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Surfaces with maximal constant mean curvature

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In this note we consider asymptotically flat manifolds with non-negative scalar curvature and an inner boundary which is an outermost minimal surface. We show that there exists an upper bound for the mean curvature of a constant mean curvature (CMC) surface homologous to a subset of the interior boundary components. This bound allows us to find a maximizer for the CMC of a surface homologous to the inner boundary.

With this maximizer at hand, we can construct an increasing family of sets with boundaries of increasing CMC. We interpret this family as a weak version of a CMC foliation.

1. Introduction

Consider a non-compact three dimensional Riemannian manifold (M, g) with compact interior boundary ∂M, which is the only minimal surface in (M, g). In this paper, we investigate how large the mean curvature H of an embedded, constant mean curvature (CMC) surface in the homology class of ∂M can be. The main result in this paper is an upper bound for this curvature. Combined with an area estimate, we then show that there exists a CMC surface which attains this maximum.

The main motivation for this work is CMC foliations. These foliations have been used successfully in general relativity to study the center of mass of isolated systems [8, 10] and the Riemannian Penrose inequality [4]. The existence result in [8] constructs a CMC foliation in the asymptotic region near infinity. The natural question arises as to how far to the interior these foliations can be extended. It is clear that in general, topological reasons imply non-existence of an entire smooth foliation. This calls for a weak version of a CMC foliation.

Let us consider a different perspective. If the interior boundary ∂M is an outermost minimal surface, that is (M, g) does not contain any other minimal surface, then it is straightforward to construct a local CMC foliation near ∂M, cf. Lemma 4.1. So another question is how far this interior foliation can be extended outward, away from ∂M. This is by far an easier question than extending the foliation inward.

The reason is the following. Roughly speaking, if we consider a potential CMC foliation reaching from ∂M to infinity, then the mean curvature has to increase near ∂M, and decrease as in Euclidean space when approaching infinity, as the surfaces of the foliation enclose increasing volume with ∂M. This has two implications. First, there is a maximal value of CMC along this foliation, and second that there are two types of behavior. The first type is portions along which CMC increases and the other is where CMC decreases. The former includes the region near ∂M, and the latter the asymptotic region.

The region in which CMC increases is easier to handle, as the maximum principle implies that the CMC surfaces along the foliation cannot touch. In the exterior region it is a lot harder to get control on the separation of the surfaces, as one can see from the many different foliations by spheres that are possible in $\mathbf{R}^3$.

This result gives a partial answer to the above questions. In Section 3 we show that there is a bound for the maximal CMC of a surface homologous to ∂M. This needs a lower bound on the scalar curvature of M, ${}^M\mathrm{Sc} \geq -C$, and uses the fact that ∂M is area-minimizing. The condition that the surface be homologous to ∂M is necessary, as close to maxima of the scalar curvature a CMC foliation exists where homologically trivial spheres have unbounded CMC [13].

In Section 4 the curvature bound is used to construct a surface which realizes the maximal CMC $H_{\max}$. This existence result needs the stronger assumption that ${}^M\mathrm{Sc} \geq 0$ in order to ensure that the area of CMC surfaces is bounded. A unique such surface can be selected by demanding that it be the innermost one.

We show that this surface bounds a region with ∂M, which can be regarded as a manifold with boundary, cf. the discussion at the end of Sections 4 and in 5. Using the outer boundary as barrier, we can construct an increasing family of sets bounded by surfaces with CMC ranging from 0 at the horizon to $H_{\max}$ at the boundary. This increasing family is a candidate for a weak version of a CMC foliation reaching up to the $H_{\max}$-surface. We explore some basic properties in the second half of Section 5.

2. Preliminaries

Let (M, g) be an asymptotically flat manifold with inner boundary ∂M which is an outermost minimal surface. Such manifolds M are called *exterior regions*. The requirement of asymptotic flatness means that there exists a compact set $K \subset M$ and a diffeomorphism $x : M \setminus K \to \mathbf{R}^3 \setminus B_1(0)$ such that in the x-coordinates the metric g approaches the Euclidean metric δ, that is there exists C such that

$$r|g-\delta| + r^2|\partial g| \le C.$$

To say that ∂M is an outermost minimal surface, means that there does not exist another minimal surface in M which is homologous to ∂M. For an asymptotically flat manifold that contains minimal surfaces, the outermost minimal surface always exists and is unique [7, Section 4]. An exterior region M is diffeomorphic to $\mathbf{R}^3 \setminus (\bigcup_{i=1}^N B_i)$, where the B_i are open balls with disjoint closure. Hence $\partial M = \bigcup_{i=1}^N S_i$, where $S_i = \partial B_i$. This restricted topology does not require any curvature assumptions. The fact that ∂M is an outermost minimal surface implies furthermore that for each $I \subset \{1,\dots,N\}$ the set $\bigcup_{i\in I} S_i$ minimizes area in its homology class, in particular ∂M is minimizing.

Let $\Sigma \subset M$ be a compact, closed, immersed and two-sided surface. We assume that one side of Σ can be identified as the outside, and denote the outward pointing normal by ν. Later we restrict ourselves to such surfaces. We denote by γ the induced metric. The mean curvature $H = \operatorname{div}\nu$ is taken with respect to the outward pointing normal as is the second fundamental form A. By ${}^\Sigma\mathrm{Sc}$ we denote the scalar curvature of Σ. The trace free part of the second fundamental form will be denoted by $\mathring{A} = A - \frac{1}{2}H\gamma$.

Consider a normal variation of Σ, that is a map $F : \Sigma \times (-\varepsilon, \varepsilon) \to M$ with $F(\cdot, 0) = \mathrm{id}_\Sigma$ and $\frac{dF}{dt}\big|_{t=0} = f\nu$. The linearization L of the operator which assigns the mean curvature to the surfaces $\Sigma_t = F(\Sigma, t)$ is given by

$$\frac{\partial}{\partial t}\bigg|_{t=0} F_t^* H(\Sigma_t) = Lf = -\Delta f - \left(\frac{1}{2}{}^M\mathrm{Sc} - \frac{1}{2}{}^\Sigma\mathrm{Sc} + \frac{1}{2}|\mathring{A}|^2 + \frac{3}{4}H^2\right) f,$$

where $F_t = F(\cdot, t) : \Sigma \to \Sigma_t$ and Δ denotes the Laplace–Beltrami operator along Σ. Here ${}^M\mathrm{Sc}$ and ${}^\Sigma\mathrm{Sc}$ denote the scalar curvature of M and Σ. L is called the *stability operator*, or Jacobi operator.

When dealing with CMC surfaces, there are two types of stability discussed in the literature. The first notion is *strong stability*, where we assume that L is a non-negative operator, that is

(2.1)
$$\int_\Sigma f^2 \left(\frac{1}{2}{}^M\mathrm{Sc} - \frac{1}{2}{}^\Sigma\mathrm{Sc} + \frac{1}{2}|\mathring{A}|^2 + \frac{3}{4}H^2\right) d\mu \le \int_\Sigma |\nabla f|^2\, d\mu \qquad \forall f \in C^\infty(\Sigma).$$

Here ∇f denotes the tangential gradient of f and $d\mu$ the induces area measure. Note that strong stability means that the principal eigenvalue of L, that is the smallest eigenvalue, is non-negative. The second notion, simply called *stability* comes from the fact that the CMC equation is the Euler–Lagrange equation for the isoperimetric problem, that is for minimizing the area of Σ, while keeping enclosed volume constant. Minimizers of this variational principle satisfy the stability inequality

$$\int_\Sigma f^2 \left(\frac{1}{2}{}^M\mathrm{Sc} - \frac{1}{2}{}^\Sigma\mathrm{Sc} + \frac{1}{2}|\mathring{A}|^2 + \frac{3}{4}H^2\right) d\mu \leq \int_\Sigma |\nabla f|^2 d\mu$$
$$\forall f \in C^\infty(\Sigma) \text{ with } \int_\Sigma f\, d\mu = 0.$$

Hence, strong stability implies stability, but not vice versa. For the following discussion only strong stability plays a role.

Subsequently, we deal with surfaces that are not necessarily connected. We say that such a surface is strongly stable if each of its components is strongly stable, and thus if a surface is not strongly stable, it means that at least one of its components is not strongly stable.

The surfaces Σ in question will be homologous to ∂M. In the case that Σ does not touch ∂M this means that there exists an open set Ω such that $\partial\Omega$ is the disjoint union $\partial\Omega = \partial M \cup \Sigma$. As we orient ∂M with the normal pointing into M, the correct orientation of Σ corresponds to the normal vector pointing out of Ω. We will make this assumption subsequently without further notice.

3. An upper bound for CMC

This section is devoted to derive an upper bound for the CMC of a compact, smooth, embedded CMC surface homologous to ∂M. Note that Σ need not be connected for the subsequent arguments. This upper bound only requires a lower bound on the scalar curvature of M, that is ${}^M\mathrm{Sc} \geq -C$ for some $C \geq 0$.

Before we can approach the main theorem, we review an existence theorem for prescribed mean curvature surfaces [3, Theorem 5.1]. This theorem implies the following existence theorem for strongly stable CMC surfaces.

Theorem 3.1. *Let (Ω, g) be a compact Riemannian manifold with smooth boundary $\partial\Omega$ which is the disjoint union $\partial\Omega = \partial^-\Omega \cup \partial^+\Omega$, where $\partial^\pm\Omega$ are smooth, non-empty and without boundary. Assume that $\partial^-\Omega$ has mean curvature H^-, where H^- is taken with respect to the normal pointing into Ω, and $\partial^+\Omega$ has mean curvature H^+, where H^+ is taken with respect to the* *normal pointing out of* Ω. *Let* h *be such that* $\max_{\partial^-\Omega} H^- \leq h \leq \min_{\partial^+\Omega} H^+$. *Then there exists a compact, smooth, embedded, strongly stable CMC surface* $\Sigma \subset \Omega$, *homologous to* $\partial^-\Omega$ *with* $H(\Sigma) = h$.

Proof. This is a direct consequence of [3, Theorem 5.1]. To adopt the notation of the reference, let K be a symmetric bilinear form on Ω. Then for a surface Σ one can define $P = \operatorname{tr}_\Sigma K = \operatorname{tr}_M K - K(\nu,\nu)$ and $\theta^+(\Sigma) = H + P$, where H is the mean curvature of Σ as usual. From the proof of [3, Theorem 5.1] we infer that if $\theta^+(\partial^-\Omega) \leq 0$ and $\theta^+(\partial^+\Omega) \geq 0$ then there exists a compact, smooth, embedded surface Σ homologous to $\partial^-\Omega$ with $\theta^+(\Sigma) = 0$.

In the reference the theorem is first proved under the hypothesis that the strict barriers hold, that is $\theta^+(\partial^-\Omega) > 0$ and $\theta^+(\partial^+\Omega) < 0$. Then it is argued that the strict inequality can be relaxed at the interior boundary. In the same way the strict inequality can also be relaxed at the outer boundary, so that the statement above follows.

We apply this construction to the data $(\Omega, g, K = -\frac{1}{2}hg)$ such that for any surface Σ we have $\theta^+(\Sigma) = H - h$. Thus $\theta^+(\partial^-\Omega) = H^- - h \leq 0$ and $\theta^+(\partial^+\Omega) = H^+ - h \geq 0$, and the existence of a surface Σ with $H(\Sigma) = h$ follows from the existence part of [3, Theorem 5.1]. The reference also establishes that the resulting surface is also stable in the sense of $\theta^+ = 0$ surfaces, which means that the smallest eigenvalue of the operator

$$\begin{aligned}\tilde{L}f = -\Delta f + 2S(\nabla f) + f\Big(&\operatorname{div}_\Sigma S - \frac{1}{2}|A + K^\Sigma|^2 \\ &- |S|^2 + \frac{1}{2}{}^\Sigma\mathrm{Sc} - \mu - J(\nu)\Big)\end{aligned} \tag{3.1}$$

is non-negative. Here $K^\Sigma = K|_\Sigma$, $S = K(\nu,\cdot)^T$, where T denotes tangential projection, $\mu = \frac{1}{2}({}^M\mathrm{Sc} - |K|^2 + (\operatorname{tr}_M K)^2)$ and $J = \operatorname{div} K - {}^M\nabla \operatorname{tr}_M K$. On a surface with $\theta^+ = 0$ we have for our choice of K that $S = 0$, $J = 0$, $|A + K^\Sigma|^2 = |A|^2 - \frac{1}{2}H^2 = |\mathring{A}|^2$ and $\mu = \frac{1}{2}{}^M\mathrm{Sc} + \frac{3}{4}H^2$. This notion of stability has been introduced in [1]. Calculation reveals that $\tilde{L}$ is nothing but the stability operator L and non-negativity of its first eigenvalue means strong stability. □

Remark 3.1. A similar existence theorem could be derived by analyzing the functional

$$J_h(F) = |\partial^* F| - h\operatorname{Vol}(F)$$

for sets F with finite perimeter. We shall use this functional in Section 5.

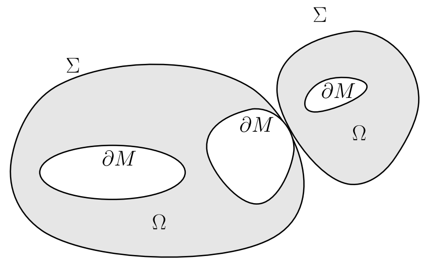

Figure 1: If a surface Σ homologous to ∂M intersects ∂M, then there is one component of Σ which intersects ∂M such that the outer normals point in the same direction.

Using the previous existence theorem together with the fact that ∂M can be used as inner barrier, we infer the following lemma.

Lemma 3.1. *If $\Sigma \subset M$ is an embedded CMC surface homologous to ∂M with $H > 0$, then there exists a strongly stable CMC surface Σ' in the same homology class with $H(\Sigma') = H(\Sigma)$.*

Proof. First, note that Σ cannot touch ∂M (cf. figure 1). As Σ is homologous to ∂M, there exists a set Ω and a set $I \subset \{1, \ldots, N\}$ such that $\partial\Omega = \bigcup_{i\in I} S_i \cup \Sigma_0$, and $\Sigma = \Sigma_0 \cup \bigcup_{i\notin I} S_i$. As $H(\Sigma) > 0$, we have must have $I = \{1, \ldots, N\}$, and thus $\Sigma_0 = \Sigma$ and $\partial M \subset \bar{\Omega}$. Thus if ∂M and Σ intersect, there exists a component Σ_1 of Σ which intersects ∂M at a point where the normals of Σ_1 and ∂M point in the same direction. This is impossible, since the maximum principle would imply that $\Sigma_1 \subset \partial M$.

Hence, Σ lies completely in the interior of M and we can apply Theorem 3.1 with the so constructed Ω where $\partial M = \partial^-\Omega$, $\Sigma = \partial^+\Omega$ and $h = H(\Sigma)$. Thus we obtain Σ', a smooth, embedded strongly stable CMC surface. □

Remark 3.2. Note that we can in fact show that the constructed Σ' cannot touch any component of Σ which is not strongly stable, as these components can be deformed in direction of $-\nu$, that is into Ω in such a way that their mean curvature increases.

It is now a simple matter to derive the claimed bound on the CMC from strong stability. As Σ is not necessarily connected, we must make use of the fact that ∂M is outermost to get a lower bound on the area of at least one component of Σ.

Lemma 3.2. *Let (M, g) be asymptotically flat with inner boundary ∂M, which is an outermost minimal surface in M. Assume that*

$$^M\mathrm{Sc} \geq -C.$$

Denote the components of ∂M by S_i, $i = 1, \ldots, N$ and let $A := \max\{|S_i| : i = 1, \ldots, N\}$. If $\Sigma \subset M$ is an embedded CMC surface homologous to ∂M, then

$$H(\Sigma)^2 \leq \frac{16\pi}{3A} + \frac{2}{3}C.$$

Remark 3.3. An obvious modification yields a similar bound if Σ is homologous to $\bigcup_{i \in I} S_i$, where $I \subset \{1, \ldots, N\}$ and A is replaced by $A(I) = \max_{i \in I} |S_i|$.

Proof. Assume that $A = |S_1|$. As M is topologically equivalent to $\mathbf{R}^3 \setminus \bigcup_{i=1}^N B_i$, as explained in Section 2, Σ can be regarded as a surface embedded in $\mathbf{R}^3$. Then any component of Σ bounds in $\mathbf{R}^3$, and since Σ is homologous to $\bigcup_{i=1}^N S_i$, we infer that there exists one component Σ_1 of Σ which is homologous to $S_1 \cup \bigcup_{i \in J} S_i$, where $J \subset \{2, \ldots, N\}$ may be empty. Since $S_1 \cup \bigcup_{i \in J} S_i$ is minimizing in its homology class in M we find that $|\Sigma_1| \geq |S_1 \cup \bigcup_{i \in J} S_i| \geq |S_1| = A$. Pick a test function $f \in C^\infty(\Sigma)$ with $f = 1$ on Σ_1 and $f = 0$ on all other components. Plugging f into the strong stability inequality (2.1), we find that

$$\int_{\Sigma_1} \frac{1}{2}|\mathring{A}|^2 + \frac{3}{4}H^2 \, d\mu \leq \int_{\Sigma_1} \frac{1}{2}{}^\Sigma\mathrm{Sc} - \frac{1}{2}{}^M\mathrm{Sc} \, d\mu.$$

From Gauss–Bonnet we infer that

$$\int_{\Sigma_1} \frac{1}{2}{}^\Sigma\mathrm{Sc} \, d\mu = 4\pi(1 - \mathrm{genus}(\Sigma_1)) \leq 4\pi.$$

As H is constant, combining the above inequality with the lower bound on ${}^M\mathrm{Sc}$ yields

$$H^2|\Sigma_1| \leq \frac{16\pi}{3} + \frac{2}{3}C|\Sigma_1|, \tag{3.2}$$

or after dividing by $|\Sigma_1|$,

$$H^2 \leq \frac{16\pi}{3|\Sigma_1|} + \frac{2}{3}C,$$

which implies the claim, since $|\Sigma_1| \geq A$. $\square$

Remark 3.4. The spatial Schwarzschild manifold of mass m is given by $(\mathbf{R}^3 \setminus \{0\}, \phi^4 g^{\mathrm{e}})$ where $\phi = 1 + \frac{m}{2r}$ and g^{e} denotes the Euclidean metric on $\mathbf{R}^3$. It is scalar flat and if $m > 0$ it has an outermost minimal surface at $r = \frac{m}{2}$. Thus $(\mathbf{R}^3 \setminus B_{\frac{m}{2}}, \phi^4 g^{\mathrm{e}})$ satisfies the assumptions of Lemma 3.2 with $C = 0$. The spheres $S_r(0)$ have CMC $H_r = \frac{2}{R}\frac{2r-m}{2r+m}$ where $R = \phi^2 r$ is the geometric area radius of S_r with respect to g^S, that is $|S_r| = 4\pi R^2$. H_r assumes its maximum where $R = 3m$ and equals $\frac{2}{3\sqrt{3}m}$ there. Thus, the estimate of Equation (3.1) is sharp in this case, whereas the assertion of Lemma 3.2 is not.

4. Existence of surfaces with maximal CMC

In this section, we construct a surface with maximal CMC. In fact, for (M, g) as before, we can let

$$H_{\max} := \sup\{H(\Sigma) : \Sigma \text{ an embedded CMC surface homologous to } \partial M\}.$$

As we have seen in the previous section, $H_{\max}$ is finite. Subsequently, we show that $H_{\max}$ is attained at a strongly stable surface. We start by showing that $H_{\max} > 0$.

Lemma 4.1. *There exists a foliation of a neighborhood of ∂M by CMC surfaces Γ_s, $s \in [0, \varepsilon)$ with $H(\Gamma_s) > 0$.*

Proof. We construct the foliation near each component of ∂M separately. Let S_i be such a component. Note that S_i is stable as a minimal surface, as ∂M is outermost. This means that the smallest eigenvalue λ of L on S_i satisfies $\lambda \geq 0$. Recall that this eigenvalue is simple and that the corresponding eigenfunction ϕ does not change sign.

If $\lambda > 0$, then L is invertible. In this case a foliation by CMC surfaces can be constructed by a simple application of the implicit function theorem. We shall not elaborate this case, as it is similar to the more difficult case $\lambda = 0$ discussed in detail below.

Thus assume $\lambda = 0$ from now on. Let $\phi > 0$ denote the corresponding eigenfunction of L. In this case a CMC foliation can be constructed as in [5]. We repeat the argument here for convenience. Consider the operator

$$\mathcal{H} : C^\infty(S_i) \times \mathbf{R} \to C^\infty(S_i) \times \mathbf{R} : (u, h) \mapsto \left(H(\operatorname{graph} u) - h, \int_{S_i} u\phi \, d\mu \right),$$

where $\operatorname{graph} u = F_u(S_i)$ and $F_u(p) = \exp_p(u(p)\nu_p)$, where $p \in S_i$ and exp is the exponential map of M. Then $H(\operatorname{graph} u)$ denotes the mean curvature of $\operatorname{graph} u$ pulled-back to S_i via F_u.

We can compute the linearization of $\mathcal{H}$ at $(u, h) = (0, 0)$ in direction $(v, s) \in C^\infty(S_i) \times \mathbf{R}$ to be

$$\mathcal{M}(v, s) := D\mathcal{H}|_{(0,0)}(v, s) = \left(Lv - s, \int_{S_i} v\phi \, d\mu\right).$$

Obviously $\mathcal{M}$ is invertible since $\ker L = \operatorname{span}\{\phi\}$ and the equation $Lv = g$ is uniquely solvable if $\int_{S_i} g\phi \, d\mu = 0$ and $\int_{S_i} v\phi \, d\mu = 0$.

By the inverse function theorem applied on suitable Banach spaces, say $\mathcal{H} : C^{2,\alpha}(S_i) \to C^{0,\alpha}(S_i)$, there exist $u(t)$ and $h(t)$ for small t such that

$$\mathcal{H}(u(t), h(t)) = (0, t). \tag{4.1}$$

This implies that the surfaces $\operatorname{graph} u(t)$ have CMC and regularity theory yields the smoothness of $u(t)$. Differentiating Equation (4.1) with respect to t yields that

$$\left(Lu'(0) - h'(0), \int_{S_i} u'(0)\phi \, d\mu\right) = (0, 1) \tag{4.2}$$

and hence that $h'(0) \in \operatorname{im} L \perp \ker L$, that is $\int_{S_i} h'(0)\phi \, d\mu = 0$. Since $h'(0)$ is a constant and $\phi > 0$, we infer $h'(0) = 0$. Then $u'(0) \in \ker L$ and $u'(0) = \alpha\phi$ where $\alpha > 0$, by (4.2). Thus, the $\operatorname{graph} u(t)$ form a foliation near S_i.

As ∂M is outermost, we must have that $h(t) > 0$ for all t, and we thus found the foliation near S_i. As $h(t)$ is smooth, there exists a t_0 such that h is increasing on $[0, t_0)$.

Thus we can find the required CMC foliation near each component of ∂M separately and join it to give a CMC foliation near ∂M. □

Remark 4.1. A different way to see that $H_{\max} > 0$ is to use asymptotic flatness to conclude that there exists a surface in the asymptotic end with positive mean curvature. An application of Theorem 3.1 then yields a CMC surface with positive CMC. However, the previous lemma emphasizes that not only the asymptotic behavior near infinity, but also the local geometry near ∂M gives a lower bound on $H_{\max}$.

Standard arguments show that there are uniform bounds on the second fundamental form of strongly stable CMC surfaces.

Lemma 4.2. *If Σ is a strongly stable CMC surface then there exists a constant $C = C(\|{}^M\mathrm{Rm}\|_{C^0}, \mathrm{inj}(M,g)^{-1}, \sup_\Sigma |H|)$ such that*

$$\sup_\Sigma |A| \leq C.$$

Proof. First, there exists $0 < r_0 = r_0(\|{}^M\mathrm{Rm}\|_{C^0}, \sup_\Sigma |H|)$ such that for all $r < r_0$ and $p \in \Sigma$ the area of the intrinsic balls $B^\Sigma(p,r)$ around p with radius r is bounded

$$|B^\Sigma(p,r)| \leq 6\pi r^2.$$

See for example [2, Theorem 8.1], which goes back to [9]. With this local bound on area, the usual argument for deriving curvature bounds yields the desired estimate (cf. [12]), we refer to [2, Section 6] for a detailed proof in a slightly more general setting. Note that the claimed constant in the estimate does not depend on the derivatives of the curvature. This improvement over [12] can be found in [2]. □

Before we can attempt the construction of surfaces realizing $H_{\max}$, we need a diameter bound for strongly stable CMC surfaces [11].

Lemma 4.3. *Let (M,g) be a complete Riemannian 3-manifold with ${}^M\mathrm{Sc} \geq 0$ and let $\Sigma \subset M$ be a closed, compact, connected, strongly stable CMC surface with $H(\Sigma) \neq 0$. Then*

$$\mathrm{diam}(\Sigma) \leq \frac{2\pi}{3H}.$$

Proof. This estimate is a direct consequence of [11, Theorem 1]. Note that in the reference an upper bound on the scalar curvature is assumed, which is relevant only along Σ. Since we assume that Σ be compact, this bound is automatic. □

Theorem 4.1. *Let (M,g) be an asymptotically flat Riemannian manifold with ${}^M\mathrm{Sc} \geq 0$ and a non-empty inner boundary ∂M, which is an outermost minimal surface. Assume that $\|{}^M\mathrm{Rm}\|_{C^0}$ is finite and $\mathrm{inj}(M,g)$ is non-zero. Then $H_{\max}$ is attained at a compact, immersed, strongly stable surface Σ homologous to ∂M. Σ is a union of spheres.*

Proof. Let $\{\Sigma^n\}_{\{n\geq 1\}}$ be family of embedded CMC surfaces homologous to ∂M with

$$H(\Sigma^n) \to H_{\max}.$$

We show that after suitable modification, the sequence Σ^n allows the extraction of a convergent subsequence.

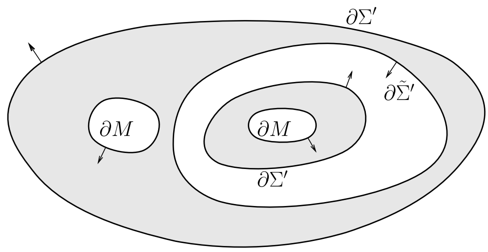

Figure 2: All components of Σ' are needed to shield ∂M from infinity. Otherwise there is a component $\tilde{\Sigma}'$ that bounds a compact region on its outside, relative to a subset of ∂M.

In view of Lemma 3.1 we can assume that the Σ^n are strongly stable. Due to Lemma 4.1 we can furthermore assume that $H(\Sigma^n) \geq \varepsilon$ for some suitably chosen $\varepsilon > 0$.

Fix an arbitrary n and denote $\Sigma := \Sigma^n$. Let Σ_j be the components of Σ, $j = 1, \ldots, N_\Sigma$. For $j = 1, \ldots, N_\Sigma$ let f_j be the test function which is equal to 1 on Σ_j and 0 on the other components. Plugging f_j into the strong stability inequality (2.1) yields that Σ_j is a sphere, as $\int_{\Sigma_j} H(\Sigma)^2 \, d\mu > 0$.

Let

$$J := \{j : \Sigma_j \text{ does not bound a compact region on its inside in } M\}$$

and delete all components Σ_j from Σ where $j \notin J$. The surface

$$\Sigma' := \bigcup_{j \in J} \Sigma_j$$

is homologous to ∂M and thus separates ∂M from infinity. Recall that M is diffeomorphic to $\mathbf{R}^3 \setminus \bigcup_{i=1}^N B_i$ and consider $\Sigma' \subset \mathbf{R}^3$.

Let $U \subset \mathbf{R}^3$ be such that $\mathbf{R}^3 \setminus U$ is the non-compact component of $\mathbf{R}^3 \setminus \Sigma'$. Note that ∂U consists of the components of Σ' needed to shield ∂M from infinity (figure 2). Indeed $\partial U = \Sigma'$. Otherwise there exists one component Σ_j in U which bounds a compact region Ω_j on its outside, relative to a subset of ∂M as illustrated in figure 2. This is clearly impossible as the boundary of $M' := M \setminus \Omega_j$ has $H(\partial M') \leq 0$ and $H(\partial M') \neq 0$. This would imply the existence of a minimal surface outside of ∂M, contradicting the assumption that ∂M is an outermost minimal surface.

Thus $\Sigma' = \partial U$ has at most N components, each of which is homologous to $\bigcup_{I_j} S_i$, where $I_j \subset \{1, \ldots, N\}$ is non-empty, and $I_j \cap I_{j'} = \emptyset$ for $j \neq j'$. To see this, let U_j be the compact region in $\mathbf{R}^3$ bounded by Σ_j. Then U_j contains at least one of the B_i, so it is clear that $I_j \neq \emptyset$. Since we have $\Sigma' = \partial U$, no component of Σ' is separated from infinity by another component of Σ', in particular the outer normal direction of Σ' agrees with the outer normal to ∂U. As all the B_i are contained in U and the components of Σ' cannot intersect, this implies that the each B_i can be in at most one U_j. Thus the I_j are mutually disjoint. For subsequent use we relabel the $(\Sigma^n)'$ as Σ^n.

This construction yields a sequence $\{\Sigma^n\}$ of CMC surfaces with $H(\Sigma^n) \to H_{\max}$, where each of the Σ^n is a strongly stable CMC surface with at most N components, and each component is homologous to a non-empty union of components of ∂M.

As $H(\Sigma^n) \geq \varepsilon$, Lemma 4.3 implies that each component of Σ^n has bounded diameter. Such a component of Σ^n encloses at least one of the S_i. We thus infer that there exists a compact set $B \subset M$ such that $\Sigma^n \subset B$ for all n. Furthermore, the curvature estimates from Lemma 4.2 imply uniform curvature bounds for Σ^n. Therefore, the Ricci curvature of Σ^n is bounded below and standard volume comparison shows that each component of Σ^n has bounded area. As there are at most N components the Σ^n have uniformly bounded area.

These three uniform estimates, area, curvature and the fact that the Σ^n are contained in a compact set, imply that there exists a convergent subsequence and a limiting surface Σ, which has CMC and consists of strongly stable components. A detailed presentation of this classical theorem can be found in [2, Section 8]. Note that the limit Σ might not be embedded. Nevertheless, Σ has an outward pointing normal vector field ν which is the limit of the outward pointing normal vector fields of the subsequence of Σ_n. □

We now examine the limiting $H_{\max}$-surface Σ more closely. As Σ is the limit of embedded surfaces, Σ can fail to be embedded only if Σ touches itself, transversal self-intersections are impossible. Let $p \in M$ denote such a point. Then at p multiple sheets of Σ can come together. Since, we have bounded curvature and bounded area, there are at most finitely many such sheets Σ^p_k, $k = 1, \ldots, n(p)$, as the curvature bounds imply that each sheet takes up a fixed amount of area in the neighborhood of p.

Around p there are coordinates $\{x^i\}$ of M such that the Σ^p_k are C^∞ graphs over an open subset U^p of the x^1, x^2-plane. That is $\Sigma^p_k = \{x : x^3 = u_k(x^1, x^2)\}$. We can assume that $u_k \leq u_l$ whenever $k \leq l$ since Σ is the limit of embedded surfaces which bound a region with respect to ∂M. Each of the sheets comes with a normal vector field ν^p_k with respect to which $H = H_{\max}$. This can be either the downward or upward pointing normal to graph u_k, and this direction alternates in k.

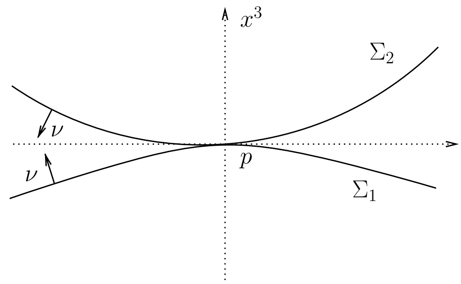

Figure 3: Two sheets touching on the outside.

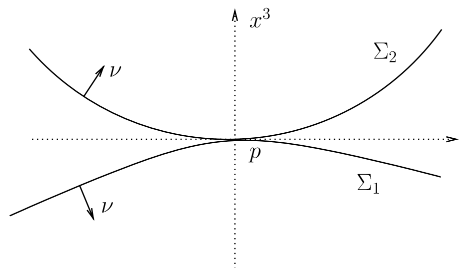

Figure 4: Two sheets touching on the inside.

We say that two sheets Σ_1 and Σ_2 touch on the *outside* at a point p, if the representing functions $u_1 \leq u_2$ of these sheets are so that the outward normal of Σ points upward along u_1 and downward along u_2 (cf. figure 3). On the other hand, if the normal along u_1 points downward, and upward along u_2, we say that Σ touches itself on the *inside* (cf. figure 4).

The following theorem is a direct consequence of the strong maximum principle for surfaces with prescribed mean curvature.

Theorem 4.2. *Let Σ be the $H_{\max}$-surface constructed in Theorem* 4.1. *Then if Σ is not embedded, Σ can only touch itself on the outside, and no more than two sheets of Σ can meet at one point of M.*

Proof. Let Σ_1 and Σ_2 be two sheets of Σ which meet on the inside, and let $u_1 \leq u_2$ be the representing functions as described above. Instead of the upward normal, consider u_2 equipped with the downward normal. Then the mean curvature of graph u_2 is $-H_{\max} < 0$ with respect to the downward normal, and the mean curvature of graph u_1 is $H_{\max}$ with respect to the downward unit normal. As graph u_1 and graph u_2 touch, we immediately obtain a contradiction to the strong maximum principle.

At any point in M where three sheets of Σ meet, Σ must touch itself on the inside, thus this is ruled out by the above argument. □

We now want to add a few remarks about the uniqueness of the $H_{\max}$-surfaces. Indeed, we can single out one particular $H_{\max}$-surface in (M, g) by choosing the *innermost* $H_{\max}$-surface.

Theorem 4.3. *Let (M, g) be as in Theorem 4.1. Then there exists a unique innermost surface in (M, g) which is homologous to ∂M and has CMC $H_{\max}$. The assertion of Theorem 4.2 holds for Σ.*

Proof. The construction of this surface is similar to the construction of the outermost MOTS in [3, Section 7]. Thus we mention only the key points for the construction.
Compactness: As in the proof of Theorem 4.1, we infer compactness of the class of $H_{\max}$-surfaces by throwing away components which bound compact regions.
Monotonicity: Let Σ_i, with $i = 1, 2$ be two $H_{\max}$-surfaces for which the assertion of Theorem 4.2 holds, and which are homologous to ∂M and bound sets Ω_i with ∂M. Then $\Omega_1 \cap \Omega_2$ contains a strongly stable $H_{\max}$-surface Σ homologous to ∂M which satisfies the assertion of Theorem 4.2.

Monotonicity allows us to construct a sequence of surfaces Σ_k bounding Ω_k together with ∂M, such that the Ω_k are descending. By compactness we find a limiting set Ω_∞ bounded by an $H_{\max}$-surface Σ_∞ and ∂M. □

5. A proposal for a weak CMC foliation

In this section, we propose a weak version of a foliation by CMC surfaces of the interior region of (M, g). There is more than one way to introduce such a foliation, and it is not clear whether the possibility discussed below is best suited for applications.

Let (M, g) be asymptotically flat with ∂M an outermost minimal surface. Assume that ${}^M\mathrm{Sc} \geq 0$ and let $\Sigma_{\max}$ be the $H_{\max}$-surface homologous to ∂M constructed in Section 4. The *interior region* Ω of M is defined as the union of the components of $M \setminus \Sigma_{\max}$ which meet components of ∂M (cf. figure 5). As $\Sigma_{\max}$ does not touch itself on the inside, Ω can be equipped with the structure of a smooth manifold with boundary $\partial M \cup \Sigma_{\max}$, where we identify $\Sigma_{\max}$ and ∂M with the points added by the metric completion

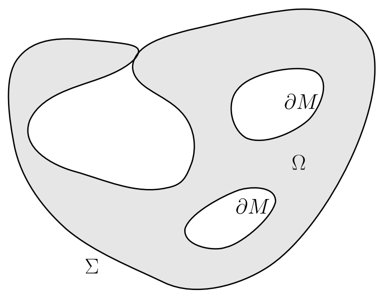

Figure 5: The interior region Ω.

of Ω. In this way, we separate the points of $\Sigma_{\max}$ which are mapped to the same point in the immersion of $\Sigma_{\max}$ into M. Note that the interior Ω is not a submanifold with smooth boundary in M if $\Sigma_{\max}$ is not embedded. The boundary of Ω consists of ∂M on the inside, subsequently denoted by $\partial^-\Omega$, and $\Sigma_{\max}$ on the outside, subsequently denoted by $\partial^+\Omega$.

5.1. Construction

To construct a weak CMC foliation for this new manifold Ω, we introduce the following notion.

Definition 5.1. Let $\Sigma \subset \Omega$ be a smooth, embedded surface homologous to $\partial^-\Omega$, with CMC $h \in (0, H_{\max})$. Denote by U the region bounded by Σ and $\partial^+\Omega$. If there does not exist a smooth embedded surface Σ' in U with the same CMC h, then Σ_h is called *outermost*.

By Andersson and Metzger [3, Section 7], for each $h \in (0, H_{\max})$ there exists a smooth, embedded surface Σ_h, homologous to $\partial^-\Omega$, which has CMC h and is outermost in the sense of Definition 5.1. We denote by Ω_h the open region bounded by $\partial^-\Omega$ and Σ_h. We define this family $\{\Sigma_h\}_{h\in[0,H_{\max}]}$ of sets to be the candidate for our weak CMC foliation of Ω.

A useful side-effect of this definition is that the constructed sets are related to a variational principle. Consider sets F of finite perimeter in Ω. We will assume that $F \supset \Omega_h$ for some $h > 0$. Thus F has one boundary component which agrees with $\partial^-\Omega$. In accordance with the above notation, we denote by ∂^+F the reduced boundary of F without $\partial^-\Omega$, that is $\partial^+F = \partial^*F \cap \Omega^\circ$, where Ω° denotes the interior of Ω.

5.2. Basic properties

Consider the functional J_h, defined on the collection of sets F of bounded perimeter in Ω,

$$J_h(F) := |\partial^+ F| - h \operatorname{Vol}(F).$$

The critical points of J_h are surfaces with CMC h, so it is natural to consider this functional here.

We say that a set E minimizes J_h on the outside, if for all sets $F \supset E$ we have

$$J_h(E) \leq J_h(F).$$

Lemma 5.1. *For each $h \in (0, H_{\max})$, the set Ω_h defined above minimizes J_h on the outside.*

Proof. If Ω_h does not minimize J_h on the outside, then there exists a minimizer E_h for J_h outside of Ω_h, with $E_h \neq \Omega_h$ (cf. [6]). The outer boundary E_h is a $C^{1,\alpha}$-surface, satisfying $H \geq h$ in a distributional sense (cf. [7, Theorem 1.3]). It is smooth with $H = h$ where it does not touch Σ_h. By the strong maximum principle (which applies here as $\partial^+ E_h$ is $C^{1,\alpha}$, see also [14, Section 3]) all components of $\partial^+ E_h$ touching Σ_h are contained in Σ_h. Thus, $\partial^+ E_h$ is a smooth surface with CMC h and lies on the outside of Σ_h. As Σ_h is outermost, $E_h = \Omega_h$ as claimed. □

This lemma implies that Σ_h minimizes area on the outside.

Lemma 5.2. *For all $h \in (0, H_{\max})$ and all sets of finite perimeter $\Omega_h \subset F \subset \Omega$, we have*

$$|\Sigma_h| \leq |\partial^+ F|,$$

in particular

$$|\Sigma_h| \leq |\partial^+ \Omega|.$$

Proof. As Ω_h minimizes J_h on the outside,

$$|\Sigma_h| + h(\operatorname{Vol}(F) - \operatorname{Vol}(\Omega_h)) \leq |\partial^+ F|.$$

□

To conclude, we mention two other properties, which follow from the construction.

Lemma 5.3. (1) *The sets* Ω_h *are increasing, that is* $\Omega_{h_1} \subset \Omega_{h_2}$ *if* $h_1 < h_2$.

(2) *If* $h \in (0, H_{\max})$ *is fixed and* $h_k \in (0, H_{\max})$ *a sequence with* $h_k \geq h$ *and* $\lim_k h_k = h$, *then*

$$\bar{\Omega}_h = \bigcap_{k \geq 1} \Omega_{h_k}.$$

Here $\bar{\Omega}_h$ *denotes the closure of* Ω_h *in* Ω.

Proof. Property (1) follows from Theorem 3.1, as we can always use Σ_{h_1} and $\partial^+\Omega$ as inner and outer barriers for the construction of a surface with CMC h_2 outside. Note that this requires the strong maximum principle to conclude that Σ_{h_1} is disjoint from $\partial^+\Omega$.

To prove property (2), note that clearly $\bar{\Omega}_h \subset \bigcap_{k \geq 1} \Omega_{h_k}$, as there is a positive distance between Σ_h and $\Sigma_{h'}$ if $h < h'$. On the other hand, in view of the curvature bound on Σ_h and the area estimate, Lemma 5.2, we can assume that the Σ_{h_k} converge to a smooth surface Σ' with CMC h. By construction, Σ' lies on the outside of Σ_h and hence must agree with Σ_h, as Σ_h is outermost. □

5.3. Level-set formulation

Clearly, the sets Ω_h constructed above can be recognized as the sub-level sets of a function u. For $x \in \Omega$, we can define $u(x)$ as follows:

$$u(x) := \inf\{h : x \in \Omega_h\}. \tag{5.1}$$

We denote the sub-level sets by

$$E_h := \{x \in \Omega : u(x) < h\},$$
$$E_h^+ := \{x \in \Omega : u(x) \leq h\}.$$

We can say the following about these level sets.

Lemma 5.4. *For all* $h \in [0, H_{\max}]$ *we have that* $E_h \subset \Omega_h$ *and* $E_h^+ = \bar{\Omega}_h$.

Proof. If $x \in E_h$ then $u(x) < h$ which implies $x \in \Omega_h$ by the definition of u, hence $E_h \subset \Omega_h$.

Let $x \in E_h^+$, that is $u(x) \leq h$. Then for all $h' > h$ we have that $x \in \Omega_{h'}$. As the intersection of all $\Omega_{h'}$ with $h' > h$ is $\bar{\Omega}_h$ by property (2) of Lemma 5.3, we infer that $E_h^+ \subset \bar{\Omega}_h$. To see the other inclusion, note that by the proof of lemma 5.3, if $x \in \bar{\Omega}_h$ then $x \in \Omega_{h'}$ for all $h' > h$. □

Lemma 5.5. *If u is as in Equation* (5.1)*, then $u \in \mathrm{BV}(\Omega) \cap C^0(\Omega)$, where $BV(\Omega)$ denotes the space of functions with bounded variation and $C^0(\Omega)$ denotes the space of bounded continuous functions.*

Proof. First note that $u(x) \in [0, H_{\max}]$ and thus u is bounded.

We show that u is continuous. First, note that since E_h^+ is closed, we have that $\{u > h\} = \Omega \setminus E_h^+$ is open. Furthermore,

$$\{u = h\} = \bar{\Omega}_h \setminus \bigcup_{h' < h} \Omega_{h'}$$

hence $\{u < h\} = \bigcup_{h'<h} \Omega_{h'}$ and thus $\{u < h\}$ is also open. These two properties imply the continuity of u.

Furthermore, for all $k \in \mathbf{N}$ we can choose values $0 = h_0^k < \cdots < h_{N(k)}^K = H_{\max}$ such that $|h_i^k - h_{i-1}^k| < 1/k$ for $i = 1, \ldots, N(k)$. Let

$$u_k := \sum_{i=1}^{N(k)} (h_i^k - h_{i-1}^k) \chi_{E^+_{h_i^k}},$$

where χ_E denotes the characteristic function of a set E. Note that the u_k converge uniformly to u as $k \to \infty$ since u is continuous. Furthermore, all u_k have their BV-norm bounded by $|\Sigma_{H_{\max}}| H_{\max}$, and thus contain a subsequence that converges weakly to a limit $u_\infty \in \mathrm{BV}$. As the u_k converge uniformly to u we have that $u = u_\infty$ and hence u is in BV and has BV-norm bounded by $|\Sigma_{H_{\max}}| H_{\max}$. □

Huisken and Ilmanen [7] introduced a notion of weak solutions to the level-set inverse mean curvature flow. This notion motivates the following definition of a self-referencing functional on sets F of bounded variation:

$$J_v(F) := |\partial^+ F| - \int_F v \, dx.$$

Based on this functional we introduce the notion of weak CMC foliations.

Definition 5.2. We say that v is a weak (respectively, sub-, super-) solution to the CMC foliation problem, if the sets $E_h^+ := \{x \in M : v(x) \le h\}$ minimize J_v (from the outside, inside, respectively).

With respect to the above definition, we show the following theorem.

Theorem 5.1. *The function u, as defined in Equation* (5.1), *is a weak sub-solution to the CMC foliation problem.*

Proof. Let $F \supset \bar\Omega_h = E_h^+$ be any subset of finite perimeter. Fix $\varepsilon > 0$ and pick $h_i \in (0, H_{\max})$ such that

$$h = h_0 < h_1 < \cdots < h_N,$$

$h_i - h_{i-1} < \varepsilon$ and h_N is such that $F \subset \Omega_{h_N}$. For each h_i we know that Ω_{h_i} minimizes J_{h_i} from the outside. Hence we can compare with the set $F_i := (\Omega_{h_{i+1}} \cap F) \cup \bar\Omega_{h_i}$ and find that

$$J_{h_i}(\Omega_{h_i}) \leq J_{h_i}(F_i).$$

Expanding this out, we obtain

$$\begin{aligned} |\Sigma_{h_i}| - h_i \operatorname{Vol}(\Omega_{h_i}) \leq |\partial^+ F \cap (\bar\Omega_{h_{i+1}} \setminus \bar\Omega_{h_i})| + |\Sigma_{h_{i+1}} \cap F| + |\Sigma_{h_i} \setminus F| \\ - h_i \operatorname{Vol}(\Omega_{h_i}) - h_i \operatorname{Vol}(F \cap (\bar\Omega_{h_{i+1}} \setminus \bar\Omega_{h_i})). \end{aligned}$$

Sorting terms, this implies that

$$|\Sigma_{h_i} \cap F| - |\Sigma_{h_{i+1}} \cap F| \leq |\partial^+ F \cap (\bar\Omega_{h_{i+1}} \setminus \bar\Omega_{h_i})| - h_i \operatorname{Vol}(F \cap (\bar\Omega_{h_{i+1}} \setminus \bar\Omega_{h_i})).$$

Taking the sum, we find that

$$\sum_{i=0}^{N-1} (|\Sigma_{h_i} \cap F| - |\Sigma_{h_{i+1}} \cap F|) \leq |\partial^+ F| - \sum_{i=0}^{N-1} h_i \operatorname{Vol}(F \cap (\bar\Omega_{h_{i+1}} \setminus \bar\Omega_{h_i})).$$

Since $\Omega_h \subset F$ and $F \subset \Omega_{h_N}$ we have that $|\Sigma_{h_0} \cap F| = |\Sigma_h|$ and $|\Sigma_{h_N} \cap F| = 0$. So the above implies

$$|\Sigma_h| \leq |\partial^+ F| - \sum_{i=0}^{N-1} h_i \operatorname{Vol}(F \cap (\bar\Omega_{h_{i+1}} \setminus \bar\Omega_{h_i})). \tag{5.2}$$

As $h_i \leq u \leq h_{i+1}$ on $\Omega_{h_{i+1}} \setminus \Omega_{h_i}$ we can estimate

$$\begin{aligned} \int_F u\,dx - \int_{\Omega_h} u\,dx &= \sum_{i=0}^{N-1} \int_{F \cap (\Omega_{h_{i+1}} \setminus \Omega_{h_i})} u\,dx \\ &\leq \sum_{i=0}^{N-1} h_{i+1} \operatorname{Vol}(F \cap (\Omega_{h_{i+1}} \setminus \Omega_{h_i})) \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=0}^{N-1} (h_i + \varepsilon) \operatorname{Vol}(F \cap (\Omega_{h_{i+1}} \setminus \Omega_{h_i})) \\
&\leq \sum_{i=0}^{N-1} h_i \operatorname{Vol}(F \cap (\Omega_{h_{i+1}} \setminus \Omega_{h_i})) + \varepsilon \operatorname{Vol}(F \setminus \Omega_h).
\end{aligned}
$$

Combining this estimate with Equation (5.2) from above, we arrive at

$$
\int_F u\, dx - \int_{\Omega_h} u\, dx \leq |\partial^+ F| - |\Sigma_h| + \varepsilon \operatorname{Vol}(F \setminus \Omega_h)
$$

This implies that

$$
J_u(\Omega_h) \leq J_u(F) + \varepsilon \operatorname{Vol}(F \setminus \Omega_h)
$$

as ε was arbitrary, this yields the claim. □

Remark 5.1. We arrived at a weak sub-solution to the CMC foliation problem in the interior region by taking the outermost sets with curvature Ω_h. Analogously, we can construct the sets $\tilde{\Omega}_h$ bounded by the innermost surfaces with CMC h. Then the procedure above will result in surfaces minimizing J_h from the inside, which in turn implies that the corresponding level-set function $\tilde{u}$ is a super-solution to the weak CMC foliation problem.

Having these sub- and super-solutions at hand it should be possible to construct a weak solution of the CMC foliation problem in the sense as defined above. This is research in progress, details of which will appear elsewhere.

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