Ricci curvature on Alexandrov spaces and rigidity theorems

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In this paper, we introduce a new notion for lower bounds of Ricci curvature on Alexandrov spaces, and extend Cheeger–Gromoll splitting theorem and Cheng's maximal diameter theorem to Alexandrov spaces under this Ricci curvature condition.

1. Introduction

Alexandrov spaces with curvature bounded below generalize successfully the concept of lower bounds of sectional curvature from Riemannian manifolds to singular spaces. The seminal paper [4] and the 10th chapter in the text book [3] provide excellent introductions to this field. Many important theorems in Riemannian geometry had been extended to Alexandrov spaces, such as Synge's theorem [30], diameter sphere theorem [28], Toponogov splitting theorem [19], etc.

However, many fundamental results in Riemannian geometry (for example, Bishop–Gromov volume comparison theorem, Cheeger–Gromoll splitting theorem and Cheng's maximal diameter theorem) assume only the lower bounds on *Ricci* curvature, not on sectional curvature. Therefore, it is a very interesting question how to generalize the concept of lower bounds of Ricci curvature from Riemannian manifolds to singular spaces.

Perhaps the first concept of lower bounds of Ricci curvature on singular spaces was given by Cheeger and Colding (see Appendix 2 in [7]). They, in [6, 7], studied Gromov–Hausdorff limit spaces of Riemannian manifolds with Ricci curvature (uniformly) bounded below. Among other results in [6], they proved the following rigidity theorem:

Theorem 1.1 (Cheeger–Colding). *Let M_i be a sequence of Riemannian manifolds and M_i converges to X in sense of Gromov–Hausdorff.*

> *(1) If X contains a line and $\mathrm{Ric}(M_i) \geqslant -\epsilon_i$ with $\epsilon_i \to 0$, then X is isometric to a direct product $\mathbb{R} \times Y$ over some length space Y.*

(2) *If* $\mathrm{Ric}(M_i) \geqslant n-1$ *and diameter of* M_i $\mathrm{diam}(M_i) \to \pi$, *then* X *is isometric to a spherical suspension* $[0,\pi] \times_{\sin} Y$ *over some length space* Y.

In [33], Petrunin considered to generalize the lower bounds of Ricci curvature for singular spaces via subharmonic functions.

Recently, in terms of L^2-Wasserstein space and optimal mass transportation, Sturm [37, 38] and Lott–Villani [16, 17] have given a generalization of "Ricci curvature has lower bounds" for metric measure spaces,[1] independently. They call that curvature-dimension conditions, denoted by $\mathrm{CD}(n,k)$ with $n \in (1,\infty]$ and $k \in \mathbb{R}$. For the convenience of readers, we repeat their definition of $\mathrm{CD}(n,k)$ in the Appendix of this paper. On the other hand, Sturm in [38] and Ohta in [20] introduced another definition of "Ricci curvature bounded below" for metric measure spaces, the measure contraction property $\mathrm{MCP}(n,k)$, which is a slight modification of a property introduced earlier by Sturm in [39] and in a similar form by Kuwae and Shioya in [14, 15]. The condition $\mathrm{MCP}(n,k)$ is indeed an infinitesimal version of the Bishop–Gromov relative volume comparison condition. For a metric measure space, Sturm [38] proved that $\mathrm{CD}(n,k)$ implies $\mathrm{MCP}(n,k)$ provided it is non-branching.[2] Note that any Alexandrov space with curvature bounded below is non-branching. Recently, Petrunin [31] proved that any n-dimensional Alexandrov space with curvature $\geqslant 0$ must satisfy $\mathrm{CD}(n,0)$ and claimed the general statement that the condition curvature $\geqslant k$ (for some $k \in \mathbb{R}$) implies the condition $\mathrm{CD}(n,(n-1)k)$ can be also proved along the same lines.

Let M be a Riemannian manifold with Riemannian distance d and Riemannian volume vol. Lott and Villani in [16] and von Renesse and Sturm in [36, 40] proved that (M,d,vol) satisfies $\mathrm{CD}(\infty,k)$ if and only if $\mathrm{Ric}(M) \geqslant k$. Indeed, they proved a stronger weighted version (see Theorem 7.3 in [16] and Theorem 1.1 in [36], Theorem 1.3 in [40]). Let ϕ be a smooth function on M with $\int_M e^{-\phi} d\mathrm{vol} = 1$. Lott and Villani in [17] proved that $(M,d,e^{-\phi} \cdot \mathrm{vol})$ satisfies $\mathrm{CD}(n,k)$ if and only if weighted Ricci curvature $\mathrm{Ric}_n(M) \geqslant k$ (see Definition 4.20 – the definition of Ric_n – and Theorem 4.22 in [17]). A similar result was proved by Sturm in [38] (see Theorem 1.7 in [38]). In particular, they proved that (M,d,vol) satisfies $\mathrm{CD}(n,k)$ if and only if $\mathrm{Ric}(M) \geqslant k$ and $\dim(M) \leqslant n$. If $\dim(M) = n$, Ohta in [20] and Sturm in [38] proved, independently, that M satisfies $\mathrm{MCP}(n,k)$ is equivalent to $\mathrm{Ric}(M) \geqslant k$.

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[1]A metric measure space is a metric space equipped a Borel measure.

[2]A geodesic space is called non-branching if for any quadruple points z, x_0, x_1, x_2 with z being the midpoint of x_0 and x_1 as well as the midpoint of x_0 and x_2, it follows that $x_1 = x_2$.

Nevertheless, since n-dimensional norm spaces $(V^n, \| \cdot \|_p)$ satisfy $CD(n,0)$ for every $p > 1$ (see, for example, in [41, p. 892]), it is impossible to show Cheeger–Gromoll splitting theorem under $CD(n,0)$ for general metric measure spaces. Furthermore, it was shown by Ohta in [22] that on a Finsler manifolds M, the curvature-dimension condition $CD(n,k)$ is equivalent to the weighted Finsler Ricci curvature condition $\mathrm{Ric}_n(M) \geqslant k$ (see also [23] or [25], refer to [23] for the definition Ric_n in Finsler manifolds). That says, the curvature-dimension condition is somewhat a Finsler geometry character. Seemly, it is difficult to show the rigidity theorems, such as Cheng's maximal diameter theorem and Obata's theorem, under $CD(n, n-1)$ for general metric measure spaces.

As a compensation, Watanabe [42] proved that if a metric measure space M satisfies $CD(n,0)$ or $MCP(n,0)$ then M has at most two ends. Ohta [21] proved that a non-branching compact metric measure space with $MCP(n, n-1)$ and diameter $= \pi$ is homeomorphic to a spherical suspension.

Alexandrov spaces with curvature bounded below have richer geometric information than general metric measure spaces. In particular, a finite dimensional norm space with curvature bounded below must be an inner-product space. Naturally, one would expect that Cheeger–Gromoll splitting theorem still holds on Alexandrov spaces with suitable non-negative "Ricci curvature condition."

Recently in [12], Kuwae and Shioya proved the following topological splitting theorem for Alexandrov spaces under the $MCP(n,0)$ condition:

Theorem 1.2 (Kuwae–Shioya). *Let M^n be an n-dimensional Alexandrov space. Assume that M^n contains a line.*

(1) *If M satisfies $MCP(n,0)$, then M^n is homeomorphic to a direct product space $\mathbb{R} \times Y$ over some topological space Y.*

(2) *If the singular set of M^n is closed and the non-singular set is an (incomplete) C^∞ Riemannian manifold of $\mathrm{Ric} \geqslant 0$, then M^n is isometric to a direct product space $\mathbb{R} \times Y$ over some Alexandrov space Y.*

We remark that Kuwae and Shioya actually obtained a more general weighted measure version of the above theorem in [13].

In the following, inspired by Petrunin's second variation of arc length [30], we will introduce a new notion of the Ricci curvature bounded below for Alexandrov spaces.

Let M be an n-dimensioal Alexandrov space of curvature bounded from below *locally* without boundary. It is well known in [34] or [32] that, for

any $p \in M$ and $\xi \in \Sigma_p$, there exists a *quasi-geodesic* starting at p along direction ξ. (See [34] or [32, Section 5] for the definition and properties of quasi-geodesics.) According to [30], the exponential map $\exp_p : T_p \rightarrow M$ is defined as follows. For any $v \in T_p$, $\exp_p(v)$ is a point on some quasi-geodesic of length $|v|$ starting point p along $v/|v| \in \Sigma_p$. If the quasi-geodesic is not unique, we take one of them as the definition of $\exp_p(v)$.

Let $\gamma : [0, \ell) \rightarrow M$ be a geodesic. Without loss of generality, we may assume that a neighborhood U_γ of γ has curvature $\geqslant k_0$ for some $k_0 < 0$.

According to Section 7 in [4], the tangent cone $T_{\gamma(t)}$ at an interior point $\gamma(t)$ ($t \in (0, \ell)$) can be split into a direct metric product. We denote

$$L_{\gamma(t)} = \{\xi \in T_{\gamma(t)} \mid \angle(\xi, \gamma^+(t)) = \angle(\xi, \gamma^-(t)) = \pi/2\},$$
$$\Lambda_{\gamma(t)} = \{\xi \in \Sigma_{\gamma(t)} \mid \angle(\xi, \gamma^+(t)) = \angle(\xi, \gamma^-(t)) = \pi/2\}.$$

In [30], Petrunin proved the following second variation formula of arc-length.

Proposition 1.1 (Petrunin). *Given any two points $q_1, q_2 \in \gamma$, which are not end points, and any positive number sequence $\{\varepsilon_j\}_{j=1}^\infty$ with $\varepsilon_j \rightarrow 0$, there exists a subsequence $\{\widetilde{\varepsilon}_j\} \subset \{\varepsilon_j\}$ and an isometry $T : L_{q_1} \rightarrow L_{q_2}$ such that*

$$|\exp_{q_1}(\widetilde{\varepsilon}_j u), \ \exp_{q_2}(\widetilde{\varepsilon}_j T v)| \leqslant |q_1 q_2| + \frac{|uv|^2}{2|q_1 q_2|} \cdot \widetilde{\varepsilon}_j^2$$
$$- \frac{k_0 \cdot |q_1 q_2|}{6} \cdot \left(|u|^2 + |v|^2 + \langle u, v \rangle\right) \cdot \widetilde{\varepsilon}_j^2 + o(\widetilde{\varepsilon}_j^2)$$

for any $u, v \in L_{q_1}$.

We remark that for a two-dimensional Alexandrov space, Cao *et al.* in [8] improved the second variation formula such that the above inequality holds for all $\{\varepsilon_j\}_{j=1}^\infty$. But for higher dimensions, to the best of our knowledge, we do not know whether the parallel translation T in the above second variation formula can be chosen independent of the sequences $\{\varepsilon_j\}$.

Based on this second variation formula, we can propose a condition which resembles the lower bounds for the radial curvature along the geodesic γ.

Let $\{g_{\gamma(t)}\}_{0<t<\ell}$ be a family of functions, where for each t, $g_{\gamma(t)}$ is a continuous function on $\Lambda_{\gamma(t)}$. For simplicity, we call $\{g_{\gamma(t)}\}_{0<t<\ell}$ to be a *continuous function family*.

Definition 1.1. A continuous function family $\{g_{\gamma(t)}\}_{0<t<\ell}$ is said to satisfy *Condition (RC)*, if for any $\epsilon > 0$ and any $t_0 \in (0, \ell)$, there exists a neighborhood $I_{t_0} := (t_0 - \tau^*, t_0 + \tau^*) \subset (0, \ell)$ with the following property. For any

two number $s, t \in I_{t_0}$ with $s < t$ and for any sequence $\{\theta_j\}_{j=1}^{\infty}$ with $\theta_j \to 0$ as $j \to \infty$, there exists an isometry $T : \Lambda_{\gamma(t)} \to \Lambda_{\gamma(s)}$ and a subsequence $\{\delta_j\}$ of $\{\theta_j\}$ such that

$$| \exp_{\gamma(s)}(\delta_j l_1 T\xi), \ \exp_{\gamma(t)}(\delta_j l_2 \xi)|$$
$$\leqslant |s - t| + \frac{(l_1 - l_2)^2}{2|s - t|} \cdot \delta_j^2$$
(1.1)
$$- \frac{(g_{\gamma(t)}(\xi) - \epsilon) \cdot |s - t|}{6} \cdot (l_1^2 + l_1 \cdot l_2 + l_2^2) \cdot \delta_j^2 + o(\delta_j^2)$$

for any $l_1, l_2 \geqslant 0$ and any $\xi \in \Lambda_{\gamma(t)}$.

Let $\mathcal{F}$ denote the set all of continuous function families$\{g_{\gamma(t)}\}_{0<t<\ell}$, which satisfy Condition (RC).

Clearly, the above proposition shows that $\{g_{\gamma(t)} = k_0\}_{0<t<\ell} \in \mathcal{F}$.

Definition 1.2. We say that M has *Ricci curvature bounded below by* $(n-1)K$ *along* γ, if

(1.2)
$$\rho := \sup_{\{g_{\gamma(t)}\} \in \mathcal{F}} \inf_{0<t<\ell} \fint_{\Lambda_{\gamma(t)}} g_{\gamma(t)}(\xi) \geqslant K,$$

where $\fint_{\Lambda_x} g_x(\xi) = \frac{1}{vol(\Lambda_x)} \int_{\Lambda_x} g_x(\xi)d\xi$.

We say M has *Ricci curvature bounded below by* $(n-1)K$ *on an open set* $U \subset M$, if for each point $p \in U$, there is a neighborhood U_p of p with $U_p \subset U$ such that M has Ricci curvature bounded below by $(n-1)K$ along every geodesic $\gamma : [0, \ell) \to U_p$. When $U = M$, we say M has Ricci curvature bounded below by $(n-1)K$ and denote $\mathrm{Ric}(M) \geqslant (n-1)K$.

Remark 1.1. (i) When M is a smooth Riemannian manifold, by the second variation of formula of arc-length, it is easy to see Condition (RC) is equivalent to

$$\sec_M(\Pi_t) \geqslant g_{\gamma(t)}(\xi),$$

where $\Pi_t \subset T_{\gamma(t)}$ is any two-dimensional subspace, spanned by $\gamma'(t)$ and a $\xi \in \Lambda_{\gamma(t)}$. Thus in a Riemannian manifold, our definition on Ricci curvature bounded below by $(n-1)K$ is exactly the classical one.

(ii) Let M be an n-dimensional Alexandrov space with curvature $\geqslant K$. The above Proposition 1.1 shows that $\mathrm{Ric}(M) \geqslant (n-1)K$.

(iii) Recall that Petrunin in [31] proved any n-dimensional Alexandrov space M with curvature $\geqslant K$ must satisfy the curvature-dimension condition $\text{CD}(n,(n-1)K)$. In the Appendix, by modifying Petrunin's proof in [31], we will show that any n-dimensional Alexandrov space M with $\text{Ric}(M) \geqslant (n-1)K$ also satisfies $\text{CD}(n,(n-1)K)$.

(iv) At the present stage, we do not know if the Ricci curvature condition $\text{Ric}(M) \geqslant (n-1)K$ is equivalent to the curvature-dimension condition $\text{CD}(n,(n-1)K)$. We will investigate this question in future.

Our main results in this paper are the following splitting theorem and maximal diameter theorem.

Theorem 1.3 (Splitting theorem). *Let M be an n-dimensional complete non-compact Alexandrov space with non-negative Ricci curvature and $\partial M = \varnothing$. If M contains a line, then M is isometric to a direct metric product $\mathbb{R} \times N$ for some Alexandrov space N with non-negative Ricci curvature.*

Theorem 1.4 (Maximal diameter theorem). *Let M be an n-dimensional compact Alexandrov space with Ricci curvature bounded below by $n-1$ and $\partial M = \varnothing$. If the diameter of M is π, then M is isometric to a spherical suspension over an Alexandrov space with curvature $\geqslant 1$.*

An open question for the curvature-dimension condition $\text{CD}(n,k)(k \neq 0)$ is "from local to global" (See, for example, the 30th chapter in [41]). In particular, given a metric measure space which admits a covering and satisfies $\text{CD}(n,k)$ ($k \neq 0$), we do not know if the covering space with pullback metric still satisfies $\text{CD}(n,k)$.

One advantage of our definition of the Ricci curvature bounded below on Alexandrov spaces is that the definition is purely local. In particular, any covering space of an n-dimensional Alexandrov space with Ricci curvature bounded below by $(n-1)K$ still satisfies the condition $\text{Ric} \geqslant (n-1)K$. Meanwhile, we note that Bishop–Gromov volume comparison theorem also holds on an Alexandrov space with Ricci curvature bounded below (see Corollary A.1 in the Appendix). Consequently, the same proofs as in Riemannian manifold case (see [1] and, for example,[27, pp. 275–276]) give the following estimates on the fundamental group and the first Betti number.

Corollary 1.1. *Let M be a compact n-dimensional Alexandrov space with non-negative Ricci curvature and $\partial M = \varnothing$. Then its fundamental group has a finite index Bieberbach subgroup.*

Corollary 1.2. *Let M be an n-dimensional Alexandrov space with non-negative Ricci curvature and $\partial M = \varnothing$. Then any finitely generated subgroup of $\pi_1(M)$ has polynomial growth of degree $\leqslant n$. If some finitely generated subgroup of $\pi_1(M)$ has polynomial growth of degree $= n$, then M is compact and flat.*

Corollary 1.3. *Let M be an n-dimensional Alexandrov space with $\partial M = \varnothing$.*

(1) If $\mathrm{Ric}(M) \geqslant (n-1)K > 0$, then its fundamental group is finite.

(2) If $\mathrm{Ric}(M) \geqslant (n-1)K$ and diameter of $M \leqslant D$, then

$$b_1(M) \leqslant C(n, K^2 \cdot D)$$

for some function $C(n, K^2 \cdot D)$.

Moreover, there exists a constants $\kappa(n) > 0$ such that if $K^2 \cdot D \geqslant -\kappa(n)$, then $b_1(M) \leqslant n$.

The paper is organized as follows. In Section 2, we recall some necessary materials for Alexandrov spaces. In Section 3, we will define a new representation of Laplacian along a geodesic and will prove the comparison theorem for the newly defined representation of Laplacian (see Theorem 3.1). In Section 4, we will discuss the rigidity part of the comparison theorem. The maximal diameter theorem and the splitting theorem will be proved in Sections 5 and 6, respectively. In the Appendix, we give a modification of Petrunin's proof in [31] to show that the condition on Ricci curvature bounded below implies the curvature-dimension condition (see Proposition A.1).

2. Preliminaries

A metric space $(X, |\cdot, \cdot|)$ is called a length space if for any two point $p, q \in X$, the distance between p and q is given by

$$|pq| = \inf_{\gamma, \gamma \text{ connect } p, q} \mathrm{Length}(\gamma).$$

A length space X is called a *geodesic* space if for any two point $p, q \in X$, there exists a curve γ connecting p and q such that $\mathrm{Length}(\gamma) = |pq|$. Such a curve is called a shortest curve. A geodesic is a *unit-speed* shortest curve.

Recall that a length space X has curvature $\geqslant k$ in an open set $U \subset X$ if for any quadruple $(p; a, b, c) \subset U$, there holds

$$\widetilde{\angle}_k apb + \widetilde{\angle}_k bpc + \widetilde{\angle}_k cpa \leqslant 2\pi,$$

where $\widetilde{\angle}_k apb, \widetilde{\angle}_k bpc$ and $\widetilde{\angle}_k cpa$ are the comparison angles in the k-plane. A length space M is called an *Alexandrov space with curvature bounded from below locally* (for short, we say M to be an *Alexandrov space*), if it is locally compact and any point in M has an open neighborhood $U \subset M$ such that M has curvature $\geqslant k_U$ in U, for some $k_U \in \mathbb{R}$.

Let M be an Alexandrov space without boundary and $U \subset M$ be an open set. A locally Lipschitz function u on U is said to be λ-*concave* on U if for any geodesic $\gamma \subset U$, the one-variable function

$$u \circ \gamma(t) - \lambda t^2 / 2$$

is concave. A function u on M is said to be *semi-concave* if for any point $x \in M$ there is a neighborhood $U_x \ni x$ and a real number λ_x such that the restriction $u|_{U_x}$ is λ_x-concave.

Let $\psi : \mathbb{R} \to \mathbb{R}$ be a continuous function. A function u on M called $\psi(u) - concave$ if for any point $x \in M$ and any $\varepsilon > 0$ there is a neighborhood $U_x \ni x$ such that $u|_{U_x}$ is $(\psi \circ u(x) + \varepsilon)$-concave.

If M has curvature $\geqslant k$ in U, then it is well known that the function $u = \varrho_k \circ \mathrm{dist}_p$ is $(1 - ku)$-concave in $U \backslash \{p\}$, where

$$\varrho_k(v) = \begin{cases} \dfrac{1}{k}\big(1 - \cos(\sqrt{k}v)\big) & \text{if} \quad k > 0, \\[2mm] \dfrac{v^2}{2} & \text{if} \quad k = 0, \\[2mm] \dfrac{1}{k}\big(\cosh(\sqrt{-k}v) - 1\big) & \text{if} \quad k < 0, \end{cases}$$

(see, for example,[32, Section 1]).

Let u be a semi-concave function on M. For any point $p \in M$, there exists a u-*gradient curve* starting at p. Hence u generates a *gradient flow* $\Phi_u^t : M \to M$, which is a locally Lipschitz map. (Actually, it is just a semi-flow, because backward flow Φ_u^{-t} is not always well-defined.) Particularly, if u is concave, the gradient flow is a 1-Lipschitz map. We refer to Sections 1 and 2 in [32] for the details on semi-concave functions, gradient curves and gradient flows.

3. Laplacian comparison theorem

Let M be an n-dimensional Alexandrov space without boundary. A canonical Dirichlet form $\mathcal{E}$ is defined by

$$\mathcal{E}(u, v) := \int_M \langle \nabla u, \nabla v \rangle \, d\,\text{vol}, \qquad \text{for} \quad u, v \in W_0^{1,2}(M).$$

(see [11]). The Laplacian associated to the canonical Dirichlet form is given as follows. Let $u : U \subset M \to \mathbb{R}$ be a λ-concave function. The (canonical) Lapliacian of u as a sign-Radon measure is defined by

$$\int_M \phi \, d\Delta u = -\mathcal{E}(u, \phi) = -\int_M \langle \nabla \phi, \nabla u \rangle \, d\,\text{vol}$$

for all Lipschitz function ϕ with compact support in U. In [31], Petrunin proved

$$\Delta u \leqslant n\lambda \cdot \text{vol},$$

in particular, the singular part of Δu is non-positive. If M has curvature $\geqslant K$, then any distance function $\text{dist}_p(x) := d(p, x)$ is $\cot_K \circ \text{dist}_p$-concave on $M \backslash \{p\}$, where the function $\cot_K(s)$ is defined by

$$\cot_K(s) = \begin{cases} \dfrac{\sqrt{K} \cdot \cos(\sqrt{K}s)}{\sin(\sqrt{K}s)} & \text{if} \quad K > 0, \\[2ex] \dfrac{1}{s} & \text{if} \quad K = 0, \\[2ex] \dfrac{\sqrt{-K} \cdot \cosh(\sqrt{-K}s)}{\sinh(\sqrt{-K}s)} & \text{if} \quad K < 0. \end{cases}$$

It is a solution of the ordinary differential equation $\chi'(s) = -K - \chi^2(s)$. Therefore the above inequality $\Delta u \leqslant n\lambda \cdot \text{vol}$ gives a Laplacian comparison theorem for the distance function on Alexandrov spaces.

In [12], by using the DC-structure (see [29]), Kuwae–Shioya defined a distributional Laplacian for a distance function dist_p by

$$\Delta \text{dist}_p = D_i \big(\sqrt{\det(g_{ij})} g^{ij} \partial_j \text{dist}_p \big)$$

on a local chart of $M \backslash \mathcal{S}_\epsilon$ for sufficiently small positive number ϵ, where

$$\mathcal{S}_\epsilon := \{ x \in M : \ \text{vol}(\Sigma_x) \leqslant \text{vol}(\mathbb{S}^{n-1}) - \epsilon \}$$

and D_i is the distributional derivative. Note that the union of all $\mathcal{S}_\epsilon$ has zero measure. One can view the distributional Laplacian $\Delta \mathrm{dist}_p$ as a sign-Radon measure. In [11], Kuwae, Machigashira and Shioya proved that the distributional Laplacian is actually a representation of the previous (canonical) Laplacian on $M \backslash \mathcal{S}_\epsilon$. Moreover in [12], Kuwae and Shioya extended the Laplacian comparison theorem under the weaker condition BG(k).

Both of the above canonical Laplacian and its DC representation (i.e., the distributional Laplacian) make sense up to a set which has zero measure.

In Riemannian geometry, according to Calabi, the Laplacian comparison theorem holds in barrier sense, not just in distribution sense. In this section, we will try to give a new representation of the above canonical Laplacian of a distance function, which makes sense in W_p, the set of points $z \in M$ such that the geodesic pz can be extended beyond z. We will also prove a comparison theorem for the new representation under our Ricci curvature condition.

Let M denote an n-dimensional complete Alexandrov space without boundary. Fix a geodesic $\gamma : [0, \ell] \to M$ with $\gamma(0) = p$ and denote $f = \mathrm{dist}_p$. Let $x \in \gamma \backslash \{p\}$ and L_x, Λ_x be as above in Section 1. Clearly, we may assume that M has curvature $\geqslant k_0$ (for some $k_0 < 0$) in a neighborhood U_γ of γ.

Perelman in [29] defined a Hessian for a semi-concave function u on almost all point $x \in M$, denoted by $\mathrm{Hess}_x u$. It is a bi-linear form on T_x ($= \mathbb{R}^n$). But for the given geodesic γ, we cannot insure that the Hessian is well defined along γ.

We now define a version of Hessian and Laplacian for the distance function f along the geodesic γ as follows. Note that the tangent space at an interior point $x \in \gamma$ can be split to $L_x \times \mathbb{R}$ and $f \circ \gamma$ is linear. So we only need to define the Hessian on the set of orthogonal directions Λ_x.

Throughout this paper, $\mathcal{S}$ will always denote the set of all sequences $\{\theta_j\}_{j=1}^\infty$ with $\theta_j \to 0$ as $j \to \infty$ and $\theta_{j+1} \leqslant \theta_j$.

Definition 3.1. Let $x \in \gamma \backslash \{p\}$. Given a sequence $\theta := \{\theta_j\}_{j=1}^\infty \in \mathcal{S}$, we define a function $H_x^\theta f : \Lambda_x \to \mathbb{R}$ by

$$H_x^\theta f(\xi) \overset{\text{def}}{=} \limsup_{s \to 0, \ s \in \theta} \frac{f \circ \exp_x(s \cdot \xi) - f(x)}{s^2/2};$$

and

$$\Delta^\theta f(x) \overset{\text{def}}{=} (n-1) \cdot \fint_{\Lambda_x} H_x^\theta f(\xi).$$

Since U_γ has curvature $\geqslant k_0$, we know that f is $\cot_{k_0}(|px|)$-concave and $\mathrm{dist}_{\gamma(\ell)}$ is $\cot_{k_0}(|x\gamma(\ell)|)$-concave near x, which imply

$$(3.1) \qquad\qquad H_x^\theta f \leqslant \cot_{k_0}(|px|)$$

for any sequence $\theta \in \mathcal{S}$, and

$$|\gamma(\ell)\ \exp_x(s \cdot \xi)| \leqslant |x\gamma(\ell)| + \cot_{k_0}(|x\gamma(\ell)|) \cdot s^2/2 + o(s^2)$$

for any $\xi \in \Lambda_x$. Then by triangle inequality, we have

$$(3.2) \qquad\qquad H_x^\theta f \geqslant -\cot_{k_0}(|x\gamma(\ell)|).$$

Thus $H_x^\theta f$ is well defined and bounded. It is easy to see that $H_x^\theta f$ is measurable on Λ_x and thus it is integrable.

If there exists Perelman's Hessian of f at a point x (see [29]), then $H_x^\theta f(\xi) = \mathrm{Hess}_x f(\xi, \xi)$ for all $\xi \in \Lambda_x$ and $\theta \in \mathcal{S}$.

Denote by Reg_f the set of points $z \in M$ such that there exists Perelman's Hessian of f at z. If we write the Lebesgue decomposition of the canonical Laplacian $\Delta f = (\Delta f)^{\mathrm{sing}} + (\Delta f)^{\mathrm{ac}} \cdot \mathrm{vol}$, with respect to the n-dimension Hausdorff measure vol, then $(\Delta f)^{\mathrm{ac}}(x) = \mathrm{Tr}\,\mathrm{Hess}_x f = \Delta^\theta f(x)$ for all $x \in W_p \cap \mathrm{Reg}_f$ and $\theta \in \mathcal{S}$. It was shown in [24, 29] that $\mathrm{Reg}_f \cap W_p$ has full measure in M. Thus $\Delta^\theta f(x)$ is actually a representation of the absolutely continuous part of the canonical Laplacian Δf on W_p.

Note from the definition that if $\theta_1 \subset \theta_2$, then

$$H_x^{\theta_1} f \leqslant H_x^{\theta_2} f \quad \text{and} \quad \Delta^{\theta_1} f(x) \leqslant \Delta^{\theta_2} f(x).$$

The following lemma is a discrete version of the propagation equation of the Hessian of f along the geodesic γ.

Lemma 3.1. *Let* $f = \mathrm{dist}_p$. *Given* $\epsilon > 0$, *a continuous functions family* $\{g_{\gamma(t)}\}_{0<t<\ell}$ *and a sequence* $\{\theta_j\}_{j=1}^\infty \in \mathcal{S}$. *Let* $y, z \in \gamma$ *with* $|py| < |pz|$. *We assume that a isometry* $T: \Lambda_z \to \Lambda_y$ *and the subsequence* $\delta := \{\delta_j\} \subset \{\theta_j\}$ *such that (1.1) holds. Then*

$$(3.3) \qquad H_z^\delta f(\xi) \leqslant l^2 \cdot H_y^\delta f(\eta) + \frac{(l-1)^2}{|yz|} - \frac{l^2+l+1}{3} \cdot |yz| \cdot \big(g_z(\xi) - \epsilon\big)$$

for any $l \geqslant 0$ *and any* $\xi \in \Lambda_z$, $\eta = T\xi \in \Lambda_y$.

Proof. For any $\xi \in \Lambda_z$, we can choose a subsequence $\{\delta'_j\} \subset \{\delta_j\}$ such that

$$H^\delta_z f(\xi) = \lim_{j\to\infty} \frac{f(\exp_z(\delta'_j \xi)) - f(z)}{\delta'^2_j / 2}.$$

Then, we have

(3.4) $$f(\exp_z(\delta'_j \xi)) = f(z) + \frac{\delta'^2_j}{2} H^\delta_z f(\xi) + o(\delta'^2_j)$$

for any $l \geqslant 0$. By definition, we have

(3.5) $$f(\exp_y(\delta'_j \cdot l\eta)) \leqslant f(y) + \frac{(l\delta'_j)^2}{2} H^\delta_y f(\eta) + o(\delta'^2_j).$$

Note that

(3.6) $$f(z) - f(y) = |yz|$$

and

(3.7) $$f(\exp_z(\delta'_j \xi)) - f(\exp_y(\delta'_j \cdot l\eta)) \leqslant |\exp_z(\delta'_j \xi), \exp_y(\delta'_j \cdot l\eta)|.$$

By combining (3.4)–(3.7) and using (1.1) with $l_1 = l, l_2 = 1$, we have

$$\frac{\delta'^2_j}{2} \left(H^\delta_z f(\xi) - l^2 \cdot H^\delta_y f(\eta) \right) + o(\delta'^2_j)$$

$$\leqslant \delta'^2_j \cdot \left(\frac{(l-1)^2}{2|yz|} - \frac{g_r(\xi) - \epsilon}{6} \cdot |yz| \cdot (l^2 + l + 1) \right) + o(\delta'^2_j),$$

for any $l \geqslant 0$. Hence

$$H^\delta_z f(\xi) - l^2 \cdot H^\delta_y f(\eta) \leqslant \frac{(l-1)^2}{|yz|} - \frac{l^2 + l + 1}{3} \cdot |yz| \cdot (g_z - \epsilon).$$

This completes the proof of the lemma. $\square$

The following result is the comparison for the above defined representation of Laplacian.

Theorem 3.1. *Let $f = \mathrm{dist}_p$ and $x \in \gamma \backslash \{p\}$. If M has Ricci $\geqslant (n-1)K$ along the geodesic $\gamma(t)$, then, given any sequence $\{\theta_j\}^\infty_{j=1} \in \mathcal{S}$, there exists a*

subsequence $\delta = \{\delta_j\}$ *of* $\{\theta_j\}$ *such that*

$$\Delta^\delta f(x) \leqslant (n-1) \cdot \cot_K(|px|).$$

(If $K > 0$, we add assumption $|px| < \pi/\sqrt{K}$.)

Proof. Arbitrarily fix two constants $\epsilon > 0$ and $K' < K$ with $10\epsilon < K - K'$.

We can choose a point $y \in px$ such that $|py| > \epsilon$ and

(3.8) $$\cot_{k_0}(|py|) \leqslant \cot_{K'}(|py| - \epsilon).$$

By our definition of Ricci curvature $\geqslant (n-1)K$ along γ, there exists a continuous function family $\{g_{\gamma(t)}\}_{0 < t < \ell} \in \mathcal{F}$ such that

$$\oint_{\Lambda_{\gamma(t)}} g_{\gamma(t)} \geqslant K - \epsilon, \quad \forall t \in (0, \ell).$$

We take a sufficiently small number $\omega > 0$.

For any $t_0 \in [|py|, |px|]$, there is a neighborhood I_{t_0} coming from Condition (RC) such that $|I_{t_0}| < \omega$. All of these neighborhoods form an open covering of $[|py|, |px|]$. Let $I_1, I_2, \ldots, I_N$ be a finite sub-covering of $[|py|, |px|]$. We take $x_a \in I_a \cap I_{a+1}$ for all $1 \leqslant a \leqslant N - 1$ and set $y = x_0$, $x = x_N$. We can assume that $|px_a| < |px_{a+1}|$ for all $0 \leqslant a \leqslant N - 1$.

By Condition (RC), we can find a subsequence $\{\delta_{1,j}\} \subset \{\theta_j\}$ and an isometry $T_1 : \Lambda_{x_1} \to \Lambda_{x_0}$ such that (1.1) holds. Next, we can find a further subsequence $\{\delta_{2,j}\} \subset \{\delta_{1,j}\}$ and an isometry $T_2 : \Lambda_{x_2} \to \Lambda_{x_1}$ such that (1.1) holds. After a finite step of these procedures, we get a subsequence $\delta = \{\delta_j\} \subset \{\delta_{N-1,j}\} \subset \cdots \subset \{\theta_j\}$ and a family isometries $\{T_{a+1}\}_{a=0}^{N-1}$, $T_{a+1} : \Lambda_{x_{a+1}} \to \Lambda_{x_a}$ such that, for each $a = 0, 1, \ldots, N - 1$,

$$| \exp_{x_a}(\delta_j l_1 T_{a+1}\xi), \ \exp_{x_{a+1}}(\delta_j l_2 \xi)|$$
$$\leqslant |x_a x_{a+1}| + \frac{(l_1 - l_2)^2}{2 \cdot |x_a x_{a+1}|} \cdot \delta_j^2$$
$$- \frac{(g_t(\xi) - \epsilon) \cdot |x_a x_{a+1}|}{6} \cdot (l_1^2 + l_1 \cdot l_2 + l_2^2) \cdot \delta_j^2 + o(\delta_j^2)$$

for any $l_1, l_2 \geqslant 0$ and any $\xi \in \Lambda_{x_{a+1}}$.

Claim: For all $0 \leqslant a \leqslant N - 1$, we have

$$\oint_{\Lambda_{x_a}} H_{x_a}^\delta f \leqslant \cot_{K'}(|px_a| - \epsilon),$$

as ω is sufficiently small.

We will prove the claim by induction argument with respect to a.

Firstly, we know from (3.8) that the case $a = 0$ is held.

Set $q = x_a$, $r = x_{a+1}$, $\mu = |x_a x_{a+1}|$ and $T = T_{a+1}$. Now we suppose that the claim is held for the case a, i.e.,

$$\oint_{\Lambda_q} H_q^\delta f \leqslant \cot_{K'}(|pq| - \epsilon).$$

We need to show the claim is also held for the case $a + 1$.

Consider the functions on Λ_r

$$(3.9) \qquad F_l(\xi) = l^2 \cdot H_q^\delta f(T(\xi)) + \frac{(l-1)^2}{\mu} - \frac{l^2 + l + 1}{3} \cdot \mu \cdot \big(g_r(\xi) - \epsilon\big).$$

From Lemma 3.1 above, we have

$$(3.10) \qquad\qquad\qquad H_r^\delta f \leqslant F_l$$

for any $l \geqslant 0$.

On the other hand, from (3.9),

$$\oint_{\Lambda_r} F_l = l^2 \cdot \oint_{\Lambda_r} H_q^\delta f \circ T + \frac{(l-1)^2}{\mu} - \frac{l^2 + l + 1}{3} \cdot \mu \cdot \left(\oint_{\Lambda_r} g_r(\xi) - \epsilon\right)$$

$$(3.11) \qquad \leqslant l^2 \cdot \big(\cot_K(|pq| - \epsilon)\big) + \frac{(l-1)^2}{\mu} - \frac{l^2 + l + 1}{3} \cdot \mu \cdot \bar{K}$$

for any $l \geqslant 0$, where $\bar{K} = K - 2\epsilon$.

By setting

$$C_1 = \max_{|py| \leqslant t \leqslant |px|} |\cot_{K'}''(t - \epsilon)|,$$

we have

$$(3.12) \quad \cot_{K'}(|pq| - \epsilon) \leqslant \cot_{K'}(|pr| - \epsilon) + \mu\big(K' + \cot_{K'}^2(|pq| - \epsilon)\big) + C_1 \mu^2.$$

Thus by combining (3.11) and (3.12), we get

$$(3.13) \qquad\qquad \oint_{\Lambda_r} F_l \leqslant \cot_{K'}(|pr| - \epsilon) + A_\mu(l),$$

where

$$A_\mu(l) = \mu\big(K' + \cot^2_{K'}(|pq| - \epsilon)\big) + C_1\mu^2 + (l^2 - 1)\cot_{K'}(|pq| - \epsilon)$$
$$+ \frac{(l-1)^2}{\mu} - \frac{l^2 + l + 1}{3} \cdot \mu \cdot \bar{K}.$$

Denote by $B = 1/\mu - \mu\bar{K}/3$ and $\cot = \cot_{K'}(|pq| - \epsilon)$. Note that

$$\cot_{K'}(|px| - \epsilon) \leqslant \cot \leqslant \cot_{K'}(|py| - \epsilon).$$

Since ω is small and $\mu \leqslant \omega$, we can assume that $\cot + B > 0$. Choose $\widetilde{l} = -(B + \mu\bar{K}/2)/(\cot + B)$. Then we get

$$A_\mu(\widetilde{l}) = \frac{-\big(B + \mu\bar{K}/2\big)^2 + \Big(\mu\big(K' + \cot^2\big) + C_1\mu^2 - \cot + B\Big) \cdot \big(\cot + B\big)}{\cot + B}$$
$$\leqslant \frac{K' - \bar{K} + C_2\mu + C_3\mu^2}{\cot + B},$$

where C_2, C_3 are positive constants independent of μ, ω (may be depending on ϵ, K', x and y). Using $\mu \leqslant \omega$, we get

$$A_\mu(\widetilde{l}) \leqslant \frac{K' - \bar{K} + C_2\omega + C_3\omega^2}{\cot + B} \leqslant 0$$

as ω is sufficiently small. Hence, by combining (3.10), (3.13) and $A_\mu(\widetilde{l}) \leqslant 0$, we get

$$\oint_{\Lambda_r} H^\delta_r f \leqslant \oint_{\Lambda_r} F(\widetilde{l}) \leqslant \cot_{K'}(|pr| - \epsilon).$$

This completes the proof of the claim. In particular, we have

$$\oint_{\Lambda_x} H^\delta_x f \leqslant \cot_{K'}(|px| - \epsilon).$$

Thus by the arbitrariness of ϵ and K' and a standard diagonal argument, we obtain a subsequence of δ, denoted again by δ, such that

$$\Delta^\delta f(x) \leqslant (n - 1) \cdot \cot_K(|px|).$$

Therefore, we have completed the proof of the theorem. $\qquad\square$

4. Rigidity estimates

We continue to consider an n-dimensional complete Alexandrov space M without boundary. Fix a geodesic $\gamma : [0, \ell] \to M$ with $\gamma(0) = p$ and denote $f = \mathrm{dist}_p$.

Let $x \in \gamma \backslash \{p\}$ and L_x, Λ_x be as above. We still assume that a neighborhood U_γ of γ has curvature $\geqslant k_0$ (for some constant $k_0 < 0$).

Lemma 4.1. *Assume M has $\mathrm{Ricci} \geqslant (n-1)K$ along the geodesic $\gamma(t)$. Let $f = \mathrm{dist}_p$ and x be an interior point on the geodesic $\gamma(t)$. Given a sequence $\theta = \{\theta_j\}_{j=1}^\infty \in \mathcal{S}$, if*

$$(4.1) \qquad \Delta^{\theta'} f(x) = (n-1) \cdot \cot_K(|px|)$$

for any subsequence $\theta' = \{\theta'_j\}$ of θ, then there exists a subsequence $\delta = \{\delta_j\}$ of θ such that

$$(4.2) \qquad H_x^\delta f(\xi) = \cot_K(|px|)$$

almost everywhere $\xi \in \Lambda_x$.
   (If $K > 0$, we add assumption $|px| < \pi / \sqrt{K}$.)

Proof. At first, we will prove the following claim:
Claim. For any $\epsilon > 0$, we can find a subsequence $\{\delta_j\}$ of θ and an integrable function h on Λ_x such that

$$H_x^\delta f \leqslant h \quad \text{and} \quad \fint_{\Lambda_x} \left(h - \cot_K(|px|)\right)^2 \leqslant \left(3 + 2|\cot_K(|px|)|\right)\epsilon.$$

By our definition of Ricci curvature $\geqslant (n-1)K$ along γ, there exists a continuous function family $\{g_{\gamma(t)}\}_{0 < t < \ell} \in \mathcal{F}$ such that

$$\fint_{\Lambda_{\gamma(t)}} g_{\gamma(t)} \geqslant K - \epsilon, \quad \forall t \in (0, \ell).$$

We may assume $g_x \geqslant k_0$, otherwise, we replace it by $\max\{g_x, k_0\}$.

By the definition of Condition (RC), we have a neighborhood $I(\subset (0, \ell))$ of $\gamma^{-1}(x)$ such that for arbitrarily taking a point $w \in \gamma(I)$ with $|pw| < |px|$, there exists a subsequence $\widetilde{\delta} = \{\widetilde{\delta}_j\}$ of θ and an isometric $T : \Lambda_x \to \Lambda_w$

such that (1.1) holds. By using Lemma 3.1 and choosing $l = 1$, we have

$$(4.3) \qquad (g_x - \epsilon) \cdot |xw| \leqslant H_w^{\widetilde{\delta}} f - H_x^{\widetilde{\delta}} f.$$

By (3.2) and the fact that f is $\cot_{k_0}(|p \cdot|)$-concave, we have

$$H_x^{\widetilde{\delta}} f \geqslant -\cot_{k_0}(|x\gamma(\ell)|) \quad \text{and} \quad H_w^{\widetilde{\delta}} f \leqslant \cot_{k_0}(|pw|).$$

Thus by combining these with (4.3) and the fact $g_x \geqslant k_0$, we get

$$(4.4) \qquad |g_x| \leqslant C_4$$

for some constant C_4, which may depend on ϵ, x and $|I|$.

Choose a point $z \in \gamma(I)$ with $|px|/2 < |pz| < |px|$ and $|xz| \ll \min\{\epsilon, |I|\}$. Then, by Condition (RC), there exists a subsequence $\{\delta_j'\}$ of θ and an isometry $T : \Lambda_x \to \Lambda_z$ satisfying (1.1). From Theorem 3.1, we can find a subsequence $\{\delta_j\} \subset \{\delta_j'\}$ such that

$$(4.5) \qquad \Delta^{\delta} f(z) \leqslant (n-1) \cdot \cot_K(|pz|).$$

We set, for any $\xi \in \Lambda_x$,

$$\mu = |xz|,$$
$$l = l(\xi) = \left(1/\mu + \frac{\mu}{6}(g_x - \epsilon)\right) \cdot \left(1/\mu - \mu(g_x - \epsilon)/3 + H_z^{\delta} f(T\xi)\right)^{-1}$$

and

$$(4.6) \qquad h_{xz}(\xi) = l^2 \cdot H_z^{\delta} f(T\xi) + \frac{(l-1)^2}{\mu} - \frac{l^2 + l + 1}{3} \mu(g_x - \epsilon).$$

By noting (4.4) and that

$$-\cot_{k_0}(|x\gamma(\ell)|) \leqslant -\cot_{k_0}(|z\gamma(\ell)|) \leqslant H_z^{\delta} f \leqslant \cot_{k_0}(|pz|) \leqslant \cot_{k_0}(|px|/2),$$

we get $l(\xi) > 0$ for μ is sufficiently small. Thus by Lemma 3.1, we have

$$H_x^{\delta} f \leqslant h_{xz}.$$

Consequently,

$$H_x^{\delta} f \leqslant h, \quad \text{on } \Lambda_x,$$

where $h = \min\{h_{xz}, \cot_{k_0}(|px|)\}$. Then, by combining this with (4.1), we get

$$\text{(4.7)} \qquad \oint_{\Lambda_x} h \geqslant \cot_K(|px|).$$

Therefore, by (4.5) and (4.7), there holds

$$\oint_{\Lambda_x} h - \oint_{\Lambda_z} H_z^\delta f \geqslant \cot_K(|px|) - \cot_K(|pz|)$$

$$\text{(4.8)} \qquad\qquad\qquad \geqslant -\mu\big(K + \cot_K^2(|px|)\big) - C_5 \mu^2,$$

where

$$C_5 = \max_{|pz| \leqslant t \leqslant |px|} |\cot_K''(t)| \leqslant \max_{|px|/2 \leqslant t \leqslant |px|} |\cot_K''(t)|.$$

On the other hand, rewriting Equation (4.6), we have

$$\Big(1/\mu - \mu(g_x - \epsilon)/3 + H_z^\delta f \circ T\Big) \cdot h_{xz}$$

$$= -(g_x - \epsilon) + H_z^\delta f \circ T \cdot \Big(1/\mu - \mu(g_x - \epsilon)/3\Big) + \big(\mu(g_x - \epsilon)\big)^2/12.$$

By the facts that $h \leqslant h_{xz}$ and $1/\mu - \mu(g_x - \epsilon)/3 + H_z^\delta f \circ T > 0$, we get

$$\Big(1/\mu - \mu(g_x - \epsilon)/3 + H_z^\delta f \circ T\Big) \cdot h$$

$$\leqslant -(g_x - \epsilon) + H_z^\delta f \circ T \cdot \Big(1/\mu - \mu(g_x - \epsilon)/3\Big) + \big(\mu(g_x - \epsilon)\big)^2/12.$$

That is,

$$\text{(4.9)} \quad \Big(1/\mu - D\Big) \cdot (h - H_z^\delta f \circ T) \leqslant -(g_x - \epsilon) - h^2 + \big(\mu(g_x - \epsilon)\big)^2/12,$$

where $D = \mu(g_x - \epsilon)/3 - h$.

Denote that $C_6 = \max |D| = \max |h + \mu(g_x - \epsilon)/3|$, which is independent of μ. Thus we get

$$\oint_{\Lambda_x} \frac{\epsilon - g_x}{1/u - D} = \oint_{\Lambda_x} \frac{(\epsilon - g_x)^+}{1/u - D} - \oint_{\Lambda_x} \frac{(\epsilon - g_x)^-}{1/u - D}$$

$$\leqslant \frac{\oint_{\Lambda_x} (\epsilon - g_x)^+}{1/u - C_6} - \frac{\oint_{\Lambda_x} (\epsilon - g_x)^-}{1/u + C_6}$$

$$\text{(4.10)} \qquad\qquad = \frac{1/\mu \oint_{\Lambda_x} (\epsilon - g_x) + C_6 \oint_{\Lambda_x} |g_x - \epsilon|}{1/\mu^2 - C_6^2}.$$

By (4.4), (4.10) and the Ricci curvature condition that $\oint_{\Lambda_x} g_x \geqslant K - \epsilon$, we have

$$(4.11) \qquad \oint_{\Lambda_x} \frac{\epsilon - g_x}{1/u - D} \leqslant \mu(2\epsilon - K) + C_7 \mu^2,$$

here and in the following of this proof, all of constants C_7, C_8, C_9 and C_{10} are independent of μ.

From (4.9) and (4.4), we get

$$(4.12)$$
$$\oint_{\Lambda_x} h - \oint_{\Lambda_x} H_z^\delta f \circ T \leqslant \mu(2\epsilon - K) + C_7 \mu^2 - \frac{\oint_{\Lambda_x} h^2}{1/\mu + C_6} + \frac{(C_4 + \epsilon)^2 \mu^2}{1/\mu - C_6}.$$

By combining (4.8), (4.12) and noting that T is an isometry, we have

$$\oint_{\Lambda_x} h^2 \leqslant \cot_K^2(|px|) + 2\epsilon + C_8 \mu,$$

Therefore,

$$(4.13) \qquad \oint_{\Lambda_x} h^2 \leqslant \cot_K^2(|px|) + 3\epsilon$$

as μ suffices small.

Note that (4.12) implies

$$\oint_{\Lambda_x} h \leqslant \oint_{\Lambda_x} H_z^\delta f \circ T + C_9 \mu,$$

Using (4.5) and noting that T is an isometry, we have

$$\oint_{\Lambda_x} h \leqslant \cot_K(|pz|) + C_9 \mu \leqslant \cot_K(|px|) + \mu\big(K + \cot_K^2(|pz|)\big) + C_9 \mu.$$

Since $|px|/2 < |pz| < |px|$, we have

$$\oint_{\Lambda_x} h \leqslant \cot_K(|px|) + C_{10} \mu,$$

Thus, when μ is sufficiently small, we get

$$(4.14) \qquad \oint_{\Lambda_x} h \leqslant \cot_K(|px|) + \epsilon.$$

By combining (4.7) and (4.14), we obtain

$$(4.15) \qquad \cot_K(|px|) \cdot \oint_{\Lambda_x} h \geqslant \cot_K^2(|px|) - \epsilon \cdot |\cot_K(|px|)|.$$

Hence, by (4.13) and (4.15), we have

$$\oint_{\Lambda_x} \big(h - \cot_K(|px|) \big)^2 \leqslant \big(3 + 2|\cot_K(|px|)| \big) \cdot \epsilon.$$

This completes the proof of the claim.

Now let us continue the proof of the lemma.

Given any $\epsilon_1 > 0$, the above claim implies that the measure

$$\nu\big(\{\xi \in \Lambda_x : \ H_x^\delta f \geqslant \cot_K +\epsilon_1\}\big)$$
$$\leqslant \nu\big(\{\xi \in \Lambda_x : \ |h - \cot_K(|px|)| \geqslant \epsilon_1\}\big) \leqslant \big(3 + 2|\cot_K(|px|)|\big)\epsilon/\epsilon_1^2.$$

Letting $\epsilon \to 0^+$, by a standard diagonal argument, we can obtain a subsequence of δ, still denoted by δ, such that

$$\nu\big(\{\xi \in \Lambda_x : \ H_x^\delta f \geqslant \cot_K +\epsilon_1\}\big) = 0.$$

By the arbitrariness of ϵ_1, after a further diagonal argument, we obtain a subsequence of δ, denoted by δ again, such that

$$\nu\big(\{\xi \in \Lambda_x : \ H_x^\delta f > \cot_K\}\big) = 0.$$

Thus we have

$$H_x^\delta f \leqslant \cot_K(|px|)$$

almost everywhere in Λ_x.

Finally, by combining (4.1) and the definition of $\Delta^\delta f$, we conclude that

$$H_x^\delta f = \cot_K(|px|)$$

almost everywhere in Λ_x. Therefore we have completed the proof of the lemma. $\qquad \square$

In order to deal with the zero-measure set in the above lemma, we need the following *segment inequality* of Cheeger and Colding [6]. See also [35] for a statement that is stronger than the following proposition.

Proposition 4.1 (Segment inequality). *Let M be an n-dimensional Alexandrov space with curvature $\geqslant k_0$ (for some constant $k_0 < 0$). Let A_1, $A_2 \subset M$ be two open sets, and let γ_{y_1,y_2} be a geodesic from y_1 to y_2 with arc-parametrization. Assume $W \subset M$ is an open set with*

$$\bigcup_{y_1 \in A_1, \ y_2 \in A_2} \gamma_{y_1,y_2} \subset W.$$

If e be a non-negative integrable function on W, then

(4.16)
$$\int_{A_1 \times A_2} \int_0^{|y_1 y_2|} e(\gamma_{y_1,y_2}(s)) ds \leqslant C(n, k_0, D) \cdot D \cdot \big(vol(A_1) + vol(A_2)\big) \int_W e,$$

where $D = \sup_{y_1 \in A_1, \ y_2 \in A_2} |y_1 y_2|$ and

$$C(n, k_0, D) = \big(\sinh(\sqrt{-k_0}D)/\sinh(\sqrt{-k_0}D/2)\big)^{n-1}.$$

We now define the *upper Hessian* of f, $\overline{\text{Hess}}_x f : T_x \to \mathbb{R} \cup \{-\infty\}$ by

(4.17)
$$\overline{\text{Hess}}_x f(v, v) \overset{\text{def}}{=} \limsup_{s \to 0} \frac{f \circ \exp_x(s \cdot v) - f(x) - d_x f(v) \cdot s}{s^2/2}$$

for any $v \in T_x$.

Clearly, this definition also works for any semi-concave function on M. If u is a λ-concave function, then its upper Hessian $\overline{\text{Hess}}_x u(\xi, \xi) \leqslant \lambda$ for any $\xi \in \Sigma_x$.

For a semi-concave function u, we denote its regular set Reg_u by

$$\text{Reg}_u := \big\{x \in M : \text{there exists Perelman's Hessian of } u \text{ at } x\big\}.$$

It was showed in [29] that Reg_u has full measure for any semi-concave function u. It is clear that $\overline{\text{Hess}}_x u = \text{Hess}_x u$ for any $x \in \text{Reg}_u$.

Definition 4.1. Let $p \in M$. The *cut locus* of p, denoted by Cut_p, is defined to be the set all of points x in M such that geodesic px, from p to x, can not be extended.

It was shown in [24] that Cut_p has zero (Hausdorff) measure (see also [26]).

Set $W_p = M \backslash (\{p\} \cup \text{Cut}_p)$. For any two points $x, y \in M$ with $x \neq y$, a direction from x to y is denoted by $\uparrow_x^y$.

The following two lemmas are concerned with the rigidity part of Theorem 3.1.

Lemma 4.2. *Let M be an n-dimensional Alexandrov space with Ricci curvature $\geqslant (n-1)K$ and let $f = \mathrm{dist}_p$. Suppose that $B_p(R)\backslash\{p\} \subset W_p$ for some $0 < R \leqslant \pi/\sqrt{K}$ (if $K \leqslant 0$, we set $\pi/\sqrt{K}$ to be $+\infty$). Assume that for each $x \in B_p(R)\backslash\{p\}$, there exists a sequence $\theta := \{\theta_j\}_{j=1}^{\infty} \in \mathcal{S}$ such that*

$$\Delta^{\theta'} f(x) = (n-1) \cdot \cot_K(|px|)$$

for any subsequence $\theta' \subset \theta$.

Then the function $\varrho_K \circ f$ is $(1 - K \cdot \varrho_K \circ f)$-concave in $B_p(R)\backslash\{p\}$.

Proof. It suffices to show one variable function $h_p := \varrho_K \circ f \circ \gamma(s)$ satisfies that

$$h_p'' \leqslant 1 - K h_p$$

for any geodesic $\gamma(s) \subset B_p(R)\backslash\{p\}$. Let $\chi(s)$ be an continuous function on an open interval (a,b). Here and in the sequel we write $\chi''(s) \leqslant B$ for $s \in (a,b)$ if $\chi(s+\tau) \leqslant \chi(s) + A \cdot \tau + B \cdot \tau^2/2 + o(\tau^2)$ for some $A \in \mathbb{R}$. $\chi''(s) < +\infty$ means that $\chi''(s) \leqslant B$ for some $B \in \mathbb{R}$. If χ_1 is another continuous function on (a,b), then $\chi'' \leqslant \chi_1$ means $\chi''(s) \leqslant \chi_1(s)$ for all $s \in (a,b)$.

Fix a geodesic $\gamma \subset B_p(R)\backslash\{p\}$. Let $x = \gamma(0)$, $y = \gamma(l)$. Without loss of generality, we can assume that γ is the unique geodesic from x to y and

$$|px| + |py| + |xy| < 2R.$$

We consider the function $u : W_p \to \mathbb{R}^+ \cup \{0\}$,

$$(4.18) \qquad u(z) = \sup_{\xi \in \Sigma_z} \left| \overline{\mathrm{Hess}}_z f(\xi, \xi) - \cot_K(|pz|) \cdot \sin^2(|\xi, \uparrow_z^p|) \right|.$$

For any point $z \in \mathrm{Reg}_f \cap B_p(R)$, $\overline{\mathrm{Hess}}_z f$ is a bilinear form on T_z and $\overline{\mathrm{Hess}}_z f(\uparrow_z^p, \uparrow_z^p) = 0$. Let

$$\Lambda_z = \{\xi \in \Sigma_z : \angle(\xi, \uparrow_z^p) = \pi/2\}.$$

By Lemma 4.1, we have $\overline{\mathrm{Hess}}_z f(\xi, \xi) = H_z^\delta f = \cot_K(|pz|)$ on Λ_z for some subsequence δ of θ, and hence $u(z) = 0$.

Since Reg_f has full measure in $B_p(R)$, we conclude that $u \equiv 0$ almost everywhere in $B_p(R)$.

Given any positive number $\epsilon > 0$ such that

$$\epsilon \ll \min\big\{|px|, |py|, |xy|, 2R - (|px| + |py| + |xy|)\big\}.$$

Let $x_1 \in B_x(\epsilon)$ and $y_1 \in B_y(\epsilon)$, and let $\gamma_{x_1,y_1}(s)$ be a geodesic from x_1 to y_1. By triangle inequality, it is easy to see

$$|px_1| + |x_1 y_1| + |py_1| < 2R$$

as ϵ is sufficiently small. Thus $\gamma_{x_1,y_1} \in B_p(R)$.

Set $u_{x_1,y_1}(s) = u(\gamma_{x_1,y_1}(s))$. By applying Proposition 4.1 to $A_1 = B_x(\epsilon)$, $A_2 = B_y(\epsilon)$, $W = B_p(R)$ and function u, we know that there exist two points $x_1 \in B_x(\epsilon)$ and $y_1 \in B_y(\epsilon)$ such that $u_{x_1,y_1}(s) = 0$ almost everywhere on $(0, |x_1 y_1|)$.

Consider a $s_0 \in (0, |x_1, y_1|)$ such that $u_{x_1,y_1}(s_0) = 0$. Set $z = \gamma_{x_1,y_1}(s_0)$, $\zeta^+ = \gamma^+_{x_1,y_1}(s_0)$ and $\zeta^- = \gamma^-_{x_1,y_1}(s_0)$. Then we have

$$\overline{\mathrm{Hess}}_z f(\zeta^+, \zeta^+) = \overline{\mathrm{Hess}}_z f(\zeta^-, \zeta^-) = \cot_K(|pz|) \sin^2(|\zeta^+, \uparrow^p_z|).$$

Therefore, for function $\widetilde{f}(s) = f \circ \gamma_{x_1,y_1}(s)$, we get

(4.19)
$$\begin{aligned}
\widetilde{f}(h + s_0) &\leqslant \widetilde{f}(s_0) + h\widetilde{f}^+(s_0) + F(s_0) \cdot h^2/2 + o(h^2), \\
\widetilde{f}(-h + s_0) &\leqslant \widetilde{f}(s_0) - h\widetilde{f}^-(s_0) + F(s_0) \cdot h^2/2 + o(h^2),
\end{aligned}$$

for any $h > 0$, where

$$F(s_0) = \cot_K(|pz|) \cdot \sin^2(|\zeta^+, \uparrow^p_z|) = \cot_K(|pz|) \cdot \big(1 - \cos^2(|\zeta^+, \uparrow^p_z|)\big).$$

By the first variation formula of arc-length, we have

$$\widetilde{f}^+(s) = -\cos(|\zeta^+, \uparrow^p_z|) \quad \text{and} \quad \widetilde{f}^-(s) = -\cos(|\zeta^-, \uparrow^p_z|).$$

Note that $\gamma_{x_1,y_1} \in W_p$,

$$|\zeta^+, \uparrow^p_z| + |\zeta^-, \uparrow^p_z| = \pi,$$

which implies that $\widetilde{f}(s)$ is continuously differential. Then by combining this with (4.19), we have

$$\widetilde{f}''(s) \leqslant F(s) = \cot_K \widetilde{f}(s) \cdot \big(1 - \widetilde{f'}^2(s)\big)$$

for almost everywhere $s \in (0, |x_1 y_1|)$. Thus the function $\widetilde{h}(s) = \varrho_K \circ \widetilde{f}(s)$ satisfies

$$\widetilde{h}''(s) \leqslant 1 - K\widetilde{h}(s)$$

for almost everywhere $s \in (0, |x_1 y_1|)$. On the other hand, the fact f is semi-concave implies that $\widetilde{h}''(s) < +\infty$ for all $s \in (0, |x_1 y_1|)$. Thus, from 1.3(3) in [34], we have

$$\widetilde{h}'' \leqslant 1 - K\widetilde{h}.$$

Letting $\epsilon \to 0^+$, we can get point sequences $\{x_i\}$ and $\{y_i\}$ such that $x_i \to x$, $y_i \to y$ and

$$\widetilde{h}_i'' \leqslant 1 - K\widetilde{h}_i,$$

where $\widetilde{h}_i = \varrho_K \circ f \circ \gamma_{x_i, y_i}(s)$. Since the geodesic from x to y is unique, there exists a subsequence of geodesics γ_{x_i, y_i}, which converges to geodesic γ uniformly. Hence $\widetilde{h}_i$ converges to h uniformly, and the desired result follows from 1.3(4) in [34]. Therefore, we have completed the proof. $\square$

Lemma 4.3. *Let $\sigma(t)$ and $\varsigma(t)$ be two geodesics in $B_p(R)$ with $\sigma(0) = \varsigma(0) = p$, and let*

$$\varphi(\tau, \tau') = \widetilde{\angle}_K \sigma(\tau) p \varsigma(\tau')$$

be the comparison angle of $\angle \sigma(\tau) p \varsigma(\tau')$ in the K-plane. Then, under the same assumptions as Lemma 4.2, we have $\varphi(\tau, \tau')$ is non-increasing with respect to τ and τ'.
(If $K > 0$, we add the assumption that $\tau + \tau' + |\sigma(\tau)\varsigma(\tau')| < 2\pi/\sqrt{K}$.)

Proof. Firstly, we claim that for any triangle $\triangle pxy$ (if $K > 0$, we assume that $|px| + |py| + |xy| < 2\pi/\sqrt{K}$), there exists a comparison triangle $\triangle \bar{p}\bar{x}\bar{y}$ in the K-plane M_K^2 such that

$$(4.20) \qquad \angle \bar{p}\bar{x}\bar{y} \leqslant \angle pxy, \quad \angle \bar{p}\bar{y}\bar{x} \leqslant \angle pyx.$$

Indeed, for any triangle $\triangle pxy \in B_p(R)$, there exists a triangle $\triangle \widehat{p}\widehat{x}\widehat{y}$ in M_K^2 such that

$$|\widehat{p}\widehat{x}| = |px|, \quad |\widehat{x}\widehat{y}| = |xy|, \quad \angle \widehat{p}\widehat{x}\widehat{y} = \angle pxy,$$

and by Lemma 4.2, we have

$$|\widehat{\bar{p}\bar{y}}| \geqslant |py|.$$

So by an obvious reason, we get the required triangle $\triangle \bar{p}\bar{x}\bar{y}$.

Fix $\tau' > 0$ and write $\varsigma = \varsigma(\tau')$. We only need to show $\varphi(\tau) := \varphi(\tau, \tau')$ is non-increasing with respect to τ.

Let $\triangle \bar{\sigma}(\tau)\bar{p}\bar{\varsigma}$ be a comparison triangle of $\triangle \sigma(\tau)p\varsigma$ in the K-plane M_K^2 and extend the geodesic $\bar{p}\bar{\sigma}(\tau)$ slightly longer to $\bar{\sigma}(\tau+s)$ for small $s > 0$.

Since the function dist_ς is λ-concave for some number $\lambda \in \mathbb{R}$, we have

$$(4.21) \qquad |\varsigma\sigma(\tau+s)| \leqslant |\varsigma\sigma(\tau)| + s \cdot \left(-\cos\angle\sigma(\tau+s)\sigma(\tau)\varsigma\right) + s^2\lambda/2.$$

On the other hand, we have

$$(4.22) \quad |\bar{\varsigma}\bar{\sigma}(\tau+s)| = |\bar{\varsigma}\bar{\sigma}(\tau)| + s \cdot \left(-\cos\angle\bar{\sigma}(\tau+s)\bar{\sigma}(\tau)\bar{\varsigma}\right) + s^2\bar{\lambda}/2 + o(s^2)$$

for some number $\bar{\lambda} \in \mathbb{R}$. Note from (4.20) that

$$\angle\bar{\sigma}(\tau+s)\bar{\sigma}(\tau)\bar{\varsigma} \geqslant \angle\sigma(\tau+s)\sigma(\tau)\varsigma.$$

By combining this with (4.21), (4.22) and $|\varsigma\sigma(\tau)| = |\bar{\varsigma}\bar{\sigma}(\tau)|$, we have

$$(4.23) \qquad |\varsigma\sigma(\tau+s)| \leqslant |\bar{\varsigma}\bar{\sigma}(\tau+s)| + (-\lambda + \bar{\lambda})s^2 + o(s^2).$$

Now, if $K > 0$, by cosine law in M_K^2, we have

$$\cos\widetilde{\angle}_K\sigma(\tau+s)p\varsigma - \cos\widetilde{\angle}_K\sigma(\tau)p\varsigma$$
$$= \frac{\cos(\sqrt{K}|\varsigma\sigma(\tau+s)|) - \cos(\sqrt{K}|\bar{\varsigma}\bar{\sigma}(\tau+s)|)}{\sin(\sqrt{K}|p\sigma(\tau+s)|) \cdot \sin(\sqrt{K}|p\varsigma|)}$$
$$\geqslant \frac{-(\lambda+\bar{\lambda})}{\sin(\sqrt{K}|p\sigma(\tau+s)|) \cdot \sin(\sqrt{K}|p\varsigma|)} \cdot s^2.$$

Hence, we get

$$\frac{d^+}{d\tau}\cos\widetilde{\angle}_K\sigma(\tau)p\varsigma \geqslant 0.$$

If $K \leqslant 0$, using a similar argument, we can get $\frac{d^+}{d\tau}\widetilde{\angle}_K\sigma(\tau)p\varsigma \leqslant 0$. Therefore we have completed the proof of the lemma. $\qquad\square$

5. Maximal diameter theorem

The main purpose of this section is to prove Theorem 1.4.

Bonnet–Myers' theorem asserts that if an n-dimensional Riemannian manifold has $\mathrm{Ric} \geqslant n - 1$, then its diameter $\leqslant \pi$. Furthermore, its fundamental group is finite.

The first assertion, the diameter estimate, has been extend to metric measure space with $\mathrm{CD}(n, n-1)$ (see [38]) or $\mathrm{MCP}(n, n-1)$ (see [20]). Since our condition $\mathrm{Ric} \geqslant n - 1$ implies the curvature-dimension condition $\mathrm{CD}(n, n-1)$, the first assertion of Bonnet–Myers' theorem also holds on an n-dimensional Alexandrov space M with $\mathrm{Ric}(M) \geqslant n - 1$ and $\partial M = \varnothing$.

Now we consider the second assertion: finiteness of the fundamental group.

Proposition 5.1. *Let M be an n-dimensional Alexandrov space without boundary and $\mathrm{Ric}(M) \geqslant n - 1$. The order of fundamental group of M, $\mathrm{ord}\pi_1(M)$, satisfies*

$$\mathrm{ord}\pi_1(M) \leqslant \frac{\omega_n}{vol(M)}$$

where ω_n is the volume of n-dimensional standard sphere $\mathbb{S}^n$. In particular, if add assumption $\mathrm{vol}(M) > \omega_n/2$, M is simply connected.

Proof. Let $\widetilde{M}$ be the universal covering of M. We have $\mathrm{Ric}(\widetilde{M}) \geqslant n - 1$. Therefore, by Bishop–Gromov volume comparison theorem (see Corollary A.1 in the Appendix), we get

$$\mathrm{ord}\pi_1(M) \cdot \mathrm{vol}(M) = \mathrm{vol}(\widetilde{M}) \leqslant \omega_n.$$

This completes the proof.   □

Now, we are in position to prove Theorem 1.4. We rewrite it as following

Theorem 5.1. *Let M be an n-dimensional Alexandrov space with $\mathrm{Ric}(M) \geqslant n - 1$ and $\partial M = \varnothing$. If $\mathrm{diam}(M) = \pi$, then M is isometric to suspension $[0, \pi] \times_{\sin} N$, where N is an Alexandrov space with curvature $\geqslant 1$.*

Proof. Takes two points $p, q \in M$ such that $|pq| = \pi$.

Exactly as in Riemannian manifold case, by using Bishop–Gromov volume comparison theorem, we have the following assertions:

Fact: (i) For any point $x \in M$, there holds $|px| + |qx| = \pi$. This implies $W_p = W_q = M \backslash \{p, q\}$.

(ii) For any $x \in M$, we can extend the geodesic px to a geodesic from p to q. We will denote it by pxq.

(iii) For any non-degenerate triangle $\triangle pxy$, we have $|px| + |py| + |xy| < 2\pi$.

(iv) For any direction $\xi \in \Sigma_p$, there exists a geodesic γ_ξ such that $\gamma_\xi(0) = p$, $\gamma_\xi^+(0) = \xi$ and its length is equal to π.

Indeed, the first assertion (i) is an immediate consequence of Bishop–Gromov volume comparison theorem (see, for example,[27, p. 271]). Gluing geodesics px and qx, the result curve has Length $= \pi = |pq|$. Thus it is a geodesic. This proves the second assertion (ii). The third assertion (iii) follows directly from triangle inequality

$$|px| + |py| + |xy| < |px| + |py| + |qx| + |qy| \leqslant 2\pi.$$

To show (iv), we consider a sequence of direction $\xi_i \in \Sigma_p$ such that $\xi_i \to \xi$ and there exists geodesics α_i with $\alpha_i(0) = p$ and $\alpha_i^+(0) = \xi_i$. From (ii), we can extend each α_i to a new geodesic with Length $= \pi$, denoted by α_i again. By Arzela–Ascoli Theorem, we can take a limit from some subsequence of α_i. Clearly, the limit is the desired geodesic. This proves the last assertion (iv).

Let $f = \text{dist}_p$ and $\bar{f} = \text{dist}_q$. For any point $x \neq p, q$, we set $\Lambda_x \subset \Sigma_x$ all of directions which are vertical with the geodesic pxq.

Fix a sequence $\theta = \{\theta_j\}_{j=1}^{\infty} \in \mathcal{S}$. By Theorem 3.1, we can find a subsequence $\delta \subset \theta$ such that

(5.1) $\quad \Delta^\delta f(x) \leqslant (n-1) \cdot \cot(|px|)$ and $\Delta^\delta \bar{f}(x) \leqslant (n-1) \cdot \cot(|qx|).$

The above fact (i) implies $f + \bar{f} = \pi$. Thus

(5.2) $$H_x^\delta \bar{f}(\xi) = - \liminf_{\substack{s \to 0 \\ s \in \delta}} \frac{f \circ \exp_x(s \cdot \xi) - f(x)}{s^2/2}.$$

By Definition 3.1, we have $H_x^{\delta'} f \geqslant -H_x^\delta \bar{f}$ for any subsequence $\delta' \subset \delta$. Hence, by combining this with (5.1) and the definition of $\Delta^\delta f$, we get

$$\Delta^{\delta'} f(x) \geqslant -\Delta^\delta \bar{f}(x) \geqslant -(n-1) \cot(|qx|) = (n-1) \cdot \cot(|px|).$$

Note also that

$$\Delta^{\delta'} f(x) \leqslant \Delta^\delta f(x).$$

By combining this with (5.1), this implies that

$$\Delta^{\delta'} f(x) = (n-1)\cot(|px|)$$

for any subsequence $\delta' \subset \delta$.

From Lemma 4.2, $-\cos f$ is $\cos f$-concave in $B_p(\pi)\backslash\{p\} = W_p$. Given any geodesic $\sigma(s) : [0, L] \to W_p$ with $L < \pi$, we have

$$(5.3) \qquad\qquad (-\cos f \circ \sigma)''(s) \leqslant \cos f \circ \sigma(s), \quad \forall s \in (0, L).$$

Similarly, $-\cos \bar{f}$ is $\cos \bar{f}$-concave in $W_q = W_p$ and

$$(5.4) \qquad\qquad (-\cos \bar{f} \circ \sigma)''(s) \leqslant \cos \bar{f} \circ \sigma(s), \quad \forall s \in (0, L).$$

Since $f + \bar{f} = \pi$, $\cos f = -\cos \bar{f}$, by combining this with (5.3) and (5.4), we get

$$(5.5) \qquad\qquad (-\cos f \circ \sigma)''(s) = \cos f \circ \sigma(s), \quad \forall s \in (0, L).$$

Denote by

$$M^+ = \{x \in M : \ f(x) \leqslant \pi/2\}, \qquad M^- = \{x \in M : \ f(x) \geqslant \pi/2\}$$

and $N = M^+ \cap M^- = \{x \in M : \ f(x) = \pi/2\}$. Set

$$v_x = (\text{geodesic } pxq) \cap N,$$

which is consisting of a single point.

We claim that N is totally geodesic in M.

Indeed, take any two points $v_1, v_2 \in N$ with $|v_1 v_2| < \pi$. Let $\sigma(s)$ be a geodesic connected v_1 and v_2. By (5.5) and noting that

$$\cos f(v_1) = \cos f(v_2) = 0,$$

we have $\cos f \circ \sigma(s) \equiv 0$. This tells us $\sigma \subset N$ and N is totally geodesic.

Now we are ready to prove that M is isometric to suspension $[0, \pi] \times_{\sin} N$. Consider any two points $x, y \in M\backslash\{p, q\}$.

If $x, y \in M^+$, we know from Lemma 4.3 that

$$(5.6) \qquad \widetilde{\angle}_1 xpy \geqslant \widetilde{\angle}_1 v_x p v_y \quad \text{and} \quad \widetilde{\angle}_1 xqy \leqslant \widetilde{\angle}_1 v_x q v_y.$$

Note from Fact (i) that

$$\widetilde{\angle}_1 xpy = \widetilde{\angle}_1 xqy.$$

Thus we obtain

$$(5.7) \qquad \widetilde{\angle}_1 xpy = \widetilde{\angle}_1 v_x p v_y.$$

Clearly, if $x, y \in M^-$, the same argument also deduces the equality (5.7). While if $x \in M^+$ and $y \in M^-$, by Lemma 4.3 again, we have

$$\widetilde{\angle}_1 xpy \geqslant \widetilde{\angle}_1 v_x py = \widetilde{\angle}_1 v_x p v_y \quad \text{and} \quad \widetilde{\angle}_1 xpy \leqslant \widetilde{\angle}_1 xp v_y = \widetilde{\angle}_1 v_x p v_y,$$

which implies the equality (5.7).

Then by applying the cosine law to the comparison triangle, we get

$$\cos(|xy|) = \cos(|px|) \cdot \cos(|py|) + \sin(|px|) \cdot \sin(|py|) \cos \widetilde{\angle}_1 v_x p v_y.$$

This proves that M is isometric to suspension $[0, \pi] \times_{\sin} N$.

It remains to show that N has curvature $\geqslant 1$.

We define a map $\Phi : N \to \Sigma_p$ by

$$\Phi(v) = \uparrow_p^v, \quad \forall v \in N.$$

Since $N \subset W_p$ and $|pv| = \pi/2$ for all $v \in N$, Φ is well defined.

Given two points $v_1, v_2 \in N$, for any $x_1 \in M$ lies in geodesic pv_1q and any $x_2 \in M$ lies in geodesic pv_2q, the equality (5.7) implies

$$\widetilde{\angle}_1 x_1 py_1 = \widetilde{\angle}_1 v_1 p v_2 = |v_1 v_2|.$$

Since $\angle v_1 p v_2 = \lim_{x_1 \to p, x_2 \to p} \widetilde{\angle}_1 x_1 py_1$, we have

$$| \uparrow_p^{v_1} \uparrow_p^{v_2} |_{\Sigma_p} = |v_1 v_2|.$$

This shows that Φ is an isometrical embedding. On the other hand, by Fact (iv), Φ is surjective. Therefore, Φ is an isometry. Thus N has curvature $\geqslant 1$. Therefore, we have completed the proof of the theorem. $\square$

Corollary 5.1. *Let M be an n-dimensional Alexandrov space with $\mathrm{Ric}(M)$ $\geqslant n-1$ and $\partial M = \varnothing$. If $\mathrm{rad}(M) = \pi$, then M is isometric to the sphere $\mathbb{S}^n$ with standard metric.*

Proof. For any point $p \in M$, there exists a point q such that $|pq| = \pi$. From the proof of Theorem 5.1, we have that $-\cos \mathrm{dist}_p$ is $\cos \mathrm{dist}_p$-concave in $B_p(\pi) \backslash \{p\}$. Thus M has curvature $\geqslant 1$. It is well known (see, for example, [3, Lemma 10.9.10]) that an n-dimensional Alexandrov space with curvature $\geqslant 1$ and $rad = \pi$ must be isometric to the sphere $\mathbb{S}^n$ with standard metric.
$\square$

Remark 5.1. Colding in [5] had proved the corollary for limit spaces of Riemannian manifolds. That is, if M_i is a sequence of m-dimensional Riemannian manifolds with $\mathrm{Ric}_{M_i} \geq m - 1$ and converging to a metric space X with $\mathrm{rad}_X = \pi$, then X is isometric to the sphere $\mathbb{S}^{m'}$ with standard metric for some integer $m' \leqslant m$.

6. Splitting theorem

In this section, M will always denote an n-dimensional Alexandrov space with curvature bounded below locally, $\mathrm{Ric}(M) \geqslant 0$ and $\partial M = \varnothing$. The main purpose of this section is to prove Theorem 1.3.

A curve $\gamma : [0, +\infty) \to M$ is called a *ray* if $|\gamma(s)\gamma(t)| = s - t$ for any $0 \leqslant t < s < +\infty$. A curve $\gamma : (-\infty, +\infty) \to M$ is called a *line* if $|\gamma(s)\gamma(t)| = s - t$ for any $-\infty < t < s < +\infty$. For a line γ, obviously, $\gamma|_{[0,+\infty)}$ and $\gamma|_{(-\infty,0]}$ form two rays.

Given a ray $\gamma(t)$, we define the *Busemann function* b_γ for γ on M by

$$b_\gamma(x) = \lim_{t \to +\infty} \big(t - |x\gamma(t)|\big).$$

Clearly, it is well defined and is a 1-Lipschitz function.

From now on, in this section, we fix a line $\gamma(t)$ in M and set $\gamma_+ = \gamma|_{[0,+\infty)}$, $\gamma_- = \gamma|_{(-\infty,0]}$. Let b_+ and b_- be the Busemann functions for rays γ_+ and γ_-, respectively.

Let us recall what is the proof of the splitting theorem in the smooth case. When M is a smooth Riemannian manifold, Cheeger–Gromoll in [9] used the standard Laplacian comparison and the maximum principle to conclude that b_+ and b_- are harmonic on M. Then the elliptic regularity theory implies that they are smooth. The important step is to use Bochner formula to show that both ∇b_+ and ∇b_- are parallel. Consequently, the

splitting theorem follows directly from de Rham decomposition theorem. In [10], Eschenburg–Heintze gave a proof avoiding the elliptic regularity, while the Bochner formula is essentially used. But for the general Alexandrov spaces case, the main difficulty is the lack of Bochner formula.

We begin with a lemma which was proved by Kuwae and Shioya for Alexandrov spaces with $\mathrm{MCP}(n,0)$ and hence for Alexandrov spaces with non-negative Ricci curvature. (See Lemma 6.5 and the proof of Theorem 1.3 in [12]).

Lemma 6.1. $b_+(x) + b_-(x) \equiv 0$, on M.

Lemma 6.2. *For any point $x \in M$, there exists a unique line γ_x such that $x = \gamma_x(0)$ and $b_+ \circ \gamma_x$ is a linear function with $(b_+ \circ \gamma_x)' = 1$.*

Proof. Existence. If $x \in \gamma$, then we can write $x = \gamma(t_0)$. Hence we set $\gamma_x(t) = \gamma(t + t_0)$, which is a desired line.

We then consider the case $x \notin \gamma$. Let $\sigma_{t,+}(s)$ be a geodesic from x to $\gamma_+(t)$. By using Arzela–Ascoli Theorem, we can take a sequence $t_j \to +\infty$ such that $\sigma_{t_j,+}$ converges to a limit curve $\sigma_{\infty,+}(s) : [0, +\infty) \to M$. It is easy to check (see, for example,[27, p. 286]) that $\sigma_{\infty,+}$ is 1-Lipschitz and

$$b_+ \circ \sigma_{\infty,+}(s) = s + b_+ \circ \sigma_{\infty,+}(0) = s + b_+(x) \quad \text{for all } s \geqslant 0.$$

By a similar construction, we can obtain a 1-Lipschitz curve $\sigma_{\infty,-}(s') : (-\infty, 0] \to M$ such that $\sigma_{\infty,-}(0) = x$ and

$$b_- \circ \sigma_{\infty,-}(s') = -s' + b_-(x) \quad \text{for all } s' \leqslant 0.$$

Let $\sigma_\infty = \sigma_{\infty,+} \cup \sigma_{\infty,-} : (-\infty, +\infty) \to M$. This is a 1-Lipschitz curve. By Lemma 6.1, we have

(6.1) $\qquad b_+ \circ \sigma_\infty(s) = s + b_+(x) \quad \text{for all } s \in (-\infty, +\infty).$

Then for any $-\infty < t < s < \infty$, by (6.1), we get

$$s - t = b_+ \circ \sigma_\infty(s) - b_+ \circ \sigma_\infty(t) \leqslant |\sigma_\infty(s) \, \sigma_\infty(t)| \leqslant s - t.$$

Thus σ_∞ is a line. Equation (6.1) shows that it is a desired line.

Uniqueness. Argue by contradiction. Suppose that there exist two such lines $\gamma_1, \, \gamma_2$.

The equations $(b_+ \circ \gamma_1)' = (b_+ \circ \gamma_2)' = 1$ implies

$$b_+ \circ \gamma_1(-1) = b_+(x) - 1 \quad \text{and} \quad b_+ \circ \gamma_2(1) = b_+(x) + 1$$

Hence

$$b_+ \circ \gamma_2(1) - b_+ \circ \gamma_1(-1) = 2.$$

Since b_+ is 1-Lipschitz, we get

(6.2) $|\gamma_1(-1) \, \gamma_2(1)| \geqslant b_+ \circ \gamma_2(1) - b_+ \circ \gamma_1(-1) = 2.$

On the other hand,

$$\text{Length}\big(\gamma_1([-1,0]) \cup \gamma_2([0,1])\big) = 2.$$

Thus $\gamma_1([-1,0]) \cup \gamma_2([0,1])$ is a geodesic. This contradicts to that M is non-branching. The proof of the lemma is completed. $\square$

For any point $x \in M$, we take the line γ_x in Lemma 6.2. Let

$$L_x = \{\xi \in T_x \mid \angle(\xi, \gamma_x^+(0)) = \angle(\xi, \gamma_x^-(0)) = \pi/2\},$$
$$\Lambda_x = \{\xi \in \Sigma_x \mid \angle(\xi, \gamma_x^+(0)) = \angle(\xi, \gamma_x^-(0)) = \pi/2\}.$$

Given a sequence $\theta := \{\theta_j\} \in \mathcal{S}$, we define a function $H_x^\theta b_+ : \Lambda_x \to \mathbb{R}$ by

$$H_x^\theta b_+(\xi) \overset{\text{def}}{=} \limsup_{s \to 0, \ s \in \theta} \frac{b_+ \circ \exp_x(s \cdot \xi) - b_+(x)}{s^2/2}$$

and

$$\Delta^\theta b_+(x) \overset{\text{def}}{=} (n-1) \cdot \fint_{\Lambda_x} H_x^\theta b_+(\xi).$$

In the following Lemma 6.3, we will prove that both b_+ and b_- are semi-concave. Thus, by Lemma 6.1, $H_x^\theta b_+$ is well defined and is locally bounded. It is easy to see that $H_x^\theta b_+$ is measurable, so $\Delta^\theta b_+(x)$ is also well defined.

Lemma 6.3. $b_+(x)$ *is a semi-concave function in M. Moreover, for any point $x \in M$ and any sequence $\theta = \{\theta_j\} \in \mathcal{S}$, there exists a subsequence $\delta \subset \theta$ such that $\Delta^\delta b_+(x) \leqslant 0$.*

Proof. Fix a point $x \in M$, we will construct a semi-concave support function for b_+ near x.

We take the line γ_x in Lemma 6.2 and choose a point $p \in \gamma_x$ such that $b_+(p) \ll b_+(x)$.

The equation $(b_+ \circ \gamma_x)' = 1$ implies

$$(6.3) \qquad b_+(x) - b_+(p) = |px|.$$

On the other hand, since b_+ is 1-Lipschitz, we have

$$(6.4) \qquad b_+(y) - b_+(p) \leqslant |py|$$

for any $y \in M$. By combining (6.3) and (6.4), we know that function $\mathrm{dist}_p(\cdot) + b_+(p)$ supports b_+ near x.

This tells us b_+ is a semi-concave function. Furthermore, from Theorem 3.1, we can find a subsequence $\widetilde{\delta} \subset \theta$ such that $\Delta^{\widetilde{\delta}} b_+(x) \leqslant (n-1)/|px|$. By letting $|px| \to \infty$ and a diagonal argument, we can choose a subsequence $\delta \subset \widetilde{\delta}$ such that $\Delta^{\delta} b_+(x) \leqslant 0$. Therefore the proof of the lemma is completed. $\qquad\square$

The following lemma is similar to Lemma 4.2.

Lemma 6.4. *Assume that for each point $x \in M$, there exists a sequence $\theta := \{\theta_j\} \in \mathcal{S}$ such that $\Delta^{\theta'} b_+(x) = 0$ for any subsequence $\theta' \subset \theta$. Then b_+ is a concave function in M.*

Proof. It suffices to show that b_+ is concave on an arbitrarily given bounded open set $\Omega \subset M$. Clearly, we may assume M has curvature $\geqslant k_\Omega$ in Ω for some constant k_Ω.

In the following, we divide the proof into three steps.

Step 1. Let γ_x be the line in Lemma 6.2. Replacing Equations (3.6) and (3.7) by the facts that $|b_+(y) - b_+(z)| = |yz|$ for any $y, z \in \gamma_x$ and b_+ is 1-Lipschitz, the same proof in Lemma 3.1 shows that the lemma also holds when we replace $f = \mathrm{dist}_p$ by b_+.

Step 2. Similar as Lemma 4.1, we want to show $H_x^\delta b_+ = 0$ almost everywhere in Λ_x, for some subsequence $\delta = \{\delta_j\} \subset \theta$.

We now follow the proof of Lemma 4.1. Firstly, from Lemma 6.3, we know that both b_+ and b_- are semi-concave. In turn, Lemma 6.1 gives a bound for $H_x^\theta b_+$. Secondly, we use Lemma 3.1 for b_+ (i.e., the above Step 1) and replace Theorem 3.1 by the above Lemma 6.3 in the proof of Lemma 4.1.

We repeat the same proof of Lemma 4.1 to get $H_x^\delta b_+ = 0$ almost everywhere in Λ_x, for some subsequence $\delta \subset \theta$.

Step 3. Following the proof of Lemma 4.2, we then deduce that $b_+(x)$ is concave in Ω. Therefore $b_+(x)$ is concave in M and the proof of the lemma is completed. $\square$

Now, we are in a position to prove Theorem 1.3.

Proof of Theorem 1.3. Given a sequence $\theta = \{\theta_j\} \in \mathcal{S}$, from Lemma 6.3, we can find a subsequence $\delta \subset \theta$ such that

$$(6.5) \qquad \Delta^\delta b_+(x) \leqslant 0 \quad \text{and} \quad \Delta^\delta b_-(x) \leqslant 0.$$

By the definition of $\Delta^\delta b_+(x)$ and $\Delta^\delta b_-(x)$, we have

$$\Delta^{\delta'} b_+(x) \leqslant \Delta^\delta b_+(x) \quad \text{and} \quad \Delta^{\delta'} b_-(x) \leqslant \Delta^\delta b_-(x)$$

for any subsequence $\delta' \subset \delta$. So (6.5) holds for any subsequence $\delta' \subset \delta$.

On the other hand, by Lemma 6.1 and the definition of $\Delta^\theta b_+(x)$, we have

$$\Delta^\vartheta b_+(x) + \Delta^\vartheta b_-(x) \geqslant 0$$

for any sequence $\vartheta = \{\vartheta_j\} \in \mathcal{S}$. Therefore, by combining with (6.5), we get

$$\Delta^{\delta'} b_+(x) = 0 \quad \text{and} \quad \Delta^{\delta'} b_-(x) = 0$$

for any subsequence $\delta' \subset \delta$.

Then we can apply Lemma 6.4 to conclude that both b_+ and b_- are concave. By using Lemma 6.1 again, we deduce that $b_+ \circ \varsigma(s)$ is a linear function on any geodesic $\varsigma(s)$ in M. In particular, the level surfaces $\mathcal{L}(a) := b_+^{-1}(a)$ are totally geodesic for all $a \in \mathbb{R}$.

Set $N = \mathcal{L}(0) = b_+^{-1}(0)$. It is an Alexandrov space with curvature bounded below locally.

When M is an Alexandrov space with curvature $\geqslant -\kappa^2$ for some $\kappa > 0$. Mashiko, in [18], proved that if there exists a function u such that $u \circ \gamma$ is a linear function for any geodesic $\gamma \subset M$ and $u \in D^{2,2}$ (see [18] for the definition of the class of $D^{2,2}$), then M is isometric to a direct product $\mathbb{R} \times Y$ over an Alexandrov space Y has curvature $\geqslant -\kappa^2$. Later in [2], Alexander and Bishop removed the condition $u \in D^{2,2}$.

Since we do not assume that M has a *uniform* lower curvature bound, we adapt Mashiko's argument as follows.

For any $x \in N$ and any $a \in \mathbb{R}$, let γ_x be the line obtained in Lemma 6.2.

Note that $(b_+ \circ \gamma_x)(s)' = 1$ which implies $\nabla b_+(\gamma_x(s)) = \gamma_x^+(s)$. Thus γ_x is a gradient curve of b_+.

It is easy to check that $\gamma_x \cap \mathcal{L}(a)$ is a set of single point. We define $\Phi_a : N \to \mathcal{L}(a)$ by $\Phi_a(x) = \gamma_x \cap \mathcal{L}(a)$. Φ_a and Φ_a^{-1} are the gradient flows of b_+ and b_-, respectively. Since a gradient flow of a concave function is non-expanding, we have that Φ_a is an isometry.

Now we are ready to show that M is isometric to the direct product $\mathbb{R} \times N$. Consider any two points $x, y \in M$.

Without loss of generality, we may assume that $x \in N$ and $y \in \mathcal{L}(a)$ with $a > 0$. Let $z = \gamma_y \cap N$, where γ_y comes from Lemma 6.2.

We take a C^1 curve $\sigma(s) \subset N$ with $\sigma(0) = x$ and $\sigma(\text{Length}(\sigma)) = z$, $|\sigma'(s)| = 1$. Define a new curve $\bar{\sigma}(s)$ by

$$\bar{\sigma}(s) = \gamma_{\sigma(s)}\Big(\frac{a}{\text{Length}(\sigma)} \cdot s\Big).$$

Clearly, we have $\bar{\sigma}(0) = x$, $\bar{\sigma}(\text{Length}(\sigma)) = \gamma_z(a) = y$ and

$$(6.6) \qquad\qquad b_+(\bar{\sigma}(s)) = \frac{a}{\text{Length}(\sigma)} \cdot s.$$

Fixed any $s \in (0, \text{Length}(\sigma))$, we set $u = \sigma(s)$ and $v = \bar{\sigma}(s)$.

We claim that

$$(6.7) \qquad\qquad \angle(\nabla_u b_+, \sigma^+(s)) = \angle(\uparrow_u^v, \sigma^+(s)) = \pi/2.$$

Indeed,

$$|v\sigma(s')| \geqslant b_+(v) - b_+(\sigma(s')) = b_+(v) = |vu|$$

for any $s' \in (0, \text{Length}(\sigma))$. Then by the first variation formula of arc-length, we have

$$(6.8) \qquad \angle(\uparrow_u^v, \sigma^+(s)) \geqslant \pi/2 \quad \text{and} \quad \angle(\uparrow_u^v, \sigma^-(s)) \geqslant \pi/2.$$

On the other hand,

$$(6.9) \qquad\qquad \angle(\uparrow_u^v, \sigma^+(s)) + \angle(\uparrow_u^v, \sigma^-(s)) = \pi.$$

Thus the desired (6.7) follows from (6.8) and (6.9).

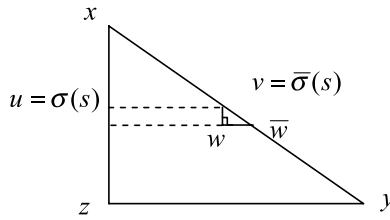Now let us calculate the length of the curve $\bar{\sigma}$.

Figure 1: Isometric splitting.

Clearly, we may assume that a neighborhood of $\bar{\sigma}$ has curvature $\geqslant k$ (for some $k < 0$).

Fixed $s \in (0, \text{Length}(\sigma))$. Let $h > 0$ be a small number. We set $\bar{w} = \bar{\sigma}(s+h)$ and $w = \gamma_{\sigma(s+h)}\left(\frac{a}{\text{Length}(\sigma)} \cdot s\right)$ (see figure 1).

By cosine law in 0−plane $\mathbb{R}^2$, we have

(6.10)
$$|\bar{\sigma}(s+h)\bar{\sigma}(s)|^2 = |v\bar{w}|^2 = |vw|^2 + |w\bar{w}|^2 - 2|w\bar{w}| \cdot |vw| \cdot \cos \widetilde{\angle}_0 vw\bar{w}.$$

Note that

(6.11) $\qquad |v\bar{w}| = |\sigma(s)\sigma(s+h)| = |\sigma^+(s) \cdot h + o(h)| = h + o(h),$

(6.12) $\qquad |w\bar{w}| = (b_+(\bar{w}) - b_+(w)) = \dfrac{a}{\text{Length}(\sigma)} \cdot h.$

By using Lemma 11.2 in [4], we have

(6.13) $\qquad\qquad\qquad \widetilde{\angle}_k vw\bar{w} \to \angle vw\bar{w} = \pi/2$

as $h \to 0$. On the other hand, note that

(6.14) $\qquad\qquad\qquad \widetilde{\angle}_0 vw\bar{w} - \widetilde{\angle}_k vw\bar{w} \to 0$

as $h \to 0$. We have $\cos \widetilde{\angle}_0 vw\bar{w} \to 0$ as $h \to 0$.

Combining this and (6.10)–(6.12), we have

(6.15) $\qquad |\bar{\sigma}(s+h)\bar{\sigma}(s)|^2 = \left(1 + \left(\dfrac{a}{\text{Length}(\sigma)}\right)^2\right) \cdot h^2 + o(h^2).$

Hence,

$$|\bar{\sigma}(s)|^+ = \left(1 + \left(\frac{a}{\text{Length}(\sigma)}\right)^2\right)^{1/2}.$$

Similarly, we can get

$$|\bar{\sigma}(s)|^- = \left(1 + \left(\frac{a}{\text{Length}(\sigma)}\right)^2\right)^{1/2}.$$

So

$$(6.16) \qquad \text{Length}(\bar{\sigma}) = \int_0^{\text{length}(\sigma)} |\bar{\sigma}|' ds = \left(a^2 + \left(\text{Length}(\sigma)\right)^2\right)^{1/2}.$$

If we take σ_1 to be a geodesic xz, we get, from (6.16), that

$$(6.17) \qquad |xy|^2 \leqslant (\text{Length}(\bar{\sigma}_1))^2 = |xz|^2 + a^2 = |xz|^2 + |yz|^2.$$

While if we take σ_2 to be the projection of a geodesic xy to N, we get, from (6.16), that

$$(6.18) \qquad |xy|^2 = (\text{Length}(\sigma_2))^2 + a^2 \geqslant |xz|^2 + |yz|^2.$$

The combination of (6.17) and (6.18) implies that

$$(6.19) \qquad |xy|^2 = |xz|^2 + |yz|^2.$$

This says that M is isometric to the direct product $N \times \mathbb{R}$.

Lastly, we need prove that N has non-negative Ricci curvature.

Let $\gamma(t) : (-\ell, \ell) \to N$ be a geodesic in N. Assume that N has curvature $\geqslant K$ in a neighborhood of γ and for some $K < 0$. Otherwise, there is nothing to prove. Hence M has curvature $\geqslant K$ in a neighborhood of γ in M.

Let p and q be two interior points in γ. We denote the tangent spaces, exponential map in N (or M, resp.) by $T_p N$, $\exp_p^N$ (or $T_p M = T_{(p,0)} M$, $\exp_p^M = \exp_{(p,0)}^M$, resp.) and

$$\Lambda_p^N = \{\xi \in T_p^N : \langle \xi, \gamma' \rangle = 0\}.$$

Let $\Lambda_p^M := \Lambda_{(p,0)}^M = \{\xi \in T_p^M : \langle \xi, \gamma' \rangle = 0\}$. Then $\Lambda_p^M = S(\Lambda_p^N)$ with vertex $\zeta^\pm$, where $\zeta^\pm$ are the directions along factor $\mathbb{R}$ in $M = N \times \mathbb{R}$. For any

$\xi \in T_p N$, we have

$$(6.20) \qquad \exp_p^M(\xi, t) = (\exp_p^N(\xi), t).$$

Suppose that a family of continuous functions $\{g_{(\gamma(t),0)}(\xi, \eta)\}_{-\ell < t < \ell}$ on $\Lambda_p^M = S(\Lambda_p^N)$ satisfies Condition (RC) on geodesic $(\gamma(t), 0)$ in M and

$$\int_{\Lambda_p^M} g_{(\gamma(t),0)}(\xi, \eta) d\,\mathrm{vol}_{\Lambda_p^M} \geqslant -\epsilon$$

for a given small number $\epsilon > 0$.

Given a sequence $\{\widetilde{s}_j\} \in \mathcal{S}$, the isometry $T: L_{(p,0)}^M \to L_{(q,0)}^M$ and subsequence $\{s_j\} \subset \{\widetilde{s}_j\}$ come from the definition of Condition (RC). Recall Petrunin's construction for T, we can assume that $T(\zeta^+) = \zeta^+$, hence $T: L_p^N \subset T_p^N \to L_q^N \subset T_q^N$.

Given a quasi-geodesic $\sigma(s)$ in N, setting $\bar{\sigma}(s) = (\sigma(as), bs)$ for any two number $a, b \in \mathbb{R}$ with $a^2 + b^2 = 1$, we will prove that $\bar{\sigma}(s)$ is a quasi-geodesic in M.

Let $u(z, r)$ be a λ-concave function, defined in a neighborhood of γ in $M = N \times \mathbb{R}$. So function $u(\cdot, r)$ is λ-concave in N and $u(z, \cdot)$ is λ-concave in $\mathbb{R}$ for all $r \in \mathbb{R}$ and $z \in N$. Since σ is quasi-geodesic in N, we have

$$u''\big(\sigma(as), r\big) \leqslant a^2 \cdot \lambda$$

for all $r \in \mathbb{R}$. Now

$$u''\big(\sigma(as), bs\big) \leqslant (a^2 + b^2) \cdot \lambda = \lambda.$$

By definition of quasi-geodesic [32], we get that $\bar{\sigma}(s)$ is a quasi-geodesic in M.

Fix any non-negative number l_1 and l_2. Let $\xi \in \Lambda_p^N$. For any constant $A \in \mathbb{R}$, we have (see figure 2)

$$(6.21)$$

$$
\begin{aligned}
&|\exp_p^N(s_j \cdot l_1 \xi), \exp_q^N(s_j \cdot l_2 T\xi)|^2 \\
&= \big|\big(\exp_p^N(s_j \cdot l_1 \xi), s_j l_1 A \zeta^+\big), \big(\exp_q^N(s_j \cdot l_2 T\xi), s_j l_2 A \zeta^+\big)\big|^2 - A^2(l_1 - l_2)^2 \cdot s_j^2 \\
&= \big|\exp_p^M\big(s_j \cdot l_1(\xi, A\zeta^+)\big), \big(\exp_q^M\big(s_j \cdot l_2(T\xi, A\zeta^+)\big)\big|^2 - A^2(l_1 - l_2)^2 \cdot s_j^2 \\
&\leqslant |pq|^2 + s_j^2 \cdot \Big((l_1 - l_2)^2 - \frac{g_p(\xi, A\zeta^+) \cdot (1 + A^2)}{3}|pq|^2(l_1^2 + l_1 \cdot l_2 + l_2^2)\Big) \\
&\quad + o(s_j^2).
\end{aligned}
$$

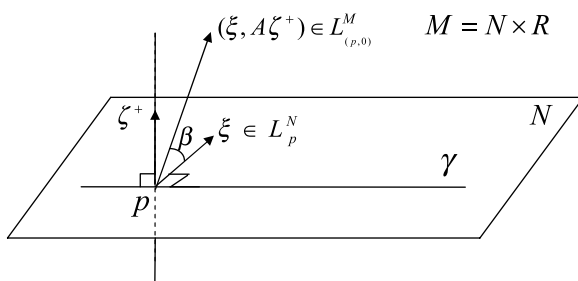Figure 2: Curvature of factor N.

We set $\beta = \angle\big((\xi, 0), (\xi, A\zeta^+)\big)$ and then $A = \tan\beta$, $\beta \in (-\pi/2, \pi/2)$.

For each $t \in (-\ell, \ell)$ and $A \in \mathbb{R}$, we define a function $g_{A,\gamma(t)} : \Lambda^N_{\gamma(t)} \to \mathbb{R}$ by

$$
\begin{aligned}
(6.22) \qquad g_{A,\gamma(t)}(\xi) :&= g_{(\gamma(t),0)}(\xi, A\zeta^+) \cdot (1 + A^2) \\
&= g_{(\gamma(t),0)}(\xi, A\zeta^+)/\cos^2\beta.
\end{aligned}
$$

From (6.21), for any $A \in \mathbb{R}$, the family of continuous functions $\{g_{A,\gamma(t)}(\xi)\}_{-\ell < t < \ell}$ satisfies Condition (RC) on γ.

On the other hand, we have

$$
\begin{aligned}
-\epsilon \leqslant & \int_{\Lambda^M_p} g_{(\gamma(t),0)}(\xi, \eta)\, d\,\mathrm{vol}_{\Lambda^M_p} \\
= & \int_{\Lambda^N_p} \int_{-\pi/2}^{\pi/2} g_{(\gamma(t),0)}(\xi, \eta) \cos^{n-2}\beta\, d\beta\, d\,\mathrm{vol}_{\Lambda^N_p} \\
= & \int_{\Lambda^N_p} \int_{-\pi/2}^{\pi/2} g_{A,\gamma(t)}(\xi) \cos^n\beta\, d\beta\, d\,\mathrm{vol}_{\Lambda^N_p} \\
(6.23) \qquad = & \int_{-\pi/2}^{\pi/2} \int_{\Lambda^N_p} g_{A,\gamma(t)}(\xi) \cos^n\beta\, d\,\mathrm{vol}_{\Lambda^N_p}\, d\beta.
\end{aligned}
$$

Thus, we can choose some $A \in \mathbb{R}$ such that

$$
\int_{\Lambda^N_p} \big(g_{A,\gamma(t)}(\xi)\big) d\,\mathrm{vol}_{\Lambda^N_p} \geqslant -c_n \cdot \epsilon,
$$

for some constant c_n. This completes the proof that N has non-negative Ricci curvature. Therefore the proof of Theorem 1.3 is completed. $\qquad\square$

7. Appendix A

In the Appendix, we will recall the definition of *curvature-dimension condition* $CD(n, k)$ which is given by Sturm [38] and Lott–Villani [16] (see also [41]). After that we will present a proof, due to Petrunin [31], for the statement that an n-dimensional Alexandrov space with Ricci curvature $\geqslant (n-1)K$ and with $\partial M = \varnothing$ must satisfy $CD(n, (n-1)K)$.

Let (X, d, m) be a metric measure space, where (X, d) is a complete separable metric space.

Given two measures μ and ν on X, a measure q on $X \times X$ is called a *coupling* (or *transference plan*) of μ and ν if

$$q(A \times X) = \mu(A) \quad \text{and} \quad q(X \times A) = \nu(A)$$

for all measurable $A \subset X$.

The L^2-*Wasserstein distance* between two measures μ, ν is defined by

$$d_W^2(\mu, \nu) = \inf_q \int_{X \times X} d^2(x, y) dq(x, y),$$

where infimum runs over all coupling q of μ and ν. If $\mu(X) \neq \nu(X)$, we set $d_W(\mu, \nu) = +\infty$.

Let $\mathcal{P}_2(X)$ be the space of all probability measures ν on X with finite second moments:

$$\int_X d^2(o, x) d\nu(x) < \infty$$

for some (hence all) point $o \in X$.

L^2-Wasserstein space is a complete metric space $(\mathcal{P}_2(X), d_W)$. (see [37] for the geometry of L^2-Wasserstein space.) Fix a Borel measure m on X. We denote L^2-Wasserstein space by $\mathcal{P}_2(X, d)$ and its subspace of m-absolutely continuous measures is denoted by $\mathcal{P}_2(X, d, m)$.

Given $k \in \mathbb{R}$, $n \in (1, \infty]$, $t \in [0, 1]$ and two points $x, y \in X$, we define $\beta_t^{(k,n)}$ as follows:

(1)   If $0 < t \leqslant 1$, then

$$\beta_t^{(k,n)}(x,y)$$

$$:= \begin{cases} \exp\left(\dfrac{k}{6}(1-t^2) \cdot d^2(x_0,x_1)\right) & \text{if } n = \infty, \\[2mm] \infty & \text{if } n < \infty,\ k > 0\ \text{ and } \alpha \geqslant \pi, \\[2mm] \left(\dfrac{\sin(t\alpha)}{t\sin\alpha}\right)^{n-1} & \text{if } n < \infty,\ k > 0\ \text{ and } \alpha \in [0,\pi), \\[2mm] 1 & \text{if } n < \infty,\ k = 0, \\[2mm] \left(\dfrac{\sinh(t\alpha)}{t\sinh\alpha}\right)^{n-1} & \text{if } n < \infty,\ k < 0, \end{cases}$$

where $\alpha = d(x,y) \cdot \sqrt{|k|/(n-1)}$.

(2) $\beta_0^{(k,n)}(x,y) = 1$.

The curvature-dimension condition $\mathrm{CD}(n,k)$ is defined as follows (see [41, 29.8, 30.32]):

Definition A.1. Let (X,d,m) be a non-branching locally compact complete separable geodesic space equipped with a locally finite measure m.[3]

Given two real numbers k and n with $n > 1$, the metric measure space (X,d,m) is said to satisfy the *curvature-dimension condition* $\mathrm{CD}(n,k)$ if and only if for each pair compactly supported $\mu_0, \mu_1 \in \mathcal{P}_2(X,d,m)$ there exists an optimal coupling q of $\mu_0 = \varrho_0 m$ and $\mu_1 = \varrho_1 m$, and a geodesic path[4] $\mu_t : [0,1] \to \mathcal{P}_2(X,d)$ connecting μ_0 and μ_1, with

$$H_n(\mu_t|m) \leqslant -(1-t)\int_{X\times X}\left(\frac{\varrho_0(x)}{\beta_{1-t}^{(k,n)}(x,y)}\right)^{-1/n} dq(x,y)$$

(A.1)
$$-t\int_{X\times X}\left(\frac{\varrho_1(y)}{\beta_t^{(k,n)}(x,y)}\right)^{-1/n} dq(x,y)$$

for all $t \in [0,1]$, where $H_n(\cdot|m) : \mathcal{P}_2(X,d) \to \mathbb{R}$ is *Rényi entropy functional* with respect to m,

$$H_n(\mu|m) := -\int_X \varrho^{-1/n} d\mu$$

and ϱ denotes the density of the absolutely continuous part in the Lebesgue decomposition $\mu = \varrho m + \mu^c$ of μ.

---

[3]Lott–Villani and Sturm defined curvature dimension condition on general metric measure spaces.

[4]Constant-speed shortest curve defined on $[0,1]$.

From now on, in the Appendix, M will always denote an n-dimensional Alexandrov space with $\mathrm{Ric}(M) \geqslant (n-1)K$ and $\partial M = \varnothing$.

Our purpose of this Appendix is to prove the following proposition, which is essentially due to Petrunin [31].

Proposition A.1. *Let M be an n-dimensional Alexandrov space without boundary and $\mathrm{Ric}(M) \geqslant (n-1)K$. Let vol denote the n-dimensional Hausdorff measure on M. Then the metric measure space $(M, |\cdot\cdot|, vol)$ satisfies $\mathrm{CD}(n, (n-1)K)$.*

From [38], we know that the curvature-dimension condition $\mathrm{CD}(n, (n-1)K)$ implies Bishop–Gromov volume comparison theorem. Consequently, we get the following

Corollary A.1. *Let M be as in above proposition. Then the function, for any $p \in M$,*

$$\frac{\mathrm{vol}\, B_p(r)}{\mathrm{vol}\, B_K^n(r)}$$

is non-increasing in $r > 0$, where $B_K^n(r)$ is a geodesic ball of radius r in the n-dimensional simply connected Riemannian manifold with constant sectional curvature K.

Before beginning the proof of Proposition A.1, let us review some indispensable materials.

For a continuous function f, we define its *Hamilton–Jacobi shift* $\mathcal{H}_t f$ for time $t > 0$ by

$$\mathcal{H}_t f \overset{\mathrm{def}}{=} \inf_{y \in M} \left\{ f(y) + \frac{1}{2t}|xy|^2 \right\}.$$

Denote by $f_t = \mathcal{H}_t f$. A solution of $\alpha^+(t) = \nabla_{\alpha(t)} f_t$ is called a f_t-*gradient curve*.

Refer to [31] for the existence and uniqueness of f_t-gradient curve and basic propositions of Hamilton–Jacobi shifts. Now we list only facts that is necessary for us to prove Proposition A.1.

Fact A: Let $f : M \to \mathbb{R}$ be bounded and continuous function and $f_t = \mathcal{H}_t f$. Assume $\gamma : (0, 1) \to M$ is a f_t-gradient curve which is also a constant-speed shortest curve. We have

(i) $f_{t_1}(x) \leqslant f_{t_0}(y) + \frac{|xy|^2}{2(t_1 - t_0)}$ for any $t_1 > t_0 > 0$ and $x, y \in M$;

(ii) $f_{t_1}(\gamma(t_1)) = f_{t_0}(\gamma(t_0)) + \frac{|\gamma(t_0)\gamma(t_1)|^2}{2(t_1 - t_0)}$;

(iii) $\nabla f_t = \gamma^+$ and $|\nabla f_t| = \frac{|\gamma(t_0)\gamma(t_1)|}{t_1 - t_0} = |\gamma(0)\gamma(1)|$.

The following result is a modification of [31, Proposition 2.2], where we replace the condition curvature $\geqslant K$ by the condition $\mathrm{Ric}(M) \geqslant (n-1)K$.

Proposition A.2. *Let M be an n-dimensional Alexandrov space with Ricci curvature $\geqslant (n-1)K$. $f : M \to \mathbb{R}$ be bounded and continuous function and $f_t = \mathcal{H}_t f$. Assume $\gamma : (0,1) \to M$ is a f_t-gradient curve which is also a constant-speed shortest curve. Suppose that the bilinear form $\mathrm{Hess}_{\gamma(t)} f_t$ is defined for almost all $t \in (0,1)$.*

Then

$$h'_T \leqslant -h_T^2,$$

$$h'_V \leqslant -(n-1)K|\gamma(0)\gamma(1)|^2 - \frac{h_V^2}{n-1}$$

in the sense of distributions, where

$$h_T(t) \overset{\mathrm{def}}{=} \mathrm{Hess}_{\gamma(t)} f_t \left(\frac{\gamma^+}{|\gamma^+|}, \frac{\gamma^+}{|\gamma^+|} \right)$$

and h_V is the trace of $\mathrm{Hess}_{\gamma(t)} f_t$ in the vertical space $L_{\gamma(t)}$, i.e.,

$$h_V(t) \overset{\mathrm{def}}{=} \mathrm{Trace}_L \mathrm{Hess}_{\gamma(t)} f_t.$$

Proof. Since the bilinear form $\mathrm{Hess}_{\gamma(t)} f_t$ is defined for almost all $t \in (0,1)$, we know from [30] that all $T_{\gamma(t)}$, $t \in (0,1)$, are isometric to n-dimensional Euclidean space. In particular, all $L_{\gamma(t)}$, $t \in (0,1)$, are isometric to $\mathbb{R}^{n-1}$.

Take two points $0 < t_0 < t_1 < 1$, we may assume that $\mathrm{Hess}_{\gamma(t)} f_t$ is defined at t_0 and t_1.

Denote by the direction $\xi_t = \gamma^+(t)/|\gamma^+(t)|$, $t \in (0,1)$. Then we have

$$f_{t_0}\big(\gamma(t_0 + s)\big) = f_{t_0}\big(\gamma(t_0)\big) + s \cdot \langle \nabla f_{t_0}, \gamma^+(t_0) \rangle$$
$$+ \frac{s^2}{2} \cdot \mathrm{Hess}_{\gamma(t_0)} f_{t_0}(\xi_{t_0}, \xi_{t_0}) \cdot |\gamma^+(t_0)|^2 + o(s^2)$$

and

$$f_{t_1}\big(\gamma(t_1 + ls)\big) = f_{t_1}\big(\gamma(t_1)\big) + ls \cdot \langle \nabla f_{t_1}, \gamma^+(t_1) \rangle$$
$$+ \frac{(ls)^2}{2} \cdot \mathrm{Hess}_{\gamma(t_1)} f_{t_1}(\xi_{t_1}, \xi_{t_1}) \cdot |\gamma^+(t_1)|^2 + o(s^2)$$

for any $l \geqslant 0$. Combining these and the Fact A, we get

$$l^2 \cdot h_T(t_1) - h_T(t_0) \leqslant \frac{(l-1)^2}{t_1 - t_0}$$

for any $l \geqslant 0$.

Thus, by choosing $l = \big(1 - (t_1 - t_0)h_T(t_1)\big)^{-1}$ (when $t_1 - t_0$ suffices small, $1 - (t_1 - t_0)h_T(t_1)$ is positive), we get

$$\frac{h_T(t_1) - h_T(t_0)}{t_1 - t_0} \leqslant -h_T(t_1) \cdot h_T(t_0).$$

That is,

$$h'_T \leqslant -h_T^2.$$

Fix arbitrary $\epsilon > 0$. By our definition of Ricci curvature $\geqslant (n-1)K$ along γ, there exists a continuous function family $\{g_{\gamma(t)}\}_{0 < t < 1} \in \mathcal{F}$ such that

$$(A.2) \qquad \fint_{\Lambda_{\gamma(t)}} g_{\gamma(t)} \geqslant K - \epsilon, \quad \forall t \in (0, 1).$$

We may assume $t_1 - t_0$ so small that we can use Equation (1.1) for some isometry $T : \Lambda_{\gamma(t_1)} \to \Lambda_{\gamma(t_0)}$ and some sequence $\{s_j\} \in \mathcal{S}$.

Given any direction $\eta \in \Lambda_{\gamma(t_1)}$, by setting $\sigma_0(s) = \exp_{\gamma(t_0)}(s \cdot T\eta)$ and $\sigma_1(s) = \exp_{\gamma(t_1)}(s\eta)$, we know from (1.1) that

(A.3)
$$|\sigma_0(s_j)\ \sigma_1(ls_j)|^2 \leqslant |\gamma(t_0)\gamma(t_1)|^2$$
$$+ \left((l-1)^2 - \frac{(g_{\gamma(t_1)} - \epsilon) \cdot |\gamma(t_0)\gamma(t_1)|^2}{3}(l^2 + l + 1) \right) \cdot s_j^2 + o(s_j^2).$$

Note that

$$(A.4) \qquad \begin{aligned} f_{t_0}(\sigma_0(s)) &= f_{t_0}(\sigma_0(0)) + \frac{s^2}{2} \cdot \mathrm{Hess}_{\gamma(t_0)} f_{t_0}(T\eta, T\eta) + o(s^2), \\ f_{t_1}(\sigma_1(ls)) &= f_{t_1}(\sigma_1(0)) + \frac{(ls)^2}{2} \cdot \mathrm{Hess}_{\gamma(t_1)} f_{t_1}(\eta, \eta) + o(s^2) \end{aligned}$$

for any $l \geqslant 0$. By combining (A.3), (A.4) and Fact A, we get

(A.5)
$$l^2 \cdot \mathrm{Hess}_{\gamma(t_1)} f_{t_1}(\eta, \eta) - \mathrm{Hess}_{\gamma(t_0)} f_{t_0}(T\eta, T\eta)$$
$$\leqslant \frac{(l-1)^2}{t_1 - t_0} - (t_1 - t_0)|\gamma^+|^2 \cdot \left(g_{\gamma(t_1)}(\eta) - \epsilon\right) \cdot \frac{l^2 + l + 1}{3}$$

for any $l \geqslant 0$. Set $\tau = t_1 - t_0$ and $G = |\gamma^+(t_1)|^2 \cdot (g_{\gamma(t_1)}(\eta) - \epsilon)$. By choosing

$$l = \left(1/\tau + \tau G/6\right) \cdot \left(1/\tau - \tau G/3 - \mathrm{Hess}_{\gamma(t_1)} f_{t_1}(\eta, \eta)\right)^{-1}$$

(when τ suffices small, $1/\tau - \tau G/3 - \mathrm{Hess}_{\gamma(t_1)} f_{t_1}(\eta, \eta)$ and l are positive), we get

(A.6)
$$\left(\frac{1}{\tau} - \tau G/3\right) \cdot \left(\mathrm{Hess}_{\gamma(t_1)} f_{t_1}(\eta, \eta) - \mathrm{Hess}_{\gamma(t_0)} f_{t_0}(T\eta, T\eta)\right)$$
$$\leqslant -\mathrm{Hess}_{\gamma(t_1)} f_{t_1}(\eta, \eta) \cdot \mathrm{Hess}_{\gamma(t_0)} f_{t_0}(T\eta, T\eta) - G + \tau^2 G^2/12.$$

Note the simple fact that for a bilinear form $\beta(a, a)$ on a m-dimensional inner product space V^m,

$$\mathrm{trace}_{V^m} \beta = \frac{m}{vol(S)} \int_S \beta(a, a) da,$$

where S is the unit sphere of V^m with canonical measure. By taking trace for $\mathrm{Hess}_{\gamma(t_0)} f_{t_0}$ (and $\mathrm{Hess}_{\gamma(t_1)} f_{t_1}$) in $L_{\gamma(t_0)}$ (and $L_{\gamma(t_1)}$, respectively), we get, from (A.2) and (A.6), that

(A.7)
$$\frac{h_V(t_1) - h_V(t_0)}{\tau} \leqslant -\frac{1}{2(n-1)}\left(h_V^2(t_0) + h_V^2(t_1)\right)$$
$$- (n-1)(K - 2\epsilon)|\gamma^+(t_1)|^2 + o(1)$$

when we fix t_1 and let $t_0 \to t_1$.

On the other hand, by setting $l = 1$ in (A.5) and taking trace, we have

$$\frac{h_V(t_1) - h_V(t_0)}{\tau} \leqslant -(n-1)(K - 2\epsilon)|\gamma^+(t_1)|^2.$$

This and (A.7) tell us that h_V is locally Lipschitz almost everywhere in $(0, 1)$.

By using (A.7), the arbitrariness of ϵ and Fact A (iii), we get

$$h_V' \leqslant -(n-1)K|\nabla f_t|^2 - \frac{h_V^2}{n-1}.$$

Therefore, we have completed the proof of this proposition. □

Now we can follow Petrunin's argument in [31] to prove the above Proposition A.1.

Proof of Proposition A.1. Let $\mu_0, \mu_1 \in \mathcal{P}(M, d, m)$ with compactly supported sets $\mathrm{spt}(\mu_0), \mathrm{spt}(\mu_1)$ and $\mu_t \in \mathcal{P}(M, d)$ be a geodesic path. We have

$$\mathrm{spt}(\mu_t) \subset \bigcup_{x \in \mathrm{spt}(\mu_0),\ y \in \mathrm{spt}(\mu_1)} \gamma_{x,y} \qquad \forall t \in [0, 1],$$

where $\gamma_{x,y}$ is any one geodesic path between x and y. Thus we can choose a big enough ball B such that $\mathrm{spt}(\mu_t) \subset B$ for all $t \in [0, 1]$. We can find a negative constant k such that M has curvature $\geqslant k$ in B.

As shown in [41, 7.22], there is a probability measure Π on the space of all geodesic paths in M such that if $\Gamma = \mathrm{spt}(\Pi)$ and $e_t : \Gamma \to M$ is *evaluation* map $e_t(\gamma) = \gamma(t)$ then $\mu_t = (e_t)_\# \Pi$. Let Γ be equipped a metric

$$|\gamma\ \gamma'|_\Gamma := \max_{t \in [0,1]} |\gamma(t)\gamma'(t)|.$$

According to [41, 5.10], there are a pair of optimal price functions ϕ and ψ on M such that

$$\phi(y) - \psi(x) \leqslant \frac{1}{2}|xy|^2$$

for any $x, y \in M$ and equality holds for any $(x, y) \in \mathrm{spt}\big((e_0, e_1)_\# \Pi\big)$.

By considering the Hamilton–Jacobi shifts

$$\psi_t = \mathcal{H}_t \psi \quad \text{and} \quad \phi_t = \mathcal{H}_{1-t}(-\psi),$$

Petrunin in [31] proved that, for any $t \in (0, 1)$, μ_t is absolutely continuous and the evaluation map e_t is bi-Lipschitz (where the bi-Lipschitz constant depends on k). Hence for any measure χ on M, there is uniquely determined one-parameter family of pull-back measures χ_t^* on Γ such that $\chi_t^*(E) = \chi(e_t E)$ for any Borel subset $E \subset \Gamma$ (refer to [31] for details).

Fix the measure $\widetilde{\nu} = \mathrm{vol}_{t_0=1/2}^*$ on Γ. We write $\mathrm{vol}_t^* = e^{w_t} \cdot \widetilde{\nu}$ for some Borel function $w_t : \Gamma \to \mathbb{R}$, since e_t is bi-Lipschitz and vol_t^* is absolutely continuous with respect to $\widetilde{\nu}$ for any $t \in (0, 1)$.

In [31], Petrunin proved that, for $\Pi-$a.e. $\gamma \in \Gamma$,

$$(A.8) \qquad w_t = \int_{t_0}^t \frac{\partial w_s}{\partial s} ds \quad \text{a.e. } t \in (0,1)$$

and

$$(A.9) \qquad \frac{\partial w_t}{\partial t} = h_t \quad \text{a.e. } t \in (0,1)$$

where

$$(A.10) \qquad h_t(\gamma) = \text{TraceHess}_{\gamma(t)} \phi_t.$$

Noting that $h_t = h_T(t) + h_V(t)$, we set

$$w_t^{(1)} = \int_{t_0}^t h_T(s) ds, \quad B_1(t) = \exp(w_t^{(1)})$$

and

$$w_t^{(2)} = \int_{t_0}^t h_V(s) ds, \quad B_2(t) = \exp\Big(\frac{w_t^{(2)}}{n-1}\Big).$$

By applying Proposition A.2, we get

$$(A.11) \qquad \begin{aligned} &B_1(t) \geqslant (1-t)B_1(0) + tB_1(1), \\ &B_2(t) \geqslant (1-t)\beta_{1-t}^{1/(n-1)} B_2(0) + t\beta_t^{1/(n-1)} B_2(1), \end{aligned}$$

where

$$\beta_t = \beta_t^{\big((n-1)K,n\big)}(\gamma(0),\gamma(1)).$$

Setting $D(t) = \exp(w_t/n)$ and using Hölder inequality

$$(a+b)^{1/n} \cdot (c+d)^{(n-1)/n} \geqslant a^{1/n} \cdot c^{(n-1)/n} + b^{1/n} \cdot d^{(n-1)/n} \quad \forall a,b,c,d > 0,$$

we have

(A.12)
$$D(t) = B_1^{1/n} \cdot B_2^{(n-1)/n}$$
$$\geqslant \left((1-t)B_1(0) + tB_1(1)\right)^{\frac{1}{n}} \cdot \left((1-t)\beta_{1-t}^{\frac{1}{n-1}}B_2(0) + t\beta_t^{\frac{1}{n-1}}B_2(1)\right)^{\frac{n-1}{n}}$$
$$\geqslant (1-t)\beta_{1-t}^{1/n}B_1(0)B_2(0)^{(n-1)/n} + t\beta_t^{1/n}B_1(1)B_2(1)^{(n-1)/n}$$
$$= (1-t)\beta_{1-t}^{1/n}D(0) + t\beta_t^{1/n}D(1).$$

Note that Petrunin in [31] had represented $H_n(\mu_t|m)$ in terms of $w_t(\gamma)$ as follows:

$$H_n(\mu_t|m) = -\int_\Gamma \exp(w_t(\gamma)/n) \cdot a d\Pi$$

for some non-negative Borel function $a : \Gamma \to \mathbb{R}$. The combination of this with (A.12) implies the desired inequality (A.1) in the definition of $\text{CD}(n, (n-1)K)$. Therefore we have completed the proof of Proposition A.1. $\qquad\square$

Acknowledgments

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