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Q-curvature flow on S^n

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In this paper, we study the Q-curvature flow on the standard sphere S^n and prove that the flow converges exponentially for all initial data.

1. Introduction

The Q-curvature is a notion introduced by Branson initially defined on manifolds of dimension four, and is a direct generalization of the Gaussian curvature on compact surface. If Σ is a compact surface with Riemannian metric g, under the conformal change of metric $\tilde{g} = e^{2w}g$, we have

$$\Delta_{\tilde{g}} = e^{-2w}\Delta_g \quad \text{and} \quad -\Delta_g w + K_g = K_{\tilde{g}}e^{2w},$$

where Δ_g and K_g (respectively $\Delta_{\tilde{g}}$ and $K_{\tilde{g}}$) are the Laplace–Beltrami operator and the Gaussian curvature of the metric g (respectively of $\tilde{g}$). This implies the invariance of the integral

$$\int_\Sigma K_{\tilde{g}}\, dV_{\tilde{g}} = \int_\Sigma K_g e^{-2w}\, dV_{\tilde{g}} = \int_\Sigma K_g\, dV_g$$

under conformal changes of the metric. In fact, the Gauss–Bonnet Theorem asserts that

$$\int_\Sigma K_g\, dV_g = 2\pi\chi(\Sigma),$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ.

Now, consider a compact manifold M of dimension four with Riemannian metric g. The Q-curvature is defined by

$$Q_g = -\frac{1}{6}(\Delta_g R_g - R_g^2 + 3|\mathrm{Ric}_g|_g^2),$$

where R_g and Ric_g denote the scalar curvature and the Ricci tensor of g. Moreover, the Paneitz operator associated with the metric g acts on a smooth function f on M via

$$P_g f = \Delta_g^2 f + d_g^* \left[\left(\frac{2}{3} R_g g - 2\mathrm{Ric}_g \right) df \right].$$

The Q-curvature plays an important role in conformal geometry, see [4, 8–10]. Indeed, it enjoys similar properties as the Gaussian curvature in dimension two. Under a conformal change of metric $\tilde{g} = \mathrm{e}^{2w} g$, the Q-curvature of $\tilde{g}$ can be written as

$$Q_{\tilde{g}} = \mathrm{e}^{-4w}(P_g w + Q_g),$$

and the Paneitz operator associated with the metric $\tilde{g}$ is related to the Paneitz operator associated with the metric g by

$$P_{\tilde{g}} f = \mathrm{e}^{-4w} P_g f.$$

Moreover, we have the following Gauss–Bonnet–Chern theorem:

$$\int_M \left(Q_g + \frac{1}{4}|W|^2 \right) dV_g = 8\pi^2 \chi(M).$$

Here W denotes the Weyl tensor of M.

Fefferman and Graham [16, 17] generalized the concept of Q-curvature to higher dimensional manifolds. They showed that on an even-dimensional Riemannian manifold (M^n, g) there exists a self-adjoint operator P_g with leading term $(-\Delta_g)^{n/2}$ such that the Q-curvature Q_g transforms according to $Q_{\tilde{g}} = \mathrm{e}^{-nw}(P_g w + Q_g)$ under the conformal change of metric $\tilde{g} = \mathrm{e}^{2w} g$.

As in the two-dimensional case, it is natural to ask whether on an even-dimensional Riemannian manifold (M^n, g) there exists a conformal metric of constant Q-curvature. This problem has been studied in [5, 10, 12, 20]. More generally, one can ask the following prescribing Q-curvature problem: Given a smooth function f on M^n, find a conformal metric $\tilde{g} = \mathrm{e}^{2w} g$ for which $Q_{\tilde{g}} = f$. This problem has been studied in [1, 2, 6, 7, 13–15, 19, 21]. The flow technique has been introduced to tackle these problems. In [5], Brendle studied the Q-curvature flow on S^4:

$$\frac{\partial}{\partial t} g(t) = -(Q_{g(t)} - \overline{Q}_{g(t)}) g(t), \tag{1.1}$$

where $\overline{Q}_{g(t)}$ denotes the mean value of $Q_{g(t)}$. He proved that if the initial metric is conformally equivalent to the standard metric on S^4, then the Q-curvature flow converges exponentially to a metric having constant sectional curvature. In [6], Brendle obtained related results. On the even-dimensional Riemannian manifold (M^n, g), he considered the following:

$$\frac{\partial}{\partial t} g(t) = -\left(Q_{g(t)} - \frac{\overline{Q}_{g(t)} f}{\overline{f}} \right) g(t),$$

where $\overline{Q}_{g(t)}$ and $\overline{f}$ are the mean values of $Q_{g(t)}$ and f, respectively. He proved the existence of a solution of the flow for all time and convergence to a metric g_∞ with $Q_{g_\infty}/f = \overline{Q}_{g_\infty}/\overline{f}$, provided that the operator P_g associated to the initial metric g is weakly positive with kernel consisting of the constant functions, and $\int_M Q_g \, dV_g < (n-1)! \, \omega_n$, where ω_n is the volume of the standard sphere S^n. See also [2, 19].

Following the arguments of Brendle in [5], we prove the following:

Theorem 1.1. *Suppose that the initial metric is conformally equivalent to the standard metric on S^n. Then the Q-curvature flow* (1.1) *exists for all time, and converges exponentially to a limiting metric. The limiting metric has constant sectional curvature and is obtained from the standard metric by pull-back along a conformal diffeomorphism.*

We remark that Theorem 1.1 is not covered in [6] since $\int_M Q_g \, dV_g = (n-1)!\omega_n$ for the metric g being conformally equivalent to the standard metric on S^n. The organization of the paper is as follows. In Section 2, we give some properties of the Q-curvature flow. In Section 3, we modify the Q-curvature flow by conformal transformations. This allows us to prove that the solution of the flow is bounded in $W^{\frac{n}{2},2}$ on every finite time interval. In Section 4, we prove that the solution of the flow is bounded in $W^{n,2}$ on every finite time interval, which implies that the flow exists for all time. In Section 5, we prove that the solution of the flow is uniformly bounded in $W^{n,2}$. We show that the flow converges exponentially to a metric of constant Q-curvature as $t \to \infty$.

Notation. All norms we use are taken with respect to the standard metric g_0 of S^n. For example, $\|f\|_{L^p}^p = \int_{S^n} |f|^p dV_{g_0}$. The letter C represents a generic constant which may vary from line to line.

2. Some properties of the Q-curvature flow

From now on, we denote g_0 the standard metric on the sphere S^n. The explicit formula for P_{g_0} on S^n is given by (see [3])

$$P_{g_0} = \prod_{k=0}^{(n-2)/2} (-\Delta_{g_0} + k(n-k-1)), \tag{2.1}$$

where Δ_{g_0} is the Laplace–Beltrami operator of g_0. The Q-curvature of g_0 is equal to

$$Q_{g_0} = (n-1)!.$$

Let g be a metric on S^n which is conformally equivalent to the standard metric g_0. If we write $g = \mathrm{e}^{nw} g_0$, then

$$P_g = \mathrm{e}^{-nw} P_{g_0}, \tag{2.2}$$

and the Q-curvature of g is given by

$$Q_g = \mathrm{e}^{-nw} (Q_{g_0} + P_{g_0} w) = \mathrm{e}^{-nw} ((n-1)! + P_{g_0} w) . \tag{2.3}$$

It follows that the quantity $\int_{S^n} Q_g \, dV_g$ is conformally invariant. The Q-curvature flow is defined as

$$\frac{\partial}{\partial t} g(t) = -(Q_{g(t)} - \overline{Q}_{g(t)}) g(t). \tag{2.4}$$

Here $\overline{Q}_{g(t)}$ denotes the mean values of $Q_{g(t)}$, that is

$$\overline{Q}_{g(t)} = \frac{\int_{S^n} Q_{g(t)} \, dV_{g(t)}}{\int_{S^n} dV_{g(t)}}. \tag{2.5}$$

Suppose that $g(t)$ is a solution of the Q-curvature flow, and that the initial metric is conformally equivalent to standard metric g_0 on S^n. Then the metric $g(t)$ can be written as $g(t) = \mathrm{e}^{2w(t)} g_0$, where $w(t)$ is a real-valued function on S^n. It follows that the equation of the Q-curvature flow (2.4) can be written as

$$\frac{\partial}{\partial t} w(t) = -\frac{1}{2} (Q_{g(t)} - \overline{Q}_{g(t)}). \tag{2.6}$$

First, we have the following:

Proposition 2.1. *The volume of S^n does not change along the Q-curvature flow.*

Proof. We have

$$\frac{d}{dt}\left(\int_{S^n} dV_{g(t)}\right) = \frac{d}{dt}\left(\int_{S^n} \mathrm{e}^{nw(t)} dV_{g_0}\right) = \int_{S^n} n\mathrm{e}^{nw(t)} \frac{\partial}{\partial t} w(t)\, dV_{g_0}$$
$$= -\frac{n}{2}\int_{S^n} (Q_{g(t)} - \overline{Q}_{g(t)})\, dV_{g(t)} = 0,$$

where we have used (2.5) and (2.6). □

We claim that $\overline{Q}_{g(t)}$ is independent of t. To see this, we note that $\int_{S^n} Q_{g(t)}\, dV_{g(t)}$ is conformally invariant, which implies that $\int_{S^n} Q_{g(t)}\, dV_{g(t)} = \int_{S^n} Q_{g_0}\, dV_{g_0} = (n-1)! \int_{S^n} dV_{g_0}$. On the other hand, the volume does not change along the Q-curvature flow by Proposition 2.1. Hence, if we let

$$q = \frac{(n-1)! \int_{S^n} dV_{g_0}}{\int_{S^n} dV_{g(0)}},$$

then

$$\overline{Q}_{g(t)} = \frac{\int_{S^n} Q_{g(t)}\, dV_{g(t)}}{\int_{S^n} dV_{g(t)}} = \frac{\int_{S^n} Q_{g_0}\, dV_{g_0}}{\int_{S^n} dV_{g(t)}} = \frac{(n-1)! \int_{S^n} dV_{g_0}}{\int_{S^n} dV_{g(0)}} = q. \tag{2.7}$$

In particular, the equation of the Q-curvature flow (2.6) can be written as

$$\frac{\partial}{\partial t} w(t) = -\frac{1}{2}(Q_{g(t)} - q). \tag{2.8}$$

The Q-curvature can be viewed as a gradient flow to a certain functional. This functional is given by

$$E_{g_0}[w] = \frac{n}{2}\int_{S^n} wP_{g_0}w\, dV_{g_0} + n! \int_{S^n} w\, dV_{g_0}$$
$$- (n-1)! \int_{S^n} dV_{g_0} \log\left(\frac{\int_{S^n} \mathrm{e}^{nw} dV_{g_0}}{\int_{S^n} dV_{g_0}}\right). \tag{2.9}$$

The functional E_{g_0} is non-increasing along the Q-curvature flow:

Proposition 2.2. *We have*

$$\frac{d}{dt}E_{g_0}[w(t)] = -\frac{n}{2}\int_{S^n}(Q_{g(t)}-q)^2\,dV_{g(t)}.$$

In particular, we have

$$E_{g_0}[w(t)] \leq E_{g_0}[w(0)] \quad \textit{for all } t \geq 0.$$

Proof. By (2.3), (2.7)–(2.9), we obtain

$$\begin{aligned}
\frac{d}{dt}E_{g_0}[w(t)] &= n\int_{S^n}\frac{\partial w}{\partial t}\left(P_{g_0}w+(n-1)!\right)dV_{g_0}\\
&\quad -(n-1)!\left(\int_{S^n}dV_{g_0}\right)\cdot\left(\frac{n\int_{S^n}\mathrm{e}^{nw}\frac{\partial w}{\partial t}\,dV_{g_0}}{\int_{S^n}\mathrm{e}^{nw}\,dV_{g_0}}\right)\\
&= -\frac{n}{2}\int_{S^n}(Q_{g(t)}-q)Q_{g(t)}\,dV_{g(t)}\\
&\quad -\frac{(n-1)!\int_{S^n}dV_{g_0}}{\int_{S^n}dV_{g(t)}}\left(-\frac{n}{2}\int_{S^n}(Q_{g(t)}-q)\,dV_{g(t)}\right)\\
&= -\frac{n}{2}\left(\int_{S^n}(Q_{g(t)}-q)Q_{g(t)}\,dV_{g(t)}-q\int_{S^n}(Q_{g(t)}-q)\,dV_{g(t)}\right)\\
&= -\frac{n}{2}\int_{S^n}(Q_{g(t)}-q)^2\,dV_{g(t)}.
\end{aligned}$$

Therefore, the functional E_{g_0} is non-increasing along the flow. From this the assertion follows. □

We have the following proposition which is essentially due to Wei and Xu [21]:

Proposition 2.3. *We can find positive real numbers η and C such that: for every function w satisfying $\int_{S^n}\mathrm{e}^{nw}x\,dV_{g_0}=0$, we have*

$$E_{g_0}[w] \geq \eta\int_{S^n}\left((-\Delta_{g_0})^{\frac{n}{4}}w\right)^2 dV_{g_0} - C.$$

Proof. According to Theorem 2.6 in [21], there exists $0 < a < 1$ such that

$$\log\left(\frac{\int_{S^n} e^{nw}\,dV_{g_0}}{\int_{S^n} dV_{g_0}}\right) \leq \frac{n}{2(n-1)!}\left[a\left(\frac{\int_{S^n} wP_{g_0}w\,dV_{g_0}}{\int_{S^n} dV_{g_0}}\right) + 2(n-1)!\left(\frac{\int_{S^n} w\,dV_{g_0}}{\int_{S^n} dV_{g_0}}\right)\right],$$

for every function w satisfying $\int_{S^n} e^{nw}x\,dV_{g_0} = 0$. Proposition 2.3 follows from the inequality $\int_{S^n} wP_{g_0}w\,dV_{g_0} \geq C\int_{S^n}\left((-\Delta_{g_0})^{\frac{n}{4}}w\right)^2 dV_{g_0} - C$. □

3. Estimates in $W^{\frac{n}{2},2}$

Proposition 3.1. *Let w be a smooth function on S^n. Then there exists a unique vector $p \in B^n$ such that*

$$\int_{S^n} e^{nw}\left(p + \frac{1-|p|^2}{1+2\langle p,x\rangle+|p|^2}(x+p)\right)dV_{g_0} = 0.$$

The proof of Proposition 3.1 can be avoided, since it is identical to the proof of Proposition 6 in [5]. Therefore, if $g(t) = e^{2w(t)}g_0$ is a solution of the Q-curvature flow where $w(t)$ is a real-valued function on S^n, then it follows from Proposition 3.1 that for every $t \geq 0$, there exists a unique vector $p(t) \in B^n$ such that

$$\int_{S^n} e^{nw(t)}\left(p(t) + \frac{1-|p(t)|^2}{1+2\langle p(t),x\rangle+|p(t)|^2}(x+p(t))\right)dV_{g_0} = 0. \tag{3.1}$$

We define a diffeomorphism $\varphi(t) : S^n \to S^n$ by

$$\varphi(x,t) = p(t) + \frac{1-|p(t)|^2}{1+2\langle p(t),x\rangle+|p(t)|^2}(x+p(t)). \tag{3.2}$$

For every vector ξ tangent to S^n at x, we have

$$\varphi(t)_*\xi = \frac{1-|p(t)|^2}{1+2\langle p(t),x\rangle+|p(t)|^2}\left(\xi - 2\frac{\langle p(t),\xi\rangle}{1+2\langle p(t),x\rangle+|p(t)|^2}(x+p(t))\right).$$

Since $\langle x,\xi\rangle = 0$ and $|x| = 1$, it follows that

$$|\varphi(t)_*\xi|^2 = \left(\frac{1-|p(t)|^2}{1+2\langle p(t),x\rangle+|p(t)|^2}\right)^2|\xi|^2.$$

Therefore, $\varphi(t)$ is a conformal mapping. Moreover, the pull-back of the standard metric g_0 is given by

$$\varphi(t)^*g_0 = \left(\frac{1-|p(t)|^2}{1+2\langle p(t),x\rangle+|p(t)|^2}\right)^2 g_0.$$

If we consider the metric $\tilde{g}(t)=\mathrm{e}^{2\tilde{w}(t)}g_0$, where

$$\tilde{w}(\varphi(x,t),t)+\log\left(\frac{1-|p(t)|^2}{1+2\langle p(t),x\rangle+|p(t)|^2}\right)=w(x,t), \tag{3.3}$$

then the metric $\tilde{g}(t)$ is related to the metric $g(t)$ by $\varphi(t)^*\tilde{g}(t)=g(t)$.

We will show that the function $\tilde{w}(t)$ is uniformly bounded in $W^{\frac{n}{2},2}$.

Proposition 3.2. *There exists a constant C depending only on the initial data such that $\|\tilde{w}(t)\|_{W^{\frac{n}{2},2}}\leq C$ for all $t\geq 0$.*

Proof. By (3.1) and (3.2), we have

$$\int_{S^n}\mathrm{e}^{nw(t)}\varphi(x,t)\,dV_{g_0}=0, \tag{3.4}$$

which implies that

$$\int_{S^n}\mathrm{e}^{n\tilde{w}(t)}x\,dV_{g_0}=0. \tag{3.5}$$

By Proposition 2.3, we obtain

$$E_{g_0}[\tilde{w}(t)]\geq\eta\int_{S^n}\left((-\Delta_{g_0})^{\frac{n}{4}}\tilde{w}(t)\right)^2dV_{g_0}-C$$

for some constant $\eta>0$. Moreover, the functional E_{g_0} is invariant under conformal transformations (see [10], part (a) of the proof of Theorem 4.1). Therefore, by Proposition 2.2, we have

$$E_{g_0}[\tilde{w}(t)]=E_{g_0}[w(t)]\leq E_{g_0}[w(0)].$$

From this it follows that

$$\int_{S^n}\left((-\Delta_{g_0})^{\frac{n}{4}}\tilde{w}(t)\right)^2dV_{g_0}\leq C,$$

which implies that

$$\|\tilde{w}(t) - \overline{\tilde{w}(t)}\|_{W^{\frac{n}{2},2}} \leq C.$$

Here C is a constant depending only on the initial data. Using Trudinger's inequality, we obtain

$$\int_{S^n} \mathrm{e}^{n(\tilde{w}(t) - \overline{\tilde{w}(t)})}\, dV_{g_0} \leq C.$$

Since

$$\int_{S^n} \mathrm{e}^{n\tilde{w}(t)}\, dV_{g_0} = \int_{S^n} \mathrm{e}^{nw(t)}\, dV_{g_0} = \int_{S^n} dV_{g(t)} = \int_{S^n} dV_{g(0)} = \int_{S^n} \mathrm{e}^{nw(0)} dV_{g_0}$$

by Proposition 2.1, we conclude that

$$\overline{\tilde{w}(t)} \geq -C$$

for some constant C depending only on the initial data. Putting all these together, we obtain

$$\|\tilde{w}(t)\|_{W^{\frac{n}{2},2}} \leq C,$$

as required. □

Proposition 3.3. *There exists a constant C such that*

$$\left|\frac{d}{dt}\log(1-|p(t)|^2)\right| \leq \left|\int_{S^n} \mathrm{e}^{n\tilde{w}(t)}(Q_{\tilde{g}(t)} - q)x\, dV_{g_0}\right| \quad \textit{for all } t \geq 0.$$

Proof. By (3.4), we have

$$\frac{d}{dt}\left(\int_{S^n} \mathrm{e}^{nw(t)}\varphi(x,t)\, dV_{g_0}\right) = 0.$$

Hence, by (2.8), we obtain

$$\begin{aligned}\int_{S^n} \mathrm{e}^{nw(t)}\frac{\partial}{\partial t}\varphi(x,t)\, dV_{g_0} &= -n\int_{S^n} \mathrm{e}^{nw(t)}\varphi(x,t)\frac{\partial w}{\partial t}\, dV_{g_0}\\ &= \frac{n}{2}\int_{S^n} \mathrm{e}^{nw(t)}(Q_{g(t)} - q)\varphi(x,t)\, dV_{g_0}.\end{aligned}$$

On the other hand, by (3.2) and (3.4), we have

$$\begin{aligned}
&\int_{S^n} e^{nw(t)}\frac{\partial}{\partial t}\varphi(x,t)\,dV_{g_0}\\
&\quad=\int_{S^n} e^{nw(t)}\frac{2}{1-|p(t)|^2}\big[p'(t)-\langle p'(t),\varphi(x,t)\rangle\varphi(x,t)\big]\,dV_{g_0}\\
&\quad-\frac{2}{1-|p(t)|^2}\bigg[\bigg\langle p(t),\int_{S^n} e^{nw(t)}\varphi(x,t)\,dV_{g_0}\bigg\rangle p'(t)\\
&\quad-\bigg\langle p'(t),\int_{S^n} e^{nw(t)}\varphi(x,t)\,dV_{g_0}\bigg\rangle p(t)\bigg]\\
&\quad=\int_{S^n} e^{nw(t)}\frac{2}{1-|p(t)|^2}\big[p'(t)-\langle p'(t),\varphi(x,t)\rangle\varphi(x,t)\big]\,dV_{g_0}.
\end{aligned}$$

Putting these facts together, we obtain

$$\begin{aligned}
&\int_{S^n} e^{nw(t)}\frac{2}{1-|p(t)|^2}\big[p'(t)-\langle p'(t),\varphi(x,t)\rangle\varphi(x,t)\big]\;dV_{g_0}\\
&\quad=\frac{n}{2}\int_{S^n} e^{nw(t)}(Q_{g(t)}-q)\varphi(x,t)\,dV_{g_0}.
\end{aligned}$$

This implies

$$\begin{aligned}
&\int_{S^n} e^{nw(t)}\frac{2}{1-|p(t)|^2}\big[|p'(t)|^2-\langle p'(t),\varphi(x,t)\rangle^2\big]\,dV_{g_0}\\
&\quad=\frac{n}{2}\int_{S^n} e^{nw(t)}(Q_{g(t)}-q)\langle p'(t),\varphi(x,t)\rangle\,dV_{g_0}.
\end{aligned}$$

From this it follows that

$$\begin{aligned}
&\int_{S^n} e^{n\tilde{w}(t)}\frac{2}{1-|p(t)|^2}\big[|p'(t)|^2-\langle p'(t),x\rangle^2\big]\,dV_{g_0}\\
&\quad=\frac{n}{2}\int_{S^n} e^{n\tilde{w}(t)}(Q_{\tilde{g}(t)}-q)\langle p'(t),x\rangle\,dV_{g_0}.
\end{aligned}\tag{3.6}$$

Therefore if we define

$$\alpha=\int_{S^n}\sqrt{1-x_{n+1}^2}dV_{g_0}=\int_{S^n}\sqrt{x_1^2+x_2^2+\cdots+x_n^2}\,dV_{g_0},$$

then we have

$$\begin{aligned}
\alpha^2|p'(t)|^2 &= \left(\int_{S^n} \sqrt{|p'(t)|^2 - \langle p'(t), x\rangle^2}\, dV_{g_0}\right)^2 \\
&\leq \left(\int_{S^n} \mathrm{e}^{n\tilde{w}(t)} \left(|p'(t)|^2 - \langle p'(t), x\rangle^2\right) dV_{g_0}\right) \left(\int_{S^n} \mathrm{e}^{-n\tilde{w}(t)} dV_{g_0}\right) \\
&= (1 - |p(t)|^2) \left(\frac{n}{4} \int_{S^n} \mathrm{e}^{n\tilde{w}(t)} (Q_{\tilde{g}(t)} - q)\langle p'(t), x\rangle\, dV_{g_0}\right) \\
&\quad \times \left(\int_{S^n} \mathrm{e}^{-n\tilde{w}(t)}\, dV_{g_0}\right) \\
&\leq \frac{n}{4}(1 - |p(t)|^2)|p'(t)| \left|\int_{S^n} \mathrm{e}^{n\tilde{w}(t)} (Q_{\tilde{g}(t)} - q)x\, dV_{g_0}\right| \\
&\quad \times \left(\int_{S^n} \mathrm{e}^{-n\tilde{w}(t)}\, dV_{g_0}\right).
\end{aligned}$$

Here we have used (3.6). Using Proposition 3.2 and Trudinger's inequality, we obtain $\int_{S^n} \mathrm{e}^{-n\tilde{w}(t)} dV_{g_0} \leq C$. Thus, we conclude that

$$|p'(t)| \leq C(1 - |p(t)|^2) \left|\int_{S^n} \mathrm{e}^{n\tilde{w}(t)} (Q_{\tilde{g}(t)} - q)x\, dV_{g_0}\right|.$$

From this the assertion follows. □

Proposition 3.4. *For every real number $T \geq 0$, there exists a constant $C(T)$ such that $\frac{1}{1 - |p(t)|^2} \leq C(T)$ for all $0 \leq t \leq T$.*

Proof. Integration by parts gives

$$\begin{aligned}
\int_{S^n} \mathrm{e}^{n\tilde{w}(t)} (Q_{\tilde{g}(t)} - q)x\, dV_{g_0} &= \int_{S^n} (P_{g_0}\tilde{w}(t) + (n-1)! - q\mathrm{e}^{n\tilde{w}(t)})x\, dV_{g_0} \\
&= \int_{S^n} (P_{g_0}\tilde{w}(t))\, x\, dV_{g_0} \\
&= \int_{S^n} \tilde{w}(t) P_{g_0} x\, dV_{g_0} \\
&= \int_{S^n} \tilde{w}(t) \prod_{k=0}^{(n-2)/2} (-\Delta_{g_0} + k(n-k-1))x\, dV_{g_0} \\
&= \int_{S^n} \tilde{w}(t) \prod_{k=0}^{(n-2)/2} (n + k(n-k-1))x\, dV_{g_0}
\end{aligned}$$

Here we have used (2.1), (2.3), (3.5) and the fact that the coordinate functions are eigenfunctions of Δ_{g_0}. This implies $\left|\int_{S^n} e^{n\tilde{w}(t)}(Q_{\tilde{g}(t)} - q)x\, dV_{g_0}\right| \leq C$ by Proposition 3.2. Using Proposition 3.3, we obtain $\left|\frac{d}{dt}\log(1 - |p(t)|^2)\right| \leq C$. From this the assertion follows. □

Corollary 3.1. *Given any $T \geq 0$, there exists a constant $C(T)$ such that*

$$\|w(t)\|_{W^{\frac{n}{2},2}} \leq C(T)$$

for all $0 \leq t \leq T$.

Proof. This follows from Propositions 3.2 and 3.4. □

4. Global existence

Following the proof in Section 4 of [6], we have the following:

Proposition 4.1. *Given any $T \geq 0$, there exists a constant $C(T)$ such that $\|w(t)\|_{W^{n,2}} \leq C(T)$ for all $0 \leq t \leq T$.*

Proof. We define

$$\begin{aligned} v(t) &= -\frac{1}{2}e^{\frac{n}{2}w(t)}(Q_{g(t)} - q) \\ &= e^{\frac{n}{2}w(t)}\frac{\partial}{\partial t}w(t) \\ &= -\frac{1}{2}e^{-\frac{n}{2}w(t)}P_{g_0}w(t) - \frac{1}{2}e^{-\frac{n}{2}w(t)}Q_{g_0} + \frac{1}{2}qe^{\frac{n}{2}w(t)}. \end{aligned}$$

Here we have used (2.3) and (2.8). This implies that

$$\begin{aligned} \frac{\partial}{\partial t}w(t) &= e^{-\frac{n}{2}w(t)}v(t), \\ P_{g_0}w(t) &= -2e^{\frac{n}{2}w(t)}v(t) - Q_{g_0} + qe^{nw(t)}. \end{aligned}$$

From this we deduce that

$$\begin{aligned} &\frac{d}{dt}\left(\int_{S^n}(P_{g_0}w(t))^2\, dV_{g_0}\right) \\ &\quad = \int_{S^n}(2P_{g_0}w(t))\, P_{g_0}\left(\frac{\partial}{\partial t}w(t)\right) dV_{g_0} \end{aligned}$$

$$
\begin{aligned}
&= 2\int_{S^n}\left(-2\mathrm{e}^{\frac{n}{2}w(t)}v(t)-Q_{g_0}+q\mathrm{e}^{nw(t)}\right)P_{g_0}\left(\mathrm{e}^{-\frac{n}{2}w(t)}v(t)\right)dV_{g_0}\\
&= -4\int_{S^n}\mathrm{e}^{\frac{n}{2}w(t)}v(t)P_{g_0}\left(\mathrm{e}^{-\frac{n}{2}w(t)}v(t)\right)dV_{g_0}\\
&\quad -2\int_{S^n}Q_{g_0}P_{g_0}\left(\mathrm{e}^{-\frac{n}{2}w(t)}v(t)\right)dV_{g_0}\\
&\quad +2q\int_{S^n}\mathrm{e}^{nw(t)}P_{g_0}\left(\mathrm{e}^{-\frac{n}{2}w(t)}v(t)\right)dV_{g_0}.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_{S^n}(P_{g_0}w(t))^2\,dV_{g_0}\right)\\
&\quad= -4\int_{S^n}(-\Delta_{g_0})^{\frac{n}{4}}\left(\mathrm{e}^{\frac{n}{2}w(t)}v(t)\right)(-\Delta_{g_0})^{\frac{n}{4}}\left(\mathrm{e}^{-\frac{n}{2}w(t)}v(t)\right)dV_{g_0}\\
&\qquad -2\int_{S^n}Q_{g_0}P_{g_0}\left(\mathrm{e}^{-\frac{n}{2}w(t)}v(t)\right)dV_{g_0}\\
&\qquad +2q\int_{S^n}(-\Delta_{g_0})^{\frac{n}{4}}(\mathrm{e}^{nw(t)})(-\Delta_{g_0})^{\frac{n}{4}}\left(\mathrm{e}^{-\frac{n}{2}w(t)}v(t)\right)dV_{g_0}\\
&\qquad + \text{ lower order terms.}
\end{aligned}
$$

Here, we adopt the convention that

$$
(-\Delta_{g_0})^{m+\frac{1}{2}} = \nabla_{g_0}(-\Delta_{g_0})^m
$$

for all integer m. The right-hand side involves derivatives of v and w of order at most $\frac{n}{2}$. Moreover, the total number of derivatives is at most n. Therefore, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_{S^n}(P_{g_0}w(t))^2dV_{g_0}\right)\\
&\quad= -4\int_{S^n}\left((-\Delta_{g_0})^{\frac{n}{4}}v(t)\right)^2dV_{g_0}\\
&\qquad + C\sum_{k_1,\ldots,k_m}\int_{S^n}|\nabla_{g_0}^{k_1}v(t)|\cdot|\nabla_{g_0}^{k_2}v(t)|\cdot|\nabla_{g_0}^{k_3}w(t)|\cdots|\nabla_{g_0}^{k_m}w(t)|\,dV_{g_0}\\
&\qquad + C\sum_{l_1,\ldots,l_m}\int_{S^n}|\nabla_{g_0}^{l_1}v(t)|\cdot|\nabla_{g_0}^{l_2}w(t)|\cdots|\nabla_{g_0}^{l_m}w(t)|\mathrm{e}^{\alpha w(t)}\,dV_{g_0}.
\end{aligned}
$$

The first sum is taken over all m-tuples $k_1, ..., k_m$ with $m \geq 3$ satisfying the conditions

$$\begin{aligned} 0 \leq k_i \leq \frac{n}{2} &\quad \text{for } 1 \leq i \leq 2, \\ 0 \leq k_i \leq \frac{n}{2} &\quad \text{for } 3 \leq i \leq m, \\ k_1 + \cdots + k_m \leq n. & \end{aligned}$$

To estimate this term, we choose real numbers $p_1, \ldots, p_m \in [2, \infty)$ such that

$$\begin{aligned} k_i \leq \frac{n}{p_i} &\quad \text{for } 1 \leq i \leq 2, \\ \frac{n}{p_i} < k_i &\quad \text{for } 3 \leq i \leq m, \\ \frac{1}{p_1} + \cdots + \frac{1}{p_m} = 1. & \end{aligned}$$

Moreover, we define real numbers $\theta_1, ..., \theta_m$ by

$$\begin{aligned} \theta_i = \frac{k_i - \frac{n}{p_i} + \frac{n}{2}}{\frac{n}{2}} \in [0,1] &\quad \text{for } 1 \leq i \leq 2, \\ \theta_i = \frac{k_1 - \frac{n}{p_i}}{\frac{n}{2}} \in (0,1) &\quad \text{for } 3 \leq i \leq m. \end{aligned}$$

Then we have $\theta_1 + \cdots + \theta_m \leq 2$; hence $\theta_3 + \cdots + \theta_m \leq (1-\theta_1) + (1-\theta_2)$. Since $\|w(t)\|_{W^{\frac{n}{2},2}} \leq C(T)$ for all $0 \leq t \leq T$ by Corollary 3.1, this implies that for all $0 \leq t \leq T$

$$\begin{aligned} &-2 \int_{S^n} \left((-\Delta_{g_0})^{\frac{n}{4}} v(t)\right)^2 dV_{g_0} \\ &\quad + C \sum_{k_1,\ldots,k_m} \int_{S^n} |\nabla_{g_0}^{k_1} v(t)| \cdot |\nabla_{g_0}^{k_2} v(t)| \cdot |\nabla_{g_0}^{k_3} w(t)| \cdots |\nabla_{g_0}^{k_m} w(t)| \, dV_{g_0} \\ &\qquad \leq -\|v(t)\|_{W^{\frac{n}{2},2}}^2 + C \sum_{k_1,\ldots,k_m} \|\nabla_{g_0}^{k_1} v(t)\|_{L^{p_1}} \cdot \|\nabla_{g_0}^{k_2} v(t)\|_{L^{p_2}} \\ &\qquad\quad \cdot \|\nabla_{g_0}^{k_3} w(t)\|_{L^{p_3}} \cdots \|\nabla_{g_0}^{k_m} w(t)\|_{L^{p_m}} \\ &\qquad \leq -\|v(t)\|_{W^{\frac{n}{2},2}}^2 + C \sum_{k_1,\ldots,k_m} \|v(t)\|_{W^{k_1 - \frac{n}{p_1} + \frac{n}{2},2}} \cdot \|v(t)\|_{W^{k_2 - \frac{n}{p_2} + \frac{n}{2},2}} \\ &\qquad\quad \cdot \|w(t)\|_{W^{k_3 - \frac{n}{p_3} + \frac{n}{2},2}} \cdots \|w(t)\|_{W^{k_m - \frac{n}{p_m} + \frac{n}{2},2}} \end{aligned}$$

$$
\begin{aligned}
&\leq -\|v(t)\|^2_{W^{\frac{n}{2},2}} + C \sum_{k_1,\dots,k_m} \|v(t)\|_{L^2}^{(1-\theta_1)+(1-\theta_2)} \cdot \|v(t)\|_{W^{\frac{n}{2},2}}^{\theta_1+\theta_2} \\
&\quad \cdot \|w(t)\|_{W^{\frac{n}{2},2}}^{(1-\theta_3)+\cdots+(1-\theta_m)} \|w(t)\|_{W^{n,2}}^{\theta_3+\cdots+\theta_m} \\
&\leq -\|v(t)\|^2_{W^{\frac{n}{2},2}} + C(T) \sum_{k_1,\dots,k_m} \|v(t)\|_{L^2}^{(1-\theta_1)+(1-\theta_2)} \\
&\quad \cdot \|v(t)\|_{W^{\frac{n}{2},2}}^{\theta_1+\theta_2} \cdot \|w(t)\|_{W^{n,2}}^{\theta_3+\cdots+\theta_m} \\
&\leq C(T) \sum_{k_1,\dots,k_m} \|v(t)\|^2_{L^2} \cdot \|w(t)\|_{W^{n,2}}^{\frac{2(\theta_3+\cdots+\theta_m)}{(1-\theta_1)+(1-\theta_2)}} \\
&\leq C(T)\|v(t)\|^2_{L^2}(\|w(t)\|^2_{W^{n,2}} + 1).
\end{aligned}
$$

The second sum is taken over all m-tuples $l_1, ..., l_m$ with $m \geq 1$ satisfying the conditions

$$
\begin{aligned}
0 \leq l_1 &\leq \frac{n}{2}, \\
1 \leq l_i &\leq \frac{n}{2} \quad \text{for } 2 \leq i \leq m, \\
l_1 + \cdots + l_m &\leq n.
\end{aligned}
$$

To estimate this term, we choose real numbers $q_1, .., q_m \in [2, \infty)$ such that

$$
\begin{aligned}
l_1 \leq \frac{n}{q_1}, \quad \frac{n}{q_i} < l_i \quad \text{for } 2 \leq i \leq m, \\
\frac{1}{2} \leq \frac{1}{q_1} + \cdots + \frac{1}{q_m} < 1.
\end{aligned}
$$

Moreover, we define real numbers $\rho_1, ..., \rho_m$ by

$$
\begin{aligned}
\rho_1 &= \frac{l_1 - \frac{n}{q_1} + \frac{n}{2}}{\frac{n}{2}} \in [0, 1], \\
\rho_i &= \frac{l_i - \frac{n}{q_i}}{\frac{n}{2}} \in (0, 1) \quad \text{for } 2 \leq i \leq m.
\end{aligned}
$$

Then we have $\rho_1 + \cdots + \rho_m \leq 2$; hence $\rho_2 + \cdots + \rho_m \leq 2 - \rho_1$. Since $\|w(t)\|_{W^{\frac{n}{2},2}} \leq C(T)$ for all $0 \leq t \leq T$ by Corollary 3.1, this implies that for all $0 \le t \le T$

$$\begin{aligned}
-2&\int_{S^n}\left((-\Delta_{g_0})^{\frac{n}{4}}v(t)\right)^2 dV_{g_0}\\
&+C\sum_{l_1,\ldots,l_m}\int_{S^n}|\nabla_{g_0}^{l_1}v(t)|\cdot|\nabla_{g_0}^{l_2}w(t)|\cdots|\nabla_{g_0}^{l_m}w(t)|\mathrm{e}^{\alpha w(t)}dV_{g_0}\\
&\le -\|v(t)\|^2_{W^{\frac{n}{2},2}}+C(T)\sum_{l_1,\ldots,l_m}\|\nabla_{g_0}^{l_1}v(t)\|_{L^{q_1}}\\
&\quad\cdot\|\nabla_{g_0}^{l_2}w(t)\|_{L^{q_2}}\cdots\|\nabla_{g_0}^{l_m}w(t)\|_{L^{q_m}}\\
&\le -\|v(t)\|^2_{W^{\frac{n}{2},2}}+C(T)\sum_{l_1,\ldots,l_m}\|v(t)\|_{W^{l_1-\frac{n}{q_1}+\frac{n}{2},2}}\\
&\quad\cdot\|w(t)\|_{W^{l_2-\frac{n}{q_2}+\frac{n}{2},2}}\cdots\|w(t)\|_{W^{l_m-\frac{n}{q_m}+\frac{n}{2},2}}\\
&\le -\|v(t)\|^2_{W^{\frac{n}{2},2}}+C(T)\sum_{l_1,\ldots,l_m}\|v(t)\|^{1-\rho_1}_{L^2}\cdot\|v(t)\|^{\rho_1}_{W^{\frac{n}{2},2}}\\
&\quad\cdot\|w(t)\|^{(1-\rho_2)+\cdots+(1-\rho_m)}_{W^{\frac{n}{2},2}}\cdot\|w(t)\|^{\rho_2+\cdots+\rho_m}_{W^{n,2}}\\
&\le -\|v(t)\|^2_{W^{\frac{n}{2},2}}+C(T)\sum_{l_1,\ldots,l_m}\|v(t)\|^{1-\rho_1}_{L^2}\cdot\|v(t)\|^{\rho_1}_{W^{\frac{n}{2},2}}\cdot\|w(t)\|^{\rho_2+\cdots+\rho_m}_{W^{n,2}}\\
&\le C(T)\sum_{l_1,\ldots,l_m}\|v(t)\|^{\frac{2-2\rho_1}{2-\rho_1}}_{L^2}\cdot\|w(t)\|^{\frac{2(\rho_2+\cdots+\rho_m)}{2-\rho_1}}_{W^{n,2}}\\
&\le C(T)(\|v(t)\|^2_{L^2}+1)(\|w(t)\|^2_{W^{n,2}}+1).
\end{aligned}$$

Thus, we conclude that

$$\frac{d}{dt}\left(\int_{S^n}(P_{g_0}w(t))^2dV_{g_0}\right)\le C(T)(\|v(t)\|^2_{L^2}+1)(\|w(t)\|^2_{W^{n,2}}+1)$$

for all $0 \le t \le T$. Hence, by the definition that $v(t)=-\frac{1}{2}\mathrm{e}^{\frac{n}{2}w(t)}(Q_{g(t)}-q)$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\int_{S^n}(P_{g_0}w(t))^2dV_{g_0}+1\right)\\
&\quad\le C(T)\left(\int_{S^n}(Q_{g(t)}-q)^2dV_{g(t)}+1\right)\left(\int_{S^n}(P_{g_0}w(t))^2dV_{g_0}+1\right),
\end{aligned}$$

for all $0 \le t \le T$. On the other hand, we have

$$\int_0^T \int_{S^n} (Q_{g(t)} - q)^2 \, dV_{g(t)} \, dt = \frac{2}{n} E_{g_0}[w(0)] - \frac{2}{n} E_{g_0}[w(t)] \le C$$

by Proposition 2.2. Thus, we conclude that

$$\int_{S^n} (P_{g_0} w(t))^2 \, dV_{g_0} \le C(T)$$

for all $0 \le t \le T$. This completes the proof. □

Once we know that the solution is bounded in $W^{n,2}$, it is not difficult to derive uniform estimates on any fixed time interval $[0, T]$. This implies that the flow exists for all time. More precisely, we have:

Proposition 4.2. *Given any $T \ge 0$ and $k > n/2$, there exists a constant $C(T)$ such that $\|w(t)\|_{W^{2k,2}} \le C(T)$ for all $0 \le t \le T$.*

Proof. Note that

$$\frac{d}{dt} \left(\int_{S^n} |(-\Delta_{g_0})^k w(t)|^2 \, dV_{g_0} \right) \le - \int_{S^n} \mathrm{e}^{-nw(t)} |(-\Delta_{g_0})^{k + \frac{n}{4}} w(t)|^2 \, dV_{g_0} + C \sum_{k_1, \ldots, k_m} \int_{S^n} |\nabla_{g_0}^{k_1} w(t)| \cdots |\nabla_{g_0}^{k_m} w(t)| \, dV_{g_0},$$

which implies that for $0 \le t \le T$

$$\frac{d}{dt} \left(\int_{S^n} |(-\Delta_{g_0})^k w(t)|^2 \, dV_{g_0} \right) \le - \frac{1}{C(T)} \int_{S^n} |(-\Delta_{g_0})^{k + \frac{n}{4}} w(t)|^2 \, dV_{g_0} + C \sum_{k_1, \ldots, k_m} \int_{S^n} |\nabla_{g_0}^{k_1} w(t)| \cdots |\nabla_{g_0}^{k_m} w(t)| \, dV_{g_0},$$

since $\|w(t)\|_{W^{n,2}} \le C(T)$ for $0 \le t \le T$ by (4.1). Here the sum is taken over all m-tuples $k_1, \ldots, k_m$, with $m \ge 3$, which satisfy the conditions

$$1 \le k_i \le 2k + \frac{n}{2} \quad \text{and} \quad k_1 + \cdots + k_m \le 4k + n.$$

Now we choose real numbers $p_1, \ldots, p_m \in [2, \infty)$ such that

$$k_i \le 2k + \frac{n}{p_i} \quad \text{and} \quad \frac{1}{p_1} + \cdots \frac{1}{p_m} = 1.$$

Moreover, we define real numbers $\theta_1, ..., \theta_m$ by

$$\theta_i = \max\left\{\frac{k_i - \frac{n}{p_i} - \frac{n}{2}}{2k - \frac{n}{2}}, 0\right\}.$$

Since $m \geq 3$, we can choose $p_1, ..., p_m \in [2, \infty)$ such that

$$\theta_1 + \cdots + \theta_m < 2.$$

From this, it follows that for $0 \leq t \leq T$

$$\begin{aligned}
&\frac{d}{dt}\|w(t)\|^2_{W^{2k,2}} \\
&\leq -\frac{1}{C(T)}\|w(t)\|_{W^{2k+\frac{n}{2},2}} + C\sum_{k_1,...,k_m}\|\nabla^{k_1}_{g_0} w(t)\|_{L^{p_1}}\cdots\|\nabla^{k_m}_{g_0} w(t)\|_{L^{p_m}} \\
&\leq -\frac{1}{C(T)}\|w(t)\|_{W^{2k+\frac{n}{2},2}} + C\sum_{k_1,...,k_m}\|w(t)\|_{W^{k_1-\frac{n}{p_1}+\frac{n}{2},2}}\cdots\|w(t)\|_{W^{k_m-\frac{n}{p_m}+\frac{n}{2},2}} \\
&\leq -\frac{1}{C(T)}\|w(t)\|_{W^{2k+\frac{n}{2},2}} + C\sum_{k_1,...,k_m}\|w(t)\|^{(1-\theta_1)+\cdots+(1-\theta_m)}_{W^{n,2}}\|w(t)\|^{\theta_1+\cdots+\theta_m}_{W^{2k+\frac{n}{2},2}} \\
&\leq -\frac{1}{C(T)}\|w(t)\|_{W^{2k+\frac{n}{2},2}} + C(T)\sum_{k_1,...,k_m}\|w(t)\|^{\theta_1+\cdots+\theta_m}_{W^{2k+\frac{n}{2},2}} \\
&\leq -\frac{1}{C(T)}\|w(t)\|_{W^{2k+\frac{n}{2},2}} + C(T) \\
&\leq -\frac{1}{C(T)}\|w(t)\|_{W^{n,2}} + C(T).
\end{aligned}$$

Thus, we conclude that

$$\|w(t)\|_{W^{2k,2}} \leq C(T)$$

for any $k > n/2$ and for all $0 \leq t \leq T$. □

5. Uniform estimates independent of time

For brevity, let

$$F(t) = \int_{S^n}(Q_{g(t)} - q)^2\, dV_{g(t)}.$$

Proposition 5.1. *We have $F(t) \to 0$ as $t \to \infty$.*

Proof. Let $\epsilon > 0$. By Proposition 2.2, we have

$$\int_0^\infty F(t)\,dt \leq C.$$

Hence, given any $\eta > 0$, we can find $t_0 \geq 0$ such that

$$F(t_0) \leq \frac{\epsilon}{2} \quad \text{and} \quad \int_{t_0}^\infty F(t)\,dt \leq \eta. \tag{5.1}$$

We want to show that $F(t) \leq \epsilon$ for all $t \geq t_0$. To this end, we define

$$t_1 = \inf\{t \geq t_0 : F(t) \geq \epsilon\}. \tag{5.2}$$

This implies that

$$F(t) \leq \epsilon \quad \text{for all } t_0 \leq t \leq t_1.$$

Since

$$\begin{aligned} F(t) &= \int_{S^n} (Q_{g(t)} - q)^2\,dV_{g(t)} \\ &= \int_{S^n} Q^2_{g(t)}\,dV_{g(t)} - 2q\int_{S^n} Q_{g(t)}\,dV_{g(t)} + q^2\int_{S^n} dV_{g(t)} \\ &= \int_{S^n} Q^2_{g(t)}\,dV_{g(t)} - q(n-1)!\int_{S^n} dV_{g_0} \end{aligned}$$

by (2.7), we have

$$\int_{S^n} Q^2_{g(t)}\,dV_{g(t)} \leq q(n-1)!\int_{S^n} dV_{g_0} + \epsilon \tag{5.3}$$

for all $t_0 \leq t \leq t_1$. By Proposition 3.2 and Trudinger's inequality, we have

$$\int_{S^n} \mathrm{e}^{3n\tilde{w}(t)} dV_{g_0} \leq C. \tag{5.4}$$

Using (2.3), (5.3), (5.4), and Hölder's inequality, we obtain

$$\begin{aligned}
&\int_{S^n}|(n-1)!+P_{g_0}\tilde{w}(t)|^{\frac{3}{2}}\,dV_{g_0}\\
&\leq\left(\int_{S^n}\mathrm{e}^{-n\tilde{w}(t)}((n-1)!+P_{g_0}\tilde{w}(t))^2\,dV_{g_0}\right)^{\frac{3}{4}}\left(\int_{S^n}\mathrm{e}^{3n\tilde{w}(t)}\,dV_{S^n}\right)^{\frac{1}{4}}\\
&=\left(\int_{S^n}Q_{\tilde{g}(t)}^2\,dV_{\tilde{g}(t)}\right)^{\frac{3}{4}}\left(\int_{S^n}\mathrm{e}^{3n\tilde{w}(t)}\,dV_{S^n}\right)^{\frac{1}{4}}\\
&\leq\left(q(n-1)!\int_{S^n}dV_{g_0}+\epsilon\right)^{\frac{3}{4}}\left(\int_{S^n}\mathrm{e}^{3n\tilde{w}(t)}\,dV_{S^n}\right)^{\frac{1}{4}}\leq C
\end{aligned}$$

for all $t_0\leq t\leq t_1$. This implies that

$$\int_{S^n}|P_{g_0}\tilde{w}(t)|^{\frac{3}{2}}\,dV_{g_0}\leq C$$

for all $t_0\leq t\leq t_1$. Using standard elliptic regularity theory, we obtain

$$|\tilde{w}(t)|\leq C\quad\text{for all } t_0\leq t\leq t_1. \tag{5.5}$$

Here C is a constant which only depends on the initial data.

By (2.2), (2.3) and (2.8), we have

$$\begin{aligned}
\frac{\partial}{\partial t}Q_{g(t)}&=\frac{\partial}{\partial t}\left(\mathrm{e}^{-nw(t)}((n-1)!+P_{g_0}w(t))\right)\\
&=-n\mathrm{e}^{-nw(t)}((n-1)!+P_{g_0}w(t))\frac{\partial}{\partial t}w(t)+\mathrm{e}^{-nw(t)}P_{g_0}\left(\frac{\partial}{\partial t}w(t)\right)\\
&=\frac{n}{2}Q_{g(t)}(Q_{g(t)}-q)-\frac{1}{2}P_{g(t)}Q_{g(t)}.
\end{aligned}$$

From this, it follows that

$$\begin{aligned}
&\frac{d}{dt}\int_{S^n}\left(Q_{g(t)}-q\right)^2dV_{g(t)}\\
&=\int_{S^n}2\left(Q_{g(t)}-q\right)\frac{\partial}{\partial t}(Q_{g(t)})\,dV_{g(t)}+\int_{S^n}\left(Q_{g(t)}-q\right)^2\frac{\partial}{\partial t}\,dV_{g(t)}\\
&=\frac{n}{2}\int_{S^n}\left(Q_{g(t)}-q\right)^3dV_{g(t)}+nq\int_{S^n}\left(Q_{g(t)}-q\right)^2dV_{g(t)}\\
&\quad-\int_{S^n}Q_{g(t)}P_{g(t)}Q_{g(t)}\,dV_{g(t)}. \qquad (5.6)
\end{aligned}$$

Using the Gagliardo–Nirenberg inequality, we have

$$\|Q_{\tilde{g}(t)} - q\|_{L^3}^3 \leq C\|Q_{\tilde{g}(t)} - q\|_{L^2}^2 \cdot \|Q_{\tilde{g}(t)} - q\|_{W^{\frac{n}{2},2}},$$

which implies

$$\int_{S^n} \left(Q_{\tilde{g}(t)} - q\right)^3 dV_{g_0} \leq C \left(\int_{S^n} \left(Q_{\tilde{g}(t)} - q\right)^2 dV_{g_0}\right) \left(\int_{S^n} Q_{\tilde{g}(t)} P_{g_0} Q_{\tilde{g}(t)} dV_{g_0}\right)^{\frac{1}{2}}.$$

Using (2.2), we obtain

$$\begin{aligned}&\int_{S^n} \mathrm{e}^{-n\tilde{w}(t)} \left(Q_{\tilde{g}(t)} - q\right)^3 dV_{\tilde{g}(t)} \\ &\quad \leq C \left(\int_{S^n} \mathrm{e}^{-n\tilde{w}(t)} \left(Q_{\tilde{g}(t)} - q\right)^2 dV_{\tilde{g}(t)}\right) \left(\int_{S^n} Q_{\tilde{g}(t)} P_{\tilde{g}(t)} Q_{\tilde{g}(t)}\, dV_{\tilde{g}(t)}\right)^{\frac{1}{2}}.\end{aligned}$$

Since the function $w(t)$ is uniformly bounded for $t_0 \leq t \leq t_1$ by (5.5), it follows that

$$\begin{aligned}&\int_{S^n} \left(Q_{\tilde{g}(t)} - q\right)^3 dV_{\tilde{g}(t)} \\ &\quad \leq C \left(\int_{S^n} \left(Q_{\tilde{g}(t)} - q\right)^2 dV_{\tilde{g}(t)}\right) \left(\int_{S^n} Q_{\tilde{g}(t)} P_{\tilde{g}(t)} Q_{\tilde{g}(t)}\, dV_{\tilde{g}(t)}\right)^{\frac{1}{2}}\end{aligned}$$

for all $t_0 \leq t \leq t_1$. This is equivalent to

$$\begin{aligned}&\int_{S^n} \left(Q_{g(t)} - q\right)^3 dV_{g(t)} \\ &\quad \leq C \left(\int_{S^n} \left(Q_{g(t)} - q\right)^2 dV_{g(t)}\right) \left(\int_{S^n} Q_{g(t)} P_{g(t)} Q_{g(t)} dV_{g(t)}\right)^{\frac{1}{2}}\end{aligned} \tag{5.7}$$

for all $t_0 \leq t \leq t_1$. Combining (5.6) and (5.7), we have

$$\begin{aligned}&\frac{d}{dt} \int_{S^n} \left(Q_{g(t)} - q\right)^2 dV_{g(t)} \\ &\quad \leq C \int_{S^n} \left(Q_{g(t)} - q\right)^2 dV_{g(t)} + C \left(\int_{S^n} \left(Q_{g(t)} - q\right)^2 dV_{g(t)}\right)^2\end{aligned}$$

for all $t_0 \leq t \leq t_1$. Hence, there exists a constant C, which depends only on the initial data, such that

$$\frac{d}{dt} F(t) \leq C(F(t) + F(t)^2)$$

for all $t_0 \le t \le t_1$, which implies that

$$\frac{\epsilon}{2} \le F(t_1) - F(t_0) \le C \int_{t_0}^{t_1} (F(t) + F(t)^2)\, dt \le C(1+\epsilon)\eta.$$

Here we have used (5.1) and (5.2). But this is impossible if we choose η sufficiently small. Thus, we conclude that $F(t) \le \epsilon$ for all $t \ge t_0$. This proves the assertion. □

Proposition 5.2. *There exists a constant C which depends only on the initial data such that $\|\tilde{w}(t)\|_{W^{n,2}} \le C$ for all $t \ge 0$. Moreover, we have*

$$\int_{S^n} ((n-1)! + P_{g_0}\tilde{w}(t) - qe^{n\tilde{w}(t)})^2\, dV_{g_0} \to 0$$

as $t \to \infty$.

Proof. By Proposition 5.1, there exists a constant C such that $F(t) \le C$ for all $t \ge 0$. This implies that $\int_{S^n} Q_{g(t)}^2\, dV_{g(t)} \le C$ for all $t \ge 0$, hence $\int_{S^n} Q_{\tilde{g}(t)}^2\, dV_{\tilde{g}(t)} \le C$ for all $t \ge 0$. By (2.3), it is equivalent to

$$\int_{S^n} \mathrm{e}^{-n\tilde{w}(t)}((n-1)! + P_{g_0}\tilde{w}(t))^2\, dV_{g_0} \le C \quad \text{for all } t \ge 0. \tag{5.8}$$

Following the arguments in the proof of Proposition 5.1, we obtain $\int_{S^n} |(n-1)! + P_{g_0}\tilde{w}(t)|^{\frac{3}{2}} dV_{g_0} \le C$ for all $t \ge 0$. From this it follows that

$$|\tilde{w}(t)| \le C \quad \text{for all } t \ge 0. \tag{5.9}$$

Combining (5.8) and (5.9), we conclude that

$$\int_{S^n} ((n-1)! + P_{g_0}\tilde{w}(t))^2 dV_{g_0} \le C \quad \text{for all } t \ge 0.$$

Therefore, the function $\tilde{w}(t)$ is uniformly bounded in $W^{n,2}$. Moreover, we have

$$\begin{aligned}
&\int_{S^n} ((n-1)! + P_{g_0}\tilde{w}(t) - q\mathrm{e}^{n\tilde{w}(t)})^2\, dV_{g_0} \\
&\quad \le C \int_{S^n} \mathrm{e}^{-n\tilde{w}(t)}((n-1)! + P_{g_0}\tilde{w}(t) - q\mathrm{e}^{n\tilde{w}(t)})^2\, dV_{g_0} \\
&\quad = CF(t).
\end{aligned}$$

Here we have used (2.3). By Proposition 5.1, $F(t)$ converges to 0 as $t \to 0$. From this, the assertion follows. □

Proposition 5.3. *We have*

$$\left\| \tilde{w}(t) - \frac{1}{n} \log \frac{(n-1)!}{q} \right\|_{W^{n,2}} \to 0$$

as $t \to \infty$.

Proof. Suppose it is not true. Then there exists a sequence of times $\{t_k : k \in \mathbb{N}\}$ such that $t_k \to \infty$ as $k \to \infty$ and

$$\liminf_{k \to \infty} \left\| \tilde{w}(t_k) - \frac{1}{n} \log \frac{(n-1)!}{q} \right\|_{W^{n,2}} > 0.$$

By Proposition 5.2, the sequence $\{\tilde{w}(t_k) : k \in \mathbb{N}\}$ is uniformly bounded in $W^{n,2}$. Hence, by passing to a subsequence if necessary, we may assume that $\tilde{w}(t_k)$ converges to a function u in the C^0-topology. The function u is a weak solution of the equation

$$P_{g_0} u + (n-1)! = q\mathrm{e}^{nu}.$$

Standard elliptic regularity theory implies that u is smooth. According to a theorem of Chang and Yang in [11] (see also [18]), there exists a vector $p \in B^n$ such that

$$u(x) = \log \frac{1 - |p|^2}{1 + 2\langle p, x \rangle + |p|^2} + \frac{1}{n} \log \frac{(n-1)!}{q} \tag{5.10}$$

for all $x \in S^n$. Using (3.5), we obtain $\int_{S^n} \mathrm{e}^{nu} x \, dV_{g_0} = 0$. Hence, by (5.10), we have

$$\int_{S^n} \left(\frac{1 - |p|^2}{1 + 2\langle p, x \rangle + |p|^2} \right)^n \langle x, p \rangle \, dV_{g_0} = 0.$$

By a change of variable, we also have

$$\int_{S^n} \left(\frac{1 - |p|^2}{1 - 2\langle p, x \rangle + |p|^2} \right)^n \langle x, p \rangle \, dV_{g_0} = 0.$$

Hence,

$$\int_{S^n} \left[\left(\frac{1 - |p|^2}{1 - 2\langle p, x \rangle + |p|^2} \right)^n - \left(\frac{1 - |p|^2}{1 + 2\langle p, x \rangle + |p|^2} \right)^n \right] \langle x, p \rangle \, dV_{g_0} = 0.$$

Since the integrand is pointwise non-negative, it follows that $p = 0$. Thus we conclude that

$$u = \frac{1}{n}\log\frac{(n-1)!}{q}$$

by (5.10). From this, it follows that

$$\left\|\tilde{w}(t_k) - \frac{1}{n}\log\frac{(n-1)!}{q}\right\|_{C^0} \to 0$$

as $k \to \infty$. This implies

$$\left\|(n-1)! - q e^{n\tilde{w}(t_k)}\right\|_{C^0} \to 0$$

as $k \to \infty$. By Proposition 5.2, we have

$$\int_{S^n} (P_{g_0}\tilde{w}(t_k) + (n-1)! - q e^{n\tilde{w}(t_k)})^2 \, dV_{g_0} \to 0$$

as $k \to \infty$. Thus, we conclude that

$$\int_{S^n} (P_{g_0}\tilde{w}(t_k))^2 \, dV_{g_0} \to 0$$

as $k \to \infty$. From this it follows that

$$\left\|\tilde{w}(t_k) - \frac{1}{n}\log\frac{(n-1)!}{q}\right\|_{W^{n,2}} \to 0$$

as $k \to \infty$. This is a contradiction. □

Proposition 5.4. *We can find positive real numbers t_0 and C such that*

$$\left\|\tilde{w}(t) - \frac{1}{n}\log\frac{(n-1)!}{q}\right\|_{W^{n,2}} \leq C\int_{S^n} (Q_{g(t)} - q)^2 \, dV_{g(t)}$$

for all $t \geq t_0$.

Proof. For abbreviation, let

$$z(t) = \tilde{w}(t) - \frac{1}{n} \log \frac{(n-1)!}{q}.$$

By Proposition 5.3, $\|z(t)\|_{C^0} \to 0$ as $t \to \infty$. This implies

$$\begin{aligned}
&\int_{S^n} (P_{g_0} z(t) - n! z(t))^2 \, dV_{g_0} \\
&\quad \leq 2 \int_{S^n} (P_{g_0} z(t) + (n-1)! - (n-1)! \mathrm{e}^{nz(t)})^2 \, dV_{g_0} \\
&\qquad + 2 \int_{S^n} ((n-1)! - (n-1)! \mathrm{e}^{nz(t)} - n! z(t))^2 \, dV_{g_0} \\
&\quad \leq 2 \int_{S^n} (P_{g_0} \tilde{w}(t) + (n-1)! - q \mathrm{e}^{n\tilde{w}(t)})^2 \, dV_{g_0} + C \int_{S^n} z(t)^4 \, dV_{g_0} \\
&\quad \leq 2 \int_{S^n} \mathrm{e}^{n\tilde{w}(t)} (Q_{\tilde{g}(t)} - q)^2 \, dV_{\tilde{g}(t)} + o(1) \|z(t)\|_{L^2},
\end{aligned}$$

where we have used (2.3) in the last inequality. Since $\tilde{w}(t)$ is uniformly bounded by Proposition 5.2, it follows that

$$\int_{S^n} (P_{g_0} z(t) - n! z(t))^2 \, dV_{g_0} \leq C \int_{S^n} (Q_{\tilde{g}(t)} - q)^2 \, dV_{\tilde{g}(t)} + o(1) \|z(t)\|_{L^2}.$$

Moreover, we have

$$\begin{aligned}
\left| \int_{S^n} n z(t) x \, dV_{g_0} \right| &= \left| \int_{S^n} (\mathrm{e}^{nz(t)} - 1 - nz(t)) x \, dV_{g_0} \right| \\
&\leq C \int_{S^n} z(t)^2 \, dV_{g_0} \leq o(1) \|z(t)\|_{L^2}.
\end{aligned}$$

Here we have used (3.5) in the first equality. Using the estimate

$$\|z(t)\|_{W^{n,2}} \leq C \int_{S^n} (P_{g_0} z(t) - n! z(t))^2 \, dV_{g_0} + C \left| \int_{S^n} z(t) x \, dV_{g_0} \right|^2,$$

we obtain

$$\|z(t)\|_{W^{n,2}} \leq C \int_{S^n} (Q_{\tilde{g}(t)} - q)^2 \, dV_{\tilde{g}(t)} + o(1) \|z(t)\|_{L^2}.$$

From this, the assertion follows. □

Proposition 5.5. *For every $t \geq 0$, we have*

$$\int_t^\infty \int_{S^n} (Q_{g(\tau)} - q)^2 \, dV_{g(\tau)} d\tau \leq C \left\| \tilde{w}(t) - \frac{1}{n} \log \frac{(n-1)!}{q} \right\|_{W^{n,2}}.$$

Proof. By Proposition 5.3, we have $\lim_{t\to\infty} \left\| \tilde{w}(t) - \frac{1}{n} \log \frac{(n-1)!}{q} \right\|_{W^{n,2}} = 0$. This implies

$$\lim_{t\to\infty} E_{g_0}[\tilde{w}(t)] = 0.$$

Since the functional E_{g_0} is invariant under conformal transformations (see [10], part (a) of the proof of Theorem 4.1), it follows that

$$\lim_{t\to\infty} E_{g_0}[w(t)] = 0.$$

By Proposition 2.2, we obtain

$$\int_t^\infty \int_{S^n} (Q_{g(\tau)} - q)^2 \, dV_{g(\tau)} d\tau = \frac{1}{2} E_{g_0}[w(t)] = \frac{1}{2} E_{g_0}[\tilde{w}(t)].$$

On the other hand, we have

$$\begin{aligned} n! \int_{S^n} \tilde{w} \, dV_{g_0} - (n-1)! \int_{S^n} dV_{g_0} \log \left(\frac{\int_{S^n} e^{n\tilde{w}} \, dV_{g_0}}{\int_{S^n} dV_{g_0}} \right) \\ \leq n! \int_{S^n} \tilde{w} dV_{g_0} - (n-1)! \int_{S^n} dV_{g_0} \cdot \frac{\int_{S^n} \log(e^{n\tilde{w}}) \, dV_{g_0}}{\int_{S^n} dV_{g_0}} = 0 \end{aligned}$$

by Jensen's inequality. Hence, by (2.9), we have

$$E_{g_0}[\tilde{w}(t)] \leq \frac{n}{2} \int_{S^n} \tilde{w}(t) P_{g_0} \tilde{w}(t) dV_{g_0} \leq C \left\| \tilde{w}(t) - \frac{1}{n} \log \frac{(n-1)!}{q} \right\|_{W^{n,2}}.$$

Thus, we conclude that

$$\int_t^\infty \int_{S^n} (Q_{g(\tau)} - q)^2 \, dV_{g(\tau)} d\tau \leq C \left\| \tilde{w}(t) - \frac{1}{n} \log \frac{(n-1)!}{q} \right\|_{W^{n,2}}$$

as required. □

Corollary 5.1. *There exists a constant C such that*

$$\int_t^\infty \int_{S^n} (Q_{g(\tau)} - q)^2 dV_{g(\tau)} d\tau \leq C \int_{S^n} (Q_{g(t)} - q)^2 \, dV_{g(t)}$$

for $t \geq t_0$.

Proof. This follows immediately from Propositions 5.4 and 5.5. □

Corollary 5.2. *We can find positive constants C and α such that*

$$\int_t^\infty \left(\int_{S^n} (Q_{g(\tau)} - q)^2 \, dV_{g(\tau)} \right)^{\frac{1}{2}} d\tau \leq C e^{-\alpha t}$$

for all $t \geq 0$.

Proof. If we let $f(t) = \int_t^\infty \int_{S^n} (Q_{g(\tau)} - q)^2 \, dV_{g(\tau)} \, d\tau$, then by Corollary 5.1, we have

$$f(t) \leq -C \frac{df(t)}{dt} \quad \text{for all } t \geq t_0.$$

Integrating it, we obtain

$$\mathrm{e}^{-C(t-t_0)} \geq \frac{f(t)}{f(t_0)} \quad \text{for all } t \geq t_0.$$

In particular, we have

$$\int_t^\infty \int_{S^n} (Q_{g(\tau)} - q)^2 \, dV_{g(\tau)} \, d\tau \leq C \mathrm{e}^{-2\alpha t}$$

for suitable constants C, $\alpha > 0$. This implies that

$$\int_k^{k+1} \left(\int_{S^n} (Q_{g(\tau)} - q)^2 \, dV_{g(\tau)} \right)^{\frac{1}{2}} d\tau \leq C \mathrm{e}^{-\alpha k}$$

by Hölder's inequality. Summation over k gives

$$\int_k^\infty \left(\int_{S^n} (Q_{g(\tau)} - q)^2 \, dV_{g(\tau)} \right)^{\frac{1}{2}} d\tau \leq C \mathrm{e}^{-\alpha k}.$$

From this the assertion follows. □

Proposition 5.6. *There exists a uniform constant C such that $\|w(t)\|_{W^{n,2}} \leq C$ for all $t \geq 0$.*

Proof. By Proposition 3.3 and Hölder's inequality, we have

$$\begin{aligned}\left|\frac{d}{dt}\log(1-|p(t)|^2)\right| &\leq \left|\int_{S^n}(Q_{\tilde{g}(t)}-q)x\,dV_{\tilde{g}(t)}\right| \\ &\leq C\left(\int_{S^n}(Q_{\tilde{g}(t)}-q)^2\,dV_{\tilde{g}(t)}\right)^{\frac{1}{2}} \\ &= C\left(\int_{S^n}(Q_{g(t)}-q)^2\,dV_{g(t)}\right)^{\frac{1}{2}}.\end{aligned}$$

Moreover, we have

$$\int_0^\infty\left(\int_{S^n}(Q_{g(t)}-q)^2dV_{g(t)}\right)^{\frac{1}{2}}dt \leq C$$

by Corollary 5.2. Hence, there exists a constant C, which depends only on the initial data, such that $\frac{1}{1-|p(t)|^2} \leq C$ for all $t \geq 0$. On the other hand, by Proposition 5.2, we have $\|\tilde{w}(t)\|_{W^{n,2}} \leq C$ for all $t \geq 0$. Thus, we conclude that $\|w(t)\|_{W^{n,2}} \leq C$ for all $t \geq 0$. □

Proposition 5.7. *There exist positive constants C and α such that*

$$\|w(t_2)-w(t_1)\|_{L^2} \leq Ce^{-\alpha t_1}$$

for all $t_1 \leq t_2$.

Proof. By (2.8), we have

$$w(t_2)-w(t_1) = -\frac{1}{2}\int_{t_1}^{t_2}(Q_{g(\tau)}-q)\,d\tau,$$

which implies

$$\|w(t_2)-w(t_1)\|_{L^2} \leq C\int_{t_1}^{t_2}\left(\int_{S^n}(Q_{g(\tau)}-q)^2\,dV_{g(\tau)}\right)^{\frac{1}{2}}d\tau \leq C\mathrm{e}^{-\alpha t_1}$$

by Corollary 5.2. This proves the assertion. □

Since $w(t)$ is uniformly bounded in $W^{n,2}$ by Proposition 5.6, it is not difficult to derive uniform regularity estimates for $w(t)$ by following the proof of Proposition 4.2. Exponential convergence follows from Proposition 5.7. This proves Theorem 1.1.

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