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Infimum of the spectrum of Laplace–Beltrami operator on a bounded pseudoconvex domain with a Kähler metric of Bergman type

Song-Ying Li and My-An Tran

The research in paper is a continuation of the work of Li and Wang [10–12] who studied upper estimates for $\lambda_1 = \lambda_1(\Delta_g)$, the bottom of the spectrum of Laplace–Beltrami operator on a complete non-compact Kähler manifold (M^n, g) with a lower bound condition on holomorphic bisectional curvature and the work of Munteanu [16] who uses lower bound condition on Ricci curvature. In this paper, we study the problems on a bounded pseudoconvex domain D in $\mathbb{C}^n$ with a certain normalized complete Kähler metrics u on D which are called Bergman-type, we find a class of Bergman-type metrics u on D so that $\lambda_1(\Delta_u) = n^2$. We also provide a simple condition on metric u, under this condition, we obtain the sharp upper bound estimates n^2 for $\lambda_1(\Delta_u)$ for such class of Bergman-type metrics, which include Kähler–Einstein metric and Bergman metric on D.

1. Introduction

Let (M^n, g) be a Kähler manifold of dimension n with Kähler metric $g = \sum_{i,j=1}^n g_{i\bar{j}} dz_i \otimes d\bar{z}_j$. Let

(1.1)
$$\Delta_g = -4 \sum_{i,j=1}^n g^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j}$$

be the Laplace–Beltrami operator with respect to the metric g, where $[g^{i\bar{j}}]^t = [g_{i\bar{j}}]^{-1}$. Let

(1.2)
$$\lambda_1(\Delta_g, M) = \inf \left\{ 4 \int_M \sum_{i,j=1}^n g^{i\bar{j}} \frac{\partial f}{\partial z_i} \frac{\partial f}{\partial \bar{z}_j} dV_g : f \in C_0^\infty(M), \|f\|_{L^2} = 1 \right\},$$

where dV_g is the volume measure on M with respect to the Kähler metric g.

When M is compact and Δ_g is uniformly elliptic, $\lambda_1(\Delta_g)$ is the first positive eigenvalue of Δ_g with Dirichlet boundary condition. A lot of research has been done on its upper and lower bound estimates and its impact on geometry and physics (see, for example, the lecture notes of Li [8] and the paper of Udagawa [17] and references therein).

When (M^n, g) is a complete non-compact Kähler manifold, $\lambda_1(\Delta_g)$ may not be an eigenvalue of Δ_g. For example, when M is the complex hyperbolic space $\mathbb{CH}^n$, $\lambda_1(\Delta_g)$ is no longer an L^2 eigenvalue of Δ_g. However, it is the infimum of the positive spectrum of Δ_g. The problem of finding estimates for $\lambda_1(\Delta_g)$ in the complete non-compact case has been studied by many mathematicians. An important upper bound estimate was obtained by Li and Wang [10]. With the assumption that the holomorphic bisectional curvature of M is bounded below by -1, they proved that $\lambda_1(\Delta_g) \leq n^2$. Their estimate is sharp and equality is achieved by the complex hyperbolic space form $\mathbb{CH}^n$. O. Munteanu obtained another estimate in [16], where he proved that $\lambda_1(\Delta_g)$ is bounded from above by n^2 if the Ricci curvature of M is bounded from below by $-2(n+1)$ (or $R_{i\bar{j}} \geq -(n+1)g_{i\bar{j}}$). His estimate is also sharp and equality is achieved by the complex hyperbolic space form $\mathbb{CH}^n$. On the other hand, the precise information on λ_1 can be used to deduce information on the geometry of manifolds. Along this line, many works have been done by Li, Wang, Ji, Kong, Zou, and several other authors (see [5, 6, 10–12], and references therein).

The main purpose of this note is to provide more examples of complete Kähler manifolds for which the precise value of λ_1 can be computed. We consider a bounded pseudoconvex domain D in $\mathbb{C}^n$ with a Kähler metric $u_{i\bar{j}} dz_i \otimes d\bar{z}_j$, where $u_{i\bar{j}} = \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}$ with u being a strictly plurisubharmonic exhaustion function for D. If D is B_n, the unit ball in $\mathbb{C}^n$, and $u(z) = -\log(1-|z|^2)$, then $u_{i\bar{j}} dz_i \otimes d\bar{z}_j$ is both the Bergman metric and the Kähler–Einstein metric on B_n. To find the exact value of $\lambda_1(\Delta_u)$ on B_n, one approach is to estimate both the upper bound and the lower bound. We first let $f(z) = (1-|z|^2)^{n/2}$. To obtain the upper bound, we apply the Rayleigh's principle, which gives $\lambda_1 \leq \int_{B_n} |\nabla f|^2 / \int_{B_n} |f|^2 = n^2$. To obtain the lower bound, we apply Proposition 9.2 in [9], which states that $\lambda_1 \geq \mu > 0$ if there exists a positive function h such that $\Delta_u h \geq \mu h$. In fact, the function f defined above satisfies $\Delta_u f \geq n^2 f$. This implies that $\lambda_1(\Delta_u) = n^2$ on B_n. For a general bounded pseudoconvex domain D, the situation is more complicated and, therefore, more subtle arguments are required to obtain the exact value of $\lambda_1(\Delta_u)$. We will impose various conditions on the exhaustion function u on D. Under these conditions, we will estimate the upper and lower bounds for λ_1 by constructing special functions and carrying out the analysis on a specific subdomain of D.

Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ with C^2 boundary. Let $r \in C^2(\mathbb{C}^n)$ be a defining function for D so that $u(z) = -\log(-r(z))$ is strictly plurisubharmonic in D. Then the complete Kähler metric induced by u is

$$u = \sum_{i,j=1}^{n} u_{i\bar{j}} dz_i \otimes d\bar{z}_j. \tag{1.3}$$

Let

$$|\partial u|_u^2 = \sum_{i,j=1}^{n} u^{i\bar{j}} \partial_i u \partial_{\bar{j}} u, \tag{1.4}$$

where $[u^{i\bar{j}}]^t = [u_{i\bar{j}}]^{-1}$. Let

$$\beta(z) = \limsup_{w \to z} |\partial u(w)|_u^2, \quad z \in \partial D. \tag{1.5}$$

We will prove the following:

Theorem 1.1. *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ with a defining function $r(z) \in C^2(\mathbb{C}^n)$. Assume that $u(z) = -\log(-r(z))$ is strictly plurisubharmonic in D with $\beta(z) = 1$ on ∂D. Then, with the notation $\lambda_1(D) = \lambda_1(\Delta_u, D)$, the following statements hold:*

(a) *$\lambda_1(D) \le \lambda_1(D \setminus K) \le n^2$ for any compact subset K of D;*

(b) *If, in addition, $r(z)$ is plurisubharmonic in D, then $\lambda_1(D) = n^2$.*

Corollary 1.1. *Let D be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with defining function $r(z)$ and $u(z) = -\log(-r(z))$. Then*

(i) *If r is strictly plurisubharmonic in D, then $\lambda_1(D) = n^2$,*

(ii) *If $\sum_{\alpha,\beta=1}^{n} u_{\alpha\bar{\beta}} dz_\alpha \otimes d\bar{z}_\beta$ is the Kähler–Einstein metric in D, then $\lambda_1 (\Delta_u, D) \le n^2$, where u is the strictly plurisubharmonic solution of Monge–Ampère equation:*

$$\det H(u) = \mathrm{e}^{(n+1)u}, \ \text{in } D; \quad u = \infty \quad \text{on } \partial D, \tag{1.6}$$

(iii) *If $\sum_{\alpha,\beta=1}^{n} u_{\alpha\bar{\beta}} dz_\alpha \otimes d\bar{z}_\beta$ is the Bergman metric on D, then $\lambda_1 (\Delta_u, D) \le n^2$, where*

$$u_{i\bar{j}} = \frac{1}{n+1} \frac{\partial^2 \log K(z,z)}{\partial z_i \partial \bar{z}_j} \tag{1.7}$$

and $K(z, w)$ is the Bergman kernel function for the domain D.

Remark 1.1. (a) *Part* (ii) *of Corollary 1.2 was proved by Munteanu in [16], we provide an alternate proof here,*

(b) *The condition $\beta(z) = 1$ on ∂D is an analysis condition, but $\beta(z) = 1$ near ∂D, has geometric interpretation related to pseudo scalar curvature for Kähler–Einstein metric, see [13, 15] for the detail.*

This paper is organized as follows: In Section 2, we will prove several theorems for a bounded pseudoconvex domain D in $\mathbb{C}^n$ with a strictly plurisubharmonic exhaustion function u that satisfies various conditions. As a consequence of those results, we will prove Theorem 1.1 there. Corollary 1.2 will be proved in Section 3. Finally, in Section 4, we will provide examples of weakly (not strictly) pseudoconvex domains for which $\lambda_1 = n^2$. Specifically, we consider the complex ellipsoid $E_m = \{z = (z_1, z_2) \in \mathbb{C}^2 : r(z) < 0\}$, where $r(z) = |z_2|^2 - (1 - |z_1|^2)^{1/m}$ and $m > 1$. With $u(z) = -\log(-r(z))$, we will prove that $\lambda_1(\Delta_u, E_m) = 4 = 2^2$.

2. Preliminary setting and main theorems

Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ with a defining function $r \in C^2(\mathbb{C}^n)$ so that $u(z) = -\log(-r(z))$ is strictly plurisubharmonic in D. We consider the Laplace–Beltrami operator Δ_u associated to the Kähler metric $u_{i\bar{j}} dz_i \otimes d\bar{z}_j$ on D, which is given by

$$\Delta_u = -4 \sum_{i,j=1}^{n} u^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j}, \tag{2.1}$$

where $[u^{i\bar{j}}]^t = H(u)^{-1} = [u_{i\bar{j}}]^{-1}$.

We start with the following lemma.

Lemma 2.1. *Let Ω be any domain in $\mathbb{C}^n$, and let $u_{i\bar{j}} dz_i \otimes d\bar{z}_j$ be any Kähler metric on Ω, where $u_{i\bar{j}} = \partial_{i\bar{j}} u$ and $u \in C^2(\Omega)$ is strictly plurisubharmonic. Let $f(z) = \mathrm{e}^{-\alpha u(z)}$ in Ω for some $\alpha > 0$. Then the following statements hold:*

(i) *The Laplacian of f is given by*

$$\Delta_u f(z) = 4\alpha f(z)\Big(n - \alpha |\partial u|_u^2\Big), \tag{2.2}$$

where

$$|\partial u|_u^2 = \sum_{i,j=1}^n u^{i\bar{j}} u_i u_{\bar{j}} = \sum_{i,j=1}^n u^{i\bar{j}} \partial_i u \partial_{\bar{j}} u, \tag{2.3}$$

(ii) *If* $r(z) = -\mathrm{e}^{-u(z)}$ *is plurisubharmonic, then* $|\partial u|_u^2 \le 1$ *on* Ω,

(iii) *Suppose that* Ω *is bounded with* $\partial\Omega \in C^1$. *Let* $h_1, h_2 \in C^2(\Omega) \cap C^1(\overline{\Omega})$. *Then*

$$\begin{aligned}\int_\Omega \Big(h_2 \Delta_u h_1 - h_1 \Delta_u h_2\Big)\, dV_u &= \int_{\partial\Omega} \bigg[h_2 \bigg(- \sum_{i,j=1}^n u^{i\bar{j}} \nu_i \frac{\partial h_1}{\partial \bar{z}_j}\bigg) \\ &\qquad - h_1 \bigg(- \sum_{i,j=1}^n u^{i\bar{j}} \nu_{\bar{j}} \frac{\partial h_2}{\partial z_i}\bigg)\bigg] g(z)\, d\sigma(z).\end{aligned} \tag{2.4}$$

In particular, if $\Delta_u h_1(z) \ge 0$ *in* Ω, $h_1(z) = 0$ *on* $\partial\Omega$, *and* $h_2(z) \ge 0$ *on* $\partial\Omega$, *then*

$$\int_\Omega \Big(h_2 \Delta_u h_1 - h_1 \Delta_u h_2\Big) dV_u \ge 0. \tag{2.5}$$

Here,

$$g(z) = \det H(u) \quad \text{and} \quad dV_u(z) = g(z)\, dv(z), \tag{2.6}$$

and $\nu(z) = (\nu_1(z), \ldots, \nu_n(z))$ *is the complex outward normal vector to* $\partial\Omega$ *so that* $|\nu(z)|^2 = 4^2$.

Proof. Note that

$$\begin{aligned}\Delta_u f(z) &= -4 f(z) \sum_{i,j=1}^n u^{i\bar{j}} [-\alpha u_{i\bar{j}} + \alpha^2 u_i u_{\bar{j}}] \\ &= 4\alpha f(z) \Big[n - \alpha |\partial u|_u^2\Big].\end{aligned}$$

Thus (i) follows. Next we prove (ii). Straightforward computation shows that

$$u_{i\bar{j}} = \frac{1}{-r}\bigg[r_{i\bar{j}} + \frac{1}{-r} r_i r_{\bar{j}}\bigg], \quad u^{i\bar{j}} = (-r)\bigg[r^{i\bar{j}} - \frac{r^i r^{\bar{j}}}{|\partial r|_r^2 - r}\bigg], \tag{2.7}$$

where

$$|\partial r|_r^2 = \sum_{i,j=1}^n r^{i\bar{j}} r_i r_{\bar{j}}, \quad r^i = \sum_{j=1}^n r^{i\bar{j}} r_{\bar{j}} \text{ and } r^{\bar{j}} = \sum_{i=1}^n r^{i\bar{j}} r_i. \tag{2.8}$$

Thus, since $|\partial r|_r^2 \geq 0$, one has

$$|\partial u|_u^2 = \frac{1}{-r}\left[|\partial r|_r^2 - \frac{|\partial r|_r^4}{|\partial r|_r^2 - r}\right] = \frac{|\partial r|_r^2}{|\partial r|_r^2 - r} \leq 1. \tag{2.9}$$

Finally, by the Divergence Theorem in complex coordinates and the fact that $\sum_{i=1}^n \partial_i(u^{i\bar{j}} g) = \sum_{j=1}^n \partial_{\bar{j}}(u^{i\bar{j}} g) = 0$, we have

$$\begin{aligned}
&\int_\Omega \Big(h_2 \Delta_u h_1 - h_1 \Delta_u h_2\Big) dV_u \\
&= \int_\Omega \Bigg[\sum_{i=1}^n \frac{\partial}{\partial z_i}\Bigg(h_2\Big(-\sum_{j=1}^n u^{i\bar{j}} g \frac{\partial h_1}{\partial \bar{z}_j}\Big)\Bigg) \\
&\qquad - \sum_{j=1}^n \frac{\partial}{\partial \bar{z}_j}\Bigg(h_1\Big(-\sum_{i=1}^n u^{i\bar{j}} g \frac{\partial h_2}{\partial z_i}\Big)\Bigg)\Bigg] dv \\
&= \int_{\partial\Omega} \Bigg[h_2(z)\Bigg(-\sum_{i,j=1}^n u^{i\bar{j}} \nu_i \frac{\partial h_1}{\partial \bar{z}_j}\Bigg) \\
&\qquad - h_1(z)\Bigg(-\sum_{i,j=1}^n u^{i\bar{j}} \nu_{\bar{j}} \frac{\partial h_2}{\partial z_i}\Bigg)\Bigg] g(z) d\sigma(z).
\end{aligned}$$

Thus, (2.4) holds. If $\Delta_u h_1 \geq 0$ in Ω and $h_1 = 0$ on $\partial\Omega$, by the Maximum Principle, we have $-\sum_{i,j=1}^n u^{i\bar{j}} \nu_i \frac{\partial h_1}{\partial \bar{z}_j} \geq 0$ on $\partial\Omega$. By the assumption $h_2 \geq 0$ on $\partial\Omega$ and (2.4), we have that (2.5) holds. Therefore, the proof of part (iii) is complete, and so is the proof of the lemma. □

As a corollary of the previous lemma and a proposition proved by Li (see Proposition 9.2 in [9]), one has

Proposition 2.1. *Let Ω be any domain in $\mathbb{C}^n$ and let $u \in C^2(\Omega)$ be strictly plurisubharmonic. If*

$$|\partial u|_u^2 \leq \beta \quad \textit{in } \Omega, \tag{2.10}$$

for some constant $\beta > 0$, then

$$\lambda_1(\Delta_u, \Omega) \geq \frac{n^2}{\beta}, \tag{2.11}$$

where $\lambda_1(\Delta_u, \Omega)$ is the infimum of the positive spectrum of Δ_u on Ω.

Proof. Let $f(z) = \mathrm{e}^{-\alpha u(z)}$. By Lemma 2.1, one has

$$\Delta_u f(z) = 4\alpha\Big(n - \alpha|\partial u|_u^2\Big) f(z) \geq 4\alpha(n - \alpha\beta) f(z), \quad z \in \Omega. \tag{2.12}$$

By the argument of the proposition of Li [9] on Ω, one has that

$$\lambda_1(\Delta_u, \Omega) \geq 4\alpha(n - \alpha\beta), \quad \alpha > 0. \tag{2.13}$$

In fact, for any $\epsilon > 0$, let $\Omega_\epsilon \subset \Omega$ be a compact subdomain of Ω such that $\partial\Omega_\epsilon \in C^\infty$ and $\Omega_\epsilon \uparrow \Omega$ as $\epsilon \to 0^+$. Let $\lambda_1(\epsilon)$ be the first positive eigenvalue for the Dirichlet problem for Δ_u with the eigenfunction $v(z)$ on Ω_ϵ. Then the regularity of v implies that v is positive in Ω_ϵ. Furthermore, $v = 0$ on $\partial\Omega_\epsilon$. By (2.5) with $h_1 = v$ and $h_2 = f$, we have

$$\Big(\lambda_1(\epsilon) - 4\alpha(n - \alpha\beta)\Big) \int_{\Omega_\epsilon} v(z) f(z)\, dV_u(z) \geq 0.$$

Thus $\lambda_1(\epsilon) \geq 4\alpha(n - \alpha\beta)$ and

$$\lambda_1(\Delta_u, \Omega) = \inf_\epsilon \lambda_1(\epsilon) \geq 4\alpha(n - \alpha\beta).$$

We know that $4\alpha(n - \alpha\beta)$ takes its maximum at $\alpha = \frac{n}{2\beta}$. Therefore,

$$\lambda_1(\Delta_u, \Omega) \geq 4\frac{n}{2\beta}\left(n - \frac{n}{2}\right) = \frac{n^2}{\beta}$$

and the proof of the proposition is complete. □

Notations. For convenience, we will use $\lambda_1(\Delta_u, D)$, $\lambda_1(D)$, and λ_1 interchangeably to denote the infimum of the positive spectrum of Δ_u on D. We let α and β denote positive constants. In addition, dV_u is the volume measure on D with respect to the Kähler metric $u_{i\bar{j}} dz_i \otimes d\bar{z}_j$, dv is the Lebesgue volume measure on $\mathbb{C}^n$, and $d\sigma$ is the Hausdorff measure on any hypersurface in D.

Let J be the Fefferman operator defined in [3]. Then

$$J(r) = -\det \begin{bmatrix} r & \overline{\partial} r \\ (\overline{\partial} r)^* & H(r) \end{bmatrix}, \tag{2.14}$$

where

$$\overline{\partial} r = [r_{\overline{1}}, \dots, r_{\overline{n}}], \quad (\overline{\partial} r)^* = [r_1, \dots, r_n]^t, \quad \text{and} \quad H(r) = [r_{i\overline{j}}]. \tag{2.15}$$

We shall prove the following:

Theorem 2.2. *If $|\partial u|_u^2 \le \beta$ on D and $r \in C^2(D) \cap C^{0,1}(\overline{D})$ with $J(r)$ being bounded on D, then*

$$\lambda_1(D) \le \beta n^2. \tag{2.16}$$

Proof. It is known [13] that

$$\det H(u) = J(r) \left(\frac{1}{-r} \right)^{n+1}, \quad dV_u = \det H(u) dv = \frac{J(r)}{(-r)^{n+1}}\, dv. \tag{2.17}$$

Let $\alpha = \frac{n}{2} + \epsilon$ with $\epsilon > 0$ very small and $f(z) = (-r(z))^\alpha$. Then

$$\begin{aligned} \int_D |f(z)|^2 dV_u = \int_D \frac{(-r(z))^{n+2\epsilon} J(r)}{(-r(z))^{n+1}}\, dv \\ = \int_D \frac{J(r)(z)}{(-r(z))^{1-2\epsilon}} dv(z) < \infty \end{aligned} \tag{2.18}$$

since $(-r(z)) \approx \operatorname{dist}(z, \partial D)$, the (Euclidean) distance of z to ∂D, when z is near ∂D. Then

$$\begin{aligned} \lambda_1 &\le \frac{(\nabla f, \nabla f)_u}{(f, f)_u} \\ &= \frac{\int_D 4 \sum_{i,j=1}^n u^{i\overline{j}} \partial_i f \partial_{\overline{j}} f dV_u}{(f, f)_u} \end{aligned}$$

$$= 4\alpha^2 \frac{\int_D (-r)^{2\alpha} |\partial u|_u^2 dV_u}{(f,f)_u}$$
$$= 4\alpha^2 \frac{\int_D |f(z)|^2 |\partial u|_u^2 dV_u}{(f,f)_u}$$
$$\leq 4\alpha^2 \beta \frac{\int_D (-r)^{2\alpha} dV_u}{(f,f)_u}$$
$$= 4\alpha^2 \beta.$$

Letting $\alpha = \frac{n}{2} + \epsilon \to \frac{n}{2}^+$, one has

$$\lambda_1 \leq n^2 \beta.$$

The proof of the theorem is complete. $\square$

For every $\epsilon > 0$, let D_ϵ be a subdomain of D defined by

$$D_\epsilon = \{z \in D : r(z) < -\epsilon\}. \tag{2.19}$$

Note that $\partial D_\epsilon \in C^2$ and $D_\epsilon \uparrow D$ as $\epsilon \to 0^+$. Then we have the following result:

Theorem 2.3. *If* $\lim_{z \to \partial D} |\partial u|_u^2 = \beta$ *and* $\int_{\partial D_t} J(r)(z) d\sigma(z)$ *is a continuous function of* t *on* $[0,1]$, *then*

$$\lambda_1(D) \leq n^2 \beta. \tag{2.20}$$

Proof. For $0 < \epsilon_1 << \epsilon < 1$, let $\eta = s/\alpha > 0$ and let

$$f(z) = \begin{cases} (-r(z) - \epsilon_1)^\alpha (\epsilon + r(z))^s & \text{if } 0 < \epsilon_1 \leq -r(z) < \epsilon, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\lim_{z \to \partial D} |\partial u|_u^2 = \beta$ and $\lim_{t \to 0} \int_{\partial D_t} J(r)(z) d\sigma(z) = \int_{\partial D} J(r)(z) d\sigma(z)$, there exists $\delta(\epsilon) > 0$ such that $|\partial u|_u^2 \leq \beta(1 + \delta(\epsilon))$ on $D \setminus D_\epsilon$ and $| \int_{\partial D_\epsilon}$

$J(r)(z)d\sigma(z) - \int_{\partial D} J(r)(z)d\sigma(z)| \le \delta(\epsilon)$ with $\lim_{\epsilon\to 0^+}\delta(\epsilon) = 0$. Thus, for $z \in D \setminus D_\epsilon$

$$\begin{aligned}\sum_{i,j=1}^n u^{i\bar{j}} f_i f_{\bar{j}} &= |f(z)|^2 \sum_{i,j=1}^n u^{i\bar{j}} (\log f)_i (\log f)_{\bar{j}} \\ &= \alpha^2 |f(z)|^2 \left(\frac{r}{r+\epsilon_1} + \eta \frac{r}{\epsilon + r}\right)^2 \sum_{i,j=1}^n u^{i\bar{j}} u_i u_{\bar{j}} \\ &\le \alpha^2 (1+\delta(\epsilon))\beta r^2 (\epsilon + r)^{2s-2} (-\epsilon_1 - r)^{2(\alpha-1)} \\ &\quad \times ((1+\eta)r + \epsilon + \eta\epsilon_1)^2.\end{aligned}$$

Let $2\alpha = n$ and $2s > 2$ and let $C = \int_{\partial D} J(r)(z)d\sigma(z)$. Then

$$\begin{aligned}&\int_{D_{\epsilon_1}\setminus D_\epsilon} \sum_{i,j=1}^n u^{i\bar{j}} f_i f_{\bar{j}}\, dV_u \\ &\quad \le \alpha^2(1+\delta(\epsilon))\beta \int_{D_{\epsilon_1}\setminus D_\epsilon} r^2 (\epsilon + r)^{2s-2} (-\epsilon_1 - r)^{2\alpha - 2} \\ &\quad\quad \times ((1+\eta)r + \epsilon + \eta\epsilon_1)^2 \det H(u)(z) dv(z) \\ &\quad = \alpha^2(1+\delta(\epsilon))\beta \int_{\epsilon_1}^{\epsilon} \int_{\partial D_t} (\epsilon - t)^{2s-2} (-\epsilon_1 + t)^{2(\alpha-1)} \\ &\quad\quad \times ((1+\eta)t - \epsilon - \eta\epsilon_1)^2 t^{-n+1} J(r) d\sigma(z) dt \\ &\quad \le \alpha^2(1+\delta(\epsilon))\beta C \int_{\epsilon_1}^{\epsilon} (\epsilon - t)^{2s-2} (t - \epsilon_1)^{2\alpha-2} ((1+\eta)t - \epsilon - \eta\epsilon_1)^2 t^{-n+1} dt \\ &\quad\quad + C\delta(\epsilon)\alpha^2(1+\delta(\epsilon))\beta \int_{\epsilon_1}^{\epsilon} (\epsilon - t)^{2s-2} (t - \epsilon_1)^{2\alpha-2} \\ &\quad\quad \times ((1+\eta)t - \epsilon - \eta\epsilon_1)^2 t^{-n+1} dt \\ &\quad = \alpha^2(1+\delta(\epsilon))\beta C \int_{\epsilon_1}^{\frac{\epsilon}{1+\eta}} (\epsilon - t)^{2s-2} (t - \epsilon_1)^{2\alpha-2} \\ &\quad\quad \times ((1+\eta)t - \epsilon - \eta\epsilon_1)^2 t^{-n+1} dt \\ &\quad\quad + \alpha^2(1+\delta(\epsilon))\beta C \int_{\frac{\epsilon}{1+\eta}}^{\epsilon} (\epsilon - t)^{2s-2} (t - \epsilon_1)^{2\alpha-2} \\ &\quad\quad \times ((1+\eta)t - \epsilon - \eta\epsilon_1)^2 t^{-n+1} dt \\ &\quad\quad + C\delta(\epsilon)\alpha^2(1+\delta(\epsilon))\beta \int_{\epsilon_1}^{\epsilon} (\epsilon - t)^{2s-2} (t - \epsilon_1)^{2\alpha-2} \\ &\quad\quad \times ((1+\eta)t - \epsilon - \eta\epsilon_1)^2 t^{-n+1} dt\end{aligned}$$

$$
\begin{aligned}
&\le \alpha^2(1+\delta(\epsilon))\beta C\epsilon^{2s}\int_{\epsilon_1}^{\frac{\epsilon}{1+\eta}} t^{-1}dt + \alpha^2(1+\delta(\epsilon))\beta C(1+\eta^2)\epsilon^{2s}\int_{\frac{\epsilon}{1+\eta}}^{\epsilon} t^{-1}dt \\
&\quad + C\delta(\epsilon)\alpha^2(1+\delta(\epsilon))\beta(1+\eta^2)\epsilon^{2s}\int_{\epsilon_1}^{\epsilon} t^{-1}dt \\
&\le C\alpha^2(1+\delta(\epsilon))\beta \\
&\quad \times \left[\epsilon^{2s}\log\left(\frac{\epsilon}{\epsilon_1}\right) + (1+\eta^2)\epsilon^{2s}\log(1+\eta) + \delta(\epsilon)(1+\eta^2)\epsilon^{2s}\log\left(\frac{\epsilon}{\epsilon_1}\right)\right]
\end{aligned}
$$

and for $\epsilon_1 << \epsilon_2 \le \frac{1}{2}\epsilon$

$$
\begin{aligned}
&\int_{D_{\epsilon_1}\setminus D_\epsilon} |f(z)|^2 dV_u(z) \\
&= \int_{D_{\epsilon_1}\setminus D_\epsilon} (\epsilon+r)^{2s}(-\epsilon_1-r)^{2\alpha}(-r)^{-n-1}J(r)dv(z) \\
&= \int_{\epsilon_1}^{\epsilon}\int_{\partial D_t} (\epsilon-t)^{2s}(-\epsilon_1+t)^{2\alpha}t^{-n-1}J(r)d\sigma(z)dt \\
&\ge C\left[\int_{\epsilon_1}^{\epsilon}(\epsilon-t)^{2s}(t-\epsilon_1)^{2\alpha}t^{-n-1}dt \ -\delta(\epsilon)\int_{\epsilon_1}^{\epsilon}(\epsilon-t)^{2s}(t-\epsilon_1)^{2\alpha}t^{-n-1}dt\right] \\
&\ge C\left[(\epsilon-\epsilon_2)^{2s}\int_{\epsilon_1}^{\epsilon_2}(t-\epsilon_1)^n t^{-1-n}dt - \delta(\epsilon)\epsilon^{2s}\int_{\epsilon_1}^{\epsilon} t^{-1}dt\right] \\
&= C\left[(\epsilon-\epsilon_2)^{2s}\left[\sum_{k=1}^{n} C_k^n(-\epsilon_1)^k\frac{1}{k}\left(\frac{1}{\epsilon_1^k}-\frac{1}{\epsilon_2^k}\right)+\log\frac{\epsilon_2}{\epsilon_1}\right] - \delta(\epsilon)\epsilon^{2s}\int_{\epsilon_1}^{\epsilon} t^{-1}dt\right] \\
&= C\left[(\epsilon-\epsilon_2)^{2s}\left[\sum_{k=1}^{n} C_k^n(-1)^k\frac{1}{k}\left(1-\frac{\epsilon_1^k}{\epsilon_2^k}\right)+\log\left(\frac{\epsilon_2}{\epsilon_1}\right)\right] - \delta(\epsilon)\epsilon^{2s}\log\left(\frac{\epsilon}{\epsilon_1}\right)\right].
\end{aligned}
$$

Therefore, with $\epsilon_2 = \epsilon^2$,

$$
\lim_{\epsilon_1\to 0^+}\frac{4\int_{D_{\epsilon_1}\setminus D_\epsilon} u^{i\bar{j}}f_i f_{\bar{j}}dV_u}{\int_{D_{\epsilon_1}\setminus D_\epsilon}|f(z)|^2 dV_u} \le 4\alpha^2\Big(1+C_0\delta(\epsilon)\Big)\beta\frac{1}{(1-\epsilon)^{2s}-\delta(\epsilon)},
$$

where C_0 is a positive constant independent of ϵ. By the domain monotonicity of eigenvalues, for any $\epsilon > 0$, one has that

$$
\lambda_1(D) \le \lambda_1(D\setminus D_\epsilon) \le 4\alpha^2\Big(1+C_0\delta(\epsilon)\Big)\beta\frac{1}{(1-\epsilon)^{2s}-\delta(\epsilon)}.
$$

Let $\alpha = \frac{n}{2}$ and then let $\epsilon \to 0^+$. Thus, one has

$$\lambda_1(D) \le n^2 \beta.$$

The proof of the theorem is complete. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1.

Proof. Part (a) of Theorem 1.1 follows from Theorems 2.3 and 2.4.

To prove part (b) of the theorem, since r is plurisubharmonic in D, without loss of generality, we may assume $H(r)(z)$ is positive definite for $z \in D$ — otherwise, we may use $r_1(z) = r(z) + \epsilon(|z|^2 - A)$ to replace r — and carry out the following computation.

By (2.9), one has that

$$|\partial u|_u^2 = \frac{|\partial r|_r^2}{-r + |\partial r|_r^2} \le 1.$$

By Proposition 2.2, one has $\lambda_1(D) \ge n^2$. Therefore, part (b) of Theorem 1.1 follows.

Finally, by (2.9), one has

$$|\partial u|_u^2 = \frac{|\partial r|_r^2}{-r + |\partial r|_r^2}.$$

This implies that

$$\beta(z) = \limsup_{w \to z} |\partial u(w)|_u^2 = 1, \quad z \in \partial D,$$

and $|\partial u|_u^2 \le 1$ on D. By parts (a) and (b), one has proved part (c). Therefore, the proof of Theorem 1.1 is complete. □

3. Proof of Corollary 1.2

We are now ready to prove Corollary 1.2.

Proof. (i) We first consider the Laplace–Beltrami operator in Kähler–Einstein metric. Let u be strictly plurisubharmonic in D so that

$$\begin{cases} \det H(u) = \mathrm{e}^{(n+1)u} & \text{in } D, \\ u = +\infty & \text{on } \partial D \end{cases} \tag{3.1}$$

and let

$$r(z) = -\mathrm{e}^{-u(z)}. \tag{3.2}$$

Then $J(r) = 1$ and by Cheng and Yau [2], Lee and Melrose [7], one has that $r(z) \in C^{n+2-\epsilon}(\overline{D})$ for any $\epsilon > 0$. Thus $\partial r \neq 0$ on ∂D and

$$\det H(r) = \mathrm{e}^{u}\Big(1 - |\partial u|_u^2\Big) \quad \text{on } D. \tag{3.3}$$

Since $\det H(r)(z)$ is bounded on $\overline{D}$ and $u(z) \to +\infty$ as $z \to \partial D$, this implies that

$$\lim_{z \to \partial D} |\partial u|_u^2 = 1.$$

Applying Theorem 2.4 with $\beta = 1$, one has that $\lambda_1(D) \leq n^2$.

(ii) Let K be the Bergman kernel function, and let $u(z) = \frac{1}{n+1} \log K(z,z)$. Then $u(z)$ is strictly plurisubharmonic in D. Let $r(z) = -\mathrm{e}^{-u(z)}$. Then $r \in C^{n+2-\epsilon}(\overline{D})$ is a defining function for D by the result of Fefferman [4]. Let $\rho \in C^{\infty}(\overline{D})$ be any strictly plurisubharmonic defining function for D. By Fefferman [4], one has that

$$u(z) = -\log(-\rho(z)) + b(z),$$

where $b \in C^{n+2-\epsilon}(\overline{D})$. Then

$$[u^{i\bar{j}}] = [(-\log(-\rho))^{i\bar{j}}](I_n + \rho B),$$

where B is an $n \times n$ matrix with bounded entries near ∂D. Let $u^0 = -\log(-\rho)$. Then by (2.9), $\lim_{z \to \partial D} |\partial u^0|_{u^0}^2 = 1$. It is easy to see that

$$|\partial u^0|_{u^0}^2 (1 + C\rho) \leq |\partial u|_u^2 \leq |\partial u^0|_{u^0}^2 (1 - C\rho)$$

for some $C >> 1$ and z near ∂D. Therefore, $\lim_{z \to \partial D} |\partial u|_u^2 = 1$. Applying Theorem 2.4 with $\beta = 1$, one has that $\lambda_1(D) \leq n^2$ for the Laplace–Beltrami operator in Bergman metric. Therefore, the proof of Corollary 1.2 is complete. □

4. Weakly pseudoconvex domains with $\lambda_1 = n^2$

In this section, we consider the complex ellipsoid in $\mathbb{C}^2$ $(n = 2)$

$$E_m = \{(z_1, z_2) : |z_2|^{2m} + |z_1|^2 < 1\}, \quad m > 1, \tag{4.1}$$

which is a weakly (not strongly) pseudoconvex domain in $\mathbb{C}^2$. We let

$$r(z) = |z_2|^2 - (1 - |z_1|^2)^{1/m}. \tag{4.2}$$

Then $r(z) \in C^\infty(E_m) \cap C^{1/m}(E_m)$ is a strictly plurisubharmonic defining function for E_m (see [14] for more details) with

$$H(r) = \begin{bmatrix} \frac{(1-|z_1|^2)^{1/m-2}}{m^2}\Big(m - |z_1|^2\Big) & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.3}$$

Let

$$u(z) = -\log(-r(z)), \quad z \in E_m. \tag{4.4}$$

Then the following holds:

Proposition 4.1. *Let Δ_u be the Laplace–Beltrami operator associated to the Kähler metric $u_{i\bar{j}} dz_i \otimes d\bar{z}_j$ on E_m. Then $\lambda_1(\Delta_u) = 4 = 2^2$.*

Proof. By (4.3), one has

$$\begin{aligned} |\partial r|_r^2 &= \frac{m(1-|z_1|^2)^{2-1/m}}{1 - \frac{|z_1|^2}{m}} \frac{1}{m^2}(1-|z_1|^2)^{2/m-2}|z_1|^2 + |z_2|^2 \\ &= \frac{(1-|z_1|^2)^{1/m}}{m - |z_1|^2}|z_1|^2 + |z_2|^2, \end{aligned} \tag{4.5}$$

$$\begin{aligned} |\partial r|_r^2 - r(z) &= (1-|z_1|^2)^{1/m}\left(1 + \frac{|z_1|^2}{m - |z_1|^2}\right) \\ &= \frac{m(1-|z_1|^2)^{1/m}}{m - |z_1|^2}, \end{aligned} \tag{4.6}$$

$$
\begin{aligned}
|\partial u|_u^2 &= \frac{|\partial r|_r^2}{|\partial r|_r^2 - r} = \frac{m|z_2|^2 - r(z)|z_1|^2}{m(1-|z_1|^2)^{1/m}} \\
&= 1 + r(z)\frac{m-|z_1|^2}{m(1-|z_1|^2)^{1/m}}
\end{aligned}
\tag{4.7}
$$

and

$$
\begin{aligned}
J(r) &= \det H(r)\Big[|\partial r|_r^2 - r\Big] \\
&= \frac{(1-|z_1|^2)^{1/m-2}}{m}\left[1 - \frac{|z_1|^2}{m}\right] \\
&\quad \times \left[(1-|z_1|^2)^{1/m} + \frac{(1-|z_1|^2)^{1/m}}{m-|z_1|^2}|z_1|^2\right] \\
&= \frac{(1-|z_1|^2)^{2/m-2}}{m}.
\end{aligned}
\tag{4.8}
$$

Moreover,

$$
r^{1\bar{1}} = \frac{m^2}{m-|z_1|^2}(1-|z_1|^2)^{2-1/m}, \quad r^{2\bar{2}} = 1, \quad r^{1\bar{2}} = 0
\tag{4.9}
$$

and

$$
r^1 = \frac{m}{m-|z_1|^2}(1-|z_1|^2)z_1, \quad r^2 = z_2.
\tag{4.10}
$$

Therefore,

$$
\begin{aligned}
[u^{i\bar{j}}] &= (-r(z))\left[r^{i\bar{j}} - \frac{r^i r^{\bar{j}}}{|\partial r|_r^2 - r}\right] \\
&= (-r(z))\begin{bmatrix} m(1-|z_1|^2)^{2-1/m} & -(1-|z_1|^2)^{1-1/m} z_1\bar{z}_2 \\ -(1-|z_1|^2)^{1-1/m} z_2\bar{z}_1 & 1 - \frac{(m-|z_1|^2)}{m}\frac{|z_2|^2}{(1-|z_1|^2)^{1/m}} \end{bmatrix}.
\end{aligned}
\tag{4.11}
$$

Let

$$
h(z) = \beta\log(1-|z_1|^2) \quad \text{and} \quad f = \mathrm{e}^{-\alpha u + h},
\tag{4.12}
$$

where α and β are positive constants. Then

$$\begin{aligned}
&\sum_{i,j=1}^{2} u^{i\bar{j}} u_i h_{\bar{j}} \\
&= \beta\Big[m(1-|z_1|^2)^{2-1/m}\frac{1}{m}(1-|z_1|^2)^{1/m-1}\bar{z}_1 - (1-|z_1|^2)^{1-1/m} z_2\bar{z}_1\bar{z}_2\Big] \\
&\quad \times \Big[\frac{-z_1}{1-|z_1|^2}\Big] \\
&= -\beta|z_1|^2\bigg(1-\frac{|z_2|^2}{(1-|z_1|^2)^{1/m}}\bigg) \\
&= -\beta|z_1|^2(1-|z_1|^2)^{-1/m}(-r(z))
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i,j=1}^{2} u^{i\bar{j}} h_{i\bar{j}} &= m\beta r(1-|z_1|^2)^{1-1/m}\Big[(1-|z_1|^2)^{-1}|z_1|^2+1\Big] \\
&= m\beta r(1-|z_1|^2)^{-1/m}|z_1|^2 + m\beta r(1-|z_1|^2)^{1-1/m}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i,j=1}^{2} u^{i\bar{j}} h_i h_{\bar{j}} &= (-r)m(1-|z_1|^2)^{2-1/m}\beta^2(1-|z_1|^2)^{-2}|z_1|^2 \\
&= m\beta^2(-r)|z_1|^2(1-|z_1|^2)^{-1/m}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\mathrm{e}^{\alpha u-h}\Delta_u\Big(\mathrm{e}^{-\alpha u+h}\Big) \\
&\quad = 4\alpha n - 4\alpha^2|\partial u|_u^2 - 4\sum_{i,j=1}^{2} u^{i\bar{j}}(h_{i\bar{j}}+h_i h_{\bar{j}}) + 4\alpha 2\mathrm{Re}\sum_{i,j=1}^{2} u^{i\bar{j}} u_i h_{\bar{j}} \\
&\quad = 4\alpha(n-\alpha) + 4\alpha^2\frac{-r(z)}{(1-|z_1|^2)^{1/m}}\frac{m-|z_1|^2}{m} \\
&\qquad + 4m\beta\frac{-r(z)}{(1-|z_1|^2)^{1/m}}(1-\beta|z_1|^2) - 8\alpha\beta|z_1|^2(1-|z_1|^2)^{-1/m}(-r(z)) \\
&\quad = 4\alpha(n-\alpha) + 4\frac{-r(z)}{(1-|z_1|^2)^{1/m}}\Big[\beta m(1-\beta|z_1|^2) - 2\beta\alpha|z_1|^2 + \alpha^2\frac{m-|z_1|^2}{m}\Big]
\end{aligned}$$

$$= 4\alpha(n-\alpha) + 4\frac{-r(z)}{(1-|z_1|^2)^{1/m}}\left[\beta m + \alpha^2 - (m\beta^2 + 2\beta\alpha + \frac{\alpha^2}{m})|z_1|^2\right]$$

and

$$\begin{aligned}
(4.13)\quad & e^{2(\alpha u-h)}|\partial(e^{-\alpha u+h})|_u^2 \\
&= \alpha^2|\partial u|_u^2 - \alpha u^{i\bar{j}}(u_i h_{\bar{j}} + u_{\bar{j}} h_i) + u^{i\bar{j}} h_i h_{\bar{j}} \\
&= \alpha^2 - \alpha^2 \frac{-r(z)}{(1-|z_1|^2)^{1/m}} \frac{m-|z_1|^2}{m} \\
&\quad + m\beta^2 \frac{-r(z)}{(1-|z_1|^2)^{1/m}}|z_1|^2 + 2\alpha\beta|z_1|^2(1-|z_1|^2)^{-1/m}(-r(z)) \\
&= \alpha^2 + \frac{-r(z)}{(1-|z_1|^2)^{1/m}}\left[-\alpha^2\frac{m-|z_1|^2}{m} + m\beta^2|z_1|^2 + 2\alpha\beta|z_1|^2\right] \\
&= \alpha^2 + \frac{-r(z)}{(1-|z_1|^2)^{1/m}}\left[-\alpha^2 + \frac{|z_1|^2}{m}(\alpha^2 + m^2\beta^2 + 2m\alpha\beta)\right].
\end{aligned}$$

Let $2\alpha > n$ and $2\beta = 1$ with $n = 2$. Then

$$\begin{aligned}
& m\int_{E_m} |f(z)|^2(-r(z))^{-n-1}J(r)dv(z) \\
&= \int_{|z_1|<1}\int_{|z_2|^2<(1-|z_1|^2)^{1/m}} \Big((1-|z_1|^2)^{1/m} - |z_2|^2\Big)^{2\alpha-n-1} \\
&\quad \times (1-|z_1|^2)^{2/m+2\beta-2} dA(z_2)dA(z_1) \\
&= \pi\int_{|z_1|<1}(1-|z_1|^2)^{2/m+2\beta-2} \\
&\quad \times \int_0^{(1-|z_1|^2)^{1/m}} \Big((1-|z_1|^2)^{1/m} - t\Big)^{2\alpha-n-1} dt dA(z_1) \\
&= \pi\int_{|z_1|<1}(1-|z_1|^2)^{2/m+2\beta-2}\frac{1}{2\alpha-n}\Big((1-|z_1|^2)^{1/m}\Big)^{2\alpha-n} dA(z_1) \\
&= \frac{\pi}{2\alpha-n}\int_{|z_1|<1}(1-|z_1|^2)^{\frac{2+2\alpha-n}{m}+2\beta-2} dA(z_1) \\
&= \frac{\pi^2}{2\alpha-n}\int_0^1 (1-t)^{\frac{2\alpha}{m}+2\beta-2}\, dt \\
&= \frac{m\pi^2}{(2\alpha-n)2\alpha}.
\end{aligned}$$

Moreover,

$$
\begin{aligned}
& m\int_{E_m} |f(z)|^2 \frac{(-r(z))}{(1-|z_1|^2)^{1/m}} \\
& \times \quad \Big[-\alpha^2 + \frac{|z_1|^2}{m}(\alpha^2+m^2\beta^2+2m\alpha\beta)\Big](-r(z))^{-n-1}J(r)(z)dv(z) \\
&= \int_{|z_1|<1}\int_{|z_2|^2<(1-|z_1|^2)^{1/m}} \Big((1-|z_1|^2)^{1/m}-|z_2|^2\Big)^{2\alpha-n} \\
&\quad \times \frac{(\alpha^2+m^2/4+m\alpha)|z_1|^2}{m(1-|z_1|^2)^{1-1/m}} dA(z_2)dA(z_1) \\
&\quad - \int_{|z_1|<1}\int_{|z_2|^2<(1-|z_1|^2)^{1/m}} \Big((1-|z_1|^2)^{1/m}-|z_2|^2\Big)^{2\alpha-n} \\
&\quad \times \frac{\alpha^2}{(1-|z_1|^2)^{1-1/m}} dA(z_2)dA(z_1) \\
&= \frac{\pi}{2\alpha-1}\int_{|z_1|<1}(1-|z_1|^2)^{\frac{2\alpha}{m}-1}(\alpha^2/m+m/4+\alpha)|z_1|^2 dA(z_1) \\
&\quad - \frac{\pi\alpha^2}{2\alpha-1}\int_{|z_1|<1}(1-|z_1|^2)^{\frac{2\alpha}{m}-1}dA(z_1) \\
&= \frac{\pi^2}{2\alpha-1}\int_0^1 (1-t)^{\frac{2\alpha}{m}-1}(\alpha^2/m+m/4+\alpha)t\,dt - \frac{\alpha^2\pi^2}{2\alpha-1}\frac{m}{2\alpha} \\
&= \frac{\pi^2}{(2\alpha-1)2\alpha}\int_0^1 (1-t)^{\frac{2\alpha}{m}}(\alpha^2+m^2/4+m\alpha)dt - \frac{\alpha m\pi^2}{2(2\alpha-1)} \\
&= \frac{\pi^2 m}{(2\alpha-1)2\alpha(2\alpha+m)}(\alpha^2+m^2/4+m\alpha) - \frac{\alpha m\pi^2}{2(2\alpha-1)}.
\end{aligned}
$$

So (4.13) implies that

$$
\begin{aligned}
& m\int_{E_m} |\nabla f(z)|^2(-r(z))^{-n-1}J(r)dv(z) \\
&= 4\alpha^2 m\int_{E_m} |f(z)|^2(-r(z))^{-n-1}J(r)dv(z) \\
&\quad + 4\Big[\frac{\pi^2 m}{(2\alpha-1)2\alpha(2\alpha+m)}(\alpha^2+m^2/4+m\alpha) - \frac{\alpha m\pi^2}{2(2\alpha-1)}\Big].
\end{aligned}
$$

Thus

$$\lambda_1 \le 4\alpha^2 + 4\frac{(2\alpha-2)2\alpha}{m\pi^2} \times \left[\frac{\pi^2 m}{(2\alpha-1)2\alpha(2\alpha+m)}\left(\alpha^2+\frac{m^2}{4}+m\alpha\right)-\frac{\alpha m\pi^2}{2(2\alpha-1)}\right]$$
$$= 4\alpha^2 + 4\left[\frac{(\alpha+m/2)}{(2\alpha-1)\alpha 2}-\frac{\alpha}{(2\alpha-1)}\right](2\alpha-2)\alpha.$$

Letting $2\alpha \to 2^+$, we have

$$\lambda_1 \le 4.$$

On the other hand, since $|\partial u|_u^2 \le 1$ in E_m, Proposition 2.2 implies that

$$\lambda_1 \ge 2^2 = 4.$$

Therefore, $\lambda_1 = 2^2$, and the proof of the proposition is complete. □

Next we will make a remark and pose a question. Note that

$$\log\det H(u)(z) = (n+1)u + \log J(r)(z)$$
$$= (n+1)u - 2\frac{m-1}{m}\log(1-|z_1|^2) - \log m,$$

where $n = 2$ and $m \ge 1$, one has the Ricci curvature

$$R_{i\bar{j}} = -\frac{\partial^2 \log\det H(u)(z)}{\partial z_i \partial \bar{z}_j}$$
$$= -(n+1)u_{i\bar{j}} + \frac{2(m-1)}{m}\frac{-1}{(1-|z_1|^2)^2}\delta_{1i}\delta_{1j},$$

where $n = 2$ and $m \ge 1$. This leads to the following remark and question.

Remark 4.1. (i) $R_{i\bar{j}} \le -(n+1)g_{i\bar{j}}$ *on* E_m*;*

(ii) $R_{i\bar{j}} \ge -(n+1)g_{i\bar{j}}$ *if and only if* $m = 1$ *and* $E_m = B_2$.

Problem 4.1. Let D be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ with a negative defining function $r(z)$ so that $u(z) = -\log(-r(z))$ is strictly plurisubharmonic in D induced a Kähler metric $u = \sum_{i,j=1}^n u_{i\bar{j}} dz_i \otimes d\bar{z}_j$. Assume $R_{i\bar{j}} \ge -(n+1)g_{i\bar{j}}$ on (D, u) and $\lambda_1 = n^2$. What can one say about D?

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Department of Mathematics
University of California
Irvine
CA 92697–3875, USA
E-mail address: sli@math.uci.edu
E-mail address: mtran@math.uci.edu

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