathcal{O}_{\mathbb{P}^1}(d+m) \otimes \mathcal{O}_{N_d}(1)) = V_{d+m} \otimes \mathcal{O}_{N_d}(1).$$

Since $Z_{d,k}$ is the scheme of zeros of $\bigwedge^{k+2+m} \rho_{d,m}$, we see that $\pi_* \mathcal{G}(m)$ pulls back to a locally free sheaf of rank $d-k$ over $Z_{d,k} \setminus Z_{d,k-1}$ for all $0 \le k \le d-1$. Set $Z_{d,-1} := \emptyset$. Then one checks directly that the disjoint union

$$\bigsqcup_{k=0}^{d-1} (Z_{d,k} \setminus Z_{d,k-1}) \bigsqcup (N_d \setminus Z_{d,d-1})$$

is exactly the flattening stratification of $\mathcal{G}$ (cf. Lecture 8, [15]).

Now, we let

$$\iota : \mathbb{P}^1 \times (Z_{d,k} \setminus Z_{d,k-1}) \longrightarrow \mathbb{P}^1 \times N_d$$

be the inclusion. Then, the torsion sheaf $\iota^* \mathcal{G}$ has relative degree $d-k$ and is flat over $Z_{d,k} \setminus Z_{d,k-1}$. Thus, by the universality of $\mathbb{P}(V_{d-k})$, we obtain a morphism

$$Z_{d,k} \setminus Z_{d,k-1} \longrightarrow \mathbb{P}(V_{d-k}). \tag{4.10}$$

What remains is to get a morphism from $Z_{d,k} \setminus Z_{d,k-1}$ to $N_k \setminus Z_{k,k-1}$. For this, we pull back (4.6) to $\mathbb{P}^1 \times (Z_{d,k} \setminus Z_{d,k-1})$. Since $\mathcal{Q}$ is flat over N_d, we get an exact sequence

$$0 \to \iota^*\mathcal{E} \to V \otimes \mathcal{O}_{\mathbb{P}^1 \times (Z_{d,k} \setminus Z_{d,k-1})} \to \iota^*\mathcal{Q} \to 0.$$

Taking the dual to the above sequence, we obtain

$$0 \to (\iota^*\mathcal{Q})^\vee \to V^\vee \otimes \mathcal{O}_{\mathbb{P}^1 \times (Z_{d,k} \setminus Z_{d,k-1})} \to (\iota^*\mathcal{E})^\vee \to \mathcal{E}xt^1_Z(\iota^*\mathcal{Q}, \mathcal{O}) \to 0, \tag{4.11}$$

where $\mathcal{E}xt^1_Z := \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^1 \times (Z_{d,k} \setminus Z_{d,k-1})}}$. We have a canonical identification

$$\mathcal{E}xt^1_Z(\iota^*\mathcal{Q}, \mathcal{O}) = \iota^*\,\mathcal{E}xt^1(\mathcal{Q}, \mathcal{O}) = \iota^*\mathcal{G}.$$

To see this, we first dualize the universal exact sequence (4.6) to obtain

$$0 \to \mathcal{Q}^\vee \to V^\vee \otimes \mathcal{O}_{\mathbb{P}^1 \times N_d} \to \mathcal{E}^\vee \to \mathcal{E}xt^1(\mathcal{Q}, \mathcal{O}) \to 0.$$

Then, we pull it back to $\mathbb{P}^1 \times (Z_{d,k} \setminus Z_{d,k-1})$ to obtain

$$V^\vee \otimes \mathcal{O}_{\mathbb{P}^1 \times (Z_{d,k} \setminus Z_{d,k-1})} \to \iota^*(\mathcal{E}^\vee) \to \iota^*\,\mathcal{E}xt^1(\mathcal{Q}, \mathcal{O}) \to 0.$$

Since pulling-back and dualizing operations commute on locally free sheaves, we have a canonical identification $(\iota^*\mathcal{E})^\vee = \iota^*(\mathcal{E}^\vee)$ and hence a commutative diagram

$$\begin{array}{ccccccc}
V^\vee \otimes \mathcal{O}_{\mathbb{P}^1 \times (Z_{d,k} \setminus Z_{d,k-1})} & \longrightarrow & \iota^*(\mathcal{E}^\vee) & \longrightarrow & \iota^*\mathcal{E}xt^1(\mathcal{Q}, \mathcal{O}) & \longrightarrow & 0 \\
\| & & \| & & \vdots & & \\
V^\vee \otimes \mathcal{O}_{\mathbb{P}^1 \times (Z_{d,k} \setminus Z_{d,k-1})} & \longrightarrow & (\iota^*\mathcal{E})^\vee & \longrightarrow & \mathcal{E}xt^1_Z(\iota^*\mathcal{Q}, \mathcal{O}) & \longrightarrow & 0.
\end{array}$$

This gives the canonical identification in the third collumn, as desired.

We now break up sequence (4.11) into two,

$$0 \to (\iota^*\mathcal{Q})^\vee \to V^\vee \otimes \mathcal{O}_{\mathbb{P}^1 \times (Z_{d,k} \setminus Z_{d,k-1})} \to \mathcal{K} \to 0 \tag{4.12}$$

and

$$0 \to \mathcal{K} \to \iota^*\mathcal{E}^\vee \to \iota^*\mathcal{G} \to 0. \tag{4.13}$$

Since $\iota^*\mathcal{G}$ is flat over $Z_{d,k} \setminus Z_{d,k-1}$, one checks that $\mathcal{K}$ is locally free, hence $(\iota^*\mathcal{Q})^\vee$ is locally free. Now taking the dual of (4.12), we get an exact sequence of locally free sheaves

$$0 \to \mathcal{K}^\vee \to V \otimes \mathcal{O}_{\mathbb{P}^1 \times (Z_{d,k} \setminus Z_{d,k-1})} \to (\iota^*\mathcal{Q})^{\vee\vee} \to 0.$$

One easily calculates that

$$\operatorname{rank}(\iota^*\mathcal{Q})^{\vee\vee} = n \quad \text{and} \quad \deg(\iota^*\mathcal{Q})^{\vee\vee} = k.$$

Thus, by the universality of the Quot scheme N_k, we obtain a morphism

$$Z_{d,k} \setminus Z_{d,k-1} \to N_k \setminus Z_{k,k-1}.$$

Together with (4.10), this gives rise to a morphism

$$\psi_{d,k} : Z_{d,k} \setminus Z_{d,k-1} \to \mathbb{P}(V_{d-k}) \times (N_k \setminus Z_{k,k-1}),$$

which is the inverse to $\varphi'_{d,k}$. This completes the proof. □

As a consequence of this proposition, we see that $Z_{d,k} \setminus Z_{d,k-1}$ is a locally closed smooth subvariety of N_d for all $k \geq 1$.

5. Statements of main results and their proofs

5.1. Describing the successive blowups

5.1. We use induction to describe the iterated blowups of N_d along the $Z_{d,0}, \ldots, Z_{d,d-1}$. We set $N_d^{-1} := N_d$ and $Z_{d,k}^{-1} := Z_{d,k}$. For any $0 \leq l \leq d-1$, we let N_d^l be the blowup of N_d^{l-1} along $Z_{d,l}^{l-1}$, $Z_{d,k}^l$ the proper transform of $Z_{d,k}^{l-1}$ when $k \neq l$, and $Z_{d,l}^l$ the exceptional divisor of $N_d^l \longrightarrow N_d^{l-1}$. Observe that $Z_{d,k}^l$ is a divisor when $k \leq l$ and $Z_{d,k}^l$ is the blowup of $Z_{d,k}^{l-1}$ along $Z_{d,l}^{l-1}$ when $k > l$.

5.2. We denote the final blowup N_d^{d-1} by M_d. We aim to show that M_d is smooth and the boundary $M_d \setminus \overset{\circ}{N}_d$ is a divisor with normal crossings. The smoothness of M_d will follow by induction if the blowup center $Z_{d,l+1}^l \subset N_d^l$ is smooth. A technical key to prove this is to show that the total transform in N_d^l of $Z_{d,l+1}^{l-1} \subset N_d^{l-1}$ is the scheme-theoretical union of a Cartier divisor with the proper transform $Z_{d,l+1}^l \subset N_d^l$, that is, locally we have

$$I_{Z_{d,l+1}} \cdot \mathcal{O}_{N_d^l} = P \cdot I_{Z_{d,l+1}^l},$$

where P is a principal ideal. Here by I_Z, we mean the ideal sheaf of a subscheme Z. To this end, we will relate our blowups to spaces of complete collineations and apply the related results of Vainsencher [20, 21].

5.2. Using the space of complete collineations

5.3. Let E and F be vector spaces. Let $S(E,F) = \mathbb{P}(\mathrm{Hom}(E,F))$ be the space of collineations from E to F. It comes equipped with a universal homomorphism

$$u_{EF} : E \to F \otimes \mathcal{O}_{S(E,F)}(1).$$

For $k \geq 1$, set $D_k(E,F)$ to be the scheme of zeros of the section

$$\bigwedge^{k+1} u_{EF} : \mathrm{Hom}\left(\bigwedge^{k+1} E, \bigwedge^{k+1} F\right) \otimes \mathcal{O}_{S(E,F)}(k+1).$$

5.4. Let $r+1 = \min\{\dim E, \dim F\}$. Set $S^0 := S(E,F)$, $D_k^0 := D_k(E,F)$. We define the following inductively. For $1 \leq l \leq r$, let S^l be the blowup of S^{l-1} along D_l^{l-1}, D_k^l the proper transform of D_k^l for $k \neq l$, and D_l^l the exceptional divisor of $S^l \longrightarrow S^{l-1}$.

5.5. Although $D_k(E,F)$ is singular for $k \geq 2$, Vainsencher [21] shows that D_l^{l-1} is smooth, thus S^l is smooth. In particular, the final blowup space S^r is smooth. Further, he shows that S^r parameterizes "complete collineations" from E to F (see [21] for more details). The property that we need from Vainsencher's construction is the following.

Proposition 5.1. (Theorem 2.4 (8), [21]) *Assume* $1 \leq l < k$.

$$I_{D_k^{l-1}} \cdot \mathcal{O}_{S^l} = I_{D_k^l} \cdot (I_{D_l^l})^{k-l+1}.$$

5.6. The relations between our blowups as described in Section 5.1 and the spaces of "complete collineations" are as follows. Note that for any $m \geq 0$, the resultant homomorphism of (3.1)

$$\rho_{f,m} : W_m \longrightarrow V_{d+m}$$

gives rise to an embedding

$$N_d \longrightarrow S(W_m, V_{d+m}),$$
$$[f] \mapsto [\rho_{f,m}].$$

Indeed, $N_d = S(W_0, V_d)$. Further, $\rho_{d,m}$ is exactly the pullback to N_d of the universal map $u = u_{W_m V_{d+m}}$ on $S(W_m, V_{d+m})$. Consequently, $\bigwedge^l u$ pulls back to $\bigwedge^l \rho_{d,m}$ for all l. Then it follows by definition that for all $0 \leq k \leq d-1$ and $m \geq d-1$

$$N_d \cap D_{k+1+m}(W_m, V_{d+m}) = Z_{d,k}$$

scheme-theoretically. Also it is easy to check that

$$\operatorname{rank} \rho_{f,m} \geq m+1,$$

for all $m \geq 0$ and $[f] \in N_d$. Therefore,

$$N_d \cap D_1(W_m, V_{d+m}) = \cdots = N_d \cap D_m(W_m, V_{d+m}) = \emptyset. \tag{5.1}$$

Consequently, we have

Proposition 5.2. *Fix any $m \geq d-1$. N_d^l is the proper transform of N_d in $S^{m+1+l}(W_m, V_{d+m})$. In particular, M_d is the proper transform of N_d in $S^{d+m}(W_m, V_{d+m})$*

Further, we have

Lemma 5.1. *Assume $1 \leq l < k$. Then*

$$I_{Z_{d,k}^{l-1}} \cdot \mathcal{O}_{N_d^l} = I_{Z_{d,k}^l} \cdot (I_{Z_{d,l}^l})^{k-l+1}.$$

Proof. In the proof, we fix $m = d-1$ and use the embedding

$$N_d \longrightarrow S(W_{d-1}, V_{2d-1}).$$

We will use the notations introduced in 5.4 with $E = W_{d-1}$ and $F = V_{2d-1}$.

By (5.1), we have

$$N_d \cap D_1 = \cdots = N_d \cap D_{d-1} = \emptyset.$$

Thus, we have the induced embedding

$$N_d \longrightarrow S^{d-1},$$

and moreover, for $0 \le k \le d-1$, we have

$$N_d \cap D^{d-1}_{d+k} = Z_{d,k}.$$

In other words,

$$I_{D^{d-1}_{d+k}} \cdot \mathcal{O}_{N_d} = I_{Z_{d,k}}.$$

Thus, we have the following blowing-up diagram

$$\begin{array}{ccc} N_d^l & \longrightarrow & S^{d+l} \\ \downarrow & & \downarrow \\ N_d^{l-1} & \longrightarrow & S^{d+l-1} \end{array}$$

for all $0 \le l \le d-1$. Further, we have

$$N_d^l \cap D^{d+l}_{d+k} = Z^l_{d,k}, \quad \text{for } l \le k.$$

Thus for $l < k$,

$$I_{Z^{l-1}_{d,k}} \cdot \mathcal{O}_{N_d^l} = (I_{D^{d+l-1}_{d+k}} \cdot \mathcal{O}_{N_d^{l-1}}) \cdot \mathcal{O}_{N_d^l} = (I_{D^{d+l-1}_{d+k}} \cdot \mathcal{O}_{S^{d+l}}) \cdot \mathcal{O}_{N_d^l}.$$

By Proposition 5.1, we have $I_{D^{d+l-1}_{d+k}} \cdot \mathcal{O}_{S^{d+l}} = I_{D^{d+l}_{d+k}} \cdot (I_{D^{d+l}_{d+l}})^{k-l+1}$. It then follows that:

$$I_{Z^{l-1}_{d,k}} \cdot \mathcal{O}_{N_d^l} = I_{D^{d+l}_{d+k}} \cdot (I_{D^{d+l}_{d+l}})^{k-l+1} \cdot \mathcal{O}_{N_d^l} = I_{Z^l_{d,k}} \cdot (I_{Z^l_{d,l}})^{k-l+1}. \qquad \square$$

Applying the above lemma repeatedly, we obtain

Corollary 5.1. *For all* $0 \le l \le d-1$,

$$I_{Z_{d,l+1}} \cdot \mathcal{O}_{N_d^l} = (I_{Z^l_{d,l+1}}) \cdot (I_{Z^l_{d,l}})^2 \cdots (I_{Z^l_{d,1}})^{l+1} \cdot (I_{Z^l_{d,0}})^{l+2}.$$

We remark here that $Z^l_{d,t}$ are Cartier divisors when $t \le l$. This implies that the blowup of N_d^l along the proper transform $Z^l_{d,l+1}$ of $Z_{d,l+1} \subset N_d$ is the same as the blowup of N_d^l along the total transform of $Z_{d,l+1} \subset N_d$.

5.3. Main theorems and proofs

Theorem 5.1. *Let $-1 \le k \le d-1$. Then*

(1) N_d^k *is nonsingular;*

(2) N_d^k *is isomorphic to the closure of the graph of the rational map*

$$N_d \dashrightarrow \prod_{l=0}^{k} S\left(\bigwedge^{m+2+l} W_m, \bigwedge^{m+2+l} V_{d+m}\right)$$
$$[f] \mapsto \left(\left[\bigwedge^{m+2} \rho_{f,m}\right], \left[\bigwedge^{m+3} \rho_{f,m}\right], \ldots, \left[\bigwedge^{m+2+k} \rho_{f,m}\right]\right),$$

for all $m \ge d-1$;

(3) $Z_{d,k+1}^k$ *is isomorphic to $\mathbb{P}(V_{d-k-1}) \times N_{k+1}^k$.*

Proof. We prove it by induction on k. When $k=-1$, the statements of the theorem are trivial (for all $d>0$ and $m \ge d-1$). Assume the statements are true for all $\le k-1$ (for all $d>0$ and $m \ge d-1$). We now prove the k-version of the theorem.

First, by the inductive assumption, N_d^{k-1} is nonsingular; also, $Z_{d,k}^{k-1}$ is smooth because it is isomorphic to $\mathbb{P}(V_{d-k-1}) \times N_k^{k-1}$. Hence, N_d^k, as the blowup of N_d^{k-1} along $Z_{d,k}^{k-1}$, is nonsingular. So (1) holds true for k.

Next, we let

$$\pi_{[k-1]} : N_d^{k-1} \longrightarrow N_d$$

be the iterated blowing-up morphism. By Corollary 5.1, the blowup of N_d^{k-1} along the proper transform $Z_{d,k}^{k-1}$ is isomorphic to the blowup of N_d^{k-1} along the total transform $\pi_{[k-1]}^{-1}(Z_{d,k})$. Hence, by the definition of $Z_{d,k}$ (see Definition 4.1 and Proposition 4.1), N_d^k is isomorphic to the closure of the graph of the rational map

$$N_d^{k-1} \dashrightarrow S\left(\bigwedge^{k+2+m} W_m, \bigwedge^{k+2+m} V_{d+m}\right).$$

By the induction hypothesis, N_d^{k-1} is isomorphic to the closure of the graph of the rational map

$$N_d \dashrightarrow \prod_{l=0}^{k-1} S\left(\bigwedge^{m+2+l} W_m, \bigwedge^{m+2+l} V_{d+m}\right)\right).$$

It follows that N_d^k is isomorphic to the closure of graph of the rational map

$$N_d \dashrightarrow \prod_{l=0}^{k} S\left(\bigwedge^{m+2+l} W_m, \bigwedge^{m+2+l} V_{d+m}\right)\Bigg).$$

Thus (2) also holds true for k.

Finally, to prove the k-version of (3), we introduce and establish the following commutative diagram:

(5.2)

$$\begin{array}{ccc} Z_{d,k+1} \setminus Z_{d,k} & \xrightarrow{\alpha} & \mathbb{P}(V_{d-k-1}) \times N_{k+1}^{k-1} \times S\left(\bigwedge^{k+m+2} W_m, \bigwedge^{k+m+2} V_{k+m+1}\right) \\ \| & & \downarrow \beta \\ Z_{d,k+1} \setminus Z_{d,k} & \xrightarrow{\gamma} & N_d^{k-1} \times S\left(\bigwedge^{k+m+2} W_m, \bigwedge^{k+m+2} V_{d+m}\right). \end{array}$$

Here γ is the obvious embedding. The morphism α is the obvious embedding induced by the inverse of the morphism $\varphi'_{d,k+1}$ of Proposition 4.3. The morphism β is defined as follows.

For any integers $r, s \geq 1$, using the following identifications:

$$\begin{aligned} V_r &= \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(-r), \mathcal{O}_{\mathbb{P}^1}), \\ V_s &= \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(s)), \\ V_{r+s}\, & \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(-r), \mathcal{O}_{\mathbb{P}^1}(s)), \end{aligned}$$

we see that each nonzero $h \in V_r$ gives rise to an injective linear map

$$\begin{aligned} L_h : V_s &\longrightarrow V_{r+s}, \\ g &\mapsto g \circ h. \end{aligned}$$

For any $1 \leq l \leq s+1$, it induces an injective linear map

$$\bigwedge^{l} L_h : \bigwedge^{l} V_s \to \bigwedge^{l} V_{r+s},$$

which in turn induces a morphism

$$\mathbb{P}(V_r) \to S\left(\bigwedge^{l} V_s, \bigwedge^{l} V_{r+s}\right).$$

Now fix any $t \geq 1$. Then, by the means of composing with $\bigwedge^l L_h$, we obtain a morphism

$$S\left(\bigwedge^l W_t, \bigwedge^l V_s\right) \to S\left(\bigwedge^l W_t, \bigwedge^l V_{r+s}\right).$$

Since $\bigwedge^l L_h$ is injective, the above morphism is an embedding (in fact, a linear embedding).

Observe now that when $l = s+1$, $\dim \bigwedge^l V_s = 1$. Hence

$$S\left(\bigwedge^{s+1} V_s, \bigwedge^{s+1} V_{r+s}\right) \cong \mathbb{P}\left(\bigwedge^{s+1} V_{r+s}\right).$$

Then one checks directly that the morphism

$$\mathbb{P}(V_r) \to S\left(\bigwedge^{s+1} V_s, \bigwedge^{s+1} V_{r+s}\right)$$

is the Veronese embedding; further, the morphism

$$S\left(\bigwedge^{s+1} W_t, \bigwedge^{s+1} V_s\right) \times S\left(\bigwedge^{s+1} V_s, \bigwedge^{s+1} V_{r+s}\right) \longrightarrow S\left(\bigwedge^{s+1} W_t, \bigwedge^{s+1} V_{r+s}\right)$$

is the Segre embedding.

We are now ready to define β. It consists of two components (β_1, β_2). They are

$$\beta_1 : \mathbb{P}(V_{d-k-1}) \times N^{k-1}_{k+1} \to N^{k-1}_d$$
$$\beta_1\left([h], [g], \prod_{l=0}^{k-1}\left[\bigwedge^{m+2+l} \rho_{g,m}\right]\right) = \left([g \circ h], \prod_{l=0}^{k-1}\left[\bigwedge^{m+2+l} L_h \circ \bigwedge^{m+2+l} \rho_{g,m}\right]\right)$$

and

$$\beta_2 : \mathbb{P}(V_{d-k-1}) \times S\left(\bigwedge^{k+m+2} W_m, \bigwedge^{k+m+2} V_{k+m+1}\right)$$
$$\to S\left(\bigwedge^{k+m+2} W_m, \bigwedge^{k+m+2} V_{d+m}\right)$$
$$\beta_2\left([h], \left[\bigwedge^{m+2+k} \rho_{g,m}\right]\right) = \left[\bigwedge^{m+2+l} L_h \circ \bigwedge^{m+2+k} \rho_{g,m}\right].$$

Then one checks routinely that β_2 is the composition of the Veronese embedding

$$\mathbb{P}(V_{d-k+1}) \longrightarrow S\left(\bigwedge^{k+m+2} V_{k+m+1}, \bigwedge^{k+m+2} V_{d+m}\right)$$

followed by the Segre embedding

$$S\left(\bigwedge^{k+m+2} W_m, \bigwedge^{k+m+2} V_{k+m+1}\right) \times S\left(\bigwedge^{k+m+2} V_{k+m+1}, \bigwedge^{k+m+2} V_{d+m}\right)$$
$$\longrightarrow S\left(\bigwedge^{k+m+2} W_m, \bigwedge^{k+m+2} V_{d+m}\right),$$

hence an embedding itself. Since it is routine, we omit the details. For β_1, observe that when $[h]$ is fixed, it is injective on other factors. Together, this implies that β is an embedding. Thus, we finally established the embedding diagram (5.2).

From (5.2), we see that the closure of $Z_{d,k+1} \setminus Z_{d,k}$ in

$$\mathbb{P}(V_{d-k-1}) \times N_{k+1}^{k-1} \times S\left(\bigwedge^{k+m+2} W_m, \bigwedge^{k+m+2} V_{k+m+1}\right)$$

is contained in $\mathbb{P}(V_{d-k-1}) \times N_{k+1}^{k}$, and hence equals to $\mathbb{P}(V_{d-k-1}) \times N_{k+1}^{k}$ because $Z_{d+1} \setminus Z_{d,k}$ is an open subvariety in it. Because the embedding diagram (5.2) commutes, the above-mentioned closure is obviously isomorphic to the closure of $Z_{d,k+1} \setminus Z_{d,k}$ in

$$N_d^{k-1} \times S\left(\bigwedge^{k+m+2} W_m, \bigwedge^{k+m+2} V_{d+m}\right),$$

which is by definition $Z_{d,k+1}^{k}$. This proves (3) for k.

By induction, the theorem is proved. □

Theorem 5.2. *Let $-1 \leq k \leq d-1$. Then*

(1) $Z_{d,0}^{k} \cup \cdots \cup Z_{d,k}^{k}$ *is a divisor of N_d^k with simple normal crossings.*

(2) *The scheme-theoretic intersection* $Z^k_{d,k+1} \cap \bigcap_{j=1}^r Z^k_{d,i_j}$ *is isomorphic to*

$$\mathbb{P}(V_{d-k-1}) \times \bigcap_{j=1}^{r} Z^k_{k+1,i_j},$$

for any distinct integers $i_1, \ldots, i_r$ *between* 0 *and* k.

Proof. Again, we prove it by induction on k. When $k = -1$, both statements are trivial.

Assume that both statements are true for $k-1$. Then, it implies that $Z^{k-1}_{d,0} \cup \cdots \cup Z^{k-1}_{d,k-1}$ is a divisor of N^{k-1}_d with simple normal crossings. In addition, by statement (2) from the induction hypothesis, the scheme-theoretic intersection of $Z^{k-1}_{d,k}$ with an arbitrary intersection of Z^{k-1}_{d,i_j}'s is smooth. Next, it is routine to check that

$$\begin{aligned}\operatorname{codim}\left(Z^{k-1}_{d,k} \cap \bigcap_{j=1}^{r} Z^{k-1}_{d,i_j}\right) &= \dim N^{k-1}_d - \dim\left(\mathbb{P}(V_{d-k}) \times \bigcap_{j=1}^{r} Z^{k-1}_{k,i_j}\right) \\ &= \operatorname{codim} Z^{k-1}_{d,k} + \sum_{j=1}^{r} \operatorname{codim} Z^{k-1}_{d,i_j}.\end{aligned}$$

Since it is routine, we omit further details. Thus, $Z^{k-1}_{d,k}$ meets the divisors $Z^{k-1}_{d,0}, \ldots, Z^{k-1}_{d,k-1}$ transversally. Since transversality is preserved under blowup along nonsingular center, we obtain the k-version of (1).

To prove (2), we consider again the commutative diagram in (5.2), which by now can also be expressed as

$$\begin{array}{ccc} Z^k_{d,k+1} & \xrightarrow{\alpha} & \mathbb{P}(V_{d-k-1}) \times N^{k-1}_{k+1} \times S\left(\bigwedge^{k+m+2} W_m, \bigwedge^{k+m+2} V_{k+m+1}\right) \\ \Big\| & & \Big\downarrow \beta \\ Z^k_{d,k+1} & \xrightarrow{\gamma} & N^{k-1}_d \times S\left(\bigwedge^{k+m+2} W_m, \bigwedge^{k+m+2} V_{d+m}\right). \end{array}$$

Then from the diagram we have that for any $0 \le i \le k$,

$$Z^k_{d,k+1} \cap Z^k_{d,i} = \mathbb{P}(V_{d-k-1}) \times Z^k_{k+1,i}$$

scheme-theoretically and further the scheme theoretic intersection

$$Z^k_{d,k+1} \cap \bigcap_{j=1}^{r} Z^k_{d,i_j}$$

is isomorphic to

$$\mathbb{P}(V_{d-k-1}) \times \bigcap_{j=1}^{r} Z^k_{k+1,i_j}. \qquad \square$$

Theorems 5.1 and 5.2 imply.

Theorem 5.3. *Fix any $m \geq d-1$. Then $M_d = N_d^{d-1}$ is a compactification of $\mathring{N}_d$ such that the following hold.*

(1) *M_d is isomorphic to the closure of the graph of the rational map*

$$N_d \dashrightarrow \prod_{l=0}^{d-1} S\left(\bigwedge^{m+2+l} W_m, \bigwedge^{m+2+l} V_{d+m}\right);$$

(2) *M_d is a nonsingular projective variety;*

(3) *The complement $M_d \setminus \mathring{N}_d = \bigcup_{k=0}^{d-1} Z^{d-1}_{d,k}$ is a divisor with simple normal crossings.*

6. The topology of the compactification

6.1. In this section, we will work over the field of complex numbers and draw a few consequences on the topology of M_d. Recall that for any complex quasi-projective variety V there is a (virtual Hodge) polynomial $\mathbf{e}(V)$ in two variables u and v which is uniquely determined by the following properties.

(1) If V is smooth and projective, then $\mathbf{e}(V) = \sum h^{p,q}(-u)^p(-v)^q$.

(2) If U is a closed subvariety of V, then $\mathbf{e}(V) = \mathbf{e}(V \setminus U) + \mathbf{e}(U)$.

(3) If $V \to B$ is a Zariski locally trivial bundle with fiber F, then $\mathbf{e}(V) = \mathbf{e}(B)\mathbf{e}(F)$.

6.2. Fix $n > 0$. For any $i > 0$, set

$$R_i(\lambda) = \frac{\lambda^{i+1} - 1}{\lambda - 1} \cdot \frac{\lambda^{ni} - \lambda}{\lambda - 1}.$$

First, we have the following recursive formula for $\mathbf{e}_{M_d}$.

Proposition 6.1. *Set $\lambda = uv$. Then*

$$\mathbf{e}_{M_d} = \mathbf{e}_{N_d} + \sum_{k=0}^{d-1} \mathbf{e}_{M_k} R_{d-k},$$

where $\mathbf{e}_{N_d} = \frac{\lambda^{(d+1)(n+1)}-1}{\lambda-1}$.

Proof. Since $M_d = N_d^{d-1}$ is the blowup of N_d^{d-2} along $Z_{d,d-1}^{d-2}$, we have

$$\mathbf{e}_{M_d} = \mathbf{e}_{N_d^{d-2}} + \mathbf{e}_{Z_{d,d-1}^{d-2}} (\mathbf{e}_{\mathbb{P}^{(\operatorname{codim} Z_{d,d-1}-1)}} - 1),$$

that is

$$\mathbf{e}_{M_d} = \mathbf{e}_{N_d^{d-2}} + \mathbf{e}_{Z_{d,d-1}^{d-2}} \frac{\lambda^{\operatorname{codim} Z_{d,d-1}} - \lambda}{\lambda - 1}.$$

Repeat the same arguments for $\mathbf{e}_{N_d^{d-2}}$ and so on, we will eventually obtain

$$\mathbf{e}_{M_d} = \mathbf{e}_{N_d} + \sum_{k=0}^{d-1} \mathbf{e}_{Z_{d,k}^{k-1}} \frac{\lambda^{\operatorname{codim} Z_{d,k}} - \lambda}{\lambda - 1}.$$

Because $Z_{d,k}^{k-1} = \mathbb{P}(V_{d-k}) \times M_k$, $\mathbb{P}(V_{d-k}) \cong \mathbb{P}^{d-k}$ and $\operatorname{codim} Z_{d,k} = n(d-k)$, from here it is routine to obtain the formula as stated in the theorem. $\square$

6.3. We can also derive a closed formula for $\mathbf{e}_{M_d}$. For this consider $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}_{>0}^r$. Let $|\alpha|$ denote the sum $\sum_{j=1}^r \alpha_j$. Set

$$R_\alpha = \prod_{j=1}^r R_{\alpha_j} \quad \text{and} \quad R_0 = R_{0,n} = 1.$$

Proposition 6.2. *The Hodge polynomial $\mathbf{e}_{M_d}(u,v)$ is given by*

$$\mathbf{e}_{M_d} = \sum_{0 \le |\alpha| \le d} R_\alpha \mathbf{e}_{N_{d-|\alpha|}},$$

where $\mathbf{e}_{N_k} = \frac{\lambda^{(k+1)(n+1)}-1}{\lambda-1}$ *for all* $k \ge 0$.

Proof. When $d=0$, it is trivial. Assume that the formula holds for all $k<d$. By Theorem 6.1,

$$\mathbf{e}_{M_d} = \mathbf{e}_{N_d} + \sum_{k=0}^{d-1} \mathbf{e}_{M_k} R_{d-k}.$$

By inductive assumption, for all $k<d$

$$\mathbf{e}_{M_k} = \sum_{0\le|\alpha|\le k} R_\alpha \mathbf{e}_{N_{k-|\alpha|}} = \sum_{\substack{0\le|\alpha|\le k\\ |\alpha|+j=k}} R_\alpha \mathbf{e}_{N_j}.$$

Substitute $\mathbf{e}_{M_k}$ into the first formula, we have

$$\mathbf{e}_{M_d} = \mathbf{e}_{N_d} + \sum_{k=0}^{d-1} \sum_{\substack{0\le|\alpha|\le k\\ |\alpha|+j=k}} R_\alpha R_{d-k} \mathbf{e}_{N_j}.$$

Let $\beta=(\alpha, d-k)$. Then $R_\beta = R_\alpha R_{d-k}$ and $\beta = |\alpha|+d-k = d-j$ if $|\alpha| = k-j$. Then a simple counting argument shows

$$\mathbf{e}_{M_d} = \mathbf{e}_{N_d} + \sum_{\substack{|\beta|+j=d\\ 0\le j\le d-1}} R_\beta \mathbf{e}_{N_j} = \sum_{\substack{0\le|\beta|\le d\\ |\beta|+j=d}} R_\beta \mathbf{e}_{N_j} = \sum_{0\le|\beta|\le d} R_\beta \mathbf{e}_{N_{d-|\beta|}},$$

as desired. □

In addition, we have

Proposition 6.3. *Let H be the pullback (to M_d) of the hyperplane class of N_d and T_k corresponds to the exceptional divisor $Z^{d-1}_{d,k}$ for all $0\le k\le d-1$. Then*

$$A^1(M_d) = \mathbb{Z}\cdot H \oplus \bigoplus_{k=0}^{d-1} \mathbb{Z}\cdot T_k.$$

Proof. This follows from that fact that M_d is a successive blowup of N_d along nonsingular centers. □

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Department of Mathematics
University of Arizona
Tucson
AZ 85721
USA
E-mail address: yhu@math.arizona.edu

Department of Mathematics
SUNY at Canton
NY 13617
USA
E-mail address: linj@canton.edu

Department of Mathematics
University of Arizona
Tucson
AZ 85721
USA
E-mail address: yshao@math.arizona.edu

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