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On the Kähler manifolds with the largest infimum of spectrum of Laplace–Beltrami operators and sharp lower bound of Ricci or holomorphic bisectional curvatures

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The paper studies the extremal or rigidity problem associated to the largest infimum of spectrum of Laplace-Beltrami operator Δ_g on Kähler manifolds (M^n, g) under the sharp lower bound assumption on either Ricci curvature or holomorphic bisectional curvature. The paper provides some conterexamples on those rigidity problems. In particular, we consider $D(A) = \{z \in \mathbb{C}^n : |z|^2 + \mathrm{Re} \sum_{j=1}^n A_j z_j^2 < 1\}$ a convex domain in $\mathbb{C}^n$ with $n > 1$ and $A_j \in (-1, 1)$. Assuming g_0 is the Kähler-Einstein metric on $D(A)$, we prove that $\lambda_1(\Delta_{g_0}) = n^2$ on $(D(A), g_0)$, but $D(A)$ is not biholomorphic to the unit ball B_n when $A \neq 0$. Moreover, we prove that $\rho(z) = -e^u$ is strictly plurisubharmonic in $D(A)$ where u is the potential function for Kähler-Einstein metric on $D(A)$. We also construct a complete Kähler metric g_1 on $D(A)$ with holomorphic bisectional curvature $\mathcal{K}_{g_1} \geq -1$ and $\lambda_1(\Delta_{g_1}) = n^2$, but $D(A)$ is not biholomorphic to B_n when $A \neq 0$.

1. Introduction

Let (M^n, g) be a Kähler manifold of complex dimension n with Kähler metric $g = \sum_{i,j=1}^n g_{i\bar{j}} dz_i \otimes d\bar{z}_j$. Then the Laplace–Beltrami operator with respect to the metric g is defined as follows:

$$\Delta_g = -4 \sum_{i,j=1}^n g^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j}, \tag{1.1}$$

where $[g^{i\bar{j}}]^t = [g_{i\bar{j}}]^{-1}$. We define

(1.2)
$$\lambda_1(\Delta_g, M) = \inf \left\{ 4 \int_M \sum_{i,j=1}^n g^{i\bar{j}} \frac{\partial f}{\partial z_i} \frac{\partial f}{\partial \bar{z}_j} dv_g : f \in C_0^\infty(M), \|f\|_{L^2} = 1 \right\},$$

where dv_g is the volume measure on M with respect to the Kähler metric g.

When M is compact and Δ_g is uniformly elliptic, $\lambda_1(\Delta_g)$ is the first positive eigenvalue of Δ_g with Dirichlet boundary condition (see, for example, the lecture notes of Li [8] and the paper of S. Udagawa [19] and references therein). When (M^n, g) is a complete noncompact Kähler manifold, in general, $\lambda_1(\Delta_g)$ is not an eigenvalue of Δ_g (see for example, [1, 10, 11, 16, 18]). However, it is the infimum of the positive spectrum of Δ_g. There have been many researches on the analytic and geometric problems associated with λ_1. An important analytic problem is to find an sharp estimate on $\lambda_1(\Delta_g)$ with certain low bound assumptions on curvatures, which provides sharp upper bound for Poincarè inequality. An important upper bound estimate was obtained by Li and Wang in [10]. They proved that $\lambda_1(\Delta_g) \leq n^2$ if the holomorphic bisectional curvature of M is bounded below by -1. Their estimate is sharp and equality is achieved by the complex hyperbolic space $\mathbb{CH}^n$. Cheng [1] proved that $\lambda_1(\Delta_g) \leq n^2$ if Ricci curvature satisfying $R_{i\bar{j}} \geq -n g_{i\bar{j}}$. Munteanu [18] proved that $\lambda_1(\Delta_g) \leq n^2$ if the Ricci curvature $R_{i\bar{j}} \geq -(n+1)g_{i\bar{j}}$. His estimate is sharp because if $M = B_n$ and g is Kähler–Einstein metric then $\lambda_1 = n^2$ and $R_{i\bar{j}} = -(n+1)g_{i\bar{j}}$. Some more results along this line with different assumptions and estimates were obtained in [5, 6, 11, 21] and references therein. Li and Tran [16] provided some alternative conditions; the authors considered a bounded pseudoconvex domain D in $\mathbb{C}^n$ with negative defining function ρ so that $u = -\log(-\rho)$ is strictly plurisubharmonic in D, and Kähler metric $g_{i\bar{j}} = \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}$ (induced by u). They proved several theorems. In particular, the following result was proved in [16] which will be used later.

Theorem 1.1. *Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with a plurisubharmonic negative defining function $\rho \in C^2(\overline{D})$ so that $u = -\log(-\rho)$ is strictly plurisubharmonic in D. Let g be the Kähler metric induced by u. Then (D, g) is a complete Kähler manifold and $\lambda_1(\Delta_g) = n^2$.*

Geometric–type problems associated with $\lambda_1(\Delta_g)$ are rigidity problems. One may raise the following general questions:

Question 1.1. *Under the assumptions: holomorphic bisectional curvature $\mathcal{K}_g \geq -1$ and $\lambda_1(\Delta_g) = n^2$. What can one say about M?*

Question 1.2. *Under the assumptions: Ricci curvature $R_{i\bar{j}} \geq -(n+1)g_{i\bar{j}}$ and $\lambda_1(\Delta_g) = n^2$. What can one say about M?*

Questions 1.1 and 1.2 for the Riemannian case were studied by Li and Wang [11, 13], and they proved a very pretty splitting theorem. Li and Wang [10, 12] considered Kähler manifolds and also obtained a similar splitting theorem. The following theorem is their results for the Kähler case.

Theorem 1.2. *Let (M^n, g) be a complete Kähler manifold. Then*

(i) *If the Ricci curvature $R_{i\bar{j}} \geq -(n+1)g_{i\bar{j}}$ and $\lambda_1 > \frac{n+1}{2}$, then M must have one infinite volume end;*

(ii) *If the holomorphic bisectional curvature $\mathcal{K}_g \geq -1$ and $\lambda_1 = n^2$, then either M has only one end or $M = \mathbb{R} \times N$ with N being a compact manifold. Moreover, the metric on M is of the form*

$$ds_M^2 = dt^2 + e^{4t}\omega_2^2 + e^{2t}\sum_{i=3}^{2n}\omega_i^2, \tag{1.3}$$

where $\{\omega_2, \ldots, \omega_{2n}\}$ are orthonormal basis of N with $J\,dt = \omega_2$.

Munteanu [18] proved the same result under a weaker condition: $R_{i\bar{j}} \geq -(n+1)g_{i\bar{j}}$ and $\lambda_1 = n^2$. Kong *et al.* [6] considered a complete quaternonic Kähler manifold (M^{4n}, g) and proved the same theorem under the condition: the scalar curvature $S_M \geq -16n(n+2)$ and $\lambda_1 \geq (2n+1)^2$.

As we know that if $M = B_n$ is the unit ball in $\mathbb{C}^n$ and g is the Kähler–Einstein metric then

$$\lambda_1(\Delta_g) = n^2, \quad R_{i\bar{j}} = -(n+1)g_{i\bar{j}}, \quad R_{i\bar{j}k\bar{\ell}} = g_{i\bar{j}}g_{k\bar{\ell}} + g_{k\bar{j}}g_{i\bar{\ell}}, \tag{1.4}$$

which means that holomorphic bisectional curvature equals -1. Comparing Obata theorem and Cheng theorem for compact Riemannian manifolds (see [8, 9]) and result in [15]. It is natural to ask the following question.

Question 1.3. *Assume that D is a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with a complete Kähler metric g satisfying either*

$$R_{i\bar{j}} = -(n+1)g_{i\bar{j}} \tag{1.5}$$

or holomorphic bisectional curvature $\mathcal{K}_g \geq -1$. Assume that $\lambda_1(\Delta_g) = n^2$. Is D biholomorphic to the ball in $\mathbb{C}^n$?

The main purpose of this paper is to provide a way of constructing some counter examples for Question 1.3. In particular, we consider strictly a pseudoconvex domain in $\mathbb{C}^n$ whose boundary is real ellipsoid. After a linearly holomorphic change of variables, these domains can be described as follows:

$$D(A) = \{z \in \mathbb{C}^n : r(z) = |z|^2 + \mathrm{Re} \sum_{j=1}^{n} A_j z_j^2 - 1 < 0\} \tag{1.6}$$

with

$$0 \le A_1 \le A_2 \le \cdots \le A_n < 1. \tag{1.7}$$

It was proved by Webster [22] that $D(A)$ is biholomorphic to the unit ball in $\mathbb{C}^n$ if and only if $A = (A_1, \dots, A_n) = 0$. We will prove the following theorem in this paper.

Theorem 1.3. *For any $0 \le A_1 \le \cdots \le A_n < 1$, we have the following:*

(i) *If g is the Kähler–Einstein metric on $D(A)$ then $\lambda_1(\Delta_g) = n^2$.*

(ii) *There is a Kähler metric g^0 on $D(A)$ with $A_n \le 2/5$ so that the holomorphic bisectional curvature $\mathcal{K}_{g^0} \ge -1$ and $\lambda_1(\Delta_{g_0}) = n^2$.*

Combining Theorem 1.3 and the above theorem of Webster in [22], we answer Question 1.3 negatively with the counter examples: $D(A)$ with $A \neq 0$ and $n > 1$.

This paper is organized as follows: in Section 2, we provide an explicit formula to approximate the potential function of the Kähler–Einstein metric. In Section 3, we prove part (i) of Theorem 1.3. Finally, part (ii) of Theorem 1.3 is proved in Section 4.

2. Approximation formula

Let D be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$. A plurisubharmonic function u on D is said to be the potential function for the Kähler–Einstein metric on D if u is the solution of the Monge–Ampère equation:

$$\det H(u) = e^{(n+1)u} \quad \text{in } D; \quad u = +\infty \quad \text{on } \partial D. \tag{2.1}$$

The existence of such a solution for (2.1) was proved by Cheng and Yau [2], the uniqueness was proved by Fefferman [3]. When D is strictly pseudoconvex, Cheng and Yau [2] prove that $\mathrm{e}^{-u} \in C^{n+3/2}(\overline{D})$. Lee and Melrose [7] give the following asymptotic expansion for e^{-u}:

$$\rho(z) = -e^{-u(z)} = \sum_{j=0}^{\infty} a_j \rho^0(z)\Big(\rho^0(z)^{n+1}\log(-\rho^0(z))\Big)^j, \tag{2.2}$$

where $a_j \in C^\infty(\overline{D})$ and ρ^0 is any negative defining function for D with $\rho^0 \in C^\infty(\overline{D})$. Fefferman [3] gave a method of how to approximate $\rho(z)$ in terms of $\rho^0(z)$. In particular, he proved $a_0(z) = J(\rho^0)^{-1/(n+1)} + \sum_{j=1}^n a_{0,j}(\rho^0(z))^j$, where

$$J(\rho_0)(z) = -\det\begin{bmatrix} \rho_0 & \overline{\partial}\rho_0(z) \\ (\overline{\partial}\rho_0)^* & H(\rho_0) \end{bmatrix}, \tag{2.3}$$

where $\overline{\partial}\rho_0 = (\frac{\partial\rho_0}{\partial\overline{z}_1}, \ldots, \frac{\partial\rho_0}{\partial\overline{z}_n})$, $H(\rho^0)$ is the complex Hessian matrix of ρ_0 on D. Question about how to compute $a_{0,j}$ in (2.2) in terms of ρ^0 explicitly has been studied by Fefferman [3] and Graham [4]; they provided a certain iteration formula for evaluating $a_{0,j}$, respectively. Here we give an alternative formula for $a_{0,j}$ or approximation ρ in terms of ρ^0 as follows.

Theorem 2.1. *Let $r(z)$ be a smooth negative defining function for D so that $\ell(\rho) := -\log(-r(z))$ is strictly plurisubharmonic in D. Let*

$$\rho_0(z) = r(z), \quad \rho_{j+1}(z) = \rho_j(z)J(\rho_j)^{-1/(n+1)}e^{-B_j} \tag{2.4}$$

with

$$B_j(z) = \frac{\operatorname{tr}(H(\ell(\rho_j))^{-1}H(\log J(\rho_j))}{(j+2)(n-j)(n+1)}. \tag{2.5}$$

Then

$$J(\rho_{j+1})(z) = 1 + O(\delta(z)^{j+2}), \quad j = 0, 1, \ldots, n-1 \tag{2.6}$$

and

$$\delta(z) = \operatorname{dist}(z, \partial D), \quad a_0(z) = \frac{\rho_n(z)}{\rho_0(z)} \tag{2.7}$$

Moreover, if

$$B_n = \frac{\operatorname{tr}(H(\ell(\rho_n))^{-1}H(\log J(\rho_n)))}{(n+2)(n+1)}\ell(\rho_n(z)) \tag{2.8}$$

then

$$J(\rho_{n+1}) = 1 + O(\delta(z)^{n+2} \log \delta(z)). \tag{2.9}$$

Proof. Let $K = J(\rho_j)\exp((n+1)b_j\rho_j)$ with $B_j = b_j\rho_j(z)$. Then $\rho_{j+1} = \rho_j$ $K^{-1/(n+1)}$ and

(2.10)

$$\begin{aligned}
J(\rho_{j+1})&\\
&= (-\rho_{j+1}(z))^{n+1} \det H(\ell(\rho_{j+1}))(z)\\
&= \frac{J(\rho_j)}{K} \det \Big[I_n + \frac{1}{(n+1)} H(\ell(\rho_j))^{-1} H(\log K)(z)\Big]\\
&= \mathrm{e}^{-(n+1)B_j} \det \Big[I_n + \frac{1}{n+1} H(\ell(\rho_j))^{-1} (H(\log J(\rho_j))(z) + (n+1)H(b_j\rho_j))\Big]\\
&= \mathrm{e}^{-(n+1)B_j} \det \Big[I_n + \frac{1}{n+1} H(\ell(\rho_j))^{-1} (H(\log J(\rho_j))(z)\\
&\quad + H(\ell(\rho_j))^{-1}(b_j H(\rho_j) + (\overline{\partial} b_j)^*(\overline{\partial}\rho_j) + (\overline{\partial}\rho_j)^*(\overline{\partial} b_j) + \rho_j(z)H(b_j))\Big].
\end{aligned}$$

Let $X = [x_{ij}]$ be an $n \times n$ matrix . Then

$$\det[I_n + X] = 1 + \mathrm{tr}(X) + O(\delta(z)), \quad \text{if} \ \ x_{ij} = O(\delta(z)). \tag{2.11}$$

Notice that

$$\begin{aligned}
&\sum_{j=1}^{n} \left(\rho^{i\bar{j}} - \frac{\rho^i \rho^{\bar{j}}}{|\partial\rho|_\rho^2 - \rho}\right) \partial_{\bar{j}}\rho = \frac{\rho^i(-\rho)}{|\partial\rho|_\rho^2 - \rho},\\
&\times \sum_{i,j=1}^{n} \left(\rho^{i\bar{j}} - \frac{\rho^i \rho^{\bar{j}}}{|\partial\rho|_\rho^2 - \rho}\right) \partial_{\bar{j}}\rho\partial_i\rho = \frac{|\partial\rho|_\rho^2(-\rho)}{|\partial\rho|_\rho^2 - \rho},
\end{aligned} \tag{2.12}$$

one has

(2.13)

$$\begin{aligned}
&\mathrm{tr}\Big(H(\ell(\rho_0))^{-1}(b_0 H(\rho_0) + (\overline{\partial} b_0)^*(\overline{\partial}\rho_0) + (\overline{\partial}\rho_0)^*(\overline{\partial} b_0) + \rho_0(z)H(b_0))\Big)\\
&\quad = -(n-1)b_0\rho_0(z) + O(\delta(z)^2).
\end{aligned}$$

If we let

$$b_0(z)\rho_0(z) = B_0(z) = \frac{1}{2n(n+1)} \mathrm{tr}(H(\ell(\rho_0))^{-1} H(\log J(\rho_0))(z), \tag{2.14}$$

then by (2.10), (2.12) and (2.13), one has

$$
J(\rho_1)(z) = 1 + \frac{\operatorname{tr}(H(\ell(\rho_0))^{-1}H(\log J(\rho_0))(z)}{(n+1)} \\
- (n-1)b_0\rho_0(z) - (n+1)B_0(z) + O(\delta(z)^2) \\
= 1 + O(\delta(z)^2). \tag{2.15}
$$

Assume that $J(\rho_j) = 1 + O(\delta(z)^{j+1})$ for some $1 \le j \le n-1$. Then by (2.12), one has

$$
\operatorname{tr}(H(\ell(\rho_j))^{-1}H(\log J(\rho_j)) = O(\delta(z)^{j+1}). \tag{2.16}
$$

If one assumes (a priori) that

$$
B_j(z) = b_j(z)\rho_j(z) = \tilde{b_0}(\rho_j(z))^{j+1}, \quad \tilde{b_0} \in C^\infty(\overline{D}) \tag{2.17}
$$

and uses (2.12), then one has

$$
\operatorname{tr}\Big(H(\ell(\rho_j))^{-1}(b_jH(\rho_j) + (\overline{\partial}b_j)^*(\overline{\partial}\rho_j) + (\overline{\partial}\rho_j)^*(\overline{\partial}b_j) + \rho_j(z)H(b_j))\Big) \\
= -(n-1)B_j(z) + 2jB_j - j(n-j)B_j + O(\delta(z)^{n+2}). \tag{2.18}
$$

Similar to (2.15), if we let

$$
B_j = \frac{\operatorname{tr}\Big(H(\ell(\rho_j))^{-1}\log J(\rho_j))}{(2+j)(n-j)(n+1)}, \tag{2.19}
$$

then

$$
J(\rho_{j+1}) = 1 + O(\delta(z)^{j+2}) \tag{2.20}
$$

for $j = 1, 2, \ldots, n-1$.

Remark. We must notice here that in the proof of (2.12), we use $H(\rho)$ as positive definite near ∂D. When $H(\rho_j)$ is not strictly positive definite, the conclusion remains true with the following argument: let $\tilde{\rho}_j = \rho_j + C\rho_j^2$ with some positive number C so that $H(\tilde\rho_j)$ is positive definite near ∂D. Then

$$\begin{aligned} H(\ell(\rho_j))^{-1} &= (1+c\rho)^{-1}[H(\ell(\tilde\rho_j)) + \frac{c(1+2c\rho)}{1+c\rho}(\partial\rho_j)\otimes(\overline\partial\rho_j)]^{-1} \\ &= (1+O(\delta(z))H(\ell(\tilde\rho_j)^{-1}. \end{aligned}$$

With the help of the above formula, all arguments in proof of (2.12) remain true.

When $j=n$, if we let

(2.21)
$$\rho_{n+1}(z) = \rho_n J(\rho_n)^{-1/(n+1)}\mathrm{e}^{-B_n(z)}, \quad B_n(z) = b_n\rho_n(z)(-\log(-\rho_n(z)),$$

then

$$\begin{aligned} J(\rho_{n+1}) &= \mathrm{e}^{-(n+1)B_n}\det\Big[I_n + \frac{1}{n+1}H(\ell(\rho_n))^{-1}(H(\log J(\rho_n))(z) \\ &\quad + (n+1)H(b_n\rho_n\ell(\rho_n))\Big] \\ &= \mathrm{e}^{-(n+1)B_n}\det\Big[I_n + \frac{1}{n+1}H(\ell(\rho_n))^{-1}H(\log J(\rho_n))(z) \\ &\quad + \ell(\rho_n)H(\ell(\rho_n))^{-1}(b_nH(\rho_n) + (\overline\partial b_n)^*(\overline\partial\rho_n) + (\overline\partial\rho_n)^*(\overline\partial b_n) \\ &\quad + \rho_n(z)H(b_n)) + H(\ell(\rho_n))^{-1}(\rho_n(z)b_nH(\ell(\rho_n) - (\overline\partial b_n)^*(\overline\partial\rho_n) \\ &\quad -(\overline\partial\rho_n)^*(\overline\partial b_n)) + \frac{2b_n}{-\rho_n}(\overline\partial\rho_n)^*(\overline\partial\rho_n)]\Big] \\ &= \mathrm{e}^{-(n+1)B_n}\det\Big[I_n + \frac{1}{n+1}H(\ell(\rho_n))^{-1}H(\log J(\rho_n))(z) \\ &\quad + \ell(\rho_n)b_n(n+1)H(\ell(\rho_n))^{-1}(H(\rho_n) + \frac{n}{\rho_n(z)}(\overline\partial\rho_n)^*(\overline\partial\rho_n)\Big] \\ &\quad + \ell(\rho_n)H(\ell(\rho_n))^{-1}((\rho_n(z))^n(n+1)[(\overline\partial\tilde b_n)^*(\overline\partial\rho_n) \\ &\quad + (\overline\partial\rho_n)^*(\overline\partial\tilde b_n)] + (\rho_n(z))^{n+1}H(\tilde b_n))] \\ &\quad + H(\ell(\rho_n))^{-1}[\rho_n(z)b_nH(\ell(\rho_n) + 2(n+1)\frac{b_n}{-\rho_n}(\overline\partial\rho_n)^*(\overline\partial\rho_n)] \\ &\quad - H(\ell(\rho_n))^{-1}[\rho_n(z))^n((\overline\partial\tilde b_n)^*(\overline\partial\rho_n) + (\overline\partial\rho_n)^*(\overline\partial\tilde b_n))]\Big] \\ &= 1 + O(\delta(z)^{n+2}\log(\delta(z))) \end{aligned}$$

if we choose

$$b_n(z)\rho_n(z) = \frac{1}{(n+2)(n+1)}\mathrm{tr}(H(\ell(\rho_n))^{-1}H(\log J(\rho_n))). \tag{2.22}$$

This completes the proof of Theorem 2.1. □

3. Domain whose boundary is real ellipsoid

We consider a class of strictly pseudoconvex domains in $\mathbb{C}^n$ with the real ellipsoid as their boundary. After a linearly holomorphic change of variables, those domains can be written as

$$D(A) = \{z \in \mathbb{C}^n : r(z) < 0\}, \tag{3.1}$$

where

$$r(z) = |z|^2 + \mathrm{Re}\sum_{j=1}^{n} A_j z_j^2 - 1, \quad 0 \le A_1 \le A_2 \le \cdots \le A_n < 1. \tag{3.2}$$

Then $H(r) = I_n$ and

$$J(r) = -r(z) + \sum_{j=1}^{n}|r_j(z)|^2 = \mathrm{Re}\sum_{j=1}^{n} A_j z_j^2 + \sum_{j=1}^{n} A_j^2|z_j|^2 + 1. \tag{3.3}$$

Let

$$\rho^1(z) = \mathrm{e}^{-b_0(z)r(z)}J(r)^{-\frac{1}{n+1}}r(z), \tag{3.4}$$

where

$$b_0(z)r(z) = \frac{1}{2n(n+1)}\mathrm{tr}(H(\ell(r))^{-1}H(\log J(r))). \tag{3.5}$$

Notice that

$$H(J(r)) = \mathrm{diag}(A_1^2, \ldots, A_n^2), \quad \partial_i J(r) = A_i \partial_{\bar{i}} r, \tag{3.6}$$

one has

(3.7)

$$
\begin{aligned}
&b_0(z)r(z)\\
&\quad= \frac{-r(z)}{2n(n+1)J(r)}\mathrm{tr}\left(\left[I_n - \frac{(\partial r)\otimes(\overline{\partial} r)}{J(r)}\right]\left[H(J(r)) - \frac{(\partial J(r))\otimes(\overline{\partial} J(r))}{J(r)}\right]\right)\\
&\quad= \frac{-r(z)}{2n(n+1)J(r)}\left(\|A\|^2 - 2\frac{\sum_{j=1}^n A_j^2|\partial_j r(z)|^2}{J(r)} + \frac{|\sum_{j=1}^n A_j(\partial_j r)^2|^2}{J(r)^2}\right).
\end{aligned}
$$

Therefore,

(3.8)

$$
b_0(z) = \frac{-1}{2n(n+1)J(r)}\left(\|A\|^2 - 2\frac{\sum_{j=1}^n A_j^2|\partial_j r(z)|^2}{J(r)} + \frac{\left|\sum_{j=1}^n A_j(\partial_j r)^2\right|^2}{J(r)^2}\right).
$$

By Theorem 2.1, one can easily see that if $\rho(z) = -\mathrm{e}^{-u(z)}$ with u is the potential function Kähler–Einstein metric then

$$
\det H(\rho) = \det H(\rho^1) \quad \text{on } \partial D(A), \tag{3.9}
$$

since $\rho(z) = \rho^1(z) + A_3(z)$ with $A_3(z) = O(\delta(z)^3)$ on $D(A)$ and $H(A_3)(z) = 0$ on $\partial D(A)$.

In order to calculate $\det H(\rho^1)$ on $\partial D(A)$, we need the following lemma.

Lemma 3.1. *Let $A = (A_1, \dots, A_n)$ and $B = (B_1, \dots, B_n)$ be two row vectors in $\mathbb{C}^n$. Then*

$$
\det(I_n - A^*B - B^*A) = |1 - \langle A, B\rangle|^2 - |A|^2|B|^2. \tag{3.10}
$$

Moreover, if $\langle B, A\rangle \neq 1$ then

$$
(I_n - A^*B)^{-1} = I_n + \frac{1}{1 - \langle B, A\rangle}A^*B. \tag{3.11}
$$

Proof. First, we consider the case $A = |A|(0, \dots, 1) = |A|e_n$. Then

$$
\begin{aligned}
\det(I_n - A^*B - B^*A) &= 1 - |A|(B_n + \overline{B}_n) - |A|^2\sum_{j=1}^{n-1}|B_j|^2\\
&= 1 - |A|(B_n + \overline{B}_n) + |A|^2|B_n|^2 - |A|^2|B|^2\\
&= |1 - \langle A, B\rangle|^2 - |A|^2|B|^2.
\end{aligned}
$$

For a general row vector A, we choose a unitary matrix $U = [\alpha_{ij}]$ so that $AU = |A|e_n$. Then

$$\begin{aligned}\det(I_n - A^*B - B^*A) &= \det(I_n - (AU)^*(BU) - (BU)^*(AU)) \\ &= |1 - \langle AU, BU\rangle|^2 - |AU|^2|BU|^2 \\ &= |1 - \langle A, B\rangle|^2 - |A|^2|B|^2.\end{aligned}$$

Since

$$\begin{aligned}(I_n - A^*B)^{-1} &= I_n + \sum_{j=1}^{\infty}(A^*B)^j = I_n + \sum_{j=1}^{\infty}(\langle B, A\rangle^{j-1}A^*B \\ &= I_n + \frac{1}{1 - \langle B, A\rangle}A^*B.\end{aligned}$$

So, the proof of the lemma is complete. □

The main purpose of this section is to prove the following theorem.

Theorem 3.1. *Let u be the potential function for the Kähler–Einstein metric on $D(A)$, and let $\rho(z) = -\mathrm{e}^{-u(z)}$ on D. Then $\rho(z)$ is strictly plurisubharmonic in $D(A)$.*

Proof. We first prove $\det H(\rho) \geq 0$ on $\partial D(A)$. By (3.9), it suffices to prove

$$\det H(\rho^1) \geq 0 \quad \text{on} \quad \partial D(A). \tag{3.12}$$

Let

$$B = (n+1)J(r)b_0\overline{\partial}r + \overline{\partial}J(r) \tag{3.13}$$

be the row vector. Notice

$$|\partial J|^2 = \sum_{j=1}^{n} A_j^2|\partial_j r(z)|^2, \quad \overline{R}J(r) = \sum_{j=1}^{n} A_j(\partial_j r(z))^2 \tag{3.14}$$

and $\det H(a + br(z)^k) = \det H(a)$ on $\partial D(A)$ for any $k \geq 3$. Moreover, by the fact that $|\partial r|^2 = J(r)$ on $\partial D(A)$ and Lemma 3.1, for any $z \in \partial D(A)$, one has

$$\begin{aligned}&\det H(\rho^1)J(r)^{n/(n+1)} \\ &\quad = J(r)^{n/(n+1)} \det H\left(\frac{r - b_0r^2}{J(r)^{1/(n+1)}}\right)\end{aligned}$$

$$
\begin{aligned}
&= \det\Big[H(r) - 2b_0(\partial r)\otimes(\overline{\partial} r) - \frac{((\partial J)\otimes(\overline{\partial} r) + (\partial r)\otimes(\overline{\partial} J))}{(n+1)J(r)}\Big]\\
&= \det\Big[I_n - \frac{(B\otimes(\partial r) + (\partial r)\otimes B}{(n+1)J(r)}\Big]\\
&= \Big|1 - \frac{\langle B, \overline{\partial} r\rangle}{(n+1)J(r)}\Big|^2 - \frac{\|B\|^2\|\partial r|^2}{(n+1)^2 J(r)^2}\\
&= \Big|1 - \frac{(n+1)b_0 J(r)|\partial r|^2 + \overline{R}J}{(n+1)J(r)}\Big|^2\\
&\quad - \frac{(n+1)^2 J(r)^2 b_0^2|\partial r|^2 + 2(n+1)J(r)b_0 \mathrm{Re}\,(\overline{R}J) + |\partial J(r)|^2}{(n+1)^2 J(r)}\\
&= 1 - 2b_0|\partial r|^2 - \frac{2\mathrm{Re}\,(\overline{R}J)}{(n+1)J(r)} + b_0^2|\partial r|^4 + 2\frac{b_0|\partial r|^2}{(n+1)J(r)}\mathrm{Re}\,(\overline{R}J(r))\\
&\quad + \frac{\Big|\overline{R}J(r)\Big|^2}{(n+1)^2 J(r)^2} - b_0(z)^2 J(r)|\partial r|^2 - \frac{2b_0}{(n+1)}\mathrm{Re}\,(\overline{R}J(r)) - \frac{|\partial J(r)|^2}{(n+1)^2 J(r)}\\
&= 1 - 2b_0|\partial r|^2 - \frac{2\mathrm{Re}\,(\overline{R}J)}{(n+1)J(r)} + \frac{\Big|\overline{R}J(r)\Big|^2}{(n+1)^2 J(r)^2} - \frac{|\partial J(r)|^2}{(n+1)^2 J(r)}\\
&\geq \Big(1 - \frac{|\overline{R}J(r)|}{(n+1)J(r)}\Big)^2 - \frac{A_n^2}{(n+1)^2} - 2b_0 J(r)\\
&\geq \Big(1 - \frac{|\overline{R}J(r)|}{(n+1)J(r)}\Big)^2 - \frac{A_n^2}{(n+1)^2}\\
&\quad + \frac{\|A\|^2 - 2\sum_{j=1}^n A_j^2|\partial_j r|^2 J(r)^{-1} + \Big|\sum_{j=1}^n A_j(\partial_j r)^2\Big|^2 J(r)^{-2}}{n(n+1)}\\
&\geq \Big(1 - \frac{A_n}{(n+1)}\Big)^2 - \frac{A_n^2}{(n+1)^2}\\
&\quad + \frac{\|A\|^2 - 2A_n^2 + \Big|\sum_{j=1}^n A_j(\partial_j r)^2\Big|^2 J(r)^{-2}}{n(n+1)}
\end{aligned}
$$

$$\geq 1 - \frac{2}{n+1}A_n - \frac{A_n^2}{n(n+1)} + \frac{\left|\sum_{j=1}^n A_j(\partial_j r)^2\right|^2 J(r)^{-2}}{n(n+1)}$$

$$> 0$$

if $n \geq 2$ (in the above inequalities, we use the assumption: $0 \leq A_1 \leq \cdots \leq A_n$). When $n = 1$, we pick up the term $\left|\sum_{j=1}^n A_j(\partial_j r)^2\right|^2 J(r)^{-2} = A_n^2$ in the estimation of $\det H(\rho^1)$ on $\partial D(A)$; we have

$$\det H(\rho) \geq 1 - \frac{2A_n}{n+1} = 1 - A_n > 0 \quad \text{on } \partial D(A). \tag{3.15}$$

Notice that $J(\rho) = 1$ on $D(A)$, then

$$\det H(\rho) = \frac{\det H(\rho)}{J(\rho)} = \mathrm{e}^{u(z)}(1 - |\partial u|_u^2). \tag{3.16}$$

It was proved by the author in [14] that $\det H(\rho)$ attains its minimum over $\overline{D(A)}$ at the some point on $\partial D(A)$. By the first step, we have $\det H(\rho) > 0$ on $\partial D(A)$. Therefore, $\det H(\rho) > 0$ on $D(A)$ and the proof of the theorem is complete. □

As a corollary of Theorems 3.2 and 1.1, we have proved the following result.

Corollary 3.1. *If g is the Kähler–Einstein metric on $D(A)$ then $\lambda_1(\Delta_g) = n^2$.*

This gives the proof of part (i) of Theorem 1.3.

4. Holomorphic bisectional curvatures

If (M, g) is a Kähler manifold with Kähler metric g, then it is well known (see [20]) that the curvature tensor is given by the following formula:

$$R_{i\bar{j}k\bar{\ell}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_\ell} + g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z_k} \frac{\partial g_{p\bar{j}}}{\partial \bar{z}_\ell}. \tag{4.1}$$

Assumes that the Kähler metric g is induced by a strictly plurisubharmonic potential function u on M with $u = +\infty$ on ∂M:

$$g_{i\bar{j}} = \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} = u_{i\bar{j}}, \quad \rho(z) = -e^{-u(z)}, \tag{4.2}$$

where ρ becomes a negative defining function for M. We first provide a formula for the curvature tensor with the components of holomorphic bisectional curvature plus some derivatives of the defining function ρ.

Proposition 4.1. *Let $u = -\log(-\rho)$ be strictly plurisubharmonic in D. Then*

$$R_{i\bar{j}k\bar{\ell}} = -(u_{i\bar{j}}u_{k\bar{\ell}} + u_{k\bar{j}}u_{i\bar{\ell}}) + E_{i\bar{j}k\bar{\ell}}, \tag{4.3}$$

where

$$\begin{aligned} E_{i\bar{j}k\bar{\ell}} = &-\frac{\rho_{i\bar{j}k\bar{\ell}}}{-\rho} + \frac{1}{\rho^2}g^{p\bar{q}}\Big[\rho_{i\bar{q}k}\rho_{p\bar{j}\,\bar{\ell}} + \rho_{i\bar{q}k}u_p\rho_{\bar{j}\,\bar{\ell}} + \rho_{p\bar{j}\,\bar{\ell}}u_{\bar{q}}\rho_{ik}\Big] \\ &- (1-|\partial u|_g^2)\frac{\rho_{ik}\rho_{\bar{j}\,\bar{\ell}}}{\rho^2}. \end{aligned} \tag{4.4}$$

Note: A theorem of Lu Qi-Keng [17] states that if g is a Bergman metric for a bounded domain in D in $\mathbb{C}^n$ then D is biholomorphic to the unit ball in $\mathbb{C}^n$ if and only if $R_{i\bar{j}k\bar{\ell}} = -(u_{i\bar{j}}u_{k\bar{\ell}} + u_{k\bar{j}}u_{i\bar{\ell}})$ on D (or $E_{i\bar{j}k\bar{\ell}} = 0$ on D). Next, we prove Proposition 4.1.

Proof. Notice that

$$u = -\log(-\rho), \quad u_i = \frac{\rho_i}{-\rho},\ u_{i\bar{j}} = \frac{\rho_{i\bar{j}}}{-\rho} + u_iu_{\bar{j}},\ u_{ik} = \frac{\rho_{ik}}{-\rho} + u_iu_k, \tag{4.5}$$

$$\begin{aligned} u_{i\bar{j}k} &= \frac{\rho_{i\bar{j}k}}{-\rho} + \frac{\rho_{i\bar{j}}}{-\rho}u_k + u_{ik}u_{\bar{j}} + u_iu_{k\bar{j}} \\ &= \frac{\rho_{i\bar{j}k}}{-\rho} + \frac{\rho_{i\bar{j}}}{-\rho}u_k + \frac{\rho_{ik}}{-\rho}u_{\bar{j}} + \frac{\rho_{k\bar{j}}}{-\rho}u_i + 2u_iu_ku_{\bar{j}} \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} u_{i\bar{j}k\bar{\ell}} = &\frac{\rho_{i\bar{j}k\bar{\ell}}}{-\rho} + \frac{\rho_{i\bar{j}k}}{-\rho}u_{\bar{\ell}} + \frac{\rho_{i\bar{j}\,\bar{\ell}}}{-\rho}u_k + \frac{\rho_{i\bar{j}}}{-\rho}u_ku_{\bar{\ell}} + \frac{\rho_{i\bar{j}}}{-\rho}u_{k\bar{\ell}} \\ &+ \frac{\rho_{ik\bar{\ell}}}{-\rho}u_{\bar{j}} + \frac{\rho_{ik}}{-\rho}u_{\bar{j}}u_{\bar{\ell}} + \frac{\rho_{ik}}{-\rho}u_{\bar{j}\,\bar{\ell}} + \frac{\rho_{k\bar{j}\,\bar{\ell}}}{-\rho}u_i + \frac{\rho_{k\bar{j}}}{-\rho}u_iu_{\bar{\ell}} + \frac{\rho_{k\bar{j}}}{-\rho}u_{i\bar{\ell}} \\ &+ 2u_{i\bar{\ell}}u_ku_{\bar{j}} + 2u_iu_{k\bar{\ell}}u_{\bar{j}} + 2u_iu_ku_{\bar{j}\,\bar{\ell}} \\ = &\frac{\rho_{i\bar{j}k\bar{\ell}}}{-\rho} + \frac{\rho_{i\bar{j}k}}{-\rho}u_{\bar{\ell}} + \frac{\rho_{i\bar{j}\,\bar{\ell}}}{-\rho}u_k + \frac{\rho_{ik\bar{\ell}}}{-\rho}u_{\bar{j}} + \frac{\rho_{k\bar{j}\,\bar{\ell}}}{-\rho}u_i \\ &+ u_{i\bar{j}}(u_ku_{\bar{\ell}} + u_{k\bar{\ell}}) + u_{ik}(u_{\bar{j}}u_{\bar{\ell}} + u_{\bar{j}\,\bar{\ell}}) + u_{k\bar{j}}(u_iu_{\bar{\ell}} + u_{i\bar{\ell}}) \\ &- 3u_iu_{\bar{j}}u_ku_{\bar{\ell}} + u_{i\bar{\ell}}u_ku_{\bar{j}} + u_iu_{k\bar{\ell}}u_{\bar{j}} + u_iu_ku_{\bar{j}\,\bar{\ell}}. \end{aligned} \tag{4.7}$$

By (4.5) and (4.6), one has

$$u_{i\bar{q}k} = \frac{\rho_{i\bar{q}k}}{-\rho} + u_{i\bar{q}}u_k - u_i u_{\bar{q}} u_k + u_{ik}u_{\bar{q}} + u_i u_{k\bar{q}}$$

and

$$u_{p\bar{j}\,\bar{\ell}} = \frac{\rho_{p\bar{j}\,\bar{\ell}}}{-\rho} + u_{p\bar{j}}u_{\bar{\ell}} - u_p u_{\bar{j}} u_{\bar{\ell}} + u_{p\bar{\ell}}u_{\bar{j}} + u_p u_{\bar{j}\,\bar{\ell}}.$$

Thus,

$$\begin{aligned}
(4.8)\quad & g^{p\bar{q}} u_{i\bar{q}k} u_{p\bar{j}\,\bar{\ell}} \\
&= \left(g^{p\bar{q}}\frac{\rho_{i\bar{q}k}}{(-\rho)} + \delta_{ip}u_k - g^{p\bar{q}}u_i u_{\bar{q}} u_k + u_{ik}g^{p\bar{q}}u_{\bar{q}} + u_i\delta_{kp}\right) \\
&\quad \times \left(\frac{\rho_{p\bar{j}\,\bar{\ell}}}{(-\rho)} + u_{p\bar{j}}u_{\bar{\ell}} - u_p u_{\bar{j}} u_{\bar{\ell}} + u_{p\bar{\ell}}u_{\bar{j}} + u_p u_{\bar{j}\,\bar{\ell}}\right) \\
&= g^{p\bar{q}}\frac{\rho_{i\bar{q}k}}{(-\rho)}\frac{\rho_{p\bar{j}\,\bar{\ell}}}{(-\rho)} + g^{p\bar{q}}\frac{\rho_{i\bar{q}k}}{(-\rho)}(u_{p\bar{j}}u_{\bar{\ell}} - u_p u_{\bar{j}} u_{\bar{\ell}} + u_{p\bar{\ell}}u_{\bar{j}} + u_p u_{\bar{j}\,\bar{\ell}}) \\
&\quad + \frac{\rho_{p\bar{j}\,\bar{\ell}}}{(-\rho)}(\delta_{ip}u_k - g^{p\bar{q}}u_i u_{\bar{q}} u_k + u_{ik}g^{p\bar{q}}u_{\bar{q}} + u_i\delta_{kp}) \\
&\quad + u_{i\bar{j}}u_k u_{\bar{\ell}} - u_i u_k u_{\bar{j}} u_{\bar{\ell}} + u_{i\bar{\ell}}u_k u_{\bar{j}} + u_i u_k u_{\bar{j}\,\bar{\ell}} \\
&\quad - u_i u_{\bar{j}} u_k u_{\bar{\ell}} + |\partial u|_g^2 u_i u_{\bar{j}} u_k u_{\bar{\ell}} - u_i u_{\bar{j}} u_k u_{\bar{\ell}} - |\partial u|_g^2 u_i u_k u_{\bar{j}\,\bar{\ell}} \\
&\quad + u_{ik}u_{\bar{j}}u_{\bar{\ell}} - |\partial u|_g^2 u_{ik}u_{\bar{j}}u_{\bar{\ell}} + u_{ik}u_{\bar{j}}u_{\bar{\ell}} + |\partial u|_g^2 u_{ik}u_{\bar{j}\,\bar{\ell}} \\
&\quad + u_{k\bar{j}}u_i u_{\bar{\ell}} - u_i u_{\bar{\ell}} u_k u_{\bar{j}} + u_i u_{\bar{j}} u_{k\bar{\ell}} + u_i u_k u_{\bar{j}\,\bar{\ell}} \\
&= g^{p\bar{q}}\frac{\rho_{i\bar{q}k}}{(-\rho)}\frac{\rho_{p\bar{j}\,\bar{\ell}}}{(-\rho)} + g^{p\bar{q}}\frac{\rho_{i\bar{q}k}}{(-\rho)}(-u_p u_{\bar{j}} u_{\bar{\ell}} + u_p u_{\bar{j}\,\bar{\ell}}) + \frac{\rho_{i\bar{j}k}}{-\rho}u_{\bar{\ell}} + \frac{\rho_{ik\bar{\ell}}}{-\rho}u_{\bar{j}} \\
&\quad + \frac{\rho_{i\bar{j}\,\bar{\ell}}}{(-\rho)}u_k + \frac{\rho_{k\bar{j}\,\bar{\ell}}}{-\rho}u_i + \frac{\rho_{p\bar{j}\,\bar{\ell}}}{-\rho}(-g^{p\bar{q}}u_i u_{\bar{q}} u_k + u_{ik}g^{p\bar{q}}u_{\bar{q}}) \\
&\quad + u_{i\bar{j}}u_k u_{\bar{\ell}} + u_{i\bar{\ell}}u_k u_{\bar{j}} + u_i u_k u_{\bar{j}\,\bar{\ell}} + u_{ik}u_{\bar{j}}u_{\bar{\ell}} + u_{k\bar{j}}u_i u_{\bar{\ell}} + u_i u_{\bar{j}} u_{k\bar{\ell}} \\
&\quad - 3u_i u_k u_{\bar{j}} u_{\bar{\ell}} - (1 - |\partial u|_g^2)u_i u_{\bar{j}} u_k u_{\bar{\ell}} \\
&\quad + (1 - |\partial u|_g^2)u_{ik}u_{\bar{j}}u_{\bar{\ell}} + |\partial u|_g^2 u_{ik}u_{\bar{j}\,\bar{\ell}} + (1 - |\partial u|_g^2)u_i u_k u_{\bar{j}\,\bar{\ell}}.
\end{aligned}$$

Therefore, by (4.1), (4.7), (4.8) and (4.5)

$$\begin{aligned}
R_{i\bar{j}k\bar{\ell}} &= -(u_{i\bar{j}}u_{k\bar{\ell}} + u_{k\bar{j}}u_{i\bar{\ell}}) - \frac{\rho_{i\bar{j}k\bar{\ell}}}{-\rho} + g^{p\bar{q}}\frac{\rho_{i\bar{q}k}}{-\rho}\frac{\rho_{p\bar{j}\,\bar{\ell}}}{-\rho} \\
&\quad + \frac{\rho_{i\bar{q}k}}{-\rho}g^{p\bar{q}}u_p(-u_{\bar{j}}u_{\bar{\ell}} + u_{\bar{j}\,\bar{\ell}}) + \frac{\rho_{p\bar{j}\,\bar{\ell}}}{-\rho}g^{p\bar{q}}u_{\bar{q}}(-u_i u_{\bar{q}} u_k + u_{ik}) \\
&\quad - (1 - |\partial u|_g^2)\Big(u_{ik}u_{\bar{j}\,\bar{\ell}} + u_i u_{\bar{j}} u_k u_{\bar{\ell}} - u_i u_k u_{\bar{j}\,\bar{\ell}} - u_{ik}u_{\bar{j}}u_{\bar{\ell}}\Big)
\end{aligned}$$

$$= -(u_{i\bar{j}}u_{k\bar{\ell}} + u_{k\bar{j}}u_{i\bar{\ell}}) - \frac{\rho_{i\bar{j}k\bar{\ell}}}{-\rho}$$
$$+ \frac{1}{\rho^2}g^{p\bar{q}}\Big[\rho_{i\bar{q}k}\rho_{p\bar{j}\,\bar{\ell}} + \rho_{i\bar{q}k}u_p\rho_{\bar{j}\,\bar{\ell}} + \rho_{p\bar{j}\,\bar{\ell}}u_{\bar{q}}\rho_{ik}\Big] - (1 - |\partial u|_g^2)\frac{\rho_{ik}\rho_{\bar{j}\,\bar{\ell}}}{\rho^2}.$$

This completes the proof of the proposition. □

Next we construct a Kähler metric g on $D(A)$ so that its holomorphic bisectional curvature $\mathcal{K}_g \geq -1$ and $\lambda_1(\Delta_g) = n^2$. On $D(A)$, we let

$$\rho(z) = 2r(z) - \frac{\alpha}{2}r(z)^2, \quad 1 \leq \alpha \leq 2, \tag{4.9}$$

where $r(z)$ is given by (3.2). Notice

$$H(r) = I_n, \quad H(\rho) = (2 - \alpha r(z))I_n - \alpha(\partial r) \otimes (\overline{\partial} r). \tag{4.10}$$

It is easy to show that $r(z)$ and $-\log(-\rho(z))$ both are strictly plurisubharmonic negative defining functions for $D(A)$. Let

$$u(z) = -\log(-\rho), \quad g = g_{i\bar{j}}dz_i \otimes d\overline{z}_j = \frac{\partial^2 u}{\partial z_i \partial \overline{z}_j}dz_i \otimes d\overline{z}_j \tag{4.11}$$

is a Kähler metric on $D(A)$ induced by u. It is easy to prove that $(D(A), g)$ is a complete Kähler manifold. We will prove the following theorem.

Theorem 4.1. *Let g be a Kähler metric defined by* (4.9) *and* (4.11) *on $D(A)$. If $A_n \leq 2/5$, then the holomorphic bisectional curvature $\mathcal{K}_g \geq -1$ on $D(A)$.*

Proof. For simplicity, we write

$$\rho(z) = 2r(z) - \frac{\alpha}{2}r(z)^2 =: \phi(r(z)). \tag{4.12}$$

Then

$$\rho_i(z) = \phi'(r)r_i, \quad \rho_{i\bar{j}} = \phi'(r)r_{i\bar{j}} - \alpha r_i r_{\bar{j}}, \quad \rho_{ik} = \phi'(r)r_{ik} - \alpha r_i r_k \tag{4.13}$$

$$\rho_{ik\bar{q}} = -\alpha\Big[r_{\bar{q}}r_{ik} + r_{i\bar{q}}r_k + r_i r_{k\bar{q}}\Big] \tag{4.14}$$

and

(4.15)
$$\begin{aligned}\rho_{i\bar{j}k\bar{\ell}} &= -\alpha\frac{\partial}{\partial \bar{z}_\ell}[r_k r_{i\bar{j}} + r_{ik}r_{\bar{j}} + r_i r_{k\bar{j}}]\\ &= -\alpha[r_{k\bar{\ell}}r_{i\bar{j}} + r_{ik}r_{\bar{j}\,\bar{\ell}} + r_{i\bar{\ell}}r_{k\bar{j}}]\\ &= -\alpha(\delta_{ij}\delta_{k\ell} + A_iA_j\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{kj}).\end{aligned}$$

From (4.13), one has

(4.16)
$$\rho_{i\bar{j}} = \phi'(r)\left(\delta_{ij} - \frac{\alpha}{\phi'(r)}r_i r_{\bar{j}}\right), \quad \rho^{i\bar{j}} = \frac{1}{\phi'(r)}\left[\delta_{ij} + \frac{\alpha}{\phi'(r) - \alpha|\partial r|^2}r_{\bar{i}}r_j\right].$$

From (4.11) and (4.13), one has

(4.17)
$$\begin{aligned}g_{i\bar{j}} &= \frac{1}{-\rho}\left[\rho_{i\bar{j}} + \frac{1}{-\rho}\rho_i\rho_{\bar{j}}\right]\\ &= \frac{1}{-\rho}\left[\phi'(r)\delta_{ij} - \alpha r_i r_{\bar{j}} + \frac{\phi'(r)^2}{-\rho}r_i r_{\bar{j}}\right]\\ &= \frac{\phi'(r)}{-\rho}\left[\delta_{ij} + \frac{\phi'(r)^2 + \alpha\rho}{\phi'(r)(-\rho)}r_i r_{\bar{j}}\right]\end{aligned}$$

and

(4.18)
$$g^{i\bar{j}} = \frac{-\rho}{\phi'(r)}[\delta_{ij} - b r_{\bar{i}} r_j] \quad \text{and} \quad b = \frac{\phi'(r)^2 + \alpha\rho}{\phi'(r)(-\rho) + (\phi'(r)^2 + \alpha\rho)|\partial r|^2}.$$

Therefore,

(4.19)
$$\begin{aligned}\sum_{q=1}^{n} g^{p\bar{q}}\rho_{ik\bar{q}} &= \frac{-\rho}{\phi'(r)}[\delta_{pq} - b r_{\overline{p}} r_q](-\alpha[r_{\overline{q}}r_{ik} + r_{i\overline{q}}r_k + r_i r_{k\overline{q}}])\\ &= -\alpha\frac{-\rho}{\phi'(r)}\Big(r_{\overline{p}}r_{ik} + r_{i\overline{p}}r_k + r_i r_{k\overline{p}} - b r_{\overline{p}}(|\partial r|^2 r_{ik} + 2r_i r_k)\Big).\end{aligned}$$

Hence,

(4.20)
$$\begin{aligned}&\sum_{p,q=1}^{n} g^{p\overline{q}}\rho_{ik\overline{q}}u_p\\ &= -\alpha\frac{-\rho}{\phi'(r)}\Big(r_{\overline{p}}r_{ik} + r_{i\overline{p}}r_k + r_i r_{k\overline{p}} - b r_{\overline{p}}(|\partial r|^2 r_{ik} + 2r_i r_k\Big)\frac{\phi'(r)}{-\rho}r_p\\ &= -\alpha(1 - b|\partial r|^2)(|\partial r|^2 r_{ik} + 2r_i r_k).\end{aligned}$$

(4.21)
$$\frac{1}{\rho^2}\sum_{p,q=1}^n g^{p\overline{q}}u_p\rho_{ik\overline{q}}\rho_{\overline{j}\,\overline{\ell}} = -\alpha\frac{(1-b|\partial r|^2)}{\rho^2}(|\partial r|^2 r_{ik}+2r_ir_k)[\phi'(r)r_{\overline{j}\,\overline{\ell}}-\alpha r_{\overline{j}}r_{\overline{\ell}}]$$

and

(4.22)
$$\frac{1}{\rho^2}\sum_{p,q=1}^n g^{p\overline{q}}u_{\overline{q}}\rho_{p\overline{j}\,\overline{\ell}}\rho_{ik} = -\alpha\frac{(1-b|\partial r|^2)}{\rho^2}[|\partial r|^2 r_{\overline{j}\,\overline{\ell}}+2r_{\overline{j}}r_{\overline{\ell}}][\phi'(r)r_{ik}-\alpha r_ir_k].$$

Using

(4.23)
$$\begin{aligned}(|\partial r|^2 r_{ik}&+2r_ir_k)[\phi'(r)r_{\overline{j}\,\overline{\ell}}-\alpha r_{\overline{j}}r_{\overline{\ell}}]+[|\partial r|^2 r_{\overline{j}\,\overline{\ell}}+2r_{\overline{j}}r_{\overline{\ell}}][\phi'(r)r_{ik}-\alpha r_ir_k]\\ &= -4\alpha r_ir_{\overline{j}}r_kr_{\overline{\ell}}+2\phi'(r)|\partial r|^2 r_{ik}r_{\overline{j}\,\overline{\ell}}+2\phi'(r)(r_{\overline{j}\,\overline{\ell}}r_ir_k+r_{ik}r_{\overline{j}}r_{\overline{\ell}})\\ &\quad -\alpha|\partial r|^2(r_{ik}r_{\overline{j}}r_{\overline{\ell}}+r_{\overline{j}\,\overline{\ell}}r_ir_k),\end{aligned}$$

(4.21), (4.22) and (4.23), one has

$$\begin{aligned}\frac{1}{\rho^2}&\sum_{p,q=1}^n g^{p\overline{q}}[u_p\rho_{ik\overline{q}}\rho_{\overline{j}\,\overline{\ell}}+u_{\overline{q}}\rho_{p\overline{j}\,\overline{\ell}}\rho_{ik}]\\ &= -2\alpha\frac{(1-b|\partial r|^2)}{\rho^2}\Big[\phi'(r)|\partial r|^2 r_{ik}r_{\overline{j}\,\overline{\ell}}+\Big(\phi'(r)-\frac{\alpha}{2}|\partial r|^2\Big)\\ &\quad\times(r_{ik}r_{\overline{j}}r_{\overline{\ell}}+r_{\overline{j}\,\overline{\ell}}r_ir_k)-2\alpha r_ir_{\overline{j}}r_kr_{\overline{\ell}}\Big]\\ &= -2\alpha\frac{(1-b|\partial r|^2)}{\rho^2}\Big[\phi'|\partial r|^2 A_i\delta_{ik}A_j\delta_{j\ell}+\Big(\phi'-\frac{\alpha}{2}|\partial r|^2\Big)\\ &\quad\times(A_i\delta_{ik}r_{\overline{j}}r_{\overline{\ell}}+A_j\delta_{j\ell}r_ir_k)-2\alpha r_ir_{\overline{j}}r_kr_{\overline{\ell}}\Big].\end{aligned}$$

Since

$$|\partial u|_g^2=\sum_{p,q=1}^n g^{p\overline{q}}u_pu_{\overline{q}}=\frac{-\rho}{\phi'(r)}(\delta_{pq}-br_{\overline{p}}r_q)\frac{\phi'(r)^2}{\rho^2}r_pr_{\overline{q}}=\frac{\phi'(r)}{-\rho}|\partial r|^2(1-b|\partial r|^2),$$

one has

$$\begin{aligned}1-b|\partial r|^2 &= 1-\frac{(\phi'(r)^2+\alpha\rho)|\partial r|^2}{\phi'(r)(-\rho)+(\phi'(r)^2+\alpha\rho)|\partial r|^2}\\ &= \frac{\phi'(r)(-\rho)}{\phi'(r)(-\rho)+(\phi'(r)^2+\alpha\rho)|\partial r|^2}\end{aligned}$$

$$= \frac{\phi'(r)(-\rho)b}{(\phi'(r)^2 + \alpha\rho)}.$$

Thus,

(4.24)
$$1 - |\partial u|_g^2 = 1 - \frac{\phi'(r)^2|\partial r|^2}{\phi'(r)(-\rho) + (\phi'(r)^2 + \alpha\rho)|\partial r|^2} = \frac{(-\rho)(\phi'(r) - \alpha|\partial r|^2)b}{(\phi'(r)^2 + \alpha\rho)}$$

and

$$\begin{aligned}
&\frac{1}{\rho^2}(1 - |\partial u|_g^2)\rho_{ik}\rho_{\bar{j}\,\bar{\ell}} \\
&\quad = \frac{(\phi'(r) - \alpha|\partial r|^2)b}{(-\rho)(\phi'(r)^2 + \alpha\rho)}(\phi'(r)A_i\delta_{ik} - \alpha r_i r_k)(\phi'(r)A_j\delta_{j\ell} - \alpha r_{\bar{j}} r_{\bar{\ell}}) \\
&\quad = \frac{(\phi'(r) - \alpha|\partial r|^2)b}{(-\rho)(\phi'(r)^2 + \alpha\rho)}(\phi'(r)^2 A_j A_i \delta_{ik}\delta_{j\ell} - \alpha\phi'(r) \\
&\qquad \times (A_j\delta_{j\ell} r_i r_k + A_i\delta_{ik} r_{\bar{j}} r_{\bar{\ell}}) + \alpha^2 r_i r_{\bar{j}} r_k r_{\bar{\ell}}).
\end{aligned}$$

Therefore,

(4.25)
$$\begin{aligned}
&\frac{(-\rho)}{\rho^2}\sum_{p,q=1}^n g^{p\bar{q}}[u_p\rho_{ik\bar{q}}\rho_{\bar{j}\,\bar{\ell}} + u_{\bar{q}}\rho_{p\bar{j}\,\bar{\ell}}\rho_{ik}] - \frac{(-\rho)}{\rho^2}(1 - |\partial u|_g^2)\rho_{ik}\rho_{\bar{j}\,\bar{\ell}} \\
&\quad = \frac{-2\alpha\phi'(r)b}{(\phi'(r)^2 + \alpha\rho)}\Big[\phi'|\partial r|^2 A_i\delta_{ik}A_j\delta_{j\ell} + (\phi' - \frac{\alpha}{2}|\partial r|^2) \\
&\quad\quad \times (A_i\delta_{ik} r_{\bar{j}} r_{\bar{\ell}} + A_j\delta_{j\ell} r_i r_k) - 2\alpha r_i r_{\bar{j}} r_k r_{\bar{\ell}}\Big] - \frac{(\phi'(r) - \alpha|\partial r|^2)b}{(\phi'(r)^2 + \alpha\rho)} \\
&\quad\quad \times \Big(\phi'(r)^2 A_j A_i\delta_{ik}\delta_{j\ell} - \alpha\phi'(r)(A_j\delta_{j\ell} r_i r_k + A_i\delta_{ik} r_{\bar{j}} r_{\bar{\ell}}) + \alpha^2 r_i r_{\bar{j}} r_k r_{\bar{\ell}}\Big) \\
&\quad = -\frac{(\alpha|\partial r|^2 + \phi'(r))\phi'(r)^2 b}{(\phi'(r)^2 + \alpha\rho)}A_i A_j\delta_{ik}\delta_{j\ell} - \frac{\alpha\phi'(r)^2 b}{(\phi'(r)^2 + \alpha\rho)} \\
&\quad\quad \times (A_j\delta_{j\ell} r_i r_k + A_i\delta_{ik} r_{\bar{j}} r_{\bar{\ell}}) + \frac{\alpha^2 b(3\phi'(r) + \alpha|\partial r|^2)}{(\phi'(r)^2 + \alpha\rho)} r_i r_{\bar{j}} r_k r_{\bar{\ell}}.
\end{aligned}$$

Therefore, by (4.4), (4.15) and (4.25), for any $\xi, \eta \in \mathbb{C}^n$, one has

(4.26)
$$\begin{aligned}
&(-\rho)E_{i\bar{j}k\bar{\ell}}\xi_i\bar{\xi}_j\eta_k\bar{\eta}_\ell \\
&\quad \geq \alpha\Big(|\xi|^2|\eta|^2 + \Big|\sum_{i=1}^n A_i\xi_i\eta_i\Big|^2 + \Big|\sum_{i=1}^n \xi_i\bar{\eta}_i\Big|^2\Big)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{(\alpha|\partial r|^2 + \phi'(r))\phi'(r)^2 b}{(\phi'(r)^2 + \alpha\rho)} \Big| \sum_{i=1}^n A_i \xi_i \eta_i \Big|^2 \\
&- \frac{2\alpha\phi'(r)^2 b}{(\phi'(r)^2 + \alpha\rho)} \mathrm{Re} \Big(\Big(\sum_{j=1}^n A_j \bar{\xi}_j \overline{\eta}_j \Big) \Big(\sum_{i=1}^n \xi_j r_i \Big) \Big(\sum_{k=1}^n \eta_k r_k \Big) \Big) \\
&+ \frac{\alpha^2 b(3\phi'(r) + \alpha|\partial r|^2)}{(\phi'(r)^2 + \alpha\rho)} \Big| \sum_{i=1}^n r_i \xi_i \Big|^2 \Big| \sum_{k=1}^n r_k \eta_k \Big|^2 \\
\geq{}& \alpha \Big(|\xi|^2 |\eta|^2 + \Big| \sum_{i=1}^n A_i \xi_i \eta_i \Big|^2 + \Big| \sum_{i=1}^n \xi_i \overline{\eta}_i \Big|^2 \Big) \\
&- \frac{(\alpha|\partial r|^2 + \phi'(r))\phi'(r)^2 b}{(\phi'(r)^2 + \alpha\rho)} \Big| \sum_{i=1}^n A_i \xi_i \eta_i \Big|^2 \\
&- \frac{1}{4} \Big[\frac{2\alpha\phi'(r)^2 b}{(\phi'(r)^2 + \alpha\rho)} \Big]^2 \frac{(\phi'(r)^2 + \alpha\rho)}{\alpha^2 b(3\phi'(r) + \alpha|\partial r|^2)} \Big| \sum_{j=1}^n A_j \bar{\xi}_j \overline{\eta}_j \Big|^2 \\
={}& \alpha \left(|\xi|^2 |\eta|^2 + \Big| \sum_{i=1}^n \xi_i \overline{\eta}_i \Big|^2 \right) + \Big[\alpha - \frac{(\alpha|\partial r|^2 + \phi'(r))\phi'(r)^2 b}{(\phi'(r)^2 + \alpha\rho)} \\
&- \frac{\phi'(r)^4 b}{(\phi'(r)^2 + \alpha\rho)} \frac{1}{(3\phi'(r) + \alpha|\partial r|^2)} \Big] \Big| \sum_{j=1}^n A_j \bar{\xi}_j \overline{\eta}_j \Big|^2 \\
={}& \alpha \left(|\xi|^2 |\eta|^2 + \Big| \sum_{i=1}^n \xi_i \overline{\eta}_i \Big|^2 \right) \\
&+ \alpha \left(1 - \frac{\phi'(r)^2 b((4/\alpha)\phi'(r)^2 + \alpha|\partial r|^4 + 4\phi'(r)|\partial r|^2)}{(\phi'(r)^2 + \alpha\rho)(3\phi'(r) + \alpha|\partial r|^2)} \right) \Big| \sum_{j=1}^n A_j \bar{\xi}_j \overline{\eta}_j \Big|^2 \\
={}& \alpha \left(|\xi|^2 |\eta|^2 + \Big| \sum_{i=1}^n \xi_i \overline{\eta}_i \Big|^2 \right) \\
&+ \alpha \Big(1 - \frac{\alpha\phi'(r)^2 (\phi'(r)(2/\alpha) + |\partial r|^2)^2}{(\phi'(r)(-\rho) + (\phi'(r)^2 + \alpha\rho)|\partial r|^2)(3\phi'(r) + \alpha|\partial r|^2)} \Big) \Big| \sum_{j=1}^n A_j \bar{\xi}_j \overline{\eta}_j \Big|^2 .
\end{aligned}
$$

Notice

$$
16 \geq \phi'(r)^2 \geq \phi'(r)^2 + \alpha\rho = 4 - 4r(z) + 2r(z)^2 \geq 2\phi'(r) \geq 4 \tag{4.27}
$$

and since $|z|^2(1 - A_n) < 1$ on $D(A)$, one has

$$
\begin{aligned}
&-\rho(z) + 2|\partial r(z)|^2 \\
&\geq -2r(z) + 2|\partial r(z)|^2
\end{aligned} \tag{4.28}
$$

$$\begin{aligned}
&\geq 2(1 - |z|^2 - \sum_{j=1}^n A_j \mathrm{Re}\, z_j^2 + |z|^2 + 2\sum_{j=1}^n A_j \mathrm{Re}\, z_j^2 + \sum_{j=1}^n A_j^2 |z_j|^2) \\
&= 2(1 + \sum_{j=1}^n A_j \mathrm{Re}\, z_j^2 + \sum_{j=1}^n A_j^2 |z_j|^2) \\
&\geq 2(1 - \sum_{j=1}^n A_j^2 |z_j|^2 - \frac{1}{4}|z|^2 + \sum_{j=1}^n A_j^2 |z_j|^2) \\
&\geq 2(1 - \frac{|z|^2}{4}) \\
&\geq 2(1 - \frac{1}{4(1-A_n)}) \\
&= \frac{3-4A_n}{2(1-A_n)}.
\end{aligned}$$

Therefore, with $A_n < 3/4$

$$\begin{aligned}
\phi'(r)(-\rho) + (\phi'(r)^2 + \alpha\rho)|\partial r|^2 &\geq \phi'(r)(-\rho) + 2\phi'(r)|\partial r|^2 \\
&= \phi'(r)[-\rho + 2|\partial r|^2] \\
&\geq \frac{(3-4A_n)\phi'(r)}{2(1-A_n)}.
\end{aligned} \tag{4.29}$$

Since $|\partial r(z)|^2 \leq \phi'(r)$ and computation in (4.29), one has $\phi'(r) + |\partial r(z)|^2 = 2 - 2r(z) + |\partial r(z)|^2 \leq 4$. Thus, since $\alpha \in [1,2]$, one has

$$\frac{(\phi'(r)(2/\alpha) + |\partial r|^2)^2}{3\phi'(r) + \alpha|\partial r|^2} \leq (\phi'(r) + |\partial r|^2) \leq 4.$$

Therefore, if $A_n \leq 2/5$, then

$$\begin{aligned}
&1 - \frac{2\phi'(r)^2(\phi'(r)(2/\alpha) + |\partial r|^2)^2}{(\phi'(r)(-\rho) + (\phi'(r)^2 + \alpha\rho)|\partial r|^2)(3\phi'(r) + \alpha|\partial r|^2)} \\
&\quad \geq 1 - 4\frac{4\phi'(r)(1-A_n)}{(3-4A_n)} \\
&\quad \geq 1 - \frac{16(1-A_n)}{(3-4A_n)} \\
&\quad = -\frac{13-12A_n}{(3-4A_n)} \\
&\quad \geq -\frac{41}{7}.
\end{aligned}$$

Therefore, if $0 \le A_n \le 2/5$, then

$$(4.30)\qquad (-\rho)E_{i\bar{j}k\bar{\ell}}\xi_i\bar{\xi}_j\eta_k\bar{\eta}_\ell \ge \alpha(1-\frac{41}{7}A_n^2)|\xi|^2|\eta|^2 \ge \alpha\frac{11}{175}|\xi|^2|\eta|^2 \ge 0.$$

Therefore, the proof of the theorem is complete. □

Finally, we will prove part (ii) of Theorem 1.3.

Proof of part (ii) of Theorem 1.3.

Proof. Let $r(z) = |z|^2 + \sum_{j=1}^n \mathrm{Re}\, A_j z_j^2 - 1$ on $D(A)$, and let

$$\rho(z) = 2r(z) - \frac{1}{2}r(z)^2, \quad u(z) = -\log(-\rho(z)), \quad g = \sum_{i,j=1}^n \frac{\partial u}{\partial z_i \partial \bar{z}_j} dz_i \otimes d\bar{z}_j.$$

If $A_n \le 2/5$ then $|r(z)| \le 5/3$. By (4.10), it is easy to show that $\rho(z)$ is strictly plurisubharmonic in $D(A)$. By Theorem 1.1, we have $\lambda_1(\Delta_g) = n^2$. By Theorem 4.2, we have $\mathcal{K}_g \ge -1$ on $D(A)$. Therefore, the proof of part (ii) of Theorem 1.3 is complete. □

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