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Global existence for the Seiberg–Witten flow

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We introduce the gradient flow of the Seiberg–Witten functional on a compact, orientable Riemannian 4-manifold and show the global existence of a unique smooth solution to the flow. The flow converges uniquely in C^∞ up to gauge to a critical point of the Seiberg–Witten functional.

1. Introduction

In his ground-breaking work, Donaldson applied Yang–Mills theory to construct a new invariant for 4-manifolds and proved that there exist topological 4-manifolds which do not admit smooth structures, and topological 4-manifolds that admit an infinite number of distinct smooth structures (e.g., [2]). A decade later, Seiberg and Witten, again using considerations from gauge theory, produced some surprisingly simple equations which have been used to produce simpler proofs of many results from Donaldson theory, and also some new results [22]. In particular, the new equations are first order and have gauge group $U(1)$. Because of its ease of computation, Seiberg–Witten theory has effectively succeeded Donaldson theory in many cases.

Computing the Seiberg–Witten invariant for a given manifold involves finding non-trivial solutions to the Seiberg–Witten equations (1.2), called Seiberg–Witten monopoles. Therefore, an important problem in Seiberg–Witten theory is the formulation of necessary and/or sufficient conditions for the existence of monopoles. In [19], for instance, Taubes proved that when a symplectic structure exists on M, there exists a monopole for a particular canonical spin^c structure. For an elementary introduction to spin geometry and the Seiberg–Witten functional, see [8]. For a longer exposition of Seiberg–Witten theory, see [10–12, or 14]

Let M be a compact oriented Riemannian 4-manifold with a spin^c structure $\mathfrak{s}$. Denote by $\mathcal{S} = W \otimes \mathcal{L}$ the corresponding spinor bundle and by $\mathcal{S}^\pm = W^\pm \otimes \mathcal{L}$ the half spinor bundles, and by $\mathcal{L}^2$ the corresponding determinant line bundle. Recall that the bundle $\mathcal{S}^+$ has fibre $\mathbb{C}^2$. Let A be a unitary connection on $\mathcal{L}^2$. Note that we can write $A = A_0 + a$, where A_0 is some fixed connection and $a \in i\Lambda^1 M$ with $i = \sqrt{-1}$. Denote by $F_A = dA \in i\Lambda^2 M$ the curvature of the line bundle connection A. Let $\{e_j\}$ be an orthonormal basis of $\mathbb{R}^4$. A $\mathrm{Spin}(4)^{\mathrm{c}}$-connection on the bundles $\mathcal{S}$ and $\mathcal{S}^{\pm}$ is locally defined by

$$\nabla_A = d + \frac{1}{2}(\omega + A), \tag{1.1}$$

where $\omega = \omega_{jk} e_j e_k$ is induced by the Levi–Civita connection matrix ω_{jk} and $e_j e_k$ acts by Clifford multiplication (see [8]). We denote the curvature of ∇_A by Ω_A. The Dirac operator $D_A : \Gamma(S) \to \Gamma(S)$ is given by

$$D_A \varphi = e_j \nabla_A^j \varphi,$$

where ∇_A^j denotes covariant differentiation along the tangent vector e_j, and e_j acts via Clifford multiplication. We define the configuration space $\Gamma(\mathcal{S}^+) \times \mathscr{A}$, where $\mathscr{A}$ is the space of unitary connections on $\mathcal{L}^2$, and let $(\varphi, A) \in \Gamma(\mathcal{S}^+) \times \mathscr{A}$.

The Seiberg–Witten equations are

$$D_A^+ \varphi = 0, \quad F_A^+ = \frac{1}{4} \langle e_j e_k \varphi, \varphi \rangle \, e^j \wedge e^k. \tag{1.2}$$

Solutions with $\varphi = 0$ are called reducible (or trivial) solutions. Non-trivial solutions are called (Seiberg–Witten) monopoles.

The heat flow for the Yang–Mills equations, suggested by Atiyah and Bott, has played an important role in Yang–Mills theory. The first contribution was made by Donaldson [1] in the case of a holomorphic vector bundle. He used the Yang–Mills heat flow to establish an important relationship between Hermitian Yang–Mills connections and stable holomorphic vector bundles. How to formulate a heat flow for the Seiberg–Witten equations and use it to establish a relationship between Seiberg–Witten monopoles and spinor bundles is a challenging question.

In order to answer this question, we introduce the gradient flow of the Seiberg–Witten functional. The Seiberg–Witten functional $\mathcal{SW} : \Gamma(\mathcal{S}^+) \times \mathscr{A} \to \mathbb{R}$ is given by

$$\mathcal{SW}(\varphi, A) = \int_M |D_A \varphi|^2 + \left| F_A^+ - \frac{1}{4} \langle e_j e_k \varphi, \varphi \rangle \, e^j \wedge e^k \right|^2 dV.$$

Using the Weitzenböck formula (e.g., [8] or [10])

$$D_A^2 \varphi = -\nabla_A^* \nabla_A \varphi + \frac{S}{4} \varphi + \frac{1}{4} F_{A,jk}(e_j e_k \varphi), \tag{1.3}$$

the Seiberg–Witten functional can be written in the following form:

$$\mathcal{SW}(\varphi, A) = \int_M |\nabla_A \varphi|^2 + \left|F_A^+\right|^2 + \frac{S}{4}|\varphi|^2 + \frac{1}{8}|\varphi|^4 \, dV, \tag{1.4}$$

where S is the scalar curvature of M. The Seiberg–Witten functional is invariant under the action of a gauge group. The group of gauge transformations is

$$\mathscr{G} = \{g : M \to U(1)\}.$$

$\mathscr{G}$ acts on elements of the configuration space via

$$g^*(\varphi, A) = (g^{-1}\varphi, A + 2g^{-1}dg).$$

It is easily seen that (1.2) and (1.4) are invariant under the action of the gauge group.

Using the relation

$$\|F_A\|_{L^2} = 2\left\|F_A^+\right\|_{L^2} - 4\pi^2 c_1(\mathcal{L})^2, \tag{1.5}$$

where $c_1(\mathcal{L})$ is the first Chern class of $\mathcal{L}$ (see [14]), one can also write the functional in the form

$$\mathcal{SW}(\varphi, A) = \int_M |\nabla_A \varphi|^2 + \frac{1}{2}|F_A|^2 + \frac{S}{4}|\varphi|^2 + \frac{1}{8}|\varphi|^4 \, dV + \pi^2 c_1(\mathcal{L})^2. \tag{1.6}$$

Note that the term $\pi^2 c_1(\mathcal{L})^2$ is constant along the flow and does not affect the flow equations. Thus in this paper, it can usually be neglected. The Euler–Lagrange equations for the Seiberg–Witten functional are

$$-\nabla_A^* \nabla_A \varphi - \frac{1}{4}\left[S + |\varphi|^2\right]\varphi = 0, \tag{1.7}$$

$$-d^* F_A - i \operatorname{Im}\langle \nabla_A \varphi, \varphi\rangle = 0. \tag{1.8}$$

The Euler–Lagrange equations for the Seiberg–Witten functional were first investigated by Jost *et al.* in [9]. They proved a number of properties including the Palais–Smale condition, compactness and the smoothness of weak solutions to the system (1.7)–(1.8). Note that Equations (1.7)–(1.8) always admit the trivial solutions with $\varphi = 0$, but among the solutions to (1.7)–(1.8) are also any non-trivial solutions, including the Seiberg–Witten monopoles (solutions of (1.2)).

Given the above functional setting, the natural evolution equation to choose for finding critical points is the gradient flow. Therefore, we define the Seiberg–Witten flow by

$$\frac{\partial \varphi}{\partial t} = -\nabla_A^* \nabla_A \varphi - \frac{1}{4}\left[S + |\varphi|^2\right]\varphi, \tag{1.9}$$

$$\frac{\partial A}{\partial t} = -d^* F_A - i \,\mathrm{Im}\,\langle \nabla_A \varphi, \varphi\rangle \tag{1.10}$$

with initial data

$$(\varphi(0), A(0)) = (\varphi_0, A_0).$$

Note that since the connection ∇_A respects the splitting $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$, for initial data $\varphi_0 \in \Gamma(S^+)$, we have $\varphi(t) \in \Gamma(S^+)$ for each t. In this paper we establish that these flow equations admit a smooth solution for all time, which converges to a critical point of the functional (1.4).

Theorem 1.1. *For any given smooth (φ_0, A_0), the system (1.9)–(1.10) admits a unique global smooth solution on $M \times [0,\infty)$ with initial data (φ_0, A_0).*

We show the existence of a local solution to (1.9) and (1.10) following an idea of Donaldson for the Yang–Mills flow (e.g., [2]) which considers a gauge equivalent flow. The critical question for the global existence of the Seiberg–Witten flow turns out to be whether or not the energy concentrates, as in the Yang–Mills and Yang–Mills–Higgs flows (see [17] and [4]). While this question remains unresolved for the Yang–Mills and Yang–Mills–Higgs flows in four dimensions (see, e.g., [6]), we fortunately show that concentration does not occur in general for the Seiberg–Witten flow at any time $T \leq \infty$.

Concerning the limiting behaviour of the flow, we show the following theorem.

Theorem 1.2. *As $t \to \infty$, the solution $(\varphi(t), A(t))$ converges smoothly, up to gauge transformations, to a unique limit $(\varphi_\infty, A_\infty)$, where $(\varphi_\infty, A_\infty)$ is a smooth solution of Equations (1.7)–(1.8). There are constants C_k and $\frac{1}{2} < \gamma < 1$ such that*

$$\|(\varphi(t), A(t)) - (\varphi_\infty, A_\infty)\|_{H^k} \leqslant C_k t^{-(1-\gamma)/(2\gamma-1)}. \tag{1.11}$$

Moreover, for any $\lambda > 0$, $(\varphi_0, A_0) \to (\varphi_\infty, A_\infty)$ defines a continuous map on the space $\{(\varphi_0, A_0) : SW(\varphi(t), A(t)) \to \lambda\}$ as $t \to \infty$.

Analogous results were proven for the Yang–Mills flow in two and three dimensions by Råde [13], and extended to the Yang–Mills–Higgs functional on a Riemann surface by Wilkin [21]. Both of these extend the work of Simon [15].

Let $\mathcal{A} = \Gamma(\mathcal{S}^+) \times \mathscr{A}$ be the configuration space and let $\mathcal{M}$ be the subspace of critical points of the Seiberg–Witten functional. We define $\Lambda := \{\mathrm{SW}(\varphi_\infty, A_\infty) : (\varphi_\infty, A_\infty) \in \mathcal{M}\}$. By the compactness result in [9] and Lemma 5.3, we know that Λ is discrete. For each $\lambda \in \Lambda$, let $\mathcal{M}_\lambda$ be the subset of critical points $(\varphi_\infty, A_\infty)$ with $\mathcal{SW}(\varphi_\infty, A_\infty) = \lambda$, and $\mathcal{A}_\lambda$ the subset of $\mathcal{A}$ such that $\mathrm{SW}(\varphi(t), A(t)) \to \lambda$. Then $\mathcal{A} = \cup_{\lambda\in\Lambda}\mathcal{A}_\lambda$ and $\mathcal{M} = \cup_{\lambda\in\Lambda}\mathcal{M}_\lambda$. As a consequence of Theorem 1.2, the Seiberg–Witten flow defines a continuous $\mathscr{G}$-equivariant flow. Furthermore, the Seiberg–Witten flow defines a deformation retraction $\Phi : [0,\infty] \times \mathcal{A}_\lambda \to \mathcal{A}_\lambda$ of $\mathcal{A}_\lambda$ onto $\mathcal{M}_\lambda$.

It is a very interesting question when the unique limit $(\varphi_\infty, A_\infty)$ of the Seiberg–Witten flow for some initial data is a Seiberg–Witten monopole. By Lemma 5.5, if the initial data (φ_0, A_0) is sufficiently close to a non-trivial Seiberg–Witten monopole, the flow will converge to a non-trivial Seiberg–Witten monopole which is close to the original non-trivial monopole. If the scalar curvature S is everywhere non-negative, the Seiberg–Witten equations (1.2) admit only the trivial solutions $\varphi = 0$ and $F_A^+ = 0$, and equations (1.7)–(1.8) admit only trivial-type solutions with $\varphi = 0$. Thus, the flow can only converge to a trivial critical point.

The paper is organized as follows: In Section 2, we establish some preliminary estimates. In Section 3, we show the local existence of the flow. In Section 4, we show global existence and complete the proof of Theorem 1.1. In Section 5, we consider the limiting behaviour of the flow and prove Theorem 1.2. Finally, in Section 6, we present a brief note about analogous results for the flow of the perturbed Seiberg–Witten functional.

2. Preliminary estimates

The familiar Sobolev spaces of functions on Euclidean spaces can be extended to the geometrical context. Given a connection $\nabla^E : \Omega^0(E) \to \Omega^1(E)$ on a vector bundle E, we can extend it to the well-known exterior covariant derivative $d_A : \Omega^p(E) \to \Omega^{p+1}(E)$. There is another extension of ∇^E, called the iterated covariant derivative

$$\nabla : \otimes^p T^*M \otimes E \to \otimes^{p+1} T^*M \otimes E.$$

We then define

$$\|\varphi\|_{W^{k,p}(M)} = \left(\sum_{n=0}^{k}\int_M \left|\nabla^{(n)}_{ref}\varphi\right|^p \, dV\right)^{\frac{1}{p}},$$

where ∇_{ref} is a given reference connection and $\nabla^{(n)}_{\mathrm{ref}}$ denotes n iterations of ∇_{ref} (we use the exponent n without the brackets to denote the nth component). It is a straightforward calculation to show that different choices of reference connection lead to equivalent norms. We define $\|A\|_{W^{k,p}}$ similarly, where the reference connection is simply the standard connection on forms induced by the Levi–Civita connection. We define, as usual, $H^k = W^{k,2}$. We also have the parabolic spaces $L^p([0,T];W^{k,p}(M))$, which require that the function $t \to \|\varphi(t)\|_{W^{k,p}}$ is in L^p over $[0,T]$. In particular,

$$\|\varphi\|^2_{L^2([0,T];L^2(M))} = \int_{M\times[0,T]} |\varphi|^2 \, dV dt.$$

We make use of another Weitzenböck formula on p-forms (one that is distinct from (1.3)). We have the covariant Laplacian $\nabla_M^*\nabla_M$ and the Hodge Laplacian $\Delta = (dd^* + d^*d)$ (which has opposite sign to the standard Laplace operator on M). They are related by

$$\nabla_M^*\nabla_M\beta - \Delta\beta = R_M \# \beta, \tag{2.1}$$

where β is any p-form, R_M is the curvature of the Levi–Civita connection, and $\#$ represents some multilinear map with smooth coefficients (so that importantly $|R_M\#\beta| \leqslant c\,|R_M|\,|\beta|$). See [8] for details.

We first establish a bound on $|\varphi|$. Let $S_0 = \min\{S(x) : x \in M\}$. Of course, if $S_0 \geq 0$, the Seiberg–Witten equations admit only the trivial (reducible) solutions $\varphi = 0$ and $F_A^+ = 0$.

Lemma 2.1. *Let (φ, A) be a solution of (1.9)–(1.10) on $M \times [0,T)$, and write $m = \sup_{x\in M}|\varphi_0|$. Then for all $t \in [0,T)$, we have*

$$\sup_{x\in M}|\varphi(x,t)| \leqslant \max\{m, \sqrt{|S_0|}\}. \tag{2.2}$$

Proof. We note the following identity:

$$\Delta\,|\varphi|^2 = 2\,\mathrm{Re}\,\langle\nabla_A^*\nabla_A\varphi, \varphi\rangle - 2\,|\nabla_A\varphi|^2\,, \tag{2.3}$$

which holds for any metric connection ∇_A (see [8, 3.2.7]). Using this identity, we have

$$\begin{aligned}\frac{\partial}{\partial t}|\varphi|^2 &= 2\ \mathrm{Re}\left\langle \frac{\partial \varphi}{\partial t}, \varphi \right\rangle \\ &= 2\ \mathrm{Re}\left\langle -\nabla_A^*\nabla_A\varphi - \frac{1}{4}\left[S+|\varphi|^2\right]\varphi, \varphi\right\rangle \\ &= -\Delta|\varphi|^2 - 2|\nabla_A\varphi|^2 - \frac{1}{2}\left[S+|\varphi|^2\right]|\varphi|^2.\end{aligned}$$

Let b be any constant with $0 < b < T$. Suppose $\phi(x,t)$ attains its maximum point at $(x_0, t_0) \in M \times [0,b]$ such that t_0 is the first time the maximum is reached, i.e.,

$$|\varphi(x_0,t_0)| = \max_{x\in M, 0\le t\le b} |\varphi(x,t)|$$

If $|\varphi(x_0,t_0)| \le \max\{m, \sqrt{|S_0|}\}$, the claim is proved. Otherwise,

$$|\varphi(x_0,t_0)| > \max\{m, \sqrt{|S_0|}\}.$$

By the continuity of φ on $M \times [0,b]$, there is a parabolic cylinder $U \times [t_1, t_2]$ inside $M \times [0,b]$ with $t_1 < t_0 \le t_2$ such that

$$|\varphi(x,t)| \ge \max\{m, \sqrt{|S_0|}\}, \quad \forall (x,t) \in U \times [t_1,t_2].$$

Then for all $(x,t) \in U \times [t_1,t_2]$ we have

$$\frac{\partial}{\partial t}|\varphi|^2 + \Delta|\varphi|^2 \le 0.$$

By the strong parabolic maximum principle, $|\phi(x,t)|$ must be a constant. This is impossible. $\square$

We have the following energy inequality.

Lemma 2.2. *Let (φ, A) be a solution of (1.9)–(1.10) on $M \times [0,T)$. Then*

$$\frac{d}{dt}\mathcal{SW}(\varphi(t),A(t)) = -\int_M \left[2\left|\frac{\partial\varphi}{\partial t}\right|^2 + \left|\frac{\partial A}{\partial t}\right|^2\right] \leqslant 0. \tag{2.4}$$

Proof. For any ψ, we compute

$$\begin{aligned}
&\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathcal{SW}(\varphi+\varepsilon\psi, A) \\
&\quad = 2\int_M \left(\operatorname{Re}\langle \nabla_A^*\nabla_A\varphi, \psi\rangle + \frac{1}{4}\left[S+|\varphi|^2\right]\operatorname{Re}\langle\varphi,\psi\rangle\right) \\
&\quad = 2\int_M \operatorname{Re}\left\langle \nabla_A^*\nabla_A\varphi + \frac{1}{4}\left[S+|\varphi|^2\right]\varphi, \psi\right\rangle,
\end{aligned}$$

and for $B \in i\Lambda^1 M$,

$$\begin{aligned}
&\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathcal{SW}(\varphi, A+\varepsilon B) \\
&= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_M \langle \nabla_{A+\varepsilon B}\,\varphi, \nabla_{A+\varepsilon B}\,\varphi\rangle + \left\langle F^+_{A+\varepsilon B}, F^+_{A+\varepsilon B}\right\rangle + \frac{S}{4}|\varphi|^2 + \frac{1}{8}|\varphi|^4 \\
&= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_M \bigg(\left\langle \nabla_A\varphi + \frac{1}{2}\varepsilon B\varphi, \nabla_A\varphi + \frac{1}{2}\varepsilon B\varphi\right\rangle \\
&\qquad + \left\langle F_A^+ + \varepsilon(dB)^+, F_A^+ + \varepsilon(dB)^+\right\rangle\bigg) \\
&= 2\int_M \left(\frac{1}{2}\operatorname{Re}\langle \nabla_A\varphi, B\varphi\rangle + \left\langle F_A^+, (dB)^+\right\rangle\right) \\
&= 2\int_M \left(\left\langle \frac{i}{2}\operatorname{Im}\langle\nabla_A\varphi,\varphi\rangle, B\right\rangle + \left\langle F_A^+, dB\right\rangle\right) \\
&= 2\int_M \left(\left\langle \frac{i}{2}\operatorname{Im}\langle\nabla_A\varphi,\varphi\rangle + \frac{1}{2}d^*F_A, B\right\rangle\right),
\end{aligned}$$

where we have used that $d^*(dA)^+ = \frac{1}{2}d^*dA$. Noting that $\frac{\partial\varphi}{\partial t} = \psi$ and $\frac{\partial A}{\partial t} = B$, the result follows. $\square$

Next, integrating (2.4) in time gives

$$\int_0^T \left[2\left\|\frac{\partial\varphi}{\partial t}\right\|_{L^2}^2 + \left\|\frac{\partial A}{\partial t}\right\|_{L^2}^2\right] = \mathcal{SW}(\varphi_0, A_0) - \mathcal{SW}(\varphi(T), A(T)). \tag{2.5}$$

That is,

$$\frac{\partial\varphi}{\partial t}, \frac{\partial A}{\partial t} \in L^2([0,T]; L^2(M)).$$

From the Seiberg–Witten functional (1.4) we see that

$$\begin{aligned}
&\|\nabla_A\varphi\|^2_{L^2} + \left\|F_A^+\right\|^2_{L^2} + \frac{1}{4}\int_M S\,|\varphi|^2 \leqslant \mathcal{SW}(\varphi, A) \leqslant \mathcal{SW}(\varphi_0, A_0)\\
&\quad\Rightarrow \|\nabla_A\varphi\|^2_{L^2} + \left\|F_A^+\right\|^2_{L^2} \leqslant \mathcal{SW}(\varphi_0, A_0) - \frac{1}{4}\int_M S\,|\varphi|^2\\
&\quad\Rightarrow \|\nabla_A\varphi\|^2_{L^2} + \left\|F_A^+\right\|^2_{L^2} \leqslant c,
\end{aligned}$$

since S and $|\varphi|$ are bounded. This implies that

$$\nabla_A\varphi \in L^\infty([0,T]; L^2(M)).$$

Furthermore, since from (1.5), $\left\|F_A^+\right\|^2_{L^2} = \frac{1}{2}\|F_A\|^2_{L^2} + c$, we also have

$$F_A \in L^\infty([0,T]; L^2(M)).$$

3. Local existence

In this section, we show the existence of a classical (smooth) solution of the system (1.9)–(1.10) on $M \times [0,T)$ for some $T > 0$. What we would like to do is to make the system parabolic by adding the term dd^*A to (1.10), since this would give us the Laplacian ΔA. Note that $\Delta = dd^* + d^*d$ denotes the Hodge Laplacian. Fortunately, this extra term points along the gauge orbit of A since it is the derivative of a function on M.

In local coordinates, we write

$$d_{\tilde{A}} = d + \tilde{A} = d_{A_0} + \tilde{a}, \quad d_A = d_{A_0} + a.$$

Then we consider the following system of equations:

$$\frac{\partial\tilde{\varphi}}{\partial t} = -\nabla^*_{\tilde{A}}\nabla_{\tilde{A}}\tilde{\varphi} - \frac{1}{4}\left[S + |\tilde{\varphi}|^2\right]\tilde{\varphi} + \frac{1}{2}d^*\tilde{a}\tilde{\varphi}, \tag{3.1}$$

$$\frac{\partial\tilde{a}}{\partial t} = -d^*F_{\tilde{A}} - i\,\mathrm{Im}\left\langle\nabla_{\tilde{A}}\tilde{\varphi}, \tilde{\varphi}\right\rangle - dd^*\tilde{a}, \tag{3.2}$$

with initial value $\tilde{a}(0) = 0$ and $\tilde{\varphi}(0) = \varphi_0$.

Since $F_{\tilde{A}} = F_{A_0} + d\tilde{a}$ and

$$-\nabla^*_{\tilde{A}}\nabla_{\tilde{A}}\tilde{\varphi} = -\nabla^*_{A_0}\nabla_{A_0}\tilde{\varphi} + \tilde{a}\#\nabla_{A_0}\tilde{\varphi} + \tilde{a}\#\tilde{a}\tilde{\varphi} + \nabla_{A_0}\tilde{a}\#\tilde{\varphi},$$

the system (3.1) and (3.2) is a quasilinear parabolic system. Thus by standard partial differential equation (PDE) theory there is a unique local smooth solution $(\tilde{\varphi}, \tilde{a})$ on $M \times [0, T)$ for some $T > 0$, given smooth initial data. See for instance [3, §III.4]. However, since $\tilde{a}$ is not bounded, we do not yet have global existence for the system (3.1) and (3.2).

We next claim that the system (3.1)–(3.2) is gauge equivalent to our original system (1.9)–(1.10). Note that locally on the manifold we can write an element $g \in \mathscr{G}$ as $g = e^{if}$ for some real-valued function f on M. By standard ODE theory, there is a local smooth solution f to the equation

$$g^{-1}\frac{dg}{dt} = -d^*\tilde{a},$$
$$g(0) = I,$$

where $(\tilde{\varphi}, \tilde{a})$ is the unique solution to (3.1) and (3.2). Since $d^*\tilde{a}$ is an imaginary-valued function on M, it is easy to check that g^{-1} and $\tilde{g}^t$ satisfy

$$\frac{dG}{dt} = d^*\tilde{a}G, \quad G(0) = I.$$

Therefore $g^{-1} = \tilde{g}^t$. Hence, g is a gauge transformation. One can check that $d(g^{-1}dg) = 0$, and g also satisfies the equation

$$2\frac{\partial}{\partial t}\left(g^{-1}dg\right) = -dd^*\tilde{a}. \tag{3.3}$$

Given our local solution $(\tilde{\varphi}, \tilde{a})$ to (3.1)–(3.2) on $M \times [0, T)$, we solve equation (3.3) to obtain our gauge transformation $g(t)$. Set

$$(\tilde{\varphi}, d_{\tilde{A}}) = (g^*(\varphi), g^*(d_A)) = (g^{-1}\varphi, d_A - 2g^{-1}dg).$$

Applying this gauge transformation, we obtain a local solution

$$(\varphi, d_A) = (g\tilde{\varphi}, d_{\tilde{A}} - 2g^{-1}dg)$$

to our original system (1.9) and (1.10) on $M \times [0, T)$, as shown below.

Note that $F_{\tilde{A}} = F_A$ since $d(g^{-1}dg) = 0$, and that

$$g\nabla^*_{\tilde{A}} \circ g^{-1} = \nabla^*_A, \quad g\nabla_{\tilde{A}} \circ g^{-1} = \nabla_A, \quad |\varphi| = |\tilde{\varphi}|.$$

Then we have

$$\operatorname{Im}\left\langle \nabla_{\tilde{A}}\tilde{\varphi}, \tilde{\varphi}\right\rangle = \operatorname{Im}\left\langle g\nabla_{\tilde{A}}\tilde{\varphi}, g\tilde{\varphi}\right\rangle = \operatorname{Im}\left\langle \nabla_A\varphi, \varphi\right\rangle.$$

We compute

$$\begin{aligned}\frac{\partial A}{\partial t} &= \frac{\partial \tilde{a}}{\partial t} - 2\frac{\partial}{\partial t}(g^{-1}dg)\\ &= -d^*F_{\tilde{A}} - i\,\mathrm{Im}\left\langle \nabla_{\tilde{A}}\tilde{\varphi}, \tilde{\varphi}\right\rangle - dd^*\tilde{a} - 2\frac{\partial}{\partial t}\left(g^{-1}dg\right)\\ &= -d^*F_{\tilde{A}} - i\,\mathrm{Im}\left\langle \nabla_{\tilde{A}}\tilde{\varphi}, \tilde{\varphi}\right\rangle\\ &= -d^*F_A - i\,\mathrm{Im}\left\langle \nabla_A\varphi, \varphi\right\rangle\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \varphi}{\partial t} &= \frac{\partial}{\partial t}(g\tilde{\varphi}) = \frac{\partial g}{\partial t}\tilde{\varphi} + g\frac{\partial \tilde{\varphi}}{\partial t}\\ &= \frac{\partial g}{\partial t}\tilde{\varphi} + g\left(-\nabla^*_{\tilde{A}}\nabla_{\tilde{A}}\tilde{\varphi} - \frac{1}{4}\left[S + |\tilde{\varphi}|^2\right]\tilde{\varphi} + \frac{1}{2}d^*\tilde{a}\tilde{\varphi}\right)\\ &= -g\nabla^*_{\tilde{A}}\circ g^{-1}\circ g\nabla_{\tilde{A}}\circ g^{-1}g\tilde{\varphi} - \frac{1}{4}\left[S + |\tilde{\varphi}|^2\right]g\tilde{\varphi}\\ &= -\nabla^*_A\nabla_A\varphi - \frac{1}{4}\left[S + |\varphi|^2\right]\varphi.\end{aligned}$$

Conversely, let (φ, A) be a solution to (1.9)–(1.10) with $A = A_0 + a$. By the standard theory of PDEs, there is a unique solution g satisfying the following parabolic equations:

$$\begin{aligned}\frac{dg}{dt} &= -gd^*\tilde{a} = -gd^*(a - 2g^{-1}dg),\\ g(0) &= I.\end{aligned}$$

Then, we obtain a solution $(\tilde{\varphi}, \tilde{a})$ of (3.1)–(3.2) by the above gauge transformation. Therefore, we have shown the existence of the local solution of (1.9)–(1.10).

Lemma 3.1. *For any given smooth initial data (φ_0, A_0), equations (1.9) and (1.10) admit a unique local smooth solution on $M \times [0, T)$ for some $T > 0$.*

We suppose that T is maximal, that is, the solution cannot be smoothly extended beyond time T, and contradict this assumption in the next section.

4. Global existence

In this section, we show that our local solution can be extended to a global solution, without restrictions on the manifold, bundles, or initial data. The obstruction to extending the local solution of (1.9)–(1.10) on $M \times [0,T)$ to a global solution on $M \times [0,\infty)$ is that it may cease to be smooth in finite time. Throughout this section, (φ, A) will represent our smooth local solution to the flow on $M \times [0,T)$. For notational simplicity, we adopt the convention that c and its variants denote positive constants, which can change from line to line.

We next compute an estimate for $\frac{\partial}{\partial t}\left(|\nabla_A\varphi|^2 + |F_A|^2\right)$.

Lemma 4.1. *There exist positive constants c, c' such that the following estimate holds:*

$$\frac{\partial}{\partial t}\left(|\nabla_A\varphi|^2 + |F_A|^2\right) + \Delta\left(|\nabla_A\varphi|^2 + |F_A|^2\right) \\ \leqslant -c'\left(\left|\nabla_A^2\varphi\right|^2 + |\nabla F_A|^2\right) + c\left(|F_A|+1\right)\left(|\nabla_A\varphi|^2 + |F_A|^2 + 1\right).$$

Proof. We first consider $|\nabla_A\varphi|^2$.

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla_A\varphi|^2 &= 2\operatorname{Re}\left\langle \nabla_A\frac{\partial\varphi}{\partial t} + \left(\frac{\partial}{\partial t}\nabla_A\right)\varphi, \nabla_A\varphi\right\rangle \\ &= -2\operatorname{Re}\langle\nabla_A\nabla_A^*\nabla_A\varphi, \nabla_A\varphi\rangle - \frac{1}{2}\operatorname{Re}\left\langle\nabla_A\left[S + |\varphi|^2\right]\varphi, \nabla_A\varphi\right\rangle \\ &\quad + \operatorname{Re}\left\langle\frac{\partial A}{\partial t}\varphi, \nabla_A\varphi\right\rangle. \end{aligned} \tag{4.1}$$

Recall that we denote the curvature of the induced connection on $\mathcal{S}^+$ by Ω_A with $A = A_0 + a$, $a \in i\Lambda^1 M$. We have the well-known Ricci formula

$$\nabla_A^{(n)}\nabla_A^*\nabla_A\varphi = \nabla_A^*\nabla_A\nabla_A^{(n)}\varphi + \sum_{j+k=n}\left(\nabla_M^{(j)}R_M + \nabla_M^{(j)}\Omega_A\right)\#\nabla_A^{(k)}\varphi, \tag{4.2}$$

where R_M represents the Riemannian curvature of M (see, e.g., [7, 2.2]). Then

$$\begin{aligned}-2\operatorname{Re}\langle\nabla_A\nabla_A^*\nabla_A\varphi, \nabla_A\varphi\rangle &\leqslant -2\operatorname{Re}\langle\nabla_A^*\nabla_A\nabla_A\varphi, \nabla_A\varphi\rangle + c\,|F_A|\,|\nabla_A\varphi|^2 \\ &\quad + c\,|\nabla_M F_A|\,|\nabla_A\varphi| + c\,|\nabla_A\varphi|^2 + c\,|\nabla_A\varphi|, \end{aligned} \tag{4.3}$$

where we note that the non-constant portion of Ω_A is F_A. We deal with the first term in (4.1) by applying (2.3) to $\nabla_A\varphi$:

$$-2\operatorname{Re}\langle\nabla_A^*\nabla_A\nabla_A\varphi, \nabla_A\varphi\rangle = -\Delta|\nabla_A\varphi|^2 - 2\left|\nabla_A^{(2)}\varphi\right|^2.$$

Considering now the second term in (4.1), we note that by the metric compatibility we have

$$d\,|\varphi|^2 = \langle \nabla_A\varphi, \varphi\rangle + \langle \varphi, \nabla_A\varphi\rangle = 2\,\mathrm{Re}\,\langle \nabla_A\varphi, \varphi\rangle$$

and so

$$\begin{aligned}
&-\frac{1}{2}\,\mathrm{Re}\left\langle \nabla_A\left[S+|\varphi|^2\right]\varphi, \nabla_A\varphi\right\rangle \\
&= -\frac{1}{2}\,\mathrm{Re}\left\langle \left[S+|\varphi|^2\right]\nabla_A\varphi + dS\varphi + d\,|\varphi|^2\,\varphi, \nabla_A\varphi\right\rangle \\
&= -\frac{1}{2}\left[S+|\varphi|^2\right]|\nabla_A\varphi|^2 - \frac{1}{2}\,\mathrm{Re}\,\langle dS\varphi, \nabla_A\varphi\rangle - \mathrm{Re}\,\langle \mathrm{Re}\,\langle \nabla_A\varphi,\varphi\rangle\,\varphi, \nabla_A\varphi\rangle \\
&= -\frac{1}{2}\left[S+|\varphi|^2\right]|\nabla_A\varphi|^2 - \frac{1}{2}\,\mathrm{Re}\,\langle dS\varphi, \nabla_A\varphi\rangle - |\mathrm{Re}\,\langle \nabla_A\varphi,\varphi\rangle|^2 \\
&\leqslant c\,|\nabla_A\varphi|^2 + c\,|\nabla_A\varphi|\,,
\end{aligned}$$

where we have used that

$$\begin{aligned}
-\,\mathrm{Re}\,\langle \mathrm{Re}\,\langle \nabla_A\varphi,\varphi\rangle\,\varphi, \nabla_A\varphi\rangle &= -\,\mathrm{Re}\left\langle \mathrm{Re}\left\langle \nabla_A^j\varphi,\varphi\right\rangle\varphi, \nabla_A^j\varphi\right\rangle \\
= -\,\mathrm{Re}\left\langle \nabla_A^j\varphi,\varphi\right\rangle \mathrm{Re}\left\langle \varphi, \nabla_A^j\varphi\right\rangle &= -\,|\mathrm{Re}\,\langle \varphi, \nabla_A\varphi\rangle|^2\,.
\end{aligned}$$

Finally, for the third term in (4.1),

$$\begin{aligned}
\mathrm{Re}\left\langle \frac{\partial A}{\partial t}\varphi, \nabla_A\varphi\right\rangle &= -\,\mathrm{Re}\,\langle d^*F_A\varphi, \nabla_A\varphi\rangle - \mathrm{Re}\,\langle i\,\mathrm{Im}\,\langle \nabla_A\varphi,\varphi\rangle\,\varphi, \nabla_A\varphi\rangle \\
&\leqslant c\,|\nabla_M F_A|\,|\nabla_A\varphi| + c\,|\nabla_A\varphi|^2\,.
\end{aligned}$$

Combining all of the above we ultimately find

$$\begin{aligned}
\frac{\partial}{\partial t}\,|\nabla_A\varphi|^2 \leqslant{}& -\Delta\,|\nabla_A\varphi|^2 - \left|\nabla_A^2\varphi\right|^2 + c\,|\nabla_M F_A|\,|\nabla_A\varphi| \\
&+ c\,|F_A|\,|\nabla_A\varphi|^2 + c\,|\nabla_A\varphi|^2 + c\,|\nabla_A\varphi|\,.
\end{aligned} \tag{4.4}$$

We next consider $|F_A|^2$.

$$\begin{aligned}
\frac{\partial}{\partial t}\,|F_A|^2 = \frac{\partial}{\partial t}\,|dA|^2 &= 2\left\langle d\frac{\partial A}{\partial t}, dA\right\rangle \\
&= 2\,\langle d\left[-d^*F_A - i\ \mathrm{Im}\,\langle \nabla_A\varphi,\varphi\rangle\right], F_A\rangle \\
&= 2\,\langle -\Delta F_A - id\ \mathrm{Im}\,\langle \nabla_A\varphi,\varphi\rangle\,, F_A\rangle \\
&= -2\,\langle \Delta F_A, F_A\rangle - 2\,\langle id\ \mathrm{Im}\,\langle \nabla_A\varphi,\varphi\rangle\,, F_A\rangle\,,
\end{aligned}$$

where we have utilized the Bianchi identity $dF_A = 0$, giving $dd^* F_A = \Delta F_A$. Applying the Weitzenböck formula (2.1) and recalling (2.3),

$$\begin{aligned} 2\langle \Delta F_A, F_A\rangle &\leqslant 2\langle \nabla_M^* \nabla_M F_A, F_A\rangle + c\,|F_A|^2 \\ &= -\Delta\,|F_A|^2 - 2\,|\nabla_M F_A|^2 + c\,|F_A|^2 . \end{aligned}$$

Then using metric compatibility

$$\begin{aligned} d\ \mathrm{Im}\,\langle \nabla_A\varphi, \varphi\rangle &= d\left(\mathrm{Im}\left\langle \nabla_A^j \varphi, \varphi\right\rangle dx^j\right) \\ &= d_k(\mathrm{Im}\left\langle \nabla_A^j\varphi, \varphi\right\rangle) dx^k \wedge dx^j \\ &= \sum_{k>j}\left(d_k(\mathrm{Im}\left\langle \nabla_A^j\varphi,\varphi\right\rangle) - d_j(\mathrm{Im}\left\langle \nabla_A^k\varphi,\varphi\right\rangle)\right) dx^k\wedge dx^j \\ &= \sum_{k>j}(\mathrm{Im}\left\langle (\nabla_A^k\nabla_A^j - \nabla_A^j\nabla_A^k)\varphi,\varphi\right\rangle + \mathrm{Im}\left\langle \nabla_A^j\varphi, \nabla_A^k\varphi\right\rangle \\ &\qquad - \mathrm{Im}\left\langle \nabla_A^k\varphi,\nabla_A^j\varphi\right\rangle) dx^k\wedge dx^j \\ &= \sum_{k>j}\left(\mathrm{Im}\left\langle \Omega_A^{kj}\varphi,\varphi\right\rangle + 2\,\mathrm{Im}\left\langle \nabla_A^j\varphi,\nabla_A^k\varphi\right\rangle\right) dx^k\wedge dx^j, \end{aligned} \tag{4.5}$$

so that

$$2\langle id\ \mathrm{Im}\,\langle\nabla_A\varphi,\varphi\rangle, F_A\rangle \leqslant c\,|F_A|\,|\nabla_A\varphi|^2 + c\,|F_A|^2 + c\,|F_A| .$$

Finally, we have

$$\frac{\partial}{\partial t}|F_A|^2 \leqslant -\Delta\,|F_A|^2 - 2\,|\nabla F_A|^2 + c\,|F_A|\,|\nabla_A\varphi|^2 + c\,|F_A|^2 + c\,|F_A| . \tag{4.6}$$

We now combine (4.4) and (4.6):

$$\begin{aligned} &\frac{\partial}{\partial t}\left(|\nabla_A\varphi|^2 + |F_A|^2\right) \\ &\quad\leqslant -\Delta\left(|\nabla_A\varphi|^2 + |F_A|^2\right) - 2\left(\left|\nabla_A^{(2)}\varphi\right|^2 + |\nabla_M F_A|^2\right) \\ &\qquad + c\,|\nabla_M F_A|\,|\nabla_A\varphi| + c\,(|F_A|+1)\left(|\nabla_A\varphi|^2 + |F_A|^2\right) + c, \end{aligned} \tag{4.7}$$

where the first powers of $|F_A|$ and $|\nabla_A\varphi|$ can be incorporated into a constant since if they are larger than one, they are bounded by the second powers. We next have to deal with the derivatives of the curvature that appear in (4.7). Fortunately, they can be controlled by the term $-2\,|\nabla_M F_A|^2$ using

Young's inequality

$$|\nabla_M F_A|\,|\nabla_A\varphi| \leqslant \frac{1}{2}\varepsilon\,|\nabla_M F_A|^2 + \frac{1}{2\varepsilon}\,|\nabla_A\varphi|^2.$$

Then if we choose ε sufficiently small we have the desired result. □

Using local coordinates, let

$$P_R(y,s) = \{(x,t) \in M \times (0,T) : |x-y| < R, \quad s - R^2 < t < s\}$$

be a parabolic cylinder of radius R centered at (y,s).

Lemma 4.2. *Suppose $(\varphi, A) \in C^\infty(P_R(y,s))$ satisfies (1.9)–(1.10). Then there exist constants δ and R_0 such that if $R \leqslant R_0$ and*

$$\sup_{0<t<s} \int_{B_R(y)} \left(|\nabla_A\varphi|^2 + |F_A|^2\right) dV < \delta,$$

then

$$\sup_{P_{R/2}(y,s)} \left(|\nabla_A\varphi|^2 + |F_A|^2\right) \leqslant 256R^{-4}.$$

Proof. The proof is similar to one in [5], but there are some differences. For completeness, we give details here. We begin by choosing $r_0 < R$ so that

$$\begin{aligned}(R-r_0)^4 &\sup_{P_{r_0}(y,s)} \left(|\nabla_A\varphi|^2 + |F_A|^2\right)\\ &= \max_{0\leqslant r\leqslant R}\left[(R-r)^4 \sup_{P_r(y,s)} \left(|\nabla_A\varphi|^2 + |F_A|^2\right)\right].\end{aligned} \tag{4.8}$$

Let

$$e_0 = \sup_{P_{r_0}(y,s)} \left(|\nabla_A\varphi|^2 + |F_A|^2\right) = \left(|\nabla_A\varphi|^2 + |F_A|^2\right)(x_0,t_0)$$

for some $(x_0,t_0) \in \bar{P}_{r_0}(y,s)$. We claim that

$$e_0 \leqslant 16(R-r_0)^{-4}. \tag{4.9}$$

Then

$$\begin{aligned}(R-r)^4 \sup_{P_r(y,s)} \left(|\nabla_A\varphi|^2 + |F_A|^2\right) &\leqslant (R-r_0)^4 \sup_{P_{r_0}(y,s)} \left(|\nabla_A\varphi|^2 + |F_A|^2\right)\\ &\leqslant 16(R-r_0)^4(R-r_0)^{-4} = 16\end{aligned}$$

for any $r < R$. Choosing $r = \frac{1}{2}R$ in the above, we have the required result.

We now prove (4.9). Define $\rho_0 = e_0^{-1/4}$ and suppose by contradiction that $\rho_0 \leqslant \frac{1}{2}(R - r_0)$. We rescale variables via $x = x_0 + \rho_0 \tilde{x}$ and $t = t_0 + \rho_0^2 \tilde{t}$ and set

$$\begin{aligned} \psi(\tilde{x}, \tilde{t}) &= \varphi(x_0 + \rho_0 \tilde{x}, t_0 + \rho_0^2 \tilde{t}), \\ B(\tilde{x}, \tilde{t}) &= \rho_0 A(x_0 + \rho_0 \tilde{x}, t_0 + \rho_0^2 \tilde{t}), \end{aligned}$$

giving

$$\begin{aligned} |\nabla_B \psi|^2 &= \rho_0^2 |\nabla_A \varphi|^2, \\ |F_B|^2 &= \rho_0^4 |F_A|^2. \end{aligned}$$

We define

$$e_{\rho_0}(\tilde{x}, \tilde{t}) = |F_B|^2 + \rho_0^2 |\nabla_B \psi|^2 = \rho_0^4 \left(|\nabla_A \varphi|^2 + |F_A|^2 \right)$$

so that

$$e_{\rho_0}(\tilde{x}, \tilde{t}) \leqslant e_{\rho_0}(0, 0) = 1.$$

We compute

$$\begin{aligned} \sup_{\tilde{P}_1(0,0)} e_{\rho_0}(\tilde{x}, \tilde{t}) &= \rho_0^4 \sup_{P_{\rho_0}(x_0, t_0)} \left(|\nabla_A \varphi|^2 + |F_A|^2 \right) \\ &\leqslant \rho_0^4 \sup_{P_{\frac{R+r_0}{2}}(y,s)} \left(|\nabla_A \varphi|^2 + |F_A|^2 \right) \\ &= \rho_0^4 \left(\frac{R - r_0}{2} \right)^{-4} \left(R - \frac{R + r_0}{2} \right)^4 \sup_{P_{\frac{R+r_0}{2}}(y,s)} \left(|\nabla_A \varphi|^2 + |F_A|^2 \right) \\ &\leqslant \rho_0^4 \left(\frac{R - r_0}{2} \right)^{-4} (R - r_0)^4 e_0 = 16, \end{aligned}$$

where we have used that $P_{\rho_0}(x_0, t_0) \subset P_{\frac{R+r_0}{2}}(y, s)$, and to get to the last line we have used (4.8). This implies that

$$e_{\rho_0} = \rho_0^4 \left(|\nabla_A \varphi|^2 + |F_A|^2 \right) \leqslant 16$$

on $\bar{P}_1(0, 0)$. By Lemma 4.1,

$$\left(\frac{\partial}{\partial t} + \Delta \right) \left(|\nabla_A \varphi|^2 + |F_A|^2 + 1 \right) \leqslant c \left(|F_A| + 1 \right) \left(|\nabla_A \varphi|^2 + |F_A|^2 + 1 \right).$$

Then

$$\left(\frac{\partial}{\partial \tilde{t}} + \tilde{\Delta}\right)(e_{\rho_0} + \rho_0^4) = \rho_0^6 \left(\frac{\partial}{\partial t} + \Delta\right)\left(|\nabla_A \varphi|^2 + |F_A|^2\right)$$
$$\leqslant c\rho_0^6 \left(|F_A| + 1\right)\left(|\nabla_A \varphi|^2 + |F_A|^2 + 1\right)$$

on $\bar{P}_1(0,0)$. Note that by assumption $\rho_0 < R$, and thus $\rho_0^2 |F_A|$ is bounded by a constant. Then

$$\left(\frac{\partial}{\partial \tilde{t}} + \tilde{\Delta}\right)(e_{\rho_0} + \rho_0^4) \leqslant c\left(e_{\rho_0} + \rho_0^4\right)$$

for a constant $c > 0$. We apply Moser's Harnack inequality to give

$$\begin{aligned}
1 + \rho_0^4 = e_{\rho_0}(0,0) + \rho_0^4 &\leqslant c \int_{\tilde{P}_1(0,0)} e_{\rho_0} d\tilde{x} d\tilde{t} + c\rho_0^4 \\
&= c\rho_0^{-2} \int_{P_{\rho_0}(x_0,t_0)} \left(|\nabla_A \varphi|^2 + |F_A|^2\right) dx dt + c\rho_0^4 \\
&\leqslant c \sup_{0 \leqslant t \leqslant s} \int_{B_R(y)} \left(|\nabla_A \varphi|^2 + |F_A|^2\right) + cR^4 \\
&< c\delta + cR^4,
\end{aligned}$$

where we have used that $\rho_0 < R$. Now if we choose R_0 and δ sufficiently small, we have the desired contradiction. □

Lemma 4.3. *Let (φ, A) be a solution to (1.9)–(1.10). Writing*

$$\mathcal{SW}_{B_R(x_0)}(\varphi, A) = \int_{B_R(x_0)} |\nabla_A \varphi|^2 + \frac{1}{2}|F_A|^2 + \frac{S}{4}|\varphi|^2 + \frac{1}{8}|\varphi|^4,$$

we have for any $x_0 \in M$ and ball of radius R,

$$\sup_{t_1 \leqslant t \leqslant t_2} \mathcal{SW}_{B_R(x_0)}(\varphi, A) \leqslant \mathcal{SW}_{B_{2R}(x_0)}(\varphi(t_1), A(t_1)) + C_1(t_2 - t_1)R^{-2},$$

where C_1 is a constant.

Proof. Let ϕ be a smooth test function with $\phi \equiv 1$ on $B_R(x_0)$ and zero outside of $B_{2R}(x_0)$. We can choose ϕ so that $0 \leqslant \phi \leqslant 1$ and $|d\phi| \leqslant cR^{-1}$.

We compute

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_M \phi^2 |F_A|^2 &= \int_M \left\langle \phi^2 d\frac{\partial A}{\partial t}, F_A \right\rangle \\ &= \int_M \left\langle \phi^2 \frac{\partial A}{\partial t}, d^* F_A \right\rangle - \int_M \left\langle d\phi^2 \wedge \frac{\partial A}{\partial t}, F_A \right\rangle \\ &\leqslant \int_M \left\langle \phi^2 \frac{\partial A}{\partial t}, d^* F_A \right\rangle + 2\int_M \phi |d\phi| \left|\frac{\partial A}{\partial t}\right| |F_A| \\ &\leqslant \int_M \left\langle \phi^2 \frac{\partial A}{\partial t}, d^* F_A \right\rangle + \int_M \phi^2 \left|\frac{\partial A}{\partial t}\right|^2 + \int_M |d\phi|^2 |F_A|^2,\end{aligned}$$

and similarly

$$\begin{aligned}\frac{d}{dt}\int_M \phi^2 |\nabla_A \varphi|^2 &= 2\int_M \phi^2 \operatorname{Re}\left\langle \nabla_A \frac{\partial \varphi}{\partial t}, \nabla_A \varphi \right\rangle + \int_M \phi^2 \operatorname{Re}\left\langle \frac{\partial A}{\partial t}\varphi, \nabla_A \varphi \right\rangle \\ &= 2\int_M \phi^2 \operatorname{Re}\left\langle \frac{\partial \varphi}{\partial t}, \nabla_A^* \nabla_A \varphi \right\rangle - 2\int_M \operatorname{Re}\left\langle d\phi^2 \otimes \frac{\partial \varphi}{\partial t}, \nabla_A \varphi \right\rangle \\ &\quad + \int_M \phi^2 \operatorname{Re}\left\langle \frac{\partial A}{\partial t}\varphi, \nabla_A \varphi \right\rangle.\end{aligned}$$

Furthermore, in the above

$$\begin{aligned}-2\int_M \operatorname{Re}\left\langle d\phi^2 \otimes \frac{\partial \varphi}{\partial t}, \nabla_A \varphi \right\rangle &\leqslant 4\int_M \phi |d\phi| \left|\frac{\partial \varphi}{\partial t}\right| |\nabla_A \varphi| \\ &\leqslant 2\int_M \phi^2 \left|\frac{\partial \varphi}{\partial t}\right|^2 + 2\int_M |d\phi|^2 |\nabla_A \varphi|^2\end{aligned}$$

and

$$\begin{aligned}&2\int_M \phi^2 \operatorname{Re}\left\langle \frac{\partial \varphi}{\partial t}, \nabla_A^* \nabla_A \varphi \right\rangle \\ &\quad = -2\int_M \phi^2 \left|\frac{\partial \varphi}{\partial t}\right|^2 - 2\int_M \phi^2 \operatorname{Re}\left\langle \frac{\partial \varphi}{\partial t}, \frac{1}{4}\left[S + |\varphi|^2\right]\varphi \right\rangle \\ &\quad = -2\int_M \phi^2 \left|\frac{\partial \varphi}{\partial t}\right|^2 - \frac{d}{dt}\int_M \phi^2 \left[\frac{S}{4}|\varphi|^2 + \frac{1}{8}|\varphi|^4\right].\end{aligned}$$

Thus

$$\begin{aligned}\frac{d}{dt}\int_M \phi^2 |\nabla_A\varphi|^2 \leqslant -\frac{d}{dt}\int_M \phi^2\left[\frac{S}{4}|\varphi|^2+\frac{1}{8}|\varphi|^4\right] \\ + 2\int_M |d\phi|^2 |\nabla_A\varphi|^2 + \int_M \phi^2 \operatorname{Re}\left\langle \frac{\partial A}{\partial t}\varphi, \nabla_A\varphi\right\rangle.\end{aligned}$$

We next note that

$$\begin{aligned}\phi^2 \operatorname{Re}\left\langle \frac{\partial A}{\partial t}\varphi, \nabla_A\varphi\right\rangle &= \phi^2 \operatorname{Im}\frac{\partial A_k}{\partial t}\operatorname{Im}\left\langle \nabla_A^k\varphi, \varphi\right\rangle \\ &= \phi^2\left\langle \frac{\partial A}{\partial t}, i\operatorname{Im}\langle\nabla_A\varphi,\varphi\rangle\right\rangle\end{aligned} \tag{4.10}$$

so that

$$\phi^2 \operatorname{Re}\left\langle \frac{\partial A}{\partial t}\varphi, \nabla_A\varphi\right\rangle + \int_M\left\langle \phi^2\frac{\partial A}{\partial t}, d^*F_A\right\rangle = -\int_M \phi^2\left|\frac{\partial A}{\partial t}\right|^2.$$

From all of the above, we finally have

$$\begin{aligned}&\frac{d}{dt}\int_M \phi^2\left(|\nabla_A\varphi|^2+\frac{1}{2}|F_A|^2+\frac{S}{4}|\varphi|^2+\frac{1}{8}|\varphi|^4\right) \\ &\quad\leqslant cR^{-2}\int_M\left(|\nabla_A\varphi|^2+\frac{1}{2}|F_A|^2\right).\end{aligned}$$

The result follows by integrating on $[t_1,t]$ and taking the supremum over $t_1 \leqslant t \leqslant t_2$. □

Lemma 4.4. *Let (φ, A) be a solution to (1.9)–(1.10) in $M\times[0,T)$ with initial values (φ_0, A_0). Suppose $|\nabla_A\varphi| \leqslant K_1$ and $|F_A| \leqslant K_1$ in $M\times[0,T)$ for some constant $K_1 > 0$. Then for any positive integer $n \geq 1$, there is a constant K_{n+1} independent of T such that*

$$\left|\nabla_A^{(n+1)}\varphi\right| \leqslant K_{n+1}, \quad \left|\nabla_M^{(n)}F_A\right| \leqslant K_{n+1} \quad \text{in } M\times[0,T).$$

Proof. We prove Lemma 4.4 by induction. We first claim that

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\left|\nabla_A^{(k+1)}\varphi\right|^2+\left|\nabla_M^{(k)}F_A\right|^2\right)+c_k'\left(\left|\nabla_A^{(k+2)}\varphi\right|^2+\left|\nabla_M^{(k+1)}F_A\right|^2\right)\\
&\quad\leqslant -\Delta\left(\left|\nabla_A^{(k+1)}\varphi\right|^2+\left|\nabla_M^{(k)}F_A\right|^2\right)\\
&\qquad + c_k\left(\left|\nabla_A^{(k+1)}\varphi\right|^2+\left|\nabla_M^{(k)}F_A\right|^2+1\right)
\end{aligned}
\tag{4.11}
$$

for all non-negative integers $k = 0, 1, 2, 3, \ldots$.

From Lemma 4.1 with the assumption of Lemma 4.4, (4.11) holds for $k = 0$. Now, assume that (4.11) is true for $k = n-1$ and $\left|\nabla_A^{(k+1)}\varphi\right| \leqslant K_{k+1}$ and $\left|\nabla_M^{(k)}F_A\right| \leqslant K_{k+1}$ for non-negative integers $k \leq n-1$. Then we will show (4.11) and Lemma 4.4 are also true for all n.

From (1.9), we have

$$
\begin{aligned}
\frac{\partial}{\partial t}\left|\nabla_A^{(n+1)}\varphi\right|^2 &= 2\,\mathrm{Re}\left\langle \frac{\partial}{\partial t}\left(\nabla_A^{(n+1)}\varphi\right), \nabla_A^{(n+1)}\varphi\right\rangle\\
&= -2\,\mathrm{Re}\left\langle \nabla_A^{(n+1)}\nabla_A^*\nabla_A\varphi, \nabla_A^{(n+1)}\varphi\right\rangle\\
&\quad -\frac{1}{2}\,\mathrm{Re}\left\langle \nabla_A^{(n+1)}\left[S+|\varphi|^2\right]\varphi, \nabla_A^{(n+1)}\varphi\right\rangle\\
&\quad +2\,\mathrm{Re}\left\langle \left(\frac{\partial}{\partial t}\nabla_A^{(n+1)}\right)\varphi, \nabla_A^{(n+1)}\varphi\right\rangle.
\end{aligned}
\tag{4.12}
$$

From the Ricci formula (4.2) we have

$$
\begin{aligned}
&-2\,\mathrm{Re}\left\langle \nabla_A^{(n+1)}\nabla_A^*\nabla_A\varphi, \nabla_A^{(n+1)}\varphi\right\rangle\\
&\quad\leqslant -2\,\mathrm{Re}\left\langle \nabla_A^*\nabla_A\nabla_A^{(n+1)}\varphi, \nabla_A^{(n+1)}\varphi\right\rangle\\
&\qquad + c\left|\nabla_M^{(n+1)}F_A\right|\left|\nabla_A^{(n+1)}\varphi\right| + c\left|\nabla_A^{(n+1)}\varphi\right|^2 + c\left|\nabla_A^{(n+1)}\varphi\right|,
\end{aligned}
$$

where we recall that the non-constant part of Ω_A is F_A. From (2.3),

$$
-2\,\mathrm{Re}\left\langle \nabla_A^*\nabla_A\nabla_A^{(n+1)}\varphi, \nabla_A^{(n+1)}\varphi\right\rangle = -\Delta\left|\nabla_A^{(n+1)}\varphi\right|^2 - 2\left|\nabla_A^{(n+2)}\varphi\right|^2.
$$

Next, applying metric compatibility $n+1$ times, we find that the $(n+1)$th order term of $\partial^{(n+1)}|\varphi|^2$ is $2\,\mathrm{Re}\left\langle \nabla_A^{(n+1)}\varphi, \varphi\right\rangle$ and

$$
\mathrm{Re}\left\langle -\nabla_A^{(n+1)}\frac{1}{4}\left[R+|\varphi|^2\right]\varphi, \nabla_A^{(n+1)}\varphi\right\rangle \leqslant c\left|\nabla_A^{(n+1)}\varphi\right|^2 + c\left|\nabla_A^{(n+1)}\varphi\right|.
$$

For the final term in (4.12), noting that $\frac{\partial}{\partial t}\nabla_A = \frac{1}{2}\frac{\partial A}{\partial t}$ involves derivatives of F_A and $\nabla_A\varphi$ and utilizing the product rule we find

$$\begin{aligned}
2\,\mathrm{Re}\left\langle \left(\frac{\partial}{\partial t}\nabla_A^{(n+1)}\right)\varphi, \nabla_A^{(n+1)}\varphi\right\rangle &= \mathrm{Re}\left\langle \sum_{j+k=n} \nabla_A^{(j)}\frac{\partial A}{\partial t}\nabla_A^{(k)}\varphi, \nabla_A^{(n+1)}\varphi\right\rangle \\
&\leqslant c\left|\nabla_M F_A\right|\left|\nabla_A^{(n+1)}\varphi\right| + c\left|\nabla_M^{(n+1)}F_A\right|\left|\nabla_A^{(n+1)}\varphi\right| + c\left|\nabla_A^{(n+1)}\varphi\right|^2 \\
&\quad + c\left|\nabla_A^{(n+1)}\varphi\right|,
\end{aligned}$$

where $\nabla_M F_A$ is equal to $\nabla_M^{(n)}F_A$ for the case $n=1$ and bounded for cases $n \geqslant 2$. Thus

$$\begin{aligned}
\frac{\partial}{\partial t}\left|\nabla_A^{(n+1)}\varphi\right|^2 &= -\Delta\left|\nabla_A^{(n+1)}\varphi\right|^2 - 2\left|\nabla_A^{(n+2)}\varphi\right|^2 + c\left|\nabla_M^{(n+1)}F_A\right|\left|\nabla_A^{(n+1)}\varphi\right| \\
&\quad + c\left|\nabla_M F_A\right|\left|\nabla_A^{(n+1)}\varphi\right| + c\left|\nabla_A^{(n+1)}\varphi\right|^2 + c\left|\nabla_A^{(n+1)}\varphi\right|.
\end{aligned} \tag{4.13}$$

Similarly, from (1.10),

$$\begin{aligned}
\frac{\partial}{\partial t}\left|\nabla_M^{(n)}F_A\right|^2 &= \frac{\partial}{\partial t}\left|\nabla_M^{(n)}dA\right|^2 = 2\left\langle \nabla_M^{(n)}d\frac{\partial A}{\partial t}, \nabla_M^{(n)}dA\right\rangle \\
&= 2\left\langle \nabla_M^{(n)}d\left[-d^*F_A - i\,\mathrm{Im}\left\langle\nabla_A\varphi,\varphi\right\rangle\right], \nabla_M^{(n)}F_A\right\rangle \\
&\leqslant 2\left\langle -\nabla_M^{(n)}\nabla_M^*\nabla_M F_A - i\nabla_M^{(n)}d\,\mathrm{Im}\left\langle\nabla_A\varphi,\varphi\right\rangle, \nabla_M^{(n)}F_A\right\rangle \\
&\quad + c\left|\nabla_M^{(n)}F_A\right|^2 + c\left|\nabla_M^{(n)}F_A\right| \\
&\leqslant -\Delta\left|\nabla_M^{(n)}F_A\right|^2 - 2\left|\nabla_M^{(n+1)}F_A\right|^2 - \left\langle i\nabla_M^{(n)}d\,\mathrm{Im}\left\langle\nabla_A\varphi,\varphi\right\rangle, \nabla_M^{(n)}F_A\right\rangle \\
&\quad + c\left|\nabla_M^{(n)}F_A\right|^2 + c\left|\nabla_M^{(n)}F_A\right|,
\end{aligned}$$

where we have used the Weitzenböck formula (2.1), the Ricci formula (4.2) and (2.3). Using (4.5), we have

$$\nabla_M^{(n)}d\,\mathrm{Im}\left\langle\nabla_A\varphi,\varphi\right\rangle = \nabla_M^{(n)}\sum_{k>j}\left(\left\langle\Omega_A^{kj}\varphi,\varphi\right\rangle + 2\,\mathrm{Im}\left\langle\nabla_A^j\varphi,\nabla_A^k\varphi\right\rangle\right)dx^k\wedge dx^j.$$

From this and metric compatibility we find

$$\begin{aligned}
\left|\left\langle i\nabla_M^{(n)}d\,\mathrm{Im}\left\langle\nabla_A\varphi,\varphi\right\rangle, \nabla_M^{(n)}F_A\right\rangle\right| &\leqslant \left|i\nabla_M^{(n)}d\,\mathrm{Im}\left\langle\nabla_A\varphi,\varphi\right\rangle\right|\left|\nabla_M^{(n)}F_A\right| \\
&\leqslant c\left|\nabla_M^{(n)}F_A\right|^2 + c\left|\nabla_M^{(n)}F_A\right| + c\left|\nabla_M^{(n)}F_A\right|\left|\nabla_A^{(n+1)}\varphi\right|.
\end{aligned}$$

Thus

$$\frac{\partial}{\partial t}\left|\nabla_M^{(n)}F_A\right|^2 \leqslant -\Delta\left|\nabla_M^{(n)}F_A\right|^2 - 2\left|\nabla_M^{(n+1)}F_A\right|^2 + c\left|\nabla_M^{(n)}F_A\right|\left|\nabla_A^{(n+1)}\varphi\right| \tag{4.14}$$
$$+ c\left|\nabla_M^{(n)}F_A\right|^2 + c\left|\nabla_M^{(n)}F_A\right|.$$

Combining now Equations (4.13) and (4.14) gives

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\left|\nabla_A^{(n+1)}\varphi\right|^2 + \left|\nabla_M^{(n)}F_A\right|^2\right)\\
&\quad\leqslant -\Delta\left(\left|\nabla_A^{(n+1)}\varphi\right|^2 + \left|\nabla_M^{(n)}F_A\right|^2\right) - 2\left(\left|\nabla_A^{(n+2)}\varphi\right|^2 + \left|\nabla_M^{(n+1)}F_A\right|^2\right)\\
&\qquad + c\left(\left|\nabla_A^{(n+1)}\varphi\right|^2 + \left|\nabla_M^{(n)}F_A\right|^2\right) + c\left|\nabla_M^{(n+1)}F_A\right|\left|\nabla_A^{(n+1)}\varphi\right|\\
&\qquad + c\left|\nabla_M F_A\right|\left|\nabla_A^{(n+1)}\varphi\right| + c.
\end{aligned}$$

Utilizing Young's inequality, we obtain (4.11) for $k = n$. We now complete the proof of Lemma 4.4.

Case 1. Assume $T \le 1$. Multiplying (4.11) by $e^{-c_n t}$, the maximum principle yields

$$\max_{x\in M, 0\le t\le T}\left(|\nabla_A^{(n+1)}\varphi|^2 + |\nabla_M^{(n)}F_A|^2\right) \le e^{c_n}\left(|\nabla_A^{(n+1)}\varphi_0|^2 + |\nabla_M^{(n)}F_{A_0}|^2 + 1\right).$$

The required result is proved.

Case 2. Assume $T > 1$. Let t_0 be any time with $0 \le t_0 \le T$. For any $t_0 \le 1$, the result follows from Case 1. For any $t_0 > 1$, integrating (4.11) over M for $k = n-1$, we have

$$\begin{aligned}
&\frac{d}{dt}\int_M |\nabla_A^{(n)}\varphi|^2 + |\nabla_M^{(n-1)}F_A|^2\,dV + c'_{n-1}\int_M |\nabla_A^{(n+1)}\varphi|^2 + |\nabla_M^{(n)}F_A|^2\,dV\\
&\quad\le c_{n-1}\int_M (|\nabla_A^{(n)}\varphi|^2 + |\nabla_M^{(n-1)}F_A|^2 + 1)\,dV.
\end{aligned}$$

Integrating in t on $[t_0-1, t_0]$ yields

$$\int_{t_0-1}^{t_0}\int_M |\nabla_A^{(n+1)}\varphi|^2 + |\nabla_M^{(n)}F_A|^2\,dV\,dt \le \frac{(c_{n-1}+1)(2K_n^2+1)|M|}{c'_{n-1}}.$$

Then, using Moser's Harnack inequality in (4.11) with $k = n$, the required result follows. □

Corollary 4.1. *Let (φ, A) be a solution to (1.9)–(1.10). Suppose $\left|\nabla_A^{(j)}\varphi\right| \leqslant K_n$ and $\left|\nabla_M^{(j-1)}F_A\right| \leqslant K_n$ in $P_1(x_0,t_0)$ for each $1 \leqslant j \leqslant n$ and some constant K_n. Then there is a positive constant K_{n+1} such that*

$$\left|\nabla_A^{(n+1)}\varphi\right| \leqslant K_{n+1}, \quad \left|\nabla_M^{(n)}F_A\right| \leqslant K_{n+1} \quad \text{in } P_{1/2}(x_0,t_0).$$

Proof. Let ξ be a smooth cut-off function $C^\infty(P_1)$ satisfying $|\xi| \le 1$ and $|\nabla\xi| + |\Delta\xi| + |\partial_t\xi| \le C$ in P_1 for some constant $C > 0$, and $\xi \equiv 1$ in $P_{3/4}$, $\xi \equiv 0$ on the parabolic boundary of P_1. Multiplying (4.11) by ξ^2 for $k = n-1$ and integrating on P_1, we have

$$\begin{aligned}
&c'_{n-1}\int_{P_1}\xi^2(|\nabla_A^{(n+1)}\varphi|^2 + |\nabla^{(n)}F_A|^2)\,dV\,dt\\
&\le 2\int_{P_1}(|\xi_t| + |\Delta\xi| + |\nabla\xi|^2)(|\nabla_A^{(n)}\varphi|^2 + |\nabla^{(n-1)}F_A|^2)\,dV\,dt\\
&\quad + c_{n-1}\int_{P_1}(|\nabla_A^{(n)}\varphi|^2 + |\nabla^{(n-1)}F_A|^2 + 1)\,dV\,dt\\
&\le |B_1|(2K_n^2+1)(4C + 2C^2 + c_{n-1}).
\end{aligned}$$

Applying Moser's Harnack inequality to (4.11) with $k = n$ in $P_{3/4}$, the required result follows. □

As mentioned in Section 1, we can show that concentration does not occur in general for the Seiberg–Witten flow. We say that the energy concentrates at a point x_0 at time $t = T$ if there are constants δ and R_0 such that

$$\limsup_{t\to T}\int_{B_R(x_0)}\left(|\nabla_A\varphi|^2 + |F_A|^2\right)\,dV \geqslant \delta$$

for all $R \in (0, R_0]$. That is, as $t \to T$ we have energy δ concentrating in smaller and smaller balls. Recall that $\delta > 0$ is the constant defined in Lemma 4.2. Concentrations of amounts of energy less than delta are ruled out by Lemma 4.2. Using Lemma 4.3 and that the energy is bounded, it follows from the proof of Struwe (see [16, 18]) that concentration can occur at no more than a finite number of points at $t = T$.

Lemma 4.5. *The energy does not concentrate at any $T \le \infty$.*

Proof. We assume by contradiction that the energy concentrates at a point x_0. We choose $R_0 > 0$ sufficiently small so that $B_{R_0}(x_0)$ contains no concentration points other than x_0. Then there exist sequences $x_m \to x_0$, $t_m \to T$

and a sequence of balls $B_{R_m}(x_m)$ with $R_m \to 0$ such that

$$\begin{aligned}\delta &> \mathcal{SW}_{B_{R_m}(x_m)}(\varphi(t_m), A(t_m)) \\ &= \sup_{0\leqslant t\leqslant t_m,\, x\in B_{R_0}(x_0)} \mathcal{SW}_{B_{R_m}(x)}(\varphi(t), A(t)) > \frac{3\delta}{4}\end{aligned} \tag{4.15}$$

for each m. Choosing $C_2 = \frac{\delta}{4C_1}$, where C_1 is the constant from Lemma 4.3, and applying Lemma 4.3 to the time interval $[t, t_m]$ for some $t \in [t_m - C_2 R_m^2, t_m]$ gives

$$\mathcal{SW}_{B_{2R_m}(x_m)}(\varphi(t), A(t)) \geqslant \frac{3\delta}{4} - C_1(t_m - t)R_m^{-2} \geqslant \frac{3\delta}{4} - \frac{\delta}{4} = \frac{\delta}{2}. \tag{4.16}$$

Define

$$\mathcal{D}_m = \{(y, s) : R_m y + x_m \in B_{R_0}(x_0),\ s \in [-C_2, 0]\}.$$

Note that as $m \to \infty$, $R_m \to 0$ and $\mathcal{D}_m$ will expand to cover $\mathbb{R}^4 \times [-C_2, 0]$. Similarly to the proof of Lemma 4.2, we rescale the data to

$$\begin{aligned}\varphi_m(y, s) &= \varphi(R_m y + x_m, R_m^2 s + t_m), \\ A_m(y, s) &= R_m A(R_m y + x_m, R_m^2 s + t_m),\end{aligned}$$

so that φ_m and A_m are defined on $\mathcal{D}_m$ and

$$\begin{aligned}|\nabla_{A_m}\varphi_m|^2 &= R_m^2 |\nabla_A \varphi|^2, \\ |F_{A_m}|^2 &= R_m^4 |F_A|^2.\end{aligned}$$

We next show that $R_m\varphi_m$ and A_m converge locally to $\tilde{\varphi}$ and $\tilde{A}$, respectively, where $\tilde{\varphi}$ and $\tilde{A}$ are defined on $\mathbb{R}^4 \times [-C_2, 0]$. We consider the rescaled equations

$$\frac{\partial R_m\varphi_m}{\partial s} = R_m^3 \frac{\partial \varphi}{\partial t} = -\nabla_{A_m}^* \nabla_{A_m} R_m\varphi_m - \frac{1}{4}\left[R_m^2 S + |R_m\varphi_m|^2\right] R_m\varphi_m, \tag{4.17}$$

$$\frac{\partial A_m}{\partial s} = R_m^3 \frac{\partial A}{\partial t} = -d^* F_{A_m} - i\,\mathrm{Im}\,\langle \nabla_{A_m} R_m\varphi_m, R_m\varphi_m\rangle. \tag{4.18}$$

Note that the following argument mirrors that presented in Lemma 4.2 and Lemma 4.4 for the original equations. From (4.15) and Lemma 4.2,

$$R_m^2 |\nabla_{A_m}\varphi_m|^2 + |F_{A_m}|^2 \leqslant K_1 \tag{4.19}$$

locally in $B_R(0) \times [-C_2, 0]$ uniformly in m where K_1 is independent of m. Noting the similarity of these equations to (1.9) and (1.10), we use (4.19) and results identical to Lemma 4.4 and Corollary 4.1 to find

$$\left|\nabla_{A_m}^{(n+1)} R_m \varphi_m\right|^2 + \left|\nabla_M^{(n)} F_{A_m}\right|^2 \leqslant K_{n+1} \tag{4.20}$$

in $B_R(0) \times [-C_2, 0]$ uniformly in m for each $n \geq 0$.

If we choose our local coordinates on $B_{R_0}(x_0)$ to be orthonormal coordinates, then the metric on the rescaled space is simply $g_{ij} = \delta_{ij}$. From (2.5), we know $\partial_t A \in L^2([0,\infty); L^2(M))$ so that

$$\int_{\mathcal{D}_m} |\partial_s A_m|^2 dyds \leqslant \int_{M \times [t_m - C_2 R_m^2, t_m]} |\partial_t A|^2 dV dt \to 0.$$

Then from (4.18), there exists some $\tau_m \in [-C_2, 0]$ such that

$$\int_{\mathcal{D}_m(s=\tau_m)} |d^* F_{A_m}|^2 dy \to 0.$$

By a result of Uhlenbeck in [20] ([20, Theorem 1.3], see also [5]), passing to a subsequence (without changing notation) and in an appropriate gauge, $A_m(\tau_m) \to \tilde{A}$ and $R_m \varphi_m \to \tilde{\varphi}$ in C^∞, where $d^* F_{\tilde{A}} = 0$ in $\mathbb{R}^4$ and $\int_{\mathbb{R}^4} |F_{\tilde{A}}|^2 \, dy < C$ for some $C > 0$, and $\tilde{\varphi} = 0$ by the boundedness of φ_m. Next, from (4.16),

$$\int_{B_2(0)} R_m^2 \left|\nabla_{A_m} \varphi_m\right|^2 + \left|F_{A_m}\right|^2 + \frac{1}{4} R_m^4 \left|\varphi_m\right|^2 (S + |\varphi_m|^2) dy \geqslant \frac{\delta}{2}. \tag{4.21}$$

Since $R_m \varphi_m \to 0$, the first and third terms of (4.21) go to zero. Then we must have

$$\int_{B_2(0)} \left|F_{\tilde{A}}\right|^2 dy \geqslant \frac{\delta}{2}. \tag{4.22}$$

We now derive a contradiction with (4.22). Since $F_{\tilde{A}}$ is harmonic in $\mathbb{R}^4$, the well-known mean value formula implies that for any $x_0 \in \mathbb{R}^4$ and $R > 0$, we have

$$|F_{\tilde{A}}|(x_0) \leq \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |F_{\tilde{A}}| \, dy \leq \left(\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |F_{\tilde{A}}|^2 \, dy \right)^{1/2}$$

Letting $R \to \infty$, $F_{\tilde{A}} = 0$ for any $x_0 \in \mathbb{R}^4$, which contradicts (4.22), as required. $\square$

Next we complete a proof of Theorem 1.1.

Proof of Theorem 1.1. By the non-concentration of the energy (Lemma 4.5) at any $T \leq \infty$, there exists $R > 0$ such that for any point $x \in M$ and $t \in [0, T)$,

$$\int_{B_R(x)} \left(|\nabla_A \varphi|^2 + |F_A|^2\right)(\cdot, t)\, dV < \delta.$$

Then by Lemma 4.2, $|\nabla_A \varphi|^2 + |F_A|^2$ is uniformly bounded on $P_{R/2}(x,t)$. Since x and t are arbitrary, $|\nabla_A \varphi|^2 + |F_A|^2$ is uniformly bounded under the flow. From this fact and Lemma 4.4 we have for each $n \in \mathbb{N}$

$$\sup_{M \times [0,\infty)} \left(\left|\nabla_A^{(n)} \varphi\right|^2 + \left|\nabla_M^{(n-1)} F_A\right|^2 \right) \leqslant K_n.$$

Note that Equations (1.9)–(1.10) depend only on these bounded quantities. It is then elementary to show using the Sobolev embedding theorem that $(\varphi(t), A(t))$ converges to smooth data $(\varphi(T), A(T))$ as $t \to T$. In conjunction with local existence, this shows Theorem 1.1. □

5. Convergence

In this section we prove Theorem 1.2. That is, we show that the flow (1.9)–(1.10) converges uniquely to a critical point of the functional (1.4). Since convergence is only possible up to gauge, throughout this section we assume an appropriate choice of gauge. We denote a critical point of the Seiberg–Witten functional by $(\varphi_\infty, A_\infty)$, and write $\varphi = \varphi_\infty + \psi$ and $A = A_\infty + a$, where (φ, A) denotes a solution to the flow. For simplicity, we denote $\|\varphi\| + \|A\|$ by $\|(\varphi, A)\|$ for any norm $\|\cdot\|$. The proof depends on the following lemmas.

Lemma 5.1. *For each $k > 0$, there exist sequences $\{t_n\}$ and $\{g_n\} \subset \mathscr{G}$ with $t_n \to \infty$ such that $g_n \cdot (\varphi(t_n), A(t_n))$ converges in H^k to a critical point $(A_\infty, \varphi_\infty)$.*

Proof. Integrating the energy inequality we find

$$\int_0^\infty \left\| \left(\frac{\partial \varphi}{\partial t}, \frac{\partial A}{\partial t} \right) \right\|_{L^2} \leqslant c.$$

It follows that there exists a sequence $\{t_n\}$ such that

$$\left\| \left(\frac{\partial \varphi}{\partial t}(t_n), \frac{\partial A}{\partial t}(t_n) \right) \right\|_{L^2} \to 0. \tag{5.1}$$

Next, recall from Lemma 4.4 that we have uniform bounds on the quantities $\|\varphi\|_{H^k}$ and $\|F_A\|_{H^k}$ for each $k \geq 0$. It follows from a theorem of Uhlenbeck [20, Theorem 1.3] that in an appropriate (time varying) gauge, we also have uniform bounds on $\|A\|_{H^k}$ for each $n \geq 0$. For each $k \geq 0$, from the Rellich–Kondrachov theorem we can pass to a subsequence of $\{t_n\}$ (without changing notation) such that $(\varphi(t_n), A(t_n))$ converges in H^k up to gauge to a point $(\varphi_\infty, A_\infty)$. It remains to show that $(\varphi_\infty, A_\infty)$ is a critical point. From (5.1) we have the required result. □

Lemma 5.2. *On any finite time interval, the solution to the flow depends continuously on the initial conditions. That is, if $(\varphi_1(t), A_1(t))$ and $(\varphi_2(t), A_2(t))$ are two solutions to the flow with different initial values, then for any $T > 0$ there exists a constant c such that*

$$\begin{aligned} &\|(\varphi_1(T), A_1(T)) - (\varphi_2(T), A_2(T))\|_{H^k} \\ &\quad \leqslant c \,\|(\varphi_1(0), A_1(0)) - (\varphi_2(0), A_2(0))\|_{H^k} . \end{aligned} \tag{5.2}$$

Proof. Recall that in the gauge of [20, Theorem 1.3], we have uniform bounds on φ, A, and all of their derivatives. In this gauge, we also know that $d^*A = 0$. Using these facts and the expansion

$$\nabla_A^* \nabla_A \varphi = -\nabla_{A_\infty}^* \nabla_{A_\infty} \varphi + a \# \nabla_{A_\infty} \varphi + \nabla_M a \# \varphi + a \# a \# \varphi,$$

we can write

$$\frac{\partial}{\partial t}(\varphi_1 - \varphi_2) = -\nabla_{A_\infty}^* \nabla_{A_\infty} (\varphi_1 - \varphi_2) + f, \tag{5.3}$$

$$\frac{\partial}{\partial t}(A_1 - A_2) = -\Delta(A_1 - A_2) + g, \tag{5.4}$$

where f and g comprise the lower order terms from (1.9) and (1.10), and $\|f\|_{H^k}$ and $\|g\|_{H^k}$ are both bounded by $c\,\|(\varphi_1 - \varphi_2, A_1 - A_2)\|_{H^k}$. When $f = g = 0$, we simply have the heat equation, whose solution depends continuously on the intial data in the H^k norm. When the data are small in the H^k norm, the perturbations f and g will be small in the H^k norm also. Thus φ and A depend continuously on their initial values. □

Lemma 5.3 (Lojasiewicz's inequality). *Let $(\varphi_\infty, A_\infty)$ be a critical point of the Seiberg–Witten functional. There exist constants $\varepsilon_1 > 0$, $\frac{1}{2} < \gamma < 1$ and $c > 0$ such that if*

$$\|(\varphi, A) - (\varphi_\infty, A_\infty)\|_{H^1} \leq \varepsilon_1,$$

then

$$\left\| \left(\frac{\partial \varphi}{\partial t}, \frac{\partial A}{\partial t} \right) \right\|_{L^2} \geq c \left| \mathcal{SW}(\varphi, A) - \mathcal{SW}(\varphi_\infty, A_\infty) \right|^\gamma . \tag{5.5}$$

Note that Lemma 5.3 is analogous to [15, Theorem 3], since

$$\|\mathrm{Grad}(\mathcal{SW})\|_{L^2} = \left\| \left(2\frac{\partial \varphi}{\partial t}, \frac{\partial A}{\partial t} \right) \right\|_{L^2},$$

where the factor of 2 arises due to the factor of 2 introduced in our definition of the flow equations.

Proof. The proof of this lemma is analogous to that of [13, Proposition 7.2] and [21, Proposition 3.5]. While Wilkin considers in Section 3 of [21] the Yang–Mills–Higgs functional, he allows in the proof of this lemma a very general functional $f : Q \to \mathbb{R}$, where Q is a Hilbert manifold and f is invariant under the action of some gauge group $\mathscr{G}$. To apply [21, Proposition 3.20], we need only check that the operator

$$H_{\mathcal{SW}} + \rho_\infty \rho_\infty^* : T_\infty H \to T_\infty H$$

is elliptic. Here $H_{\mathcal{SW}}$ represents the Hessian of the Seiberg–Witten functional at the point $(\varphi_\infty, A_\infty)$, and $\rho_\infty : \mathrm{Lie}(\mathscr{G}) \to T_\infty M$ is the infinitesimal action of the gauge group $\mathscr{G}$. The operator ρ_∞^* is defined by

$$\langle \rho_\infty^* X, u \rangle_{\mathrm{Lie}(\mathscr{G})} = \int_M \langle X, \rho_\infty u \rangle,$$

for $X \in T_\infty H$ and $u \in \mathrm{Lie}(\mathscr{G})$. We begin by computing the operator

$$H_{\mathcal{SW}}(\psi, a) = \left. \frac{d}{ds} \right|_{s=0} Grad(\mathcal{SW})(s\psi, sa)$$

where Grad($\mathcal{SW}$) represents the gradient operator of the Seiberg–Witten functional. There holds

$$\begin{aligned}
&\frac{\partial}{\partial s}\bigg|_{s=0}\bigg(\nabla^*_{A_\infty+sa}\nabla_{A_\infty+sa}(\varphi_\infty+s\psi)+\frac{S}{4}(\varphi_\infty+s\psi)\\
&\quad+\frac{1}{4}\left|\varphi_\infty+s\psi\right|^2(\varphi_\infty+s\psi)\bigg)\\
&\qquad=\nabla^*_{A_\infty}\nabla_{A_\infty}\psi+\frac{S}{4}\psi+\frac{1}{4}\left|\varphi_\infty\right|^2\psi+\frac{1}{2}\operatorname{Re}\langle\varphi_\infty,\psi\rangle\varphi_\infty\\
&\quad+\left\langle\frac{1}{2}i\operatorname{Im}\langle\varphi_\infty,\nabla_{A_\infty}\psi\rangle+\frac{1}{2}i\operatorname{Im}\langle\psi,\nabla_{A_\infty}\varphi_\infty\rangle,a\right\rangle,
\end{aligned}$$

where we use a relationship analogous to that in Equation (4.10). Similarly,

$$\begin{aligned}
&\frac{\partial}{\partial s}\bigg|_{s=0}\frac{1}{2}d^*d(A_\infty+sa)\\
&\quad+\frac{1}{2}i\operatorname{Im}\left\langle\nabla_{A_\infty}(\varphi_\infty+s\psi)+\frac{1}{2}sa(\varphi_\infty+s\psi),(\varphi_\infty+s\psi)\right\rangle\\
&\qquad=\frac{1}{2}d^*da+\frac{1}{2}i\operatorname{Im}\langle\nabla_{A_\infty}\varphi_\infty,\psi\rangle+\frac{1}{2}i\operatorname{Im}\langle\nabla_{A_\infty}\psi,\varphi_\infty\rangle+\frac{1}{2}a\left|\varphi_\infty\right|^2.
\end{aligned}$$

Using the above and recalling the calculations in the proof of Lemma 2.2, the Hessian at the point $(\varphi_\infty, A_\infty)$ is given by

$$\begin{aligned}
H_{\mathcal{SW}}(\psi,a)=\bigg(&\nabla^*_{A_\infty}\nabla_{A_\infty}\psi+\frac{1}{2}\operatorname{Re}\langle\varphi_\infty,\psi\rangle\varphi_\infty+\frac{S}{4}\psi+\frac{1}{4}\left|\varphi_\infty\right|^2\psi\,,\\
&\frac{1}{2}d^*da+\frac{1}{4}\left|\varphi_\infty\right|^2a+i\operatorname{Im}\langle\varphi_\infty,\nabla_{A_\infty}\psi\rangle+i\operatorname{Im}\langle\psi,\nabla_{A_\infty}\varphi_\infty\rangle\bigg).
\end{aligned}\tag{5.6}$$

In the following, we continue to use $(\cdot,\cdot)$ to denote an element of the configuration space, i.e., $(\psi, a) \in \Gamma(S^+) \times \mathscr{A}$. Now, note that if $g(t)$ represents a path through the gauge group $\mathscr{G}$ with $g(0) = I$, then

$$\begin{aligned}
\rho_\infty(g'(0))&=\frac{1}{\sqrt{2}}\frac{\partial}{\partial t}\bigg|_{t=0}(g(t)^*(\varphi_\infty,A_\infty))\\
&=\frac{1}{\sqrt{2}}\frac{\partial}{\partial t}\bigg|_{t=0}(g(t)^{-1}\varphi_\infty,A_\infty+2g(t)^{-1}dg(t))\\
&=\frac{1}{\sqrt{2}}(-g'(0)\varphi_\infty,2dg'(0)),
\end{aligned}$$

where we write $g(t) = e^{i\theta}$ for some function $\theta(t,x)$ defined locally on the manifold M, so that

$$\left.\frac{\partial}{\partial t}\right|_{t=0} 2g(t)^{-1}dg(t) = \left.\frac{\partial}{\partial t}\right|_{t=0} 2id\theta = 2dg'(0).$$

It follows that

$$\begin{aligned}\left\langle \rho_\infty^*(\psi,a), g'(0)\right\rangle_{\mathrm{Lie}(\mathscr{G})} &= \frac{1}{\sqrt{2}}\int_M \left\langle \psi, -g'(0)\varphi_\infty\right\rangle + \left\langle a, 2dg'(0)\right\rangle \\ &= \frac{1}{\sqrt{2}}\left\langle \left\langle \psi,\varphi_\infty\right\rangle, g'(0)\right\rangle_{\mathrm{Lie}(\mathscr{G})} + \frac{1}{\sqrt{2}}\left\langle \frac{1}{2}d^*a, g'(0)\right\rangle_{\mathrm{Lie}(\mathscr{G})},\end{aligned}$$

that is,

$$\rho_\infty^*(\psi,a) = \frac{1}{\sqrt{2}}\left(\left\langle \psi,\varphi_\infty\right\rangle, \frac{1}{2}d^*a\right)$$

and

$$\rho_\infty\rho_\infty^*(\psi,a) = \frac{1}{2}\left(-\left\langle \psi,\varphi_\infty\right\rangle\varphi_\infty, dd^*a\right). \tag{5.7}$$

Comparing (5.7) with (5.6), we find that $H_{\mathcal{SW}} + \rho_\infty\rho_\infty^*$ is an elliptic operator, as required. Then, the required result follows from the same arguments as for [21, Theorem 3.19]. □

Lemma 5.4. *There exists a constant c such that if $T \geq 0$ and $S > 1$ are such that $0 \leq T \leq S-1$, then*

$$\int_{T+1}^{S} \left\|\left(\frac{\partial\varphi}{\partial t}, \frac{\partial A}{\partial t}\right)\right\|_{H^k} \leqslant c\int_T^S \left\|\left(\frac{\partial\varphi}{\partial t}, \frac{\partial A}{\partial t}\right)\right\|_{L^2}. \tag{5.8}$$

Proof. We define $G = (G_1, G_2) = \left(\frac{\partial\varphi}{\partial t}, \frac{\partial A}{\partial t}\right)$. Noting that

$$\nabla_A^*\nabla_A\varphi = -\nabla_{A_\infty}^*\nabla_{A_\infty}\varphi + a\#\nabla_{A_\infty}\varphi + \nabla_M a\#\varphi + a\#a\#\varphi,$$

we have

$$\begin{aligned}\frac{\partial G_1}{\partial t} = &-\nabla_{A_\infty}^*\nabla_{A_\infty}G_1 + G_2\#\nabla_{A_\infty}\varphi + a\#\nabla_{A_\infty}G_1 + \nabla_M G_2\#\varphi + \nabla_M a\#G_1 \\ &+ G_2\#a\#\varphi + a\#a\#G_1 - \frac{S}{4}G_1 + \varphi\#\varphi\#G_1,\end{aligned}$$

and

$$\frac{\partial G_2}{\partial t} = -d^* dG_2 + G_2\#\varphi\#\varphi + a\#\varphi\#G_1 + \nabla_{A_\infty}\varphi\#G_1 + \varphi\#\nabla_{A_\infty}G_1.$$

Using the Bianchi identify and the Weitzenböck formula (2.1) we can write

$$\begin{aligned}-d^* dG_2 &= -\Delta G_2 - idd^* \operatorname{Im}\langle\nabla_A\varphi,\varphi\rangle \\ &= -\nabla_M^*\nabla_M G_2 - idd^* \operatorname{Im}\langle\nabla_A\varphi,\varphi\rangle + R_M\#G_2.\end{aligned}$$

Using metric compatibility and Equation (1.9), we compute in normal coordinates

$$\begin{aligned}-idd^* \operatorname{Im}\langle\nabla_A\varphi,\varphi\rangle &= id * d\left(\operatorname{Im}\left\langle\nabla_A^j\varphi,\varphi\right\rangle * dx^j\right) \\ &= id*\left(\left[\operatorname{Im}\left\langle\nabla_A^k\nabla_A^j\varphi,\varphi\right\rangle\right.\right. \\ &\quad \left.\left.+\operatorname{Im}\left\langle\nabla_A^j\varphi,\nabla_A^k\varphi\right\rangle\right] dx^k\wedge *dx^j\right) \\ &= id*\left(\operatorname{Im}\left\langle\nabla_A^j\nabla_A^j\varphi,\varphi\right\rangle dV\right) = -id\operatorname{Im}\langle\nabla_A^*\nabla_A\varphi,\varphi\rangle \\ &= -i\operatorname{Im}\left(\langle\nabla_A\nabla_A^*\nabla_A\varphi,\varphi\rangle + \langle\nabla_A^*\nabla_A\varphi,\nabla_A\varphi\rangle\right) \\ &= -i\frac{1}{4}\left[S+|\varphi|^2\right]\operatorname{Im}\left(\langle\nabla_A\varphi,\varphi\rangle + \langle\varphi,\nabla_A\varphi\rangle\right) \\ &\quad - i\frac{1}{4}\operatorname{Im}\left\langle\left[dS + d|\varphi|^2\right]\varphi,\varphi\right\rangle \\ &\quad + \varphi\#\nabla_{A_\infty}G_1 + \nabla_{A_\infty}\varphi\#G_1 + a\#\varphi\#G_1 \\ &= \varphi\#\nabla_{A_\infty}G_1 + \nabla_{A_\infty}\varphi\#G_1 + a\#\varphi\#G_1,\end{aligned}$$

where the second term in line two and the first two terms in the second to last expression are zero. Thus recalling the uniform bounds on φ, A and their derivatives (see the proof of Lemma 5.1), we can combine all of the above in the compact form

$$\frac{\partial G}{\partial t} + \nabla^*\nabla G = V_0\#G + V_1\#\nabla G,$$

where the V_j are smooth vectors having all derivatives uniformly bounded, and ∇ acts as ∇_{A_∞} on sections of $\mathcal{S}^+$ and as ∇_M on forms. This equation is of the same form as the equation in the proof of Proposition 3.6 of [21], and the rest of the proof is the same. Note that since we have uniform bounds on all derivatives, we do not need to require the assumption $\|(\varphi(T),A(T)) - (\varphi_\infty,A_\infty)\|_{H^k} < \varepsilon$ as in [13] and [21]. $\square$

Lemma 5.5. *Let $(\varphi_\infty, A_\infty)$ is an arbitrary critical point of the* SW *functional. For any positive integer k, there exists $\varepsilon > 0$ such that if for some $T > 0$*

$$\|(\varphi(T), A(T)) - (\varphi_\infty, A_\infty)\|_{H^k} < \varepsilon, \tag{5.9}$$

then either $\mathcal{SW}(\varphi(t), A(t)) < \mathcal{SW}(\varphi_\infty, A_\infty)$ for some $t > T$, or $(\varphi(t), A(t))$ converges in H^k to a critical point $(\varphi'_\infty, A'_\infty)$ satisfying $\mathcal{SW}(\varphi'_\infty, A'_\infty) = \mathcal{SW}(\varphi_\infty, A_\infty)$ and

$$\left\|(\varphi'_\infty, A'_\infty) - (\varphi_\infty, A_\infty)\right\|_{H^k} \leqslant c \,\|(\varphi(T), A(T)) - (\varphi_\infty, A_\infty)\|_{H^k}^{2(1-\gamma)}, \tag{5.10}$$

where γ is as in Lemma 5.3. We also have the following convergence estimate:

$$\left\|(\varphi(t), A(t)) - (\varphi'_\infty, A'_\infty)\right\|_{H^k} \leqslant c(t-T)^{-(1-\gamma)/(2\gamma-1)}. \tag{5.11}$$

(Note that the constants throughout this section and in Lemma 5.5 in particular are considered as constants along the flow, i.e., independent of time t. They can depend on the initial data (φ_0, A_0) and the manifold M. Moreover, if we assume that $(\varphi_\infty, A_\infty)$ is a limit of the flow, then $(\varphi_\infty, A_\infty) = (\varphi'_\infty, A'_\infty)$ in Lemma 5.5.)

Proof. We set

$$\Delta\mathcal{SW}(t) = \mathcal{SW}(\varphi(t), A(t)) - \mathcal{SW}(\varphi_\infty, A_\infty).$$

Then, we can assume that $\Delta\mathcal{SW}(t) \geq 0$ for all t. Otherwise, the required result is proved.

We note

$$\begin{aligned}
&\int_M |\nabla_A\varphi|^2 - |\nabla_{A_\infty}\varphi_\infty|^2 \\
&\quad = \int_M \left|\nabla_{A_\infty}\psi + \frac{1}{2}a\varphi_\infty + \frac{1}{2}a\psi\right|^2 \\
&\qquad + 2\,\mathrm{Re}\left\langle \nabla_{A_\infty}\varphi_\infty, \nabla_{A_\infty}\psi + \frac{1}{2}a\varphi_\infty + \frac{1}{2}a\psi\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \int_M |\nabla_{A_\infty}\psi|^2 + \left|\frac{1}{2}a\varphi_\infty\right|^2 + \left|\frac{1}{2}a\psi\right|^2 + 2\operatorname{Re}\left\langle \nabla_{A_\infty}\psi, \frac{1}{2}a\varphi_\infty\right\rangle \\
&\quad + 2\operatorname{Re}\left\langle \nabla_{A_\infty}\psi, \frac{1}{2}a\psi\right\rangle + 2\operatorname{Re}\left\langle \frac{1}{2}a\varphi_\infty, \frac{1}{2}a\psi\right\rangle + 2\operatorname{Re}\left\langle \nabla_{A_\infty}\varphi_\infty, \nabla_{A_\infty}\psi\right\rangle \\
&\quad + 2\operatorname{Re}\left\langle \nabla_{A_\infty}\varphi_\infty, \frac{1}{2}a\varphi_\infty\right\rangle + 2\operatorname{Re}\left\langle \nabla_{A_\infty}\varphi_\infty, \frac{1}{2}a\psi\right\rangle \\
&= \int_M |\nabla_{A_\infty}\psi|^2 + \frac{1}{4}|\varphi_\infty|^2|a|^2 + \frac{1}{4}|\psi|^2|a|^2 + \left\langle a, i\operatorname{Im}\left\langle \nabla_{A_\infty}\psi, \varphi_\infty\right\rangle\right\rangle \\
&\quad + \left\langle a, i\operatorname{Im}\left\langle \nabla_{A_\infty}\psi, \psi\right\rangle\right\rangle + \frac{1}{2}|a|^2\operatorname{Re}\left\langle \varphi_\infty, \psi\right\rangle + 2\operatorname{Re}\left\langle \nabla^*_{A_\infty}\nabla_{A_\infty}\varphi_\infty, \psi\right\rangle \\
&\quad + \left\langle a, i\operatorname{Im}\left\langle \nabla_{A_\infty}\varphi_\infty, \varphi_\infty\right\rangle\right\rangle + \left\langle a, i\operatorname{Im}\left\langle \nabla_{A_\infty}\varphi_\infty, \psi\right\rangle\right\rangle,
\end{aligned}
$$

where we again use a relationship analogous to that in Equation (4.10). It is easy to see

$$
\int_M \frac{1}{2}|F_A|^2 - \frac{1}{2}|F_{A_\infty}|^2 = \int_M \frac{1}{2}|da|^2 + \left\langle dA_\infty, da\right\rangle.
$$

We have also

$$
\int_M \frac{S}{4}|\varphi|^2 - \frac{S}{4}|\varphi_\infty|^2 = \int_M \frac{S}{4}\left(|\psi|^2 + 2\operatorname{Re}\left\langle \psi, \varphi_\infty\right\rangle\right)
$$

and

$$
\begin{aligned}
\frac{1}{8}\int_M |\varphi|^4 - |\varphi_\infty|^4 &= \frac{1}{8}\int_M \left(|\varphi_\infty|^2 + |\psi|^2 + 2\operatorname{Re}\left\langle \varphi_\infty, \psi\right\rangle\right)^2 - |\varphi_\infty|^4 \\
&= \frac{1}{8}\int_M |\psi|^4 + 4\operatorname{Re}\left\langle \varphi_\infty, \psi\right\rangle^2 + 2|\varphi_\infty|^2|\psi|^2 + 4|\varphi_\infty|^2\operatorname{Re}\left\langle \varphi_\infty, \psi\right\rangle \\
&\quad + 4|\psi|^2\operatorname{Re}\left\langle \varphi_\infty, \psi\right\rangle.
\end{aligned}
$$

Recalling that $(\varphi_\infty, A_\infty)$ satisfies the critical point Equations (1.7) and (1.8), we have

$$
\int_M \left\langle \nabla^*_{A_\infty}\nabla_{A_\infty}\varphi_\infty + \frac{S}{4}\varphi_\infty + \frac{1}{4}|\varphi_\infty|^2\varphi_\infty, \psi\right\rangle = 0
$$

and

$$
\int_M \left\langle d^*dA_\infty + i\operatorname{Im}\left\langle \nabla_{A_\infty}\varphi_\infty, \varphi_\infty\right\rangle, a\right\rangle = 0,
$$

Combining above estimates, we have

$$\begin{aligned}\Delta\mathcal{SW}(t) = \int_M & |\nabla_{A_\infty}\psi|^2 + \frac{1}{2}\operatorname{Re}\langle\varphi_\infty,\psi\rangle^2 + \frac{S}{4}|\psi|^2 + \frac{1}{4}|\varphi_\infty|^2\left(|\psi|^2+|a|^2\right)\\ &+ \frac{1}{2}|da|^2 + \langle a, i\operatorname{Im}\langle\nabla_{A_\infty}\psi,\varphi_\infty\rangle\rangle\\ &+ \langle a, i\operatorname{Im}\langle\nabla_{A_\infty}\varphi_\infty,\psi\rangle\rangle + O(3),\end{aligned} \tag{5.12}$$

where

$$\begin{aligned}O(3) = \int_M & \frac{1}{2}\left(|\psi|^2+|a|^2\right)\operatorname{Re}\langle\varphi_\infty,\psi\rangle + \langle a, i\operatorname{Im}\langle\nabla_{A_\infty}\psi,\psi\rangle\rangle\\ &+ \frac{1}{8}|\psi|^4 + \frac{1}{4}|\psi|^2|a|^2.\end{aligned}$$

Since $\Delta\mathcal{SW}(t)$ is a polynomial functional and $(A_\infty,\varphi_\infty)$ is a critical point, the first-order terms of $\Delta\mathcal{SW}(t)$ vanish. Then for ε small enough we have

$$\begin{aligned}\Delta\mathcal{SW}(T) &\leqslant c\,\|(\varphi(T),A(T)) - (\varphi_\infty,A_\infty)\|_{H^1}^2\\ &\leqslant c\,\|(\varphi(T),A(T)) - (\varphi_\infty,A_\infty)\|_{H^k}^2.\end{aligned} \tag{5.13}$$

From the continuous dependence on initial conditions (Lemma 5.2), for ε in (5.9) sufficiently small we have for $t\in[T,T+1]$,

$$\|(\varphi(t),A(t)) - (\varphi_\infty,A_\infty)\|_{H^k} < \frac{1}{2}\varepsilon_1. \tag{5.14}$$

We claim that if ε is sufficiently small, then for all $t\geq T$ we have

$$\|(\varphi(t),A(t)) - (\varphi_\infty,A_\infty)\|_{H^k} < \varepsilon_1. \tag{5.15}$$

Suppose by contradiction that $S>T$ is the smallest number such that $\|(\varphi(S),A(S)) - (\varphi_\infty,A_\infty)\|_{H^k} \geqslant \varepsilon_1$. From Lemma (5.3) we have for $T\leq t\leq S$,

$$\begin{aligned}\frac{d}{dt}(\Delta\mathcal{SW}(t))^{1-\gamma} &= -c(1-\gamma)(\Delta\mathcal{SW}(t))^{-\gamma}\left\|\left(\frac{\partial\varphi}{\partial t},\frac{\partial A}{\partial t}\right)\right\|_{L^2}^2\\ &\leq -c\left\|\left(\frac{\partial\varphi}{\partial t},\frac{\partial A}{\partial t}\right)\right\|_{L^2}.\end{aligned} \tag{5.16}$$

Integrating (5.16) in time gives

$$\int_T^S \left\| \left(\frac{\partial\varphi}{\partial t}, \frac{\partial A}{\partial t} \right) \right\|_{L^2} \leq c\Delta(\mathcal{SW}(T))^{1-\gamma}. \tag{5.17}$$

Recalling (5.13),

$$\begin{aligned} \int_T^S \left\| \left(\frac{\partial\varphi}{\partial t}, \frac{\partial A}{\partial t} \right) \right\|_{L^2} &\leqslant c \left\| (\varphi(T), A(T)) - (\varphi_\infty, A_\infty) \right\|_{H^k}^{2(1-\gamma)} \\ &\leqslant c\varepsilon^{2(1-\gamma)}. \end{aligned} \tag{5.18}$$

From (5.14) we know that $S > T + 1$, and then

$$\begin{aligned} \int_{T+1}^S \left\| \left(\frac{\partial\varphi}{\partial t}, \frac{\partial A}{\partial t} \right) \right\|_{H^k} &\geqslant \left\| \int_{T+1}^S \left(\frac{\partial\varphi}{\partial t}, \frac{\partial A}{\partial t} \right) \right\|_{H^k} \\ &\geqslant \|(\varphi(S), A(S)) - (\varphi_\infty, A_\infty)\|_{H^k} \\ &\quad - \|(\varphi(T+1), A(T+1)) - (\varphi_\infty, A_\infty)\|_{H^k} \\ &\geqslant \varepsilon_1 - \frac{1}{2}\varepsilon_1. \end{aligned}$$

Then using our results above and Lemma 5.4, we find

$$\frac{1}{2}\varepsilon_1 \leqslant c\varepsilon^{2(1-\gamma)},$$

which is impossible for ε small enough. Thus, as claimed, for ε small enough we have $\|(\varphi(t), A(t)) - (\varphi_\infty, A_\infty)\|_{H^k} < \varepsilon_1$ for all $t \geq T$.

Finally, letting $S \to \infty$ in Lemma 5.4 and (5.17) we have

$$\begin{aligned} \int_{t_1+1}^\infty \left\| \left(\frac{\partial\varphi}{\partial t}, \frac{\partial A}{\partial t} \right) \right\|_{H^k} &\leq c \int_{t_1}^\infty \left\| \left(\frac{\partial\varphi}{\partial t}, \frac{\partial A}{\partial t} \right) \right\|_{L^2} \\ &\leq c(\Delta\mathcal{SW}(t_1))^{1-\gamma} \end{aligned} \tag{5.19}$$

for any $t_1 \geq T$. From Lemma 5.1 and Lemma 5.3, we have

$$\int_{t_1}^\infty \left\| \left(\frac{\partial\varphi}{\partial t}, \frac{\partial A}{\partial t} \right) \right\|_{H^k} \to 0$$

as $t_1 \to \infty$. This establishes unique convergence of the flow in the H^k norm to a point $(\varphi'_\infty, A'_\infty)$, provided that $\|(\varphi(T), A(T)) - (\varphi_\infty, A_\infty)\|_{H^k} < \varepsilon$ for some T.

As in Lemma 5.2, it follows that $(\varphi'_\infty, A'_\infty)$ is a critical point, and it follows from Lemma 5.3 that $\mathcal{SW}(\varphi'_\infty, A'_\infty) = \mathcal{SW}(\varphi_\infty, A_\infty)$. Then from (5.19) and (5.13) we have

$$\begin{aligned} &\left\|(\varphi(T+1), A(T+1)) - (\varphi'_\infty, A'_\infty)\right\|_{H^k} \\ &\quad \leq \int_{T+1}^{\infty} \left\| \left(\frac{\partial \varphi}{\partial t}, \frac{\partial A}{\partial t} \right) \right\|_{H^k} \leq c(\Delta \mathcal{SW}(T))^{1-\gamma} \\ &\quad \leq c \left\|(\varphi(T), A(T)) - (\varphi_\infty, A_\infty)\right\|_{H^k}^{2(1-\gamma)} \\ &\quad \leq c \left\|(\varphi(T), A(T)) - (\varphi_\infty, A_\infty)\right\|_{H^k}, \end{aligned} \tag{5.20}$$

since $\gamma \in (\frac{1}{2}, 1)$. Then from Lemma 5.2,

$$\|(\varphi(T+1), A(T+1)) - (\varphi_\infty, A_\infty)\|_{H^k} \leqslant c \|(\varphi(T), A(T)) - (\varphi_\infty, A_\infty)\|_{H^k}.$$

The estimate (5.10) follows from the above two inequalities. It remains to show (5.11). As in (5.20), for $t \geq T$ we have

$$\left\|(\varphi(t+1), A(t+1)) - (\varphi'_\infty, A'_\infty)\right\|_{H^k} \leqslant c(\Delta \mathcal{SW}(t))^{1-\gamma}.$$

Then from Lemma 5.3 we have

$$\frac{d}{dt} \Delta \mathcal{SW}(t) = -c \left\| \left(\frac{\partial \varphi}{\partial t}, \frac{\partial A}{\partial t} \right) \right\|_{L^2}^2 \leqslant -c(\Delta SW(t))^{2\gamma},$$

which implies that

$$\Delta \mathcal{SW}(t) \leqslant c(t-T)^{-1/(2\gamma-1)}. \tag{5.21}$$

Thus combining the above, for $t \geq T+1$ we find

$$\left\|(\varphi(t), A(t)) - (\varphi'_\infty, A'_\infty)\right\|_{H^k} \leqslant c(t-T-1)^{-(1-\gamma)/(2\gamma-1)}. \tag{5.22}$$

Note that since the left-hand side is bounded under the flow, by adjusting the constant c if necessary, we can drop the constant 1, and (5.11) follows. $\square$

We now prove Theorem 1.2.

Proof of Theorem 1.2. From the convergence of a subsequence $\{t_k\}$ of the flow to a critical point $(\varphi_\infty, A_\infty)$ (Lemma 5.1), we know that there exists a T such that $\|(\varphi(T), A(T)) - (\varphi_\infty, A_\infty)\|_{H^k} < \varepsilon$. We can then apply Lemma 5.5. Note that in deriving (1.11), as for (5.22), by adjusting the constant c if necessary we can drop the constant T.

Finally, we show that the limit depends continuously on the initial data in the space $\{(\varphi_0, A_0) : SW(\varphi(t), A(t)) \to \lambda\}$ as $t \to \infty$. Let $(\varphi(t), A(t))$ be a solution to the flow which converges to $(\varphi_\infty, A_\infty)$ as $t \to \infty$. Let $(\varphi'(t), A'(t))$ be another solution to the flow with initial data $(\varphi'(0), A'(0))$ with

$$\lim_{t\to\infty} \mathcal{SW}(\varphi'(t), A'(t)) = \mathcal{SW}(\varphi'_\infty, A'_\infty) = \mathcal{SW}(\varphi_\infty, A_\infty).$$

From Lemma 5.5, for any $\beta_1 > 0$ there exists a $\beta_2 > 0$ such that if for some $T \geq 0$,

$$\left\|(\varphi'(T), A'(T)) - (\varphi_\infty, A_\infty)\right\|_{H^k} \leqslant \beta_2,$$

then $(\varphi'(t), A'(t))$ converges in H^k as $t \to \infty$ to a critical point $(\varphi'_\infty, A'_\infty)$, and further $\|(\varphi'_\infty, A'_\infty) - (\varphi_\infty, A_\infty)\|_{H^k} \leqslant \beta_1$. Choose T such that

$$\|(\varphi(T), A(T)) - (\varphi_\infty, A_\infty)\|_{H^k} \leqslant \frac{\beta_2}{2}.$$

From Lemma 5.2, there exists $\beta_3 > 0$ such that if

$$\left\|(\varphi'(0), A'(0)) - (\varphi(0), A(0))\right\|_{H^k} \leqslant \beta_3,$$

then $\|(\varphi(T), A(T)) - (\varphi'(T), A'(T))\|_{H^k} \leqslant \frac{\beta_2}{2}$. Applying the triangle inequality, for any $\beta_1 > 0$ there exists a $\beta_3 > 0$ such that if

$$\left\|(\varphi'(0), A'(0)) - (\varphi(0), A(0))\right\|_{H^k} \leq \beta_3,$$

then

$$\left\|(\varphi'_\infty, A'_\infty) - (\varphi_\infty, A_\infty)\right\|_{H^k} \leq \beta_1.$$

This completes the proof of Theorem 1.2. □

6. Perturbed functional

One can also consider the perturbed Seiberg–Witten equations

$$D_A\varphi = 0, \quad F_A^+ = \frac{1}{4} \langle e_j e_k \varphi, \varphi\rangle \, e^j \wedge e^k + \mu \tag{6.1}$$

and the corresponding perturbed Seiberg–Witten functional

$$\begin{aligned}\mathcal{SW}_\mu(\varphi, A) &= \int_M |D_A\varphi|^2 + \left|F_A^+ - \frac{1}{4}\langle e_j e_k \varphi, \varphi\rangle e^j \wedge e^k - \mu\right|^2. \\ &= \int_M |\nabla_A\varphi|^2 + |F_A^+|^2 + \frac{S}{4}|\varphi|^2 + \frac{1}{8}|\varphi|^4 \\ &\quad + \frac{1}{2}\langle \mu\cdot\varphi, \varphi\rangle - 2\langle F_A^+, \mu\rangle + |\mu|^2, \end{aligned} \tag{6.2}$$

where μ is some fixed imaginary-valued self-dual 2-form and $\mu\cdot\varphi$ represents Clifford multiplication. Then, we define the perturbed flow equations to be

$$\frac{\partial\varphi}{\partial t} = -\nabla_A^*\nabla_A\varphi - \frac{1}{4}\left[S + |\varphi|^2\right]\varphi - \frac{1}{2}\mu\cdot\varphi, \tag{6.3}$$

$$\frac{\partial A}{\partial t} = -d^*F_A - i\,\mathrm{Im}\,\langle\nabla_A\varphi, \varphi\rangle + d^*\mu. \tag{6.4}$$

The purpose of this section is to show that our global existence and convergence results extend to these perturbed equations. Rather than duplicate each proof, we will simply outline the differences. In Lemma 2.1, we have instead the equation

$$\frac{\partial}{\partial t}|\varphi|^2 = -\Delta|\varphi|^2 - 2|\nabla_A\varphi|^2 - \frac{1}{2}\left[S + |\varphi|^2\right]|\varphi|^2 - \mathrm{Re}\,\langle\mu\cdot\varphi, \varphi\rangle,$$

where the additional term satisfies $-\mathrm{Re}\,\langle\mu\cdot\varphi,\varphi\rangle \leqslant \left(\max_{x\in M}|\mu(x)|\right)|\varphi|^2$. Since

$$-\frac{1}{2}\left[S + \max_{x\in M}|\mu(x)| + |\varphi|^2\right]|\varphi|^2 \leq 0$$

for $|\varphi|$ sufficiently large, the same argument as before yields a uniform bound on $\sup\{|\varphi(x,t)| : x\in M\}$. The other estimates in Section 2 also continue to hold. For the proof of local existence in Section 3, we note that the additional terms are zeroth order and do not change the parabolicity of the gauge transformed equations. In Section 4, in the proof of Lemma 4.1, we have additional terms of

$$2\,\mathrm{Re}\left\langle -\frac{1}{2}\nabla_A(\mu\cdot\varphi), \nabla_A\varphi\right\rangle \leqslant c|\nabla_A\varphi|^2 + c|\nabla_A\varphi|$$

and

$$2\langle dd^*\mu, dA\rangle \leqslant c|F_A|,$$

and the lemma continues to hold. The proof of Lemma 4.2 relies only on Lemma 4.1, and is unchanged. For Lemma 4.3, noting that

$$-2 \int_M \phi^2 \operatorname{Re} \left\langle \frac{\partial \varphi}{\partial t}, \frac{1}{2}\mu \cdot \varphi \right\rangle = -\frac{d}{dt} \int_M \phi^2 \frac{1}{2} \langle \mu \cdot \varphi, \varphi \rangle,$$

the proof is entirely analogous. Lemma 4.4 continues to hold for the same reason as Lemma 4.1, as does its corollary. In Lemma 4.5, the new terms in (4.21) are multiplied by factors of R_m, and become negligible in the limit. This establishes global existence. In Section 5, the proofs of Lemmas 5.1 and 5.2 are unchanged. In Lemma 5.3, as for local existence, the additional terms are of order zero and do not affect parabolicity. Finally, in Lemma 5.4, the additional terms lead to an equation of the same form. The remaining arguments in this section are unchanged. Thus the analogues of Theorems 1.1 and 1.2 hold also for the perturbed equations (6.3) and (6.4), for an arbitrary perturbation parameter μ.

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