Isoresonant conformal surfaces with cusps and boundedness of the relative determinant

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We study the isoresonance problem on non-compact surfaces of finite area that are hyperbolic outside a compact set. Inverse resonance problems correspond to inverse spectral problems in the non-compact setting. We consider a conformal class of surfaces with hyperbolic cusps where the deformation takes place inside a fixed compact set. Inside this compactly supported conformal class we consider isoresonant metrics i.e., metrics for which the set of resonances is the same, including multiplicities. We prove that sets of isoresonant metrics inside the conformal class are sequentially compact. We use relative determinants, splitting formulae for determinants and the result of Osgood, Phillips and Sarnak about compactness of sets of isospectral metrics on closed surfaces. In the second part, we study the relative determinant of the Laplace operator on a hyperbolic surface as function on the moduli space. We consider the moduli space of hyperbolic surfaces of fixed genus and fixed number of cusps. We consider the relative determinant of the Laplace operator and a model operator defined on the cusps. We prove that the relative determinant tends to zero as one approaches the boundary of the moduli space.

1. Introduction

In this paper we consider two problems. We first focus on the isoresonance problem for a surface with cusps and negative Euler characteristic, restricting our attention to a suitable conformal class of metrics. The second problem, in Section 5, is the study of the relative determinant of the Laplacian compared to a fixed model operator on the moduli space of hyperbolic surfaces of fixed conformal type as one approaches the boundary.

We study the inverse resonance problem inside a conformal class of metrics whose "conformal factors" have support in a fixed compact set. We prove that given a fixed compact set $K \subset M$, inside a "K-compactly supported" conformal class, sets of isoresonant metrics are compact in the C^∞-topology. With this we partially generalize the result of Osgood, Phillips

and Sarnak (OPS) in [24] that states that on a closed surface every set of isometry classes of isospectral metrics is sequentially compact in the C^∞-topology. We use the results of W. Müller about scattering theory for admissible surfaces in [20].

Isospectral problems go back to 1960 when Leon Green asked if a Riemannian manifold was determined by its spectrum. The question was rephrased by Kac for planar domains in the very suggestive way: "Can one hear the shape of a drum?" see [13]. An important result is the well known existence of non-isometric manifolds that are isospectral, see [28] and the references therein. We also refer to [31] for a comprehensive survey of inverse spectral problems in geometry.

The compactness theorem of OPS in [24] uses the fact that if two metrics are isospectral, i.e., spectra of the Laplacians are the same including multiplicities, then the heat invariants and the determinant of the Laplace operator have the same values at each metric. The authors note in the paper that the use of the regularized determinant of the Laplacian is essential in order to obtain compactness, since the heat invariants are not enough. On planar domains the problem has been studied by Melrose in [18] and OPS in [25] and for compact surfaces with boundary by Y. Kim in [14].

The isospectral problem also makes sense for certain non-compact manifolds. Then scattering theory comes into play and we need to deal with inverse scattering theory. The spectrum of the Laplacian is not enough, one also has to consider resonances. For example, on exterior planar domains the isospectral problem was studied by Hassell and Zelditch in [11]. There two exterior planar domains are called isophasal if they have the same scattering phase. Hassell and Zelditch prove that each class of isophasal exterior planar domains is sequentially compact in the C^∞-topology. In the proof they define a regularized determinant of the Laplacian that plays a fundamental role. More recently, Borthwick *et al.* in [3] used determinants to prove sequential compactness of sets of isopolar (same scattering phase) surfaces of infinite volume under certain conditions. In a later work [4], Borthwick and Perry studied the inverse resonance problem for infinite volume manifolds of finite dimension that are hyperbolic outside a compact set. In dimension 2 they improved the result of [3] and proved compactness of isoresonant surfaces without cusps that are isometric at infinity.

We study the isospectral problem inside a conformal class of a given metric in a surface with cusps. In this setting, two metrics are isospectral if the resonances are the same for both metrics including multiplicities. Because of this we use the terminology "isoresonant" instead of "isospectral". For hyperbolic surfaces of finite area, W. Müller proved in [20] that

the resonance set associated to the surface determines the surface up to finitely many possibilities. Our result in this part is the following theorem:

Theorem 4.1. *Let (M, g) be a surface with cusps with negative Euler characteristic, $\chi(M) < 0$, let $K \subset M$ be compact, and let $[g]_K = \{e^{2\varphi}g | \varphi \in C_c^\infty(M), \operatorname{supp}(\varphi) \subset K\}$ be the K-compactly supported conformal class of g. Then isoresonant sets in $[g]_K$ are compact in the C^∞-topology.*

There are two strong restrictions in this theorem. First, we consider deformations only with compact support because of the lack of results in the theory of resonances of surfaces with asymptotically hyperbolic cusp ends. The second restriction is to consider deformations only inside a conformal class. This is due to the fact that the proof of Theorem 4.1 relies on a splitting formula for the relative determinant to reduce the problem to the compact case. The splitting formula relates $\det(\Delta_g, \Delta_{\beta,0})$ (with β big enough) to the determinant of the Dirichlet-to-Neumann operator acting on a submanifold of M homeomorphic to S^1. To relate the determinants of the Dirichlet-to-Neumann maps associated to different metrics we use the conformal variation of the Laplacian. If the metrics are not conformal, it is not clear how the different Dirichlet-to-Neumann operators are related. We prove this formula in Section 3. The main difficulty to treat the isoresonance problem on surfaces with cusps is that the injectivity radius of these surfaces vanishes and the Sobolev embeddings do not hold anymore.

In the second part, we study the relative determinant as a function on the moduli space of hyperbolic surfaces with cusps. We work over $\mathcal{M}_{p,m}$, the moduli space of compact Riemann surfaces of genus p with m punctures. Each such a surface can be decomposed as the union of a compact part and m cusps, as it is explained at the beginning of sections 2 and 5. In order to define the relative determinant, we use a global model operator. We define the free Laplacian as being the Dirichlet Laplacian $\bar{\Delta}_{1,0}$, as in Definition 2.1, associated to the union of m cusps all starting at 1, i.e., each cusp is taken as $[1, \infty) \times S^1$ with the hyperbolic metric on it. In particular, $\bar{\Delta}_{1,0}$ is independent of $[g]$. Hence the relative determinant defines a function on the moduli space: $[g] \in \mathcal{M}_{p,m} \mapsto \det(\Delta_g, \bar{\Delta}_{1,0}) \in \mathbb{R}^+$, where $g \in [g]$ is hyperbolic. We prove Theorem 5.1 that establishes that the relative determinant $\det(\Delta_g, \bar{\Delta}_{1,0})$ tends to zero as $[g]$ approaches the boundary of the moduli space. Points at the boundary of $\mathcal{M}_{p,m}$ can be reached through a degenerating family of metrics. The degeneration arises from closed geodesics whose length converges to zero. The proof of Theorem 5.1 relies strongly on the results of Bers in [2] and of Jorgenson and Lundelius (JL) in [12]. We

remark that the hyperbolic determinant of JL also tends to zero as the metric approaches the boundary of the moduli space. However, they do not state it explicitly in [12]. In an earlier work [17], Lundelius considers a relative determinant for admissible surfaces. He studies the behavior of the relative weight (minus the logarithm of his relative determinant) of a continuous family of hyperbolic surfaces of finite volume that degenerates by pinching geodesics; but again there is no mention to the moduli space. Our contribution in this part consists in using the results of [12] and [2] to make a statement about the behavior of the relative determinant $\det(\Delta, \bar{\Delta}_{1,0})$ as function on the moduli space $\mathcal{M}_{p,m}$. Although our remarks on this are straightforward consequences of these results, they are worth mentioning explicitly in light of future investigations on isospectral compactness problems.

2. Surfaces with cusps, Laplacians and relative determinants

A surface with cusps is a two-dimensional Riemannian manifold (M, g) that is complete, non-compact, has finite volume and is hyperbolic in the complement of a compact set. It admits a decomposition of the form

$$M = M_0 \cup Z_1 \cup \cdots \cup Z_m,$$

where M_0 is a compact surface with smooth boundary and for each $i = 1, \ldots, m$ we assume that

$$Z_i \cong [a_i, \infty) \times S^1 \ni (y_i, x_i), \quad g|_{Z_i} = y_i^{-2}(dy_i^2 + dx_i^2), \quad a_i > 0.$$

The subsets Z_i are called cusps. Sometimes we denote Z_i by Z_{a_i} to indicate the "starting point" a_i. Instances of surfaces with cusps are quotients of the form $\Gamma(N)\backslash\mathbb{H}$, where $\mathbb{H}$ is the upper half plane and $\Gamma(N) \subseteq \mathrm{SL}_2(\mathbb{Z})$ is a congruence subgroup.

To any surface with cusps (M, g) we can associate a compact surface $\overline{M}$ such that (M, g) is diffeomorphic to the complement of m points in $\overline{M}$. Let p denote the genus of the compact surface $\overline{M}$; then the pair (p, m) is called the conformal type of M.

For any oriented Riemannian manifold (M, g) the Laplace–Beltrami operator on functions is defined as $\Delta f = -\mathrm{div}\,\mathrm{grad}\,f$. It is equal to $\Delta = d^*d$. If we want to emphasize the dependence on the metric we denote the Laplacian by Δ_g. We consider positive Laplacians. If (M, g) is complete, Δ_g has a unique closed extension that is denoted in the same way.

Let us consider some Laplacians that are naturally associated to the cusps:

Definition 2.1. Let $a > 0$, let $\Delta_{a,0}$ denote the self-adjoint extension of the operator

$$-y^2 \frac{\partial^2}{\partial y^2} : C_c^\infty((a, \infty)) \to L^2([a, \infty), y^{-2}dy)$$

obtained after imposing Dirichlet boundary conditions at $y = a$. The domain of $\Delta_{a,0}$ is given by $\mathrm{Dom}(\Delta_{a,0}) = H_0^1([a, \infty)) \cap H^2([a, \infty))$, where $H_0^1([a, \infty)) = \{f \in H^1([a, \infty)) : f(a) = 0\}$.

Let $\bar{\Delta}_{a,0} = \oplus_{j=1}^m \Delta_{a_j,0}$ be defined as the direct sum of the self-adjoint operators $\Delta_{a_j,0}$ defined above. The operator $\bar{\Delta}_{a,0}$ acts on a subspace of $\oplus_{j=1}^m L^2([a_j, \infty), y_j^{-2}dy_j)$.

The kernel of the heat operator associated to $\Delta_{a_j,0}$ is described in [7, Section 14.2] and it is given by the equation:

$$(2.1) \quad p_a(y, y', t) = \frac{e^{-t/4}}{\sqrt{4\pi t}}(yy')^{1/2}\{e^{-(\log(y/y'))^2/4t} - e^{-(\log(yy')-\log(a^2))^2/4t}\},$$

for $y, y' > a$, and for $1 \le y \le a$, $p_a(y, y', t) = 0$. We extend it in the obvious way to see it as a function of $z \in M$.

Now, let $a > 0$, let Z_a be endowed with the hyperbolic metric g and let $\Delta_{Z_a,D}$ be the self-adjoint extension of

$$-y^2 \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} \right) : C_c^\infty((a, \infty) \times S^1) \to L^2(Z_a, dA_g)$$

obtained after imposing Dirichlet boundary conditions at $\{a\} \times S^1$.

Let us describe a decomposition of the operator $\Delta_{Z_a,D}$ that is very useful in our case: the space $L^2(Z_a, dA_g)$ can be decomposed using the isomorphism

$$L^2(Z_a, dA_g) \cong L^2([a, \infty), y^{-2}dy) \oplus L_0^2(Z_a),$$

with $\quad L_0^2(Z_a) = \{f \in L^2(Z_a, dA_g) | \int_{S^1} f(y, x)dx = 0 \text{ for a. e. } y \ge a\}$. This decomposition is invariant under $\Delta_{Z_a,D}$; in terms of it, we can write $\Delta_{Z_a,D} = \Delta_{a,0} \oplus \Delta_{Z_a,1}$ where $\Delta_{Z_a,1}$ acts on $L_0^2(Z_a)$.

For the spectral theory of manifolds with cusps we refer the reader to Müller in [19, 20], to Colin de Verdière in [8], and to the references therein. The results in [19] hold for any dimension. For surfaces in particular we refer to [20].

On a surface with cusps (M, g), the spectrum of the Laplacian $\sigma(\Delta_g)$ is the union of the point spectrum σ_p and the continuous spectrum σ_c. The

point spectrum consist of a sequence of eigenvalues

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots.$$

Each eigenvalue has finite multiplicity, and the counting function $N(\Lambda) = \#\{\lambda_j | \lambda_j \leq \Lambda^2\}$ for $\Lambda > 0$ satisfies $\limsup N(\Lambda)\Lambda^{-2} \leq A_g(4\pi)^{-1}$, where A_g denotes the area of (M, g). Depending on the metric, the set of eigenvalues may be infinite or not.

The continuous spectrum σ_c of Δ_g is the interval $[\frac{1}{4}, \infty)$ with multiplicity equal to the number of cusps of M. The spectral decomposition of the absolutely continuous part of Δ_g is described by the generalized eigenfunctions $E_j(z, s)$, for $j = 1, \dots, m$ with $z \in M$, $s \in \mathbb{C}$. Let us recall some of their properties as well as the definition of the scattering matrix that we will use; for the details see [19, 20]. To each cusp there is associated a generalized eigenfunction that satisfies:

$$\Delta_g E_i(z, s) = s(1 - s)E_i(z, s), \quad \text{for } z \in M.$$

Each $E_i(z, s)$ is a meromorphic function of $s \in \mathbb{C}$ with poles contained in $\{s | \operatorname{Re}(s) < 1/2\} \cup (1/2, 1]$. The zeroth Fourier coefficient of the expansion of $E_i(s, z)$ in a Fourier series on the cusp $Z_j = [a_j, \infty) \times S^1$ has the form

$$\delta_{ij} y_j^s + C_{ij}(s) y_j^{1-s}, \quad \text{for } y_j \geq a_j.$$

Using this expression we can define the scattering matrix as the $m \times m$ matrix given by:

$$C(s) = (C_{ij}(s)).$$

It is a meromorphic function of $s \in \mathbb{C}$ and all its poles are contained in $\{s \mid \operatorname{Re}(s) < 1/2\} \cup (1/2, 1]$. The scattering matrix also satisfies:

$$C(s)C(1 - s) = \operatorname{Id}, \quad \overline{C(s)} = C(\bar{s}), \quad C(s)^* = C(\bar{s}).$$

A quantity of interest is the determinant of the scattering matrix which we denote by $\phi(s) = \det C(s)$. It satisfies the following equations:

$$\phi(s)\phi(1 - s) = 1, \quad \overline{\phi(s)} = \phi(\bar{s}), \quad s \in \mathbb{C}.$$

The poles of $\phi(s)$ will be called resonances. They will be the complementary quantities to the eigenvalues that we will need to study "isospectral" surfaces.

In [21] Müller defines the relative determinant for pairs of operators in a general setting. Let us recall the definition since we will use it. Let H_1 and H_0 be two self-adjoint, nonnegative linear operators in a separable Hilbert space $\mathcal{H}$ satisfying the following assumptions:

1. For each $t > 0$, $\mathrm{e}^{-tH_1} - \mathrm{e}^{-tH_0}$ is a trace class operator.

2. As $t \to 0$, there is an asymptotic expansion of the relative trace of the form:
$$\mathrm{Tr}(\mathrm{e}^{-tH_1} - \mathrm{e}^{-tH_0}) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{k(j)} a_{jk} t^{\alpha_j} \log^k t,$$
where $-\infty < \alpha_0 < \alpha_1 < \cdots$ and $\alpha_k \to \infty$. Moreover, if $\alpha_j = 0$ we assume that $a_{jk} = 0$ for $k > 0$.

3. $\mathrm{Tr}(\mathrm{e}^{-tH_1} - \mathrm{e}^{-tH_0}) = h + O(\mathrm{e}^{-ct})$, as $t \to \infty$, where $h = \dim \mathrm{Ker}\, H_1 - \dim \mathrm{Ker}\, H_0$.

The relative spectral zeta function is defined as:
$$\zeta(s; H_1, H_0) = \frac{1}{\Gamma(s)} \int_0^{\infty} (\mathrm{Tr}(\mathrm{e}^{-tH_1} - \mathrm{e}^{-tH_0}) - h) t^{s-1} dt.$$

Thanks to the properties given above, it has a meromorphic extension to the complex plane that is meromorphic at $s = 0$. The relative determinant is then defined as:
$$\det(H_1, H_0) := \mathrm{e}^{-\zeta'(0; H_1, H_0)}.$$

This determinant is multiplicative. If the determinant of each operator can be defined separately, then their relative determinant is the quotient of the determinants. In this paper we work with the relative determinant of the following pairs: $(\Delta_g, \bar{\Delta}_{a,0})$, $(\Delta_g, \Delta_{Z_a,D})$. The good definition of these relative determinants is guaranteed by the results of Müller in [19, 21].

3. Splitting formula

Splitting formulae for determinants have been widely studied. They have been proved in the setting of compact manifold by Burghelea *et al.* in [5], and in other settings by many other authors. For example, for manifolds with cylindrical ends they were studied by Müller and Müller in [22] and Loya and Park in [16]. In this section we use the Dirichlet-to-Neumann operator for the Laplacian on a manifold with cusps to obtain a splitting formula for the relative determinant $\det(\Delta_g, \bar{\Delta}_{\beta,0})$.

3.1. Dirichlet-to-Neumann operator for Δ_g

Let us start by recalling the definition of the Dirichlet-to-Neumann operator $\mathcal{N}(z)$ and its main properties. Then, we study the limit operator as the parameter z goes to zero.

Let us assume that (M, g) has only one cusps and that we can decompose it as $M = M_0 \cup Z_\alpha$ where $\alpha \geq 1$ and Z_α is isometric to $[\alpha, \infty) \times S^1$ with the hyperbolic metric.

Let $\beta \geq \alpha$, then M may be decomposed as $M = M_\beta \cup Z_\beta$, with $M_\beta = M_0 \cup [\alpha, \beta] \times S^1$, $Z_\beta = [\beta, \infty) \times S^1$, and $\Sigma_\beta = \{\beta\} \times S^1 = \partial M_\beta = \partial Z_\beta$. Let Δ_{M_β} denote the Laplace operator acting on $C^\infty(M_\beta)$ and $\Delta_{M_\beta, D}$ denote its self-adjoint extension with respect to Dirichlet boundary conditions at Σ_β. Let $\Delta_{Z_\beta, D}$ be as it was defined in Section 2. We will explicitly compute the part of the Dirichlet-to-Neumann operator $\mathcal{N}(z)$ on Σ_β, for any value of $\beta > \alpha$ coming from the cusps Z_β. The metric on Σ_β is given by $g_{\Sigma_\beta} = \beta^{-2} dx^2$, the eigenvalues of the Laplacian Δ_{Σ_β} are $\{4\pi^2 n^2 \beta^2\}_{n \in \mathbb{Z}}$ and the corresponding eigenfunctions are $\{\beta \exp(2\pi i n x)\}_{n \in \mathbb{Z}}$.

Let z be in the resolvent set of $\Delta_g, \rho(\Delta_g)$. Then the Dirichlet-to-Neumann operator,

$$\mathcal{N}(z) : C^\infty(\Sigma_\beta) \rightarrow C^\infty(\Sigma_\beta),$$

is defined as follows: Let $f \in C^\infty(\Sigma_\beta)$ and let $\tilde{f}$ be the unique square integrable solution to the problem

$$\begin{cases} (\Delta_g - z)\tilde{f} = 0 & \text{in } M \setminus \Sigma_\beta \\ \tilde{f} = f & \text{on } \Sigma_\beta. \end{cases}$$

Let n^+ denote the inwards unit normal vector field at Σ_β on M_β and n^- the one on Z_β. Then $\mathcal{N}(z)f$ is defined by the following equation

$$\mathcal{N}(z)f := -\left(\frac{\partial}{\partial n^+}(\tilde{f}|_{M_\beta}) + \frac{\partial}{\partial n^-}(\tilde{f}|_{Z_\beta}) \right).$$

Theorem 2.1 of Carron in [6] establishes that for $z \in \mathbb{C} \setminus [0, \infty)$, the Dirichlet-to-Neumann operator is a first-order elliptic, invertible, pseudo-differential operator whose principal symbol is a scalar, $\text{sym}_p(\mathcal{N}(z))(x, \eta) = 2\sqrt{g_x(\eta, \eta)}$, $(x, \eta) \in T^* M$. In addition, the function $z \mapsto \mathcal{N}(z)$ is holomorphic as function of z. In particular, $\mathcal{N}(z) : C^\infty(\Sigma_\beta) \rightarrow C^\infty(\Sigma_\beta)$ has continuous extensions to the Sobolev spaces, $H^1(\Sigma_\beta) \rightarrow L^2(\Sigma_\beta) \rightarrow H^{-1}(\Sigma_\beta)$. Then we

can think of $\mathcal{N}(z)$ as an operator on $L^2(\Sigma_\beta)$ by $\mathcal{N}(z) : H^1(\Sigma_\beta) \subset L^2(\Sigma_\beta) \to L^2(\Sigma_\beta)$. Furthermore, for $f \in C^\infty(\Sigma_\beta)$ we have that:

$$\mathcal{N}(z)^{-1} f(x) = \int_{\Sigma_\beta} G(x, y, z) f(y) d\mu(y), \tag{3.1}$$

where $G(x, y, z)$ is the Schwartz kernel of $(\Delta_g - z)^{-1}$ on M, see Theorem 2.1 in [6]. This expression is equivalent to:

$$\mathcal{N}(z)^{-1} f = \rho_{\Sigma_\beta} \circ (\Delta_g - z)^{-1} \circ i_{\Sigma_\beta}(f), \tag{3.2}$$

where ρ_{Σ_β} denotes the restriction to Σ_β and $i_{\Sigma_\beta}(f) = f \otimes \delta_{\Sigma_\beta}$ in the distributional sense, this means $f \otimes \delta_{\Sigma_\beta}(\varphi) = \int_{\Sigma_\beta} \varphi \cdot f$ for any $\varphi \in C^\infty(M)$.

Now, remember that $0 \in \sigma(\Delta_g)$ is an isolated eigenvalue. Thus the Dirichlet-to-Neumann operator $\mathcal{N}(z)$ is actually defined for z in a neighborhood of zero and it makes sense to consider its limit as z approaches zero. Indeed, it exists for $z = 0$ and the dependence on z is continuous. In order to prove this, we split the problem in the classical way letting $\mathcal{N}(z) = \mathcal{N}_1(z) + \mathcal{N}_2(z)$, where for $i = 1, 2$ $\mathcal{N}_i(z)$ is defined as follows:

Let $f \in C^\infty(\Sigma_\beta)$, then let $\varphi_1 \in C^\infty(M_\beta \setminus \Sigma_\beta) \cap C^0(M_\beta)$ be the unique solution to the problem

$$\begin{cases} (\Delta - z)\varphi_1 = 0 & \text{in } M_\beta \setminus \Sigma_\beta \\ \varphi_1 = f & \text{on } \Sigma_\beta. \end{cases}$$

Put $\mathcal{N}_1(z)f = -\frac{\partial \varphi_1}{\partial n^+}$. Similarly, let $\varphi_2 \in C^\infty(Z_\beta) \cap L^2(Z_\beta)$ be the unique square integrable solution to the problem:

$$\begin{cases} (\Delta - z)\varphi_2 = 0 & \text{in } Z_\beta \\ \varphi_2 = f & \text{on } \Sigma_\beta. \end{cases}$$

Put $\mathcal{N}_2(z)f = -\frac{\partial \varphi_2}{\partial n^-}$.

Using the usual method of separation of variables in the cusp we can compute the operator $\mathcal{N}_2(z)$ explicitly. The explicit expression of $\mathcal{N}_2(z)$ is useful to compute the limit of the operator as $z \to 0$.

Lemma 3.1. *Let $f \in C^\infty(\Sigma_\beta)$. Write $z = s(1-s)$. If $\mathrm{Re}(s) > \frac{1}{2}$ then*

$$\mathcal{N}_2(s(1-s))f = -(1-2s)c_0(f)\beta - sf + \beta\sqrt{\Delta_{\Sigma_\beta}} \frac{K_{s+1/2}}{K_{s-1/2}}\left(\beta\sqrt{\Delta_{\Sigma_\beta}}\right)f, \tag{3.3}$$

where $c_0(f)$ is the projection of f on the kernel of Δ_{Σ_β} and K_ν is the modified Bessel function of order ν. In the case $\mathrm{Re}(s) < \frac{1}{2}$,

$$(3.4) \qquad \mathcal{N}_2(s)f(x) = -sf(x) + \beta\sqrt{\Delta_{\Sigma_\beta}}\frac{K_{s+1/2}}{K_{s-1/2}}\left(\beta\sqrt{\Delta_{\Sigma_\beta}}\right)f(x).$$

If $\mathrm{Re}(s) = \frac{1}{2}$, $f \in \mathrm{Dom}(\mathcal{N}_2(z))$ only if it its zero Fourier coefficient vanishes, $\int_{\Sigma_\beta} f dA_{\Sigma_\beta} = 0$. In this case we have:

$$(3.5) \qquad \mathcal{N}_2(s)f = -sf + \beta\sqrt{\Delta_{\Sigma_\beta}}\frac{K_{s+1/2}}{K_{s-1/2}}\left(\beta\sqrt{\Delta_{\Sigma_\beta}}\right)f.$$

Proof. Take the Fourier expansion of φ_2 and f on the cusp, $\varphi_2(y,x) = \sum_{n\in\mathbb{Z}} a_n(y)\beta e^{2\pi inx}$ and $f(x) = \sum_{n\in\mathbb{Z}} c_n\beta e^{2\pi inx}$. Then, using separation of variables the problem becomes

$$\begin{cases} (-y^2\frac{d^2}{dy^2} + y^2 4\pi^2 n^2\beta^2 - z)a_n(y) = 0 \\ a_n(\beta) = c_n, & \text{for } n \in \mathbb{Z}. \end{cases}$$

Set $z = s(1-s)$ with $s \in \mathbb{C}$. Then for $n \neq 0$, two linear independent solutions of the equation

$$(3.6) \qquad \left(-y^2\frac{d^2}{dy^2} + 4\pi^2 n^2\beta^2 y^2 - s(1-s)\right)a_n(y) = 0$$

are $y^{1/2}K_{s-1/2}(2\pi|n|\beta y)$ and $y^{1/2}I_{s-1/2}(2\pi|n|\beta y)$, where $K_{s-1/2}$ and $I_{s-1/2}$ are the modified Bessel functions. The function $I_{s-1/2}$ is discarded because it is not square integrable on $[1,\infty)$ for any value of s. Thus,

$$\varphi_2(y,x) = b_{0,1}y^s\beta + b_{0,2}y^{1-s}\beta + \sum_{n\neq 0} b_n y^{1/2}K_{s-1/2}(2\pi|n|\beta y)\beta e^{2\pi inx}.$$

Then for $n \neq 0$, $a_n(y) = b_n y^{1/2}K_{s-1/2}(2\pi|n|\beta y)$, where b_n and $b_{0,1}, b_{0,2}$ are constants determined by the boundary and the square integrable conditions.

Case $\mathrm{Re}(s) > \frac{1}{2}$. In this case $b_{0,1} = 0$ and $y^{1/2}K_{s-1/2}(2\pi|n|\beta y)$ is square integrable on $[1,\infty[$. Then we have:

$$\varphi_2(y,x) = b_{0,2}y^{1-s}\beta + \sum_{n\neq 0} b_n y^{1/2}K_{s-1/2}(2\pi|n|\beta y)\beta e^{2\pi inx},$$

where $a_0(y) = b_{0,2}y^{1-s}$ and $a_n(y) = b_n y^{1/2}K_{s-1/2}(2\pi|n|\beta y)$. The boundary condition $\varphi_2(\beta, x) = f(x)$ is equivalent to $a_n(\beta) = c_n$. Thus $b_{0,2} = c_0\beta^{s-1}$

and

$$b_n = \frac{c_n}{\beta^{1/2} K_{s-1/2}(2\pi|n|\beta^2)}.$$

In this way we obtain:

$$\varphi_2(y, x) = c_0 \beta^s y^{1-s}$$
$$+ \sum_{n \neq 0} \frac{c_n}{\beta^{1/2} K_{s-1/2}(2\pi|n|\beta^2)} y^{1/2} K_{s-1/2}(2\pi|n|\beta y)\beta e^{2\pi i n x},$$

after differentiation, evaluation at $y = \beta$ gives:

$$y\frac{\partial}{\partial y}\varphi_2(y, x)\Big|_{y=\beta} = (1-s)c_0\beta + \beta\sum_{n\neq 0}c_n$$
$$\times \left(s\beta^{-1} - 2\pi|n|\beta\frac{K_{s+1/2}(2\pi|n|\beta^2)}{K_{s-1/2}(2\pi|n|\beta^2)}\right)\beta e^{2\pi i n x}$$
$$= (1-2s)c_0\beta + sf(x) - \beta\sqrt{\Delta_{\Sigma_\beta}}\frac{K_{s+1/2}}{K_{s-1/2}}(\beta\sqrt{\Delta_{\Sigma_\beta}})f(x),$$

where we have chosen the positive square root of the eigenvalues to define the operator $\sqrt{\Delta_{\Sigma_\beta}}$.

Case $\mathrm{Re}(s) = \frac{1}{2}$. The computations are the same as in the previous case but the square integrability condition implies that the zero term in the Fourier expansion of the solution φ_2 should be null, thus

$$\varphi_2(y, x) = \sum_{n\neq 0} b_n y^{1/2} K_{s-1/2}(2\pi|n|\beta y)\beta e^{2\pi i n x}.$$

In addition, the condition $a_0 = c_0$ gives $c_0 = 0$. This means that only in the case when $c_0 = 0$ will there exist a solution to the problem. Hence for f to be in the domain of $\mathcal{N}_2(s(1-1))$, f should satisfy $c_0(f) = \int_{\Sigma_\beta} f dA_{\Sigma_\beta} = 0$. For such functions f equation (3.5) holds.

The case when $\mathrm{Re}(s) < \frac{1}{2}$ is similar. □

Remark 3.1. Let $z < 0$, then the operator $\mathcal{N}(z)$ is positive. This follows from the non-negativity of the Laplacian Δ_g and the definition of $\mathcal{N}(z)$. Remember that the Schwartz kernel of $\mathcal{N}(z)^{-1}$ is the same as the Schwartz kernel of $(\Delta_g - z)^{-1}$. We have $\Delta_g \geq 0$. If $z < 0$, then $(\Delta_g - z) > 0$, and $(\Delta_g - z)^{-1} > 0$. Therefore $\mathcal{N}(z)^{-1} > 0$. In addition, in this case $\mathcal{N}(z)$ is also self-adjoint and by the work of Kontsevich and Vishik in [15] we know that its zeta determinant is well defined.

The existence of the Dirichlet-to-Neumann operator when $z = 0$ is given by the following lemma:

Lemma 3.2. *For every $f \in C^\infty(\Sigma_\beta)$ there exists a unique solution $\tilde{f} \in C^\infty(M \setminus \Sigma_\beta) \cap C^0(M)$ and $\tilde{f}|_{Z_\beta} \in L^2$, to the problem:*

$$\begin{cases} \Delta_g \tilde{f} = 0 & \text{in } M \setminus \Sigma_\beta \\ \tilde{f} = f & \text{on } \Sigma_\beta. \end{cases}$$

In addition, using the notation introduced above we have that:

$$\mathcal{N}_2 f := -\left. y \frac{\partial}{\partial y} \varphi_2(y, x) \right|_{y=\beta} = \beta \sqrt{\Delta_{\Sigma_\beta}} f.$$

Proof. In the same way as in the proof of Lemma 3.1 in Müller and Müller [22], the uniqueness of the solution $\varphi_1 \in C^\infty(M_\beta \setminus \Sigma_\beta) \cap C^0(M_\beta)$ of the Dirichlet problem on M_β follows from the invertibility of $\Delta_{M_\beta, D}$. The uniqueness of the solution on Z_β also follows from the invertibility of $\Delta_{Z_\beta, D}$. To see the existence on Z_β more explicitly let us follow the same procedure of the proof of Lemma 3.1 but taking $z = 0$. One way to obtain $z = 0$ is to take $s = 1$ in equation (3.6). In this case the square integrable condition gives

$$\varphi_2(y, x) = \sum_{n \in \mathbb{Z}} a_n(y) e^{2\pi i n x} = b_{0,2} \beta + \sum_{n \neq 0} b_n y^{1/2} K_{1/2}(2\pi |n| \beta y) \beta e^{2\pi i n x}.$$

We know that $K_{1/2}(r) = \sqrt{\frac{\pi}{2}} r^{-1/2} e^{-r}$. Then for $n \neq 0$ we have $a_n(y) = \frac{b_n}{2\sqrt{|n|\beta}} e^{-2\pi|n|\beta y}$. The boundary condition $\varphi_2(\beta, x) = f(x)$, which is equivalent to $a_n(\beta) = c_n$, gives $b_0 = c_0$ and $b_n = c_n 2\sqrt{|n|\beta} e^{2\pi|n|\beta^2}$. Then

$$\varphi_2(y, x) = c_0 \beta + \sum_{n \neq 0} c_n e^{2\pi|n|\beta^2} e^{-2\pi|n|\beta y} \beta e^{2\pi i n x}.$$

Taking the inward derivative we obtain:

$$\left. y \frac{\partial}{\partial y} \varphi_2(y, x) \right|_{y=\beta} = \beta \sum_{n \neq 0} -2\pi|n|\beta c_n \beta e^{2\pi i n x} = -\beta \sqrt{\Delta_{\Sigma_\beta}} f.$$

The other way to obtain $z = 0$ is taking $s = 0$ in equation (3.6). In this case we have:

$$\varphi_2(y, x) = b_{0,1}\beta + \sum_{n \neq 0} b_n y^{1/2} K_{-1/2}(2\pi|n|\beta y)\beta e^{2\pi inx}$$

$$= c_0\beta + \sum_{n \neq 0} c_n e^{2\pi|n|\beta^2} e^{-2\pi|n|\beta y}\beta e^{2\pi inx},$$

where we have used that $K_{-1/2} = K_{1/2}$. Thus for $s = 0$ and for $s = 1$, the solutions of the Dirichlet problem on Z_β are the same. Since $\varphi_1|_{\Sigma_\beta} = \varphi_2|_{\Sigma_\beta}$, we have that the solution $\tilde{f}$ is continuous on M. $\qquad\square$

Remark 3.2. For $z \in \rho(\Delta_g)$, the resolvent set of Δ_g, it is well known that $\mathcal{N}_1(z)$ is a first order invertible elliptic pseudodifferential operator. The limit, $\mathcal{N}_1$, as $z \to 0$, it is well known to be a first order elliptic pseudodifferential operator, but it is non-invertible, see for example Burghelea *et al.* in [5] and Taylor in [29] Section 7.11. Therefore the operator $\mathcal{N} = \mathcal{N}_1 + \mathcal{N}_2$ is non-invertible. However it is non-negative and $\dim(\text{Ker}(\mathcal{N})) = 1$.

Proposition 3.1. *Let $f \in C^\infty(\Sigma_\beta)$. Then $\mathcal{N}(z)f$ depends continuously of z in a small enough neighborhood of $z = 0$, and*

$$\lim_{z \to 0} \mathcal{N}(z)f = \mathcal{N}f.$$

Proof. The proof of $\lim_{z \to 0} \mathcal{N}_1(z)f = \mathcal{N}_1 f$ is the same as the proof of Lemma 3.3 in [22]. For the convenience of the reader we repeat here the argument with our notation. For $f \in C^\infty(\Sigma_\beta)$, let $\varphi_1(z)$ be the unique function in $C^\infty(M_\beta \setminus \Sigma_\beta)$ satisfying $(\Delta_g - z)\varphi_1(z) = 0$, $\varphi_1(z)|_{\Sigma_\beta} = f$ and

$$\varphi_1(z) = \tilde{f} - (\Delta_{M_\beta,D} - z)^{-1}((\Delta_{M_\beta} - z)(\tilde{f})),$$

where $\tilde{f} \in C^\infty(M_\beta)$ is any extension of f. Since $\Delta_{M_\beta,D}$ is invertible, the formula also holds for $z = 0$. From this representation of $\varphi_1(z)$, it follows immediately that $\mathcal{N}_1(z)f$ converges to $\mathcal{N}_1 f$ as $z \to 0$.

Now let us take the limit of $\mathcal{N}_2(z)$ as $s \to 1$. To do that we use equation (3.3) to obtain:

$$\lim_{s \to 1} \mathcal{N}_2(s(1-s))f = c_0\beta - f + \beta\sqrt{\Delta_{\Sigma_\beta}}\frac{K_{3/2}}{K_{1/2}}(\sqrt{\Delta_{\Sigma_\beta}})f.$$

Using the expression $K_{3/2}(u) = \sqrt{\frac{\pi}{2}} u^{-3/2} e^{-u}(u+1)$, we have that
$\frac{K_{3/2}(2\pi|n|\beta^2)}{K_{1/2}(2\pi|n|\beta^2)} = \frac{2\pi|n|\beta^2+1}{2\pi|n|\beta^2}$. Thus,

$$\lim_{s\to 1} \mathcal{N}_2(s(1-s))f = \beta \sum_{n\neq 0} 2\pi|n|\beta c_n \beta e^{2\pi inx} = \beta\sqrt{\Delta_{\Sigma_\beta}} f = \mathcal{N}_2 f.$$

For the limit when $s \to 0$ we have:

$$\lim_{s\to 0} \mathcal{N}_2(s(1-s))f = \lim_{s\to 0} -sf(x) + \beta \sum_{n\neq 0} 2\pi|n|\beta \frac{K_{s+1/2}(2\pi|n|\beta^2)}{K_{s-1/2}(2\pi|n|\beta^2)} c_n \beta e^{2\pi inx}$$

$$= \beta \sum_{n\neq 0} 2\pi|n|\beta c_n \beta e^{2\pi inx} = \beta\sqrt{\Delta_{\Sigma_\beta}} f.$$

Thus it follows that

$$\lim_{s\to 1} \mathcal{N}_2(s(1-s))f = \lim_{s\to 0} \mathcal{N}_2(s(1-s))f = \mathcal{N}_2(0)f = \beta\sqrt{\Delta_{\Sigma_\beta}} f = \mathcal{N}_2 f.$$

$$\square$$

3.2. Splitting formula for the relative determinant

We want to have a splitting formula for the relative determinant that relates $\det(\Delta_g, \Delta_{\beta,0})$ to the regularized determinant of the operator $\mathcal{N}$. We will use this formula in Section 4 to prove Theorem 4.1. For $z \in \rho(\Delta_g)$ Corollary 4.6 in Carron [6] establishes the following splitting formula for complete surfaces, which we rewrite using his notation:

$$(3.7)\qquad\qquad \det(\mathcal{L} - z, \mathcal{L}_{0,D} - z) = \det \mathcal{N}(z),$$

where $\mathcal{L}$ is the self-adjoint extension of the Laplacian on M and $\mathcal{L}_{0,D}$ is the self-adjoint extension of the Laplacian on $M \setminus \Sigma$ with Dirichlet boundary conditions on Σ. Let $\lambda > 0$, put $z = -\lambda$ and let us denote $\mathcal{N}(-\lambda)$ by $R(\lambda)$. Then $R(\lambda) > 0$ and it has the same properties as $\mathcal{N}(-\lambda)$. In our case equation (3.7) has the form:

(3.8)
$$\det(\Delta_g + \lambda, \Delta_{Z_{\beta,D}} + \lambda)(\det(\Delta_{M_{\beta,D}} + \lambda))^{-1} = \det \mathcal{N}(-\lambda) = \det R(\lambda).$$

Both sides of equation (3.8) diverge as $\lambda \to 0^+$, we study how is this divergence.

Lemma 3.3. *As $\lambda \to 0^+$, the left hand side of equation (3.8) has the following decomposition:*

$$\log \det(\Delta_g + \lambda, \Delta_{Z_\beta,D} + \lambda) - \log \det(\Delta_{M_\beta,D} + \lambda)$$
$$= \log \lambda + \log \det(\Delta_g, \Delta_{Z_\beta,D}) - \log \det \Delta_{M_\beta,D} + o(1).$$

Proof. Let us go back to the definition of the relative determinant and use the definition of the relative zeta functions for $(\Delta_g, \Delta_{Z_\beta,D})$ and $(\Delta_g + \lambda, \Delta_{Z_\beta,D} + \lambda)$. From them we have:

$$\zeta(s; \Delta_g + \lambda, \Delta_{Z_\beta,D} + \lambda) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}(\mathrm{e}^{-t\Delta_g} - \mathrm{e}^{-t\Delta_{Z_\beta,D}})\mathrm{e}^{-t\lambda}t^{s-1}dt$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty (\mathrm{Tr}(\mathrm{e}^{-t\Delta_g} - \mathrm{e}^{-t\Delta_{Z_\beta,D}}) - 1)\mathrm{e}^{-t\lambda}t^{s-1}dt + \frac{1}{\Gamma(s)}\Gamma(s)\lambda^{-s}$$

$$= \lambda^{-s} + \zeta(s, \Delta_g, \Delta_{Z_\beta,D}) + \frac{\lambda}{\Gamma(s)} \int_0^\infty (\mathrm{Tr}(\mathrm{e}^{-t\Delta_g} - \mathrm{e}^{-t\Delta_{Z_\beta,D}}) - 1)$$

$$\times \frac{\mathrm{e}^{-t\lambda} - 1}{\lambda}t^{s-1}dt.$$

The last integral converges in a half plane. Therefore due to the asymptotic expansions of the relative heat trace for small and large t, it has an analytic continuation that is holomorphic at $s = 0$. So, as $\lambda \to 0^+$ we obtain:

$$\frac{d}{ds}\zeta(s; \Delta_g + \lambda, \Delta_{Z_\beta,D} + \lambda)\Big|_{s=0} = -\log \lambda + \frac{d}{ds}\zeta(s; \Delta_g, \Delta_{Z_\beta,D})\Big|_{s=0} - o(1),$$
$$-\log \det(\Delta_g + \lambda, \Delta_{Z_\beta,D} + \lambda) = -\log \lambda - \log \det(\Delta_g, \Delta_{Z_\beta,D}) - o(1),$$

as desired. Similarly, from the definition of $\zeta_{\Delta_{M_\beta,D}+\lambda}(s)$ it follows that

$$\log \det(\Delta_{M_\beta,D} + \lambda) = \log \det(\Delta_{M_\beta,D}) + o(1),$$

as $\lambda \to 0^+$. This finishes the proof of the lemma. $\square$

In order to study the asymptotic behavior of the right hand side of equation (3.8), we need some preliminaries.

Let $\lambda > 0$ and $R(\lambda)$ be as above. Recall that $R \geq 0$, $\mathrm{Ker}\, R = \mathbb{C}$ and $\lim_{\lambda \to 0} R(\lambda) = R$. It is not difficult to prove that R is self-adjoint; therefore the regularized determinant of R, $\det^* R$, may defined by the meromorphic continuation of

$$\zeta_R^*(s) = \sum_{\mu_i > 0} \mu_i^{-s},$$

where the sum runs over the positive eigenvalues of R.

Now, let μ_1 be the first non-zero eigenvalue of R, let $0 < \mu < \mu_1$, and let P_μ be the spectral projection of the Laplacian Δ_g on $[0, \mu]$. Then by equation (3.2), $R(\lambda)^{-1}$ can be decomposed as:

$$R(\lambda)^{-1} = \rho_{\Sigma_\beta} \circ P_\mu (\Delta_g + \lambda)^{-1} \circ i_{\Sigma_\beta}$$
$$+ \rho_{\Sigma_\beta} \circ (I - P_\mu)(\Delta_g + \lambda)^{-1} \circ i_{\Sigma_\beta} =: Q_\mu(\lambda) + \widetilde{Q}_\mu(\lambda).$$

The kernel of $Q_\mu(\lambda)$ in terms of the spectral decomposition of Δ_g on M is given by:

$$K_{Q_\mu(\lambda)}(x, x', \lambda) = \sum_{0 \leq \lambda_j \leq \mu} \frac{1}{\lambda_j + \lambda} \varphi_j(x) \overline{\varphi_j(x')}$$
$$+ \frac{1}{2\pi} \int_0^\mu \frac{1}{\lambda + 1/4 + r^2} E\left(x, \frac{1}{2} + ir\right) E\left(x', \frac{1}{2} - ir\right) dr,$$

for $x, x' \in \Sigma_\beta$. We can further decompose $Q_\mu(\lambda)$ as $Q_{\mu,1}(\lambda) + Q_{\mu,2}(\lambda)$, where $Q_{\mu,1}(\lambda)$ is given by:

$$Q_{\mu,1}(\lambda) f = \frac{1}{\lambda} \frac{1}{A_g} \int_{\Sigma_\beta} f(x) d\mu(x), \quad \text{with } K_{Q_{\mu,1}(\lambda)}(x, x', \lambda) = \frac{1}{\lambda} \frac{1}{A_g},$$

and $K_{Q_{\mu,2}(\lambda)} = K_{Q_\mu(\lambda)} - K_{Q_{\mu,1}(\lambda)}$. Taking the limit as $\lambda \to 0$ of $K_{Q_{\mu,2}(\lambda)}$ we obtain:

$$\lim_{\lambda \to 0} K_{Q_{\mu,2}(\lambda)}(x, x', \lambda) = \sum_{0 < \lambda_j \leq \mu} \frac{1}{\lambda_j} \varphi_j(x) \overline{\varphi_j(x')}$$
$$+ \frac{1}{2\pi} \int_0^\mu \frac{1}{1/4 + r^2} E\left(x, \frac{1}{2} + ir\right) E\left(x', \frac{1}{2} - ir\right) dr.$$

Thus $\|Q_{\mu,2}(\lambda)\|$ remains bounded as $\lambda \to 0^+$. In the same way as in [22, Lemma 3.5], we can prove that there is a constant $C > 0$, depending only on μ, such that for all $\lambda > 0$:

$$(3.9) \qquad \|\widetilde{Q}_\mu(\lambda)\| = \|\rho_{\Sigma_\beta} \circ (I - P_\mu)(\Delta_g + \lambda)^{-1} \circ i_{\Sigma_\beta}\| \leq C.$$

Therefore the operator $R(\lambda)^{-1}$ can be written as:

$$(3.10) \qquad\qquad R(\lambda)^{-1} = Q_{\mu,1}(\lambda) + K_\mu(\lambda),$$

with $\|K_\mu(\lambda)\|$ uniformly bounded as $\lambda \to 0^+$.

Now, just note that $\rho_{\Sigma_\beta}(\mathrm{Ker}(\Delta_g)) = \mathrm{Ker}(R)$. Let $0 < \mu_1(\lambda) \leq \mu_2(\lambda) \leq \mu_3(\lambda) \leq \ldots$ be the eigenvalues of $R(\lambda)$. Then from the discussion above it is clear that:

$$\mu_1(\lambda) \to 0, \quad \text{as } \lambda \to 0,$$
$$\mu_i(\lambda) \geq c > 0, \quad \text{for } i \geq 2, \ \lambda \geq 0.$$

Lemma 3.4. *There is the following asymptotic expansion as $\lambda \to 0^+$:*

(3.11) $$\log \det R(\lambda) = \log \mu_1(\lambda) + \log \mathrm{det}^* R + o(1).$$

Proof. Let $\mathrm{Ker}(R)$ be the kernel of R, $\mathcal{H} = (\mathrm{Ker}(R))^\perp$ be its orthogonal complement, and $P : L^2(\Sigma_\beta) \to \mathrm{Ker}(R)$ and $P^\perp : L^2(\Sigma_\beta) \to \mathcal{H}$ be the corresponding orthogonal projections. By definition:

$$\log \det R(\lambda) := -\left.\frac{d}{ds}\right|_{s=0} \zeta_{R(\lambda)}(s) = -\left.\frac{d}{ds}\right|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}(\mathrm{e}^{-tR(\lambda)}) t^{s-1} dt.$$

The first thing to do is to separate the first eigenvalue. For that, let γ be a contour in $\mathbb{C}$ contained in the resolvent set of $R(\lambda)$, $\rho(R(\lambda))$, and surrounding the spectrum of $R(\lambda)$, for all $\lambda \geq 0$ small enough. Then:

$$\mathrm{e}^{-tR(\lambda)} = \frac{1}{2i\pi} \int_\gamma \mathrm{e}^{-t\xi} (R(\lambda) - \xi)^{-1} d\xi$$
$$= \frac{1}{2i\pi} \int_{\gamma_1} \mathrm{e}^{-t\xi} (R(\lambda) - \xi)^{-1} d\xi + \frac{1}{2i\pi} \int_{\gamma_2} \mathrm{e}^{-t\xi} (R(\lambda) - \xi)^{-1} d\xi,$$

where γ_1 is a contour surrounding $\{\mu_1(\lambda), 0\}$ and γ_2 surrounds the half line $[c, \infty)$, where $\mu_2(\lambda) \geq c$ for all $\lambda > 0$. The curves γ_1 and γ_2 can be chosen without overlapping and independently of λ. It is clear that:

$$\frac{1}{2i\pi} \int_{\gamma_1} \mathrm{e}^{-t\xi} (R(\lambda) - \xi)^{-1} d\xi = \mathrm{e}^{-t\mu_1(\lambda)} P(\lambda),$$

where $P(\lambda)$ is the orthogonal projection on the $\mu_1(\lambda)$-eigenspace. Therefore

$$\zeta_{R(\lambda)}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{e}^{-t\mu_1(\lambda)} t^{s-1} dt$$
$$+ \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}\left(\frac{1}{2i\pi} \int_{\gamma_2} \mathrm{e}^{-t\xi} (R(\lambda) - \xi)^{-1} d\xi\right) t^{s-1} dt.$$

The family $R(\lambda)$ acting on a subspace of $L^2(\Sigma_\beta)$ into $L^2(\Sigma_\beta)$ depends continuously on λ. The resolvent of $R(\lambda)$ depends continuously of λ too. Since R has 0 as eigenvalue, the resolvent $(R - \xi)^{-1}$ has a pole at $\xi = 0$ and can be written as:

$$(R - \xi)^{-1} = -\xi^{-1}P + A(\xi),$$

with $A(\xi)$ a holomorphic operator in ξ. On the other hand, $\mu_1(\lambda) > 0$ for $\lambda > 0$. Therefore $(R(\lambda) - \xi)^{-1}$ is continuous in λ close to 0 and holomorphic in ξ far from $\sigma(R(\lambda))$. When integrating over γ_2 we are actually dealing with the operators $P(\lambda)^\perp R(\lambda)$ or $P^\perp R(\lambda)$. From general results about resolvents we have that $(P(\lambda)^\perp R(\lambda) - \xi)^{-1}$ converges continuously to $(P^\perp R - \xi)^{-1}$ as $\lambda \to 0^+$, for $\xi \in \rho(R(\lambda))$. This fact in addition to the following expressions

$$\mathrm{e}^{-tP(\lambda)^\perp R(\lambda)} = \frac{1}{2i\pi} \int_{\gamma_2} \mathrm{e}^{-t\xi}(P(\lambda)^\perp R(\lambda) - \xi)^{-1}d\xi, \text{ and,}$$

$$\mathrm{e}^{-tP^\perp R} = \frac{1}{2i\pi} \int_{\gamma_2} \mathrm{e}^{-t\xi}(P^\perp R - \xi)^{-1}d\xi,$$

imply that $\mathrm{e}^{-tP(\lambda)^\perp R(\lambda)}$ converges to $\mathrm{e}^{-tP^\perp R}$. Therefore $\mathrm{Tr}(\mathrm{e}^{-tP(\lambda)^\perp R(\lambda)})$ depends continuously on λ and so does the zeta function. In this way we obtain:

$$\log \det P(\lambda)^\perp R(\lambda) = \log \det P^\perp R + o(1), \text{ as } \lambda \to 0^+.$$

This finishes the proof of equation (3.11). $\square$

We consider now the behavior of the term $\log \mu_1(\lambda)$ in equation (3.11):

Lemma 3.5. *As $\lambda \to 0^+$*

(3.12) $$\log \mu_1(\lambda) = \log \lambda + \log\left(\frac{A_g}{\ell_\beta}\right) + o(1),$$

where $A_g = area(M)$ and $\ell_\beta = length(\Sigma_\beta)$.

Proof. First observe that $\frac{1}{\mu_1(\lambda)} = \|R(\lambda)^{-1}\|$, where the norm is the operator norm in L^2. From equations (3.10) and $\|Q_{\mu,1}(\lambda)\| = \frac{1}{\lambda}\frac{\ell_\beta}{A_g}$, it follows that:

$$\|R(\lambda)^{-1}\| = \frac{\ell_\beta}{\lambda A_g} + O(1) = \frac{1}{\mu_1(\lambda)}.$$

The expansion for the logarithm applied to $\frac{1}{\mu_1(\lambda)} = \frac{\ell_\beta}{\lambda A_g} + u(\lambda)$, with $u(\lambda) = O(1)$ implies that:

$$\log\left(\frac{1}{\mu_1(\lambda)}\right) = -\log(\mu_1(\lambda)) = \log\left(\frac{\ell_\beta}{A_g}\right) - \log\lambda + O(\lambda) \text{ as } \lambda \to 0^+.$$

This finishes the proof of equation (3.12).   □

Putting everything together we obtain the following splitting formula:

Theorem 3.1. *For the relative determinant of the Laplace operator on a surface with cusps (M, g), and the regularized determinant of the Dirichlet-to-Neumann operator R on $\Sigma_\beta = \{\beta\} \times S^1 \subset M$, we have the following splitting formula:*

$$\frac{\det(\Delta_g, \Delta_{Z_\beta,D})}{\det(\Delta_{M_\beta,D})} = \frac{A_g}{\ell_\beta}\det{}^* R,$$

where A_g denotes the area of M and ℓ_β denoted the length of Σ_β.

Proof. We start with the splitting formula for $\lambda > 0$, and $\lambda \in \rho(\Delta_g)$:

$$\log\det(\Delta_g + \lambda, \Delta_{Z_\beta,D} + \lambda) - \log\det(\Delta_{M_\beta,D} + \lambda) = \log\det R(\lambda).$$

From the previous lemmas we have that:

$$\log\det(\Delta_g, \Delta_{Z_\beta,D}) + \log\lambda - \log\det(\Delta_{M_\beta,D} + \lambda) + o(1)$$
$$= \log\mu_1(\lambda) + \log\det{}^* R + o(1)$$
$$= \log\lambda + \log\left(\frac{A_g}{\ell_\beta}\right) + \log\det{}^* R + o(1).$$

Letting $\lambda \to 0$, we finally obtain:

$$\log\det(\Delta_g, \Delta_{Z_\beta,D}) - \log\det(\Delta_{M_\beta,D}) = \log\left(\frac{A_g}{\ell_\beta}\right) + \log\det{}^* R.$$

□

Remark 3.3. If we further decompose the operator $\Delta_{Z_\beta,D}$ as $\Delta_{\beta,0} \oplus \Delta_{Z_{\beta,1}}$ we obtain:

(3.13)
$$\log\det(\Delta_g, \Delta_{\beta,0}) - \log\det(\Delta_{Z_{\beta,1}}) - \log\det(\Delta_{M_\beta,D}) = \log\left(\frac{A_g}{\ell_\beta}\right) + \log\det{}^* R.$$

4. Compactness

In this section we use results of Müller in [20] and of OPS in [24]. We refer
the reader to these references for all the details.

In [24] OPS proved that sets of isospectral isometry classes of metrics on
closed surfaces are sequentially compact in the C^∞-topology. Let us recall
some of the main steps of the proof in the case $\chi(M) < 0$. In that setting, two
metrics g_1 and g_2 are called isospectral if the spectra of the Laplacians Δ_{g_1}
and Δ_{g_2} are the same including multiplicities. In particular, the regularized
determinant and the heat invariants have the same values at g_1 and g_2.

To define the notion of convergence they fix a background metric g_0.
Associated to g_0, there is the Levi-Civita connection and the covariant
derivative that allow to differentiate in the whole tensor algebra. A sequence
of metrics $\{g_n\}_{n\in\mathbb{N}}$ converges to a metric g in C^k if $\|g_n - g\|_{C^k} \to 0$, as
$n \to \infty$. A sequence of isometry classes of metrics $\hat{g}_n$ converges to an isom-
etry class $\hat{g}$ if there are representatives $h_n \in \hat{g}_n$, $h \in \hat{g}$, such that $h_n \to h$,
as $n \to \infty$. Now, let $\{\rho_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $C^k(M)$ and let
σ be a fixed metric on M. Then $\rho_n\sigma \to \rho\sigma$ in C^k as metrics if and only if
$\rho_n \to \rho$ in C^k as functions. Moreover, if the metrics $\sigma_n \to \sigma$ in C^∞, and the
function $\rho_n \to \rho$ in C^k, then the metrics $\rho_n\sigma_n \to \rho\sigma$ in C^k.

After defining convergence and isospectrality, OPS consider a sequence of
isospectral isometry classes of metrics $\{\hat{g}_n\}_{n\in\mathbb{N}}$ and pick representatives g_n.
For each g_n there is a metric of constant curvature τ_n such that $g_n = e^{2\varphi_n}\tau_n$.
In this way, they associate to each $\hat{g}_n$ a hyperbolic isometry class $\hat{\tau}_n$. They
use that for each n, $\det\Delta_{\hat{\tau}_n} \geq \det\Delta_{\hat{g}_n} = \text{constant} > 0$ and Mumford's com-
pactness theorem to prove that there exists a subsequence of $\{\hat{\tau}_n\}_{n\in\mathbb{N}}$ that
converges to an element $\hat{\tau}$ in the moduli space. To have compactness of the
conformal factors $\{\varphi_n\}_{n\in\mathbb{N}}$, they prove that for each $k \in \mathbb{N}$ the kth Sobolev
norms $\|\varphi_n\|_k$ are uniformly bounded. Compactness in the C^∞-topology fol-
lows then from Rellich's Lemma and the Sobolev embedding theorems on
M. The constant value of the determinant and Polyakov's formula for the
regularized determinant (see [23]) are used to prove uniform boundedness of
the first Sobolev norm. For the higher Sobolev norms, they use the constant
values of the heat invariants.

If we restrict the proof of OPS to a conformal class, we only need the con-
stant values of the determinant of the Laplacians and of the heat invariants
associated to the metrics.

Now, let (M, g) be a surface of fixed genus p and fixed number of cusps
m. We usually take $m = 1$ to make the proofs simpler but the statements
hold for general m. We take g as the background Riemannian metric. Let

us decompose M as $M = M_0 \cup_{\Sigma_\alpha} Z_\alpha$ where M_0 is compact with boundary Σ_α and the metric on $Z_\alpha = [\alpha, \infty) \times S^1$ is the usual hyperbolic metric.

Let K be a compact subset of M and let us define the "K-compactly supported" conformal class of g as the set

$$(4.1) \qquad [g]_K = \{\mathrm{e}^{2\varphi}g \mid \varphi \in C_c^\infty(M), \ \mathrm{supp}\,\varphi \subset K\}.$$

Since K is compact, there is a $\beta \geq \alpha$ such that $K \subset M_\beta$ and such that $K \cap \Sigma_\beta = \emptyset$. From now on, when K is given we consider β fixed. Then for every metric in $h \in [g]_K$, (M, h) is a surface with cusps and the cusp is contained in $M \setminus M_\beta$.

For $s > 0$ and $f \in H^s(M, g)$, the s-Sobolev norm is given by $\|f\|_{H^s} := \|(\Delta + I)^{s/2}f\|_{L^2}$.

Since we restrict to a conformal class the notion of convergence of metrics reduces to the convergence of the conformal factors:

Definition 4.1. A sequence of metrics $\{g_n\}_{n \in \mathbb{N}}$, with $g_n = \mathrm{e}^{2\varphi_n}g$ converges to a metric h in C^k if and only if the sequence of function $\{\varphi_n\}_{n \in \mathbb{N}}$ converges to a function φ in C^k.

As we explained in the introduction, in the setting of surfaces with cusps the concept of isospectrality is not enough to study inverse problems and it should be replaced by the concept of isoresonance. This is motivated by the close relation between eigenvalues and resonances. The traditional approach to resonances defines them as the poles of certain meromorphic extension of the resolvent. However, in this paper we rather work with another approach. We use the definition of the resonance set as it is given in [20] because our work relies on trace formulae stated there.

Let us recall the precise definition of the resonance set: In [20, p.287], Müller starts assigning to each $\eta \in \mathbb{C}$ a multiplicity $m(\eta)$:

1. If $\mathrm{Re}(\eta) \geq 1/2$ and $\eta \neq 1/2$, $m(\eta)$ is the dimension of the eigenspace of Δ_g for the eigenvalue $\eta(1 - \eta)$.

2. If $\mathrm{Re}(\eta) < 1/2$. Let $E_{\eta(1-\eta)}$ denote the eigenspace of Δ_g for the eigenvalue $\eta(1 - \eta)$. If $\phi(s)$ has a pole at η of order n, then $m(\eta) = \dim(E_{\eta(1-\eta)}) + n$. If $\phi(s)$ has a zero at η of order $\tilde{n}$, then $m(\eta) = \dim(E_{\eta(1-\eta)}) - \tilde{n}$. By the spectral properties of the Laplacian and the properties of the scattering phase, we know that $m(\eta) \geq 0$.

3. For $\eta = \frac{1}{2}$ we have that $m(\frac{1}{2}) = \frac{1}{2}(\text{Tr}(C(\frac{1}{2})) + m) + 2\dim(E_{1/4})$, where $E_{1/4}$ is the 1/4-eigenspace.

4. In any other case, $m(\eta) = 0$.

Definition 4.2. [20]. The resonance set of Δ_g is the set of all $\eta \in \mathbb{C}$ such that $m(\eta) > 0$. Each element in the set is counted with its multiplicity.

In this way, the resonance set is the union of the poles and some of the zeros of the scattering phase $\phi(s)$ in the half-plane $\{s \mid \text{Re}(s) < 1/2\}$, the set $\{s_j \in \mathbb{C} \mid s_j(1 - s_j) \text{ is an eigenvalue } of\,\Delta_g\}$ and $\{\frac{1}{2}\}$. Each element carries its multiplicity. In particular, the definition implies that the resonance set carries the information of the value of $\text{Tr}(C(\frac{1}{2}))$. Since $C(\frac{1}{2})$ is a real symmetric matrix with $C(\frac{1}{2})^2 = I$, its eigenvalues are ± 1. Then $\text{Tr}(C(\frac{1}{2})) = 2\ell - m$, where ℓ is the algebraic multiplicity of the eigenvalue $+1$. In this way, the resonance set determines the value of ℓ and the algebraic multiplicity of the eigenvalue -1. Therefore it determines $\phi(\frac{1}{2})$.

We are now ready to define isoresonant surfaces with cusps:

Definition 4.3. Two cusp metrics g_1 and g_2 on M are isoresonant if their resonance sets are the same including the multiplicities.

Remark 4.1. The scattering phases of two isoresonant surfaces with cusps (M, g_1) and (M, g_2) are the same. This follows from Theorem 3.31 in [20], that expresses the determinant of the scattering matrix as the Weierstrass product:

$$(4.2) \qquad \phi(s) = \phi(1/2)q^{s-1/2} \prod_{\rho} \frac{s - 1 + \bar{\rho}}{s - \rho},$$

where ρ runs over all poles of $\phi(s)$, counted with the order and q is a well-determined constant. Indeed, equation (5.17) in [20] implies that the constant q is determined by the resonance set.

Proposition 4.1. *Let (M, g_1) and (M, g_2) be two surfaces with cusps that are isoresonant. Let $\bar{\Delta}_{a,0}$ be the Laplacian given in Definition 2.1 for any $a = (a_1, \ldots, a_m)$ with $\min\{a_j, 1 \le j \le m\}$ big enough. Then the corresponding relative heat traces coincide, i.e.,*

$$(4.3) \qquad \text{Tr}(e^{-t\Delta_{g_1}} - e^{-t\bar{\Delta}_{a,0}}) = \text{Tr}(e^{-t\Delta_{g_2}} - e^{-t\bar{\Delta}_{a,0}}),$$

and so do the relative determinants:

(4.4) $$\det(\Delta_{g_1}, \bar{\Delta}_{a,0}) = \det(\Delta_{g_2}, \bar{\Delta}_{a,0}).$$

Proof. The proof of this proposition follows straight forward from the results of [20]. Let (M, g_1) and (M, g_2) be isoresonant. The trace formula for the relative heat operators in ([20, equation (2.2)]) establishes:

$$\mathrm{Tr}(e^{-t\Delta_g} - e^{-t\bar{\Delta}_{a,0}}) = \int_M (K_g(z, z, t) - \sum_{j=1}^m p_{a_j}(z, z, t))dA_g(z)$$

$$= \sum_k e^{-\lambda_k t} - \frac{1}{4\pi} \int_{-\infty}^{\infty} e^{-(1/4+\lambda^2)t} \frac{\phi'}{\phi}(1/2 + i\lambda)d\lambda$$

(4.5) $$+ \frac{1}{4} e^{-t/4}(\mathrm{Tr}(C_g(1/2)) + m) + \frac{e^{-t/4}}{\sqrt{4\pi t}} \sum_{j=1}^m \log(a_j),$$

where $p_{a_j}(z, z, t))$ is given by equation (2.1). The term $\frac{m}{4} e^{-t/4}$ on the right hand side of (4.5) is missing in ([20, equation (2.2)]) because of a missprint. This term comes from the boundary condition of the model operator $\bar{\Delta}_{a,0}$. Now, by Theorem 5.11 in [20] the integral that involves the logarithmic derivative of the scattering matrix can be rewritten as follows:

(4.6)
$$-\frac{1}{4\pi} \int_{-\infty}^{\infty} e^{-(1/4+\lambda^2)t} \frac{\phi'}{\phi}(1/2 + i\lambda)d\lambda = \frac{-\log(q)}{(4\pi)^{3/2}} \frac{e^{-t/4}}{\sqrt{t}}$$
$$+ \frac{1}{4} \sum_{\rho} n(\rho)\{e^{-t\rho(1-\rho)}\mathrm{Erfc}((\sqrt{t}(1/2 - \rho)) + e^{-t\bar{\rho}(1-\bar{\rho})}\mathrm{Erfc}(\sqrt{t}(1 - \bar{\rho})))\},$$

where ρ runs over all zeros and poles of $\phi(s)$ in $\mathrm{Re}(s) < 1/2$, $n(\rho)$ denotes either the order of the pole ρ or the negative of the order of the zero ρ, q is the same constant as in equation (4.2), and Erfc is the complementary error function, see [20, (5.13)]. In addition, it is clear that the eigenvalues of the Laplacians coincide. Then, equations (4.5) and (4.6) imply equation (4.3). Equation (4.4) follows straightforward from the definition of the relative determinant. □

We are ready to state the main theorem of this section:

Theorem 4.1. *Let (M, g) be a surface with cusps and with $\chi(M) < 0$, let $K \subset M$ be a fixed compact subset of M and let $[g]_K$ be the K-compactly*

supported conformal class of g. Then isoresonant sets in $[g]_K$ are compact in the C^∞-topology.

The proof of the theorem consists in reducing to the compact case and apply the result of OPS in [24] restricted to a conformal class.

Proof. First of all we need to compactify M to a Riemannian manifold that contains K isometrically. It is convenient at this point to change coordinates in the cusp, we first identify $z = (y, x) \in [\alpha, \infty) \times S^1$ with $z = x + iy \in S^1 \times i[\alpha, \infty) \subset \mathbb{C}$ and then we apply the transformation $z \to w = e^{iz}$. Then Z_α becomes $\{w \in \mathbb{C} : 0 < |w| \le e^{-\alpha}\} =: D_{e^{-\alpha}}^*$ and the metric on it becomes

$$g|_{D_{e^{-\alpha}}^*} = \log(|w|^{-1})^{-2}|w|^{-2}|dw|^2.$$

Let us keep the old notation in these new coordinates. Then for any $b \ge \alpha$, $M_b = M_0 \cup (D_{e^{-\alpha}}^* \setminus D_{e^{-b}}^*) \cup \Sigma_b$ and we could also denote $D_{e^{-b}}^*$ by Z_b. Let $\beta > \alpha$ be fixed, as it was explained after equation (4.1). Let $f \in C^\infty(M)$ satisfy:

(4.7)
$$f(w) := \begin{cases} |\log(|w|)||w| & \text{if } w \in D_{e^{-\beta-2}}^* (\cong Z_{\beta+2}) \\ 1 & \text{if } w \in M_{\beta+1}, \end{cases}$$

and put:

(4.8)
$$\sigma = f(z)^2 \cdot g.$$

Then take $\widetilde{M} = M \cup \{0\}$ the one-point compactification of M (m-point compactification if M has m cusps). The metric σ on M extends to a smooth metric on $\widetilde{M}$ which we denote again by σ. Thus $(\widetilde{M}, \sigma)$ is a closed manifold that contains M_β isometrically and that has the same genus as M. In particular, $K \subset M_0 \cup (D_{e^{-\alpha}}^* \setminus D_{e^{-\beta}}^*)$.

Now let $\{g_n\}_{n\in\mathbb{N}} \subset [g]_K$ be a sequence of isoresonant metrics. Notice that since the metrics in the sequence are isoresonant, they have all the same zeroth heat invariant, therefore their areas A_{g_n} have the same value. Since $g_n \in [g]_K$, there exists a function $\varphi_n \in C_c^\infty(M)$ such that $g_n = e^{2\varphi_n}g$ and $\operatorname{supp}\varphi_n \subset K$, for each $n \in \mathbb{N}$. Now put:

$$\widetilde{g}_n := e^{2\varphi_n}\sigma.$$

Then the metrics $\widetilde{g}_n$ are conformal to σ on $\widetilde{M}$. The fact that $K \subsetneq M_{\beta+1} = M \setminus D_{e^{-\beta-1}}^*$ and $\sigma|_{M_{\beta+1}} = g|_{M_{\beta+1}}$ imply that the values $A_{g_n} - A_g(D_{e^{\beta+1}}^*)$ are

constant. Then the areas $A_{\widetilde{g}_n}$ of $(\widetilde{M}, \widetilde{g}_n)$ have all the same value; this follows from:

$$A_{\widetilde{g}_n} = A_{g_n} - A_g(D^*_{\mathrm{e}^{\beta+1}}) + A_\sigma(\widetilde{M} \setminus M_{\beta+1}).$$

Therefore we can renormalize the metrics $\widetilde{g}_n$ such that $A_{\widetilde{g}_n} = 1$.

In addition, the definitions of K and σ, the condition $\mathrm{supp}\,\varphi_n \subset K$ for all $n \in \mathbb{N}$, and the locality of the Laplacians Δ_g and Δ_σ imply that

(4.9)
$$\|\varphi_n\|^2_{H^k(\widetilde{M},\sigma)} = \|\varphi_n\|^2_{H^k(M,g)}.$$

Notice that compactness of $\{\varphi_n\}_{n\in\mathbb{N}}$ in $C^\infty(\widetilde{M},\sigma)$ together with $\mathrm{supp}\,\varphi_n \subset K \Subset M$ for all $n \in \mathbb{N}$, imply compactness of $\{\varphi_n\}_{n\in\mathbb{N}}$ in $C^\infty(M,g)$. Therefore, in order to prove compactness in $C^\infty(M,g)$ we need to prove uniform boundedness of the sequence $\{\|\varphi_n\|_{H^k(\widetilde{M},\sigma)}\}_{n\in\mathbb{N}}$ of the kth Sobolev norms for each $k \geq 1$, i.e., we need to prove that

$$\|\varphi_n\|_{H^k(\widetilde{M},\sigma)} \leq C(k), \quad \text{for all } n \in \mathbb{N},$$

where $C(k)$ is a constant that may depend on k.

In Lemmas 4.1 and 4.2 we prove that if $\{g_n\}_{n\in\mathbb{N}}$ is isoresonant then $\det \Delta_{\widetilde{g}_n}$ is constant and the heat invariants of the metrics $\widetilde{g}_n$ are the same for all n.

Then the theorem follows from the results of OPS in [24] since the uniform bound of the Sobolev norms of the functions $\{\varphi_n\}_{n\in\mathbb{N}}$ in $(\widetilde{M},\sigma)$, restricted to our case, only requires that the determinants, $\det \Delta_{\widetilde{g}_n}$, the areas, $A_{\widetilde{g}_n}$, and the heat invariants, $a_j(g_n)$, are constants independent of n. $\square$

Lemma 4.1. *Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of isoresonant metrics in a conformal class $[g]_K$. Let $\{\widetilde{g}_n\}_{n\in\mathbb{N}}$ be the associated sequence of metrics on $\widetilde{M}$ defined above. Then the regularized determinants $\det \Delta_{\widetilde{g}_n}$ are constant, i.e., their value is independent of n.*

Proof. Let h be any metric in $[g]_K$. Remember the construction we did in the proof of Theorem 4.1. Recall that $\widetilde{M} = M \cup \{0\}$, the one-point compactification of M, is endowed with a smooth Riemannian metric σ obtained from equation (4.8). Let $\widetilde{h}$ be the metric on $\widetilde{M}$ corresponding to h via the process described in the proof of Theorem 4.1. Then for the relative determinant of $(\Delta_h, \Delta_{\beta,0})$ and the determinant of $\Delta_{\widetilde{h}}$ we have the following splitting

formulas:

$$\log \det(\Delta_h, \Delta_{\beta,0}) - \log \det \Delta_{Z_\beta,1} - \log \det \Delta_{(M_\beta,h),D}$$
$$= \log \left(\frac{A_h(M)}{\ell(\Sigma_\beta, h)} \right) + \log \det{}^* R_h$$

and

$$\log \det \Delta_{(\widetilde{M},\widetilde{h})} - \log \det \Delta_{(M_\beta,\widetilde{h}),D} - \log \det \Delta_{(\widetilde{M}\backslash M_\beta,\widetilde{h}),D}$$
$$= \log \left(\frac{A_{\widetilde{h}}(\widetilde{M})}{\ell(\Sigma_\beta, \widetilde{h})} \right) + \log \det{}^* R_{\widetilde{h}},$$

where the first formula was proved in Theorem 3.1 (equation (3.13)), and the second formula is the well known splitting formula for a closed surface, as in [5]. Subtracting the equations we obtain:

$$\log \det \Delta_{(\widetilde{M},\widetilde{h})} - \log \det(\Delta_h, \Delta_{\beta,0}) + \log \det \Delta_{Z_\beta,1} - \log \det \Delta_{(\widetilde{M}\backslash M_\beta,\widetilde{h}),D}$$
$$= \log \left(\frac{A_{\widetilde{h}}(\widetilde{M})}{\ell(\Sigma_\beta, \widetilde{h})} \right) - \log \left(\frac{A_h(M)}{\ell(\Sigma_\beta, h)} \right) + \log \det{}^* R_{\widetilde{h}} - \log \det{}^* R_h.$$

From the definition of f we have that $\widetilde{h} = h$ on $M_{\beta+1}$, and $f \equiv 1$ in a neighborhood of Σ_β. So we have that $\ell(\Sigma_\beta, h) = \ell(\Sigma_\beta, \widetilde{h})$.

Now, let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of isoresonant metrics in $[g]_K$ satisfying the hypothesis of this lemma, and let $\{\widetilde{g}_n\}_{n\in\mathbb{N}}$ be the corresponding sequence in $\widetilde{M}$. If we take $h = g_n$, the Dirichlet-to-Neumann operators are the same for all n. To see this, notice that given a function $u \in C^\infty(\Sigma_\beta)$, the unique solution to the problem $\Delta_g \widetilde{u} = 0$ on $M \setminus \Sigma_\beta$ with $\widetilde{u}|_{\Sigma_\beta} = u$ will also be a solution of $\Delta_{g_n} \widetilde{u} = e^{-2\varphi_n} \Delta_g \widetilde{u} = 0$ on $M \setminus \Sigma_\beta$ satisfying the same boundary condition. Then, it follows from Lemma 3.2, Proposition 3.1, and the fact that the metrics coincide in a neighborhood of the curve Σ_β that the operators R_{g_n} are the same for all n. Therefore, $\det{}^* R_{g_n} = c_1$, for all $n \in \mathbb{N}$. The same argument applied to the sequence $\{\widetilde{g}_n\}_{n\in\mathbb{N}}$ gives $\det{}^* R_{\widetilde{g}_n} = c_2$, for all $n \in \mathbb{N}$. In this way, we obtain:

$$\log \det \Delta_{(\widetilde{M},\widetilde{g}_n)} - \log \det(\Delta_{g_n}, \Delta_{\beta,0}) - \log \det \Delta_{(\widetilde{M}\backslash M_\beta,\widetilde{g}_n),D}$$
$$= \log(A_{\widetilde{g}_n}(\widetilde{M})) - \log(A_{g_n}(M)) + c$$

where c is a constant that does not depend on n.

Recall that $A_{g_n}(M)$, $A_{\widetilde{g}_n}(\widetilde{M})$, $\det(\Delta_{g_n}, \Delta_{\beta,0})$ are constants independent of n. Moreover, $\widetilde{g}_n|_{\widetilde{M}\backslash M_\beta} = \sigma|_{\widetilde{M}\backslash M_\beta}$. Therefore $\det \Delta_{(\widetilde{M}\backslash M_\beta, \widetilde{g}_n), D}$ is also constant. Thus,

$$\log \det(\Delta_{\widetilde{g}_n}) = \text{constant}.$$

$\square$

Lemma 4.2. *The heat invariants corresponding to the metrics of the sequence $\{\widetilde{g}_n\}_{n\in\mathbb{N}}$ are the same for any $n \in \mathbb{N}$ if we start with an isoresonant sequence $\{g_n\}_{n\in\mathbb{N}}$.*

Proof. Let h be any of the metrics g_n that we are considering. Let us start by constructing the kernel of a parametrix H_h for the heat operator $\mathrm{e}^{-t\Delta_h}$ on the surface with cusps (M, h), as it was done in [19, p.245]. Namely we use the standard method of gluing the heat kernel on the complete hyperbolic cusp $(0, \infty) \times S^1$, denoted by K_1 and independent of the choice of h, with the heat kernel on $(\widetilde{M}, \widetilde{h})$, denoted by $K_{2,\widetilde{h}}$, restricted to $M_{\beta+2}$. Let us recall briefly the definition of the gluing functions: For any two constants $1 < b < c$, let $\phi_{(b,c)}$ be such that $\phi_{(b,c)}(y, x) = 0$ for $y \le b$, and $\phi_{(b,c)}(y, x) = 1$ for $y \ge c$. Let $\psi_1 = \phi_{(\beta+5/4, \beta+2)}$, and $\psi_2 = 1 - \psi_1$; then $\{\psi_1, \psi_2\}$ is a partition of unity on $[\beta + 1, \beta + 2] \times S^1$. Let $\phi_1 = \phi_{(\beta, \beta+1)}$ and $\phi_2 = 1 - \phi_{(\beta+5/2, \beta+3)}$, so that $\phi_i = 1$ on the support of ψ_i, $i = 1, 2$. Then the function:

$$H_h(z, z', t) = \phi_1(z)K_1(z, z', t)\psi_1(z') + \phi_2(z)K_{2,\widetilde{h}}(z, z', t)\psi_2(z').$$

is a parametrix, see [19]. It is not difficult to prove that there exist constants $C, c > 0$ such that:

$$\int_M |K_h(z, z, t) - H_h(z, z, t)| dA_h(z) \le C\mathrm{e}^{-(c/t)}$$

for $0 < t \le 1$, see for example [1]. Then for small t we can replace the heat kernel K_h for the parametrix H_h. Let $p_{\beta+1}(z, z', t)$ be as in equation (2.1). We can obtain the analog to equation (8.14) in [19, p.283], exactly in the same way as it is done there, this is:

$$\int_M (K_h(z, z, t) - p_{\beta+1}(z, z, t))\, dA_h(z)$$

$$= \int_{Z_{\beta+1}} (K_1(z, z, t) - p_{\beta+1}(z, z, t))\, dA_h(z)$$

(4.10)
$$+ \int_{M_{\beta+1}} K_{2,\widetilde{h}}(z, z, t)\, dA_h(z) + O(\mathrm{e}^{-c/t}), \text{ as } t \to 0.$$

For the convenience of the reader, we give some explicit steps of the proof of equation (4.10). In order to keep the notation simple, if there is no place to confusion, we drop the variable inside the integrals. From the definition of the cutoff functions we have that

$$\int_M (K_h - p_{\beta+1}) \, dA_h = \int_{Z_{\beta+1}} (K_1 \psi_1 - p_{\beta+1}) \, dA_h$$
$$+ \int_{M_{\beta+2}} K_{2,\widetilde{h}} \psi_2 \, dA_h + O(e^{-c/t}), \text{ as } t \to 0.$$

On the other hand,

$$\int_{Z_{\beta+1}} (K_1 \psi_1 - p_{\beta+1}) \, dA_h$$
$$= \int_{Z_{\beta+1}} (K_1 - p_{\beta+1}) \, dA_h - \int_{[\beta+1,\beta+2] \times S^1} K_1 \psi_2 \, dA_h$$

Therefore,

$$\int_M (K_h - p_{\beta+1}) \, dA_h$$
$$= \int_{Z_{\beta+1}} (K_1 - p_{\beta+1}) \, dA_h + \int_{M_{\beta+1}} K_{2,\widetilde{h}} \, dA_h$$
$$- \int_{[\beta+1,\beta+2] \times S^1} (K_1 - K_{2,\widetilde{h}}) \psi_2 \, dA_h + O(e^{-c/t}), \text{ as } t \to 0.$$

Now, Proposition 3.24 in [19] implies that the coefficients of the asymptotic expansions of $K_1(z, z, t)$ and $K_2(z, z, t)$, as $t \to 0$, coincide on $[\beta + 1, \beta + 2] \times S^1$. Then equation (4.10) follows.

For a metric $\widetilde{g}_n$ on $\widetilde{M}$ the heat invariants are, by definition, the coefficients in the asymptotic expansion of the trace of the heat kernel as $t \to 0$:

$$\int_{\widetilde{M}} K_{2,\widetilde{g}_n}(z, z, t) \, dA_{\widetilde{g}_n}(z) \sim \frac{1}{t} \sum_{j=0}^{\infty} a_j(\widetilde{g}_n) t^j, \qquad \text{as } t \to 0.$$

The goal of this lemma is to prove that $a_j(\widetilde{g}_n) = a_j(\widetilde{g}_m)$ for any $n, m \in \mathbb{N}$, and for all $j \geq 0$. This will follow from the equality of the asymptotic expansions for small values of t of the integrals

(4.11) $\displaystyle\int_{\widetilde{M}} K_{2,\widetilde{g}_n}(z, z, t) \, dA_{\widetilde{g}_n}(z)$ and $\displaystyle\int_{\widetilde{M}} K_{2,\widetilde{g}_m}(z, z, t) \, dA_{\widetilde{g}_m}(z),$

for any $n, m \in \mathbb{N}$. We can split the integral over $\widetilde{M}$ as an integral over $M_{\beta+1}$ and one over $\widetilde{M} \setminus M_{\beta+1}$. Given two metrics g_n and g_m as in the statement of the lemma, we have that on $\widetilde{M} \setminus M_{\beta+1}, \widetilde{g}_n = \widetilde{g}_m$. Since relative to any coordinate system, the coefficients of the asymptotic expansion of the heat kernel are given by universal polynomials in terms of the metric tensor and its covariant derivatives, we have that $a_j(z, \widetilde{g}_n) = a_j(z, \widetilde{g}_m)$, for $z \in \widetilde{M} \setminus M_{\beta+1}$. On $\widetilde{M} \setminus M_{\beta+1}$ we have that $dA_{\widetilde{g}_n} = dA_{\widetilde{g}_m}$. Therefore:

$$\int_{\widetilde{M} \setminus M_{\beta+1}} K_{2,\widetilde{g}_n}(z, z, t) \, dA_{\widetilde{g}_n}(z) = \int_{\widetilde{M} \setminus M_{\beta+1}} K_{2,\widetilde{g}_m}(z, z, t) \, dA_{\widetilde{g}_n}(z).$$

By assumption, K_1 and $p_{\beta+1}$ are independent of g_n and g_m. Therefore, by equation (4.10) we have:

$$\int_{M_{\beta+1}} K_{2,\widetilde{g}_n} \, dA_{\widetilde{g}_n} - \int_{M_{\beta+1}} K_{2,\widetilde{g}_m} \, dA_{\widetilde{g}_m} \sim_{t \to 0} \int_M (K_{g_n} - p_{\beta+1}) \, dA_{g_n}$$

$$- \int_{Z_{\beta+1}} (K_1 - p_{\beta+1}) \, dA_{g_n} - \int_M (K_{g_m} - p_{\beta+1}) \, dA_{g_m}$$

$$+ \int_{Z_{\beta+1}} (K_1 - p_{\beta+1}) \, dA_{g_m}$$

$$= \int_M (K_{g_n} - p_{\beta+1}) \, dA_{g_n} - \int_M (K_{g_m} - p_{\beta+1}) \, dA_{g_m} = 0,$$

where the last equality follows from the fact that the metrics are isoresonant and from Proposition 4.1. So, we have proved that the asymptotic expansions as $t \to 0$ for the integrals in (4.11) are the same. From the definition of the heat invariants it follows that:

$$a_j(\widetilde{g}_n) = a_j(\widetilde{g}_m), \quad \text{for all } j \geq 0, \text{ and } n, m \in \mathbb{N}.$$

$\square$

5. Boundedness of the relative determinant as function on the moduli space of hyperbolic surfaces with cusps

In this section we restrict to surfaces with cusps that are hyperbolic. Let (M, τ) be a Riemann surface of genus q with m cusps, where τ is a hyperbolic metric of constant negative unitary curvature. To each element $[\tau] \in \mathcal{M}_{q,m}$ we associate the relative determinant $\det(\Delta_\tau, \bar{\Delta}_{1,0})$, where the operator $\bar{\Delta}_{1,0}$ is given in Definition 2.1 with $a = 1$ and it acts on a subspace of

$\oplus_{j=1}^m L^2([1,\infty), y_j^{-2}dy_j)$. If (M, τ) can be decomposed as $M = M_0 \cup Z_{a_1} \cup \cdots Z_{a_m}$, with $a_j \geq 1$; then the difference $e^{-t\Delta_\tau} - e^{-\bar\Delta_{1,0}}$ is taken in the extended L^2 space given by:

$$L^2(M, dA_\tau) \oplus \oplus_{j=1}^m L^2([1,a_j], y^{-2}dy)$$
$$= L^2(M_0, dA_\tau) \oplus \oplus_{j=1}^m (L_0^2(Z_{a_j}) \oplus L^2([1,\infty), y^{-2}dy)).$$

Our result is:

Theorem 5.1. *As function on the moduli space of hyperbolic surfaces of fixed genus q with m cusps, $\mathcal{M}_{q,m}$, the relative determinant $\det(\Delta_\tau, \bar\Delta_{1,0})$ tends to zero as $[\tau]$ approaches the boundary; where by boundary we mean the set $\overline{\mathcal{M}_{q,m}} \setminus \mathcal{M}_{q,m}$.*

This theorem implies that the relative determinant is bounded as a function on the moduli space. In addition, it also implies that it is a proper function.

We use Selberg's trace formula and the work of Bers in [2] and of Jorgenson and Lundelius in [12]. In [12] the authors define a hyperbolic determinant for Laplacians on hyperbolic Riemann surfaces of finite volume, non-connected in general. We compare both determinants and use their results together with the results in [2] about degeneration of surfaces.

Let us start by recalling Selberg's trace formula [27] as it is presented by Iwaniec in [10], applied to the function $h(r) = e^{-t(1/4+r^2)}$ and its Fourier transform $g(u) = \frac{1}{\sqrt{4\pi t}} e^{-t/4} e^{-(u^2/4t)}$.

Let Γ be a Fuchsian group of the first kind. Let $\Gamma \setminus \mathbb{H} = M$ be the associated surface, let Δ_τ be the Laplacian on M and let $\lambda_j = \frac{1}{4} + r_j^2$ be the sequence of eigenvalues of Δ_τ. We do not include the contribution of the elliptic elements, because we consider groups without elliptic elements. In this case Selberg's trace formula (see [27] or [10]) applied to the heat operator takes the form:

(5.1)
$$\sum_j e^{-t(1/4+r_j^2)} - \frac{1}{4\pi}\int_{\mathbb{R}} e^{-t(1/4+\lambda^2)}\frac{\phi'}{\phi}\left(\frac{1}{2}+i\lambda\right)d\lambda + \frac{e^{-t/4}}{4}\mathrm{Tr}\left(C\left(\frac{1}{2}\right)\right)$$
$$= \frac{\mathrm{Area}(M)}{4\pi}\int_{\mathbb{R}} e^{-t(1/4+\lambda^2)}\lambda\tanh(\pi\lambda)d\lambda$$

$$+ \frac{\mathrm{e}^{-t/4}}{\sqrt{4\pi t}} \sum_{k=1}^{\infty} \sum_{\{\gamma\}_{\Gamma}} \frac{\ell(\gamma)}{2\sinh\left(k\ell(\gamma)/2\right)} \mathrm{e}^{-(k\ell(\gamma))^2/4t}$$

$$- \frac{m}{\pi} \int_{\mathbb{R}} \mathrm{e}^{-t(1/4+\lambda^2)} \frac{\Gamma'}{\Gamma}(1+i\lambda)d\lambda + \frac{m}{4}\mathrm{e}^{-t/4} - m\log(2)\frac{\mathrm{e}^{-t/4}}{\sqrt{4\pi t}},$$

where the sum runs over the primitive hyperbolic conjugacy classes γ with length $\ell(\gamma)$, m is the number of inequivalent cusps, and as before $C(s)$ is the scattering matrix and $\phi(s) = \det C(s)$.

In the notation of [12] the hyperbolic heat trace $\mathrm{HTr}K_M(t)$ and the regularized trace $\mathrm{STr}K_M(t)$ are given by:

$$\mathrm{HTr}\,K_M(t) = \frac{\mathrm{e}^{-t/4}}{\sqrt{16\pi t}} \sum_{k=1}^{\infty} \sum_{\{\gamma\}\Gamma} \frac{\ell(\gamma)}{\sinh\left(k\ell(\gamma)/2\right)} \mathrm{e}^{-(k\ell(\gamma))^2/4t},$$

$$\mathrm{STr}K_M(t) = \mathrm{HTr}\,K_M(t) + \mathrm{Area}(M)K_{\mathbb{H}}(t,0),$$

where

$$K_{\mathbb{H}}(t,0) = \frac{1}{4\pi} \int_{\mathbb{R}} \mathrm{e}^{-t(1/4+\lambda^2)} \lambda \tanh(\pi\lambda)d\lambda.$$

With these expressions, they define the hyperbolic zeta function and the hyperbolic determinant as

$$\zeta_{M,\mathrm{hyp}}(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} (\mathrm{STr}K_M(t) - d)t^{s-1}dt \quad \text{and}$$

$$\det_{\mathrm{hyp}}\Delta_\tau := \exp(-\zeta'_{\mathrm{hyp}}(0)),$$

where d is the number of connected components of M as well as the dimension of $\mathrm{Ker}(\Delta_\tau)$. Let $Z(s)$ be the Selberg zeta function associated to $M = \Gamma \setminus \mathbb{H}$, then there is the following relation between the hyperbolic determinant and the derivative at $s = 1$ of the Selberg zeta function:

$$\det_{\mathrm{hyp}}\Delta_\tau = Z'_M(1)\mathrm{e}^{\chi(M)(-2\zeta'_{\mathrm{R}}(-1)+(1/4)-(\log(2\pi)/2))},$$

where ζ_{R} denotes the Riemann zeta function. This formula was proven for the hyperbolic determinant on Riemann surfaces of finite volume by JL in [12], as a generalization of the corresponding formula on compact Riemann surfaces given in [9, 26].

We want to see the relation between the hyperbolic determinant $\det_{\mathrm{hyp}}$ Δ_τ and the relative determinant of $(\Delta_\tau, \bar{\Delta}_{1,0})$. In order to do that we consider $P(t)$, the contribution of the parabolic elements to the trace formula.

We know that $P(t)$ is given by

$$P(t) = \int_{\mathbb{R}} e^{-t(1/4+r^2)} \frac{\Gamma'}{\Gamma}(1+ir) \, dr,$$

for which we have the following lemma:

Lemma 5.1. *$P(t)$ has the following asymptotic expansions:*

$$P(t) \sim -\frac{\pi}{2}\frac{\log(t)}{t} + \frac{\sqrt{\pi}}{2\sqrt{t}}(-B_1 + \gamma - \log(4) + \pi)$$
$$+ t^{-1/2} \sum_{j=1}^{\infty} b_j t^{j/2}, \quad \text{as } t \to 0,$$

where B_1 is the first Bernoulli number and γ in this case denotes the Euler constant. As $t \to \infty$, we have that $P(t) = O(e^{-t/4})$.

Proof. The proof of Lemma 5.1 easily follows from the formula

$$\frac{\Gamma'(z+1)}{\Gamma(z+1)} = \frac{1}{2z} + \log(z) - \int_0^\infty \left(\frac{1}{2} - \frac{1}{u} + \frac{1}{e^u - 1} \right) du,$$

for $\mathrm{Re}(z) > 0$, and from Stirling's formula:

$$\log(\Gamma(z)) = \left(z - \frac{1}{2} \right) \log(z) - z + \frac{1}{2}\log(2\pi) + \sum_{r=1}^{\infty} \frac{(-1)^{r-1}B_r}{2r(2r-1)z^{2r-1}},$$

for $|\arg(z)| \leq \frac{\pi}{2} - \theta$, where B_r is the rth Bernoulli number. $\square$

Proposition 5.1. *For the relative determinant and the hyperbolic determinant we have the following relation:*

$$\det(\Delta_\tau, \bar{\Delta}_{1,0}) = \tilde{A} \, \det_{\mathrm{hyp}}(\Delta_\tau),$$

where $\tilde{A}$ is a constant that depends only on the number of cusps of M. In particular, $\det(\Delta_\tau, \bar{\Delta}_{1,0}) = A \, Z'_M(1)$, where A depends only on the topology of M.

Proof. We know that for any $a > 1$ and $t > 0$ the operator $e^{-t\Delta_{a,0}} - e^{-t\Delta_{1,0}}$ acting on $L^2([1,\infty), y^{-2}dy)$ is trace class and the trace is given by

$$\mathrm{Tr}(e^{-t\Delta_{a,0}} - e^{-t\Delta_{1,0}}) = -\frac{1}{\sqrt{4\pi t}}\, e^{-t/4} \log(a),$$

see [1, Proposition 2.6]. This fact together with equation (4.5), ([20, equation (2.2)]), and the linearity of the trace imply that

$$\mathrm{Tr}(e^{-t\Delta_\tau} - e^{-t\bar\Delta_{1,0}}) = \sum_j e^{-t\lambda_j} - \frac{1}{4\pi}\int_\mathbb{R} e^{-t(1/4+r^2)}\frac{\phi'}{\phi}\left(\frac{1}{2} + ir\right) dr$$

$$+ \frac{e^{-t/4}}{4}\left(\mathrm{Tr}\left(C\left(\frac{1}{2}\right)\right) + m\right).$$

Putting this equation together with Selberg's trace formula we obtain:

(5.2)
$$\mathrm{Tr}(e^{-t\Delta_\tau} - e^{-t\bar\Delta_{1,0}}) - \mathrm{STrK}_M(t) = -\frac{m}{\pi}P(t) - \frac{m\log(2)}{\sqrt{4\pi t}}e^{-t/4} + \frac{m}{2}e^{-t/4}.$$

Let us consider the following auxiliary function:

(5.3) $$\xi(s) = \frac{m}{\Gamma(s)}\int_0^\infty \left\{-\frac{1}{\pi}P(t) + e^{-\frac{t}{4}}\left(\frac{1}{2} - \frac{\log(2)}{\sqrt{4\pi t}}\right)\right\} t^{s-1}dt.$$

Then we have that $\zeta(s; \Delta_\tau, \bar\Delta_{1,0}) = \zeta_{M,\mathrm{hyp}}(s) + \xi(s)$. On the other hand, Lemma 5.1 implies that the function $\xi(s)$ has a meromorphic continuation to $\mathbb{C}$ that is analytic at $s = 0$. Thus,

$$\det(\Delta_\tau, \bar\Delta_{1,0}) = e^{-\xi'(0)}\det_{\mathrm{hyp}}(\Delta_\tau).$$

The constant $\tilde A = e^{-\xi'(0)}$ depends only on the number of cusps of M. □

Let us now recall how one can approach the boundary of the moduli space. For this we refer to Bers in [2]. Let us recall the notation and the result in [2] that we use here. Let $G = \mathrm{SL}(2,\mathbb{R})/\{\pm I\}$. Every Fuchsian group Γ satisfying the condition $\mathrm{mes}(G/\Gamma) < \infty$, has a signature $\sigma = (p, n; \nu_1, \cdots, \nu_n)$, where p and n are integers, the ν_j are integers or the symbol ∞, and $p \geq 0$, $n \geq 0$, $2 \leq \nu_1 \leq \cdots \leq \nu_n \leq \infty$. In the quotient $\Gamma\backslash\mathbb{H}$, the number p corresponds to the genus and n corresponds to the number of "singular" points. The values $\nu_j < \infty$ correspond to elliptic points, and $\nu_j = \infty$ correspond to

cusps. Since we do not consider elliptic points, all ν_j are equal to infinity. Let

$$X(\sigma)$$
$$= \{[\Gamma] : [\Gamma] \text{ is a conjugacy class of Fuchsian groups } \Gamma \text{ with signature } \sigma\}$$

The spaces $X(\sigma)$, with their natural topologies, are metrizable. The topology of $X(\sigma)$ can be derived from the Teichmüller topology. The theorem that is of our interest is the following:

Theorem 5.2. (Bers [2]). *The subset of $X(\sigma)$ corresponding to groups Γ such that $\ell(\gamma) \geq 2 + \epsilon > 2$ for all hyperbolic $\gamma \in \Gamma$ is compact.*

This implies that the only possible deformations reaching the boundary of the moduli space are obtained by deforming hyperbolic elements in the group, i.e., by pinching smallest geodesics.

As we already mentioned, the proof of Theorem 5.1 relies strongly on the results of Jorgenson and Lundelius in [12]. Let us recall them: let $\{M_l\}_{l \in I \subset \mathbb{R}_+^p}$ be a degenerating family of hyperbolic Riemann surfaces of finite volume (each surface M_l is assumed to have m cusps and to be connected) with p pinching geodesics. This means that for each $l = (l_1, \ldots, l_p) \in I$ the cutoff cylinders $C_{l_k, \epsilon}$ are embedded in M_l for every $0 < \epsilon < 1/2$. From Gauss–Bonnet we know that the area of the surfaces is kept invariant during the deformation. Let Γ_l be the group corresponding to M_l, let $\mathrm{H}(\Gamma_l)$ denote a set of representatives of primitive non-conjugated hyperbolic classes in Γ_l, and let $\mathrm{DH}(\Gamma_l) \subset \mathrm{H}(\Gamma_l)$ be the subset corresponding to the geodesics that we are pinching. Proposition 2.1 in [12] yields that the degenerating heat trace for $t > 0$ equals:

$$\mathrm{DTr} K_{M_l}(t) = \frac{\mathrm{e}^{-t/4}}{\sqrt{16\pi t}} \sum_{\mathrm{DH}(\Gamma_l)} \sum_{n=1}^{\infty} \frac{\ell(\gamma)}{\sinh(n\ell(\gamma)/2)} \mathrm{e}^{-(n\ell(\gamma))^2/4t}.$$

Let M be the Riemann surface that is the limit of the degenerating family $\{M_l\}$ then M is not necessarily connected and the number of cusps of M is $m + 2p$. Theorem 2.2 in [12] states that:

$$\lim_{l \to 0}(\mathrm{HTr} K_{M_l}(t) - \mathrm{DTr} K_{M_l}(t)) = \mathrm{HTr} K_M(t).$$

Their next step is to separate (in the trace) the small eigenvalues of the Laplacian on M_l. Let $\{\lambda_{n,l}\}_n$ denote the eigenvalues of Δ_{M_l} and $\{\lambda_j\}_j$

denote the eigenvalues of Δ_M. Let $0 < \alpha < 1/4$ be such that α is not an eigenvalue of the Laplacian on M and consider:

$$\mathrm{HTr}K_{M_l}^\alpha(t) := \mathrm{HTr}K_{M_l}(t) - \sum_{\lambda_{n,l} \leq \alpha} \mathrm{e}^{-\lambda_{n,l}t},$$

From this definition we have that: $\mathrm{STr}K_{M_l}^\alpha(t) = \mathrm{STr}K_{M_l}(t) - \sum_{\lambda_{j,l} \leq \alpha} \mathrm{e}^{-t\lambda_{j,l}}$, and $\mathrm{STr}K_M^\alpha(t) = \mathrm{HTr}K_M(t) - \sum_{\lambda_j(M) \leq \alpha} \mathrm{e}^{-t\lambda_j(M)} + AK_{\mathbb{H}}(t,0)$, where A denotes the area of the limit surface $\bar{M}$. For a given hyperbolic surface M_*, JL consider the truncated hyperbolic zeta function:

$$\zeta_{\mathrm{hyp}\ M_*}^\alpha(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{STr}K_{M_*}^\alpha(t)t^{s-1}\ dt$$

and the corresponding determinant $\det_{\mathrm{hyp}}^\alpha \Delta_{M_*}$ is defined in the usual way. Let us see now how $\det_{\mathrm{hyp}}^\alpha \Delta_{M_l}$ relates to $\det(\Delta_{M_l}, \bar{\Delta}_{1,0})$. Notice that the operator $\bar{\Delta}_{1,0}$ remains constant through the degeneration. At the moment we are not concerned with the relative determinant of the limiting surface but rather with the behavior of the relative determinant of the degenerating surfaces. Equation (5.2) applied to M_l can be rewritten as:

$$\mathrm{Tr}(\mathrm{e}^{-t\Delta_{M_l}} - \mathrm{e}^{-t\bar{\Delta}_{1,0}}) - \mathrm{STr}K_{M_l}^\alpha(t)$$
$$= m\left(-\frac{1}{\pi}P(t) + \left(\frac{1}{2} - \frac{\log(2)}{\sqrt{4\pi t}}\right)\mathrm{e}^{-t/4}\right) + \sum_{\lambda_{j,l} \leq \alpha} \mathrm{e}^{-t\lambda_{j,l}}$$

Writing this in terms of zeta functions we obtain:

$$\zeta(s, \Delta_{M_l}, \bar{\Delta}_{1,0}) - \zeta_{\mathrm{hyp}\ M_l}^\alpha(s) = \xi(s) + \sum_{\lambda_{j,l} \leq \alpha} \lambda_{j,l}^{-s},$$

where $\xi(s)$ is as in equation (5.3). Taking the meromorphic continuations and differentiating we obtain that:

$$(5.4) \qquad \log \det_{\mathrm{hyp}}^\alpha \Delta_{M_l} = \log \det(\Delta_{M_l}, \bar{\Delta}_{1,0}) + mc - \sum_{\lambda_{j,l} \leq \alpha} \log(\lambda_{j,l}),$$

where for $\xi(s)$ we used again Lemma 5.1 and the fact that from equation (5.3) is clear that $\xi'(0) = c\ m$, where c is a constant independent of l.

Due to a misprint in a sign in Corollary 4.3 in [12] we do not use it directly. Instead we refer to their Theorem 4.6 and keep track of the signs.

Theorem 4.1 in [12] establishes that for all $s \in \mathbb{C}$

$$(5.5) \quad \lim_{l \to 0} \left(\zeta_{\text{hyp } M_l}^{\alpha}(s) - \frac{1}{\Gamma(s)} \int_0^\infty \text{DTr} K_{M_l}(t) t^{s-1} dt - \zeta_{\text{hyp } M}^{\alpha}(s) \right) = 0,$$

and the convergence is uniform in any half plane $\text{Re}(s) > C > -\infty$. In order to deal with the second term in the left-hand side of equation (5.5) we follow Remark 4.2 in [12] to obtain:

$$\frac{d}{ds} \frac{1}{\Gamma(s)} \int_0^\infty \text{DTr} K_{M_l}(t) t^{s-1} dt \bigg|_{s=0} = \sum_{\text{DH}(\Gamma_l)} \sum_{n=1}^{\infty} \frac{e^{-n\ell(\gamma)}}{n(1 - e^{-n\ell(\gamma)})},$$

This together with equation (5.5) gives:

$$\lim_{l \to 0} \left(\log \det_{\text{hyp}}^{\alpha} \Delta_{M_l} + \sum_{\gamma \in \text{DH}(\Gamma_l)} \sum_{n=1}^{\infty} \frac{e^{-n\ell(\gamma)}}{n(1 - e^{-n\ell(\gamma)})} \right) = \log \det_{\text{hyp}}^{\alpha} \Delta_M.$$

Let us replace $\log \det_{\text{hyp}}^{\alpha} \Delta_{M_l}$ in the expression above using equation (5.4):

$$\lim_{l \to 0} \left(\log \det(\Delta_{M_l}, \bar{\Delta}_{1,0}) + mc + \sum_{\gamma \in \text{DH}(\Gamma_l)} \sum_{n=1}^{\infty} \frac{e^{-n\ell(\gamma)}}{n(1 - e^{-n\ell(\gamma)})} \right.$$

$$(5.6) \qquad \left. - \sum_{0 < \lambda_{j,l} \leq \alpha} \log(\lambda_{j,l}) \right) = \log \det_{\text{hyp}}^{\alpha} \Delta_M.$$

In order to study the behavior of $\log \det(\Delta_{M_l}, \bar{\Delta}_{1,0})$ we need to know the behavior of the series in the left-hand side of equation (5.6) as $l \to 0$; recall that $\ell(\gamma) \to 0$ as $l \to 0$. This series was already studied by Wolpert in [30, p.308], if $\text{Re}(s) > 0$ then as $\ell(\gamma) \to 0^+$ we have:

$$\sum_{n=1}^{\infty} \frac{e^{-ns\ell(\gamma)}}{n(1 - e^{-n\ell(\gamma)})} = \left(\frac{\pi^2}{6\ell(\gamma)} + \left(s - \frac{1}{2} \right) \log \left(1 - e^{-s\ell(\gamma)} \right) \right) + O(1).$$

Taking $s = 1$ we see that

$$\lim_{l \to 0} \sum_{\gamma \in \text{DH}(\Gamma_l)} \sum_{n=1}^{\infty} \frac{e^{-n\ell(\gamma)}}{n(1 - e^{-n\ell(\gamma)})} = \infty.$$

For the sum involving the logarithm of the small eigenvalues we know that some of the small eigenvalues of the family $\{M_l\}$ may degenerate. For

the eigenvalues of M, $0 = \lambda_j(M)$, that come from degeneration we know that for any $0 < \alpha < \frac{1}{4}$, α not an eigenvalue of M, there is a l_0 such that for all $0 < l \leq l_0$, $\lambda_{l,j} \leq \alpha$. This is due to the convergence of any finite number of eigenvalues. Thus $\lim_{l \to 0} \sum_{0 < \lambda_{j,l} \leq \alpha} \log(\lambda_{j,l}) = -\infty$. In this way we have:

$$\lim_{l \to 0} \sum_{\gamma \in \mathrm{DH}(\Gamma_l)} \sum_{n=1}^{\infty} \frac{-\mathrm{e}^{-n\ell(\gamma)}}{n(1 - \mathrm{e}^{-n\ell(\gamma)})} - \sum_{0 < \lambda_{j,l} \leq \alpha} \log(\lambda_{j,l}) = \infty,$$

since the term cm and the hyperbolic α-regularized determinant of the limit surface are both finite, it follows that

$$\lim_{l \to 0} \log(\det(\Delta_{M_l}, \bar{\Delta}_{1,0})) = -\infty.$$

This finishes the proof of Theorem 5.1.

Acknowledgments

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