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On multiply twisted knots that are Seifert fibered or toroidal

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We consider knots whose diagrams have a high amount of twisting of multiple strands. By encircling twists on multiple strands with unknotted curves, we obtain a link called a generalized augmented link. Dehn filling this link gives the original knot. We classify those generalized augmented links that are Seifert fibered, and give a torus decomposition for those that are toroidal. In particular, we find that each component of the torus decomposition is either "trivial," in some sense, or homeomorphic to the complement of a generalized augmented link. We show this structure persists under high Dehn filling, giving results on the torus decomposition of knots with generalized twist regions and a high amount of twisting. As an application, we give lower bounds on the Gromov norms of these knot complements and of generalized augmented links.

1. Introduction

This paper continues a program to understand the geometry of knot and link complements in S^3, given only a diagram of the knot or link. Each knot complement decomposes uniquely along incompressible tori into hyperbolic and Seifert fibered pieces, by work of Jaco–Shalen [16] and Johannson [17]. By Mostow–Prasad rigidity, the metric on the hyperbolic pieces is unique. Thus this geometric information on the complement is completely determined by a diagram of the knot. However, reading geometric information of a diagram seems to be difficult.

In recent years, techniques have been developed to relate geometric properties to a diagram for classes of knots and links admitting particular types of diagrams, such as alternating [19], and highly twisted knots and links [13, 22, 23]. However, many links of interest to knot theorists and hyperbolic geometers do not admit these types of diagrams. These include Berge knots [4–6], twisted torus knots and Lorenz knots [7], which contain many of the smallest volume hyperbolic knots [10]. These knots admit diagrams that are highly non-alternating, that have few twists per twist region, but contain regions where multiple strands of the diagram twist around each other some number of times. The ideas of this paper and a companion paper [21] grew out of a desire to understand geometric properties of these "multiply twisted" knots and links, given only a diagram. The results here give a first step towards such an understanding.

In this paper, we consider a class of multiply twisted knots in S^3, described below, and classify those which are *not* hyperbolic. We completely determine those which are Seifert fibered, and describe the unique torus decomposition (the JSJ decomposition) for those which are toroidal. Recall that a knot or link is *toroidal* if its complement contains an embedded essential torus. We obtain our results by *augmenting* diagrams of the multiply twisted knots, that is, encircling regions of the diagram where multiple strands twist about each other by a simple closed curve, called a *crossing circle*. This generalizes a construction of Adams [1]. Since all knots are obtained by Dehn filling some generalized augmented link, the geometric properties of these links are also interesting. In [21], we describe geometric properties of those generalized augmented links which are known to be hyperbolic. Combining these results leads to the JSJ decomposition of certain multiply twisted knots.

To state our results carefully, we need some definitions.

1.1. Generalized augmented links

First, although our main results relate to knots in S^3, in fact many results in this paper apply more generally to knots and links in a 3-manifold M. We will assume M is compact, orientable, with (possibly empty) boundary consisting of tori, and M admits an orientation reversing involution fixing a surface S. For example, S^3 is such a manifold, taking S to be a separating 2-sphere. A solid torus V is another example, with S a Möbius band or annulus.

Let K be a knot or link in M that can be ambient isotoped into a neighborhood of S. We define a *diagram* of the knot or link K with respect to the surface S to be a projection of K to S yielding a 4-valent graph on S with over–under crossing information at each vertex.

Given a diagram, we may define twisting, twist regions, and generalized twist regions exactly as in [21], whether or not our link is in S^3. We review the definitions briefly here. More precise statements are found in [21].

A *twist region* of a diagram is a region in which two strands twist about each other maximally, as in figure 1(a). Note that the two strands bound a "ribbon surface" between them. A *generalized twist region* is a region of a

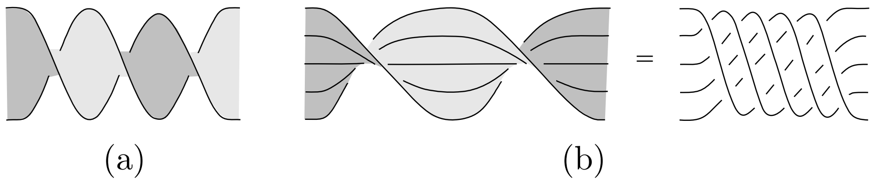

Figure 1: (a) A twist region. (b) A generalized twist region. Multiple strands lie on the twisted ribbon surface.

diagram in which multiple strands twist about each other maximally, as in figure 1(b). Note that all strands lie on the ribbon surface bounded between the outermost strands. A *half-twist* of a generalized twist region consists of a single crossing of the two outermost strands, which flips the ribbon surface over once. Figure 1(b) shows a single full-twist, or two half-twists of five strands.

We may group all crossings of a diagram into generalized twist regions, so that each crossing is contained in exactly one generalized twist region. This is called a *maximal twist region selection*, and is not necessarily unique. For example, in figure 1(b), we could group all crossings into one generalized twist region on five strands, or group each crossing into its own twist region on two strands.

Given a maximal twist region selection, at each generalized twist region insert a *crossing circle*, i.e., a simple closed curve C_i encircling the strands of the generalized twist region, bounding a disk D_i perpendicular to the projection plane of the diagram. The complement of the resulting link is homeomorphic to the complement of the link whose diagram is obtained by removing all full-twists from each generalized twist region. This is illustrated in figure 2. The resulting link, with crossing circles added and all full-twists removed, is defined to be a *generalized augmented link*. We will always assume that such a link contains at least one crossing circle, to avoid trivial cases.

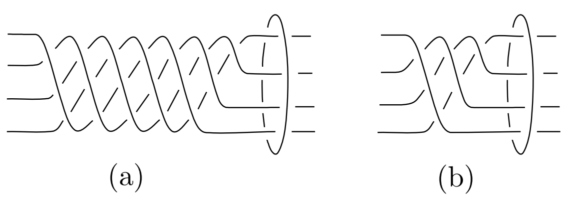

Figure 2: (a) Encircle each twist region with a crossing circle. (b) Link L given by removing full-twists from the diagram.

Note that if L is a generalized augmented link, obtained by augmenting a knot K in M, then $M \smallsetminus K$ is obtained from $M \smallsetminus L$ by Dehn filling. Let $\mathcal{N}(C_i)$ denote a small embedded tubular neighborhood of C_i in M. Let μ_i be (the isotopy class of) the meridian of $\mathcal{N}(C_i)$ (i.e., μ_i bounds a disk in $\mathcal{N}(C_i)$), and let $\lambda_i = \partial D_i$ be the longitude. Suppose n_i full-twists were removed at C_i to go from the diagram of $K \cup (\cup C_j)$ to that of L. Then Dehn filling along the slope $\mu_i + n_i\lambda_i$ on $\partial\mathcal{N}(C_i)$, for each i, yields $M \smallsetminus K$. See, for example, Rolfsen [26] for a more complete description of this process.

We refer to this type of Dehn filling as *twisting* along the disk D_i, or along C_i, or, when D_i or C_i are understood, simply as *twisting*. Note any link in S^3 is obtained by twisting some generalized augmented link.

Finally, we wish to use diagrams of knots that do not involve unnecessary twisting. That is, we wish them to be *reduced* in the sense of the following definition, which we will use to generalize Lackenby's definition of twist reduced [19], and Menasco–Thistlethwaite's definition of standard [20].

Definition 1.1. A generalized augmented link with knot strands K_j and crossing circles C_i is said to be *reduced* if the following hold:

(1) (Minimality of twisting disks) The twisting disk D_i with boundary C_i intersects $\cup K_j$ in m_i points, where $m_i \geq 2$, and m_i is minimal over all disks in M bounded by C_i. That is, if E_i is another disk embedded in M with boundary C_i, disjoint from all other crossing circles, then $|E_i \cap (\cup K_j)| \geq m_i$.

(2) (No redundant twisting) There is no annulus embedded in $M \smallsetminus L$ with one boundary component isotopic to ∂D_j on $\partial\mathcal{N}(C_j)$ and the other isotopic to ∂D_i on $\partial\mathcal{N}(C_i)$, $i \neq j$.

(3) (No trivial twisting) There is no annulus embedded in $M \smallsetminus L$ with one boundary component isotopic to ∂D_j on $\partial\mathcal{N}(C_j)$ and the other boundary component on ∂M.

These conditions allow us to rule out unnecessary crossing circles and generalized twist regions. For example, condition (1) prohibits "nugatory" twist regions, such as those shown in figure 3. Condition (2) rules out redundant generalized twist regions and their associated crossing circles, such as shown in figure 4, where two crossing circles encircle the same generalized twist region.

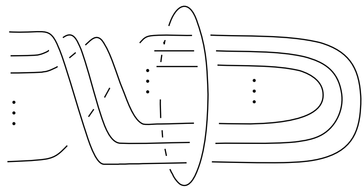

Figure 3: Untwist at a homotopically trivial crossing circle.

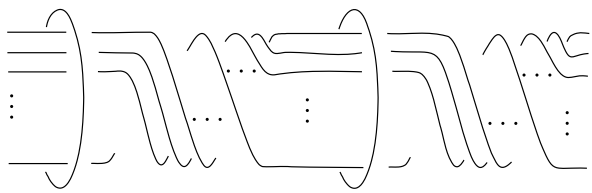

Figure 4: Concatenate twists when crossing circles bound an annulus.

1.2. Results

We are now ready to state the main results of this paper.

In Section 3, we classify all reduced augmented links which are Seifert fibered. This is the content of Theorem 3.1. As a consequence, we obtain the following result.

Corollary 3.2. *Let K be a knot in S^3 which has a diagram D whose augmentation is a Seifert fibered reduced augmented link. Then K is a $(2, q)$ torus knot.*

In Section 4, we describe a torus decomposition of generalized augmented links. In particular, in Theorem 4.1, we show that components of such a torus decomposition are either in some sense trivial, or atoroidal reduced augmented links. Applying this to links in S^3, using results on hyperbolic generalized augmented links of [21], we obtain a torus decomposition for all links in S^3 obtained by high Dehn fillings of toroidal generalized augmented links. This is Theorem 4.2. In particular, if at least six half-twists are inserted when we twist along C_i, then the torus decomposition of $M \smallsetminus K$ will agree with that of $M \smallsetminus L$, aside from "trivial" pieces.

We wish to apply these results to as many knots in S^3 as possible. In Section 5, we show that any knot in S^3 admits a diagram whose augmentation is reduced. Thus the above results will apply at least to their augmentations. This is Theorem 5.1.

In Section 6, we apply these results to knots in S^3, to determine the JSJ decomposition of multiply twisted knots. Our main result is the following.

Theorem 6.1. *Let K be a knot in S^3 which is toroidal, with a twist-reduced diagram and a maximal twist region selection with at least six half-twists in each generalized twist region. Let L denote the corresponding augmentation. Then there exists a sublink $\hat{L}$ of L, possibly containing fewer crossing circles, such that:*

(1) *The essential tori of the JSJ decomposition of $S^3 \smallsetminus K$ are in one-to-one correspondence with those of $S^3 \smallsetminus \hat{L}$.*

(2) *Corresponding components of the torus decompositions have the same geometric type, i.e., are hyperbolic or Seifert fibered.*

(3) *Essential tori of $S^3 \smallsetminus \hat{L}$ and $S^3 \smallsetminus K$ form a collection of nested tori, each bounding a solid torus in S^3 which contains K, and is fixed under a reflection of $S^3 \smallsetminus L$.*

Putting this theorem with the results on hyperbolic geometry of generalized augmented links in [21], we obtain as an application a lower bound on the Gromov norm of such knots.

Theorem 7.2. *Let K be a knot in S^3 which is toroidal, with a twist-reduced diagram at least seven half-twists in each generalized twist region. Let L denote the corresponding augmentation, and let $\hat{L}$ denote the sublink of Theorem 6.1. Let t denote the number of crossing circles of $\hat{L}$. Then the Gromov norm of $S^3 \smallsetminus K$ satisfies*

$$\|[S^3 \smallsetminus K]\| \geq 0.65721\,(t-1).$$

1.3. Comments and additional questions

The results of this paper give geometric information based purely on diagrammatical properties of extensive classes of knots. However, because of the high amount of twisting required, for example in Theorem 6.1, these classes still do not include examples of many knots. Considering knots which are not included leads to two interesting remaining questions.

First, can the results of Theorem 6.1 be sharpened to require fewer half-twists? We can construct examples of atoroidal knots K whose geometric type (hyperbolic or Seifert fibered) does not agree with that of the corresponding reduced augmented link L. However, in all these examples, at least one generalized twist region contains fewer than three half-twists. In [3], Aït-Nouh, Matignon, and Motegi, working on a related question, show that when exactly one crossing circle is inserted into the diagram of an unknot, and then the unknot is twisted, inserting at least four half-twists, the geometric type of the resulting knot (Seifert fibered, toroidal, or hyperbolic) agrees with that of the union of the unknot and the crossing circle. While these results do not apply to generalized augmented links, the result requiring only four half-twists is intriguing.

Secondly, given an arbitrary diagram of a knot, is there a way to optimize the maximal twist region selection? There are many ways to choose a maximal twist region selection. For example, figure 1(b) could either be seen as a single full-twist of five strands, or as 20 half-twists, each of two strands. To apply the results of this paper, it seems we would want to select generalized twist regions to maximize the number of half-twists in each twist region. Is there an algorithm that, given a diagram of a knot K, produces a reduced diagram and a maximal twist region selection with the highest number of half-twists per generalized twist region possible for K? Results along these lines would be interesting.

2. Reflection

A generalized augmented link in a manifold M admits a reflection through a surface fixed pointwise, just as in the case of links in S^3 [24, Proposition 3.1]. This reflection is necessary for many of the results that follow, and so we state it first.

Proposition 2.1. *Let M be an orientable 3-manifold with torus boundary which admits an orientation reversing involution fixing a surface S pointwise. Let K be a link in M which may be isotoped to lie in a neighborhood of S. Finally, let L be an augmentation of a diagram of K, with crossing circles $\{C_1, \dots, C_n\}$ and knot strands $\{K_1, \dots, K_m\}$. Then $M \setminus (\cup C_i)$ admits an orientation reversing involution σ which fixes a surface P pointwise, and each K_j is embedded in P. In particular, $M \setminus L$ admits an orientation reversing involution σ.*

Proof. Isotope crossing circles to be orthogonal to S, preserved by the reflection of M through S. If there are no half-twists in the diagram of L, then all components K_j of L are embedded in S. Hence the reflection in S preserves each K_j as well as each C_j, so $P = S$ and the involution is the restriction of the involution of M to $M \setminus L$.

If there are half-twists in the diagram of L, then the reflection of M through the surface S gives a new link L' in which all the directions of the crossings at each half-twist have been reversed. Let τ be the homeomorphism of $M \setminus (\cup C_j)$ which twists exactly one full time in the opposite direction of these half-twists at each corresponding crossing circle. Applying τ changes the diagram to one in which the crossings of half-twists have been reversed again, hence to the diagram of L. So $M \setminus L'$ is homeomorphic to $M \setminus L$, and the orientation reversing involution σ of $M \setminus (\cup C_j)$ is given by reflection of M in S followed by the homeomorphism τ.

Finally, we describe the surface P fixed pointwise by σ in the case of half-twists. In this case, P is equal to S outside a neighborhood of those crossing disks for which the corresponding crossing circle C_i bounds a half-twist. Inside such a neighborhood, the surface P follows the ribbon surface of the half-twist between the outermost knot strands. Between C_i and the outermost knot strands, P runs over the overcrossing, under the undercrossing, and meets up with the surface S on the opposite side of the link. Its boundary $P \cap \partial\mathcal{N}(C_i)$ runs twice along the meridian of $\partial\mathcal{N}(C_i)$, once along the longitude, as in figure 5. $\square$

Note that we may take ∂D_i to be fixed by σ. In [21, Lemma 3.1], we showed that in this setting, the slopes ∂D_i on $\partial\mathcal{N}(C_i)$ will meet the surface P of Proposition 2.1 exactly twice. We will use this fact here, and so we state it as a lemma.

Lemma 2.1. *Let L be a generalized augmented link in M, with reflective surface P from Proposition 2.1 and twisting disks D_i. Then for each i, ∂D_i meets P exactly twice on $\partial\mathcal{N}(C_i)$.*

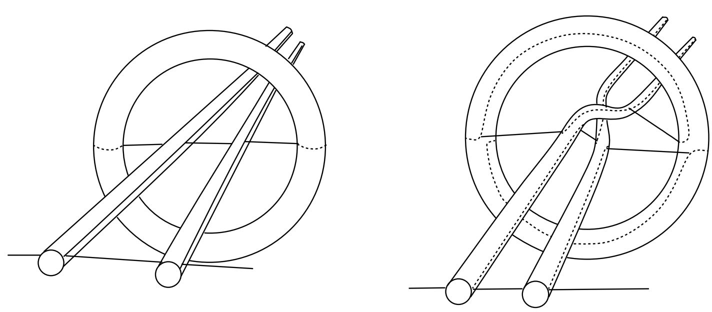

Figure 5: Left: $P \cap \partial\mathcal{N}(C_i)$ consists of two meridians when there is no half-twist. Right: Under a half-twist, $P \cap \partial\mathcal{N}(C_i)$ has boundary shown by the dotted lines.

3. The Seifert fibered case

In this section, we classify Seifert fibered augmented links.

3.1. Incompressibility of surfaces

Let $M \setminus L$ be the complement of a generalized augmented link. By Proposition 2.1, $M \setminus L$ admits an involution σ which fixes a surface P. We show that P and the (punctured) twisting disks $D_i \setminus (D_i \cap L) \subset M \setminus L$ are incompressible.

Lemma 3.1. *Let N be an irreducible 3-manifold with torus boundary components which admits an orientation reversing involution σ fixing a surface P. Then P is incompressible.*

Proof. Suppose not. Suppose D is a compressing disk for P. Then ∂D lies on P, so is fixed by σ, and $\sigma(D)$ is a disk whose boundary agrees with that of D. We first show we can assume D and $\sigma(D)$ are disjoint except on their boundaries. If not, consider intersections of D and $\sigma(D)$. These consist of closed curves on D and on $\sigma(D)$ — not arcs, since ∂D is contained on P, and σ acts as a reflection in a small neighborhood of P.

Let γ_1 be a circle of intersection of D and $\sigma(D)$ which is innermost on $\sigma(D)$. Then γ_1 lies on D and $\sigma(D)$, hence $\gamma_2 = \sigma(\gamma_1)$ lies on $\sigma(D)$ and D, and γ_2 is innermost on D. Surger: Replace D by replacing the disk bounded by γ_1 on D with the disk bounded by γ_1 on $\sigma(D)$, and push off $\sigma(D)$ slightly. Call this new disk D'.

We claim that the number of intersections $|D' \cap \sigma(D')|$ is now less than $|D \cap \sigma(D)|$. Outside a neighborhood of the disk bounded by γ_1, D' agrees with D. Hence outside a neighborhood of the disk bounded by $\gamma_2 = \sigma(\gamma_1)$, $\sigma(D')$ agrees with $\sigma(D)$.

There are two cases to consider. First, if γ_2 is outside the disk bounded by γ_1 on D, then γ_2 and the disk it bounds in D are still contained in D' (since D' agrees with D outside γ_1). Similarly, γ_1 will be outside the disk bounded by γ_2 on $\sigma(D)$, so γ_1 and the disk bounded by γ_1 on $\sigma(D)$ will still remain on $\sigma(D')$. An example of this is illustrated in figure 6.

When we push off $\sigma(D)$ to form D', we may do so equivariantly. So D' does not intersect $\sigma(D')$ in a neighborhood of the disk bounded by γ_1 on $\sigma(D')$, and $\sigma(D')$ does not intersect D' in a neighborhood of the disk bounded by γ_2 on D'. Elsewhere, D and D' agree, so other intersections

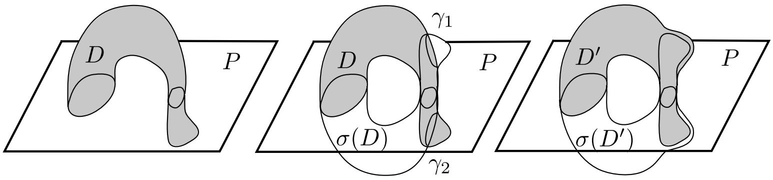

Figure 6: Right: D (shaded). Middle: D and $\sigma(D)$ intersect at γ_1 and γ_2. Left: Replace the interior of γ_1 and push off to reduce the number of intersections by 2.

have not changed. Thus the number of intersections has decreased under this operation.

If instead γ_2 is inside the disk E_1 bounded by γ_1 on D, then when we surger, even before pushing off, γ_1 now bounds a disk E on D', whose image under σ is bounded by γ_2 in E_1, hence is contained in E_1. But now E_1 is disjoint from D'. Since D and D' agree elsewhere, D' and $\sigma(D')$ have fewer intersections.

Repeating this process a finite number of times, we obtain a compressing disk D such that D and $\sigma(D)$ are disjoint.

Then $D \cup \sigma(D)$ is a sphere S in N. Since N is irreducible, S bounds a ball B in N whose boundary S is invariant under σ. Since S meets P, so does B, and hence B must be preserved by the involution σ.

Now we have an orientation reversing involution of a ball B which fixes a circle ∂D on the boundary of the ball and swaps the disks on the boundary. Double the ball across its boundary and extend σ. This gives an orientation reversing involution of S^3 with fixed point set a surface. It follows from work of Smith in the 1930s that the fixed point set must be a 2-sphere containing ∂D, and B intersected with this fixed point set must therefore be a disk. But the fixed point set of σ is P, so P contains a disk with boundary ∂D. This contradicts the fact that D was a compressing disk for P. $\square$

Lemma 3.2. *Let L be a reduced generalized augmented link in M such that $M \setminus L$ is irreducible, with twisting disks D_i. Then each $D_i \setminus (L \cap D_i)$ is incompressible in $M \setminus L$.*

Proof. Suppose, by way of contradiction, that the punctured disk $D_i \setminus (L \cap D_i)$ is compressible. Let D be a compressing disk in $M \setminus L$. Consider ∂D on D_i. This bounds a disk E on D_i which must meet strands of K_j. Consider the disk obtained by replacing D_i by replacing E with D. We have a new disk embedded in M with boundary on C_i, intersecting $\cup K_j$ in fewer points than does D_i. This contradicts the fact that L is reduced, specifically part (1) of Definition 1.1. □

3.2. Annuli

Next, we show a series of results on annuli that are admitted in a generalized augmented link.

Lemma 3.3 [11, Lemma 2.5.3]**.** *Let M be an irreducible 3-manifold with torus boundary components, not homeomorphic to $T^2 \times I$. Let T_1 and T_2 be boundary components which are incompressible. Suppose A_1 and A_2 are properly embedded annuli in M with $\partial A_i = c_{i1} \cup c_{i2}$, with c_{ij} on T_j. Then c_{1j} is isotopic to c_{2j} on T_j, $j = 1, 2$.*

Lemma 3.3 is actually not as general as [11, Lemma 2.5.3]. Because the statement of that lemma is a little different from Lemma 3.3, we reproduce the proof here for convenience.

Proof. Assume A_1 and A_2 are in general position. Let Δ_j, $j = 1, 2$ denote the number of intersections of c_{1j} and c_{2j}. Suppose one is non-zero.

There is an isotopy of the A_i such that $\Delta_1 = \Delta_2$, and any arc of intersection of $A_1 \cap A_2$ runs from one torus to the other. Otherwise, an arc of intersection runs from one torus back to itself. By an innermost arc argument, using the irreducibility of M and the incompressibility of T_i, we can isotope A_1 and A_2 to remove this arc of intersection.

Now, we claim we can replace A_2, if necessary, so that c_{1j} and c_{2j} intersect just once. Suppose c_{1j} and c_{2j} intersect at least twice. Choose two arcs adjacent to each other on A_2, say a and a'. That is, a and a' bound a disk E_2 on A_2 whose interior is disjoint from A_1. The arcs a and a' will also bound a disk E_1 on A_1, whose interior is not necessarily disjoint from A_2, but must be disjoint from E_2 by choice of E_2.

The boundaries of E_1 are a, an arc on c_{11} which we denote b_1, a', and an arc on c_{12}, which we denote d_1. Similarly, the boundaries of E_2 are a, b_2 on c_{21}, a', and d_2 on c_{22}. Then $E_1 \cup E_2$ gives an annulus A embedded in M with boundary components $b_1 \cup b_2$ on T_1 and $d_1 \cup d_2$ on T_2. Since $\mathrm{int}(b_2 \cap A_1) = \emptyset$ (because E_2 is disjoint from A_1), A has slope r_0, say, on T_1, where $\Delta(r_0, c_{i1}) = 1$. Hence after isotopy, $A \cap A_1$ will consist of a single arc (see [11, Figure 2.3]). Replace A_2 with A.

Now, $A_1 \cup A_2$ is homeomorphic to $X \times I$, where $X = X \times \{0\}$ is the union of two simple loops on T_1 which intersect transversely in a single point. It has a regular neighborhood homeomorphic to $N \times I$, where $N = N \times \{0\}$

is a regular neighborhood of X on T_1. But now ∂N bounds a disk D_1 on T_1, $\partial N \times \{1\}$ bounds a disk D_2 on T_2, so $D_1 \cup \partial N \times I \cup D_2$ is a 2-sphere, which bounds a 3–ball B in M since M is irreducible. Then $M = N \times I \cup B$ is homeomorphic to $T \times I$. $\square$

Corollary 3.1. *Suppose M is an irreducible 3-manifold with torus boundary components, not homeomorphic to $T^2 \times I$, which admits an orientation reversing involution σ which fixes a surface P meeting incompressible components T_1 and T_2 of ∂M. Suppose A is an annulus embedded in M with boundary components lying on T_1 and T_2. Then the slopes of ∂A on T_i, $i = 1, 2$, are preserved by the involution σ.*

Proof. If σ does not preserve one of the slopes $(\partial A)_i$ on T_i, then A and $\sigma(A)$ are two distinct annuli embedded in M with non-isotopic boundary components. This contradicts Lemma 3.3. $\square$

We now apply these results to generalized augmented links. First, we need to rule out the case that a generalized augmented link might have complement in M homeomorphic to $T^2 \times I$.

Lemma 3.4. *Suppose L is a reduced generalized augmented link in M and $M \smallsetminus L$ is irreducible. Then $M \smallsetminus L$ is not homeomorphic to $T^2 \times I$.*

Proof. Suppose not. Since $T^2 \times I$ has just two boundary components, and since we assume any generalized augmented link has at least one crossing circle C_1, one boundary component of M corresponds to C_1 and the other to a knot strand. Since the punctured D_1 is incompressible by Lemma 3.2 and 2-sided, it must be either horizontal or vertical in a Seifert fibering of $T^2 \times I$. But then $D_1 \smallsetminus L$ must be an annulus, contradicting the fact that $m_1 \geq 2$. $\square$

Using this fact, we can rule out annuli embedded in $M \smallsetminus L$.

Lemma 3.5. *Let L be a reduced generalized augmented link in M such that $M \smallsetminus L$ is irreducible. Then there is no annulus embedded in $M \smallsetminus L$ with one boundary component on $\partial \mathcal{N}(C_i)$ parallel to $P \cap \partial \mathcal{N}(C_i)$, and the other boundary component disjoint from $\partial \mathcal{N}(C_i)$.*

Proof. Suppose A is an annulus embedded in $M \smallsetminus L$ with boundary component $(\partial A)_i$ on $\partial \mathcal{N}(C_i)$ parallel to $P \cap \partial \mathcal{N}(C_i)$.

Consider the intersection of D_i with A. Since $(\partial A)_i$ is parallel to $P \cap \partial \mathcal{N}(C_i)$, and ∂D_i meets P twice by Lemma 2.1, we may isotope A so that

$\partial D_i \cap A$ consists of one or two points, depending on whether $P \cap \partial\mathcal{N}(C_i)$ consists of two or one components, as illustrated on the left and right of figure 5, respectively.

If $A \cap \partial D_i$ consists of one point, then $P \cap \partial\mathcal{N}(C_i)$ has two meridional components, and $(\partial A)_i$ is a meridian on $\partial\mathcal{N}(C_i)$. But now consider $A \cap D_i$. This consists of arcs and curves of intersection. Any arc has two endpoints on $A \cap \partial D_i$, so there must be an even number of points of intersection of $A \cap \partial D_i$. However, we are assuming there is just one such point. This is a contradiction.

Thus $A \cap \partial D_i$ must consist of two points, and $A \cap D_i$ must have a single arc component running from ∂D_i to ∂D_i. This arc bounds a disk E in D_i, and a disk E' in A, since the other boundary component of A is disjoint from $\partial\mathcal{N}(C_i)$. We may assume the interiors of E and E' are disjoint by an innermost curve argument, for intersections must be simple closed curves in both, since $A \cap D_i$ has just one arc component. By incompressibility of $D_i \smallsetminus (L \cap D_i)$ (Lemma 3.2), any simple closed curve on $D_i \smallsetminus (L \cap D_i)$ bounding a disk in $M \smallsetminus L$ also bounds a disk in $D_i \smallsetminus (L \cap D_i)$, and hence if E and E' are not disjoint we may isotope them off of each other using irreducibility of $M \smallsetminus L$. So $E \cup E'$ is a disk in $M \smallsetminus L$ with boundary on $\partial\mathcal{N}(C_i)$. This must bound a disk on $\partial\mathcal{N}(C_i)$, using the incompressibility of $\partial\mathcal{N}(C_i)$ (which follows from the fact that L is reduced, particularly Definition 1.1(1)). Again by irreducibility of $M \smallsetminus L$, we may therefore isotope A to have no intersections with ∂D_i, contradicting the fact that $(\partial A)_i$ is parallel to $P \cap \partial\mathcal{N}(C_i)$. $\square$

Lemma 3.6. *If L is a reduced generalized augmented link in M such that $M \smallsetminus L$ is irreducible, then there is no annulus embedded in $M \smallsetminus L$ with boundary components on $\partial\mathcal{N}(C_i)$, $\partial\mathcal{N}(C_j)$, for $i \neq j$.*

Proof. By Definition 1.1(1), each $\partial\mathcal{N}(C_i)$ is incompressible. Thus by Corollary 3.1, any embedded annulus A must have boundary components fixed by σ. At most two slopes on $\partial\mathcal{N}(C_i)$ are fixed by σ. These are the slopes of $P \cap \partial\mathcal{N}(C_i)$ and of $D_i \cap \partial\mathcal{N}(C_i) = \partial D_i$.

By Lemma 3.5, no boundary component of A is parallel to $P \cap \partial\mathcal{N}(C_i)$ or to $P \cap \partial\mathcal{N}(C_j)$. By Definition 1.1(2), we cannot have $(\partial A)_i$ and $(\partial A)_j$ parallel to ∂D_i and ∂D_j. Thus no such annulus exists. $\square$

3.3. Seifert fibered augmented links

We may now classify all Seifert fibered reduced generalized augmented links.

Theorem 3.1. *If $M \setminus L$ is irreducible and Seifert fibered, where L is a reduced generalized augmented link in M, then L has just one crossing circle component C_1, $M \setminus C_1$ is a solid torus, P is an embedded annulus or Möbius band in $M \setminus C_1$, and the knot strands are embedded on P. In particular, if P is an annulus, there are at least two knot strand components.*

Proof. Suppose $M \setminus L$ is Seifert fibered. First, by Lemma 3.6, there can be no annuli between link components C_i and C_j. This implies that there cannot be more than one link component C_1.

Now, $D_1 \setminus (L \cap D_1)$ is incompressible by Lemma 3.2. Since $D_1 \setminus (L \cap D_1)$ is 2-sided, it is horizontal or vertical in $M \setminus L$. If vertical, it must be an annulus, contradicting the fact that $m_1 \geq 2$. So $D_1 \setminus (L \cap D_1)$ is horizontal. Then the meridians of the knot strands (i.e., the curves on $\partial \mathcal{N}(K_j)$ which bound disks in $\mathcal{N}(K_j)$) cannot be Seifert fibers, so the Seifert fibering of $M \setminus L$ extends to $M \setminus C_1$. The base orbifold of $M \setminus C_1$ is branch covered by the horizontal surface D_1, hence it is a disk with one singular point. Thus $M \setminus C_1$ must be a solid torus.

Since P is incompressible by Lemma 3.1, if it is orientable, then it is an annulus. Because the knot strands are embedded in P and non-trivial, they must be parallel to the core of the solid torus $M \setminus C_1$. In particular, if there is just one knot strand component, then $M \setminus L$ is homeomorphic to $T^2 \times I$, contradicting Lemma 3.4. So in this case there are at least two knot strand components.

If P is non-orientable, then by work of Frohman [12] and Rannard [25], P is *pseudo-vertical* in a solid torus, meaning, in this case, it is a punctured non-orientable surface in the solid torus $M \setminus C_1$. These were classified by Tsau [28], and have boundary of the form $P \cap \partial \mathcal{N}(C_1) = \alpha = q\mu + (2k)\lambda$, where μ is a meridian of the solid torus, λ is a longitude, $k \geq 1$, and q is an odd integer.

In our case, we know which boundary slopes α can occur, because of the existence of the involution σ. In particular, since ∂D_1 intersects P exactly twice, by Lemma 2.1, $k = 1$. Then by untwisting, we may assume $q = 1$, and so α is the slope $\mu + 2\lambda$, and P is a Möbius band in the solid torus $M \setminus C_1$.

The knot strands are embedded in the Möbius band P and non-trivial. If there is just one knot strand, the link L is as in figure 7. More precisely, it has complement homeomorphic to the complement of the link of figure 7 in S^3. $\square$

We obtain the following immediate consequence of this result.

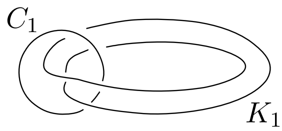

Figure 7: The only Seifert fibered reflective augmented link with one knot strand component.

Corollary 3.2. *Let K be a knot in S^3 which has a diagram whose augmentation is a Seifert fibered reduced generalized augmented link. Then K is a $(2, q)$ torus knot.*

Proof. Let L denote the augmentation of K. First we show $S^3 \smallsetminus L$ is irreducible. The manifold $S^3 \smallsetminus L$ is homeomorphic to $(S^3 \smallsetminus K) \smallsetminus (\cup C_i)$, where C_i are crossing circles encircling generalized twist regions. Let S be a sphere in $S^3 \smallsetminus L$, and assume it does not bound a ball in $S^3 \smallsetminus L$. Then the image of S under the homeomorphism is a sphere in $(S^3 \smallsetminus K) \smallsetminus (\cup C_i)$ which does not bound a ball. Since $S^3 \smallsetminus K$ is irreducible, S must bound a ball B in $S^3 \smallsetminus K$. Hence some C_i lies in B. But C_i is unknotted, hence bounds a disk in B. This contradicts property (1) of the definition of reduced, Definition 1.1.

By Theorem 3.1, the diagram D can have only one generalized twist region. The augmentation is the link shown in figure 7, since there is just one component K. Thus when we twist to obtain $S^3 \smallsetminus K$, we remove C_1 from the diagram and add an even number of crossings at the twist region it bounds. This is a $(2, q)$ torus knot. □

4. Essential tori and augmented links

In this section, we consider reduced generalized augmented links L such that $M \smallsetminus L$ is toroidal, i.e., contains an embedded essential torus. We show that the torus decomposition of $M \smallsetminus L$ satisfies some nice properties.

Recall that by work of Jaco and Shalen [16] and Johannson [17], every irreducible three-manifold N with (possibly empty) torus boundary contains a pairwise disjoint collection of embedded essential tori $\mathbb{T}$, unique up to isotopy, such that the closure of a component of $N \smallsetminus \mathbb{T}$ is either atoroidal or Seifert fibered. If N admits an orientation reversing involution σ, then by the equivariant torus theorem, first proved by Holzmann [15], each incompressible torus in N is isotopic to one which is preserved by σ or taken off itself. Then σ applied to $\mathbb{T}$ gives a new torus decomposition of N, which by uniqueness must agree with $\mathbb{T}$. Thus the closure of each component of $N \smallsetminus \mathbb{T}$ is either fixed by σ, or taken off itself. This is the equivariant torus decomposition of Bonahon and Siebenmann [8]. We refer to this equivariant torus decomposition as the JSJ decomposition.

Theorem 4.1. *Let L be a reduced generalized augmented link in M, such that $M \smallsetminus L$ is irreducible. Then there exists a collection $\mathcal{T}$ of tori, incompressible in $M \smallsetminus L$, such that if U is a component of $M \smallsetminus \mathcal{T}$, U satisfies one of the following:*

- *U does not meet any crossing circles of L.*
- *U meets exactly one crossing circle C_i, U is homeomorphic to $T^2 \times I$, and C_i is isotopic to a simple closed curve on ∂U.*
- *$U \smallsetminus (L \cap U)$ is the complement of a reduced generalized augmented link.*

The collection of tori $\mathcal{T}$ may be larger than the minimal collection of the JSJ decomposition. However, we find $\mathcal{T}$ by adding incompressible tori to the JSJ decomposition.

Lemma 4.1. *Let L be a reduced generalized augmented link in M such that $M \smallsetminus L$ is irreducible. Let $\mathbb{T}$ denote the tori of the JSJ decomposition for $M \smallsetminus L$. Suppose there is a component U of $M \smallsetminus \mathbb{T}$ that contains a crossing circle C_j and an embedded annulus A with one boundary component on some $T_0 \subset \partial U$ and one on C_j. Let T_j be the torus obtained by taking the boundary of a small regular neighborhood of the union of T_0, A, and C_j. Then T_j is incompressible in $M \smallsetminus L$.*

The collection $\mathcal{T}$ of Theorem 4.1 will be obtained by adding to $\mathbb{T}$ any T_j of Lemma 4.1 which are not isotopic to tori already in the collection.

Before proving the lemma, we illustrate by example a situation in which the lemma will apply. Consider the link in figure 8. The heavy line in that figure shows the location of an incompressible torus T_0. When we cut along T_0, we obtain a hyperbolic generalized augmented link on the outside, homeomorphic to the complement of the Borromean rings. On the inside, C_1 and T_0 bound an annulus. Take the boundary of a regular neighborhood of the union of this annulus with T_0 and C_1, and we obtain an incompressible torus T_1 as in Lemma 4.1. Cutting along T_1, we split the inside into two components, one homeomorphic to $(T^2 \times I) \smallsetminus C_1$, and the other (in this example) another copy of the Borromean rings complement.

Recall that we are interested in links obtained by twisting generalized augmented links. When we do twisting along C_1 in the above example, the component $(T^2 \times I) \smallsetminus C_1$ becomes the manifold $T^2 \times I$. Thus the two

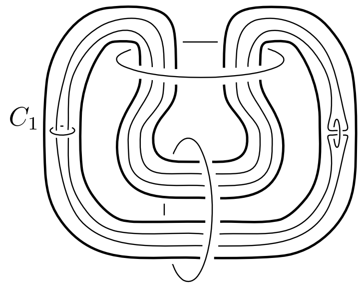

Figure 8: C_1 and the torus denoted by the thick line bound an annulus.

incompressible tori T_0 and T_1 become isotopic to each other after twisting. By the results in [21], sufficiently high twisting along the remaining crossing circles in each gives a manifold which remains hyperbolic. Thus the torus decomposition of the twisted link contains just one of T_0 and T_1. We will see this in Theorem 4.2.

Proof of Lemma 4.1. Suppose by way of contradiction that T_j is compressible in $M \setminus L$. A compressing disk can be isotoped to lie in $U \setminus (L \cap U)$, else we obtain a compressing disk for some torus in $\mathbb{T}$, which is impossible.

Surger along a compressing disk for T_j in $U \setminus (L \cap U)$ to obtain a sphere S embedded in $U \setminus (L \cap U)$. Note that $U \setminus (L \cap U)$ is irreducible, since $M \setminus L$ is irreducible and T_0 is incompressible. Thus S bounds a ball B in $U \setminus (L \cap U)$. Now, B cannot be on the side of S containing A since this side contains boundary components T_0 and C_j of $U \setminus (L \cap U)$. Thus T_j bounds a solid torus V in $U \setminus (L \cap U)$. Recall that T_j was formed by taking the boundary of a regular neighborhood of the union of T_0, an annulus A, and C_j in U. Since T_j bounds a solid torus V in $U \setminus (L \cap U)$, it must be the case that $U \setminus (L \cap U)$ is homeomorphic to V union a regular neighborhood of the annulus A. This has just two boundary components: T_0 and C_j. We claim this is impossible.

Since U contains C_j, the surface P of Proposition 2.1 meets U, and the torus boundary components of U are preserved by σ. Hence $U \setminus (\cup C_i)$ is preserved by σ, where the union is over C_i in U. The annulus A has one boundary component, A_1, say, on $\partial\mathcal{N}(C_j)$, taken by σ to $-A_1$. Hence it is isotopic to an annulus which meets P in two arcs and is preserved under σ. Since tori T_0 and $\partial\mathcal{N}(C_j)$ are also preserved under σ, the solid torus V is preserved under σ. Then $V \cap P$ must be an annulus, a Möbius band, or two meridional disks in V.

We form $U \setminus (L \cap U)$ by attaching a thickened annulus to V. This thickened annulus is attached along some slope μ on ∂V. Since A is taken to itself with reversed orientation by σ, the slope μ must be taken to $-\mu$ by σ. There are very few possibilities for μ.

In case $V \cap P$ is an annulus or Möbius band, μ must bound a disk in V. Attaching an annulus to V along two meridians gives a manifold with compressible boundary, but neither C_j nor T_0 is compressible.

Thus $V \cap P$ consists of two meridional disks, and μ must be some longitude of ∂V. When we attach a thickened annulus to longitude slopes, the resulting manifold is homeomorphic to $T^2 \times I$. Then T_0 is parallel to C_j, contradicting the fact that T_0 is essential in $M \setminus L$. $\square$

Form the collection $\mathcal{T}$ of Theorem 4.1 by starting with $\mathbb{T}$, and adding tori T_j of Lemma 4.1 which are not isotopic to a torus already in the collection. The closure of each component of $(M \setminus L) \setminus \mathcal{T}$ is still either atoroidal or Seifert fibered, but $\mathcal{T}$ may no longer be the minimal such collection. By construction, components of $M \setminus \mathcal{T}$ either do not contain crossing circles of L; contain a single crossing circle C_j sandwiched between incompressible tori T_0 and T_j of Lemma 4.1, in which case the second possibility of Theorem 4.1 holds; or U contains crossing circles of L, but none of these bound annuli with boundary on ∂U. For the proof of Theorem 4.1, we need to examine these components, and show that any such $U \setminus (L \cap U)$ is homeomorphic to the complement of a reduced generalized augmented link.

Generalized augmented links are defined to lie in a 3-manifold admitting a reflection through a surface S. First, we define a manifold N which will play the role of this underlying 3-manifold.

Let U be a component of $M \setminus \mathcal{T}$ which contains at least one crossing circle C_i, but does not contain an embedded annulus with boundary on ∂U and on C_i, for any $C_i \subset U$. Consider $D_i \cap U$. In the manifold M, D_i may intersect essential tori of $\mathcal{T}$. Then D_i will meet boundary components of U. Let $\ell_{i_1}, \dots, \ell_{i_k}$ be the boundary components of that component of $D_i \cap U$ which meets C_i. Replace U by the manifold N_i obtained by Dehn filling V along the slopes ℓ_{i_j}, $j = 1, \dots, k$. Let $\hat{K}_{i_1}, \dots, \hat{K}_{i_k}$ denote the solid tori attached in the Dehn filling.

Do this Dehn filling for each C_i contained in U. We obtain a new manifold N. Note N is well-defined because the D_i are disjoint; thus if T_k is met by D_i and D_j, then $D_i \cap T_k$ and $D_j \cap T_k$ must give the same slope, so the Dehn fillings along that slope are the same. Similarly, D_i might meet T_k several times, but again along the same slope.

Now, let L_U be the link in N consisting of components $C_i \cap U$ and $K_j \cap U$, as well as the cores of each distinct solid torus in the set $\{\hat{K}_k\}$. We will abuse notation slightly and continue to refer to these cores of solid tori by $\hat{K}_k$. The following is immediate.

Lemma 4.2. *For N the manifold and L_U the link in N constructed as above, $N \setminus L_U$ is homeomorphic to $U \setminus (L \cap U)$.*

We claim that L_U is a reduced generalized augmented link in N, with components $C_i \cap U$ taking the role of the crossing circles, and components $K_j \cap U$ and $\hat{K}_k$ taking the role of the knot strands.

By assumption, there is at least one C_i, bounding D_i, meeting at least one K_i or $\hat{K}_i$. So the link L_U contains at least the minimal number of necessary link components to be a generalized augmented link.

Lemma 4.3. *The manifold N admits an involution through a surface $S \subset N$, a link K is contained in a neighborhood of S, has diagram $D(K)$ and maximal twist region selection such that when we encircle generalized twist regions of $D(K)$ by crossing circles and untwist, the result is a link isotopic to L_U. That is, L_U is a generalized augmented link, given by augmenting a link diagram in N.*

Proof. The involution σ of Proposition 2.1 preserves $U \setminus (\cup C_i)$, where the union is over crossing circles C_i in U. It has fixed point set $U \cap P$, and components K_j in L_U are embedded in P. We show that the involution σ extends to the solid tori $\mathcal{N}(\hat{K}_k)$, and that the cores $\hat{K}_k$ are embedded in the surface P.

Recall that $\mathcal{N}(\hat{K}_k)$ has boundary which is an incompressible torus T_k in $M \setminus L$, and T_k is preserved by σ (by work of Holzmann [15]). Moreover, some D_j meets T_k in a meridian of $\mathcal{N}(\hat{K}_k)$. The slope $\partial D_j \cap T_k$ cannot bound a disk in $M \setminus L$ by incompressibility of T_k. Since it does bound a disk in D_j, this disk must be punctured by some component K_i in $M \setminus L$. Therefore the slope $\partial D_j \cap T_k$ must be taken by σ to $-\partial D_j \cap T_k$. This means a meridian of the solid torus $\mathcal{N}(\hat{K}_k)$ is inverted by the involution σ. Since the boundary is preserved by σ, it follows that the involution extends to give an involution of the solid torus $\mathcal{N}(\hat{K}_k)$. Also, $P \cap \mathcal{N}(\hat{K}_k)$ must be a longitude of $\partial\mathcal{N}(\hat{K}_k)$, in the sense that it intersects the meridian exactly once. Therefore the core, $\hat{K}_k$, is embedded in P.

Now note that N and L_U satisfy the conclusion of Proposition 2.1. Since σ acts on N as an extension of the involution of Proposition 2.1 acting on $M \setminus (\cup C_i)$, we know that P meets the C_i in N in the same way it meets the

C_i in M. Namely, P either meets $\mathcal{N}(C_i)$ in two meridian components, as on the left in figure 5, or in a single component as on the right in figure 5. Form the surface S by taking $\hat{S}$ to be P outside of a neighborhood of those crossing disks that meet half-twists. Inside a neighborhood of a half-twist, P will appear as on the right of figure 5. The surface $\hat{S}$, however, should run straight through the crossing circle, meeting $\mathcal{N}(C_i)$ in two meridians, as on the left of figure 5. Note there is a reflection τ in $\hat{S}$ which still preserves $N \smallsetminus (\cup C_i)$, although it reverses crossings at half-twists. Moreover, the reflection τ preserves a meridian of each crossing circle, hence extends to a reflection τ of N through the surface S, where S is obtained from $\hat{S}$ by capping off boundary components on the $\mathcal{N}(C_i)$ by disks.

Notice that the strands K_i and $\hat{K}_k$ lie in a neighborhood of S, as do the crossing circles C_i. Hence when we twist along crossing circles, we form a link K which still lies in a neighborhood of S. Moreover, we may project K to S such that the twists obtained by twisting along the C_i form distinct generalized twist regions, and so that L_U is an augmentation of a diagram of K. □

We now need to show that L_U is reduced in N. To do so, we find disjoint embedded disks bounded by the crossing circles C_j in N, and show these are minimal in the sense of part (1) of Definition 1.1.

Lemma 4.4. *Let $C_{i_1}, \dots, C_{i_k}$ be the crossing circles of L_U. Each bounds a disk D'_{i_j} in N such that the collection $\{D'_{i_j}\}$ is embedded in N, and the disks meet K_i and $\hat{K}_i$ in m_i points, where $m_i \geq 2$ and m_i is minimal over all disks in M bounded by C_i.*

Proof. Let $C_{i_1}, \dots, C_{i_k}$ be the crossing circles in U. Each C_{i_j} bounds a disk D_{i_j} in M. By construction of N, the collection $\{D_{i_1}, \dots, D_{i_k}\}$ extends to an embedded collection of disks $\{D'_{i_1}, \dots, D'_{i_k}\}$ in N.

If some m_{i_j} is not minimal, then we can find an embedded punctured disk which meets ∂U fewer times. Replace D'_{i_j} with this disk. Now each disk D'_{i_j} must meet K_i and $\hat{K}_i$ in at least two points, for otherwise we would have an annulus between some C_j and a component K_i or $\hat{K}_i$. The first cannot happen by definition of a reduced generalized augmented link in M. The second cannot happen by assumption: U is assumed to be a component of $M \smallsetminus \mathcal{T}$ which does not contain an embedded annulus with boundary components on C_i and on ∂U, and $\partial \mathcal{N} \hat{K}_i$ is a component of ∂U. □

We are now ready to complete the proof of Theorem 4.1.

Proof of Theorem 4.1. Let $\mathcal{T}$ be the collection of tori described after the proof of Lemma 4.1. Let U be a component of $M \setminus \mathcal{T}$. If U does not contain any crossing circle C_i, then we are done. If U contains a crossing circle C_i and an embedded annulus with boundary C_i and boundary on ∂U, then by construction of $\mathcal{T}$, U is homeomorphic to $T^2 \times I$, C_i is the only crossing circle contained in U, and the curve C_i is boundary parallel in U. This is the second case of the theorem.

So assume U contains a crossing circle, but does not contain any embedded annuli with boundary on ∂U and on C_i. Then by Lemma 4.2, $U \setminus (L \cap U)$ is homeomorphic to the manifold $N \setminus L_U$, which by Lemma 4.3 is a generalized augmented link complement. By Lemma 4.4, L_U satisfies condition (1) of the definition of reduced, Definition 1.1. It satisfies condition (2) as well, since any annulus embedded in $N \setminus L_U$ with boundary components on C_i and C_j is embedded in $M \setminus L$ with boundary components on C_i and C_j. Since L is reduced, no such annulus exists. The link L_U satisfies condition (3) of Definition 1.1 by assumption, given the definition of $\mathcal{T}$. □

Theorem 4.2. *Let L be a reduced generalized augmented link in M with $M \setminus L$ irreducible, and let $\mathcal{T}$ be the tori of Theorem 4.1. Let K be the link formed by twisting along all the crossing circles of L, subject to the restriction that if C_i is contained in a component of $M \setminus \mathcal{T}$ which is not homeomorphic to $T^2 \times I$, then at least six half-twists are inserted when we twist along C_i. Then there is a torus decomposition of $M \setminus K$ for which components of the decomposition*

- *are either atoroidal or Seifert fibered,*
- *are in one-to-one correspondence with the components of $M \setminus \mathcal{T}$ which are not homeomorphic to $T^2 \times I$, and*
- *have the same geometric type (hyperbolic or Seifert fibered) as the corresponding component of $M \setminus \mathcal{T}$.*

Remark 4.1. The decomposition of Theorem 4.2 may not be the JSJ decomposition. In particular, there may be two Seifert fibered components which our decomposition separates, but which are considered as one in the minimal JSJ decomposition. We will see in Section 6 that this does not happen when L comes from the augmentation of a knot in S^3, but it could happen more generally.

Proof. Let U be a component of $M \setminus \mathcal{T}$. If U contains no crossing circles, then twisting does not affect U and so the result holds.

Similarly, if U is homeomorphic to $T^2 \times I$, and contains just a single boundary parallel C_i, then twisting yields a manifold homeomorphic to $T^2 \times I$, which will not be a component of a torus decomposition.

If U contains a crossing circle, but is not $T^2 \times I$, then $U \smallsetminus (L \cap U)$ is either hyperbolic or Seifert fibered.

If it is hyperbolic, by [21, Proposition 3.5], the slope of the twisting on a horoball neighborhood of a crossing circle C_i has length at least $\sqrt{(1/4) + c_i^2}$, where c_i is the number of half-twists inserted. Since $c_i \geq 6$, this length is greater than 6, hence by the 6–Theorem [2, 18], the result of Dehn filling is hyperbolike. Perelman's work on geometrization of 3-manifolds then implies the result is actually hyperbolic.

If $U \smallsetminus (L \cap U)$ is Seifert fibered, Theorem 3.1 implies $U \smallsetminus (L \cap U)$ is homeomorphic to the complement of parallel strands embedded in an annulus or Möbius band in a solid torus. Note we may take the solid torus to be a fibered solid torus in S^3, the knot strands to be fibers, and the crossing circle to be the complement of the solid torus in S^3. After twisting, this fibered solid torus is replaced by cutting it open, rotating the bottom of the resulting fibered solid cylinder some integer number of full rotations, and then reattaching. We may still take this to be a fibered solid torus, with knot strands as fibers. The Dehn filling glues a solid torus to the exterior of this solid torus. Note that this exterior solid torus may be taken to be a fibered solid torus, such that the fibration along the common boundaries of the tori agree. In any case, the result is some number of knot strands which can be chosen to be fibers of some Seifert fibration of S^3. Work of Burde and Murasugi implies that the complement is Seifert fibered [9]. $\square$

5. Reducing knot diagrams

We wish to apply the previous results to as many knots and links as possible.

In this section, we prove that all knots in S^3 admit a diagram such that the augmentation is reduced, as in Definition 1.1. We say the diagram D of a link K is *twist reduced* if there exists a maximal twist region selection such that the corresponding augmentation of K gives a reduced generalized augmented link.

Theorem 5.1. *Let K be a knot in S^3 with diagram D and a maximal twist region selection. Then there exists a twist reduced diagram D' for K.*

We will find the diagram of Theorem 5.1 by forming the augmentation of the given diagram, and then removing unnecessary crossing circles and extracting unnecessary knot strands from crossing disks. When we do twisting on remaining crossing circles, projecting twists to the projection plane in S^3 in the usual way, we will obtain the desired diagram of the theorem.

Definition 5.1. A *standard diagram* of a generalized augmented link is a diagram such that all knot strands lie on the projection plane except at half-twists, which are contained in a neighborhood of the corresponding crossing circle. Crossing circles are perpendicular to the projection plane, and crossing disks project to straight lines running directly under the crossing circles of the diagram. For example, the portions of the diagrams in figures 2(b), 3, and 4 are standard.

The next few lemmas ensure part (1) of Definition 1.1 will hold.

Lemma 5.1. *Let L be a generalized augmented link in S^3 with standard diagram. Suppose there exists a disk E embedded in S^3, with boundary some crossing circle C_i, disjoint from the other crossing circles, and suppose that E meets the knot strands fewer times than does D_i. Then there exists such a disk F, possibly with ∂F on a different C_j, such that in addition, $F \cap (\cup\,\mathrm{int}(D_j)) = \emptyset$.*

Proof. Suppose E meets the interior of some D_j. We may assume the intersection is transverse and consists of simple closed curve components. There is some innermost disk $\hat{E}$ on E whose boundary is a curve γ on D_j. Consider the disk G constructed by taking the disk D_j outside γ, replacing D_j inside γ by $\hat{E}$. Push G off D_j slightly, so G and D_j do not intersect.

Suppose first that G meets the knot strands fewer times than does D_j. Then replace E by G, replacing C_i by C_j. This disk G has fewer intersections with $\cup\,\mathrm{int}(D_k)$ than does E.

Suppose instead G meets the knot strands at least as many times as does D_j. Replace E by replacing $\hat{E} \subset E$ with the portion of D_j bounded by γ, and push off D_j. We have decreased the number of intersections of E with D_j without increasing the number of intersections of E with knot strands.

In either case, we have a new disk which meets the interiors of the D_k fewer times. Repeat a finite number of times, and we obtain a disk F as in the statement of the lemma. □

Lemma 5.2. *Let L be a generalized augmented link in S^3 with standard diagram. Suppose there is a disk E embedded in S^3 such that E has boundary some crossing circle C_i, is disjoint from the other crossing circles, and $E \cap (\cup\,\mathrm{int}(D_k))$ is empty. Then there exists such a disk F such that in addition,*

F intersects the projection plane in a single arc γ_1, the intersections of F with knot strands all lie on γ_1, and the number of intersections of F with knot strands is at most the number of intersections of E with the knot strands.

Proof. First, we show we may assume that E does not meet the diagram of L in the neighborhood of any D_j containing a half-twist. That is, we show we may assume E does not meet any half-twists of the diagram. Assume E does run through a half-twist corresponding to D_j. The half-twist is contained in a neighborhood of D_j, which is homeomorphic to $D_j \times [-1, 1]$. Without loss of generality, assume the half-twist is contained in $D_j \times (-1, 0)$, with D_j lying at $D_j \times \{0\}$. Since E does not meet D_j, it must intersect $\partial(D_j \times (-1, 0))$ in the surfaces $D_j \times \{-1\}$ or $\partial D_j \times (-1, 0)$. If any intersections of E with $\partial(D_j \times (-1, 0))$ bound disks in $\partial(D_j \times (-1, 0)) \smallsetminus L$, then we may replace E by replacing corresponding disks in E with those in $\partial(D_j \times (-1, 0)) \smallsetminus L$, and pushing out of $D_j \times (-1, 0)$ slightly. After this replacement, we may assume that any component of $E \cap \partial(D_j \times (-1, 0))$ lies on $D_j \times \{-1\}$ and bounds punctures of $D_j \smallsetminus L$. Since $(D_j \times (-1, 0)) \smallsetminus L$ is homeomorphic to $(D_j \smallsetminus L) \times (-1, 0)$, we may replace E by replacing a disk bounded by $E \cap (D_j \times \{-1\})$ in E by the corresponding disk in D_j, and pushing out of $D_j \times (-1, 0)$ slightly. This will meet the knot strands of L at most as many times as does E, and will not intersect the neighborhood of D_j containing the half-twist.

So assume E does not meet the diagram of L in any half-twists. Since outside of half-twists, all knot strands lie on the projection plane, any intersections of E with the knot strands must lie on the projection plane. It remains to show we can assume there is just one component of intersection of E with the projection plane.

Let γ_1 be the arc of intersection of E with the projection plane whose endpoints lie on C_i. Consider S^3 cut along the projection plane. Remove neighborhoods of all the D_j for $j \neq i$, including all half-twists, as well as a neighborhood of a half-twist at D_i, if applicable, which does not contain C_i. This gives two balls, with γ_1 embedded in the boundary of each ball, and one half of C_i embedded as an arc in each ball, with endpoints meeting those of γ_1. Note no components of L intersect the interior of either ball, aside from C_i. Thus in each ball we may find an embedded disk with boundary running along C_i and along the arc γ_1 which only meets L in its boundary. These disks glue to give a new disk F which meets the knot strands exactly where γ_1 meets the knot strands, hence meets the knot strands at most as many times as does E. The disk F satisfies the conclusions of the lemma. $\square$

Lemma 5.3. *Let L be a generalized augmented link in S^3 with standard diagram. Suppose there is a disk E in S^3 with boundary some crossing circle C_i, disjoint from the other crossing circles, such that E meets the knot strands fewer times than does D_i, and $E \cap (\cup \mathrm{D_j})$ is empty. Then we may isotope L to a generalized augmented link with standard diagram with the same crossing circles, and the same number of strands running through each crossing circle, except that D_i meets the knot strands fewer times.*

Proof. Consider the intersection of the sphere $E \cup D_i$ with the projection plane. By Lemma 5.2, we may assume this intersection is a single simple closed curve γ, with one arc of the curve running along the intersection of D_i with the projection plane, and the other arc meeting the knot strands exactly in the intersections of E with the knot strands.

If C_i bounds a half-twist, we may isotope the diagram such that all the crossings of the half-twist lie outside of the sphere $E \cup D_i$, as in figure 9. In any case, the portion of the diagram containing the sphere consists of m_i strands running parallel, embedded on the projection plane, entering D_i, and $n_i < m_i$ parallel strands, embedded on the projection plane, exiting E.

Isotope to move the ball bounded by $E \cup D_i$ to the opposite side of C_i, as in figure 10. Notice that we have a new diagram, with the same crossing circles as before, and the same numbers of knot strands running through each crossing circle, except there are now n_i strands running through C_i.

If C_i does not bound a half-twist, then this is a standard diagram of a generalized augmented link, and the lemma is proved. If C_i does bound

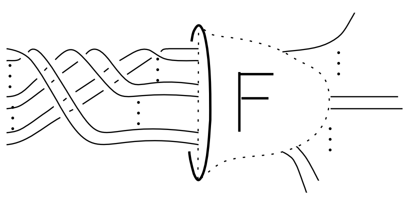

Figure 9: Isotope half-twists to lie outside of the sphere $E \cup D_i$. (The label F on the sphere is to illustrate the result of a flype in figure 11.)

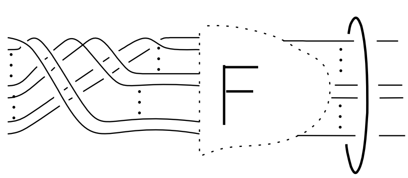

Figure 10: Isotope the ball bounded by $E \cup D_i$ to the opposite side of C_i.

a half-twist, then this is no longer a standard diagram, since the half-twist associated with C_i no longer remains in a neighborhood of C_i after the isotopy. In this case, we perform a flype on the region of the diagram bounded by γ. That is, we rotate 180° in the opposite direction of the half twist. This cancels the crossing of the "old" half-twist associated with C_i, and adds a half-twist in the immediate neighborhood of C_i after the isotopy, as in figure 11.

Finally, notice that the flype does not add any crossings, aside from those of the "new" half-twist adjacent to C_i. For within the region bounded by γ, knot strands on the projection plane are taken back to the projection plane under the flype. Crossing circles are taken to crossing circles. Half-twists rotate 180° to become identical half-twists. Outside the region bounded by γ, the diagram does not change at all, except to shift the half-twist from one side of γ to the other. Thus in this case we have established the lemma. □

Lemma 5.4. *Given a standard diagram of a generalized augmented link L in S^3, there exists a new diagram that is also standard, but for which D_i meets the knot strands in the minimal number of points.*

Proof. If the original diagram does not satisfy this property, then combining Lemmas 5.1 and 5.3 we obtain a new diagram for which the number of strands running through a single crossing circle has been reduced. Notice that there are only a finite number of crossing circles, and a finite number of strands running through each crossing circle. Thus we need only repeat a finite number of times, and we are left with a diagram for which D_i meets the knot strands in the minimal number of points. □

The next lemmas will give property (2) of Definition 1.1.

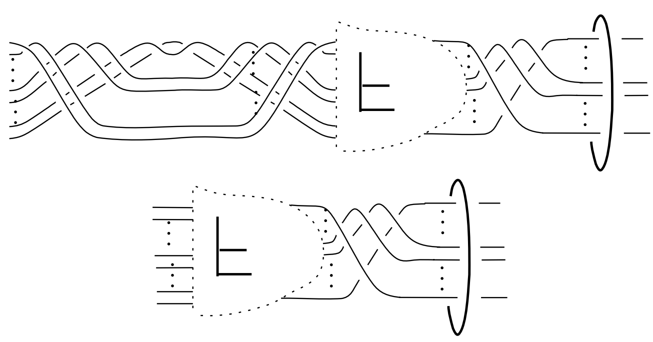

Figure 11: Perform a flype to cancel the half-twist on the left, add a half-twist on the right.

Lemma 5.5. *Let L be a generalized augmented link in S^3 with standard diagram. Suppose there is an incompressible annulus A embedded in $S^3 \smallsetminus L$ with boundary C_i and C_j, for some $i \neq j$. Then A is isotopic to an annulus which does not meet D_i or D_j, and intersects $\cup\,\mathrm{int}(D_k)$ in a (possibly empty) collection of core curves of the annulus.*

That is, any such annulus must either completely run through a twist region, or completely miss the twist region.

Proof. First, apply Lemma 5.4 to the diagram of L. We may assume that L has standard diagram for which each D_i meets the knot strands in the minimal number of points.

Consider the intersection of A with some D_k. This is a collection of closed curves. Suppose one of them, say α, bounds a disk D in A. Then the disk obtained by taking D_k outside α and D inside α must meet the knot strands the same number of times as D_k, since D_k meets the knot strands the minimal number of times. Thus α must bound a disk in D_k. Hence we can isotope off, reducing the number of intersections.

Now suppose A intersects D_i. Then by the preceding paragraph, any intersection must be an essential curve α in A. Thus we may isotope C_i along A to this curve of intersection, and then push off slightly, reducing the number of intersections of A with D_i. Repeat, until there are no intersections with D_i. Similarly for D_j. ☐

Lemma 5.6. *Let L be a generalized augmented link in S^3. Suppose there is an embedded annulus A in $S^3 \smallsetminus L$ with boundary C_i and C_j, $i \neq j$. Then there is a generalized augmented link L' in S^3 such that first, $S^3 \smallsetminus L$ is homeomorphic to $S^3 \smallsetminus L'$, second, there is a one-to-one correspondence between crossing circles and knot strands of L and L', third, twisting disks of a standard diagram of L meet the knot strands the same number of times as the corresponding twisting disks in a standard diagram of L', but the crossing circle corresponding to C_i does* not *meet the knot strands of L' in a half-twist.*

Proof. Start with a standard diagram of L, and consider C_i and C_j. If one of C_i and C_j does not encircle a half-twist, then L, possibly with C_i and C_j switched, satisfies the conclusions of the lemma.

So suppose both C_i and C_j encircle half-twists. Isotope A as in Lemma 5.5. Then $A \cup D_i \cup D_j$ is a sphere S in S^3. We may isotope C_i and C_j such that the half-twists bounded by these crossing circles are both on the outside of S.

Now perform a flype on the inside of S in the direction opposite the half-twist at C_i. We wish to analyze what happens to the diagram after the flype. First, consider the portions of the diagram inside S. These are rotated $180°$. This rotation takes strands on the projection plane back to the projection plane, and takes crossing circles to crossing circles without affecting the number of strands running through each crossing circle. Finally, the rotation takes half-twists to identical half-twists, in the same direction.

Outside of S, the flype does not affect any strands of the diagram, except that it removes the half-twist at C_i and adds a new half-twist at C_j. This will either cancel with the half-twist already at C_j, or it will add to it, putting a full-twist at C_j. In the case the flype leaves a full-twist at C_j, we replace the flyped diagram with the one in which the full twist at C_j has been removed, replaced by parallel strands with no crossings at C_j, as in figure 12. This link is the link L'. The complement of L' is homeomorphic to the complement of L.

To finish the proof of the lemma, we need to show that the given diagram of L' is the standard diagram of a generalized augmented link.

If no portion of the diagram of L on the outside of S crosses over a portion of the diagram of L on the inside of S, then this will still be true for L', and since both outside and inside of S in L' we have portions of

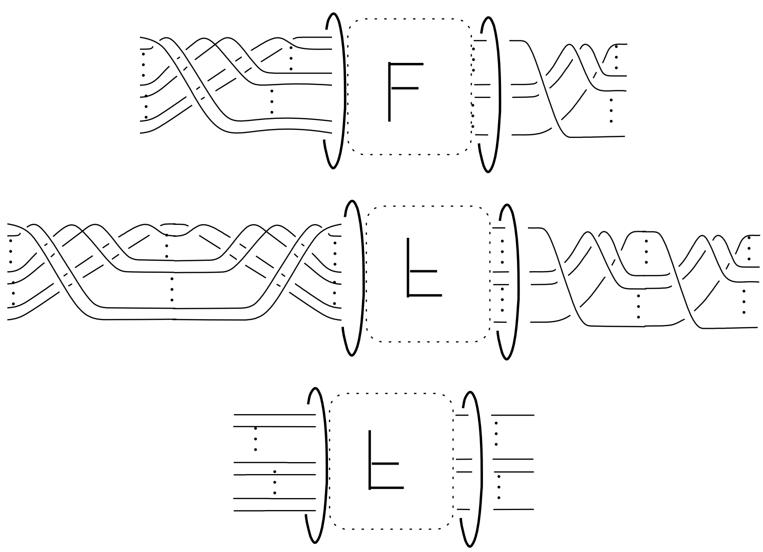

Figure 12: Top to bottom: the link L before the flype. After the flype. The link L'. (The label F is to illustrate the action of the flype.)

diagrams of generalized augmented links, their union must be the diagram of a generalized augmented link.

So suppose some portion of the diagram of L outside of S crosses over some portion diagram inside S. These will form crossings of the original standard diagram of L. All crossings are associated with half-twists. Since the crossings of the half-twists associated with C_i and C_j lie outside of S, and do not meet S, these crossings must be associated to some C_k with $k \neq i, j$. Then we may assume that S intersects the corresponding twisting disk D_k, or we may isotope S out of a neighborhood of D_i as in Lemma 5.2, reducing the number of crossings of the outside of S with the inside of S.

So assume S intersects D_k. By Lemma 5.5, D_k must meet the annulus A in its non-trivial core curve. Then the portion of the diagram of L inside S at this intersection will be a half-twist. When we perform the flype, the half-twist becomes an identical half-twist in the same direction. Thus in a neighborhood of D_k inside of S, the diagram remains unchanged after the flype. Since the diagram does not change outside of S, a neighborhood of D_k will still be a half-twist after the flype. Thus in all cases, the diagram of L' is the standard diagram of a generalized augmented link. □

Proof of Theorem 5.1. Let K be a knot in S^3 with a maximal twist region selection, and let L the corresponding generalized augmented link with standard diagram. If L is not reduced, then it fails one of properties (1), (2), or (3) of Definition 1.1. Since our knot is embedded in S^3, which has empty boundary, condition (3) will not hold, so we need only show (1) and (2).

Suppose that property (1) fails. Then by Lemma 5.4, we may replace our diagram by a new standard diagram in which D_i meets the knot strands in the minimal number of points. Note that *a priori*, there may now be a twisting disk D_i that meets the knot strands in m_i points, where $m_i \leq 1$. In this case, simply remove C_i from the diagram, since twisting along D_i when $m_i \leq 1$ gives a manifold homeomorphic to that obtained by performing meridional Dehn filling on C_i.

Suppose property (2) fails. Then there is an annulus embedded in $S^3 \smallsetminus L$ with boundary on some C_i and C_j. By Lemma 5.6, we may replace the diagram of L with a standard diagram D' of a generalized augmented link L', without increasing the numbers of crossing circles or strands running through crossing circles, such that now the crossing circle corresponding to C_i does not encircle a half-twist in the diagram D'. Consider the image of the annulus in $S^3 \smallsetminus L'$ under the homeomorphism of $S^3 \smallsetminus L$ and $S^3 \smallsetminus L'$. This is an annulus embedded in $S^3 \smallsetminus L$ with one boundary component on C_i. We may isotope C_i along this annulus to be parallel to the other boundary component, C_j. The result has diagram D'' identical to D', except that C_i has been removed and a crossing circle parallel to C_j has been added.

Now, to consider the knot K, we perform twisting on C_i and on C_j. However, note that both add full-twists to the same generalized twist region. Hence we may remove C_i from the diagram, and adjust the amount of twisting at C_j to obtain the same link K.

Perform Dehn filling along the crossing circles, adding the appropriate number of twists so that the result has complement homeomorphic to $S^3 \setminus K$. This has a diagram given by replacing the crossing circles C_k in the diagram with a number of full-twists. The diagram will be reduced, by construction. Moreover, since knots are determined by their complements [14], the diagram is of a knot isotopic to K. $\square$

Remark 5.1. Three remarks on the above proof.

First, note that when we reduced the diagram, we only removed crossing circles from the unreduced augmented link, never added. Thus the resulting diagram will have at most as many generalized twist regions as the original. This is useful for Theorem 7.2 below.

Second, what would be more useful would be if the reduced diagram had at least as many half twists per generalized twist region as the original, since many of our results in this paper require a knot with a high number of half twists per twist region. However, this may not be the case if property (2) of Definition 1.1 does not hold for the original diagram. In that case, we must concatenate two generalized twist regions into a single one. This may cancel half twists in opposite directions, reducing the total number of half twists in the new generalized twist region.

Finally, notice that the above proof used the fact that K was a knot only in the very last step. To show the same result for links in S^3, one needs to show that an isotopy of the generalized augmented link followed by the appropriate Dehn filling is equivalent to an isotopy of the original link. We believe this is true (and possibly even known), but we have not worked out the details here.

6. Applications to knots

In this section, we apply the results of the previous sections to give geometric information on knots in S^3. Throughout, we will let K be a knot in S^3, and let D be a twist reduced diagram of K. By Theorem 5.1, we may assume any knot admits such a diagram.

6.1. Torus decomposition and geometric type

Given a knot K in S^3, with twist reduced diagram D, form an augmented link L by adding crossing circles to the diagram in a maximal twist region selection. We have restrictions on essential tori in $S^3 \setminus L$.

Lemma 6.1. *Let T be an essential torus in $S^3 \setminus L$. Then T bounds a solid torus V such that*

1) *V is invariant under the involution σ,*
2) *V contains the link component K,*
3) *$V \cap P$ is nonempty, with $P \cap \partial V$ containing no meridians of V.*

Proof. Because T is essential, it must intersect the surface P. By the invariant torus theorem [15], we may assume T is invariant under σ. By the solid torus theorem, T bounds a solid torus V in S^3, which must also intersect P. Because σ fixes $P \cap V$ pointwise, σ must preserve V. This gives the first item of the lemma.

Suppose V does not contain K. At least one crossing circle must be inside V, else a meridian of V is a compressing disk of T, contradicting incompressibility of T. Let $C_i \subset V$. The circle C_i bounds D_i. If D_i does not intersect ∂V, then D_i cannot intersect K, contradicting the definition of a generalized augmented link. Thus D_i must intersect ∂V.

We may assume all such intersections are non-trivial curves on ∂V, else we may replace disk portions of D_i with disks of ∂V. This gives us a new D_i with required properties of the definition of a generalized augmented link. So we assume all intersections of D_i with ∂V are non-trivial.

Consider a curve of $D_i \cap \partial V$ that is innermost in D_i. This bounds a disk E on D_i. The disk E cannot lie in V, or it would be a compressing disk. Thus it must lie outside V. But then V is unknotted in S^3. Therefore replace V with the exterior of V in S^3. This is a solid torus that contains K. Thus we have the second item of the lemma.

Finally, suppose $\partial V \cap P$ consists of meridians of ∂V, i.e., curves that bound disks in V. Then since V is fixed by σ, $V \cap P$ must consist of two meridional disks. The knot K is contained in V, and K is contained in P by Proposition 2.1. Since the C_i are contained in a neighborhood of K, the link L must lie in a neighborhood of a meridional disk of V. But then V is compressible. Contradiction. $\square$

Lemma 6.1 allows us to classify Seifert fibered components of the torus decomposition of $S^3\smallsetminus L$. We have the following.

Proposition 6.1. *Let K be a knot in S^3 with a twist reduced diagram. Let L be a corresponding augmentation. Let $\mathcal{T}$ be the tori in the torus decomposition of Theorem 4.1. Then the tori of $\mathcal{T}$ are nested, bounding solid tori $V_1 \supset V_2 \supset \cdots \supset V_k \supset K$, and if U is a component of $S^3\smallsetminus\mathcal{T}$ which is not homeomorphic to $T^2 \times I$, such that $U\smallsetminus(L \cap U)$ is Seifert fibered, then U is the outermost component of the decomposition, and $U\smallsetminus(L \cap U)$ is homeomorphic to the manifold of figure 7.*

Proof. Since each torus of $\mathcal{T}$ is essential in $S^3\smallsetminus L$, by Lemma 6.1 each bounds a solid torus invariant under σ, containing K, with $V \cap P$ a longitudinal slope. Thus we can arrange the solid tori in order of containment:

$$K \subset V_k \subset \cdots \subset V_1.$$

By Theorem 4.1, for each component U of $S^3\smallsetminus\mathcal{T}$ which is not homeomorphic to $T^2 \times I$, $U\smallsetminus(L \cap U)$ is a reduced augmented link complement. Thus it has a boundary component corresponding to some C_j. Then any such component except the outermost must have at least three boundary components: at least one C_j, some V_i, and V_{i+1} (or K). Now, if this component is Seifert fibered, Theorem 3.1 tells us that V_i and V_{i+1} must be parallel on P, and therefore V_{i+1} cannot be a subset of V_i. This is impossible.

So only the outermost component may be Seifert fibered. In this case, again Theorem 3.1 tells us there is just one C_1 in the component $(S^3\smallsetminus L)\smallsetminus V_1$, and $(S^3\smallsetminus V_1)\smallsetminus C_1$ is a solid torus. Since there is only one other link component, namely V_1, it must be embedded in the Möbius band P, parallel to the boundary of P, as in figure 7. $\square$

Corollary 6.1. *Let K be a torus knot with a twist reduced diagram D and a maximal twist region selection in which each twist region admits at least six half-twists. Then K is a $(2,p)$ torus knot, and D has one twist region.*

Proof. $S^3\smallsetminus K$ is Seifert fibered. Since D is twist reduced, adding crossing circles to twists of D yields a reduced augmented link L in S^3. By Theorems 4.1 and 4.2, there is a sublink $\hat{L}$ of L, possibly consisting of fewer crossing circles, and a collection of tori $\hat{\mathbb{T}}$ such that components of $(S^3\smallsetminus\hat{L})\smallsetminus\hat{\mathbb{T}}$ are reduced augmented links with the same geometric type as those of $S^3 \setminus K$. Thus the components must all be Seifert fibered.

By Proposition 6.1, only the outermost component is Seifert fibered, and it is of the form of figure 7. Thus the collection of tori of $\hat{\mathbb{T}}$ is empty. When we twist to insert at least six half-twists, we obtain a $(2, p)$ torus knot. □

Theorem 6.1. *Let K be a knot in S^3 which is toroidal, with a twist–reduced diagram and a maximal twist region selection with at least six half-twists in each generalized twist region. Let L denote the corresponding augmentation. Then there exists a sublink $\hat{L}$ of L, possibly containing fewer crossing circles, such that:*

(1) *The essential tori of the JSJ decomposition of $S^3 \setminus K$ are in one-to-one correspondence with those of $S^3 \setminus \hat{L}$.*

(2) *Corresponding components of the torus decompositions have the same geometric type, i.e., are hyperbolic or Seifert fibered.*

(3) *The essential tori of $S^3 \setminus \hat{L}$ and $S^3 \setminus K$ form a collection of nested tori, each bounding a solid torus in S^3 which contains K, and which is fixed under the reflection of $S^3 \setminus L$.*

Proof. By Theorems 4.1 and 4.2, there is some sublink $\hat{L}$ of L, possibly containing fewer crossing circles (i.e., those inside any $T^2 \times I$ components), and tori $\hat{\mathbb{T}}$ such that the tori form a torus decomposition of $S^3 \setminus K$, and corresponding components of the decompositions of $S^3 \setminus K$ and $S^3 \setminus \hat{L}$ have the same geometric type. By Lemma 6.1, item (3) must hold for all essential tori.

All that remains to prove is that the decomposition given by $\hat{\mathbb{T}}$ is the JSJ decomposition, i.e., it is the unique minimal torus decomposition of $S^3 \setminus \hat{L}$ and $S^3 \setminus K$. If not, then some torus T of $\hat{\mathbb{T}}$ separates two Seifert fibered components. But by Proposition 6.1, this is impossible: only the outermost component of $S^3 \setminus \hat{L}$ can be Seifert fibered. Thus any other components must be hyperbolic, and so $\hat{\mathbb{T}}$ is the unique torus decomposition of $S^3 \setminus \hat{L}$.

Similarly, the only way $\hat{\mathbb{T}}$ can fail to be the unique torus decomposition of $S^3 \setminus K$ is if some essential torus T of $\hat{\mathbb{T}}$ splits a single Seifert fibered component into two. But again the components of $S^3 \setminus K$ have the same geometric type as those of $S^3 \setminus \hat{L}$, by Theorem 4.2. Hence $\hat{\mathbb{T}}$ must be the unique torus decomposition of $S^3 \setminus K$. □

7. Application: Gromov norms

In this section, we apply the previous results to bound the Gromov norms of many toroidal knots. The results use heavily the particular torus decompositions developed in previous sections, as well as results on hyperbolic generalized augmented links in [21].

First, we insert a bound on the volume of a hyperbolic augmented link in a solid torus, whose proof follows immediately from [21].

Proposition 7.1. *Let L be a generalized augmented link in a solid torus V, with t_i crossing circles. Suppose $V \setminus L$ is hyperbolic. Then its volume satisfies* $\mathrm{vol}(V \setminus L) \geq 2\, v_8\, t_i$, *where* $v_8 \approx 3.66386$ *is the volume of a hyperbolic regular ideal octahedron.*

Proof. By [21, Lemma 4.1], the volume of a hyperbolic generalized augmented link in S^3 with t generalized twist regions is at least $2\, v_8(t-1)$. We may use this result to bound volumes of hyperbolic generalized augmented links in a solid torus as follows. Embed the solid torus as one of the solid tori of a standard genus one Heegaard splitting of S^3. Let C be the core of the other solid torus. Then the embedding gives a generalized augmented link in S^3 where the curve C is an additional crossing circle component. Thus the volume of a hyperbolic augmented link in a solid torus is at least $2\, v_8(t_i + 1 - 1) = 2\, v_8\, t_i$, where t_i denotes the number of crossing circles. □

Alternately, Proposition 7.1 may be proved by cutting $V \setminus L$ along P and crossing disks, and using this to determine the number of vertices and edges of a one-skeleton as in the proof of [21, Lemma 4.1].

Theorem 7.1. *Let K be a knot with twist reduced diagram with t generalized twist regions. Let L be the corresponding generalized augmented link and let t_0 denote the number of components of the torus decomposition of the form $(T^2 \times I) \setminus C_i$. Then the Gromov norm of $S^3 \setminus L$ satisfies*

$$\|[S^3 \setminus L]\| \geq 2\, v_3\, v_8(t - t_0 - 1),$$

where $v_3 \approx 1.0149$ *is the volume of a regular ideal hyperbolic tetrahedron, and* $v_8 \approx 3.66386$ *is the volume of a regular ideal hyperbolic octahedron.*

Proof. By Proposition 6.1 and Theorem 4.1, $S^3 \setminus L$ admits a torus decomposition such that each piece of the decomposition is either a link in $T^2 \times I$, or a generalized augmented link in a solid torus. Moreover, the only Seifert fibered pieces each meet just one crossing circle component, either as the outermost component of the decomposition or as a link in $T^2 \times I$. Since the Gromov norm is v_3 times the sum of the volumes of the hyperbolic pieces [27], we use the results of Proposition 7.1.

Now, in case the outermost piece is hyperbolic, it is an augmented link in S^3 and so its volume is at least $2\,v_8(t_1-1)$, where t_1 denotes the number of crossing circles in the outermost piece. Then the Gromov norm is at least

$$\|[S^3 \smallsetminus L]\| \geq v_3 \left(2\,v_8(t_1-1) + \sum 2\,v_8\,t_i\right) = 2\,v_3\,v_8\,(t - t_0 - 1),$$

where the sum in the center is over all components of the torus decomposition which are homeomorphic to hyperbolic generalized augmented links in a solid torus.

If the outermost piece is *not* hyperbolic, then it contributes nothing to the Gromov norm. By Proposition 6.1, it contains just one crossing circle, and so the Gromov norm is at least

$$\|[S^3 \smallsetminus L]\| \geq v_3 \sum 2\,v_8\,t_i = 2\,v_3\,v_8\,(t - t_0 - 1),$$

where the final -1 corresponds to the single crossing circle in the outermost component of the torus decomposition. □

For knots:

Theorem 7.2. *Let K be a knot in S^3 which is toroidal, with a twist–reduced diagram at least seven half-twists in each generalized twist region. Let L denote the corresponding augmentation, and let $\hat{L}$ denote the sublink of Theorem 6.1. Let t denote the number of crossing circles of $\hat{L}$. Then the Gromov norm of $S^3 \smallsetminus K$ satisfies*

$$\|[S^3 \smallsetminus K]\| \geq 0.65721\,(t-1).$$

Proof. Since the Gromov norm is v_3 times the sum of volumes of hyperbolic pieces of the torus decomposition, we bound volumes of the hyperbolic pieces. Each is obtained by Dehn filling a generalized augmented link. Moreover, by [21, Proposition 3.5], the length of the slope of Dehn filling is at least $\sqrt{(1/4) + c_i^2}$, where c_i is the number of half-twists. Thus if there are at least seven half-twists in each twist region, then the Dehn filling slopes have length at least $\sqrt{49.25} \approx 7.0178 > 2\pi$.

Now apply [13, Theorem 1.1]. This theorem bounds the volume under Dehn filling in terms of the length of the filling slope and the volume of the unfilled manifold. In particular, for twisting a hyperbolic augmented link with t_i crossing circles in a solid torus, by Proposition 7.1 we have

$$\text{volume after twisting} \geq \left(1 - \left(\frac{2\pi}{\sqrt{49.25}}\right)^2\right)^{3/2} (2\, v_8\, t_i) > 0.64756\, t_i.$$

Only the outermost component of the torus decomposition may be Seifert fibered. If it is not, the volume of the outermost piece is at least 0.64756 $(t_1 - 1)$, where t_1 is the number of crossing circles, by [21, Theorem 4.2]. Since $\hat{L}$ is obtained from L by removing all crossing circles which lie in components $(T^2 \times I) \setminus C_j$ of the torus decomposition of $S^3 \setminus L$ (Theorem 4.1), the total number of crossing circles in $\hat{L}$ is $t_1 + \sum t_i$, where the sum is over all pieces homeomorphic to hyperbolic generalized augmented links in solid tori. Thus the Gromov norm satisfies

$$\|[S^3 \setminus K]\| \geq v_3 \left(0.64756\, (t_1 - 1) + \sum 0.64756\, t_i\right) = 0.65721\ldots\ (t-1).$$

If the outermost component is Seifert fibered, then the outermost component contains a single crossing circle component, by Proposition 6.1, and so the Gromov norm satisfies

$$\|[S^3 \setminus K]\| \geq v_3 \sum 0.64756\, t_i = 0.65721\ldots (t-1).$$

□

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References

[1] C.C. Adams, *Augmented alternating link complements are hyperbolic*, Low-dimensional topology and Kleinian groups (Coventry/Durham, 1984), London Math. Soc. Lecture Note Ser., **112**, Cambridge University Press, Cambridge, 1986, 115–130.

[2] I. Agol, *Bounds on exceptional Dehn filling*, Geom. Topol. **4** (2000), 431–449 (electronic).

[3] M. Aït-Nouh, D. Matignon and K. Motegi, *Geometric types of twisted knots*, Ann. Math. Blaise Pascal **13**(1) (2006), 31–85.

[4] K.L. Baker, *Surgery descriptions and volumes of Berge knots. I. Large volume Berge knots*, J. Knot Theory Ramifications **17**(9) (2008), 1077–1097.

[5] K.L. Baker, *Surgery descriptions and volumes of Berge knots. II. Descriptions on the minimally twisted five chain link*, J. Knot Theory Ramifications **17**(9) (2008), 1099–1120.

[6] J. Berge, *Some knots with surgery yielding lens spaces*, unpublished manuscript.

[7] J. Birman and I. Kofman, *A new twist on Lorenz links*, J. Topol. **2**(2) (2009), 227–248.

[8] F. Bonahon and L.C. Siebenmann, *The characteristic toric splitting of irreducible compact* 3*-orbifolds*, Math. Ann. **278**(1–4) (1987), 441–479.

[9] G. Burde and K. Murasugi, *Links and Seifert fiber spaces*, Duke Math. J. **37** (1970), 89–93.

[10] A. Champanerkar, I. Kofman and E. Patterson, *The next simplest hyperbolic knots*, J. Knot Theory Ramifications **13**(7) (2004), 965–987.

[11] M. Culler, C. McA. Gordon, J. Luecke and P.B. Shalen, *Dehn surgery on knots*, Ann. Math. (2) **125**(2) (1987), 237–300.

[12] C. Frohman, *One-sided incompressible surfaces in Seifert fibered spaces*, Topology Appl. **23**(2) (1986), 103–116.

[13] D. Futer, E. Kalfagianni and J.S. Purcell, *Dehn filling, volume, and the Jones polynomial*, J. Differential Geom. **78**(3) (2008), 429–464.

[14] C. McA. Gordon and J. Luecke, *Knots are determined by their complements*, J. Amer. Math. Soc. **2**(2) (1989), 371–415.

[15] W.H. Holzmann, *An equivariant torus theorem for involutions*, Trans. Amer. Math. Soc. **326**(2) (1991), 887–906.

[16] W.H. Jaco and P.B. Shalen, *Seifert fibered spaces in* 3*-manifolds*, Mem. Amer. Math. Soc. **21**(220) (1979), viii+192.

[17] K. Johannson, *Homotopy equivalences of* 3*-manifolds with boundaries*, Lecture Notes in Mathematics, **761**, Springer, Berlin, 1979.

[18] M. Lackenby, *Word hyperbolic Dehn surgery*, Invent. Math. **140**(2) (2000), 243–282.

[19] M. Lackenby, *The volume of hyperbolic alternating link complements*, Proc. London Math. Soc. (3) **88**(1) (2004), 204–224, With an appendix by Ian Agol and Dylan Thurston.

[20] W.W. Menasco and M.B. Thistlethwaite, *Surfaces with boundary in alternating knot exteriors*, J. Reine Angew. Math. **426** (1992), 47–65.

[21] J.S. Purcell, *Hyperbolic geometry of multiply twisted knots*, Comm. Anal. Geom. **18**(1) (2010).

[22] J.S. Purcell, *Volumes of highly twisted knots and links*, Algebr. Geom. Topol. **7** (2007), 93–108.

[23] J.S. Purcell, *Cusp shapes under cone deformation*, J. Differential Geom. **80**(3) (2008), 453–500.

[24] J.S. Purcell, *Slope lengths and generalized augmented links*, Comm. Anal. Geom. **16**(4) (2008), 883–905.

[25] R. Rannard, *Incompressible surfaces in Seifert fibered spaces*, Topology Appl. **72**(1) (1996), 19–30.

[26] D. Rolfsen, *Knots and links*, Mathematics Lecture Series, No. 7, Publish or Perish Inc., Berkeley, CA, 1976.

[27] T. Soma, *The Gromov invariant of links*, Invent. Math. **64**(3) (1981), 445–454.

[28] C.M. Tsau, *A note on incompressible surfaces in solid tori and in lens spaces*, Knots 90 (Osaka, 1990), de Gruyter, Berlin, 1992, 213–229.

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