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Contracting convex immersed closed plane curves with fast speed of curvature

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We study the contraction of a convex immersed plane curve along its normal vector direction with speed function $\frac{1}{\alpha}k^{\alpha}$, where $k > 0$ is the curvature of the evolving curve and $\alpha > 1$ is a constant. We show that the blow-up rate of the curvature is always of *type one* and the rescaled solution will converge to a limit that may or may not be degenerate.

1. Introduction

Let $\gamma_t, t \in [0, T)$, be a family of convex[1] closed plane curves given by smooth immersions $X_t = X(\cdot, t) : S^1 \to \mathbb{R}^2$, where S^1 is the unit circle. The curves γ_t are said to evolve (contract) under the k^{α} flow, where $\alpha > 1$ is a constant, if we have

$$\frac{\partial X}{\partial t}(u, t) = \frac{1}{\alpha}k^{\alpha}(u, t) \cdot \mathbf{N}(u, t), \quad \forall (u, t) \in S^1 \times [0, T). \tag{1.1}$$

Here $k(\cdot, t)$ is the curvature of γ_t and $\mathbf{N}(\cdot, t)$ is the unit normal vector of the curve γ_t. Throughout this paper the constant α is assumed to be $\alpha > 1$.[2] Here we use the convention that for convex plane curves the curvature $k > 0$ is positive everywhere and as for the direction of the normal $\mathbf{N}$, we choose $\mathbf{N} = (0, 1)$ at a point with minimum y-coordinate and extend it continuously to the whole curve.

The purpose of this paper is to look at the evolution under (1.1) of a given smooth convex *immersed* closed plane curve γ_0 and study its asymptotic behavior. "Immersed" means that γ_0 may have self-intersections.

The case when $\alpha = 1$ in (1.1) is the well-known *curve shortening flow.* See Gage-Hamilton [12] for the convex embedded case (i.e., γ_0 is embedded) and Angenent [5], Angenent-Velázquez [6] for the convex immersed case (i.e., γ_0

[1]When a convex closed curve has self-intersections, convex always means "locally convex." It has positive curvature everywhere.

[2]Since $\alpha > 1$, for convenience, we call (1.1) *fast speed* contraction. If $0 < \alpha < 1$, then we call (1.1) *slow speed* contraction.

is immersed). The evolution behavior for solutions to equation (1.1) for $\alpha \in (0,1), \alpha = 1, \alpha \in (1,\infty)$, are quite different. For embedded case with general $\alpha \in (0,\infty)$, one can see the nice paper by Andrews [2] and the references cited therein for the existing research results. One can also see the references in the book by Chou-Zhu [9]. It provides an excellent and unified account of many results related to the evolution of plane curves by curvature.

The only results for the immersed case, as far as we know, are those appeared in [5, 6] (both are for the case $\alpha = 1$). Other than that, very little is known. From those elaborate estimates established in [5, 6], one can sense the difficulty of the problem when $\alpha = 1$. For $\alpha \in (0,1)$, the behavior of the flow is more or less analogous to the case $\alpha = 1$; see [17]. But fortunately when $\alpha > 1$, the problem becomes much easier as we only have *type one* blow-up of the curvature (see Proposition 1.1 below), but this is not so when $\alpha = 1$ or when $\alpha \in (0,1)$.

In addition to the contrast in blow-up behavior, there is also another interesting difference: when $\alpha = 1$, if we have type one curvature blow-up, the limiting rescaled curvature is everywhere positive (described by the so-called *Abresch-Langer* functions; see Theorem A in [5]); but for $\alpha > 1$, the limiting rescaled curvature is only non-negative. It can be zero somewhere. See the discussion below.

Let γ_0 be a given smooth convex immersed closed plane curve (with rotation index $m \in \mathbb{N}$, $m \geq 1$) given by a smooth immersion $X_0 : S^1 \to \mathbb{R}^2$. Following the arguments of [5, 12, 22], there exists a unique smooth solution $X(u,t) : S^1 \times [0,T) \to \mathbb{R}^2$ to the initial value problem

$$(1.2)\qquad \begin{cases} \dfrac{\partial X}{\partial t}(u,t) = \dfrac{1}{\alpha} k^\alpha(u,t) \cdot \mathbf{N}(u,t), & \alpha > 1, \\ X(u,0) = X_0(u), & u \in S^1 \end{cases}$$

for some short time $T > 0$, with each $\gamma_t := X(\cdot,t)$ representing a smooth convex immersed closed curve (with the same rotation index m). Moreover, (1.2) is equivalent to the following scalar PDE for the curvature function $k(\theta,t)$:

(1.3)
$$\begin{cases} \dfrac{\partial k}{\partial t}(\theta,t) = k^2(\theta,t)\left[\left(\dfrac{1}{\alpha}k^\alpha\right)_{\theta\theta}(\theta,t) + \left(\dfrac{1}{\alpha}k^\alpha\right)(\theta,t)\right], & (\theta,t) \in T_m \times [0,T), \\ k(\theta,0) = k_0(\theta) > 0, & \theta \in T_m, \end{cases}$$

where $k_0(\theta)$ is the curvature of the initial curve γ_0. Here the variable $\theta \in T_m = \mathbb{R}/2m\pi\mathbb{Z}$ is *the angle of the tangent vector* of γ_t. For each $t \in [0,T)$,

$k(\theta, t)$ is a smooth function defined on T_m, which means that it is defined over $\mathbb{R}$ with period $2m\pi, m \geq 1$. One can see [2, 12] or [22] for details of the equivalence.

By parabolic theory, given a smooth $k_0(\theta) > 0$, the smooth solution $k(\theta, t)$ to Equation (1.3) exists on some maximal time interval $[0, T_{\max})$ where

$$\lim_{t \to T_{\max}} k_{\max}(t) = \infty, \quad k_{\max}(t) := \max_{x \in T_m} k(\theta, t)$$

and $k_{\min}(t)$ is increasing on $[0, T_{\max})$ due to the maximum principle. It is also known that $k_{\max}(t)$ is eventually increasing in time. Since $\alpha > 1$, it implies $T_{\max} < \infty$. If we set $v(\theta, t) = k^{\alpha}(\theta, t)$, (1.3) becomes the simpler form

$$\begin{cases} \dfrac{\partial v}{\partial t}(\theta, t) = v^p(\theta, t)[v_{\theta\theta}(\theta, t) + v(\theta, t)], & (\theta, t) \in T_m \times [0, T_{\max}), \\ v(\theta, 0) = v_0(\theta) := k_0^{\alpha}(\theta) > 0, & \theta \in T_m, \end{cases} \tag{1.4}$$

where the exponent $p = 1 + \frac{1}{\alpha}, \alpha > 1$, lies in the range $p \in (1, 2)$.

From now on we shall focus on the scalar Equation (1.4) as it is easier to handle than the system (1.2). We note that Equation (1.4) is similar to Equation (4) in [13]. A lot of estimates and methods of proof established in [13] are also applicable here. We shall refer to the paper [13] quite often.

If we let $R(t)$ be the unique solution to the ODE

$$\frac{dR}{dt} = R^{p+1}(t), \quad R(T_{\max}) = \infty, \tag{1.5}$$

then $R(t) = [p(T_{\max} - t)]^{-1/p}$ and similar to p. 155 of [13], one can use Jensen's inequality and comparison principle to derive

$$0 < v_{\min}(t) \leq \left(\frac{1}{2m\pi} \int_{-m\pi}^{m\pi} v^{1-p}(\theta, t) d\theta\right)^{\frac{1}{1-p}} \leq R(t) \leq v_{\max}(t), \quad \forall t \in [0, T_{\max}). \tag{1.6}$$

In particular, we have the estimate on the blow up time $T_{\max}$:

$$\frac{1}{2m\pi}\left(\int_{-m\pi}^{m\pi} v_0^{1-p}(\theta) d\theta\right)^{\frac{1}{1-p}} \leq R(0) = \left(\frac{1}{pT_{\max}}\right)^{\frac{1}{p}} \leq v_{\max}(0), \quad v_0(\theta) = k_0^{\alpha}(\theta). \tag{1.7}$$

Due to the special form of the equation, we also have the integral identity

$$\int_{-m\pi}^{m\pi} v^{1-p}(\theta,t)\mathrm{e}^{\mathrm{i}\theta}d\theta = \int_{-m\pi}^{m\pi} v_0^{1-p}(\theta)\mathrm{e}^{\mathrm{i}\theta}d\theta = 0, \quad \forall t\in[0,T_{\max}) \tag{1.8}$$

where $\mathrm{e}^{\mathrm{i}\theta}=\cos\theta+\mathrm{i}\sin\theta$ and the second equality holds since γ_0 is a closed curve.

We next define the following terminology:

Definition 1.1. Let $v(\theta,t)$ be the solution to (1.4) on $[0,T_{\max})$. If there exists a constant C, independent of time, such that

$$0<v_{\max}(t)\le CR(t), \quad \forall t\in[0,T_{\max}) \tag{1.9}$$

then we say $v(\theta,t)$ has *type one* blow-up. Otherwise we say it has *type two* blow-up.

A type two blow-up is definitely much more complicated than type one blow-up. Fortunately we have the following:

Proposition 1.1. *For solution $v(\theta,t)$ to (1.4) which blows up at $t=T_{\max}$, there is a constant $C>0$, independent of time, such that*

$$v_{\max}(t)\le C(T_{\max}-t)^{-1/p}, \quad \forall t\in[0,T_{\max}). \tag{1.10}$$

That is, there is only type one *blow-up in (1.4) when $p\in(1,2)$.*

Remark 1.1. If $p\ge 2$ in Equation (1.4), then estimate (1.10) fails in general. See [17].

Proof. The proof is similar to the case $p\in(0,1)$, which corresponds to the *expanding* of a convex immersed closed curve $\gamma_0\subset\mathbb{R}^2$ with speed $1/(\alpha k^\alpha)$, $\alpha>1$. See Proposition 8 and Remark 9 in [13] for details. $\square$

By (1.10), if we use flow (1.2) to contract a smooth convex immersed closed curve $\gamma_0\subset\mathbb{R}^2$, then the curvature function $k(\theta,t)$ of γ_t satisfies

$$0<k_{\max}(t)\le C(T_{\max}-t)^{-1/(1+\alpha)}, \quad \forall t\in[0,T_{\max}). \tag{1.11}$$

2. The rescaling

Our next step is to consider the rescaled curvature (or rescaled v) and study its asymptotic behavior. The type one blow-up (1.10) indicates an obvious way of rescaling given by $u(\theta,t) := v(\theta,t)/R(t), t \in [0,T_{\max})$, and if we let $\tau \in [0,\infty)$ be the new time given by $t = T_{\max}(1-\mathrm{e}^{-p\tau})$, then the function

$$u(\theta,\tau) = p^{1/p} T_{\max}^{1/p}\, \mathrm{e}^{-\tau} \cdot v(\theta, T_{\max}(1-\mathrm{e}^{-p\tau})), \quad (\theta,\tau) \in T_m \times [0,\infty) \tag{2.12}$$

will be a positive, *bounded* solution (due to (1.10)) of the rescaled equation

$$\begin{cases} \dfrac{\partial u}{\partial \tau} = u^p(u_{\theta\theta} + u - u^{1-p}), \quad p \in (1,2), \\ u(\theta,0) = u_0(\theta) := v_0(\theta)/R(0) > 0 \end{cases} \tag{2.13}$$

on $T_m \times [0,\infty)$, satisfying

$$0 < u_{\min}(\tau) \le 1 \le u_{\max}(\tau), \quad \forall \tau \in [0,\infty). \tag{2.14}$$

We note that Propositions 5–7, Lemma 10, Corollary 11 and Proposition 12 in [13] are all valid here for solution $v(\theta,t)$ to (1.4) and for solution $u(\theta,\tau)$ to (2.13). In particular, we have the following uniform gradient estimate:

$$|u_\theta(\theta,\tau)| \le C, \quad \forall (\theta,\tau) \in T_m \times [0,\infty), \tag{2.15}$$

where C is a constant depending only on γ_0 and p.

We now can conclude the following:

Proposition 2.1. *For any sequence $\tau_n \to \infty$, there is a subsequence, which we also call it τ_n, so that $u(\theta,\tau_n)$ converges uniformly on T_m to a Lipschitz function $w(\theta) \ge 0$, which is $2m\pi$-periodic over $\mathbb{R}$ and satisfies*

$$0 \le \min_{\theta \in T_m} w(\theta) \le 1 \le \max_{\theta \in T_m} w(\theta). \tag{2.16}$$

Moreover, over the open set $\Omega_+ = \{\theta \in \mathbb{R} : w(\theta) > 0\}$, $w(\theta)$ is smooth and satisfies the ODE

$$w''(\theta) + w(\theta) - w^{1-p}(\theta) = 0, \quad \forall \theta \in \Omega_+. \tag{2.17}$$

Remark 2.1. Since $p \in (1,2)$, in case Ω_+ is a proper subset of $\mathbb{R}$, then $w''(\theta)$ blows up at the boundary of Ω_+. This is different from the case $p \in (0,1)$.

Remark 2.2. By regularity theory, $u(\theta,\tau_n)$ actually converges (passing to a further subsequence if necessary) in $C^\infty(I)$ to $w(\theta)$ as $n \to \infty$, where I is any compact subset of Ω_+. The reason is that, roughly speaking, Equation

(2.13) is *uniformly parabolic* on I along the sequence of time τ_n. We shall need this result in the proof of Theorem 3.1. Also see Section 6 for details.

Proof. It suffices to show that $w(\theta)$ is smooth and satisfies the ODE (2.17) on Ω_+.For any interval $[a,b] \subset \Omega_+$ and any test function $\varphi \in C_0^\infty[a,b]$, similar to the proof of Proposition 14 in [13], we have

$$\int_a^b [w(\theta)\phi''(\theta) + w(\theta)\phi(\theta) - w^{1-p}(\theta)\phi(\theta)]d\theta = 0. \tag{2.18}$$

Hence $w(\theta)$ solves the ODE (2.17) in weak sense over the open set Ω_+ (note that since $|u_\theta(\theta,\tau)|$ is uniformly bounded, the function $w \in W^{1,2}(T_m)$). Since $w(x) > 0$ in Ω_+, by regularity theory we actually have $w(x) \in C^\infty(\Omega_+)$. □

For convenience we introduce the following terminology: we say the limit $w(\theta)$ is *non-degenerate* if $w(\theta) > 0$ everywhere over $\mathbb{R}$. Otherwise we call it *degenerate*. The reason for this terminology is that PDE (2.13) will degenerate eventually if $w(\theta)$ is degenerate. When $w(\theta)$ is non-degenerate, it gives rise to a *self-similar solution* (or call it *homothetic solution*). Such solutions are similar to the Abresch-Langer functions in curve shortening flow ($\alpha = 1$). We also know that, in curve shortening flow, degenerate limit cannot happen in type one blow-up.

The next step is to show that we actually have the convergence of $u(\theta,\tau)$ to the limit function $w(\theta)$ as $\tau \to \infty$, not just along a sequence of times $\tau_n \nearrow \infty$. We need one more property of u which will be used very often.

Lemma 2.1. *For any $\theta_0 \in T_m$, there is a $\tau_0 > 0$ so that $u_\theta(\theta_0,\tau)$ does not change sign for all $\tau > \tau_0$, i.e., the limit $\lim_{\tau\to\infty} \operatorname{sgn}(u_\theta(\theta_0,\tau))$ exists, where by definition*

$$\operatorname{sgn}(\xi) = 1 \textit{ if } \xi > 0; \quad \operatorname{sgn}(\xi) = 0 \textit{ if } \xi = 0; \quad \operatorname{sgn}(\xi) = -1 \textit{ if } \xi < 0. \tag{2.19}$$

Moreover, if $\lim_{\tau\to\infty} \operatorname{sgn}(u_\theta(\theta_0,\tau)) = 0$, then $u(\theta_0,\tau)$ is symmetric with respect to θ_0.

Proof. This is a consequence of Theorem C in p. 267 of [19]. □

3. The convergence

To prove the full convergence of $u(\theta,\tau)$ as $\tau \to \infty$, we shall rely on the result of Lemma 2.1. We first need to study more on the solution behavior of the ODE (2.17). Let $w(\theta)$ be the function from Proposition 2.1 and set $M = \max_{\theta\in\mathbb{R}} w(\theta) \geq 1, \ell = \min_{\theta\in\mathbb{R}} w(\theta) \leq 1$. We know that whenever $w(\theta)$ is positive, it must satisfy ODE (2.17). In view of this, if $M \in [1, \gamma)$, where $\gamma = [2/(2-p)]^{1/p} > 1, p \in (1, 2)$, then on the maximal connected open interval $I \subset \Omega_+$ where M is attained we have

(3.20)
$$\frac{(w'(\theta))^2}{2} + \frac{w^2(\theta)}{2} - \frac{1}{2-p}w^{2-p}(\theta) = \frac{M^2}{2} - \frac{1}{2-p}M^{2-p} \quad \text{for all } \theta \in I.$$

From it we can see that $w(\theta)$ is actually a C^2 *positive entire* solution of the ODE (2.17) over $\mathbb{R}$ when $M \in [1, \gamma)$. When $M = 1$, $w(\theta) \equiv 1$ everywhere; when $M \in (1, \gamma)$, we have $\ell \in (0, 1)$ with $F(M) = F(\ell)$, where $F(s)$ is the function given by

$$F(s) = \frac{s^2}{2} - \frac{1}{2-p}s^{2-p}, \quad s \in (0, \infty), \quad F(0) = F(\gamma) = 0. \tag{3.21}$$

When $M \in [\gamma, \infty)$, the limit $w(\theta)$ attains zero somewhere. Assume $w(0) = M$. If $M = \gamma, w(\theta)$ is a symmetric bump over $I = [-\pi/p, \pi/p]$ described by (3.24) below with $w(\theta) = w'(\theta) = 0$ at $\theta = \pm\pi/p$. If $M > \gamma, w(\theta)$ still looks like a symmetric bump but is supported on a smaller interval $I = (-\beta, \beta)$ with $w'(\beta-) = -w'((-\beta)+) < 0$, $\beta \in (\pi/2, \pi/p)$. The value of $w'(\beta-)$ is related to the maximum M via the identity

$$\frac{(w'(\beta-))^2}{2} = \frac{M^2}{2} - \frac{1}{2-p}M^{2-p} > 0, \quad M \in [\gamma, \infty). \tag{3.22}$$

Finally, we see that as $M \to \infty$, $w'(\beta-) \to -\infty$ and the interval $I = (-\beta, \beta)$ shrinks to $(-\pi/2, \pi/2)$. See [23] for details.

Another important property is that if w is a positive entire solution of (2.17) over $\mathbb{R}$, its minimal period is a monotone function of its maximum value M. See [3].

We also note that if w_1 and w_2 are two different positive entire solutions of (2.17) over $\mathbb{R}$ with different periods, one can find a point $\theta_* \in \mathbb{R}$ such that

$$w_1'(\theta_*)w_2'(\theta_*) < 0. \tag{3.23}$$

In case w_1 and w_2 are two different positive entire solutions of (2.17) but only differ by a translation, (3.23) still holds. Finally, if at any $\theta \in \mathbb{R}$ we have $w_1(\theta) = w_2(\theta)$, then the difference $(w_1'(\theta))^2 - (w_2'(\theta))^2$ is a constant independent of θ due to (3.20).

We are now ready to prove the following full convergence:

Theorem 3.1. *As $\tau \to \infty, u(\theta, \tau)$ converges uniformly on T_m to a Lipschitz function $w(\theta) \geq 0$, which is $2m\pi$-periodic over $\mathbb{R}$ (as described in Proposition 2.1).*

Proof. We prove by contradiction. By Proposition 2.1 there is a sequence $\tau_n \to \infty$ so that $u(\theta, \tau_n)$ converges uniformly on T_m to a Lipschitz function $w(\theta) \geq 0$. Let $M = \max_{\theta \in \mathbb{R}} w(\theta)$. Assume there exists another sequence $s_n \to \infty$ so that $u(\theta, s_n)$ converges uniformly to a different Lipschitz function $\bar{w}(\theta)$ and let $\bar{M} = \max_{\theta \in \mathbb{R}} \bar{w}(\theta)$. In case $M \neq \bar{M}$, say $M < \bar{M}$, then similar to the continuity result observed in p. 167 of [13] for $p \in (0,1)$, for $p \in (1,2)$ we also have the following result: for any $\tilde{M} \in (M, \bar{M})$, there exists a sequence $r_n \to \infty$ so that $u(\theta, r_n)$ converges uniformly to some Lipschitz function $\tilde{w}(\theta)$ with $\max_{\theta \in \mathbb{R}} \tilde{w}(\theta) = \tilde{M}$, and it is smooth and satisfies the ODE (2.17) whenever it is positive.

Since either $M \in [1, \gamma)$ or $M \in [\gamma, \infty)$, and the same for $\bar{M}$, there are several cases to discuss. Note that although $w(\theta)$ is different from $\bar{w}(\theta)$, it may be possible to have $M = \bar{M}$.

Case 1: $M \in [1, \gamma)$ and $\bar{M} \in [1, \gamma)$.

For $M = 1$we have $w(\theta) \equiv 1$, hence $\bar{M} \in (1, \gamma)$. By the above continuity result, one can find two sequences going to infinity such that the corresponding limits w_1, w_2 of u are two different entire positive solutions of the ODE (2.17) over $\mathbb{R}$ with different maximum values (or equivalently, different periods). Moreover, there is a point $\theta_* \in \mathbb{R}$ such that (3.23) holds. It will contradict Lemma 2.1 (see Remark 2.2 also). Similarly for $M \in (1, \gamma), \bar{M} = 1$.

For $M \in (1, \gamma)$ and $\bar{M} \in (1, \gamma)$ with $M \neq \bar{M}$, by the same reason as in the above sign consideration, it is impossible to happen. If $M = \bar{M}$, then since they are both entire positive solutions of the ODE (2.17), uniqueness implies that they only differ by a translation, i.e., $\bar{w}(\theta) = w(\theta + \theta_0)$ for some $\theta_0 \in \mathbb{R}$. But if $w(\theta)$ and $\bar{w}(\theta)$ does not coincide, one can find some point θ_* with $w'(\theta_*)\bar{w}'(\theta_*) < 0$. Again this contradicts Lemma 2.1.

Case 2: $M \in [1, \gamma)$ and $\bar{M} \in [\gamma, \infty)$ (or $M \in [\gamma, \infty)$ and $\bar{M} \in [1, \gamma)$).

In this case, one can also find two sequences going to infinity such that the corresponding limits w_1, w_2 of u are two different entire positive solutions of the ODE (2.17) over $\mathbb{R}$ with different maximum values. The same discussions as in Case 1 exclude this case.

As a result of the above, we only have to consider the case where both M and $\bar{M} \in [\gamma, \infty)$.

Case 3A: $M = \bar{M} = \gamma$.

Without loss of generality, we may assume that $w(\theta)$ attains its maximum γ at $\theta = 0$. In such a case, over the domain $[-\pi/p, \pi/p], w(\theta)$ is given explicitly by (see Section 3 in [13] also)

$$w(\theta) = \left[\frac{2}{2-p}\cos^2\left(\frac{p}{2}\theta\right)\right]^{\frac{1}{p}}, \quad \theta \in [-\pi/p, \pi/p]. \tag{3.24}$$

One can check that $w(\pm\pi/p) = w'(\pm\pi/p) = 0$. However $w''(\theta)$ blows up at $\theta = \pm\pi/p$.

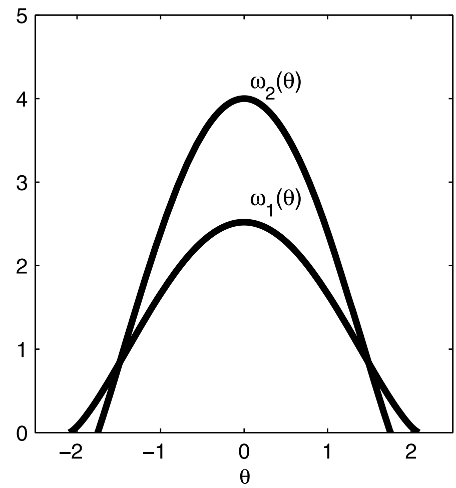
On other maximal connected open interval I where $w > 0$ on I, it also looks like the bump given by (3.24); otherwise we have $\max_{\theta\in I} w(\theta) \in (1, \gamma)$, which will force $w(\theta)$ to be entire and positive over $\mathbb{R}$. Thus $w(\theta)$ is composed of one or several bumps of the form (3.24), joined together by zero function. The same for $\bar{w}(\theta)$.

Since $w(\theta)$ is different from $\bar{w}(\theta)$, due to Lemma 2.1 there must be some interval J of the form $J = [\theta_0 - \pi/p, \theta_0 + \pi/p]$ for some θ_0 such that $w(\theta)$ is a symmetric bump over J but $\bar{w}(\theta) \equiv 0$ on J. We have $\lim_{n\to\infty} u(\theta_0, \tau_n) = \gamma$ and $\lim_{n\to\infty} u(\theta_0, s_n) = 0$. By continuity argument, for any $\tilde{M} \in (0, \gamma)$, there exists a sequence $r_n \to \infty$ so that $\lim_{n\to\infty} u(\theta_0, r_n) = \tilde{M}$. We can choose $\tilde{M} \in (1, \gamma)$ and so $u(\theta, r_n)$ converges uniformly to a function $\tilde{w}(\theta)$ with $\tilde{w}(\theta_0) = \tilde{M}$, where by Lemma 2.1 again, we have $\tilde{w}'(\theta) \geq 0$ on the left-hand side of $\theta = \theta_0$ and $\tilde{w}'(\theta) \leq 0$ on the right-hand side. Hence $\tilde{w}(\theta_0) = \tilde{M} \in (1, \gamma)$ must be a local maximum. Again this will imply $\tilde{w}(\theta)$ to be entire and positive over $\mathbb{R}$. By Case 2 it will yield a contradiction.

Case 3B: $M = \gamma$ and $\bar{M} > \gamma$ (or $M > \gamma, \bar{M} = \gamma$).

According to the above observation, in this case, without loss of generality, we may assume the existence of three sequences of time $\tau_{1n}, \tau_{2n}, \tau_{3n}$ such that along them $u(\theta, \cdot)$ converges uniformly to three different bumps (all centered at $\theta = 0$, with maximum at $\theta = 0$) w_1, w_2, w_3 with maximum values $M_1 = \gamma, M_2 \in (M_1, M_3), M_3 = \bar{M} > \gamma$. They are supported on closed intervals I_1, I_2, I_3 with $I_1 \supset I_2 \supset I_3$, where $I_1 = [-\pi/p, \pi/p]$. That is, the higher the maximum, the smaller the support. By Lemma 2.1, $w_2 = 0$ on $I_1 - I_2, w_3 = 0$ on $I_1 - I_3$.

For simplicity, we write $I_i = [-\alpha_i, \alpha_i],\ i = 1, 2, 3$, where $\alpha_1 = \pi/p > \alpha_2 > \alpha_3 > 0$. By uniqueness, any two of these three functions intersect in a transversal way. Let $\eta > 0$ be small and let $I_1^* = [-\alpha_2 - \eta, \alpha_2 + \eta]$. We have $I_2 \subset I_1^* \subset I_1$. Over I_1^*, we have $w_1(\theta) \geq \varepsilon > 0$ for some $\varepsilon > 0$, and by parabolic regularity theory, $u(\theta, \tau_{1n})$ converges to $w_1(\theta)$ in C^1 over I_1^*. Also any two of these three functions have exactly two transversal intersection points on I_1^*, symmetric with respect to the y-axis. In the following picture, we choose $p = 1.5$ and $w_1(\theta)$ is the bump function given by (3.24).



We shall consider the intersection of $w_1(\theta)$ with a small translation of $w_2(\theta)$. For fixed $|\theta_0| < \eta$, consider the function $w_2(\theta - \theta_0)$, where $\theta \in I_1^* = [-\alpha_2 - \eta, \alpha_2 + \eta]$. Then

$$\begin{cases} w_2(\theta - \theta_0) > 0 & \text{in the interior of } I_2 + \theta_0 = [-\alpha_2 + \theta_0, \alpha_2 + \theta_0], \\ w_2(\theta - \theta_0) = 0 & \text{on } I_1^* - (I_2 + \theta_0). \end{cases}$$

As $\eta > 0$ is small, we see that for any $|\theta_0| < \eta$, $w_1(\theta)$ and $w_2(\theta - \theta_0)$ still have exactly two transversal intersection points on I_1^*. We also note that, by continuity, there exists some number $\lambda > 0$, *independent of* $|\theta_0| < \eta$, such that if a translation $w_2(\theta - \theta_0)$ intersects $w_1(\theta)$ at some $\theta = \theta_*$, then

$$|w_1'(\theta_*) - w_2'(\theta_* - \theta_0)| \geq \lambda > 0. \tag{3.25}$$

Since $u(\theta, \tau_{1n})$ converges to $w_1(\theta)$ in C^1 over I_1^*, and for any $|\theta_0| < \eta$, $w_1(\theta)$ and $w_2(\theta - \theta_0)$ has exactly two intersection points (transversal in an uniform way due to (3.25)) on I_1^*, there exists a large number N, independent of $|\theta_0| < \eta$, such that $u(\theta, \tau_{1n}) - w_2(\theta - \theta_0)$ has exactly two simple zeros over I_1^* for all $n \geq N$ and all $|\theta_0| < \eta$.

Note that on I_2 both $u(\theta, \tau)$ and $w_2(\theta)$ are solutions to the PDE

$$\frac{\partial u}{\partial \tau} = u^p(u_{\theta\theta} + u - u^{1-p}), \quad \tau \in [0, \infty). \tag{3.26}$$

Their difference $u(\theta,\tau)-w_2(\theta)$ is positive (hence non-zero) on the boundary of I_2 for all time. By Angenent's result (see [5, pp. 607, 609]), the number of zeros of $u(\theta,\tau)-w_2(\theta)$, even counted with multiplicity, is non-increasing in time $\tau\in[0,\infty)$ over I_2. Moreover since $w_2(\theta-\theta_0)$ is also a solution to (3.26) on $I_2+\theta_0$, the number of zeros of $u(\theta,\tau)-w_2(\theta-\theta_0)$ is non-increasing in time $\tau\in[0,\infty)$ over $I_2+\theta_0$. Because $w_2(\theta-\theta_0)=0$ on $I_1^*-(I_2+\theta_0)$ and $u>0$ on $I_1^*-(I_2+\theta_0)$, we conclude that for any $|\theta_0|<\eta$, the number of zeros of $u(\theta,\tau)-w_2(\theta-\theta_0)$ is non-increasing in time $\tau\in[0,\infty)$ over I_1^*. On the other hand, along the sequence of times $\tau_{1N},\tau_{1N+1},\tau_{1N+2},\ldots$, we know that $u(\theta,\cdot)-w_2(\theta-\theta_0)$ has two simple zeros over I_1^* for all $|\theta_0|<\eta$. Hence there exists some time $T>0$ (for example one can choose $T=\tau_{1N}$), independent of $|\theta_0|<\eta$, so that the function $u(\theta,\tau)-w_2(\theta-\theta_0)$ has exactly two simple zeros over I_1^* for all $\tau>T$ and all $|\theta_0|<\eta$.

Since $u(0,\tau_{3n})$ converges to $w_3(0)=M_3>M_2$, there are infinitely many τ so that

$$\max_{\theta\in I_1^*}u(\theta,\tau)>\max_{\theta\in I_1^*}w_2=M_2.$$

Similarly, since $u(0,\tau_{1n})$ converges to $w_1(0)=M_1<M_2$, there are infinitely many τ so that

$$\max_{\theta\in I_1^*}u(\theta,\tau)<\max_{\theta\in I_1^*}w_2=M_2.$$

By continuity, there is an infinite sequence $\xi_n\to\infty$ so that

$$\max_{\theta\in I_1^*}u(\theta,\xi_n)=M_2,\quad \forall n. \tag{3.27}$$

By choosing subsequence if necessary, we may also assume that $u(\theta,\xi_n)$ converges to a Lipschitz function $v(\theta)$ uniformly on I_1^*, with $\max_{\theta\in I_1^*}v(\theta)=M_2$. From the above discussion, $v(\theta)$ must have a local maximum at $\theta=0$ and is a solution of the ODE (2.17) as long as $v(\theta)>0$. As $v(0)=M_2$, we actually have $v(\theta)=w_2(\theta)$ for $\theta\in I_2$.

By (3.27) and the fact that $u(\theta,\xi_n)$ converges to $w_2(\theta)$ uniformly on I_1^*, continuity argument implies the existence of some time $s>T$ so that over the interval $I_1^*, u(\theta,s)$ has a maximum at some $\theta=\theta_0$ with $|\theta_0|<\eta$, and $u(\theta_0,s)=M_2$. Hence over $I_1^*, u(\theta,s)-w_2(\theta-\theta_0)$ has a double zero at $\theta=\theta_0, |\theta_0|<\eta$, and time $s>T$, contradicting to the above observation (by definition, a function $f(\theta)$ has a double zero at θ_0 means that $f(\theta_0)=f'(\theta_0)=0$).

Case 3C: $M>\gamma$ and $\bar{M}>\gamma$.

This case can be argued in ways similar to Case 3B. We omit it.

The proof of Theorem 3.1 is now complete. □

4. Converging to a degenerate limit

As the blow-up rate is always of type one, a remaining question is whether the limit $w(\theta)$ is degenerate or not. In this section, we show the existence of an initial curve so that the limit $w(\theta)$ is degenerate, i.e., $w(\theta) = 0$ somewhere. Geometrically, it means that when we blow down the curvature by rescaling, the non-blow-up part of the curvature goes into the region where $w(\theta) = 0$.

We provide two methods. The first one is simple, based on a method similar to that in p. 630 of [5]. We consider the contraction of a symmetric cardioid-like curve (with rotation index $m = 2$) and see that its blow-up set is a proper subset of T_m (now $m = 2$). The second one is more complicated, based on period and energy considerations. In this approach, we need to assume that the rotation index $m \in \mathbb{N}$ is large enough. However, the second method can allow the initial data to be non-symmetric (see Remark 4.6).

Our first convergence result is:

Lemma 4.1. *Let $m \geq 2, m \in \mathbb{N}$, and let θ_* be any number satisfying*

$$\frac{\pi}{2} < \theta_* \leq \pi. \tag{4.28}$$

Then there exists a positive symmetric initial data $v_0(\theta), \theta \in T_m$, of (1.4) *satisfying*

$$\int_{-m\pi}^{m\pi} v_0^{1-p}(\theta) \mathrm{e}^{\mathrm{i}\theta} d\theta = 0, \quad \mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta \tag{4.29}$$

such that the corresponding rescaled solution $u(\theta, \tau)$ converges uniformly to a $2m\pi$-periodic Lipschitz function $w(\theta) \geq 0$. Moreover $w(\theta)$ is a bump-like function, symmetric with respect to $\theta = 0$, with support contained in $(-\theta_*, \theta_*)$.

Remark 4.1. Since $p \in (1, 2)$, we also have $\pi/2 < \pi/p < \pi$. Hence if one choose θ_* such that $\pi/2 < \theta_* \leq \pi/p < \pi$, then $w(\theta)$ in the above lemma cannot be given by (3.24). On the other hand, if $\pi/2 < \pi/p < \theta_* \leq \pi$, then $w(\theta)$ may be given by (3.24).

Remark 4.2. The degeneracy of $w(\theta)$ clearly holds if the positive periodic initial data $v_0(\theta), \theta \in T_m$, does *not* satisfy the integral condition (4.29) (say the real part is non-zero). This means that $v_0^{1-p}(\theta)$ does not represent the radius of curvature of a convex immersed closed plane curve. By (1.8) we have

$$\int_{-m\pi}^{m\pi} v^{1-p}(\theta,t)\cos\theta d\theta = \int_{-m\pi}^{m\pi} v_0^{1-p}(\theta)\cos\theta d\theta \neq 0 \tag{4.30}$$

for all $t \in [0, T_{\max})$. Thus it is impossible to have $\lim_{t\to T_{\max}} v(\theta,t) = \infty$ for all $\theta \in T_m$ and we must have $w(\theta) = 0$ somewhere.

Proof. We shall prove the result for $m = 2$. For $m > 2$ the proof is the same. Let $v_0(\theta)$ be a positive symmetric function satisfying $v_0(\theta) = v_0(-\theta)$, $\theta \in [-2\pi, 2\pi]$ and $v_0'(\theta) < 0$ for $\theta \in (0, 2\pi), v_0'(0) = v_0'(2\pi) = 0$ (both are simple zeros of $v_0'(\theta)$), so that

$$\int_0^{\theta_*} v_0^{1-p}(\theta)\cos\theta d\theta < 0, \quad \theta_* \in (\pi/2, \pi], \quad p \in (1,2). \tag{4.31}$$

In addition, we also require $v_0(\theta)$ to satisfy (4.29) (with $m = 2$). Geometrically, $v_0^{1-p}(\theta)$ represents the *radius of curvature* of a symmetric cardioid-like closed curve γ_0 (we assume that the point with tangent angle $\theta = 0$ is the origin) such that at the point with tangent angle $\theta = \theta_*$ its x-coordinate is negative. Such a curve is symmetric with respect to the y-axis with two vertices, one at the origin, the other lying on negative y-axis. The existence of such a function $v_0(\theta)$ can therefore be asserted.

Let $v(\theta,t)$ be the solution of (1.4) with the above $v_0(\theta)$ as initial data. By symmetry we have

$$\begin{cases} v(\theta,t) = v(-\theta,t), & \theta \in [-2\pi, 2\pi], \\ v_\theta(\theta,t) < 0 \quad \text{for} \quad \theta \in (0,2\pi); & v_\theta(0,t) = v_\theta(2\pi,t) = 0, \end{cases} \tag{4.32}$$

which means that the closure of the blow-up set for $v(\theta,t)$ must have the form $[-\beta, \beta]$ for some $\beta > \pi/2$. Moreover, the limit function $w(\theta)$ is non-increasing on $[0, 2\pi]$. Straightforward computation gives

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{1-p}\int_0^{\theta_*} v^{1-p}(\theta,t)\cos\theta d\theta\right) \\ &\quad = v(\theta_*,t)\sin\theta_* + v_\theta(\theta_*,t)\cos\theta_* > 0, \quad \theta_* \in (\pi/2,\pi]. \end{aligned} \tag{4.33}$$

Hence

$$\int_0^{\theta_*} v^{1-p}(\theta,t)\cos\theta d\theta, \quad p \in (1,2)$$

is strictly decreasing in time and is negative for all $t \in [0, T_{\max})$, where $T_{\max}$ is the blow-up time of $v(\theta, t)$. However, if $v(\theta, t) \to \infty$ as $t \to T_{\max}$ for all $\theta \in [0, \theta_*]$, then

$$\int_0^{\theta_*} v^{1-p}(\theta, t) \cos\theta d\theta \to 0 \quad \text{as} \quad t \to T_{\max}, \quad p \in (1, 2), \tag{4.34}$$

which gives a contradiction. Thus the blowup set of $v(\theta, t)$ is a subset of $(-\theta_*, \theta_*)$. This implies that the set $\{\theta : w(\theta) > 0\}$ is contained in $(-\theta_*, \theta_*)$. □

Remark 4.3. In the above proof, if the closure of the blow-up set is given by $[-\beta, \beta]$ for some $\beta \in (\pi/2, \theta_*)$, then by $v_{\min}^{1-p}(t) \cos\theta \le v^{1-p}(\theta, t) \cos\theta < 0$ for $\theta \in [\beta, \theta_*], t \in [0, T_{\max})$, we have

$$\begin{aligned} &\int_0^{\beta} v^{1-p}(\theta, t) \cos\theta d\theta + v_{\min}^{1-p}(t) \int_\beta^{\theta_*} \cos\theta d\theta \\ &\quad \le \int_0^{\theta_*} v^{1-p}(\theta, t) \cos\theta d\theta < \int_0^{\theta_*} v_0^{1-p}(\theta) \cos\theta d\theta < 0, \end{aligned}$$

which gives

$$\lim_{t \to T_{\max}} \left(v_{\min}^{1-p}(t) \int_\beta^{\theta_*} \cos\theta d\theta\right) \le \int_0^{\theta_*} v_0^{1-p}(\theta) \cos\theta d\theta < 0$$

and (note that $v_{\min}(t)$ is increasing on $[0, T_{\max})$) we have the following upper bound on $v_{\min}(t)$:

$$\lim_{t \to T_{\max}} v_{\min}(t) \le \left(\frac{\int_\beta^{\theta_*} \cos\theta d\theta}{\int_0^{\theta_*} v_0^{1-p}(\theta) \cos\theta d\theta} \right)^{\frac{1}{p-1}}. \tag{4.35}$$

Remark 4.4. The symmetry condition of the initial data is needed in the proof (see (4.32)).

To prove the second result, we need to make use of the following known result (see [3, 23]):

Lemma 4.2. *For $p \in (0, 2)$, if $w(\theta)$ is an entire positive non-constant periodic solution of the ODE*

$$w''(\theta) + w(\theta) - w^{1-p}(\theta) = 0, \quad \theta \in \mathbb{R}, \tag{4.36}$$

then the (minimal) period T of $w(\theta)$ satisfies

$$\begin{cases} T \in (2\pi/\sqrt{p}, 2\pi/p) \quad \text{if } p \in (0,1); \quad T \in (2\pi/p, 2\pi/\sqrt{p}) \quad \text{if } p \in (1,2), \\ T = 2\pi \quad \text{if } p = 1. \end{cases}$$

In such a case, $\max_{\mathbb{R}} w(\theta) \in (1, \gamma)$*, where* $\gamma = [2/(2-p)]^{1/p} > 1$.

The second result is:

Lemma 4.3. *If the rotation index $m \in \mathbb{N}$ is large enough, then there exists a positive symmetric initial data $v_0(\theta), \theta \in T_m$, of* (1.4) *satisfying* (4.29) *such that the corresponding rescaled solution $u(\theta, \tau)$ converges uniformly to a $2m\pi$-periodic Lipschitz function $w(\theta) \geq 0$. Moreover $w(\theta)$ is a bump-like function, symmetric with respect to $\theta = 0$, with support given by $[-\beta, \beta] \subset [-m\pi, m\pi]$, where $\beta \in (\pi/2, \pi/p]$.*

Remark 4.5. In the above lemma, $w(\theta)$ may or may not be given by (3.24). Since $w''(\pm\beta)$ blows up, $u(\theta, \tau)$ cannot have bounded $u_{\theta\theta\theta}(\theta, \tau)$ in $T_m \times [0, \infty)$. Moreover, if $\beta \in (\pi/2, \pi/p)$, then $w'(\theta)$ has a jump discontinuity at $\theta = \pm\beta$. It implies that $u(\theta, \tau)$ cannot have bounded $u_{\theta\theta}(\theta, \tau)$ in $T_m \times [0, \infty)$. Conversely if $u_{\theta\theta}(\theta, \tau)$ is uniformly bounded on $T_m \times [0, \infty)$, then $w(\theta)$ must be given by (3.24).

Proof. The proof is similar to the case for $p \in (0, 1)$ (see Proposition 27 in [13]). We only sketch some of the key points.

Let $v_0(\theta)$ be a positive symmetric function satisfying $v_0(\theta) = v_0(-\theta), \theta \in T_m = [-m\pi, m\pi]$ and $v_0'(\theta) < 0$ for $\theta \in (0, m\pi), v_0'(0) = v_0'(m\pi) = 0$ (both are simple zeros of $v_0'(\theta)$). We also require it to satisfy (4.29). The solution $v(\theta, t)$ of (1.4) with the above initial data will satisfy (4.32) (with 2 replaced by m) for all $t \in [0, T_{\max})$. The rescaled bounded solution $u(\theta, \tau)$ also satisfies (4.32) and it makes the Lyapunov functional

$$\mathcal{E}(u(\cdot, \tau)) = \int_{-m\pi}^{m\pi} \left(u_\theta^2 - u^2 + \frac{2}{2-p} u^{2-p} \right) d\theta, \quad p \in (1, 2)$$

decreasing in $\tau \in [0, \infty)$. If the rotation index $m \in \mathbb{N}$ is large enough, one can also adjust $v_0(\theta)$ to satisfy the additional property

$$\mathcal{E}(u(\cdot, 0)) < \mathcal{E}(1) = 2m\pi \left(\frac{2}{2-p} - 1 \right), \quad \text{where } u(\theta, 0) = \frac{v_0(\theta)}{R(0)}. \tag{4.37}$$

In doing so, the limit $w(\theta) \geq 0$ cannot be the constant $w \equiv 1$ (it is the only positive entire constant solution to the ODE (4.36)). Moreover, $w(\theta)$ is also symmetric (with respect to $\theta = 0$) and non-increasing on $(0, m\pi)$ due to (4.32). Now if $w(\theta)$ is positive everywhere on T_m, it will be an entire non-constant periodic solution to (4.36), which will force it to satisfy

$$w'(\theta) < 0 \quad \text{for } \theta \in (0, m\pi); \quad w'(0) = w'(m\pi) = 0.$$

On the other hand by Lemma 4.2 its period is at most $2\pi/\sqrt{p}$. We will obtain a contradiction if m is large enough and so $w(\theta)$ cannot be positive everywhere on T_m. □

Remark 4.6. The advantage of the above method of proof is two-fold: First, its proof is valid for both $p \in (0, 1)$ and $p \in (1, 2)$ (on the other hand, the method of proof in Lemma 4.1 is valid only for $p \in (1, 2)$, for example, (4.34) fails for $p \in (0, 1)$). Second, we can also find a *non-symmetric* initial data $v_0(\theta)$ such that if $m \in \mathbb{N}$ is large enough, then the limit function $w(\theta) \geq 0$ is *degenerate* on T_m. The symmetry condition plays no role in (4.37) (just take a small non-symmetric perturbation of $v_0(\theta)$). One simply choose the positive periodic function $v_0(\theta)$ on $[-m\pi, m\pi]$ to satisfy

$$\begin{cases} v_0'(\theta) > 0 \quad \text{for } \theta \in (-m\pi, 0); \quad v_0'(\theta) < 0 \quad \text{for } \theta \in (0, m\pi), \\ v_0'(\theta) = 0 \quad \text{at } \theta = 0, \pm m\pi \quad \text{(both are simple zeros of } v_0'(\theta)). \end{cases}$$

We also require $v_0(\theta)$ to satisfy the energy condition $\mathcal{E}(u_0) < \mathcal{E}(1)$ and condition (4.29). Under the evolution, $u(\theta, \tau)$ is positive and bounded. Since the number of zeros of $u_\theta(\cdot, \tau)$ is non-increasing in time (see the discussion in [13]), $u_\theta(\theta, \tau) = 0$ at *exactly two* points in $[-m\pi, m\pi)$ (one at global maximum point, the other at global minimum point). By energy consideration, the limit function $w(\theta)$ of $u(\theta, \tau)$ cannot be $w \equiv 1$. By Lemma 4.2, $w(\theta)$ cannot be positive everywhere on T_m.

Remark 4.7. The energy functional $\mathcal{E}(w)$ for the limit $w(\theta)$ is always positive (no matter it is degenerate or not). Since there is a negative term in the integral, this may not be so clear. To see this, on any fixed disjoint open interval $I = (a, b)$ of the open set $\Omega_+ = \{\theta \in \mathbb{R} : w(\theta) > 0\}$ we have $w(a) = w(b) = 0$ and both $w'(a+)$ and $w'(b-)$ are finite. The energy on

(a, b) becomes

$$\begin{aligned}
&\int_a^b \left((w'(\theta))^2 - w^2(\theta) + \frac{2}{2-p} w^{2-p}(\theta) \right) d\theta \\
&\quad = [w(b)w'(b-) - w(a)w'(a+)] \\
&\qquad + \int_a^b \left(-w(\theta)[w''(\theta) + w(\theta)] + \frac{2}{2-p} w^{2-p}(\theta) \right) d\theta \\
&\quad = \left(\frac{2}{2-p} - 1 \right) \int_a^b w^{2-p}(\theta) d\theta > 0, \quad p \in (1, 2).
\end{aligned}$$

Summing together over all intervals, we obtain $\mathcal{E}(w) > 0$. In particular, the initial energy $\mathcal{E}(u_0)$ for $u_0(\theta) = v_0(\theta)/R(0)$ is positive no matter what the initial condition is.

5. The shape of the degenerate limiting curve

Assume the limit $w(\theta)$ is degenerate with

$$\Omega_+ = \{\theta \in T_m : w(\theta) > 0\} = (-\beta, \beta) \quad \text{for some } \beta \in (\pi/2, \pi/p], \quad p \in (1, 2).$$

Geometrically, $w^{p-1}(\theta)$ represents the curvature $k_\infty(\theta)$ of the limiting curve γ_∞. Using $w(\theta)$ over Ω_+, the part of γ_∞ with positive curvature is determined by (up to a translation) the following parametrization (we assume the origin $O = (0, 0)$ lies on γ_∞ with tangent angle $\theta = 0$)

$$\begin{aligned}
P_\infty(\theta) = &\int_{-\beta}^{\theta} w^{1-p}(s)(\cos s, \sin s) ds \\
&- \int_{-\beta}^{0} w^{1-p}(s)(\cos s, \sin s) ds, \quad \theta \in (-\beta, \beta).
\end{aligned} \tag{5.38}$$

Outside the interval $(-\beta, \beta)$, $w(\theta) \equiv 0$ and formula (5.38) does not make sense. The endpoints of γ_∞ are

$$\begin{aligned}
P_\infty(-\beta) &= -\int_{-\beta}^{0} (w''(s) + w(s))(\cos s, \sin s) ds \\
&= (-w'(\beta)\cos\beta, w'(\beta)\sin\beta + w(0))
\end{aligned}$$

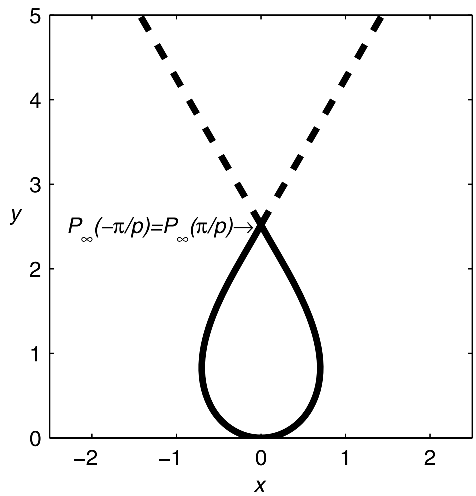
and

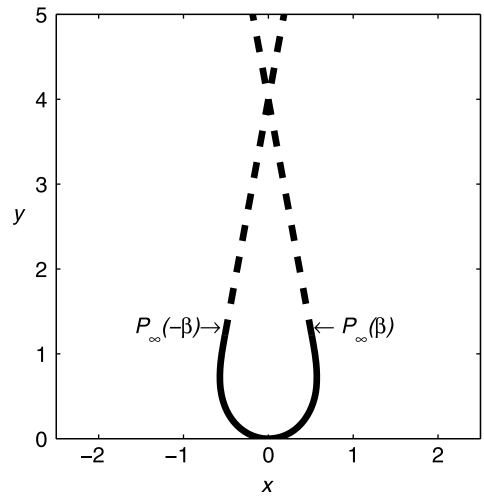
$$P_\infty(\beta) = \int_0^\beta (w''(s) + w(s))(\cos s, \sin s) ds = (w'(\beta)\cos\beta, w'(\beta)\sin\beta + w(0)).$$

By (3.22), we have $w'(\beta)\sin\beta + w(0) > 0$ for any $\beta \in (\pi/2, \pi/p]$ and

$$\begin{cases} w'(\beta)\cos\beta > 0 & \text{if } \beta \in (\pi/2, \pi/p), \\ w'(\beta)\cos\beta = w'(\beta)\sin\beta = 0 & \text{if } \beta = \pi/p. \end{cases}$$

As $\theta \to \pm\beta, w^{1-p}(\theta) \to \infty$. But $P_\infty(\theta)$ has a limiting tangent direction as $\theta \to \pm\beta$. We can attach two rays to make it a complete curve of $\mathbb{R}^2$ with continuous curvature. Since the turning angle $2\beta \in (\pi, 2\pi/p]$ is greater than π, the curve γ_∞ has a self-intersection as shown in the following pictures:





The above pictures are two possible shapes of the singularity after rescaling.

6. More on regularity estimate of $u(\theta, \tau)$; Andrews's estimate

In Remark 2.2 we point out that $u(\theta, \tau_n)$ actually converges in $C^\infty(I)$ to $w(\theta)$ as $n \to \infty$, where $I \subset\subset \Omega_+$ and the reason is that along the sequence τ_n, the rescaled equation (2.13) is *uniformly parabolic*. Perhaps this may not be so convincing because if we want to quote the regularity estimates in standard parabolic theory, the assumption is usually that equations be uniformly parabolic *for all time*, not just along a sequence. Since this is an important result and we rely on it often in the proof of Theorem 3.1, it is worthwhile to clarify it.

In below we mention an elegant regularity estimate, Theorem 6.1, which was actually due to Ben Andrews (except that we filled in some supplementary details later on). I first learned it and its proof from Andrews long time ago (see [4]) when dealing with the case $0 < p < 1$ in equation (2.13). In this range of values for p, equation (2.13) describes the *expansion* of a convex immersed closed plane curve γ_0 (with $u(\theta,\tau)$ representing power of *radius of curvature*, instead of power of curvature; see [13] for details). Hence Theorem 6.1 below was originally presented as a result for plane curves expansion, not contraction. However, a careful look at the long proof of Theorem 6.1 convinces us that it is actually valid for all $p > 0$ (we need this condition in (6.59)). In particular, it is also valid for $1 < p < 2$, which is our case here. Theorem 6.1 will immediately imply the assertions in Remark 2.2.

We first would like to give a motivation for the main estimate (6.51) below. Consider the one-dimensional *porous medium equation* (see Evans PDE [10, p. 180])

$$y_t = (y^\alpha)_{xx}, \quad y = y(x,t) \geq 0, \quad \text{where } \alpha > 1 \text{ is a constant.} \tag{6.39}$$

It has the following so-called *Barenblatt solution* (compact supported solution), given by

$$y(x,t) = \frac{1}{t^{1/(\alpha+1)}}\left(b - C(\alpha)\frac{x^2}{t^{2/(\alpha+1)}}\right)^{\frac{1}{\alpha-1}}, \quad t > 0, \quad x \in (-\infty,\infty), \tag{6.40}$$

where $b > 0$ is an arbitrary constant and $C(\alpha) = (\alpha-1)/[2\alpha(\alpha+1)]$. If we let $F = y^\alpha$, we have

$$\frac{\partial F}{\partial t} = \alpha F^{1-\frac{1}{\alpha}} F_{xx}, \quad F = y^\alpha, \ \alpha > 1. \tag{6.41}$$

By a simple rescaling in x, we may assume the equation is

$$\frac{\partial F}{\partial t} = F^p F_{xx}, \quad p = 1 - \frac{1}{\alpha} \in (0,1), \quad \alpha > 1 \tag{6.42}$$

and if we further let $G = F^p = y^{\alpha-1}$, we get the better-looking equation

$$\frac{\partial G}{\partial t} = G G_{xx} - \left(\frac{p-1}{p}\right) G_x^2, \quad p \in (0,1), \tag{6.43}$$

which is similar to Equation (6.49) below, but without the lower order term $p(G^2 - G)$. For convenience we may choose $b = 1$ and assume $C(\alpha) = 1$ in (6.40) and obtain

$$G(x,t) = \begin{cases} \dfrac{1}{t^{(\alpha-1)/(\alpha+1)}}\left(1 - \dfrac{x^2}{t^{2/(\alpha+1)}}\right), & |x| \leq t^{1/(\alpha+1)}, \\ 0, & |x| \geq t^{1/(\alpha+1)}. \end{cases} \tag{6.44}$$

For each $t > 0$, $G(x,t)$ is smooth and positive on the domain $|x| < t^{1/(\alpha+1)}$ and

$$G_{\max}(t) = \sup_{\mathbb{R}\times\{t\}} G = \frac{1}{t^{(\alpha-1)/(\alpha+1)}}, \quad \alpha > 1$$

is decreasing in $t \in (0,\infty)$ with $G_{\max}(0+) = +\infty$ (singular initial data). We can compare the derivatives of $G(x,t)$ with $G_{\max}(t)$ to get

(6.45)
$$\left|\frac{\partial G}{\partial x}(x,t)\right| \leq \frac{C}{t^{(\alpha-1)/(\alpha+1)}}\frac{t^{1/(\alpha+1)}}{t^{2/(\alpha+1)}} \leq C\sqrt{\frac{G_{\max}(t)}{t}}, \quad \text{for } |x| < t^{1/(\alpha+1)},\ t>0$$

and

$$\left|\frac{\partial^2 G}{\partial x^2}(x,t)\right| \leq \frac{C}{t^{(\alpha-1)/(\alpha+1)}}\frac{1}{t^{2/(\alpha+1)}} = \frac{C}{t}$$

and then

$$G(x,t)\left|\frac{\partial^2 G}{\partial x^2}(x,t)\right| \leq C\left(\frac{G_{\max}(t)}{t}\right), \quad \text{for } |x| < t^{1/(\alpha+1)},\ t > 0. \tag{6.46}$$

Since we are in the one-dimensional case, if we go further we get the trivial estimate

$$G^{k-1}(x,t)\left|\frac{\partial^k G}{\partial x^k}(x,t)\right| = 0, \quad \text{for } |x| < t^{1/(\alpha+1)},\ t > 0, \quad k \geq 3. \tag{6.47}$$

Now let us come back to the solution $u(\theta,\tau)$ of (2.13). We know that it is a positive, bounded solution to Equation (2.13) with

$$0 < u_{\min}(\tau) \leq 1 \leq u_{\max}(\tau), \quad |u_\theta(\theta,\tau)| \leq \max\{C,\ u_{\max}(\tau)\} \tag{6.48}$$

for all $(\theta,\tau) \in T_m \times [0,\infty)$. Here C is a constant depending only on γ_0 and p.

Let $G(\theta,\tau) = u^p(\theta,\tau)$, $p \in (1,2)$. We have

$$\frac{\partial G}{\partial \tau} = GG_{\theta\theta} - \left(\frac{p-1}{p}\right)G_\theta^2 + p(G^2 - G), \quad p \in (1,2). \tag{6.49}$$

We shall derive an estimate similar to the above Barenblatt solution. However, now we have smooth initial condition and the form of the estimate is slightly different.

The main result in this section is the following regularity estimate, which is due to Andrews [4]. Although the computation involved in the proof is quite lengthy, it has the merits of being straightforward, elementary and elegant.

Theorem 6.1 (Andrews [4])[3]. *Let $\tau_0 > 0$ be a fixed time. Suppose $u : T_m \times [0, \tau_0] \to \mathbb{R}^+$is a smooth solution of* (2.13) *with*

$$M_0 = \sup_{T_m \times [0,\tau_0]} G(\theta, \tau) < \infty, \quad G(\theta, \tau) = u^p(\theta, \tau), \quad p \in (1, 2). \tag{6.50}$$

Then for each $k \geq 1$,we have

$$G^{k-1}(\theta, \tau) \left| \frac{\partial^k G}{\partial \theta^k}(\theta, \tau) \right| = G^{k-1}(\theta, \tau) \left| G^{(k)}(\theta, \tau) \right| \leq C_k(p) \left[M_0^k + \left(\sqrt{\frac{M_0}{\tau}} \right)^k \right] \tag{6.51}$$

on $T_m \times (0, \tau_0]$, where $C_k\,(p)$ is a constant depending only on k and p.

Remark 6.1. See also Theorem II1.9 in p. 349 of Andrews [2] for a similar regularity estimate.

Remark 6.2. Note that being bounded by $C_k(p) \left[M_0^k + \left(\sqrt{\frac{M_0}{\tau}} \right)^k \right]$ is equivalent to being bounded by $C_k(p) \left[M_0 + \sqrt{\frac{M_0}{\tau}} \right]^k$.

Remark 6.3. The nice thing about these estimates is that they give good control over higher derivatives of G where G is positive, but allows them to blow up where G approaches zero. According to Remark 4.5, this is possible to happen.

Remark 6.4. Since $u(\theta, \tau)$ is positive, smooth and uniformly bounded on $T_m \times [0, \tau_0]$ for any $\tau_0 > 0$, estimate (6.51) can be replaced by

$$G^{k-1}(\theta, \tau) |G^{(k)}(\theta, \tau)| \leq C(k, p, u_0, M_0), \quad \forall (\theta, \tau) \in T_m \times [0, \tau_0], \quad \forall \tau_0 > 0, \tag{6.52}$$

[3]We point out again that although we have checked the validity of the proof for the case $p > 0$ and supply all the necessary details, Theorem 6.1 is still due to Ben Andrews, not us. Here we thank him a lot for his permission of the proof.

where u_0 is the initial condition in (2.13) and $C(k, p, u_0, M_0)$ is a constant depending only on k, p, u_0, M_0. Equation (6.52) also allows the derivatives of G to blow up where G approaches zero (as $\tau_0 \to \infty$).

Proof (Andrews [4]). By (6.48), (6.51) clearly holds for $k = 1$. However, we want to give a different proof. In the following three sections, we prove Theorem 6.1 for $k = 1$, $k = 2$, and higher k, respectively.

6.1. Proof of Theorem 6.1 for $k = 1$

The case $k = 1$ is slightly different from that for higher k. We want to show that $|G_\theta|$ is uniformly bounded for all $t > 0$. Compute

$$\frac{\partial G_\theta}{\partial \tau} = G G_{\theta\theta\theta} + \left(\frac{2-p}{p}\right) G_\theta G_{\theta\theta} + p(2G-1)G_\theta. \tag{6.53}$$

From this we see that $|G_\theta|$ grows at most exponentially, but we want to do better.

Set

$$a = \frac{1}{2\sup_{T_m \times [0,\tau_0]} G} = \frac{1}{2M_0}, \quad \frac{1}{a} = 2 \sup_{T_m \times [0,\tau_0]} G = 2M_0. \tag{6.54}$$

Then $0 < aG < 1/2$ on $T_m \times [0, \tau_0]$. We shall compute the evolution of the quantity

$$Z(\theta, \tau) := \left(\frac{G_\theta}{1 - aG}\right)(\theta, \tau), \quad \frac{1}{2} < 1 - aG(\theta, \tau) < 1. \tag{6.55}$$

We have

$$\frac{\partial}{\partial \tau}(1 - aG) = G(1 - aG)_{\theta\theta} + a\left(\frac{p-1}{p}\right) G_\theta^2 - ap(G^2 - G)$$

and by the formula

$$\frac{f\Delta h - h\Delta f}{f^2} = \Delta\left(\frac{h}{f}\right) + 2\nabla(\log f) \cdot \nabla\left(\frac{h}{f}\right), \quad f > 0 \tag{6.56}$$

and

$$Z_\theta = \frac{G_{\theta\theta}}{1 - aG} + aZ^2, \quad Z = \frac{G_\theta}{1 - aG}, \tag{6.57}$$

we get

$$\frac{\partial}{\partial \tau}\left(\frac{G_\theta}{1-aG}\right)$$
$$= \frac{1}{(1-aG)^2}\cdot\left\{\begin{array}{l}(1-aG)\left[GG_{\theta\theta\theta}+\left(\frac{2-p}{p}\right)G_\theta G_{\theta\theta}+p(2G-1)G_\theta\right]\\ -G_\theta\left[G(1-aG)_{\theta\theta}+a\left(\frac{p-1}{p}\right)G_\theta^2-ap(G^2-G)\right]\end{array}\right\}$$

and then

$$\frac{\partial}{\partial \tau}\left(\frac{G_\theta}{1-aG}\right)=\left\{\begin{array}{l}G\left(\frac{G_\theta}{1-aG}\right)_{\theta\theta}+2G[\log(1-aG)]_\theta\left(\frac{G_\theta}{1-aG}\right)_\theta\\ \quad +\frac{1}{1-aG}\left[\left(\frac{2-p}{p}\right)G_\theta G_{\theta\theta}+p\,(2G-1)\,G_\theta\right]\\ \quad -\frac{G_\theta}{(1-aG)^2}\left[a\left(\frac{p-1}{p}\right)G_\theta^2-ap(G^2-G)\right]\end{array}\right.$$

and by (6.57) we obtain

$$\frac{\partial Z}{\partial \tau}=\left\{\begin{array}{l}GZ_{\theta\theta}+\left(\frac{2-p}{p}-\frac{2aG}{1-aG}\right)G_\theta Z_\theta\\ -\frac{a}{p}(1-aG)Z^3+\left[\frac{a}{1-aG}\left(G^2-G\right)+(2G-1)\right]pZ,\end{array}\right.$$

where $Z = G_\theta/(1-aG)$. By the maximum principle, if

(6.58) $$Z(\tau)=\max_{\theta\in T_m} Z(\theta,\tau)>0,\quad \tau\in[0,\tau_0],$$

we have

(6.59) $$\begin{aligned}\frac{dZ}{d\tau}&\le -\frac{a}{p}(1-aG)Z^3+\left[\frac{a}{1-aG}\left(G^2-G\right)+(2G-1)\right]pZ\\ &\le -\frac{a}{2p}Z^3+\left(\frac{aG}{1-aG}+2\right)pGZ,\quad p\in(1,2),\quad p>0\\ &\le -\frac{C_1(p)}{M_0}Z^3+C_2(p)M_0Z,\quad a=\frac{1}{2M_0},\quad 1-aG>\frac{1}{2}\end{aligned}$$

for some constants $C_1(p)$ and $C_2(p)$ depending only on p. It follows that

$$Z(\tau) \le \max\left\{C_3(p)M_0,\ C_4(p)\sqrt{\frac{M_0}{\tau}}\right\} \le C_5(p)\left(M_0 + \sqrt{\frac{M_0}{\tau}}\right), \quad \forall \tau \in (0,\tau_0] \tag{6.60}$$

due to the following lemma (its proof is straightforward; we omit it):

Lemma 6.1. *Let $M_0 = \sup_{T_m\times[0,\tau_0]} G > 0$. If $Z(\tau) > 0$ satisfies the differential inequality*

$$\frac{dZ}{d\tau} \le -\frac{C_1(p)}{M_0}Z^3 + C_2(p)M_0 Z, \quad \forall \tau \in [0,\tau_0] \tag{6.61}$$

then

$$Z(\tau) \le \max\left\{C_3(p)M_0, C_4(p)\sqrt{\frac{M_0}{\tau}}\right\} \le C_5(p)\left(M_0 + \sqrt{\frac{M_0}{\tau}}\right), \quad \forall \tau \in (0,\tau_0]. \tag{6.62}$$

Here $C_i(p)$are constants depending only on p, $1 \le i \le 5$.

Since $Z(\theta,\tau) = \frac{G_\theta}{1-aG}$, it is comparable to $G_\theta(\theta,\tau)$. This gives the required estimate of G_θ from above. A similar argument at the minimum point of $Z(\theta,\tau)$ (consider $z(\tau) = \min_{\theta\in T_m} Z(\theta,\tau)$ and the reverse inequality of (6.61)) gives the estimate of G_θ from below.

The proof for $k = 1$ is now complete.

6.2. Proof of Theorem 6.1 for $k = 2$

The case $k = 2$ is also a little different from that for higher k. Differentiate equation (6.49) twice to get

$$\begin{aligned}\frac{\partial G_{\theta\theta}}{\partial\tau} &= \left[GG_{\theta\theta\theta} + \left(\frac{2-p}{p}\right)G_\theta G_{\theta\theta} + p(2G-1)G_\theta\right]_\theta \\ &= G(G_{\theta\theta})_{\theta\theta} + \frac{2}{p}G_\theta G_{\theta\theta\theta} + \left(\frac{2-p}{p}\right)G_{\theta\theta}^2 + p(2G-1)G_{\theta\theta} + 2pG_\theta^2.\end{aligned}$$

Compared with (6.55), the idea next is to compute the evolution of the quantity

$$Z(\theta,\tau) := \left(\frac{GG_{\theta\theta}}{1-aG_\theta^2}\right)(\theta,\tau). \tag{6.63}$$

Letting

$$Q_1 = G_\theta, \quad Q_2 = GG_{\theta\theta} \tag{6.64}$$

and by

$$\begin{cases} (Q_2)_\theta = GG_{\theta\theta\theta} + G_\theta G_{\theta\theta},\ (Q_2)_{\theta\theta} = GG_{\theta\theta\theta\theta} + 2G_\theta G_{\theta\theta\theta} + G_{\theta\theta}^2, \\ G_\theta (Q_2)_\theta = GG_\theta G_{\theta\theta\theta} + G_\theta^2 G_{\theta\theta}, \end{cases}$$

we get

(6.65)

$$\begin{aligned}
\frac{\partial Q_2}{\partial \tau} &= \begin{cases} G\left[G(G_{\theta\theta})_{\theta\theta} + \dfrac{2}{p}G_\theta G_{\theta\theta\theta} + \left(\dfrac{2-p}{p}\right)G_{\theta\theta}^2 \right. \\ \qquad \left. + p(2G-1)G_{\theta\theta} + 2pG_\theta^2\right] \\ +G_{\theta\theta}\left[GG_{\theta\theta} - \left(\dfrac{p-1}{p}\right)G_\theta^2 + p(G^2-G)\right] \end{cases} \\
&= \begin{cases} G(Q_2)_{\theta\theta} - 2G_\theta[(Q_2)_\theta - G_\theta G_{\theta\theta}] + \dfrac{2}{p}G_\theta[(Q_2)_\theta - G_\theta G_{\theta\theta}] \\ + G\left(\dfrac{2-p}{p}\right)G_{\theta\theta}^2 + pG(2G-1)G_{\theta\theta} + 2pGG_\theta^2 \\ \qquad + \left[-\left(\dfrac{p-1}{p}\right)G_\theta^2 G_{\theta\theta} + p(G^2-G)G_{\theta\theta}\right] \end{cases} \\
&= \begin{cases} G(Q_2)_{\theta\theta} + \left(\dfrac{2}{p} - 2\right)G_\theta(Q_2)_\theta + \dfrac{1}{G}\left(\dfrac{2-p}{p}\right)Q_2^2 \\ + \dfrac{1}{G}\left[\left(1-\dfrac{1}{p}\right)G_\theta^2 + p(3G^2-2G)\right]Q_2 + \dfrac{2p}{G}G^2Q_1^2, \quad p\in(1,2). \end{cases}
\end{aligned}$$

Let

$$M_1(\tau) = \max_{\theta\in T_m}|G_\theta(\theta,\tau)| = \max_{\theta\in T_m}|Q_1(\theta,\tau)|, \quad \tau\in[0,\tau_0]. \tag{6.66}$$

By the estimate already established, we have

$$M_1(\tau) \le C_1(p)\left(M_0 + \sqrt{\frac{M_0}{\tau}}\right), \quad M_0 = \sup_{T_m \times [0,\tau_0]} G < \infty. \tag{6.67}$$

We have

$$\frac{\partial Q_2}{\partial \tau} \le G(Q_2)_{\theta\theta} + \left(\frac{2}{p} - 2\right) G_\theta (Q_2)_\theta + \frac{C(p)}{G}\left[Q_2^2 + \left(M_1^2 + M_0^2\right) Q_2 + M_1^2 M_0^2\right] \tag{6.68}$$

whenever $Q_2(\theta,\tau) \ge 0$ on $T_m \times [0,\tau_0]$. Here $C(p)$ is a constant depending only on p. Note that G is positive on $T_m \times [0,\tau_0]$. Hence $\frac{C(p)}{G}$ is well defined.

Similarly, we can compute

$$\begin{aligned} \frac{\partial Q_1^2}{\partial \tau} &= 2G_\theta\left[GG_{\theta\theta\theta} + \left(\frac{2-p}{p}\right) G_\theta G_{\theta\theta} + p\left(2G-1\right) G_\theta\right], \qquad Q_1 = G_\theta \\ &= G\left(Q_1^2\right)_{\theta\theta} - \frac{2}{G} G^2 G_{\theta\theta}^2 + \left[\left(\frac{2-p}{p}\right)\frac{2G_\theta^2}{G} GG_{\theta\theta} + \frac{2p\left(2G^2 - G\right)}{G} G_\theta^2\right] \\ &\le G\left(Q_1^2\right)_{\theta\theta} - \frac{2}{G} Q_2^2 + \frac{C(p)}{G}\left(M_1^2 Q_2 + M_1^2 M_0^2\right) \end{aligned} \tag{6.69}$$

whenever $Q_2(\theta,\tau) \ge 0$.

To obtain a useful estimate at some fixed time $s \in (0,\tau_0]$, we work on the time interval $I = [s/2, s] \subset (0,\tau_0]$ so that

$$\max_I M_1(\tau) \le C_1(p)\left(M_0 + \sqrt{2}\sqrt{\frac{M_0}{s}}\right). \tag{6.70}$$

Then choose

$$a = \frac{1}{2C_1^2\left(M_0 + \sqrt{2}\sqrt{\frac{M_0}{s}}\right)^2}, \quad C_1 = C_1(p), \tag{6.71}$$

we see that

$$aQ_1^2(\theta,\tau) = \frac{Q_1^2(\theta,\tau)}{2C_1^2\left(M_0+\sqrt{2}\sqrt{\frac{M_0}{s}}\right)^2} \le \frac{M_1^2(\tau)}{2C_1^2\left(M_0+\sqrt{2}\sqrt{\frac{M_0}{s}}\right)^2}$$
$$\le \frac{1}{2} \quad \text{on } I=[s/2,s]$$

and so

$$\frac{1}{2} \le 1 - aQ_1^2(\theta,\tau) \le 1, \quad \forall(\theta,\tau)\in T_m\times I. \tag{6.72}$$

Also note that $Z(\theta,\tau) = Q_2/(1-aQ_1^2) \ge 0$ if and only if $Q_2(\theta,\tau)\ge 0$. Compute

$$\frac{\partial Z}{\partial\tau} = \frac{1}{1-aQ_1^2}\frac{\partial Q_2}{\partial\tau} + \frac{aQ_2}{(1-aQ_1^2)^2}\frac{\partial(Q_1^2)}{\partial\tau}$$
$$\le \begin{cases} \dfrac{1}{1-aQ_1^2}\left\{G(Q_2)_{\theta\theta} + \left(\dfrac{2}{p}-2\right)G_\theta(Q_2)_\theta \right. \\ \quad \left. +\dfrac{C(p)}{G}\left[Q_2^2+(M_1^2+M_0^2)Q_2+M_1^2M_0^2\right]\right\} \\ +\dfrac{aQ_2}{(1-aQ_1^2)^2}\left[G(Q_1^2)_{\theta\theta}-\dfrac{2}{G}Q_2^2+\dfrac{C(p)}{G}(M_1^2Q_2+M_1^2M_0^2)\right] \end{cases}$$

whenever $Q_2(\theta,\tau)\ge 0$. Using formula (6.56) again, we get

$$Z_{\theta\theta} = \frac{(1-aQ_1^2)(Q_2)_{\theta\theta}+aQ_2(Q_1^2)_{\theta\theta}}{(1-aQ_1^2)^2} + \frac{2a(Q_1^2)_\theta}{1-aQ_1^2}Z_\theta,$$
$$Z_\theta = \frac{(Q_2)_\theta}{1-aQ_1^2} + \frac{aQ_2(Q_1^2)_\theta}{(1-aQ_1^2)^2}.$$

Now

$$\frac{\partial Z}{\partial\tau} \le \begin{cases} G\left(Z_{\theta\theta} - \dfrac{2a(Q_1^2)_\theta}{1-aQ_1^2}Z_\theta\right) \\ +\dfrac{1}{1-aQ_1^2}\left\{\left(\dfrac{2}{p}-2\right)G_\theta(Q_2)_\theta + \dfrac{C(p)}{G}\right. \\ \quad \left.\times\left[Q_2^2+(M_1^2+M_0^2)Q_2+M_1^2M_0^2\right]\right\} \\ +\dfrac{aQ_2}{(1-aQ_1^2)^2}\left[-\dfrac{2}{G}Q_2^2+\dfrac{C(p)}{G}(M_1^2Q_2+M_1^2M_0^2)\right] \end{cases}$$

$$
= \begin{cases} GZ_{\theta\theta} - G\dfrac{2a(Q_1^2)_\theta}{1-aQ_1^2}Z_\theta + \left(\dfrac{2}{p}-2\right)G_\theta\left(Z_\theta - \dfrac{aQ_2(Q_1^2)_\theta}{(1-aQ_1^2)^2}\right) \\ + \dfrac{1}{1-aQ_1^2}\left\{\dfrac{C(p)}{G}[Q_2^2 + (M_1^2+M_0^2)Q_2 + M_1^2M_0^2]\right\} + \dfrac{aQ_2}{(1-aQ_1^2)^2} \\ \quad \times \left[-\dfrac{2}{G}Q_2^2 + \dfrac{C(p)}{G}(M_1^2Q_2 + M_1^2M_0^2)\right] \end{cases}
$$

$$
= GZ_{\theta\theta} + \left[\left(\frac{2}{p}-2\right)G_\theta - 4aG_\theta Z\right]Z_\theta + \text{the rest},
$$

where

$$
\text{the rest} = \begin{cases} -\left(\dfrac{2}{p}-2\right)G_\theta\dfrac{aQ_2(Q_1^2)_\theta}{(1-aQ_1^2)^2} + \dfrac{C(p)}{G} \\ \quad \times\left\{\dfrac{Q_2^2}{1-aQ_1^2} + \dfrac{(M_1^2+M_0^2)Q_2}{1-aQ_1^2} + \dfrac{M_1^2M_0^2}{1-aQ_1^2}\right\} \\ -\dfrac{2}{G}\dfrac{aQ_2}{(1-aQ_1^2)^2}Q_2^2 + \dfrac{aQ_2}{(1-aQ_1^2)^2}\dfrac{C(p)}{G}(M_1^2Q_2 + M_1^2M_0^2) \end{cases}
$$

$$
\le \begin{cases} -\dfrac{2a}{G}(1-aQ_1^2)Z^3 - \left(\dfrac{2}{p}-2\right)G_\theta\dfrac{2aG_\theta}{G}Z^2 \\ +\dfrac{C(p)}{G}\left\{(1-aQ_1^2)Z^2 + (M_1^2+M_0^2)Z + M_1^2M_0^2\right\} \\ \quad +\dfrac{C(p)}{G}\left(aM_1^2Z^2 + aM_0^2M_1^2\dfrac{Z}{(1-aQ_1^2)}\right) \end{cases}
$$

$$
\le -\frac{2a}{G}(1-aQ_1^2)Z^3 + \frac{C(p)}{G} \\ \times\left[(1+aM_1^2)Z^2 + (M_1^2+M_0^2+aM_0^2M_1^2)Z + M_1^2M_0^2\right].
$$

We conclude that, on $T_m \times I = T_m \times [s/2, s]$, whenever $Z(\theta,\tau) \ge 0$, we have the inequality

(6.73)

$$
\frac{\partial Z}{\partial \tau} \le \begin{cases} GZ_{\theta\theta} + \left[\left(\dfrac{2}{p}-2\right)G_\theta - 4aG_\theta Z\right]Z_\theta - \dfrac{a}{G}\left(1-aQ_1^2\right)Z^3 \\ -\dfrac{a}{G}\left(1-aQ_1^2\right)Z^3 + \dfrac{C(p)}{G}\left[\left(1+aM_1^2\right)Z^2\right. \\ \quad \left. + \left(M_1^2+M_0^2+aM_0^2M_1^2\right)Z + M_1^2M_0^2\right]. \end{cases}
$$

For convenience, let Ψ be the terms in the second line of (6.73), i.e.,

$$\Psi = \frac{1}{G}\left\{-a\left(1-aQ_1^2\right)Z^3 + C(p)\left[\left(1+aM_1^2\right)Z^2 \right.\right.$$
$$\left.\left. + \left(M_1^2+M_0^2+aM_0^2M_1^2\right)Z + M_1^2M_0^2\right]\right\}. \tag{6.74}$$

The sign of it is determined by the terms inside $\{\cdot\}$. Observe that $aM_1^2(\tau) \le 1/2$ for all $\tau \in I$, and $a/2 \le a\left(1-aQ_1^2\right) \le a$ for all $\tau \in I$. Hence

(6.75)

$$\begin{aligned}
&-a\left(1-aQ_1^2\right)Z^3 + C(p)\left[\left(1+aM_1^2\right)Z^2 + \left(M_1^2+M_0^2+aM_0^2M_1^2\right)Z + M_1^2M_0^2\right] \\
&\quad \le -\frac{a}{2}Z^3 + C(p)\left[Z^2 + \left(M_1^2+M_0^2\right)Z + M_1^2M_0^2\right] \\
&\qquad \text{(for different constant } C(p)\text{)} \\
&\quad = \left(-\frac{1}{6}aZ^3 + C(p)Z^2\right) + \left(-\frac{1}{6}aZ^3 + C(p)\left(M_1^2+M_0^2\right)Z\right) \\
&\qquad + \left(-\frac{1}{6}aZ^3 + C(p)M_1^2M_0^2\right)
\end{aligned}$$

whenever $Z(\theta,\tau) \ge 0$. Here $M_1 = M_1(\tau)$, $\tau \in I$.

For now, we can summarize the following: we have

(6.76)

$$\frac{\partial Z}{\partial \tau} \le \begin{cases} GZ_{\theta\theta} + \left[\left(\frac{2}{p}-2\right)G_\theta - 4aG_\theta Z\right]Z_\theta - \frac{a}{G}\left(1-aQ_1^2\right)Z^3 \\ \frac{1}{G}\left[-\frac{1}{6}aZ^3 + C(p)Z^2\right] + \frac{1}{G}\left[-\frac{1}{6}aZ^3 + C(p)\left(M_1^2+M_0^2\right)Z\right] \\ \quad + \frac{1}{G}\left[-\frac{1}{6}aZ^3 + C(p)M_1^2M_0^2\right] \end{cases}$$

whenever $Z(\theta,\tau) \ge 0$ on $T_m \times I$.

To proceed further, we need the following lemma (its proof is straightforward; we omit it):

Lemma 6.2. *Let $\lambda > 0$ and $b > 0$ be two constants. If $Z(\tau) > 0$ is defined on $[0,\tau_0]$ satisfying the following:*

$$\text{whenever } \ Z(\tau) \ge \lambda, \quad \text{then } \frac{dZ}{d\tau} \le -bZ^3, \tag{6.77}$$

then we have the estimate

$$Z(\tau) \le \max\left\{\lambda,\ \frac{1}{\sqrt{2b\tau}}\right\}, \quad \forall \tau \in (0, \tau_0]. \tag{6.78}$$

Remark 6.5. The lemma remains correct if we replace the interval $[0, \tau_0]$ by $[0, \infty)$. Also if we replace $[0, \tau_0]$ by $[\tau_1, \tau_2]$, $0 < \tau_1 < \tau_2$, then (6.78) should be replaced by

$$Z(\tau) \le \max\left\{\lambda,\ \frac{1}{\sqrt{2b(\tau - \tau_1)}}\right\}, \quad \forall \tau \in (\tau_1, \tau_2]. \tag{6.79}$$

Recall we have

$$|G_\theta(\theta, \tau)| \le C_1(p)\left(M_0 + \sqrt{\frac{M_0}{\tau}}\right), \quad (\theta, \tau) \in T_m \times (0, \tau_0]$$

so in particular on the interval $I = [s/2, s]$, we have

$$|G_\theta(\theta, \tau)| \le C_1(p)\left(M_0 + \sqrt{\frac{2M_0}{s}}\right), \quad (\theta, \tau) \in T_m \times I. \tag{6.80}$$

Let

$$B_1 = C_1(p)\left(M_0 + \sqrt{\frac{2M_0}{s}}\right).$$

We know

$$M_1(\tau) \le C_1(p)\left(M_0 + \sqrt{2}\sqrt{\frac{M_0}{s}}\right) = B_1, \quad \forall \tau \in I. \tag{6.81}$$

Let $Z(\tau) = \max_{\theta \in T_m} Z(\theta, \tau)$, $\tau \in I$. We find that at any time $\tau \in I$, if $Z(\tau)$ is so large such that it exceeds the constant λ below

$$\lambda := C(p) \cdot \max\left\{\frac{1}{a},\ \left(\frac{B_1^2 + M_0^2}{a}\right)^{1/2},\ \left(\frac{B_1^2 M_0^2}{a}\right)^{1/3}\right\}, \tag{6.82}$$

where $C(p)$ is some large constant depending only on p, then the three terms in the second and third lines of (6.76) will all become negative. Hence we conclude that whenever $Z(\tau) \geq \lambda > 0$ on I, then

$$\frac{dZ}{d\tau} \leq -\frac{a}{G}\left(1 - aQ_1^2\right) Z^3 \leq -\frac{a}{2M_0} Z^3, \quad \tau \in I = [s/2, s]\,. \tag{6.83}$$

Now by (6.79), we get

(6.84)

$$Z(\theta,\tau) \leq Z(\tau) \leq C(p) \max\left\{\lambda, \quad \sqrt{\frac{M_0}{a\left(\tau - \frac{s}{2}\right)}}\right\}, \quad \forall(\theta,\tau) \in T_m \times (s/2, s].$$

In particular, at time s, we get

$$\begin{aligned}
\sqrt{\frac{M_0}{a\left(s - \frac{s}{2}\right)}} &= C\sqrt{\frac{M_0}{as}} = C\sqrt{\frac{2C_1^2\left(M_0 + \sqrt{2}\sqrt{\frac{M_0}{s}}\right)^2 M_0}{s}} \\
&\leq C(p)\left(M_0 + \sqrt{\frac{M_0}{s}}\right)\sqrt{\frac{M_0}{s}} \leq C(p)\left(M_0^2 + \frac{M_0}{s}\right),
\end{aligned}$$

where we have used the inequality $ab \leq \left(a^2 + b^2\right)/2$.

As for the three terms in λ, we have

$$\frac{C(p)}{a} = C(p)\left(M_0 + \sqrt{2}\sqrt{\frac{M_0}{s}}\right)^2 \leq C(p)\left(M_0^2 + \frac{M_0}{s}\right)$$

and

$$\begin{aligned}
C(p)\left(\frac{B_1^2 + M_0^2}{a}\right)^{1/2} &\leq C(p)\left(M_0 + \sqrt{\frac{M_0}{s}}\right)(B_1 + M_0) \\
&\leq C(p)\left(M_0^2 + \frac{M_0}{s}\right)
\end{aligned}$$

and finally

$$\begin{aligned}
C(p)\left(\frac{B_1^2 M_0^2}{a}\right)^{1/3} &\leq C(p)\left[\left(M_0 + \sqrt{\frac{M_0}{s}}\right)^2\left(M_0 + \sqrt{\frac{M_0}{s}}\right)^2 M_0^2\right]^{1/3} \\
&\leq C(p)\left(M_0^2 + \frac{M_0}{s}\right).
\end{aligned}$$

Hence we conclude that

$$Z(\theta,s)\le Z(s)\le C(p)\left(M_0^2+\frac{M_0}{s}\right),\quad \forall(\theta,s)\in T_m\times(0,\tau_0], \tag{6.85}$$

which is equivalent to

$$Q_2(\theta,\tau)=G(\theta,\tau)G_{\theta\theta}(\theta,\tau)\le C(p)\left(M_0^2+\frac{M_0}{\tau}\right),\quad \forall(\theta,\tau)\in T_m\times(0,\tau_0] \tag{6.86}$$

whenever $G_{\theta\theta}(\theta,\tau)\ge 0$. A similar proof gives a lower bound estimate whenever $G_{\theta\theta}(\theta,\tau)\le 0$. We conclude

$$G(\theta,\tau)|G_{\theta\theta}(\theta,\tau)|\le C(p)\left(M_0^2+\frac{M_0}{\tau}\right)\quad \text{on } T_m\times(0,\tau_0] \tag{6.87}$$

and the proof for $k=2$ is done.

6.3. Proof of Theorem 6.1 for higher k

We will assume $k\ge 3$ from now on and use the induction method. Let

$$Q_k(\theta,\tau)=G^{k-1}(\theta,\tau)G^{(k)}(\theta,\tau),\quad \text{where } G^{(k)}(\theta,\tau)=\frac{\partial^k G}{\partial\theta^k}(\theta,\tau) \tag{6.88}$$

and assume we have estimate of the form

$$\begin{aligned}|Q_j(\theta,\tau)|&=|G^{j-1}(\theta,\tau)G^{(j)}(\theta,\tau)|\\ &\le C_j(p)\left[M_0^j+\left(\sqrt{\frac{M_0}{\tau}}\right)^j\right]\quad \text{on } T_m\times(0,\tau_0]\end{aligned} \tag{6.89}$$

for $j=1,2,\ldots,k-1$, where $C_j(p)$ is a constant depending only on j and p.

Let

$$M_j(\tau)=M_j=M_0^j+\left(\sqrt{\frac{M_0}{\tau}}\right)^j,\quad j=1,2,3\ldots. \tag{6.90}$$

Note that M_0 is a constant but $M_j=M_j(\tau)$ is a function of time $\tau\in(0,\tau_0]$ for $j=1,2,\ldots$. The evolution equation of Q_k can be computed as follows.

We first recall

$$\frac{\partial G}{\partial \tau} = GG_{\theta\theta} - \left(\frac{p-1}{p}\right) G_\theta^2 + p\left(G^2 - G\right), \quad p \in (1,2) \tag{6.91}$$

and

$$\frac{\partial G^{(1)}}{\partial \tau} = GG_{\theta\theta}^{(1)} + \left(\frac{2-p}{p}\right) G^{(1)}G_\theta^{(1)} + p(2G-1)G^{(1)} \tag{6.92}$$

and

$$\begin{aligned}\frac{\partial G^{(2)}}{\partial \tau} &= GG_{\theta\theta}^{(2)} + \frac{2}{p}G^{(1)}G_\theta^{(2)} + [C(2,p)G^{(2)} \\ &\quad + p(2G-1)]G^{(2)} + C(2,p)G^{(1)}G^{(1)},\end{aligned} \tag{6.93}$$

where $C(2,p)$ is a constant depending only on p ($C(2,p)$ in different terms may be different). If we differentiate one more time in θ, we get for $k=3$:

$$\begin{aligned}\frac{\partial G^{(3)}}{\partial \tau} &= GG_{\theta\theta}^{(3)} + \left(1+\frac{2}{p}\right) G^{(1)}G_\theta^{(3)} \\ &\quad + \left[C(3,p)G^{(2)} + p(2G-1)\right] G^{(3)} + \underline{C(3,p)G^{(1)}G^{(2)}},\end{aligned} \tag{6.94}$$

where $C(3,p)$ are constants depending only on p. To get a more clear pattern, we do $k=4$:

$$\frac{\partial G^{(4)}}{\partial \tau} = \begin{cases} GG_{\theta\theta}^{(4)} + G^{(1)}G_\theta^{(4)} + \left(1+\dfrac{2}{p}\right) G^{(1)}G_\theta^{(4)} \\ + [C(4,p)G^{(2)} + p(2G-1)]G^{(4)} \\ + \underline{[C(4,p)G^{(3)} + 2pG^{(1)}]G^{(3)} + C(4,p)G^{(2)}G^{(2)} + C(4,p)G^{(1)}G^{(3)}} \end{cases}$$

and simplify it to get

(6.95)

$$\frac{\partial G^{(4)}}{\partial \tau} = \begin{cases} GG_{\theta\theta}^{(4)} + \left(2+\dfrac{2}{p}\right) G^{(1)}G_\theta^{(4)} + [C(4,p)G^{(2)} + p(2G-1)]G^{(4)} \\ + \underline{C(4,p)G^{(3)}G^{(3)} + C(4,p)G^{(2)}G^{(2)} + C(4,p)G^{(1)}G^{(3)}}. \end{cases}$$

Also for $k = 5$, we get

(6.96)

$$\frac{\partial G^{(5)}}{\partial \tau} = \begin{cases} GG^{(5)}_{\theta\theta} + \left(3 + \dfrac{2}{p}\right) G^{(1)} G^{(5)}_{\theta} + [C(5,p)G^{(2)} + p(2G-1)]G^{(5)} \\ + \underline{C(5,p)G^{(3)}G^{(4)} + C(5,p)G^{(1)}G^{(4)} + C(5,p)G^{(2)}G^{(3)}}. \end{cases}$$

Observing the underlined terms in (6.94) to (6.96), we get for

$$\begin{aligned} k = 3: &\quad 0 + [CG^{(1)}G^{(2)}], \\ k = 4: &\quad CG^{(3)}G^{(3)} + [CG^{(1)}G^{(3)} + CG^{(2)}G^{(2)}], \\ k = 5: &\quad CG^{(3)}G^{(4)} + [CG^{(1)}G^{(4)} + CG^{(2)}G^{(3)}], \end{aligned} \tag{6.97}$$

where C are constants depending only on k and p.

Similarly we obtain for

$$\begin{aligned} k = 6: &\quad CG^{(3)}G^{(5)} + CG^{(4)}G^{(4)} + [CG^{(1)}G^{(5)} + CG^{(2)}G^{(4)} + CG^{(3)}G^{(3)}], \\ k = 7: &\quad CG^{(3)}G^{(6)} + CG^{(4)}G^{(5)} + [CG^{(1)}G^{(6)} + CG^{(2)}G^{(5)} + CG^{(3)}G^{(4)}], \\ k = 8: &\quad CG^{(3)}G^{(7)} + CG^{(4)}G^{(6)} + CG^{(5)}G^{(5)} + [CG^{(1)}G^{(7)} + CG^{(2)}G^{(6)} \\ &\quad + CG^{(3)}G^{(5)} + CG^{(4)}G^{(4)}], \\ k = 9: &\quad CG^{(3)}G^{(8)} + CG^{(4)}G^{(7)} + CG^{(5)}G^{(6)} + [CG^{(1)}G^{(8)} + CG^{(2)}G^{(7)} \\ &\quad + CG^{(3)}G^{(6)} + CG^{(4)}G^{(5)}] \\ &\vdots \end{aligned}$$

and conclude the general formula

(6.98)

$$\frac{\partial G^{(k)}}{\partial \tau} = \begin{cases} GG^{(k)}_{\theta\theta} + \left(k - 2 + \dfrac{2}{p}\right) G^{(1)} G^{(k)}_{\theta} + [C(k,p)G^{(2)} + p(2G-1)]G^{(k)} \\ + \displaystyle\sum_{j=1}^{[k/2-1]} C(k,j,p) G^{(j+2)} G^{(k-j)} + \sum_{j=1}^{[k/2]} \tilde{C}(k,j,p) G^{(j)} G^{(k-j)}, \\ k \geq 3 \end{cases}$$

for some constants $C(k,j,p)$ and $\tilde{C}(k,j,p)$.

Remark 6.6. (1) When $k=3$, the term $\sum_{j=1}^{[k/2-1]} C(k,j,p)G^{(j+2)}G^{(k-j)}$ disappears in (6.98). (2) From now on we may simply write $C(k,j,p)$ and $\tilde{C}(k,j,p)$ as C, where C may change from terms to terms.

Note that for $G^{(k)}(\theta,\tau)\geq 0$ we have

(6.99)
$$\begin{aligned}[C(k,p)G^{(2)}+p(2G-1)]G^{(k)} &= \frac{1}{G}[C(k,p)GG^{(2)}+p(2G^2-G)]G^{(k)}\\ &\leq \frac{C}{G}(M_2+M_0^2)G^{(k)}\end{aligned}$$

and by the induction hypothesis (6.89) we have

$$\begin{aligned}\sum_{j=1}^{[k/2-1]} C(k,j,p)G^{(j+2)}G^{(k-j)} \\ &= \frac{1}{G^k}\sum_{j=1}^{[k/2-1]} C(k,j,p)\cdot G^{j+1}G^{(j+2)}\cdot G^{k-j-1}G^{(k-j)}\\ &\leq \frac{C}{G^k}\sum_{j=1}^{[k/2-1]} M_{j+2}M_{k-j}\end{aligned}\tag{6.100}$$

and

(6.101)
$$\begin{aligned}\sum_{j=1}^{[k/2]}\tilde{C}(k,j,p)G^{(j)}G^{(k-j)} &= \frac{1}{G^{k-2}}\sum_{j=1}^{[k/2]}\tilde{C}(k,j,p)\cdot G^{j-1}G^{(j)}\cdot G^{k-j-1}G^{(k-j)}\\ &\leq \frac{C}{G^{k-2}}\sum_{j=1}^{[k/2]} M_jM_{k-j}.\end{aligned}$$

Combined together, we conclude for $G^{(k)}(\theta,\tau)\geq 0$:

$$\frac{\partial G^{(k)}}{\partial\tau}\leq\begin{cases} GG^{(k)}_{\theta\theta}+\left(k-2+\dfrac{2}{p}\right)G^{(1)}G^{(k)}_{\theta}+\dfrac{C}{G}(M_2+M_0^2)G^{(k)}\\ +\dfrac{C}{G^k}\displaystyle\sum_{j=1}^{[k/2-1]} M_{j+2}M_{k-j}+\dfrac{C}{G^{k-2}}\sum_{j=1}^{[k/2]} M_jM_{k-j}.\end{cases}\tag{6.102}$$

Let Λ denote the terms in the second line of (6.102). We have

$$\Lambda \leq \frac{C}{G^k} \sum_{j=1}^{[k/2-1]} M_{j+2} M_{k-j} + \frac{C}{G^k} \sum_{j=1}^{[k/2]} (M_0 M_j)(M_0 M_{k-j})$$

and by Hölder's inequality

$$\begin{aligned} M_0 M_j = M_0 \left[M_0^j + \left(\sqrt{\frac{M_0}{\tau}} \right)^j \right] &\leq M_0^{j+1} + \left[M_0 \left(\sqrt{\frac{M_0}{\tau}} \right)^j \right] \\ &\leq M_0^{j+1} + \frac{(M_0)^a}{a} + \frac{\left(\sqrt{\frac{M_0}{\tau}} \right)^{jb}}{b} \\ &\leq C \left[M_0^{j+1} + \left(\sqrt{\frac{M_0}{\tau}} \right)^{j+1} \right] = C M_{j+1}, \end{aligned}$$

where $a = j+1, b = \frac{j+1}{j}, \frac{1}{a} + \frac{1}{b} = 1$. Therefore

$$\begin{aligned} \Lambda &\leq \frac{C}{G^k} \sum_{j=1}^{[k/2-1]} M_{j+2} M_{k-j} + \frac{C}{G^k} \sum_{j=1}^{[k/2]} M_{j+1} M_{k-j+1} \\ &\leq \frac{C}{G^k} \sum_{0 \leq i,j \leq k;\ i+j \leq k+2} M_i M_j \end{aligned} \tag{6.103}$$

and we conclude for $G^{(k)}(\theta, \tau) \geq 0$ the following:

$$\begin{aligned} \frac{\partial G^{(k)}}{\partial \tau} \leq G G_{\theta\theta}^{(k)} &+ \left(k - 2 + \frac{2}{p} \right) G^{(1)} G_\theta^{(k)} + \frac{C}{G} (M_0^2 + M_2) G^{(k)} \\ &+ \frac{C}{G^k} \sum_{0 \leq i,j \leq k;\, i+j \leq k+2} M_i M_j. \end{aligned} \tag{6.104}$$

On the other hand, we have

$$\frac{\partial G^{k-1}}{\partial \tau} = (k-1) G^{k-2} \left[G G_{\theta\theta} - \left(\frac{p-1}{p} \right) G_\theta^2 + p(G^2 - G) \right]$$

and by

$$\begin{aligned} &(G^{k-1})_\theta = (k-1) G^{k-2} G_\theta, \\ &(G^{k-1})_{\theta\theta} = (k-1) G^{k-2} G_{\theta\theta} + (k-1)(k-2) G^{k-3} G_\theta^2, \end{aligned}$$

we obtain

$$\frac{\partial G^{k-1}}{\partial\tau} = G[(G^{k-1})_{\theta\theta} - (k-1)(k-2)G^{k-3}G_\theta^2] \tag{6.105}$$
$$+ (k-1)\left[-\left(\frac{p-1}{p}\right)G^{k-2}G_\theta^2 + p(G^k - G^{k-1})\right]$$
$$\le G(G^{k-1})_{\theta\theta} + (k-1)\left(1-k+\frac{1}{p}\right)G^{k-2}G_\theta^2 + (k-1)pG^k$$
$$\le G(G^{k-1})_{\theta\theta} + CG^{k-2}(M_0^2+M_1^2).$$

Combining (6.104) and (6.105), whenever $Q_k(\theta,\tau) = G^{k-1}(\theta,\tau)G^{(k)}(\theta,\tau) \ge 0$, we will have

$$\frac{\partial Q_k}{\partial\tau} = G^{k-1}\frac{\partial G^{(k)}}{\partial\tau} + G^{(k)}\frac{\partial G^{k-1}}{\partial\tau},$$
$$\le \begin{cases} G^{k-1}\left[GG_{\theta\theta}^{(k)} + \left(k-2+\frac{2}{p}\right)G^{(1)}G_\theta^{(k)}\right. \\ \qquad \left. + \frac{C}{G}(M_0^2+M_2)G^{(k)} + \frac{C}{G^k}\sum_{0\le i,j\le k;\, i+j\le k+2} M_iM_j\right] \\ + G^{(k)}[G(G^{k-1})_{\theta\theta} + CG^{k-2}(M_0^2+M_1^2)]. \end{cases}$$

One then uses

$$(Q_k)_\theta = G^{k-1}G_\theta^{(k)} + (G^{k-1})_\theta G^{(k)},$$
$$(Q_k)_{\theta\theta} = G^{k-1}G_{\theta\theta}^{(k)} + 2(G^{k-1})_\theta G_\theta^{(k)} + (G^{k-1})_{\theta\theta}G^{(k)}$$

to convert the above into

$$\frac{\partial Q_k}{\partial\tau} \le \begin{cases} G[(Q_k)_{\theta\theta} - \underline{2(G^{k-1})_\theta G_\theta^{(k)}}] \\ + \left[\underline{G^{k-1}\left(k-2+\frac{2}{p}\right)G_\theta G_\theta^{(k)}} + G^{k-1}\frac{C}{G}(M_0^2+M_2)G^{(k)}\right. \\ \left. + G^{k-1}\frac{C}{G^k}\sum_{0\le i,j\le k;\, i+j\le k+2} M_iM_j\right] \\ + G^{(k)}CG^{k-2}(M_0^2+M_1^2), \end{cases}$$

where the underlined terms are

$$
\begin{aligned}
&\underline{-2G(G^{k-1})_\theta G_\theta^{(k)} + G^{k-1}\left(k-2+\frac{2}{p}\right)G_\theta G_\theta^{(k)}} \\
&\quad = \left[-2G(G^{k-1})_\theta + G^{k-1}\left(k-2+\frac{2}{p}\right)G_\theta\right]\frac{(Q_k)_\theta - (G^{k-1})_\theta G^{(k)}}{G^{k-1}} \\
&\quad = \left[-2(k-1)G^{k-1}G_\theta + G^{k-1}\left(k-2+\frac{2}{p}\right)G_\theta\right]\frac{(Q_k)_\theta - (G^{k-1})_\theta G^{(k)}}{G^{k-1}} \\
&\quad = \left(-k+\frac{2}{p}\right)G_\theta (Q_k)_\theta - \frac{1}{G}\left(-k+\frac{2}{p}\right)(k-1)G_\theta^2 G^{k-1}G^{(k)}.
\end{aligned}
$$

Hence we conclude that

$$
\begin{aligned}
\frac{\partial Q_k}{\partial \tau} &\le G(Q_k)_{\theta\theta} + \left(-k+\frac{2}{p}\right)G^{(1)}(Q_k)_\theta \\
&\quad + \frac{C}{G}\left[(M_0^2+M_1^2+M_2)Q_k + \sum_{0\le i,j\le k;\, i+j\le k+2} M_iM_j\right]
\end{aligned}
\tag{6.106}
$$

at points (θ,τ) where $Q_k(\theta,\tau) = G^{k-1}(\theta,\tau)G^{(k)}(\theta,\tau) \ge 0$.

Remark 6.7. One can check that at points (θ,τ) where $Q_k(\theta,\tau) = G^{k-1}$ $(\theta,\tau)G^{(k)}(\theta,\tau) \le 0$ we have

$$
\begin{aligned}
\frac{\partial Q_k}{\partial \tau} &\ge G(Q_k)_{\theta\theta} + \left(-k+\frac{2}{p}\right)G^{(1)}(Q_k)_\theta \\
&\quad + \frac{C}{G}\left[(M_0^2+M_1^2+M_2)Q_k - \sum_{0\le i,j\le k;\, i+j\le k+2} M_iM_j\right],
\end{aligned}
\tag{6.107}
$$

where now the two terms inside the last brackets are negative.

Replacing k by $k-1$ in (6.106) gives

(6.108)

$$
\frac{\partial Q_{k-1}}{\partial \tau} \le \begin{cases} G(Q_{k-1})_{\theta\theta} + \left(1-k+\dfrac{2}{p}\right)G^{(1)}(Q_{k-1})_\theta \\ +\dfrac{C}{G}\left[(M_0^2+M_1^2+M_2)Q_{k-1} + \displaystyle\sum_{0\le i,j\le k-1;\, i+j\le k+1} M_iM_j\right] \end{cases}
$$

at points (θ, τ) where $Q_{k-1}(\theta, \tau) \geq 0$. Hence

$$\frac{\partial (Q_{k-1}^2)}{\partial \tau} = 2Q_{k-1}\left(\frac{\partial Q_{k-1}}{\partial \tau}\right) \leq 2Q_{k-1}(\text{the RHS of (6.108)})$$

at points (θ, τ) where $Q_{k-1}(\theta, \tau) \geq 0$. Use

$$(Q_{k-1}^2)_\theta = 2Q_{k-1}(Q_{k-1})_\theta, \quad (Q_{k-1}^2)_{\theta\theta} = 2Q_{k-1}(Q_{k-1})_{\theta\theta} + 2(Q_{k-1})_\theta^2$$

to get

$$\begin{aligned}\frac{\partial (Q_{k-1}^2)}{\partial \tau} \leq 2Q_{k-1}\Bigg\{ & G(Q_{k-1})_{\theta\theta} + \left(1 - k + \frac{2}{p}\right) G^{(1)}(Q_{k-1})_\theta \\ & + \frac{C}{G}[(M_0^2 + M_1^2 + M_2)Q_{k-1} + \sum_{0 \leq i,j \leq k-1;\, i+j \leq k+1} M_i M_j]\Bigg\} \\ \leq & \begin{cases} G(Q_{k-1}^2)_{\theta\theta} - 2G(Q_{k-1})_\theta^2 + \left(1 - k + \frac{2}{p}\right) G^{(1)}(Q_{k-1}^2)_\theta \\ + \frac{C}{G}\left[(M_0^2 + M_1^2 + M_2)M_{k-1}^2 + \sum_{0 \leq i,j \leq k-1;\, i+j \leq k+1} M_i M_j M_{k-1}\right]. \end{cases}\end{aligned}$$

At points (θ, τ) where both $Q_{k-1}(\theta, \tau) \geq 0$ and $Q_k(\theta, \tau) \geq 0$, we have

(6.109)

$$\begin{aligned} -2G(Q_{k-1})_\theta^2 &= -2G(G^{k-2}G^{(k-1)})_\theta^2 = -2G\left(\frac{Q_k}{G} + \frac{(k-2)G_\theta Q_{k-1}}{G}\right)^2 \\ &\leq -\frac{2}{G}Q_k^2 + \frac{C}{G}(G_\theta Q_{k-1})^2 + \frac{C}{G}|G_\theta Q_{k-1}|Q_k \\ &\leq -\frac{2}{G}Q_k^2 + \frac{C}{G}M_1^2 M_{k-1}^2 + \frac{C}{G}M_1 M_{k-1} Q_k \end{aligned}$$

and

(6.110)

$$\begin{aligned} \left(1 - k + \frac{2}{p}\right) G^{(1)}(Q_{k-1}^2)_\theta &= CG_\theta Q_{k-1}(Q_{k-1})_\theta \leq |C||G_\theta Q_{k-1}||(Q_{k-1})_\theta| \\ &\leq CM_1 M_{k-1} \cdot \frac{1}{G}[|G^{k-1}G^{(k)}| + (k-2)|G_\theta Q_{k-1}|] \\ &\leq \frac{C}{G}M_1 M_{k-1} Q_k + \frac{C}{G}M_1^2 M_{k-1}^2 \end{aligned}$$

and we conclude

(6.111)
$$\frac{\partial (Q_{k-1}^2)}{\partial \tau} \leq \begin{cases} G(Q_{k-1}^2)_{\theta\theta} - \dfrac{2}{G}Q_k^2 + \dfrac{C}{G}M_1 M_{k-1} Q_k \\ + \dfrac{C}{G}\left((M_0^2 + M_1^2 + M_2) M_{k-1}^2 + \displaystyle\sum_{0\leq i,j\leq k-1;\, i+j\leq k+1} M_i M_j M_{k-1} \right). \end{cases}$$

Remark 6.8. Although we have required $Q_{k-1}(\theta,\tau) \geq 0$ in obtaining (6.111), it still holds without such a requirement. By (6.107), at points (θ,τ) where $Q_{k-1}(\theta,\tau) \leq 0$ we have

(6.112)
$$\begin{aligned} \frac{\partial Q_{k-1}}{\partial \tau} &\geq G(Q_{k-1})_{\theta\theta} + \left(1 - k + \frac{2}{p}\right) G^{(1)} (Q_{k-1})_\theta \\ &\quad + \frac{C}{G}\left((M_0^2 + M_1^2 + M_2) Q_{k-1} - \sum_{0\leq i,j\leq k-1;\, i+j\leq k+1} M_i M_j \right) \end{aligned}$$

and so

$$\frac{\partial (Q_{k-1}^2)}{\partial \tau} = 2Q_{k-1}\left(\frac{\partial Q_{k-1}}{\partial \tau}\right) \leq 2Q_{k-1}(\text{the RHS of (6.112)})$$

and we conclude (6.111) again (compare with (6.69)). This observation is important. Hence we conclude that (6.111) holds at points (θ,τ) where $Q_k(\theta,\tau) \geq 0$.

Similar to (6.63), we let

$$Z(\theta,\tau) = \left(\frac{Q_k}{1 - aQ_{k-1}^2}\right)(\theta,\tau). \tag{6.113}$$

We will derive an evolution inequality at points (θ,τ), where $Z(\theta,\tau) \geq 0$. Same as before we focus on the time interval $\tau \in [s/2, s]$, $s \in [0, \tau_0]$. On this interval, we have the estimate

$$\sup_{T_m \times [s/2,s]} |Q_{k-1}| \leq C \sup_{[s/2,s]} M_{k-1}(\tau), \tag{6.114}$$

where

(6.115)
$$C \sup_{[s/2,s]} M_{k-1}(\tau) \le C\left[M_0^{k-1} + \left(\sqrt{\frac{2M_0}{s}}\right)^{k-1}\right] \le C\left[M_0^{k-1} + \left(\sqrt{\frac{M_0}{s}}\right)^{k-1}\right]$$

and so we choose

$$a = \frac{1}{2C^2\left[M_0^{k-1} + \left(\sqrt{\frac{M_0}{s}}\right)^{k-1}\right]^2} \tag{6.116}$$

to get

(6.117)
$$0 \le aQ_{k-1}^2 = \frac{Q_{k-1}^2}{2C^2\left[M_0^{k-1} + \left(\sqrt{\frac{M_0}{s}}\right)^{k-1}\right]^2} \le \frac{1}{2} \quad \text{on } T_m \times [s/2, s].$$

Compute

$$\frac{\partial Z}{\partial \tau} = \frac{1}{1-aQ_{k-1}^2}\frac{\partial Q_k}{\partial \tau} + \frac{aQ_k}{(1-aQ_{k-1}^2)^2}\frac{\partial (Q_{k-1}^2)}{\partial \tau}$$
$$\le \left\{\begin{array}{l} \dfrac{1}{1-aQ_{k-1}^2}\cdot\left\{\begin{array}{l} G(Q_k)_{\theta\theta} + \left(-k+\dfrac{2}{p}\right)G^{(1)}(Q_k)_\theta \\ +\dfrac{C}{G}\left[(M_0^2+M_1^2+M_2)Q_k \right. \\ \left. \qquad + \displaystyle\sum_{0\le i,j\le k;\ i+j\le k+2} M_iM_j\right] \end{array}\right\} \\ +\dfrac{aQ_k}{(1-aQ_{k-1}^2)^2}\cdot\left\{\begin{array}{l} G(Q_{k-1}^2)_{\theta\theta} - \dfrac{2}{G}Q_k^2 + \dfrac{C}{G}M_1M_{k-1}Q_k \\ +\dfrac{C}{G}\left[(M_0^2+M_1^2+M_2)M_{k-1}^2 \right. \\ \left. \qquad + \displaystyle\sum_{0\le i,j\le k-1;\ i+j\le k+1} M_iM_jM_{k-1}\right] \end{array}\right\} \end{array}\right.$$

whenever $Q_k(\theta,\tau)\geq 0$. Use formula (6.56) to get

$$Z_{\theta\theta}=\frac{(1-aQ_{k-1}^2)(Q_k)_{\theta\theta}+aQ_k(Q_{k-1}^2)_{\theta\theta}}{(1-aQ_{k-1}^2)^2}+\frac{2a(Q_{k-1}^2)_\theta}{1-aQ_{k-1}^2}Z_\theta$$

and also

$$Z_\theta=\frac{(Q_k)_\theta}{1-aQ_{k-1}^2}+\frac{aQ_k(Q_{k-1}^2)_\theta}{(1-aQ_{k-1}^2)^2}.$$

Now

$$\frac{\partial Z}{\partial\tau}\leq\begin{cases} G\left(Z_{\theta\theta}-\dfrac{2a(Q_{k-1}^2)_\theta}{1-aQ_{k-1}^2}Z_\theta\right) \\ +\dfrac{1}{1-aQ_{k-1}^2}\cdot\left\{\begin{array}{l}\left(-k+\dfrac{2}{p}\right)G^{(1)}(Q_k)_\theta \\ +\dfrac{C}{G}\left[(M_0^2+M_1^2+M_2)Q_k\right. \\ \left.\qquad+\displaystyle\sum_{0\leq i,j\leq k;\ i+j\leq k+2}M_iM_j\right]\end{array}\right\} \\ +\dfrac{aQ_k}{(1-aQ_{k-1}^2)^2}\cdot\left\{\begin{array}{l}-\dfrac{2}{G}Q_k^2+\dfrac{C}{G}M_1M_{k-1}Q_k \\ +\dfrac{C}{G}\left[(M_0^2+M_1^2+M_2)M_{k-1}^2\right. \\ \left.\qquad+\displaystyle\sum_{0\leq i,j\leq k-1;\ i+j\leq k+1}M_iM_jM_{k-1}\right]\end{array}\right\}\end{cases}$$

and so

$$\frac{\partial Z}{\partial\tau}\leq\begin{cases} GZ_{\theta\theta}-G\dfrac{2a(Q_{k-1}^2)_\theta}{1-aQ_{k-1}^2}Z_\theta \\ +\left(-k+\dfrac{2}{p}\right)G^{(1)}\left[Z_\theta-\dfrac{aQ_k(Q_{k-1}^2)_\theta}{(1-aQ_{k-1}^2)^2}\right] \\ -\dfrac{2a}{G}(1-aQ_{k-1}^2)Z^3+\text{the rest}\end{cases}$$

$$\leq \begin{cases} GZ_{\theta\theta} + \left[\left(-k + \frac{2}{p}\right) G^{(1)} - \frac{4aGQ_{k-1}(Q_{k-1})_\theta}{1 - aQ_{k-1}^2}\right] \\ Z_\theta - \frac{2a}{G}(1 - aQ_{k-1}^2) Z^3 \\ - \left(-k + \frac{2}{p}\right) G^{(1)} \frac{aQ_k (Q_{k-1}^2)_\theta}{(1 - aQ_{k-1}^2)^2} + \text{the rest}, \end{cases}$$

where

(6.118)

$$\begin{aligned} \text{the rest} = {} & \frac{1}{1 - aQ_{k-1}^2} \frac{C}{G} \left\{ (M_0^2 + M_1^2 + M_2) Q_k + \sum_{0 \leq i,j \leq k;\ i+j \leq k+2} M_i M_j \right\} \\ & + \frac{aQ_k}{(1 - aQ_{k-1}^2)^2} \left\{ \frac{C}{G} M_1 M_{k-1} Q_k + \frac{C}{G} \left[(M_0^2 + M_1^2 + M_2) M_{k-1}^2 \right.\right. \\ & \left.\left. + \sum_{0 \leq i,j \leq k-1;\ i+j \leq k+1} M_i M_j M_{k-1} \right] \right\} \\ \leq {} & \begin{cases} \frac{C}{G} \left[(M_0^2 + M_1^2 + M_2) Z + \sum_{0 \leq i,j \leq k;\ i+j \leq k+2} M_i M_j \right] \\ + \frac{C}{G} a M_1 M_{k-1} Z^2 + \frac{C}{G} \left[a (M_0^2 + M_1^2 + M_2) M_{k-1}^2 \right. \\ \qquad \left. + a \sum_{0 \leq i,j \leq k-1;\ i+j \leq k+1} M_i M_j M_{k-1} \right] Z. \end{cases} \end{aligned}$$

On the other hand, by (6.110) we also know

$$\begin{aligned} (6.119) \quad & - \left(-k + \frac{2}{p}\right) G^{(1)} \frac{aQ_k (Q_{k-1}^2)_\theta}{(1 - aQ_{k-1}^2)^2} \leq aC |G^{(1)} (Q_{k-1}^2)_\theta| \frac{Q_k}{(1 - aQ_{k-1}^2)^2} \\ & \leq a \left(\frac{C}{G} M_1 M_{k-1} Q_k + \frac{C}{G} M_1^2 M_{k-1}^2 \right) \frac{Q_k}{(1 - aQ_{k-1}^2)^2} \\ & \leq \frac{C}{G} a M_1 M_{k-1} Z^2 + \frac{C}{G} a M_1^2 M_{k-1}^2 Z. \end{aligned}$$

One can combine (6.119) and (6.118) to get

$$\frac{\partial Z}{\partial \tau} \leq GZ_{\theta\theta} + \left[\left(-k+\frac{2}{p}\right)G^{(1)} - \frac{4aGQ_{k-1}(Q_{k-1})_\theta}{1-aQ_{k-1}^2}\right]Z_\theta$$
$$+\begin{cases} -\dfrac{2a}{G}(1-aQ_{k-1}^2)Z^3 + \left(\dfrac{C}{G}aM_1M_{k-1}Z^2 + \dfrac{C}{G}aM_1^2M_{k-1}^2 Z\right) \\ + \dfrac{C}{G}\left[(M_0^2+M_1^2+M_2)Z + \displaystyle\sum_{0\leq i,j\leq k;\ i+j\leq k+2} M_iM_j\right] \\ + \dfrac{C}{G}aM_1M_{k-1}Z^2 + \dfrac{C}{G}\left[a(M_0^2+M_1^2+M_2)M_{k-1}^2\right. \\ \left. + a\displaystyle\sum_{0\leq i,j\leq k-1;\ i+j\leq k+1} M_iM_jM_{k-1}\right]Z \end{cases}$$

and use

$$-\frac{2a}{G}(1-aQ_{k-1}^2)Z^3 \leq -\frac{a}{G}Z^3 \quad \text{on } T_m \times [s/2, s]$$

to obtain *the final evolution inequality*: whenever $Q_k(\theta,\tau) \geq 0$, we have

$$\frac{\partial Z}{\partial \tau} \leq GZ_{\theta\theta} + \left[\left(-k+\frac{2}{p}\right)G^{(1)} - \frac{4aGQ_{k-1}(Q_{k-1})_\theta}{1-aQ_{k-1}^2}\right]Z_\theta - \frac{a}{2G}Z^3 - \frac{a}{2G}Z^3 + \frac{C}{G}\cdot\Omega \tag{6.120}$$

where Ω is a second order polynomial in Z, given by

$$\Omega = \begin{cases} aM_1M_{k-1}Z^2 \\ + \left[(M_0^2+M_1^2+M_2)(1+aM_{k-1}^2)\right. \\ \qquad \left. + a\displaystyle\sum_{0\leq i,j\leq k-1;\ i+j\leq k+1} M_iM_jM_{k-1}\right]Z \\ + \displaystyle\sum_{0\leq i,j\leq k;\ i+j\leq k+2} M_iM_j. \end{cases} \tag{6.121}$$

Remark 6.9. Note that (6.120) has exactly the same form as (6.73).

On the time interval $[s/2, s]$, at points (θ, τ) where $Z(\theta, \tau) \geq 0$ is so large that it exceeds the constant

$$\lambda := C \sup_{[s/2,s]} \max \Pi, \tag{6.122}$$

where C is some large constant and

$$\Pi = \left\{ \begin{array}{l} M_1 M_{k-1}, \quad \max\limits_{0 \leq i,j \leq k;\ i+j \leq k+2} (a^{-1} M_i M_j)^{1/3}, \\ \sqrt{(M_0^2 + M_1^2 + M_2)(a^{-1} + M_{k-1}^2)}, \\ \max\limits_{0 \leq i,j \leq k-1;\ i+j \leq k+1} \sqrt{M_i M_j M_{k-1}} \end{array} \right\},$$

then we have: (we shall split $\frac{a}{2G} Z^3$ into four $\frac{a}{8G} Z^3$)

$$-\frac{a}{8G} Z^3 + \frac{C}{G} (a M_1 M_{k-1} Z^2) = \frac{a}{G} \left(-\frac{1}{8} Z + C M_1 M_{k-1} \right) Z^2 \leq 0 \tag{6.123}$$

since

$$Z(\theta, \tau) \geq C \sup_{[s/2,s]} M_1 M_{k-1}$$

and

$$\begin{aligned} &-\frac{a}{8G} Z^3 + \frac{C}{G} ((M_0^2 + M_1^2 + M_2)(1 + a M_{k-1}^2)) Z \\ &= \frac{a}{G} \left[-\frac{1}{8} Z^2 + C \left((M_0^2 + M_1^2 + M_2) \left(\frac{1}{a} + M_{k-1}^2 \right) \right) \right] Z \leq 0 \end{aligned} \tag{6.124}$$

since

$$Z(\theta, \tau) \geq C \sup_{[s/2,s]} \sqrt{(M_0^2 + M_1^2 + M_2)(a^{-1} + M_{k-1}^2)}.$$

We also have

$$\begin{aligned} &-\frac{a}{8G} Z^3 + \frac{C}{G} \left(a \sum_{0 \leq i,j \leq k-1;\ i+j \leq k+1} M_i M_j M_{k-1} \right) Z \\ &= \frac{a}{G} \left(-\frac{1}{8} Z^2 + C \sum_{0 \leq i,j \leq k-1;\ i+j \leq k+1} M_i M_j M_{k-1} \right) Z \leq 0 \end{aligned} \tag{6.125}$$

since

$$Z(\theta, \tau) \geq C \sup_{[s/2,s]} \max_{0 \leq i,j \leq k-1;\ i+j \leq k+1} \sqrt{M_i M_j M_{k-1}}.$$

Finally

$$
\begin{aligned}
(6.126)\qquad & -\frac{a}{8G}Z^3+\frac{C}{G}\sum_{0\le i,j\le k;\ i+j\le k+2} M_iM_j \\
&= \frac{a}{G}\left(-\frac{1}{8}Z^3+\frac{C}{a}\sum_{0\le i,j\le k;\ i+j\le k+2} M_iM_j\right)\le 0
\end{aligned}
$$

since

$$
Z(\theta,\tau)\ge C\sup_{[s/2,s]}\max_{0\le i,j\le k;\ i+j\le k+2}(a^{-1}M_iM_j)^{1/3}.
$$

Now by (6.123) to (6.126), we have

$$
\frac{\partial Z}{\partial \tau}\le -\frac{a}{2G}Z^3\le -\frac{a}{2M_0}Z^3 \quad \text{on } [s/2,s] \tag{6.127}
$$

whenever at the maximum point (θ,τ) we have $Z(\theta,\tau)\ge\lambda$.

By (6.79) again, we can infer

$$
Z(\theta,\tau)\le C\max\left\{\lambda,\frac{\sqrt{M_0}}{\sqrt{a(\tau-s/2)}}\right\},\quad \forall(\theta,\tau)\in T_m\times(s/2,s]. \tag{6.128}
$$

In particular, at time s, we get

$$
Z(\theta,s)\le C\max\left\{\lambda,\sqrt{\frac{M_0}{as}}\right\},\quad \forall\theta\in T_m. \tag{6.129}
$$

Now

(6.130)

$$
\begin{aligned}
\sqrt{\frac{M_0}{as}} &\le C\sqrt{2C^2\left[M_0^{k-1}+\left(\sqrt{\frac{M_0}{s}}\right)^{k-1}\right]^2\frac{M_0}{s}} \\
&= C\left[M_0^{k-1}+\left(\sqrt{\frac{M_0}{s}}\right)^{k-1}\right]\sqrt{\frac{M_0}{s}}\le C\left[M_0^{k}+\left(\sqrt{\frac{M_0}{s}}\right)^{k}\right],
\end{aligned}
$$

where we have used the inequality $a^{k-1}b\le C(a^k+b^k)$ for $a,b>0,\ k\ge 2$.

As for the first term in λ, we have

$$\text{(6.131)}\qquad \sup_{[s/2,s]} M_1 M_{k-1} \le C\left[M_0 + \left(\sqrt{\frac{M_0}{s}}\right)\right]\left[M_0^{k-1} + \left(\sqrt{\frac{M_0}{s}}\right)^{k-1}\right] \le C\left[M_0^k + \left(\sqrt{\frac{M_0}{s}}\right)^{k}\right].$$

For the second term in λ, for each i, j with $0 \le i,\ j \le k,\ i+j \le k+2$, we have

$$\begin{aligned}
&\sup_{[s/2,s]} (a^{-1} M_i M_j)^{1/3} \\
&\le C\left(\left[M_0^{k-1} + \left(\sqrt{\frac{M_0}{s}}\right)^{k-1}\right]^2 \left[M_0^{i} + \left(\sqrt{\frac{M_0}{s}}\right)^{i}\right]\left[M_0^{j} + \left(\sqrt{\frac{M_0}{s}}\right)^{j}\right]\right)^{1/3} \\
&\le C\left(\left[M_0 + \left(\sqrt{\frac{M_0}{s}}\right)\right]^{2k-2+(i+j)}\right)^{1/3} \\
&\le C\left(\left[M_0 + \left(\sqrt{\frac{M_0}{s}}\right)\right]^{2k-2+(k+2)}\right)^{1/3} \\
&= C\left[M_0 + \left(\sqrt{\frac{M_0}{s}}\right)\right]^{k},
\end{aligned}$$

where we have used the equivalence (we have $a^p + b^p < (a+b)^p \le 2^{p-1}(a^p + b^p)$ for any $a > 0, b > 0, 1 < p < \infty$)

$$M_0^i + \left(\sqrt{\frac{M_0}{s}}\right)^i \sim \left[M_0 + \left(\sqrt{\frac{M_0}{s}}\right)\right]^i$$

and the fact that $M_0 \ge 1$ (since $G_{\max}(\tau) = u^p_{\max}(\tau) \ge 1$ for all time). Hence we conclude that

$$\text{(6.132)}\qquad \sup_{[s/2,s]} \max_{0\le i,j\le k;\ i+j\le k+2} (a^{-1} M_i M_j)^{1/3} \le C\left[M_0^k + \left(\sqrt{\frac{M_0}{s}}\right)^{k}\right].$$

For the third term in λ, first note that

$$\sup_{[s/2,s]}\sqrt{(M_0^2+M_1^2+M_2)}\le C\left[M_0+\left(\sqrt{\frac{M_0}{s}}\right)\right] \tag{6.133}$$

and

$$\sup_{[s/2,s]}\sqrt{(a^{-1}+M_{k-1}^2)}\le C\left[M_0^{k-1}+\left(\sqrt{\frac{M_0}{s}}\right)^{k-1}\right] \tag{6.134}$$

and combine (6.133), (6.134) to get

$$\sup_{[s/2,s]}\sqrt{(M_0^2+M_1^2+M_2)(a^{-1}+M_{k-1}^2)}\le C\left[M_0^k+\left(\sqrt{\frac{M_0}{s}}\right)^k\right]. \tag{6.135}$$

Finally for the fourth term in λ, we have for each i,j with $0\le i,\ j\le k-1,\ i+j\le k+1$, that

(6.136)

$$\sup_{[s/2,s]}\sqrt{M_iM_jM_{k-1}}\le C\left[M_0+\left(\sqrt{\frac{M_0}{s}}\right)\right]^{\frac{i+j+k-1}{2}}\le C\left[M_0^k+\left(\sqrt{\frac{M_0}{s}}\right)^k\right]$$

due to the same reason as in obtaining (6.132). Hence

$$\sup_{[s/2,s]}\max_{0\le i,j\le k-1;\ i+j\le k+1}\sqrt{M_iM_jM_{k-1}}\le C\left[M_0^k+\left(\sqrt{\frac{M_0}{s}}\right)^k\right]. \tag{6.137}$$

Combining (6.129)–(6.132), (6.135) and (6.137), we conclude that

$$\begin{aligned} Q_k(\theta,\tau)&=G^{k-1}(\theta,\tau)G^{(k)}(\theta,\tau)\\ &\le C(k,p)\left[M_0^k+\left(\sqrt{\frac{M_0}{\tau}}\right)^k\right]\quad \text{on } T_m\times(0,\tau_0]\end{aligned} \tag{6.138}$$

at points (θ,τ) where $Q_k(\theta,\tau)\ge 0$.

Similar argument applied to (θ, τ), where $Q_k(\theta,\tau) \le 0$, gives the following:

(6.139)
$$G^{k-1}(\theta,\tau)|G^{(k)}(\theta,\tau)| \le C(k,p)\left[M_0^k + \left(\sqrt{\frac{M_0}{\tau}}\right)^k\right] \quad \text{on } T_m \times (0, \tau_0]$$

for all $k \in \mathbb{N}$. The proof of Theorem 6.1 is now finished. □

An immediate consequence of Theorem 6.1 is the following space-time estimate of $G(\theta,\tau)$. We first observe the following:

Lemma 6.3. *For any integers $i \ge 0$ and $j > 0$, we have the formula*

$$\partial_\theta^i \partial_\tau^j G = \sum_{i_1+\cdots+i_{j+1} \le i+2j} C(\partial_\theta^{i_1} G)\cdot\cdots\cdot\left(\partial_\theta^{i_{j+1}} G\right). \tag{6.140}$$

Proof. For example, by (6.98), for $j = 1, i = k$, we have

(6.141)
$$\frac{\partial G^{(k)}}{\partial \tau} = \begin{cases} GG_{\theta\theta}^{(k)} + \left(k - 2 + \dfrac{2}{p}\right) G^{(1)} G_\theta^{(k)} + [C(k,p)G^{(2)} + p(2G-1)]G^{(k)} \\ + \displaystyle\sum_{j=1}^{[k/2-1]} C(k,j,p) G^{(j+2)} G^{(k-j)} + \sum_{j=1}^{[k/2]} \tilde{C}(k,j,p) G^{(j)} G^{(k-j)}, \ \ k \ge 3, \end{cases}$$

which is (6.140).

Note that if we just differentiate τ once, we get combinations of terms like $G^{(p)}G^{(q)}$. But if we differentiate τ twice, we will get combinations of terms like $G^{(p)}G^{(q)}G^{(r)}$. This explains why the RHS of (6.140) is $(\partial_\theta^{i_1} G)\cdots\cdot(\partial_\theta^{i_{j+1}} G)$. There remains the counting of the total number of space differentiation in G. The total number is at most $i + 2j$. The general case holds by tedious induction. We will omit it. □

As a consequence of Theorem 6.1, we can deduce the following *space-time regularity estimate* for G:

Theorem 6.2 (Andrews [4]). *For any integers $i \ge 0$ and $j > 0$, we have*

$$G^{i+j-1}(\theta,\tau)|\partial_\theta^i \partial_\tau^j G(\theta,\tau)| \le C(i,j,p)\left(M_0 + \sqrt{\frac{M_0}{\tau}}\right)^{i+2j} \tag{6.142}$$

on $T_m \times (0, \tau_0]$.

Proof. Let $C = C(i, j, p)$, where C may change from line to line. The proof of (6.142) is based on formula (6.140). By (6.139) we get

(6.143)

$$
\begin{aligned}
&|\partial_\theta^i \partial_\tau^j G| \\
&\quad \le \sum_{i_1+\cdots+i_{j+1} \le i+2j} C |\partial_\theta^{i_1} G| \cdot \cdots \cdot |\partial_\theta^{i_{j+1}} G| \\
&\quad \le \sum_{i_1+\cdots+i_{j+1} \le i+2j} C G^{1-i_1} \left(M_0 + \sqrt{\frac{M_0}{\tau}} \right)^{i_1} \cdot \cdots \cdot G^{1-i_{j+1}} \left(M_0 + \sqrt{\frac{M_0}{\tau}} \right)^{i_{j+1}} \\
&\quad \le \frac{1}{G^{i+j-1}} \sum_{i_1+\cdots+i_{j+1} \le i+2j} C G^{i+j-1} \cdot G^{j+1-(i_1+\cdots\cdot+i_{j+1})} \\
&\qquad \times \left(M_0 + \sqrt{\frac{M_0}{\tau}} \right)^{i_1+\cdots\cdot+i_{j+1}} \\
&\quad = \frac{1}{G^{i+j-1}} \sum_{i_1+\cdots+i_{j+1} \le i+2j} C G^{i+2j-(i_1+\cdots\cdot+i_{j+1})} \left(M_0 + \sqrt{\frac{M_0}{\tau}} \right)^{i_1+\cdots\cdot+i_{j+1}} \\
&\quad \le \frac{C}{G^{i+j-1}} \left(M_0 + \sqrt{\frac{M_0}{\tau}} \right)^{i+2j}.
\end{aligned}
$$

The theorem is proved. □

As a corollary of Theorem 6.1, we can state the following:

Corollary 6.3. *As $\tau \to \infty$, the solution $u(\theta, \tau)$ in* (2.13) *converges uniformly on T_m to a Lipschitz function $w(\theta) \ge 0$, which is $2m\pi$-periodic over $\mathbb{R}$ (as described in Proposition* 2.1*). Moreover, on any compact subset $I \subset\subset \Omega_+$, where $\Omega_+ = \{\theta \in \mathbb{R} : w(\theta) > 0\}$, $w(\theta)$ is smooth and $u(\theta, \tau)$ converges in $C^\infty(I)$ to $w(\theta)$ as $\tau \to \infty$. That is*

$$
\lim_{\tau \to \infty} \sup_{\theta \in I} \left\| \frac{\partial^k u}{\partial \theta^k}(\theta, \tau) - w^{(k)}(\theta) \right\| = 0 \quad \text{for any } k \in \mathbb{N}.
$$

Proof. We already know that $u(\theta, \tau)$ is uniformly bounded and converges uniformly to $w(\theta)$ as $\tau \to \infty$. By Arzela–Ascoli theorem, it suffices to show that for any fixed $k \in \mathbb{N}$, $|(\partial^k u / \partial \theta^k)(\theta, \tau)|$ is uniformly bounded on $I \times$

$[0, \infty)$. By

$$|G_\theta(\theta, \tau)| = |pu^{p-1}(\theta, \tau)u_\theta(\theta, \tau)|, \quad G = u^p, \;\; \theta \in I$$

and that $u^{p-1}(\theta, \tau)$ has positive lower bound on $I \times [0, \infty)$, $|u_\theta(\theta, \tau)|$ must be uniformly bounded on $I \times [0, \infty)$. Also by

$$|G_{\theta\theta}| = |pu^{p-1}u_{\theta\theta} + p(p-1)u^{p-2}u_\theta^2|$$

and Theorem 6.1, we know that $|G_{\theta\theta}(\theta, \tau)|$ is uniformly bounded on $I \times [0, \infty)$, which will in turn force $|u_{\theta\theta}(\theta, \tau)|$ to be uniformly bounded. Next by

$$|G_{\theta\theta\theta}| = |pu^{p-1}u_{\theta\theta\theta} + (\text{lower order terms of } u, u_\theta, u_{\theta\theta})|$$

we know that $|u_{\theta\theta\theta}(\theta, \tau)|$ is uniformly bounded on $I \times [0, \infty)$.

Keep going and the proof is done. □

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References

[1] U. Abresch and J. Langer, *The normalized curve shortening flow and homothetic solutions*, J. Differential Geom. **23** (1986), 175–196.

[2] B. Andrews, *Evolving convex curves*, Calc. Var. Partial Differential Equations **7**(4) (1998), 315–371.

[3] B. Andrews, *Classification of limiting shapes for isotropic curve flows*, J. Amer. Math. Sci. **16**(2) (2003), 443–459.

[4] B. Andrews, *Private communications*, 2001.

[5] S. Angenent, *On the formation of singularities in the curve shortening flow*, J. Differential. Geom. **33** (1991), 601–633.

[6] S. Angenent and J. J. L. Velázquez, *Asymptotic shape of cusp singularities in curve shortening*, Duke Math. J. **77**(1) (1995), 71–110.

[7] B. Chow, *Geometric aspects of Aleksandrov reflection and gradient estimates for parabolic equations*, Comm. Anal. Geom. **5**(2) (1997), 389–409.

[8] B. Chow and D. H. Tsai, *Geometric expansion of convex plane curves*, J. Differential Geom. **44** (1996), 312–330.

[9] K. S. Chou and X. P. Zhu, *The curve shortening problem*, Chapman & Hall/CRC Press, 2001.

[10] L. Evans, *Partial differential equations*, Graduate Studies in Math. **19**, AMS, Providence, RI, 1998.

[11] M. Gage, *Curve shortening makes convex curves circular*, Invent. Math. **76** (1984), 357–364.

[12] M. Gage and R. S. Hamilton, *The heat equation shrinking convex plane curves*, J. Differential Geom. **23** (1986), 69–96.

[13] T. C. Lin, C. C. Poon and D. H. Tsai, *Expanding convex immersed closed plane curves*, Calc. Var. Partial Differential Equations **34** (2009), 153–178, 2009.

[14] Y. C. Lin and D. H. Tsai, *On a simple maximum principle technique applied to equations on the circle*, J. Differential Equations **245** (2008), 377–391.

[15] Y. C. Lin and D. H. Tsai, *Evolving a convex closed curve to another one via a length-preserving linear flow*, J. Differential Equations **247** (2009), 2620–2636.

[16] H. Matano, *Convergence of solutions of one-dimensional semilinear parabolic equations*, J. Math. Kyoto Univ. **18**(2) (1978), 221–227.

[17] C. C. Poon and D. H. Tsai, *Contracting convex immersed closed plane curves with slow speed*, preprint, 2010.

[18] D. H. Tsai, *Blowup and convergence of expanding immersed convex plane curves*, Comm. Anal. Geom. **8**(4) (2000), 761–794.

[19] D. H. Tsai, *Behavior of the gradient for solutions of parabolic equations on the circle*, Cal. Var. Partial Differential Equations **23** (2005), 251–270.

[20] D. H. Tsai, *Asymptotic closeness to limiting shapes for expanding embedded plane curves*, Invent. Math. **162** (2005), 473–492.

[21] K. Tso, *Deforming a hypersurface by its Gauss–Kronecker curvature*, Comm. Pure and Appl. Math. **38** (1985), 867–882.

[22] J. Urbas, *An expansion of convex hypersurfaces*, J. Differential Geom. **33** (1991), 91–125.

[23] J. Urbas, *Convex curves moving homothetically by negative powers of their curvature*, Asian J. Math. **3**(3) (1999), 635–658.

[24] X.-P. Zhu, *Lectures on mean curvature flows*, Studies in Advanced Math. **32**, AMS, Providence, RI, 2002.

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