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Orientability in Yang–Mills theory over nonorientable surfaces

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The first two authors have constructed a gauge-equivariant Morse stratification on the space of connections on a principal $U(n)$-bundle over a connected, closed, nonorientable surface Σ. This space can be identified with the real locus of the space of connections on the pullback of this bundle over the orientable double cover of Σ. In this context, the normal bundles to the Morse strata are real vector bundles. We show that these bundles, and their associated homotopy orbit bundles, are orientable for any n when Σ is not homeomorphic to the Klein bottle, and for $n \leq 3$ when Σ is the Klein bottle. We also derive similar orientability results when the structure group is $\mathrm{SU}(n)$.

1. Introduction

Consider a finite stratification $\{\mathcal{A}_\mu\}$ of a manifold S. If each stratum $\mathcal{A}_\mu$ is a locally closed submanifold of S with codimension d_μ, and the index set is partially ordered so that for any λ,

$$\bar{\mathcal{A}}_\lambda \subset \bigcup_{\mu \geq \lambda} \mathcal{A}_\mu$$

holds, then $\{\mathcal{A}_\mu\}$ is called a Morse stratification. A Morse stratification gives a Morse polynomial $M_t(S;K) = \sum t^{d_\mu} P_t(\mathcal{A}_\mu;K)$, where $P_t(-;K)$ denotes the Poincaré polynomial with coefficients in the field K. The Morse inequalities state that there exists a polynomial $R_K(t)$ with nonnegative coefficients such that

$$M_t(S;K) = P_t(S;K) + (1+t)R_K(t).$$

Under fairly general conditions, the Morse inequalities hold for $K = \mathbb{Z}_2$. If, moreover, the normal bundle $\mathbb{N}_\mu$ to each stratum $\mathcal{A}_\mu$ is orientable, then these Morse inequalities hold for any coefficient field K.

Atiyah and Bott studied the moduli space of flat G-connections over a Riemann surface via this Morse theoretical approach when the structure

group G is compact and connected. One of their main results is the computation of the $\mathcal{G}$-equivariant Poincaré series $P_t^{\mathcal{G}}(\mathcal{A}_{\text{flat}}; K)$ for the space $\mathcal{A}_{\text{flat}}$ of flat connections on a principal bundle over a Riemann surface, where $\mathcal{G}$ is the gauge group. They used the Yang–Mills functional, which is invariant under the action of the gauge group, as a Morse-type function and constructed a gauge equivariant Morse stratification $\{\mathcal{A}_\mu\}$ on the space $\mathcal{A}$ of all connections on a principal bundle over a Riemann surface. The space $\mathcal{A}_{\text{flat}}$ of flat connections sits inside of the unique open stratum $\mathcal{A}_{\text{ss}}$ and is a deformation retract of $\mathcal{A}_{\text{ss}}$ via the Yang–Mills flow [5, 22]. Thus, $\mathcal{A}_{\text{flat}}$ and $\mathcal{A}_{\text{ss}}$ are homotopy equivalent, and $P_t^{\mathcal{G}}(\mathcal{A}_{\text{flat}}; K) = P_t^{\mathcal{G}}(\mathcal{A}_{\text{ss}}; K)$. With this Morse stratification, one can write down the $\mathcal{G}$-equivariant Morse series of the space $\mathcal{A}$ of all connections,

$$M_t^{\mathcal{G}}(\mathcal{A}; K) = \sum_{\mu \in I} t^{d_\mu} P_t^{\mathcal{G}}(\mathcal{A}_\mu; K),$$

and the $\mathcal{G}$-equivariant Morse inequalities

$$M_t^{\mathcal{G}}(\mathcal{A}; K) = P_t^{\mathcal{G}}(\mathcal{A}; K) + (1+t)R_K(t),$$

where d_μ is the codimension of the stratum $\mathcal{A}_\mu$, I is the index set of the stratification and $R_K(t)$ is a power series with nonnegative coefficients. In their construction, the normal bundles $\mathbb{N}_\mu$ are complex vector bundles, thus orientable, so K can be any field. In order to compute the $\mathcal{G}$-equivariant Poincaré series $P_t^{\mathcal{G}}(\mathcal{A}_{\text{ss}}; K)$, one needs four ingredients: $P_t^{\mathcal{G}}(\mathcal{A}; K)$, d_μ, $R_K(t)$ and $P_t^{\mathcal{G}}(\mathcal{A}_\mu; K)$ for all $\mathcal{A}_\mu \neq \mathcal{A}_{\text{ss}}$. Since the space $\mathcal{A}$ of all connections over a Riemann surface is an infinite-dimensional complex affine space and thus contractible, $P_t^{\mathcal{G}}(\mathcal{A}; K)$ is just $P_t(B\mathcal{G}; K)$, the Poincaré series of the classifying space of $\mathcal{G}$. As for $P_t^{\mathcal{G}}(\mathcal{A}_\mu)$, Atiyah and Bott found reduction formulas [1, Proposition 7.12] that reduce the question to smaller groups. The Morse index d_μ can be computed by Riemann–Roch (see [1, Equation (7.15)]). Most importantly, they showed that this stratification is $\mathcal{G}$-equivariantly perfect [1, Theorem 7.14], i.e., $R_K(t) = 0$, and

$$P_t(B\mathcal{G}; K) = P_t^{\mathcal{G}}(\mathcal{A}; K) = \sum_{\mu \in I} t^{d_\mu} P_t^{\mathcal{G}}(\mathcal{A}_\mu; K).$$

In the end, this method produces a recursive formula for $P_t^{\mathcal{G}}(\mathcal{A}_{\text{ss}}; K)$.

The first two authors defined a Yang–Mills functional on the space of connections over any *nonorientable* surface Σ in [12]. Using this Yang–Mills

functional, they constructed a $\mathcal{G}$-equivariant Morse stratification on the space of connections over Σ. To be precise, consider the orientable double cover $\pi : \tilde{\Sigma} \to \Sigma$, and let $\tilde{P} = \pi^* P$ over $\tilde{\Sigma}$ denote the pullback of a principal bundle P over Σ. The nontrivial deck transformation of $\tilde{\Sigma}$ induces an involution on the space $\tilde{\mathcal{A}}$ of connections of $\tilde{P}$ whose fixed point set is exactly the space $\mathcal{A}$ of connections of P. Ho and Liu define the Yang–Mills functional L on $\mathcal{A}$ to be the restriction of the Yang–Mills functional $\tilde{L}$ on the fixed point set of $\tilde{\mathcal{A}}$. The absolute minimum of L is zero, achieved by flat connections on P. The gradient flow of L defines a $\mathcal{G}$-equivariant Morse stratification $\{\mathcal{A}_\mu\}$ on $\mathcal{A}$. Indeed, the Morse stratification $\{\mathcal{A}_\mu\}$ is just the intersection of $\mathcal{A}$ with the Morse stratification $\{\tilde{\mathcal{A}}_\mu\}$. This procedure also tells us that the normal bundle $\mathbb{N}_\mu$ to each stratum $\mathcal{A}_\mu$ in $\mathcal{A}$ is the fixed locus of the normal bundle $\tilde{\mathbb{N}}_\mu$ to each stratum $\tilde{\mathcal{A}}_\mu$ in $\tilde{\mathcal{A}}$, which is complex (we will discuss in detail the various involutions on vector bundles in Section 3). In other words, the normal bundle $\mathbb{N}_\mu$ to each Morse stratum $\mathcal{A}_\mu$ is a $\mathcal{G}$-equivariant real vector bundle and hence is not automatically orientable.

The $\mathcal{G}$-equivariant Morse series of this stratification $\{\mathcal{A}_\mu\}$ is

$$M_t^{\mathcal{G}}(\mathcal{A}; K) = \sum_{\mu \in I} t^{d_\mu} P_t^{\mathcal{G}}(\mathcal{A}_\mu; K).$$

Since the Yang–Mills strata admit gauge-invariant tubular neighborhoods (see [23] for a construction), one can use the stratification to obtain the $\mathcal{G}$-equivariant Morse inequalities

$$M_t^{\mathcal{G}}(\mathcal{A}; K) = P_t^{\mathcal{G}}(\mathcal{A}; K) + (1 + t) R_K(t).$$

A priori, we cannot assume orientability of the normal bundles $\mathbb{N}_\mu$, so the Morse inequalities holds only for $K = \mathbb{Z}_2$. To compute the Poincaré series $P_t^{\mathcal{G}}(\mathcal{A}_{\mathrm{ss}}; K)$, we again need four ingredients: $P_t^{\mathcal{G}}(\mathcal{A}; K)$, d_μ, $R_K(t)$ and $P_t^{\mathcal{G}}(\mathcal{A}_\mu; K)$ for all $\mathcal{A}_\mu \neq \mathcal{A}_{\mathrm{ss}}$. Reduction formulas for $P_t^{\mathcal{G}}(\mathcal{A}_\mu; K)$ and a formula for d_μ were given in [12, 13]. On the other hand, the computation of $P_t^{\mathcal{G}}(\mathcal{A}; K)$ is rather difficult when $K = \mathbb{Z}_2$ due to the existence of two-torsion elements in integral cohomology (see [12, Section 5.3; 13, Section 2] for more details) and one is encouraged to consider rational coefficients. Hence we need to establish orientability of the normal bundles.

Let X_{hG} denote the homotopy orbit space $EG \times_G X$. Then $(\mathbb{N}_\mu)_{h\mathcal{G}}$ is also a real vector bundle over $(\mathcal{A}_\mu)_{h\mathcal{G}}$. In this paper, we fix the structure

group of the principal bundle P to be the unitary group $U(n)$ or the special unitary group $\mathrm{SU}(n)$. Our main result is:

Theorem 1.1. *Suppose that either* (i) $\chi(\Sigma) = 0$ (*so that* Σ *is homeomorphic to the Klein bottle*) *and* $n \leq 3$, *or* (ii) $\chi(\Sigma) \neq 0$ *and* n *is any positive integer. Then* $(\mathbb{N}_\mu)_{h\mathcal{G}}$ *is an orientable vector bundle over* $(\mathcal{A}_\mu)_{h\mathcal{G}}$ *for all* μ. *As a consequence,* $\mathbb{N}_\mu$ *is an orientable vector bundle over* $\mathcal{A}_\mu$ *for all* μ.

In [14], the first two authors discuss how far this stratification is from being perfect, i.e., what the power series $R_K(t)$ looks like. They define the notion of *antiperfection*, which leads to some conjectural formulas for $P_t^{\mathcal{G}}(\mathcal{A}_{\mathrm{ss}}; K)$.

Thomas Baird [4] has recently proven the formula conjectured in [14] for the $\mathcal{G}$-equivariant Poincaré series of the space of flat $U(3)$-connections over a nonorientable surface. His argument relies on Yang–Mills theory, and in particular uses our orientability results. Thus Baird's work may be viewed as a concrete application of the results in this paper.

2. Preliminaries

Let $\tilde{\Sigma}$ be a Riemann surface. Let $P_{\tilde{\Sigma}}^{n,k}$ denote the degree k principal $U(n)$-bundle on $\tilde{\Sigma}$. Let $\rho : U(n) \to \mathrm{GL}(n, \mathbb{C})$ be the fundamental representation, and let $E = P_{\tilde{\Sigma}}^{n,k} \times_\rho \mathbb{C}^n$ be the associated complex vector bundle over $\tilde{\Sigma}$. Then E is a rank n, degree k complex vector bundle equipped with a Hermitian metric h, and the unitary frame bundle $U(E, h)$ of the Hermitian vector bundle (E, h) is isomorphic to $P_{\tilde{\Sigma}}^{n,k}$ as a C^∞ principal $U(n)$-bundle.

2.1. Hermitian, $(0,1)$-, and $(1,0)$-connections

Let $\mathcal{A}(P_{\tilde{\Sigma}}^{n,k})$ be the space of $U(n)$-connections on $P_{\tilde{\Sigma}}^{n,k}$, which can be identified with $\mathcal{A}(E, h)$, the space of Hermitian connections on (E, h) (connections on E which are compatible with the Hermitian structure h, cf. [26, p.76]) . It is a complex affine space whose vector space of translations is $\Omega^1_{\tilde{\Sigma}}(\mathrm{ad}\, P_{\tilde{\Sigma}}^{n,k})$, where the complex structure is given by the Hodge star $*$ (cf. [1]). Let $\mathcal{C}(E)$ denote the space of $(0,1)$-connections $\bar{\partial} : \Omega^0_{\tilde{\Sigma}}(E) \to \Omega^{0,1}_{\tilde{\Sigma}}(E)$, and let $\mathcal{C}'(E)$ denote the space of $(1,0)$-connections $\partial : \Omega^0_{\tilde{\Sigma}}(E) \to \Omega^{1,0}_{\tilde{\Sigma}}(E)$. Recall that a $(0,1)$-connection (resp. $(1,0)$-connection) defines a holomorphic (resp. anti-holomorphic) structure on E if and only if $\bar{\partial}^2 = 0$ (resp. $\partial^2 = 0$) (cf. [7, Section 2.2.2]); now $\Omega^{0,2}_{\tilde{\Sigma}} = 0$ (resp. $\Omega^{2,0}_{\tilde{\Sigma}} = 0$) since $\dim_{\mathbb{C}} \tilde{\Sigma} = 1$, so the integrability condition $\bar{\partial}^2 = 0$ (resp. $\partial^2 = 0$) holds automatically. The

local holomorphic (resp. anti-holomorphic) sections are solutions to $\bar\partial s = 0$ (resp. $\partial s = 0$). $\mathcal{C}(E)$ and $\mathcal{C}'(E)$ are complex affine spaces whose vector spaces of translations are $\Omega^{0,1}_{\tilde\Sigma}(\mathrm{End}(E))$ and $\Omega^{1,0}_{\tilde\Sigma}(\mathrm{End}(E))$, respectively (cf. [1]).

Given a Hermitian connection $\nabla : \Omega^0_{\tilde\Sigma}(E) \to \Omega^1_{\tilde\Sigma}(E)$, let $\nabla' : \Omega^0_{\tilde\Sigma}(E) \to \Omega^{1,0}_{\tilde\Sigma}(E)$ and $\nabla'' : \Omega^0_{\tilde\Sigma}(E) \to \Omega^{0,1}_{\tilde\Sigma}(E)$ be the $(1,0)$ and $(0,1)$ parts of ∇. Then $\nabla \mapsto \nabla''$ and $\nabla \mapsto \nabla'$ define isomorphisms $j : \mathcal{A}(P^{n,k}_{\tilde\Sigma}) \to \mathcal{C}(E)$ and $j' : \mathcal{A}(P^{n,k}_{\tilde\Sigma}) \to \mathcal{C}'(E)$ of real affine spaces. Their differentials

$$j_* : \Omega^1_{\tilde\Sigma}(\mathrm{ad}\, P^{n,k}_{\tilde\Sigma}) \to \Omega^{0,1}_{\tilde\Sigma}(\mathrm{End}E), \quad j'_* : \Omega^1_{\tilde\Sigma}(\mathrm{ad}\, P^{n,k}_{\tilde\Sigma}) \to \Omega^{1,0}_{\tilde\Sigma}(\mathrm{End}E),$$

are complex linear and conjugate linear, respectively. More explicitly, j_* and j'_* are $C^\infty(\tilde\Sigma, \mathbb{R})$-linear, so they are induced by real vector bundle maps $\tilde j : T^*_{\tilde\Sigma} \otimes \mathrm{ad}\, P^{n,k}_{\tilde\Sigma} \to (T^*_{\tilde\Sigma})^{0,1} \otimes \mathrm{End}E$ and $\tilde j' : T^*_{\tilde\Sigma} \otimes \mathrm{ad}\, P^{n,k}_{\tilde\Sigma} \to (T^*_{\tilde\Sigma})^{1,0} \otimes \mathrm{End}E$, respectively. Given any point $x \in \tilde\Sigma$, let $dz = dx + idy$ be a local basis of $(T^*_{\tilde\Sigma})^{1,0}_p$ and let $X, Y \in \mathfrak{u}(n)$. Then $X + iY \in \mathfrak{gl}(n, \mathbb{C})$, and

$$\tilde j(X dx + Y dy) = \tfrac{1}{2}(X + iY) d\bar z, \quad \tilde j'(X dx + Y dy) = \tfrac{1}{2}(X - iY) dz.$$

The complex structure on $(T^*_{\tilde\Sigma} \otimes \mathrm{ad}\, P^{n,k}_{\tilde\Sigma})_x$ is given by the Hodge star: $*(X dx + Y dy) = -Y dx + X dy$. It is straightforward to check that $\tilde j$ is complex linear and $\tilde j'$ is conjugate linear.

Given a $(0,1)$-connection $\bar\partial$ on a Hermitian vector bundle (E, h) over $\tilde\Sigma$, there is a unique connection ∇ on E which is compatible with h and such that $\nabla'' = \bar\partial$ (see, e.g., [26, p.78]). We denote this canonical Hermitian connection by $\nabla_{h,\bar\partial}$. The map $j' \circ j^{-1} : \mathcal{C}(E) \to \mathcal{C}'(E)$ is given by $\bar\partial \mapsto (\nabla_{h,\bar\partial})'$, where $(\nabla_{h,\bar\partial})'$ is the $(1,0)$-part of $\nabla_{h,\bar\partial}$.

Let $E^\vee$ be the complex dual of E (see, e.g., [20, pp.168–169]). Then $E^\vee$ is a rank n, degree $-k$ complex vector bundle equipped with a Hermitian metric $h^\vee$ induced by h. More explicitly, if $\{e_1, \ldots, e_n\}$ is a local orthonormal frame of the Hermitian vector bundle (E, h), then its dual coframe $\{e_1^\vee, \ldots, e_n^\vee\}$ is a local orthonormal frame of the Hermitian vector bundle $(E^\vee, h^\vee)$. The map $v \mapsto h(\cdot, v)$ defines a conjugate linear bundle map $E \to E^\vee$ which induces an isomorphism $I_h : (E, h) \cong (\overline{E^\vee}, \overline{h^\vee})$ of Hermitian vector bundles. We have $U(\overline{E^\vee}, \overline{h^\vee}) \cong U(E, h) \cong P^{n,k}_{\tilde\Sigma}$.

A $(0,1)$-connection $\bar\partial$ on E induces a $(0,1)$-connection $\bar\partial^\vee$ on $E^\vee$ and a $(1,0)$-connection $\partial^\vee$ on $\overline{E^\vee}$. This gives a map $j_1 : \mathcal{C}(E) \to \mathcal{C}'(\overline{E^\vee})$. The map $v \mapsto h(\cdot, v)$ defines an isomorphism $E \cong \overline{E^\vee}$ of C^∞ complex vector bundles, which induces an isomorphism $j_2 : \mathcal{C}'(E) \to \mathcal{C}'(\overline{E^\vee})$ of complex affine spaces. It is straightforward to check that $j' \circ j^{-1} = j_2^{-1} \circ j_1 : \mathcal{C}(E) \to \mathcal{C}'(E)$.

2.2. Gauge groups

Let $\mathrm{GL}(E)$ be the frame bundle of the complex vector bundle E. Let $U(E,h)$ be the unitary frame bundle of the Hermitian vector bundle (E,h) as in the previous subsection. Then $\mathrm{GL}(E)$ is a principal $\mathrm{GL}(n,\mathbb{C})$-bundle over $\tilde{\Sigma}$, and $U(E,h)$ is a principal $U(n)$-bundle over $\tilde{\Sigma}$. Let $\mathrm{Aut}(E)$ be the (infinite-dimensional) group of complex vector bundle isomorphisms $E \to E$, and let $\mathrm{Aut}(E,h)$ be the (infinite-dimensional) group of Hermitian bundle isomorphisms $(E,h) \to (E,h)$. (See [1, Section 2] for details.) Then $\mathrm{Aut}(E) \cong \mathrm{Aut}\,\mathrm{GL}(E)$ and $\mathrm{Aut}(E,h) \cong \mathrm{Aut}\,U(E,h)$; $\mathrm{Aut}(E,h)$ is a subgroup of $\mathrm{Aut}(E)$.

$\mathrm{Aut}(E)$ acts on $\mathcal{C}(E)$ by $u \cdot \bar{\partial} = u \circ \bar{\partial} \circ u^{-1}$ and $\mathrm{Aut}(E,h)$ acts on $\mathcal{A}(E,h)$ by $u \cdot \nabla = u \circ \nabla \circ u^{-1}$. More explicitly, relative to a local orthonormal frame, a $(0,1)$-connection on E is of the form

$$\bar{\partial} = \bar{\partial}_0 + B,$$

where $\bar{\partial}_0$ is the usual Cauchy–Riemann operator and B is a $\mathfrak{gl}(n,\mathbb{C})$-valued $(0,1)$-form; a unitary connection is of the form

$$\nabla = d + A,$$

where d is the usual exterior derivative and A is an $\mathfrak{u}(n)$-valued 1-form. An element u in the gauge group $\mathrm{Aut}(E)$ is locally a $\mathrm{GL}(n,\mathbb{C})$-valued function, and acts on the form B by

$$B \mapsto uBu^{-1} - (\bar{\partial}_0 u)u^{-1}, \tag{2.1}$$

an element u in the gauge group $\mathrm{Aut}(E,h)$ is locally a $U(n)$-valued function, and acts on the form A by

$$A \mapsto uAu^{-1} - (du)u^{-1}. \tag{2.2}$$

In particular, if $u \in \mathrm{GL}(n,\mathbb{C})$ (resp. $U(n)$) is a *constant* gauge transformation, then it acts on B (resp. A) by $B \mapsto uBu^{-1}$ (resp. $A \mapsto uAu^{-1}$).

Given $u \in \mathrm{Aut}(E)$ and $x \in \tilde{\Sigma}$, $u_x : E_x \to E_x$ is a complex linear isomorphism for all $x \in \tilde{\Sigma}$. The dual of u_x is a complex linear isomorphism $(u_x)^\vee : (E_x)^\vee \to (E_x)^\vee = (E^\vee)_x$. It induces a complex linear isomorphism

$$\overline{(u_x)^\vee} : \overline{(E_x)^\vee} \to \overline{(E_x)^\vee} \cong (\overline{E^\vee})_x.$$

Define $\overline{u^\vee} \in \mathrm{Aut}(\overline{E^\vee})$ by $(\overline{u^\vee})_x = \overline{(u_x)^\vee}$. Then $u \to \overline{u}^\vee$ defines a group homomorphism $\mathrm{Aut}(E) \to \mathrm{Aut}(\overline{E^\vee})$. The isomorphism $I_h : E \cong \overline{E^\vee}$ allows us to identify $\mathrm{Aut}(E)$ with $\mathrm{Aut}(\overline{E^\vee})$. We let $\tilde{I}_h : \mathrm{Aut}(E) \to \mathrm{Aut}(\overline{E^\vee})$ be this h-dependent identification, and let $\phi_h : \mathrm{Aut}(E) \to \mathrm{Aut}(E)$ be defined by $u \mapsto \tilde{I}_h(\overline{u^\vee})$. Then ϕ_h can be described explicitly as follows. Let $u \in \mathrm{Aut}(E)$, and let $A \in \mathrm{GL}(n, \mathbb{C})$ be the matrix of $u_x : E_x \to E_x$ with respect to an orthonormal basis of (E_x, h_x). Then $\phi_h(u)_x = \overline{(A^t)^{-1}}$. Note that $\phi_h : \mathrm{Aut}(E) \to \mathrm{Aut}(E)$ is an involution, and the fixed locus $\mathrm{Aut}(E)^{\phi_h} = \mathrm{Aut}(E, h)$.

3. Involution

Let Σ be a closed nonorientable surface, and let $\pi : \tilde{\Sigma} \to \Sigma$ be its orientable double cover. Then $\tilde{\Sigma}$ is a Riemann surface, and the nontrivial deck transformation is an anti-holomorphic, anti-symplectic involution $\tau : \tilde{\Sigma} \to \tilde{\Sigma}$ such that $\pi \circ \tau = \pi$.

3.1. The action of τ on holomorphic structures

There is an anti-holomorphic, anti-symplectic map $\tau_{\mathcal{A}} : \mathcal{A}(E, h) \to \mathcal{A}(\tau^* E, \tau^* h)$ given by $\nabla \mapsto \tau^* \nabla$. Note that $\tau^* P^{n,k}_{\tilde{\Sigma}} \cong P^{n,-k}_{\tilde{\Sigma}}$, so $\mathcal{A}(\tau^* E, \tau^* h) \cong \mathcal{A}(P^{n,-k}_{\tilde{\Sigma}})$. We have

$$(\tau^* \nabla)' = \tau^*(\nabla''), \quad (\tau^* \nabla)'' = \tau^*(\nabla'),$$

so there are maps

$$\begin{aligned} &\tau^* : \mathcal{C}(E) \to \mathcal{C}'(\tau^* E), \quad \mathcal{C}(\tau^* E) \to \mathcal{C}'(E), \quad \bar{\partial} \mapsto \tau^* \bar{\partial}, \\ &\tau^* : \mathcal{C}'(E) \to \mathcal{C}(\tau^* E), \quad \mathcal{C}'(\tau^* E) \to \mathcal{C}(E), \quad \partial \mapsto \tau^* \partial, \end{aligned}$$

such that $\tau^* \circ \tau^*$ is the identity map.

Define $\tau_{\mathcal{C}} := j \circ \tau_{\mathcal{A}} \circ j^{-1} = \tau^* \circ j' \circ j^{-1} : \mathcal{C}(E) \to \mathcal{C}(\tau^* E)$. Then $\tau_{\mathcal{C}}$ is given by $\bar{\partial} \mapsto \tau^*(\nabla_{h,\bar{\partial}})'$. In the rest of this subsection, we study the effect of $\tau_{\mathcal{C}}$ on the Harder–Narasimhan filtration.

Let $\mathcal{E}$ denote E equipped with a $(0,1)$-connection (holomorphic structure), so that $\mathcal{E}$ can be viewed as a point in $\mathcal{C}(E)$. Let

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_r = \mathcal{E}$$

be the Harder–Narasimhan filtration, so $\mathcal{E}_j / \mathcal{E}_{j-1}$ is semi-stable. For $j = 1, \ldots, r$, set

$$\mathcal{D}_j = \mathcal{E}_j / \mathcal{E}_{j-1}, \quad n_j = \mathrm{rank}_{\mathbb{C}} \mathcal{D}_j, \quad k_j = \deg \mathcal{D}_j.$$

The Atiyah–Bott type of $\mathcal{E}$ is

$$\mu(\mathcal{E}) = \Big(\underbrace{\frac{k_1}{n_1},\dots,\frac{k_1}{n_1}}_{n_1},\dots,\underbrace{\frac{k_r}{n_r},\dots,\frac{k_r}{n_r}}_{n_r}\Big), \quad \text{where } \frac{k_1}{n_1} > \cdots > \frac{k_r}{n_r}. \tag{3.1}$$

Recall that $\operatorname{rank}_{\mathbb{C}} E = n$ and $\deg E = k$. Let

$$I_{n,k} = \Bigg\{ \Big(\underbrace{\frac{k_1}{n_1},\dots,\frac{k_1}{n_1}}_{n_1},\dots,\underbrace{\frac{k_r}{n_r},\dots,\frac{k_r}{n_r}}_{n_r}\Big) \;\Big|\; \frac{k_1}{n_1} > \cdots > \frac{k_r}{n_r}, \sum_{j=1}^{r} n_j = n,\ \sum_{j=1}^{r} k_j = k \Bigg\},$$

and for $\mu \in I_{n,k}$ let $\mathcal{C}_\mu = \{\mathcal{E} \in \mathcal{C}(E) \;\big|\; \mu(\mathcal{E}) = \mu\}$. The Harder–Narasimhan strata of $\mathcal{C}(E)$ are $\{\mathcal{C}_\mu \mid \mu \in I_{n,k}\}$ (cf. [1, Section 7]).

Using the isomorphism $I_h : E \cong \overline{E^\vee}$ defined in Section 2.1, we may identify $\mathcal{C}(\tau^* E)$ with $\mathcal{C}(\tau^* \overline{E^\vee})$. Then $\tau_{\mathcal{C}} : \mathcal{C}(E) \to \mathcal{C}(\tau^* \overline{E^\vee})$ is given by $\mathcal{E} \mapsto \tau^* \overline{\mathcal{E}^\vee}$, where $\mathcal{E}, \mathcal{E}^\vee, \tau^* \overline{\mathcal{E}^\vee}$ are holomorphic vector bundles over $\tilde{\Sigma}$, while $\overline{\mathcal{E}^\vee}$ is an anti-holomorphic vector bundle over $\tilde{\Sigma}$.

For $j = 0, \dots, r$, define a holomorphic subbundle $(\mathcal{E}^\vee)_{-j}$ of $\mathcal{E}^\vee$ by

$$\big((\mathcal{E}^\vee)_{-j}\big)_x = \{\alpha \in \mathcal{E}_x^\vee \mid \alpha(v) = 0 \ \forall v \in (\mathcal{E}_j)_x\}.$$

Then $(\mathcal{E}^\vee)_{-j} = (\mathcal{E}/\mathcal{E}_j)^\vee$. The Harder–Narasimhan filtration of $\mathcal{E}^\vee \in \mathcal{C}(E^\vee)$ is given by

$$0 = (\mathcal{E}^\vee)_{-r} \subset (\mathcal{E}^\vee)_{-(r-1)} \subset \cdots \subset (\mathcal{E}^\vee)_{-1} \subset (\mathcal{E}^\vee)_0 = \mathcal{E}^\vee.$$

Notice that

$$(\mathcal{E}^\vee)_{-i}/(\mathcal{E}^\vee)_{-(i+1)} \cong (\mathcal{E}_{i+1}/\mathcal{E}_i)^\vee = (\mathcal{D}_{i+1})^\vee.$$

For $j = 1, \dots, r$, set $\mathcal{H}_j = (\mathcal{E}^\vee)_{-(r-j)}/(\mathcal{E}^\vee)_{-(r-j+1)}$. Then

$$\mathcal{H}_j \cong (\mathcal{D}_{r+1-j})^\vee, \quad \operatorname{rank}_{\mathbb{C}} \mathcal{H}_j = n_{r+1-j}, \quad \deg \mathcal{H}_j = -k_{r+1-j}.$$

Hence the Atiyah–Bott type of $\mathcal{E}^\vee$ is

$$\mu = \Big(\underbrace{-\frac{k_r}{n_r},\dots,-\frac{k_r}{n_r}}_{n_r},\dots,\underbrace{-\frac{k_1}{n_1},\dots,-\frac{k_1}{n_1}}_{n_1}\Big), \quad \text{where } -\frac{k_r}{n_r} > \cdots > -\frac{k_1}{n_1}.$$

For $j = 0, \dots, r$, define a holomorphic subbundle $\tau_{\mathcal{C}}(\mathcal{E})_{-j}$ of $\tau_{\mathcal{C}}(\mathcal{E}) = \tau^*\overline{\mathcal{E}^\vee}$ by $\tau_{\mathcal{C}}(\mathcal{E})_{-j} = \tau^*\overline{(\mathcal{E}^\vee)_{-j}}$. The Harder–Narasimhan filtration of $\tau_{\mathcal{C}}(\mathcal{E})$ is given by

$$0 = \tau_{\mathcal{C}}(\mathcal{E})_{-r} \subset \tau_{\mathcal{C}}(\mathcal{E})_{-(r-1)} \subset \cdots \subset \tau_{\mathcal{C}}(\mathcal{E})_0 = \tau_{\mathcal{C}}(\mathcal{E}).$$

For $j = 1, \dots, r$, let $\mathcal{K}_j = \tau_{\mathcal{C}}(\mathcal{E})_{-(r-j)}/\tau_{\mathcal{C}}(\mathcal{E})_{-(r-j+1)}$. Then

$$\mathcal{K}_j \cong \tau^*\overline{\mathcal{H}_j} \cong \tau^*\overline{(\mathcal{D}_{r+1-j})^\vee} = \tau_{\mathcal{C}}(\mathcal{D}_{r+1-j}),$$

and

$$\operatorname{rank}_{\mathbb{C}} \mathcal{K}_j = n_{r+1-j}, \quad \deg \mathcal{K}_j = -k_{r+1-j}.$$

The Atiyah–Bott type of $\tau_{\mathcal{C}}(\mathcal{E})$ is

$$\mu = \Big(\underbrace{-\frac{k_r}{n_r}, \dots, -\frac{k_r}{n_r}}_{n_r}, \dots, \underbrace{-\frac{k_1}{n_1}, \dots, -\frac{k_1}{n_1}}_{n_1}\Big).$$

From the above discussion, we conclude:

Lemma 3.1. *Let $\tau_{\mathcal{C}} : \mathcal{C}(E) \to \mathcal{C}(\tau^*E) \cong \mathcal{C}(\tau^*\overline{E^\vee})$ be defined as above, and define $\tau_{\mathcal{C}} : \operatorname{Aut}(E) \to \operatorname{Aut}(\tau^*E)$ by $u \mapsto \tau^*\phi_h(u)$, where ϕ_h is defined as in Section 2.2. Define $\tau_0 : I_{n,k} \to I_{n,-k}$ by*

$$\Big(\underbrace{\frac{k_1}{n_1}, \dots, \frac{k_1}{n_1}}_{n_1}, \dots, \underbrace{\frac{k_r}{n_r}, \dots, \frac{k_r}{n_r}}_{n_r}\Big) \mapsto \Big(\underbrace{-\frac{k_r}{n_r}, \dots, -\frac{k_r}{n_r}}_{n_r}, \dots, \underbrace{-\frac{k_1}{n_1}, \dots, -\frac{k_1}{n_1}}_{n_1}\Big),$$

Then

1. *$\tau_{\mathcal{C}} : \mathcal{C}(E) \to \mathcal{C}(\tau^*E) \cong \mathcal{C}(\tau^*\overline{E^\vee})$ maps $\mathcal{C}_\mu$ bijectively to $\mathcal{C}_{\tau_0(\mu)}$.*
2. *$\tau_{\mathcal{C}}$ is equivariant with respect to the $\operatorname{Aut}(E)$-action on $\mathcal{C}(E)$ and $\operatorname{Aut}(\tau^*E)$-action on $\mathcal{C}(\tau^*E)$, i.e.,*

$$\tau_{\mathcal{C}}(u \cdot \bar\partial) = \tau_{\mathcal{C}}(u) \cdot \tau_{\mathcal{C}}(\bar\partial), \quad u \in \operatorname{Aut}(E), \quad \bar\partial \in \mathcal{C}(E).$$

3.2. The degree zero case

Let $P \to \Sigma$ be a principal $U(n)$-bundle, and let $\tilde{P} = \pi^* P$ be the pull back principal $U(n)$-bundle on $\tilde{\Sigma}$. We first review some facts about $\tilde{P}$ (see [12, Section 3.2] for details). The pull back bundle $\tilde{P} \cong P^{n,0}_{\tilde{\Sigma}} \cong \tilde{\Sigma} \times U(n)$ is topologically trivial. We wish to describe an involution $\tilde{\tau} : \tilde{P} \to \tilde{P}$ which is $U(n)$-equivariant, covers the involution $\tau : \tilde{\Sigma} \to \tilde{\Sigma}$, and satisfies $P = \tilde{P}/\tilde{\tau}_s$. Fixing a trivialization $\tilde{P} \cong \tilde{\Sigma} \times U(n)$, any such involution must be given by $\tilde{\tau}_s : \tilde{\Sigma} \times U(n) \to \tilde{\Sigma} \times U(n)$, $(x, h) \mapsto (\tau(x), s(x)h)$, for some C^∞ map $s : \tilde{\Sigma} \to U(n)$ satisfying $s(\tau(x)) = s(x)^{-1}$.

The topological type of a principal $U(n)$-bundle $P \to \Sigma$ is classified by $c_1(P) \in H^2(\Sigma; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Let $P^{n,+}_\Sigma$ and $P^{n,-}_\Sigma$ denote the principal $U(n)$-bundles on Σ with $c_1 = 0$ and $c_1 = 1$ in $\mathbb{Z}/2\mathbb{Z}$, respectively. Let τ_ϵ be the involution on $P^{n,0}_{\tilde{\Sigma}} = \tilde{\Sigma} \times U(n)$ defined by a constant map $s(x) = \epsilon \in U(n)$. We must have $\epsilon^2 = I_n$, so $\det \epsilon = \pm 1$. Then $P^{n,0}_{\tilde{\Sigma}}/\tau_\epsilon \cong P^{n,\pm}_\Sigma$ if $\det \epsilon = \pm 1$. We choose $\epsilon_\pm$ to be the diagonal matrix $\mathrm{diag}(\pm 1, 1, \ldots, 1)$, and define $\tau^\pm = \tau_{\epsilon_\pm}$. Then $P^{n,0}_{\tilde{\Sigma}}/\tau^\pm \cong P^{n,\pm}_\Sigma$.

Let $E = P^{n,0}_{\tilde{\Sigma}} \times_\rho \mathbb{C}^n \cong \tilde{\Sigma} \times \mathbb{C}^n$, where $\rho : U(n) \to \mathrm{GL}(n, \mathbb{C})$ is the fundamental representation. Then $\tau^\pm$ induces an involution $\tau^\pm : E \cong \tilde{\Sigma} \times \mathbb{C}^n \to E \cong \tilde{\Sigma} \times \mathbb{C}^n$ given by $(x, v) \mapsto (\tau(x), \epsilon_\pm v)$. The two involutions τ^+, τ^- give two isomorphisms $\tau^* E \cong E$, which induce isomorphisms

$$\mathcal{A}(E, h) \cong \mathcal{A}(\tau^* E, \tau^* h), \quad \mathcal{C}(E) \cong \mathcal{C}(\tau^* E), \quad \mathrm{Aut}(E) \cong \mathrm{Aut}(\tau^* E).$$

Therefore, we have involutions

$$\tau^\pm_{\mathcal{A}} : \mathcal{A}(E, h) \to \mathcal{A}(E, h), \quad \tau^\pm_{\mathcal{C}} : \mathcal{C}(E) \to \mathcal{C}(E), \quad \tau^\pm_{\mathcal{C}} : \mathrm{Aut}(E) \to \mathrm{Aut}(E),$$

and $\tau^\pm_{\mathcal{C}} : \mathcal{C}(E) \to \mathcal{C}(E)$ is $\mathrm{Aut}(E)$-equivariant with respect to the $\mathrm{Aut}(E)$-action on $\mathcal{C}(E)$. We have

$$\mathcal{A}(P^{n,\pm}_\Sigma) = \mathcal{A}(E, h)^{\tau^\pm_{\mathcal{A}}} \cong \mathcal{C}(E)^{\tau^\pm_{\mathcal{C}}}, \quad \mathrm{Aut}(P^{n,\pm}_\Sigma) \cong \mathrm{Aut}(E, h)^{\tau^\pm_{\mathcal{C}}},$$

where $\mathrm{Aut}(E, h) \subset \mathrm{Aut}(E)$ is the group of unitary gauge transformations of the Hermitian vector bundle (E, h). The following two equivariant pairs are isomorphic:

$$\left(\mathcal{A}(P^{n,\pm}_\Sigma), \mathrm{Aut}(P^{n,\pm}_\Sigma)\right) \cong \left(\mathcal{C}(E)^{\tau^\pm_{\mathcal{C}}}, \mathrm{Aut}(E, h)^{\tau^\pm_{\mathcal{C}}}\right).$$

3.3. SU(n)-connections

Let $Q^n_\Sigma \to \Sigma$ be a principal $\mathrm{SU}(n)$-bundle. Then Q^n_Σ is topologically trivial. We fix a trivialization $Q^n_\Sigma \cong \Sigma \times \mathrm{SU}(n)$, which allows us to identify the space $\mathcal{A}(Q^n_\Sigma)$ of connections on Q^n_Σ with the vector space of $\mathfrak{su}(n)$-valued 1-forms on Σ. Let $P^{n,+}_\Sigma \cong \Sigma \times U(n)$ be the trivial $U(n)$-bundle on Σ, as before. The short exact sequence of vector spaces

$$0 \to \mathfrak{su}(n) \to \mathfrak{u}(n) \xrightarrow{\mathrm{Tr}} \mathfrak{u}(1) \to 1$$

induces a short exact sequence of infinite-dimensional vector spaces

$$0 \to \mathcal{A}(Q^n_\Sigma) \to \mathcal{A}(P^{n,+}_\Sigma) \xrightarrow{\mathrm{Tr}} \mathcal{A}(P^{1,+}_\Sigma) \to 0.$$

The Yang–Mills functional on $\mathcal{A}(Q^n_\Sigma)$ is the restriction of the Yang–Mills functional on $\mathcal{A}(P^{n,+}_\Sigma)$. The Morse stratifications on $\mathcal{A}(P^{n,+}_\Sigma)$ and on $\mathcal{A}(Q^n_\Sigma)$ are given by

$$\mathcal{A}(P^{n,+}_\Sigma) = \bigcup_{\mu\in I} \mathcal{A}_\mu, \quad \mathcal{A}(Q^n_\Sigma) = \bigcup_{\mu\in I} \mathcal{A}'_\mu,$$

where $\mathcal{A}'_\mu = \mathcal{A}_\mu \cap \mathcal{A}(Q^n_\Sigma)$ is nonempty for any $\mu \in I$. Given $\mu \in I$ such that $\mathcal{A}_\mu \neq \mathcal{A}_{\mathrm{ss}}$, let $\mathbb{N}_\mu$ (resp. $\mathbb{N}'_\mu$) be the normal bundle of $\mathcal{A}_\mu$ (resp. $\mathcal{A}'_\mu$) in $\mathcal{A}(P^{n,+}_\Sigma)$ (resp. $\mathcal{A}(Q^n_\Sigma)$). Let $\iota_\mu : \mathcal{A}'_\mu \hookrightarrow \mathcal{A}_\mu$ be the inclusion. Then $\mathbb{N}'_\mu = \iota^*_\mu \mathbb{N}_\mu$. Therefore, if $\mathbb{N}_\mu$ is an orientable real vector bundle over $\mathcal{A}_\mu$ then $\mathbb{N}'_\mu$ is an orientable real vector bundle over $\mathcal{A}'_\mu$.

The short exact sequence of Lie groups

$$1 \to \mathrm{SU}(n) \to U(n) \xrightarrow{\det} U(1) \to 1$$

induces a short exact sequence of infinite-dimensional gauge groups

$$1 \to \mathrm{Map}(\Sigma, \mathrm{SU}(n)) \to \mathrm{Map}(\Sigma, U(n)) \to \mathrm{Map}(\Sigma, U(1)) \to 1$$

or equivalently,

$$1 \to \mathrm{Aut}(Q^n_\Sigma) \to \mathrm{Aut}(P^{n,+}_\Sigma) \to \mathrm{Aut}(P^{1,+}_\Sigma) \to 1. \tag{3.2}$$

In particular, $\mathcal{G}' := \mathrm{Aut}(Q^n_\Sigma)$ is a subgroup of $\mathcal{G} := \mathrm{Aut}(P^{n,+}_\Sigma)$; indeed $\mathcal{G}'$ is a closed, normal subgroup of $\mathcal{G}$. We have the following diagram:

$$\begin{array}{ccccc} E\mathcal{G} \times_{\mathcal{G}'} \mathbb{N}'_\mu & \longrightarrow & E\mathcal{G} \times_{\mathcal{G}'} \mathbb{N}_\mu & \longrightarrow & E\mathcal{G} \times_{\mathcal{G}} \mathbb{N}_\mu \\ \downarrow & & \downarrow & & \downarrow \\ E\mathcal{G} \times_{\mathcal{G}'} \mathcal{A}'_\mu & \longrightarrow & E\mathcal{G} \times_{\mathcal{G}'} \mathcal{A}_\mu & \longrightarrow & E\mathcal{G} \times_{\mathcal{G}} \mathcal{A}_\mu, \end{array}$$

which can be identified with

$$\begin{array}{ccccc} (\mathbb{N}'_\mu)_{h\mathcal{G}'} & \longrightarrow & (\mathbb{N}_\mu)_{h\mathcal{G}'} & \longrightarrow & (\mathbb{N}_\mu)_{h\mathcal{G}} \\ \downarrow & & \downarrow & & \downarrow \\ (\mathcal{A}'_\mu)_{h\mathcal{G}'} & \xrightarrow{\iota_\mu} & (\mathcal{A}_\mu)_{h\mathcal{G}'} & \xrightarrow{q_\mu} & (\mathcal{A}_\mu)_{h\mathcal{G}}, \end{array}$$

where

$$(\mathbb{N}_\mu)_{h\mathcal{G}'} = q_\mu^* \left((\mathbb{N}_\mu)_{h\mathcal{G}} \right), \quad (\mathbb{N}'_\mu)_{h\mathcal{G}'} = \iota_\mu^* \left((\mathbb{N}_\mu)_{h\mathcal{G}'} \right).$$

Therefore, if $(\mathbb{N}_\mu)_{h\mathcal{G}}$ is an orientable vector bundle over $(\mathcal{A}_\mu)_{h\mathcal{G}}$ then $(\mathbb{N}'_\mu)_{h\mathcal{G}'}$ is an orientable vector bundle over $(\mathcal{A}'_\mu)_{h\mathcal{G}'}$.

From the above discussion, if Theorem 1.1 holds for $U(n)$ then it holds for $\mathrm{SU}(n)$. In the remainder of this paper, we prove Theorem 1.1 for $U(n)$.

4. Reduction

Let Σ denote a closed, nonorientable surface. In this section, we reduce the question of orientability for normal bundles of Morse strata in $\mathcal{A}(P^{n,\pm}_\Sigma)$ to the question of orientability for certain real vector bundles $V_{n,k}$ over the representation varieties associated to central Yang–Mills connections on $P^{n,k}_{\tilde{\Sigma}}$.

The reduction will pass through a variety of gauge-theoretical spaces, most of which are not CW complexes. Hence one needs to be careful in applying the usual bundle-theoretical arguments. In the end, however, we will show that the normal bundle to each Morse stratum $\mathcal{A}_\mu$, when considered equivariantly as a bundle over $(\mathcal{A}_\mu)_{h\mathcal{G}}$, is pulled back under a weak equivalence from a bundle over the homotopy orbit space $(\mathcal{N}_\mu/\mathcal{G}_0)_{hU(n)}$ (here $\mathcal{N}_\mu$ denotes the set of type μ Yang–Mills connections). These representation varieties are analytical sets [12], and their homotopy orbit spaces are triangulable by results of Illman [11]. The fact that the normal bundle is pulled back from a bundle over a CW complex will allow us to use standard bundle-theoretical arguments. At the end of this section we will summarize

the arguments to follow, so as to make the overall strategy of the reduction clear and the proof rigorous.

4.1. Reduction to Levi subgroups

On the stratum $\mathcal{C}_\mu$, where μ is as in Equation (3.1), we will proceed to reduce the $U(n)$-gauge group to a Levi subgroup corresponding to $U(n_1) \times \cdots \times U(n_r)$. Our arguments follow [1, Section 7] closely.

Let $\mathcal{F}_\mu$ denote the space of all C^∞ filtrations of type μ. The Harder–Narasimhan filtration provides a continuous map $p : \mathcal{C}_\mu \to \mathcal{F}_\mu$. Let $E_\mu \in \mathcal{F}_\mu$ be a fixed C^∞ filtration of E and let $\mathcal{B}_\mu = p^{-1}(E_\mu)$. We choose splittings of the filtration E_μ to obtain a direct sum decomposition $E^0_\mu = D_1 \oplus \cdots \oplus D_r$ of E, and let $\mathcal{B}^0_\mu \subset \mathcal{B}_\mu$ be the space of complex structures compatible with the direct sum decomposition E^0_μ. The inclusion $\mathcal{B}^0_\mu \hookrightarrow \mathcal{B}_\mu$ splits the fibration $\mathcal{B}_\mu \to \mathcal{B}^0_\mu$, which has a vector space as fiber. Hence this inclusion is a weak equivalence. Since $\mathcal{C}_\mu$ is the extension of the $\mathrm{Aut}(E_\mu)$-space $\mathcal{B}_\mu$ to a $\mathrm{Aut}(E)$-space, we have a homeomorphism of homotopy orbit spaces $(\mathcal{C}_\mu)_{h\mathrm{Aut}(E)} \cong (\mathcal{B}_\mu)_{h\mathrm{Aut}(E_\mu)}$. Thus

$$(\mathcal{C}_\mu)_{h\mathrm{Aut}(E)} \cong (\mathcal{B}_\mu)_{h\mathrm{Aut}(E_\mu)} \sim (\mathcal{B}^0_\mu)_{h\mathrm{Aut}(E^0_\mu)} \cong \prod_{j=1}^{r} \mathcal{C}_{\mathrm{ss}}(D_i)_{h\mathrm{Aut}(D_i)}.$$

Let $\tilde{\mathbb{N}}_\mu \to \mathcal{C}_\mu$ be the normal bundle of $\mathcal{C}_\mu$ in $\mathcal{C}(E)$. Given $\mathcal{E} \in \mathcal{B}^0_\mu \subset \mathcal{C}_\mu$, $\mathcal{E}$ is a direct sum of holomorphic subbundles $\mathcal{D}_1, \ldots, \mathcal{D}_r$, and

$$(\tilde{\mathbb{N}}_\mu)_{\mathcal{E}} = \bigoplus_{i<j} H^1(\tilde{\Sigma}, \mathcal{H}om(\mathcal{D}_i, \mathcal{D}_j)).$$

The Harder–Narasimhan filtration is again a continuous map $p : \mathcal{C}_{\tau_0(\mu)} \to \mathcal{F}_{\tau_0(\mu)}$. We have $\tau^* E_\mu \in \mathcal{F}_{\tau_0(\mu)}$. Let $\mathcal{B}_{\tau_0(\mu)} = p^{-1}(\tau^* E_\mu)$ and let $\mathcal{B}^0_{\tau_0(\mu)}$ be the space of complex structures compatible with the direct sum decomposition $\tau^*(E^0_\mu) = \tau^* D_1 \oplus \cdots \oplus \tau^* D_r$. Then $\tau(\mathcal{B}^0_\mu) = \mathcal{B}^0_{\tau_0(\mu)}$.

Given a holomorphic vector bundle $\mathcal{V} \to \tilde{\Sigma}$, let $\mathcal{O}(\mathcal{V})$ be the sheaf of local holomorphic sections on $\mathcal{V}$. Then for $i = 0, 1$,

$$H^i_{\bar\partial}(\tilde{\Sigma}, \mathcal{V}) \cong H^i(\tilde{\Sigma}, \mathcal{O}(\mathcal{V})) \cong \check{H}^i(\{U_\alpha\}, \mathcal{O}(\mathcal{V})), \tag{4.1}$$

where $H^i_{\bar\partial}(\tilde{\Sigma}, \mathcal{V})$ is the Dolbeault cohomology of the holomorphic vector bundle $\mathcal{V}$, $H^i(\tilde{\Sigma}, \mathcal{O}(\mathcal{V}))$ is the sheaf cohomology of the sheaf $\mathcal{O}(\mathcal{V})$, and $\check{H}^i(\{U_\alpha\}, \mathcal{O}(\mathcal{V}))$ is the Čech cohomology with coefficient in the sheaf $\mathcal{O}(\mathcal{V})$

for a good cover $\{U_\alpha\}$ of $\tilde{\Sigma}$. Let $H^i(\tilde{\Sigma}, \mathcal{V})$ denote any of the three cohomology groups in (4.1).

Let $\tilde{\mathbb{N}}_{\tau_0(\mu)} \to \mathcal{C}_{\tau_0(\mu)}$ be the normal bundle of $\mathcal{C}_{\tau_0(\mu)}$ in $\mathcal{C}(\tau^* E)$. Then $\tau_{\mathcal{C}}(\mathcal{E}) = \tau_{\mathcal{C}}(\mathcal{D}_1) \oplus \cdots \oplus \tau_{\mathcal{C}}(\mathcal{D}_r)$, and

$$(\tilde{\mathbb{N}}_{\tau_0(\mu)})_{\tau_{\mathcal{C}}(\mathcal{E})} = \bigoplus_{i<j} H^1\left(\tilde{\Sigma}, \mathcal{H}om(\tau_{\mathcal{C}}(\mathcal{D}_j), \tau_{\mathcal{C}}(\mathcal{D}_i))\right).$$

Now, τ^* induces an anti-holomorphic map from the holomorphic vector bundle $\mathcal{H}om(\mathcal{D}_i, \mathcal{D}_j)$ to the anti-holomorphic bundle $\tau^*\mathcal{H}om(\mathcal{D}_i, \mathcal{D}_j) = \tau^*\mathcal{H}om(\mathcal{D}_j^\vee, \mathcal{D}_i^\vee)$. So τ induces an isomorphism of holomorphic vector bundles

$$\begin{aligned}\mathcal{H}om(\mathcal{D}_i, \mathcal{D}_j) \xrightarrow{\cong} \overline{\tau^*\mathcal{H}om(\mathcal{D}_j^\vee, \mathcal{D}_i^\vee)} &= \mathcal{H}om(\tau^*\overline{\mathcal{D}_j^\vee}, \tau^*\overline{\mathcal{D}_i^\vee}) \\ &= \mathcal{H}om(\tau_{\mathcal{C}}(\mathcal{D}_j), \tau_{\mathcal{C}}(\mathcal{D}_i)).\end{aligned}$$

Given a local holomorphic section of s of $\mathcal{H}om(\mathcal{D}_i, \mathcal{D}_j)|_U$, where U is an open subset of $\tilde{\Sigma}$, we let $\tau(s)$ denote the local holomorphic section of $\overline{\tau^*\mathcal{H}om(\mathcal{D}_i, \mathcal{D}_j)}\Big|_{\tau(U)}$ defined by $\tau(s)(z) = \overline{s(\tau(z))}$. Then τ defines a conjugate linear map between Čech complexes associated to $\mathcal{H}om(\mathcal{D}_i, \mathcal{D}_j)$ and $\overline{\tau^*\mathcal{H}om(\mathcal{D}_i, \mathcal{D}_j)} = \mathcal{H}om(\tau_{\mathcal{C}}(\mathcal{D}_j), \tau_{\mathcal{C}}(\mathcal{D}_i))$, and this in turn induces a conjugate linear map

$$\tau : H^1(\tilde{\Sigma}, \mathcal{H}om(\mathcal{D}_i, \mathcal{D}_j)) \to H^1\left(\tilde{\Sigma}, \mathcal{H}om(\tau_{\mathcal{C}}(\mathcal{D}_j), \tau_{\mathcal{C}}(\mathcal{D}_i))\right).$$

The direct sum of these maps (over $i < j$) is the conjugate linear map $(\mathbb{N}_\mu)\mathcal{E} \to (\mathbb{N}_{\tau_0(\mu)})_{\tau_{\mathcal{C}}(\mathcal{E})}$ induced by $\tau_{\mathcal{C}} : \mathcal{C}(E) \to \mathcal{C}(\tau^* E)$.

4.2. Degree zero case

We now specialize to the degree zero case (see [12, Section 7] for details).

Let Σ_0^ℓ be a Riemann surface of genus $\ell \geq 0$. Let Σ_1^ℓ be the connected sum of Σ_0^ℓ and $\mathbb{RP}^2$, and let Σ_2^ℓ be the connected sum of Σ_0^ℓ and a Klein bottle. Any closed connected surface is of the form Σ_i^ℓ, where $\ell \geq 0$ and $i = 0, 1, 2$. Σ_i^ℓ is orientable if and only if $i = 0$. Define

$$\begin{aligned}I_n^0 = \Big\{\mu = (\nu, \underbrace{0, \ldots, 0}_{n_0}, \tau_0(\nu))\Big|\nu = \Big(\underbrace{\frac{k_1}{n_1}, \ldots, \frac{k_1}{n_1}}_{n_1}, \ldots, \underbrace{\frac{k_r}{n_r}, \ldots, \frac{k_r}{n_r}}_{n_r}\Big) \in I_{n',k}, \\ n_0 > 0, \quad 2n' + n_0 = n, \quad \frac{k_1}{n_1} > \cdots > \frac{k_r}{n_r} > 0\Big\},\end{aligned}$$

$$I_n^{i,\pm} = \Big\{\mu = (\nu, \tau_0(\nu)) \Big| \nu = \big(\underbrace{\frac{k_1}{n_1}, \dots, \frac{k_1}{n_1}}_{n_1}, \dots, \underbrace{\frac{k_r}{n_r}, \dots, \frac{k_r}{n_r}}_{n_r}\big) \in I_{n',k}$$
$$2n' = n, \quad (-1)^{n'i+k} = \pm 1, \quad \frac{k_1}{n_1} > \cdots > \frac{k_r}{n_r} > 0\Big\},$$

where $i = 1, 2$. Then $I_{n,0}^{\tau_0} = I_n^0 \cup I_n^{i,+} \cup I_n^{i,-}$. To simplify notation, we set

$$\sigma(\mu) = (-1)^{n'i+k}$$

for any $\mu \in I_n^0$.

We have $\Sigma = \Sigma_i^\ell$ for some $\ell \geq 0$ and $i = 1, 2$, and $\tilde{\Sigma} = \Sigma_0^{2\ell+i-1}$. Let $E = P_{\tilde{\Sigma}}^{n,0} \times_\rho \mathbb{C}^n \cong \tilde{\Sigma} \times \mathbb{C}^n$. The involution $\tau^\pm : E \longrightarrow E$ defines an isomorphism $\phi^\pm : E \xrightarrow{\cong} \tau^* E$. We have $\tau_{\mathcal{C}}^\pm : \mathcal{C}(E) \to \mathcal{C}(E)$. Suppose that $\mu \in I_{n,0}$, and $\mathcal{C}_\mu^{\tau_{\mathcal{C}}^\pm}$ is nonempty. Then $\mu \in I_n^0 \cup I_n^{i,\pm}$, so μ is of the form

$$\mu = (\nu, \underbrace{0, \dots, 0}_{n_0}, \tau_0(\nu)), \quad \nu = \big(\underbrace{\frac{k_1}{n_1}, \dots, \frac{k_1}{n_1}}_{n_1}, \dots, \underbrace{\frac{k_r}{n_r}, \dots, \frac{k_r}{n_r}}_{n_r}\big) \in I_{n',k},$$

where $n_0 \geq 0$. There exist C^∞ subbundles $D_0, \dots, D_r$ of E such that

1) For $i = 0, \dots, r$, $\operatorname{rank}_{\mathbb{C}} D_i = n_i$, $\quad \deg D_i = k_i$, where $k_0 = 0$.
2) $E = D_1 \oplus \cdots \oplus D_r \oplus D_0 \oplus \tau^* D_r \oplus \cdots \oplus \tau^* D_1$.
3) $\tau^\pm$ preserves D_0 and switches D_i with $\tau^* D_i$ for $i = 1, \dots, r$.

Let

$$E_\mu^0 = D_1 \oplus \cdots \oplus D_r \oplus D_0 \oplus \tau^* D_r \oplus \cdots \oplus \tau^* D_1,$$

and define $\mathcal{B}_\mu^0$ as in Section 4.1. Then $\tau_{\mathcal{C}}^\pm$ acts on $\mathcal{B}_\mu^0$ by

$$\mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_r \oplus \mathcal{D}_0 \oplus \mathcal{D}_{-r} \oplus \cdots \oplus \mathcal{D}_{-1}$$
$$\mapsto \tau_{\mathcal{C}}(\mathcal{D}_{-1}) \oplus \cdots \oplus \tau_{\mathcal{C}}(\mathcal{D}_{-r}) \oplus \tau_{\mathcal{C}}^{\pm\sigma(\mu)}(\mathcal{D}_0) \oplus \tau_{\mathcal{C}}(\mathcal{D}_r) \oplus \cdots \oplus \tau_{\mathcal{C}}(\mathcal{D}_1).$$

Let $\mathcal{C}_{\mathrm{ss}}(D_i) \subset \mathcal{C}(D_i)$ be the semistable stratum. Any element in the fixed locus $(\mathcal{B}_\mu^0)^{\tau_{\mathcal{C}}^\pm}$ is of the form

$$\mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_r \oplus \mathcal{D}_0 \oplus \tau_{\mathcal{C}}(\mathcal{D}_r) \oplus \cdots \tau_{\mathcal{C}}(\mathcal{D}_1),$$

where $\mathcal{D}_i \in \mathcal{C}_{\mathrm{ss}}(D_i)$ for $i = 1, \dots, r$, and $\mathcal{D}_0 \in \mathcal{C}_{\mathrm{ss}}(D_0)^{\tau_{\mathcal{C}}^{\pm\sigma(\mu)}}$. Therefore,

$$(\mathcal{B}_\mu^0)^{\tau_{\mathcal{C}}^{\pm}} \cong \mathcal{C}_{\mathrm{ss}}(D_0)^{\tau_{\mathcal{C}}^{\pm\sigma(\mu)}} \times \prod_{i=1}^{r} \mathcal{C}_{\mathrm{ss}}(D_i).$$

Now, $\tau_{\mathcal{C}}^{\pm}$ acts on

$$\begin{aligned}\mathrm{Aut}(E_\mu^0) = {} & \mathrm{Aut}(D_1) \times \cdots \times \mathrm{Aut}(D_r) \times \mathrm{Aut}(D_0) \times \mathrm{Aut}(\tau^* D_r) \\ & \times \cdots \times \mathrm{Aut}(\tau^* D_1)\end{aligned}$$

by

$$\begin{aligned}&(u_1, \dots, u_r, u_0, u_{-r}, \dots, u_{-1}) \\ &\quad \mapsto (\tau_{\mathcal{C}}(u_{-1}), \dots, \tau_{\mathcal{C}}(u_{-r}), \tau_{\mathcal{C}}^{\pm\sigma(\mu)}(u_0), \tau_{\mathcal{C}}(u_r), \dots, \tau_{\mathcal{C}}(u_1)).\end{aligned}$$

To simplify notation, we write $\mathcal{G}_\mu = \mathrm{Aut}(E_\mu^0)^{\tau_{\mathcal{C}}^{\pm}}$. Then we have

$$\mathcal{G}_\mu \cong \mathrm{Aut}(D_0)^{\tau_{\mathcal{C}}^{\pm\sigma(\mu)}} \times \prod_{i=1}^{r} \mathrm{Aut}(D_i).$$

Let $\mathcal{A}_\mu \subset \mathcal{A}(P_\Sigma^{n,\pm})$ be the equivariant Morse stratum that corresponds to $\mathcal{C}_\mu^{\tau_{\mathcal{C}}^{\pm}} \subset \mathcal{C}(E)^{\tau_{\mathcal{C}}^{\pm}}$. As in the orientable case, the inclusion $i : (\mathcal{B}_\mu^0)^{\tau_{\mathcal{C}}^{\pm}} \hookrightarrow \mathcal{C}_\mu^{\tau_{\mathcal{C}}^{\pm}}$ induces a weak homotopy equivalence

$$\left((\mathcal{B}_\mu^0)^{\tau_{\mathcal{C}}^{\pm}}\right)_{h\mathcal{G}_\mu} \xrightarrow{\sim} \left(\mathcal{C}_\mu^{\tau_{\mathcal{C}}^{\pm}}\right)_{h(\mathrm{Aut}(E)^{\tau_{\mathcal{C}}^{\pm}})}.$$

We now have a sequence of maps (where $\sim$ denotes a weak homotopy equivalence)

$$\begin{aligned}(\mathcal{A}_\mu)_{h\mathrm{Aut}(P_\Sigma^{n,\pm})} &\cong \left(\mathcal{C}_\mu^{\tau_{\mathcal{C}}^{\pm}}\right)_{h(\mathrm{Aut}(E,h)^{\tau_{\mathcal{C}}^{\pm}})} \xleftarrow{\sim} \left(\mathcal{C}_\mu^{\tau_{\mathcal{C}}^{\pm}}\right)_{h(\mathrm{Aut}(E)^{\tau_{\mathcal{C}}^{\pm}})} \xleftarrow{\sim} \left((\mathcal{B}_\mu^0)^{\tau_{\mathcal{C}}^{\pm}}\right)_{h\mathcal{G}_\mu} \\ &\cong \left(\mathcal{C}_{\mathrm{ss}}(D_0)^{\tau_{\mathcal{C}}^{\pm\sigma(\mu)}}\right)_{h\left(\mathrm{Aut}(D_0)^{\tau_{\mathcal{C}}^{\pm\sigma(\mu)}}\right)} \times \prod_{j=1}^{r} \mathcal{C}_{\mathrm{ss}}(D_i)_{h\mathrm{Aut}(D_i)}.\end{aligned}$$

When $\mu \in I_n^{i,\pm}$, we do not have the first factor $\mathcal{C}_{\mathrm{ss}}(D_0)$.

Let $\tilde{\mathbb{N}}_\mu$ be the normal bundle of $\mathcal{C}_\mu$ in $\mathcal{C}(E)$. Given

$$\mathcal{E} = \mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_r \oplus \mathcal{D}_0 \oplus \tau_{\mathcal{C}}(\mathcal{D}_r) \oplus \cdots \oplus \tau_{\mathcal{C}}(\mathcal{D}_1) \in (\mathcal{B}_\mu^0)^{\tau_{\mathcal{C}}^{\pm}},$$

we have

$$\begin{aligned}(\tilde{\mathbb{N}}_\mu)_{\mathcal{E}} &= H^1(\tilde{\Sigma}, \mathcal{E}nd''(\mathcal{E})) \\ &= \bigoplus_{0<i<j} H^1\left(\tilde{\Sigma}, \mathcal{H}om(\mathcal{D}_i, \mathcal{D}_j)\right) \oplus \bigoplus_{0<i<j} H^1\left(\tilde{\Sigma}, \mathcal{H}om(\tau_{\mathcal{C}}(\mathcal{D}_j), \tau_{\mathcal{C}}(\mathcal{D}_i)\right) \\ &\quad \oplus \bigoplus_{0<i,j} H^1\left(\tilde{\Sigma}, \mathcal{H}om(\mathcal{D}_i, \tau_{\mathcal{C}}(\mathcal{D}_j))\right) \\ &\quad \oplus \bigoplus_{i>0} H^1\left(\tilde{\Sigma}, \mathcal{H}om(\mathcal{D}_i, \mathcal{D}_0)\right) \oplus \bigoplus_{i>0} H^1\left(\tilde{\Sigma}, \mathcal{H}om(\mathcal{D}_0, \tau_{\mathcal{C}}(\mathcal{D}_i))\right).\end{aligned}$$

By the discussion of Section 4.1, τ induces conjugate linear maps of complex vector spaces:

$$\begin{aligned}H^1(\tilde{\Sigma}, \mathcal{H}om(\mathcal{D}_i, \mathcal{D}_j)) &\to H^1(\tilde{\Sigma}, \mathcal{H}om(\tau_{\mathcal{C}}(\mathcal{D}_j), \tau_{\mathcal{C}}(\mathcal{D}_i))), \text{ and its inverse,} \\ H^1(\tilde{\Sigma}, \mathcal{H}om(\mathcal{D}_i, \tau_{\mathcal{C}}(\mathcal{D}_j)) &\to H^1(\tilde{\Sigma}, \mathcal{H}om(\mathcal{D}_j, \tau_{\mathcal{C}}(\mathcal{D}_i)), \\ H^1(\tilde{\Sigma}, \mathcal{H}om(\mathcal{D}_i, \mathcal{D}_0)) &\to H^1(\tilde{\Sigma}, \mathcal{H}om(\mathcal{D}_0, \tau_{\mathcal{C}}(\mathcal{D}_i))), \text{ and its inverse.}\end{aligned}$$

Let $\mathbb{N}_\mu$ be the normal bundle of $\mathcal{A}_\mu$ in $\mathcal{A}(P_\Sigma^{n,\pm})$, or equivalently, the normal bundle of $\mathcal{C}_\mu^{\tau_{\mathcal{C}}^\pm}$ in $\mathcal{C}(E)^{\tau_{\mathcal{C}}^\pm}$. Then

(4.2)

$$\begin{aligned}(\mathbb{N}_\mu)_{\mathcal{E}} &= H^1(\tilde{\Sigma}, \mathcal{E}nd''(\mathcal{E}))^\tau \\ &\cong \bigoplus_{0<i<j} H^1\left(\tilde{\Sigma}, \mathcal{H}om(\mathcal{D}_i, \mathcal{D}_j)\right) \oplus \bigoplus_{0<i<j} H^1\left(\tilde{\Sigma}, \mathcal{H}om(\mathcal{D}_i, \tau_{\mathcal{C}}(\mathcal{D}_j))\right) \\ &\quad \oplus \bigoplus_{i>0} H^1\left(\tilde{\Sigma}, \mathcal{H}om(\mathcal{D}_i, \tau_{\mathcal{C}}(\mathcal{D}_i))\right)^\tau \oplus \bigoplus_{i>0} H^1\left(\tilde{\Sigma}, \mathcal{H}om(\mathcal{D}_i, \mathcal{D}_0)\right)\end{aligned}$$

Let $i : (\mathcal{B}_\mu^0)^{\tau_{\mathcal{C}}^\pm} \hookrightarrow \mathcal{C}_\mu^{\tau_{\mathcal{C}}^\pm}$ denote the inclusion map. By (4.2), $i^*\mathbb{N}_\mu = \mathbb{N}_\mu^{\mathbb{C}} \oplus \mathbb{N}_\mu^{\mathbb{R}}$, where

$$\begin{aligned}(\mathbb{N}_\mu^{\mathbb{C}})_{\mathcal{E}} &= \bigoplus_{0<i<j} H^1\left(\tilde{\Sigma}, \mathcal{H}om(\mathcal{D}_i, \mathcal{D}_j)\right) \oplus \bigoplus_{0<i<j} H^1\left(\tilde{\Sigma}, \mathcal{H}om(\mathcal{D}_i, \tau_{\mathcal{C}}(\mathcal{D}_j))\right) \\ &\quad \oplus \bigoplus_{i>0} H^1\left(\tilde{\Sigma}, \mathcal{H}om(\mathcal{D}_i, \mathcal{D}_0)\right); \\ (\mathbb{N}_\mu^{\mathbb{R}})_{\mathcal{E}} &= \bigoplus_{i>0} H^1\left(\tilde{\Sigma}, \mathcal{H}om(\mathcal{D}_i, \tau_{\mathcal{C}}(\mathcal{D}_i))\right)^\tau.\end{aligned}$$

Note that $(\mathbb{N}_\mu^{\mathbb{C}})_{h\mathcal{G}_\mu} \to \left((\mathcal{B}_\mu^0)^{\tau_{\mathcal{C}}^{\pm}}\right)_{h\mathcal{G}_\mu}$ is a complex vector bundle, thus an oriented real vector bundle. Hence orientability of $i^*\mathbb{N}_\mu$ is equivalent to orientability of the real vector bundle

$$(\mathbb{N}_\mu^{\mathbb{R}})_{h\mathcal{G}_\mu} \to \left((\mathcal{B}_\mu^0)^{\tau_{\mathcal{C}}^{\pm}}\right)_{h\mathcal{G}_\mu} = \left(\mathcal{C}_{\mathrm{ss}}(D_0)^{\tau_{\mathcal{C}}^{\pm\sigma(\mu)}}\right)_{h\left(\mathrm{Aut}(D_0)^{\tau_{\mathcal{C}}^{\pm\sigma(\mu)}}\right)} \times \prod_{i=1}^{r} \mathcal{C}_{\mathrm{ss}}(D_i)_{h\mathrm{Aut}(D_i)}.$$

We have

$$\mathbb{N}_\mu^{\mathbb{R}} = \bigoplus_{i=1}^{r} V_i, \quad (V_i)_{\mathcal{E}} = H^1(\tilde{\Sigma}, \mathcal{H}om(\mathcal{D}_i, \tau_{\mathcal{C}}(\mathcal{D}_i)))^{\tau}.$$

Let D be a rank n, degree $k > 0$ complex vector bundle over $\tilde{\Sigma}$. Let $\mathbb{V}_{n,k}$ be the $\mathrm{Aut}(D)$-equivariant real vector bundle over $\mathcal{C}_{\mathrm{ss}}(D)$ whose fiber at $\mathcal{D}$ is $H^1(\tilde{\Sigma}, \mathcal{H}om(\mathcal{D}, \tau_{\mathcal{C}}(\mathcal{D}))^{\tau}$.

The following result will be a direct consequence of Lemma 4.2 (the $\tilde{g} = 0$ case) and Theorem 4.2 (the $\tilde{g} > 0$ case) in Section 4.3 below.

Theorem 4.1. *Let D be a rank n, degree $k > 0$ complex vector bundle over a Riemann surface of genus $\tilde{g}$. If $n = 1$ or $\tilde{g} \neq 1$ then $(\mathbb{V}_{n,k})_{h\mathrm{Aut}(D)} \to \mathcal{C}_{\mathrm{ss}}(D)_{h\mathrm{Aut}(D)}$ is orientable.*

Suppose that Σ is diffeomorphic to the Klein bottle, so that its orientable double cover $\tilde{\Sigma}$ is a Riemann surface of genus $\tilde{g} = 1$. Note that

$$I_2^{\tau_0} = \{(k, -k) \mid k \in \mathbb{Z}_{\geq 0}\}, \quad I_3^{\tau_0} = \{(k, 0, -k) \mid k \in \mathbb{Z}_{\geq 0}\}.$$

The $k = 0$ case corresponds to open strata whose normal bundles are of rank zero. From the above discussion, when $\mu = (k, -k)$ or $(k, 0, -k)$, where $k > 0$, we have

$$(\mathbb{N}_\mu^{\mathbb{R}})_{\mathcal{E}} = H^1(\tilde{\Sigma}, \mathcal{H}om(\mathcal{D}_1, \tau_{\mathcal{C}}(\mathcal{D}_1)))^{\tau},$$

where $\mathcal{D}_1$ is a rank 1, degree k holomorphic bundle over $\tilde{\Sigma}$. Therefore Theorem 4.1 implies Theorem 1.1, our main orientability theorem.

4.3. Reduction to representation varieties

We consider the following equivariant real vector bundles:

1. The $\mathrm{Aut}(E)$-equivariant vector bundle $\mathbb{V}_{n,k} \to \mathcal{C}_{\mathrm{ss}}(E)$.

2. The $\mathrm{Aut}(E,h)$-equivariant vector bundle $\mathbb{V}_{n,k} \to \mathcal{C}_{\mathrm{ss}}(E)$, or equivalently, the $\tilde{\mathcal{G}}$-equivariant vector bundle $\mathbb{V}_{n,k} \to \mathcal{A}_{\mathrm{ss}}(P_{\tilde{\Sigma}}^{n,k})$, where $\tilde{\mathcal{G}} = \mathrm{Aut}(P_{\tilde{\Sigma}}^{n,k})$ and $\mathcal{A}_{\mathrm{ss}}(P_{\tilde{\Sigma}}^{n,k})$ is the open Morse stratum.
3. The $\tilde{\mathcal{G}}$-equivariant vector bundle $i_{n,k}^*\mathbb{V}_{n,k} \to \mathcal{N}_{\mathrm{ss}}(P_{\tilde{\Sigma}}^{n,k})$, where $i_{n,k} : \mathcal{N}_{\mathrm{ss}}(P_{\tilde{\Sigma}}^{n,k}) \hookrightarrow \mathcal{A}_{\mathrm{ss}}(P_{\tilde{\Sigma}}^{n,k})$ is the inclusion of the space of central Yang–Mills connections on $P_{\tilde{\Sigma}}^{n,k}$.

The inclusion $\mathrm{Aut}(E,h) \subset \mathrm{Aut}(E)$ is a homotopy equivalence; $\mathcal{A}_{\mathrm{ss}}(P_{\tilde{\Sigma}}^{n,k})$ is the stable manifold of $\mathcal{N}_{\mathrm{ss}}(P_{\tilde{\Sigma}}^{n,k})$, and the gradient flow of the Yang–Mills functional gives a $\tilde{\mathcal{G}}$-equivariant deformation retraction $\mathcal{A}_{\mathrm{ss}}(P_{\tilde{\Sigma}}^{n,k}) \to \mathcal{N}_{\mathrm{ss}}(P_{\tilde{\Sigma}}^{n,k})$. Therefore $(\mathbb{V}_{n,k})_{h\mathrm{Aut}(E)} \to \mathcal{C}_{\mathrm{ss}}(E)_{h\mathrm{Aut}(E)}$ is orientable if and only if $(i_{n,k}^*\mathbb{V}_{n,k})_{h\tilde{\mathcal{G}}} \to \mathcal{N}_{\mathrm{ss}}(P_{\tilde{\Sigma}}^{n,k})_{h\tilde{\mathcal{G}}}$ is orientable.

We fix a base point $x_0 \in \tilde{\Sigma}$, and let $\mathrm{ev} : \tilde{\mathcal{G}} \to U(n)$ be the evaluation at x_0. Then ev is a surjective group homomorphism, and the kernel $\tilde{\mathcal{G}}_0$ is the based gauge group. Therefore, $\tilde{\mathcal{G}}_0$ is a normal subgroup of $\tilde{\mathcal{G}}$, and $\tilde{\mathcal{G}}/\tilde{\mathcal{G}}_0 = U(n)$. The group $\tilde{\mathcal{G}}_0$ acts freely on $\mathcal{N}_{\mathrm{ss}}(P_{\tilde{\Sigma}}^{n,k})$. Let $\tilde{g}$ be the genus of $\tilde{\Sigma}$. The representation variety of central Yang–Mills connections on $P_{\tilde{\Sigma}}^{n,k}$ is given by

$$X_{\mathrm{YM}}^{\tilde{g},0}(U(n))_{\frac{k}{n},\dots,\frac{k}{n}} = \{V \in U(n)^{2\tilde{g}} \mid \mathfrak{m}(V) = \mathrm{e}^{-2\pi\sqrt{-1}k/n} I_n\},$$

where $\mathfrak{m}(a_1,b_1,\dots,a_{\tilde{g}},b_{\tilde{g}}) = \prod_{i=1}^{\tilde{g}}[a_i,b_i]$. (See [12, Section 6.1] for the definition of $X_{\mathrm{YM}}^{\tilde{g},0}(U(n))_\mu$ for a general Atiyah–Bott type μ.) There is a homeomorphism

$$\mathcal{N}_{\mathrm{ss}}(P_{\tilde{\Sigma}}^{n,k})/\tilde{\mathcal{G}}_0 \cong X_{\mathrm{YM}}^{\tilde{g},0}(U(n))_{\frac{k}{n},\dots,\frac{k}{n}}.$$

The $\tilde{\mathcal{G}}$-equivariant vector bundle $i_{n,k}^*\mathbb{V}_{n,k}$ over $\mathcal{N}_{\mathrm{ss}}(P_{\tilde{\Sigma}}^{n,k})$ descends to a $U(n)$-equivariant vector bundle $V_{n,k}$ over $X_{\mathrm{YM}}^{\tilde{g},0}(U(n))_{\frac{k}{n},\dots,\frac{k}{n}}$. Therefore, orientability of $(i_{n,k}^*\mathbb{V}_{n,k})_{h\tilde{\mathcal{G}}} \to \mathcal{N}_{\mathrm{ss}}(P_{\tilde{\Sigma}}^{n,k})_{h\tilde{\mathcal{G}}}$ will follow from orientability of the bundle $(V_{n,k})_{hU(n)} \to (X_{\mathrm{YM}}^{\tilde{g},0}(U(n))_{\frac{k}{n},\dots,\frac{k}{n}})_{hU(n)}$.

We will need the following lemma regarding orientability of equivariant vector bundles.

Lemma 4.1. *Let G be a compact, connected Lie group and let X be a paracompact G-space. Then a G-equivariant real vector bundle $W \to X$ is orientable if and only if the vector bundle $EG \times_G W \to EG \times_G X$ is orientable.*

Proof. Since W is the restriction of W_{hG} to a fiber of the projection $X_{hG} \to BG$, the "if" direction is immediate. Now assume W is orientable. Then $\det W \to X$ is trivial, and it will suffice to show that

$$\det(EG \times_G W) \cong EG \times_G (\det W) \longrightarrow EG \times_G X$$

is trivial. Since X is paracompact and G is compact, we may choose a G-equivariant metric on $\det W$. The set $\det(W)_1$ of length-one vectors in $\det(W) \cong X \times \mathbb{R}$ is homeomorphic to $X \coprod X$, so there is a section $s : X \to \det(W)$ with image in $\det(W)_1$.

We claim that s is G-equivariant. Fix $x \in X$, $g \in G$. Since G is connected, there exists a path g_t from g to e, yielding paths $s(g_t \cdot x)$ and $g_t \cdot s(x)$ from $s(g \cdot x)$ and $g \cdot s(x)$ to $s(x)$. By G-invariance of the metric, these paths lie in $\det(W)_1$, so $s(g \cdot x)$ and $g \cdot s(x)$ lie in the same path component of $\det(W)_1 \cong X \coprod X$. Since both points are in the fiber over $g \cdot x$, we have $s(g \cdot x) = g \cdot s(x)$. The map $EG \times X \to EG \times_G \det(W)$ given by $(e, x) \mapsto [e, s(x)]$ now factors through $EG \times_G X$, giving a nowhere-zero section of this line bundle. □

Since $U(n)$ is compact and connected, to show that

$$(V_{n,k})_{hU(n)} \to \left(X^{\tilde{g},0}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\ldots,\frac{k}{n}}\right)_{hU(n)}$$

is orientable it suffices, by Lemma 4.1, to show that

$$V_{n,k} \to X^{\tilde{g},0}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\ldots,\frac{k}{n}}$$

is orientable.

When $\tilde{g} = 0$, the definition of $X^{\tilde{g},0}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\ldots,\frac{k}{n}}$ degenerates (the reader may wish to compare with the general definition given in [12, Section 6.1]) and we find that $X^{0,0}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\ldots,\frac{k}{n}}$ is a single point when k is a multiple of n, and is empty otherwise.

Lemma 4.2.

$$X^{0,0}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\ldots,\frac{k}{n}} = \begin{cases} \{V \in U(n)^0 | V = \mathrm{e}^{-2\pi\sqrt{-1}k/n} I_n\} = \{I_n\}, & \frac{k}{n} \in \mathbb{Z}, \\ \emptyset, & \frac{k}{n} \notin \mathbb{Z}. \end{cases}$$

So $V_{n,k} \to X^{0,0}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\ldots,\frac{k}{n}}$ is orientable whenever $X^{0,0}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\ldots,\frac{k}{n}}$ is nonempty.

We will prove the following.

Theorem 4.2. *Let $k > 0$. The real vector bundle $V_{n,k} \to X^{\tilde{g},0}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\dots,\frac{k}{n}}$ is orientable when $n = 1$ or $\tilde{g} \geq 2$.*

We now explain how exactly we deduce Theorem 1.1 from Lemma 4.2, Theorem 4.2 and the previous results and arguments in this section. We must be careful due to the fact that many of the maps we have been considering are only weak homotopy equivalences. In particular, if $f : X \to Y$ is a weak homotopy equivalence and $V \to Y$ is a real vector bundle, orientability of $f^*(V)$ does not necessarily imply orientability of V (although by the Bundle Homotopy Theorem [17, Section 4.9], this implication does hold for homotopy equivalences).

We need to prove orientability of the normal bundle $(\mathbb{N}_\mu)_{h\mathcal{G}}$ over $(\mathcal{A}_\mu)_{h\mathcal{G}}$. Letting $i : \mathcal{N}_\mu \hookrightarrow \mathcal{A}_\mu$ denote the inclusion of the critical set, this bundle is isomorphic to the pullback of the bundle $i^*((\mathbb{N}_\mu)_{h\mathcal{G}})$ under the retraction $r : \mathcal{A}_\mu \to \mathcal{N}_\mu$ provided by the Yang–Mills flow (because i and r are homotopy inverses). Moreover, $i^*((\mathbb{N}_\mu)_{h\mathcal{G}})$ is the pull back of a bundle $W_{hU(n)}$ over the representation variety $(\mathcal{N}_\mu/\mathcal{G}_0)_{hU(n)}$ (this reduction to the representation variety is analogous to the argument in Section 4.3). By the results in [12, Sections 6, 7], $\mathcal{A}_\mu/\mathcal{G}_0$ is an analytic set, and hence admits a $U(n)$-equivariant triangulation [11]. Thus the homotopy orbit space is a CW complex, and to prove Theorem 1.1 we now just need to prove orientability of the bundle $W_{hU(n)}$ over the CW complex $(\mathcal{N}_\mu/\mathcal{G}_0)_{hU(n)}$.

The various weak equivalences exhibited in this section provide a weak equivalence

$$\left((\mathcal{B}^0_\mu)^{\tau^{\pm}_c}\right)_{h\mathrm{Aut}^0_\mu} \xrightarrow{\sim} (\mathcal{N}_\mu/\mathcal{G}_0)_{hU(n)}.$$

By a standard CW approximation argument (for example, pull back over the singular complex of $((\mathcal{B}^0_\mu)^{\tau^{\pm}_c})_{h\mathrm{Aut}^0_\mu}$), orientability of the bundle $W_{hU(n)} \to (\mathcal{N}_\mu/\mathcal{G}_0)_{hU(n)}$ is implied by orientability of the pullback of this bundle to $((\mathcal{B}^0_\mu)^{\tau^{\pm}_c})_{h\mathrm{Aut}^0_\mu}$; note that this pullback is just the restriction of $(\mathbb{N}_\mu)_{h\mathcal{G}}$ to $((\mathcal{B}^0_\mu)^{\tau^{\pm}_c})_{h\mathrm{Aut}^0_\mu}$. Finally, we have seen that orientability of this restricted bundle is implied by Theorem 4.1, which follows from Theorem 4.2. In the subsequent sections, we will prove Theorem 4.2 by explicitly examining the restrictions of the bundle $V_{n,k}$ (see Theorem 4.2) to loops generating the fundamental group of $X^{\tilde{g},0}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\dots,\frac{k}{n}}$. Note that $X^{\tilde{g},0}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\dots,\frac{k}{n}}$ is again an analytic set, hence triangulable (here we do not need an equivariant triangulation, so the classical result of Łojasiewicz [19] suffices).

Remark 4.1. For the main applications we have in mind (e.g., the Morse inequalities mentioned in the introduction), it is not strictly necessary to

prove orientability of the normal bundles to the Yang–Mills strata; one simply needs Thom isomorphisms describing how each critical set contributes to the cohomology of the space $\mathcal{A}_{h\mathcal{G}}$. We now explain how to deduce these isomorphisms without resorting to Illman's equivariant triangulability results, or even the nonequivariant result of Łojasiewicz.

The partial ordering on the Yang–Mills strata defined by Atiyah and Bott [1, Section 7] can be refined to a linear ordering in which the union of each initial segment is open (see [23] for details). Let $\mathcal{A}_I$ denote the union of the strata in some initial segment I in this ordering (so $\mathcal{A}_I$ an open neighborhood of $\mathcal{A}_{\mathrm{ss}}$) and let $\mathcal{A}_\mu$ be the next stratum. Then, by excising the complement of a gauge-invariant tubular neighborhood (see [23]) and applying the Thom Isomorphism Theorem to the (orientable) normal bundle $(\mathbb{N}_\mu)_{h\mathcal{G}}$, one obtains isomorphisms

$$H^*_{\mathcal{G}}(\mathcal{A}_I \cup \mathcal{A}_\mu, \mathcal{A}_I) \cong H^*_{\mathcal{G}}(\mathbb{N}_\mu, (\mathbb{N}_\mu)_0) \cong H^{*-c(\mu)}_{\mathcal{G}}(\mathcal{A}_\mu), \tag{4.3}$$

where $(\mathbb{N}_\mu)_0$ denotes the complement of the zero section and $c(\mu)$ is the dimension of $\mathbb{N}_\mu$. The isomorphism between the first and third terms is what we need in order to compute equivariant cohomology.

Rather than applying the Thom Isomorphism directly to $(\mathbb{N}_\mu)_{h\mathcal{G}}$, one may instead pull back over a CW approximation $f : X \to (\mathcal{A}_\mu)_{h\mathcal{G}}$. Since f is a weak equivalence and both $\mathbb{N}_\mu$ and the complement of its zero section fiber over $\mathcal{A}_\mu$, we have an isomorphism

$$H^*_{h\mathcal{G}}(f^*\mathbb{N}_\mu, (f^*\mathbb{N}_\mu)_0) \cong H^*_{h\mathcal{G}}(\mathbb{N}_\mu, (\mathbb{N}_\mu)_0).$$

To establish an isomorphism between the first and third terms in (4.3), we need only deduce orientability of $f^*(\mathbb{N}_\mu)$. This follows from Theorem 4.2 by applying CW approximations throughout the previous argument; in fact we only need to know that the bundle $V_{n,k}$ in Theorem 4.2 is orientable after pulling back over a CW approximation $\alpha : K \xrightarrow{\simeq} X^{\tilde{g},0}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\ldots,\frac{k}{n}}$. We will show in subsequent sections that $V_{n,k}$ is orientable along loops $\{\gamma_i\}$ generating the fundamental group of $X^{\tilde{g},0}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\ldots,\frac{k}{n}}$. Choosing $\gamma_i' : S^1 \to K$ such that $\alpha \circ \gamma_i' \simeq \gamma_i$, the Bundle Homotopy Theorem [17, Section 4.9] implies that $\alpha^* V_{n,k}$ is orientable along the loops γ_i', which generate $\pi_1 K$. Since K is a CW complex, this implies (see Remark 6.1) that $\alpha^* V_{n,k}$ is orientable, as desired. In this approach, we do not need to use the fact that $X^{\tilde{g},0}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\ldots,\frac{k}{n}}$ is triangulable.

5. Fundamental groups

Our orientability argument requires a calculation of fundamental groups.

Proposition 5.1. *For $\tilde{g} \geq 2$, the map* det *induces an isomorphism*

$$\pi_1\left(X_{\mathrm{YM}}^{\tilde{g},0}(U(n))_{\frac{k}{n},\ldots,\frac{k}{n}}\right) \xrightarrow{\det_*} \pi_1(X_{\mathrm{YM}}^{\tilde{g},0}(U(1))_k) \cong \mathbb{Z}^{2\tilde{g}}.$$

Proof. We may assume $n \geqslant 2$. We first introduce some notation. Let $P^{m,k} = P_{\tilde{\Sigma}}^{m,k}$; note that $\det(P^{n,k}) = P^{1,k}$. Let $\mathcal{A}(m,k) = \mathcal{A}(P^{m,k})$ be the space of $U(m)$-connections on $P^{m,k}$, let $\mathcal{G}(m,k) = \mathrm{Aut}(P^{m,k})$ be the gauge group, and let $\mathcal{G}_0(m,k) \subset \mathcal{G}(m,k)$ be the base gauge group. Let $\mathcal{C}(m,k)$ be the space of holomorphic structures on $E^{m,k}$, the rank m, degree k complex vector bundle over $\tilde{\Sigma}$. Let $\mathcal{C}_{\mathrm{ss}}(m,k) \subset \mathcal{C}(m,k)$ be the semi-stable stratum.

Recall that Trace: $\mathfrak{u}(n) \to \mathfrak{u}(1)$ is the derivative of the determinant map det: $U(n) \to U(1)$ at the identity. Clearly, it is ad-invariant and it induces a map $\mathrm{ad}\,(P^{n,k}) \to \mathrm{ad}\,(P^{1,k})$, and thus a map $\mathrm{Tr} : \mathcal{A}(n,k) \to \mathcal{A}(1,k)$. The map Tr sends a Yang–Mills $U(n)$-connection to a Yang–Mills $U(1)$-connection. Since all Yang–Mills $U(1)$-connection are central, the map Tr descends to a map

$$\det : X_{\mathrm{YM}}^{\tilde{g},0}(U(n))_{\frac{k}{n},\ldots,\frac{k}{n}} \to X_{\mathrm{YM}}^{\tilde{g},0}(U(1))_k$$

(recall that $\det(\exp M) = \exp(\mathrm{Tr}M), \forall M \in \mathfrak{u}(n)$). In other words, we have a commuting diagram:

$$\begin{array}{ccc} \mathcal{N}_{\frac{k}{n},\ldots,\frac{k}{n}} & \xrightarrow{\mathrm{Tr}} & \mathcal{N}_k \\ \downarrow \mathrm{hol} & & \downarrow \mathrm{hol} \\ X_{\mathrm{YM}}^{\tilde{g},0}(U(n))_{\frac{k}{n},\ldots,\frac{k}{n}} & \xrightarrow{\det} & X_{\mathrm{YM}}^{\tilde{g},0}(U(1))_k. \end{array}$$

The determinant map $U(n) \to U(1)$ also induces a homomorphism $\phi : \mathcal{G}_0(n,k) \to \mathcal{G}_0(1,k)$, and the map $\mathrm{Tr} : \mathcal{A}(n,k) \to \mathcal{A}(1,k)$ is ϕ-equivariant. In particular, the map $\mathrm{Tr} : \mathcal{N}_{\frac{k}{n},\ldots,\frac{k}{n}} \to \mathcal{N}_k$ is ϕ-equivariant, and we have a well-defined map

$$\mathrm{Tr} : E\mathcal{G}_0(n,k) \times_{\mathcal{G}_0(n,k)} \mathcal{N}_{\frac{k}{n},\ldots,\frac{k}{n}} \longrightarrow E\mathcal{G}_0(1,k) \times_{\mathcal{G}_0(1,k)} \mathcal{N}_k,$$

which we may identify up to homotopy with the determinant map

$$\det : X_{\mathrm{YM}}^{\tilde{g},0}(U(n))_{\frac{k}{n},\ldots,\frac{k}{n}} \to X_{\mathrm{YM}}^{\tilde{g},0}(U(1))_k.$$

Moreover, the Yang–Mills flow provides a gauge-equivariant deformation retraction from the space $\mathcal{C}_{\mathrm{ss}}(m,k)$ of semi-stable bundles to the critical set $\mathcal{N}_{\frac{k}{m},\ldots,\frac{k}{m}}$ [22], so it suffices to show that the map

$$\pi_1\left(E\mathcal{G}_0(n,k)\times_{\mathcal{G}_0(n,k)}\mathcal{C}_{\mathrm{ss}}(n,k)\right)\xrightarrow{\mathrm{Tr}_*}\pi_1\left(E\mathcal{G}_0(1,k)\times_{\mathcal{G}_0(1,k)}\mathcal{C}_{\mathrm{ss}}(1,k)\right) \tag{5.1}$$

is an isomorphism.

We have an induced map of fibration sequences

$$\begin{array}{ccccc} \mathcal{C}_{\mathrm{ss}}(n,k) & \longrightarrow & E\mathcal{G}_0(n,k)\times_{\mathcal{G}_0(n,k)}\mathcal{C}_{\mathrm{ss}}(n,k) & \longrightarrow & B\mathcal{G}_0(n,k) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{C}_{\mathrm{ss}}(1,k) & \longrightarrow & E\mathcal{G}_0(1,k)\times_{\mathcal{G}_0(1,k)}\mathcal{C}_{\mathrm{ss}}(1,k) & \longrightarrow & B\mathcal{G}_0(1,k), \end{array} \tag{5.2}$$

and we claim that both fibers are simply connected. For $n=1$, all critical connections are minimal, i.e., there is only one stratum and thus the set of minimal Yang–Mills connections is a deformation retraction of the total space $\mathcal{A}(1,k)$, which is an affine space. Thus, $\mathcal{C}_{\mathrm{ss}}(1,k)=\mathcal{A}(1,k)$ is contractible. Since both n and $\tilde{g}$ are at least 2, the complement of $\mathcal{C}_{\mathrm{ss}}(n,k)$ in the contractible space $\mathcal{A}(n,k)$ may be stratified by submanifolds of (finite) real codimension at least $2(\tilde{g}-1)(n-1)+2\geqslant 4$. Transversality arguments (as in [24, Section 4] or [6]) now apply to prove simple connectivity.

Since both π_1 and π_0 of $\mathcal{C}_{\mathrm{ss}}(1,k)$ and $\mathcal{C}_{\mathrm{ss}}(n,k)$ are trivial, we may now identify the map (5.1) with the map $\pi_1(B\mathcal{G}_0(n,k))\xrightarrow{c}\pi_1(B\mathcal{G}_0(1,k))$ induced by diagram (5.2). By Atiyah and Bott [1, Section 2], we have homotopy equivalences $B\mathcal{G}_0(m,k)\simeq \mathrm{Map}_*^{P^{m,k}}(\tilde{\Sigma},BU(m))$ for any m, where $\mathrm{Map}_*^{P^{m,k}}$ denotes the subspace of based maps which induce the bundle $P^{m,k}$. Hence we may identify the map c with the determinant map

$$\pi_1(\mathrm{Map}_*^{P^{n,k}}(\tilde{\Sigma},BU(n)))\longrightarrow\pi_1(\mathrm{Map}_*^{P^{1,k}}(\tilde{\Sigma},BU(1))). \tag{5.3}$$

The splitting $U(1)\to U(n)$ of $\det: U(n)\to U(1)$ induces a splitting

$$\mathrm{Map}_*(\tilde{\Sigma},BU(1))\xrightarrow{i}\mathrm{Map}_*(\tilde{\Sigma},BU(n)) \tag{5.4}$$

of the determinant map $\mathrm{Map}_*(\tilde{\Sigma},BU(n))\longrightarrow\mathrm{Map}_*(\tilde{\Sigma},BU(1))$, and hence after restricting to components (recall that $\det(P^{n,k})=P^{1,k}$) we obtain splittings of the maps (5.3). This implies that the maps (5.3) are surjective.

To prove that the maps (5.3) are also injective, it suffices to show that their domain and range are isomorphic to $\mathbb{Z}^{2\tilde{g}}$. Note that $\tilde{\Sigma}$ is the mapping

cone of the attaching map η for its 2–cell, so we have a homotopy cofiber sequence $S^1 \xrightarrow{\eta} \bigvee_{2\tilde{g}} S^1 \to \tilde{\Sigma}$. For any $m \geqslant 1$, applying $\mathrm{Map}_*(-, BU(m))$ to this sequence gives the fibration sequence

(5.5)
$$\mathrm{Map}_*(\tilde{\Sigma}, BU(m)) \xrightarrow{r} \mathrm{Map}_*\Big(\bigvee_{2\tilde{g}} S^1, BU(m)\Big) \xrightarrow{s} \mathrm{Map}_*(S^1, BU(m)).$$

We have $\mathrm{Map}_*(\bigvee_{2\tilde{g}} S^1, BU(m)) = (\Omega BU(m))^{2\tilde{g}} \simeq U(m)^{2\tilde{g}}$ and similarly $\mathrm{Map}_*(S^1, BU(m)) \simeq U(m)$, so the fundamental groups of these spaces are $\mathbb{Z}^{2\tilde{g}}$ and $\mathbb{Z}$, respectively. Since the attaching map η can be written as a product of commutators, so can the induced map

$$s_* : \pi_1 \mathrm{Map}_*\Big(\bigvee_{2\tilde{g}} S^1, BU(m)\Big) \longrightarrow \pi_1 \,\mathrm{Map}_*(S^1, BU(m)).$$

Since these groups are abelian, we see that $s_* = 0$.

Now, a classifying map for $P^{m,k}$ gives each space in (5.5) a basepoint, and the resulting long exact sequence in homotopy is, in part,

$$\pi_2 \Omega BU(m) = 0 \longrightarrow \pi_1 \,\mathrm{Map}_*^{P^{m,k}}(\tilde{\Sigma}, BU(m)) \xrightarrow{r_*} \mathbb{Z}^{2\tilde{g}} \xrightarrow{s_*=0} \mathbb{Z}.$$

Hence r_* is an isomorphism, which completes the proof. □

6. Symmetric representation varieties

In Section 4, we reduced our main theorem (Theorem 1.1) to the orientability of a real vector bundle $V_{n,k}$ over the representation variety

$$X_{\mathrm{YM}}^{2\ell+i-1,0}(U(n))_{\frac{k}{n},\ldots,\frac{k}{n}}$$

of the central Yang–Mills $U(n)$-connection on the orientable double cover $\Sigma_0^{2\ell+i-1}$ of the nonorientable surface Σ_i^ℓ (Theorem 4.2). In this section, we will use Proposition 5.1 to write down:

(i) loops in $X_{\mathrm{YM}}^{2\ell+i-1,0}(U(n))_{\frac{k}{n},\ldots,\frac{k}{n}}$ that generate the fundamental group of $X_{\mathrm{YM}}^{2\ell+i-1,0}(U(n))_{\frac{k}{n},\ldots,\frac{k}{n}}$, and

(ii) lifts of these loops under the surjective continuous map

$$\Phi^{\ell,i} : Z_{\mathrm{YM}}^{\ell,i}(U(n))_{\frac{k}{n},\ldots,\frac{k}{n}} \longrightarrow X_{\mathrm{YM}}^{2\ell+i-1,0}(U(n))_{\frac{k}{n},\ldots,\frac{k}{n}}$$

from the symmetric representation variety.

To prove Theorem 4.2, it suffices to examine the orientability of the restrictions of the pull back bundle $W_{n,k} = (\Phi^{\ell,i})^* V_{n,k}$ to the loops in (ii). This will be carried out in Section 7.

6.1. Review of symmetric representation varieties

We recall definitions and some properties of symmetric representation varieties introduced in [12].

Given $V = (a_1, b_1, \dots, a_\ell, b_\ell) \in U(n)^{2\ell}$, let $\mathfrak{m}(V) = \prod_{i=1}^{\ell} [a_i, b_i]$. For integers k, n, where $n > 0$, we introduce symmetric representation varieties:

$$\begin{aligned} Z^{\ell,1}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\dots,\frac{k}{n}} = \big\{ & (V, c, V', c', -2\sqrt{-1}\pi \frac{k}{n} I_n) | V, V' \in U(n)^{2\ell}, c, c' \in U(n), \\ & \mathfrak{m}(V) = \mathrm{e}^{-\pi\sqrt{-1}k/n} I_n c c',\ \mathfrak{m}(V') = \mathrm{e}^{\pi\sqrt{-1}k/n} I_n c' c \big\}, \\ Z^{\ell,2}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\dots,\frac{k}{n}} = \big\{ & (V, d, c, V', d', c', -2\sqrt{-1}\pi \frac{k}{n} I_n) \mid V, V' \in U(n)^{2\ell}, \\ & d, c, d', c' \in U(n), \mathfrak{m}(V) = \mathrm{e}^{-\pi\sqrt{-1}k/n} I_n c d' c^{-1} d, \\ & \mathfrak{m}(V') = \mathrm{e}^{\pi\sqrt{-1}k/n} I_n c' d (c')^{-1} d' \big\}. \end{aligned}$$

In particular, we have homeomorphisms

$$\begin{aligned} Z^{\ell,1}_{\mathrm{YM}}(U(1))_k = \big\{ & (V, c, V', c', -2\sqrt{-1}\pi k) \mid V, V' \in U(1)^{2\ell}, \\ & c, c' \in U(1), cc' = (-1)^k \big\} \ \cong\ U(1)^{4\ell+1}, \\ Z^{\ell,2}_{\mathrm{YM}}(U(1))_k = \big\{ & (V, d, c, V', d', c', -2\sqrt{-1}\pi k) \mid V, V' \in U(1)^{2\ell}, \\ & d, c, d', c' \in U(1),\ dd' = (-1)^k \big\} \ \cong\ U(1)^{4\ell+3}. \end{aligned}$$

Given $g \in U(n)$ and $V = (a_1, b_1, \dots, a_\ell, b_\ell) \in U(n)^{2\ell}$, let

$$gVg^{-1} = (ga_1g^{-1}, gb_1g^{-1}, \dots, ga_\ell g^{-1}, gb_\ell g^{-1}).$$

With this notation, $U(n)^2$ acts on $Z^{\ell,i}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\dots,\frac{k}{n}}$ by

$$\begin{aligned} & (g_1, g_2) \cdot (V, c, V', c', -2\sqrt{-1}\pi \frac{k}{n} I_n) \\ & \quad = (g_1 V g_1^{-1}, g_1 c g_2^{-1}, g_2 V' g_2^{-1}, g_2 c' g_1^{-1}, -2\sqrt{-1}\pi \frac{k}{n} I_n), \\ & (g_1, g_2) \cdot (V, d, c, V', d', c', -2\sqrt{-1}\pi \frac{k}{n} I_n) \\ & \quad = (g_1 V g_1^{-1}, g_1 d g_1^{-1}, g_1 c g_2^{-1}, g_2 V' g_2^{-1}, g_2 d' g_2^{-1}, g_2 c' g_1^{-1}, -2\sqrt{-1}\pi \frac{k}{n} I_n). \end{aligned}$$

Define $\Phi^{\ell,i} : U(n)^{2(2\ell+i)} \times \mathfrak{u}(n) \to U(n)^{2(2\ell+i-1)} \times \mathfrak{u}(n)$ by

$$\Phi^{\ell,1}(V, c, V', c', X) = (V, c\mathfrak{r}(V')c^{-1}, X),$$
$$\Phi^{\ell,2}(V, d, c, V', d', c', X) = (V, d^{-1}c\mathfrak{r}(V')c^{-1}d, d^{-1}, cc', X),$$

where $\mathfrak{r}(a_1, b_1, \dots, a_\ell, b_\ell) = (b_\ell, a_\ell, \dots, b_1, a_1)$. Then

$$\Phi^{\ell,i}(Z^{\ell,i}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\dots,\frac{k}{n}}) = X^{2\ell+i-1,0}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\dots,\frac{k}{n}}.$$

6.2. Maps and vector bundles

In this subsection, $i = 1, 2$, and n, k are positive integers.

Given a rank n, degree k holomorphic vector bundle $\mathcal{D}$ over $\tilde{\Sigma}$, $\tau_{\mathcal{C}}(\mathcal{D})$ is a rank n, degree $-k$ holomorphic vector bundle over $\tilde{\Sigma}$, and $\mathcal{H}om(\mathcal{D}, \tau_{\mathcal{C}}(\mathcal{D})) = \mathcal{D}^\vee \otimes \tau_{\mathcal{C}}(\mathcal{D})$ is a degree $-2k$, rank n^2 holomorphic vector bundle over $\tilde{\Sigma}$. The map $\mathcal{D} \mapsto \tau_{\mathcal{C}}(\mathcal{D})$ defines

$$\tau : Z^{\ell,i}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\dots,\frac{k}{n}} \to Z^{\ell,i}_{\mathrm{YM}}(U(n))_{-\frac{k}{n},\dots,-\frac{k}{n}}. \tag{6.1}$$

The map $\mathcal{D} \mapsto \mathcal{H}om(\mathcal{D}, \tau_{\mathcal{C}}(\mathcal{D}))$ defines

$$\phi : Z^{\ell,i}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\dots,\frac{k}{n}} \to Z^{\ell,i}_{\mathrm{YM}}(U(n^2))_{-\frac{2k}{n},\dots,-\frac{2k}{n}}. \tag{6.2}$$

The map $\mathcal{M} \mapsto \tau_{\mathcal{C}}(\mathcal{M}^\vee) = \overline{\tau^*\mathcal{M}}$ defines

$$\hat{\tau} : Z^{\ell,i}_{\mathrm{YM}}(U(n^2))_{-\frac{2k}{n},\dots,-\frac{2k}{n}} \to Z^{\ell,i}_{\mathrm{YM}}(U(n^2))_{-\frac{2k}{n},\dots,-\frac{2k}{n}}. \tag{6.3}$$

There is a map $U(n) \times U(n) \to U(n^2)$ given by $(A, B) \mapsto A \otimes B$. More explicitly,

$$(A \otimes B)_{ij,pq} = A_{ip}B_{jq}, \quad 1 \le i, j, p, q \le n.$$

Note that $I_n \otimes I_n = I_{n^2}$. In particular, when $n = 1$, this map is the multiplication: $U(1) \times U(1) \to U(1)$, $(c_1, c_2) \mapsto c_1c_2$.

We introduce some notation.

(i) Given $A = (A_{ij}) \in U(n)$, let $\bar{A} = (\bar{A}_{ij})$ be the complex conjugate of A. Then $\bar{A} = (A^t)^{-1}$.

(ii) We define a complex linear involution T on $\mathbb{C}^n \otimes \mathbb{C}^n \cong \mathbb{C}^{n^2}$ by $T(u \otimes v) = v \otimes u$ for $u, v \in \mathbb{C}^n$. Then $T \in O(n^2) \subset U(n^2)$. We have

$$T_{ij,pq} = \delta_{iq}\delta_{pj}, \quad T = T^t = T^{-1}, \quad (TCT^{-1})_{ij,pq} = C_{ji,qp}.$$

(iii) Define an involution

$$f_T : Z^{\ell,i}_{\mathrm{YM}}(U(n^2))_{-\frac{2k}{n},\dots,-\frac{2k}{n}} \to Z^{\ell,i}_{\mathrm{YM}}(U(n^2))_{-\frac{2k}{n},\dots,-\frac{2k}{n}}$$

by

$$\left(V, c, V', c', 4\sqrt{-1}\pi\frac{k}{n}I_{n^2}\right) \mapsto (I_{n^2}, T)\cdot\left(V, c, V', c', 4\sqrt{-1}\pi\frac{k}{n}I_{n^2}\right),$$
$$\left(V, d, c, V', d', c', 4\sqrt{-1}\pi\frac{k}{n}I_{n^2}\right) \mapsto (I_{n^2}, T)\cdot\left(V, d, c, V', d', c', 4\sqrt{-1}\pi\frac{k}{n}I_{n^2}\right).$$

More explicitly,

$$\begin{aligned} &f_T\left(V, c, V', c', 4\sqrt{-1}\pi\frac{k}{n}I_{n^2}\right)\\ &\quad= \left(V, cT^{-1}, TV'T^{-1}, Tc', 4\sqrt{-1}\pi\frac{k}{n}I_{n^2}\right),\\ &f_T\left(V, d, c, V', d', c', 4\sqrt{-1}\pi\frac{k}{n}I_{n^2}\right)\\ &\quad= \left(V, d, cT^{-1}, TV'T^{-1}, Td'T^{-1}, Tc', 4\sqrt{-1}\pi\frac{k}{n}I_{n^2}\right). \end{aligned}$$

(iv) Given $V = (a_1, b_1, \dots, a_\ell, b_\ell)$ and $V' = (a'_1, b'_1, \dots, a'_\ell, b'_\ell)$ in $U(1)^{2\ell}$, define

$$VV' = (a_1a'_1, b_1b'_1, \dots, a_\ell a'_\ell, b_\ell b'_\ell) \in U(1)^{2\ell}.$$

(v) Given $V = (a_1, b_1, \dots, a_\ell, b_\ell)$ and $V' = (a'_1, b'_1, \dots, a'_\ell, b'_\ell)$ in $U(n)^{2\ell}$, define

$$V \otimes V' = (a_1 \otimes a'_1, b_1 \otimes b'_1, \dots, a_\ell \otimes a'_\ell, b_\ell \otimes b'_\ell) \in U(n^2)^{2\ell}.$$

(vi) Given $V = (a_1, b_1, \dots, a_\ell, b_\ell) \in U(n)^{2\ell}$, define

$$\bar{V} = (\bar{a}_1, \bar{b}_1, \dots, \bar{a}_\ell, \bar{b}_\ell) \in U(n)^{2\ell}.$$

It is straightforward to check that for $A, B \in U(n)$,

$$\overline{A \otimes B} = \bar{A} \otimes \bar{B}, \tag{6.4}$$

$$T(A \otimes B)T^{-1} = B \otimes A. \tag{6.5}$$

If A, B are diagonal with respect to the standard basis $\{e_i \mid i = 1, \dots, n\}$ of $\mathbb{C}^n$, then $A \otimes B$ is diagonal with respect to the basis $\{e_i \otimes e_j \mid i, j = 1, \dots, n\}$ of $\mathbb{C}^{n^2}$.

With the above notation, we have the following explicit description of the maps τ, ϕ, $\hat{\tau}$ in (6.1), (6.2), (6.3), respectively.

The involution $\tau : Z^{\ell,i}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\dots,\frac{k}{n}} \to Z^{\ell,i}_{\mathrm{YM}}(U(n))_{-\frac{k}{n},\dots,-\frac{k}{n}}$ is given by

$$\begin{aligned}
&\left(V, c, V', c', -2\sqrt{-1}\pi\frac{k}{n}I_n\right) \mapsto \left(V', c', V, c, 2\sqrt{-1}\pi\frac{k}{n}I_n\right), \quad i = 1,\\
&\left(V, d, c, V', d', c', -2\sqrt{-1}\pi\frac{k}{n}I_n\right) \mapsto \left(V', d', c', V, d, c, 2\sqrt{-1}\pi\frac{k}{n}I_n\right),\\
&\quad i = 2.
\end{aligned}$$

The map $\phi : Z^{\ell,i}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\dots,\frac{k}{n}} \to Z^{\ell,i}_{\mathrm{YM}}(U(n^2))_{-\frac{2k}{n},\dots,-\frac{2k}{n}}$ is given by

$$\begin{aligned}
&\left(V, c, V', c', -2\sqrt{-1}\pi\frac{k}{n}I_n\right)\\
&\quad \mapsto \left(\bar{V} \otimes V', \bar{c} \otimes c', \bar{V}' \otimes V, \bar{c}' \otimes c, 4\sqrt{-1}\pi\frac{k}{n}I_{n^2}\right), \quad i = 1,\\
&\left(V, d, c, V', d', c', -2\sqrt{-1}\pi\frac{k}{n}I_n\right)\\
&\quad \mapsto \left(\bar{V} \otimes V', \bar{d} \otimes d', \bar{c} \otimes c', \bar{V}' \otimes V, \bar{d}' \otimes d, \bar{c}' \otimes c, 4\sqrt{-1}\pi\frac{k}{n}I_{n^2}\right), \quad i = 2.
\end{aligned}$$

Letting $\phi_T = f_T \circ \phi$, we see that ϕ and ϕ_T define the same map to the quotient of $Z^{\ell,i}_{\mathrm{YM}}(U(n^2))_{-\frac{2k}{n},\dots,-\frac{2k}{n}}$ by $U(n^2)^2$. We have

(6.6)
$$\begin{aligned}
&\phi_T\left(V, c, V', c', -2\sqrt{-1}\pi\frac{k}{n}I_n\right)\\
&\quad = \left(\bar{V} \otimes V', \left(\bar{c} \otimes c'\right)T, V \otimes \bar{V}', \left(c \otimes \bar{c}'\right)T, 4\sqrt{-1}\pi\frac{k}{n}I_{n^2}\right), \quad i = 1,
\end{aligned}$$

(6.7)
$$\begin{aligned}
&\phi_T\left(V, d, c, V', d', c', -2\sqrt{-1}\pi\frac{k}{n}I_n\right)\\
&\quad = \left(\bar{V} \otimes V', \bar{d} \otimes d', \left(\bar{c} \otimes c'\right)T, V \otimes \bar{V}', d \otimes \bar{d}', \left(c \otimes \bar{c}'\right)T, 4\sqrt{-1}\pi\frac{k}{n}I_{n^2}\right),\\
&\qquad i = 2.
\end{aligned}$$

The involution $\hat{\tau} : Z^{\ell,i}_{\mathrm{YM}}(U(n^2))_{-\frac{2k}{n},\dots,-\frac{2k}{n}} \to Z^{\ell,i}_{\mathrm{YM}}(U(n^2))_{-\frac{2k}{n},\dots,-\frac{2k}{n}}$ is given by

$$\begin{aligned}
&\left(V, c, V', c', 4\sqrt{-1}\pi\frac{k}{n}I_{n^2}\right) \mapsto \left(\bar{V}', \bar{c}', \bar{V}, \bar{c}, 4\sqrt{-1}\pi\frac{k}{n}I_{n^2}\right), \quad i = 1,\\
&\left(V, d, c, V', d', c', 4\sqrt{-1}\pi\frac{k}{n}I_{n^2}\right) \mapsto \left(\bar{V}', \bar{d}', \bar{c}', \bar{V}, \bar{d}, \bar{c}, 4\sqrt{-1}\pi\frac{k}{n}I_{n^2}\right),\\
&\quad i = 2.
\end{aligned}$$

Let $Z^{\ell,i}_{\mathrm{YM}}(U(n^2))^{\hat{\tau}}_{-\frac{2k}{n},\dots,-\frac{2k}{n}}$ be the fixed locus of $\hat{\tau}$. Then

$$\phi_T\left(Z^{\ell,i}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\dots,\frac{k}{n}}\right) \subset Z^{\ell,i}_{\mathrm{YM}}(U(n^2))^{\hat{\tau}}_{-\frac{2k}{n},\dots,-\frac{2k}{n}}.$$

In particular, when $n = 1$, we have $\bar{c} = c^{-1}$ and $\bar{d} = d^{-1}$. The involution $\tau : Z^{\ell,i}_{\mathrm{YM}}(U(1))_k \to Z^{\ell,i}_{\mathrm{YM}}(U(1))_{-k}$ is given by

$$\begin{aligned}
&(V, c, V', (-1)^k\bar{c}, -2\sqrt{-1}\pi k) \mapsto (V', (-1)^k\bar{c}, V, c, 2\sqrt{-1}\pi k), \quad i = 1,\\
&(V, d, c, V', (-1)^k\bar{d}, c', -2\sqrt{-1}\pi k) \mapsto (V', (-1)^k\bar{d}, c', V, d, c, 2\sqrt{-1}\pi k),\\
&\quad i = 2.
\end{aligned}$$

The map $\phi = \phi_T : Z^{\ell,i}_{\mathrm{YM}}(U(1))_k \to Z^{\ell,i}_{\mathrm{YM}}(U(1))_{-2k}$ is given by

$$\begin{aligned}
&(V, c, V', (-1)^k\bar{c}, -2\sqrt{-1}\pi k)\\
&\quad \mapsto (\bar{V}V', (-1)^k\bar{c}^2, \bar{V}'V, (-1)^k c^2, 4\sqrt{-1}\pi k), \quad i = 1,\\
&(V, d, c, V', (-1)^k\bar{d}, c', -2\sqrt{-1}\pi k)\\
&\quad \mapsto (\bar{V}V', (-1)^k\bar{d}^2, \bar{c}c', \bar{V}'V, (-1)^k d^2, \bar{c}'c, 4\sqrt{-1}\pi k), \quad i = 2.
\end{aligned}$$

The involution $\hat{\tau} : Z^{\ell,1}_{\mathrm{YM}}(U(1))_{-2k} \to Z^{\ell,1}_{\mathrm{YM}}(U(1))_{-2k}$ is given by

$$\begin{aligned}
&(V, c, V', \bar{c}, 4\sqrt{-1}\pi k) \mapsto (\bar{V}', c, \bar{V}, \bar{c}, 4\sqrt{-1}\pi k), \quad i = 1,\\
&(V, d, c, V', \bar{d}, c', 4\sqrt{-1}\pi k) \mapsto (\bar{V}', d, \bar{c}', \bar{V}, \bar{d}, \bar{c}, 4\sqrt{-1}\pi k), \quad i = 2.
\end{aligned}$$

Definition 6.1. When $k > 0$, let $U_{n,k} \to Z^{\ell,i}_{\mathrm{YM}}(U(n^2))_{-\frac{2k}{n},\dots,-\frac{2k}{n}}$ be the complex vector bundle whose fiber at $\mathcal{M}$ is $H^1(\tilde{\Sigma}, \mathcal{M})$, where $\mathcal{M}$ is a polystable holomorphic vector bundle of rank n^2, degree $-2nk$.

From the discussion in Section 4.1, the involution $\tau : \tilde{\Sigma} \to \tilde{\Sigma}$ induces a conjugate linear map

$$\hat{\tau} : (U_{n,k})_{\mathcal{M}} = H^1(\tilde{\Sigma}, \mathcal{M}) \to (U_{n,k})_{\hat{\tau}(\mathcal{M})} = H^1(\tilde{\Sigma}, \overline{\tau^* \mathcal{M}}).$$

So the involution $\hat{\tau}$ on $Z^{\ell,i}_{\mathrm{YM}}(U(n^2))_{-\frac{2k}{n},\ldots,-\frac{2k}{n}}$ lifts to an involution $\hat{\tau}$ on $U_{n,k}$, and the fixed locus $U^{\hat{\tau}}_{n,k}$ is a real vector bundle over $Z^{\ell,i}_{\mathrm{YM}}(U(n^2))^{\hat{\tau}}_{-\frac{2k}{n},\ldots,-\frac{2k}{n}}$. We have

$$\begin{aligned} &\phi_T^* U^{\hat{\tau}}_{n,k} = W_{n,k}, \\ &\operatorname{rank}_{\mathbb{R}} W_{n,k} = \operatorname{rank}_{\mathbb{R}} U^{\hat{\tau}}_{n,k} = \operatorname{rank}_{\mathbb{C}} U_{n,k} = 2nk + n^2(2\ell + i - 2). \end{aligned}$$

Definition 6.2. When $k = 0$, let $U_n^{\mathrm{vir}} \to Z^{\ell,i}_{\mathrm{YM}}(U(n^2))_{0,\ldots,0} = Z^{\ell,i}_{\mathrm{flat}}(U(n^2))$ be the virtual complex vector bundle whose fiber at $\mathcal{M}$ is $H^1(\tilde{\Sigma}, \mathcal{M}) - H^0(\tilde{\Sigma}, \mathcal{M})$, where $\mathcal{M}$ is a polystable holomorphic vector bundle of rank n^2, degree 0.

The involution $\tau : \tilde{\Sigma} \to \tilde{\Sigma}$ induces a conjugate linear map

$$\begin{aligned} \hat{\tau} : (U_n^{\mathrm{vir}})_{\mathcal{M}} = H^1(\tilde{\Sigma}, \mathcal{M}) - H^0(\tilde{\Sigma}, \mathcal{M}) \to \\ (U_n^{\mathrm{vir}})_{\hat{\tau}(\mathcal{M})} = H^1(\tilde{\Sigma}, \hat{\tau}(\mathcal{M})) - H^0(\tilde{\Sigma}, \hat{\tau}(\mathcal{M})). \end{aligned}$$

$Z^{\ell,i}_{\mathrm{flat}}(U(n^2))$ lifts to an involution $\hat{\tau}$ on U_n^{vir}, and the fixed locus $(U_n^{\mathrm{vir}})^{\hat{\tau}}$ is a virtual real vector bundle over $Z^{\ell,i}_{\mathrm{flat}}(U(n^2))^{\hat{\tau}}$ of rank $n^2(2\ell + i - 2)$.

6.3. Loops in the symmetric representation variety

Let

$$\Psi : X^{2\ell+i-1,0}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\ldots,\frac{k}{n}} \longrightarrow X^{2\ell+i-1,0}_{\mathrm{YM}}(U(1))_k$$

be as in Proposition 5.1. In particular, Ψ is the identity map when $n = 1$.

Given $V = (a_1, b_1, \ldots, a_\ell, b_\ell) \in U(n)^{2\ell}$, define

$$\det(V) = (\det(a_1), \det(b_1), \ldots, \det(a_\ell), \det(b_\ell)) \in U(1)^{2\ell}.$$

Then $\Psi\circ\Phi^{\ell,i}: Z^{\ell,i}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\dots,\frac{k}{n}} \longrightarrow X^{2\ell+i-1,0}_{\mathrm{YM}}(U(1))_k \cong U(1)^{2(2\ell+i-1)}$ is given by

(6.8)
$$\Psi\circ\Phi^{\ell,1}(V,c,V',c',X) = \big(\det(V),\det(\mathfrak{r}(V')),\mathrm{Tr}(X)\big),$$

(6.9)
$$\Psi\circ\Phi^{\ell,2}(V,d,c,V',d',c',X) = \big(\det(V),\det(\mathfrak{r}(V')),\det(d)^{-1},\det(cc'),\mathrm{Tr}(X)\big).$$

In the rest of this subsection, we write Φ instead of $\Phi^{\ell,i}$.

Remark 6.1. The following observation will be useful. Let M be a CW complex, and let $E\to M$ be a real vector bundle. Then E is orientable if and only if $w_1(E)=0$, or equivalently, $\gamma\cap w_1(E)=0$ for all $\gamma\in H_1(M;\mathbb{Z}/2\mathbb{Z})$. (Recall here that orientability is equivalent, over any base, to triviality of the determinant line bundle, and line bundles over a CW complex X are classified by $w_1\in H^1(X;\mathbb{Z}/2\mathbb{Z})\cong[X,\mathbb{RP}^\infty=K(\mathbb{Z}/2\mathbb{Z},1)]$.) Now suppose that $\pi_1(M)$ is a free abelian group generated by loops $\gamma_1,\dots,\gamma_r$. Then the γ_i represent generators for $H_1(X;\mathbb{Z}/2\mathbb{Z})$, so E is orientable if and only if $[\gamma_i]\cap w_1(E)=0$ for $i=1,\dots,r$, or equivalently, $E\big|_{\gamma_i}$ is orientable for $i=1,\dots,r$.

6.3.1. $n=1$**,** $i=1$ When $\ell=0$, $X^{0,0}(U(1))_{k,\dots,k}$ consists of a single point. We may assume $\ell\geq 1$. For $i=1,\dots,\ell$, let α_i, β_i, α_i' and β_i' be loops on $\tilde\Sigma$ which generate the fundamental group of $\tilde\Sigma=\Sigma_0^{2\ell}$, i.e., the holonomies along α_i, β_i, α_i' and β_i' are a_i, b_i, a_i' and b_i', respectively. Let $\tilde\alpha_i: S^1\to Z^{\ell,1}_{\mathrm{YM}}(U(1))_k$ be the loop defined by

$$a_i=\mathrm{e}^{\sqrt{-1}\theta},\quad a_j=1, j\neq i;\quad b_j=a_j'=b_j'=1, j=1,\dots,\ell;$$
$$c=c'=\sqrt{-1}^k,\ X=-2\sqrt{-1}\pi k.$$

We define $\tilde\alpha_i'$, $\tilde\beta_j$, $\tilde\beta_j'$ similarly. Then $\Phi\circ\tilde\alpha_i: S^1\to X^{2\ell,0}_{\mathrm{YM}}(U(1))_k$ is a loop defined by

$$a_i=\mathrm{e}^{\sqrt{-1}\theta},\quad a_j=1, j\neq i,\quad b_j=1, j=1,\dots,2\ell,\quad X=-2\sqrt{-1}\pi k.$$

Thus the 4ℓ loops

$$\Phi\circ\tilde\alpha_i,\ \Phi\circ\tilde\beta_i,\ \Phi\circ\tilde\alpha_i',\ \Phi\circ\tilde\beta_i',\quad i=1,\dots,\ell, \tag{6.10}$$

generate the fundamental group of $X^{2\ell,0}_{\mathrm{YM}}(U(1))_k\cong U(1)^{4\ell}$.

To show that $V_{1,k} \to X_{\mathrm{YM}}^{2\ell,0}(U(1))_k$ is orientable, it suffices to show that its restriction to each of the 4ℓ loops in (6.10) is orientable, or equivalently:

Proposition 6.1. *Let $\ell \geq 1$. The restriction of $W_{1,k} \to Z_{\mathrm{YM}}^{\ell,1}(U(1))_k$ to each of the following 4ℓ loops is orientable:*

$$\tilde{\alpha}_i,\ \tilde{\beta}_i,\ \tilde{\alpha}'_i,\ \tilde{\beta}'_i, \quad i = 1, \dots, \ell.$$

Proof. See Section 7.3. □

6.3.2. $n = 1$, $i = 2$ In this subsection, we assume that $\ell \geq 0$. For $i = 1, \dots, \ell$, the holonomies along α_i, β_i, α'_i and β'_i are a_i, b_i, a'_i and b'_i respectively. Let γ be the curve from p_+ to p_-, γ' be the curve from p_- to p_+, δ be the loop starting at p_+, and δ' be the loop starting at p_- i.e., the holonomies along $\gamma, \gamma', \delta, \delta'$ are c, c', d, d', respectively. The fundamental group of $\tilde{\Sigma} = \Sigma_0^{2\ell+1}$ is generated by the $4\ell + 2$ loops

$$\alpha_i, \beta_i, \alpha'_i, \beta'_i, \quad i = 1, \dots, \ell, \quad \delta, \gamma\gamma'.$$

In particular, when $\ell = 0$, the fundamental group of $\Sigma_0^1 \cong S^1 \times S^1$ is generated by δ and $\gamma\gamma'$.

Let $\tilde{\alpha}_i : S^1 \to Z_{\mathrm{YM}}^{\ell,2}(U(1))_k$ be the loop defined by

$$\begin{aligned} &a_i = \mathrm{e}^{\sqrt{-1}\theta}, \quad a_j = 1 \text{ if } j \neq i, \quad b_j = a'_j = b'_j = 1, j = 1, \dots, \ell, \\ &c = c' = 1, \quad d = \sqrt{-1}^{k+1}, d' = \sqrt{-1}^{k-1}, \quad X = -2\sqrt{-1}\pi k. \end{aligned}$$

We define $\tilde{\beta}_i$, $\tilde{\alpha}'_i$, $\tilde{\beta}'_i$ similarly.

Let $\tilde{\delta} : S^1 \to Z_{\mathrm{YM}}^{\ell,2}(U(1))_k$ be the loop defined by

$$\begin{aligned} &d = \sqrt{-1}^{k+1}\mathrm{e}^{-\sqrt{-1}\theta}, \quad d' = \sqrt{-1}^{k-1}\mathrm{e}^{\sqrt{-1}\theta}, \quad c = c' = 1, \\ &a_j = b_j = a'_j = b'_j = 1,\ j = 1, \dots, \ell, \quad X = -2\sqrt{-1}\pi k. \end{aligned}$$

Let $\tilde{\gamma} : S^1 \to Z_{\mathrm{YM}}^{\ell,2}(U(1))_k$ be the loop defined by

$$\begin{aligned} &c = \mathrm{e}^{\sqrt{-1}\theta}, \quad c' = 1, \quad d = \sqrt{-1}^{k+1},\ d' = \sqrt{-1}^{k-1}, \\ &a_j = b_j = a'_j = b'_j = 1, j = 1, \dots, \ell, \quad X = -2\sqrt{-1}\pi k. \end{aligned}$$

Then $\Phi \circ \tilde{\alpha}_i : S^1 \to X_{\mathrm{YM}}^{2\ell+1,0}(U(1))_k$ is a loop defined by

$$\begin{aligned} &a_i = \mathrm{e}^{\sqrt{-1}\theta}, \quad a_j = 1 \text{ if } j \notin \{i, 2\ell+1\}, \quad a_{2\ell+1} = (\sqrt{-1})^{-k-1}, \\ &b_j = 1, j = 1, \dots, 2\ell+1, \quad X = -2\sqrt{-1}\pi k; \end{aligned}$$

$\Phi \circ \tilde{\delta} : S^1 \to X^{2\ell+1,0}_{\mathrm{YM}}(U(1))_k$ is a loop defined by

$$a_j = b_j = 1, j = 1, \dots, 2\ell,$$

$$a_{2\ell+1} = (\sqrt{-1})^{-k-1} \mathrm{e}^{\sqrt{-1}\theta}, \quad b_{2\ell+1} = 1, \quad X = -2\sqrt{-1}\pi k;$$

and $\Phi \circ \tilde{\gamma} : S^1 \to X^{2\ell+1,0}_{\mathrm{YM}}(U(1))_k$ is a loop defined by

$$a_j = b_j = 1, j = 1, \dots, 2\ell,$$

$$a_{2\ell+1} = (\sqrt{-1})^{-k-1}, \quad b_{2\ell+1} = \mathrm{e}^{\sqrt{-1}\theta}, \quad X = -2\sqrt{-1}\pi k.$$

Thus the $(4\ell + 2)$ loops

$$\Phi \circ \tilde{\alpha}_i,\ \Phi \circ \tilde{\beta}_i,\ \Phi \circ \tilde{\alpha}'_i,\ \Phi \circ \tilde{\beta}'_i,\ i = 1, \dots, \ell, \quad \Phi \circ \tilde{\delta},\ \Phi \circ \tilde{\gamma} \tag{6.11}$$

generate the fundamental group of $X^{2\ell+1,0}_{\mathrm{YM}}(U(1))_k \cong U(1)^{4\ell+2}$.

To show that $V_{1,k} \to X^{2\ell+1,0}_{\mathrm{YM}}(U(1))_k$ is orientable, it suffices to show that its restriction to each of the $(4\ell + 2)$ loops in (6.11) is orientable, or equivalently:

Proposition 6.2. *Let $\ell \geq 0$. The restriction of $W_{1,k} \to Z^{\ell,2}_{\mathrm{YM}}(U(1))_k$ to each of the following $(4\ell + 2)$ loops is orientable:*

$$\tilde{\alpha}_i,\ \tilde{\beta}_i,\ \tilde{\alpha}'_i,\ \tilde{\beta}'_i,\ i = 1, \dots, \ell, \quad \tilde{\delta},\ \tilde{\gamma}.$$

Proof. See Section 7.4. □

6.3.3. $n > 1$, $i = 1$ Let $\ell \geq 1$. Let $\alpha_i,\ \beta_i,\ \alpha'_i,\ \beta'_i$ be defined in Section 6.3.1, so that the holonomies along $\alpha_i,\ \beta_i,\ \alpha'_i,\ \beta'_i$ are $a_i,\ b_i,\ a'_i,\ b'_i \in U(n)$, respectively.

Let $a_\theta = \mathrm{diag}(\mathrm{e}^{\sqrt{-1}\theta}, \underbrace{1, \dots, 1}_{n-1}) \in U(n)$. By Goto's commutator theorem, the map $G^2 \to G$ defined by $(a, b) \mapsto [a, b]$ is surjective if G is semisimple (cf. [16, Theorem 9.2]). So there exist $a, b \in \mathrm{SU}(n)$ such that

$$[a, b] = \mathrm{e}^{2\pi\sqrt{-1}k/n} I_n \in \mathrm{SU}(n).$$

Let $\tilde{\alpha}_i : S^1 \to Z^{\ell,1}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\dots,\frac{k}{n}}$ be the loop defined by

$$a_i = a_\theta, \quad b_i = I_n, \quad a'_i = a, \quad b'_i = b,$$

$$a_j = b_j = a'_j = b'_j = I_n \text{ for } j \neq i, \quad c = c' = \mathrm{e}^{\frac{\pi\sqrt{-1}k}{2n}} I_n, \quad X = -2\sqrt{-1}\pi \frac{k}{n} I_n.$$

We define $\tilde{\beta}_i$, $\tilde{\alpha}'_i$, $\tilde{\beta}'_i$ similarly. From (6.8) it is clear that the fundamental group of $X^{2\ell,0}_{\mathrm{YM}}(U(1))_k \cong U(1)^{4\ell}$ is generated by the following 4ℓ loops:

$$\Psi \circ \Phi \circ \tilde{\alpha}_i,\ \Psi \circ \Phi \circ \tilde{\beta}_i,\ \Psi \circ \Phi \circ \tilde{\alpha}'_i,\ \Psi \circ \Phi \circ \tilde{\beta}'_i, \quad i = 1, \dots, \ell.$$

By Proposition 5.1, the fundamental group of $X^{2\ell,0}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\dots,\frac{k}{n}}$ is generated by the following 4ℓ loops:

$$\Phi \circ \tilde{\alpha}_i,\ \Phi \circ \tilde{\beta}_i,\ \Phi \circ \tilde{\alpha}'_i,\ \Phi \circ \tilde{\beta}'_i, \quad i = 1, \dots, \ell. \tag{6.12}$$

To show that $V_{n,k} \to X^{2\ell,0}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\dots,\frac{k}{n}}$ is orientable, it suffices to show that its restriction to each of the 4ℓ loops in (6.12) is orientable, or equivalently:

Proposition 6.3. *Let $\ell \geq 1$ and let $n > 1$. The restriction of the vector bundle $W_{n,k} \to Z^{\ell,1}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\dots,\frac{k}{n}}$ to each of the following 4ℓ loops is orientable:*

$$\tilde{\alpha}_i,\ \tilde{\beta}_i,\ \tilde{\alpha}'_i,\ \tilde{\beta}'_i, \quad i = 1, \dots, \ell.$$

Proof. See Section 7.3. □

6.3.4. $n > 1$, $i = 2$ Let $\ell \geq 1$. Let α_i, β_i, α'_i, β'_i, δ, δ', γ, γ' be defined as in Section 6.3.2, so that the holonomies along α_i, β_i, α'_i, β'_i, δ, δ', γ, γ' are a_i, b_i, a'_i, b'_i, d, d', c, $c' \in U(n)$, respectively. Define a, b, a_θ as in Section 6.3.3, so that $[a, b] = \mathrm{e}^{2\sqrt{-1}\pi k/n} I_n$.

Let $\tilde{\alpha}_i : S^1 \to Z^{\ell,2}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\dots,\frac{k}{n}}$ be the loop defined by

$$\begin{aligned}
&a_i = a_\theta, \quad b_i = I_n, \quad a'_i = a, \quad b'_i = b,\\
&a_j = b_j = a'_j = b'_j = I_n \text{ for } j \neq i, \quad c = c' = I_n,\\
&d = \sqrt{-1}\mathrm{e}^{\frac{\pi\sqrt{-1}k}{2n}} I_n, d' = -\sqrt{-1}\mathrm{e}^{\frac{\pi\sqrt{-1}k}{2n}} I_n, \quad X = -2\sqrt{-1}\pi\frac{k}{n} I_n.
\end{aligned}$$

We define $\tilde{\beta}i$, $\tilde{\alpha}'_i$, $\tilde{\beta}'_i$ similarly.

Let $\tilde{\delta} : S^1 \to Z^{\ell,2}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\dots,\frac{k}{n}}$ be the loop defined by

$$\begin{aligned}
&a_1 = b_1 = I_n, \quad a'_1 = a, \quad b'_1 = b, \quad d = \sqrt{-1}\mathrm{e}^{\frac{\pi\sqrt{-1}k}{2n}} \bar{a}_\theta, d' = -\sqrt{-1}\mathrm{e}^{\frac{\pi\sqrt{-1}k}{2n}} a_\theta,\\
&c = c' = I_n, \quad a_j = b_j = a'_j = b'_j = I_n,\ j = 2, \dots, \ell, \quad X = -2\sqrt{-1}\pi\frac{k}{n} I_n.
\end{aligned}$$

Let $\tilde{\gamma} : S^1 \to Z^{\ell,2}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\dots,\frac{k}{n}}$ be the loop defined by

$$\begin{aligned}
&a_1 = b_1 = I_n, \quad a_1' = a, \quad b_1' = b, \quad c = a_\theta, \quad c' = I_n,\\
&d = \sqrt{-1}\mathrm{e}^{\frac{\pi\sqrt{-1}k}{2n}} I_n, \quad d' = -\sqrt{-1}\mathrm{e}^{\frac{\pi\sqrt{-1}k}{2n}} I_n,\\
&a_j = b_j = a_j' = b_j' = I_n, j = 2,\dots,\ell, \quad X = -2\sqrt{-1}\pi\frac{k}{n} I_n.
\end{aligned}$$

From (6.9), it is clear that the following $4\ell+2$ loops generate the fundamental group of $X^{2\ell+1,0}_{\mathrm{YM}}(U(1))_k \cong U(1)^{4\ell+2}$:

$$\begin{aligned}
&\Psi\circ\Phi\circ\tilde{\alpha}_i,\ \Psi\circ\Phi\circ\tilde{\beta}_i,\ \Psi\circ\Phi\circ\tilde{\alpha}_i',\ \Psi\circ\Phi\circ\tilde{\beta}_i',\ i=1,\dots,\ell,\\
&\Psi\circ\Phi\circ\tilde{\delta},\ \Psi\circ\Phi\circ\tilde{\gamma}.
\end{aligned}$$

By Proposition 5.1, the fundamental group of $X^{2\ell+1,0}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\dots,\frac{k}{n}}$ is generated by the following $4\ell+2$ loops:

$$\Phi\circ\tilde{\alpha}_i,\ \Phi\circ\tilde{\beta}_i,\ \Phi\circ\tilde{\alpha}_i',\ \Phi\circ\tilde{\beta}_i',\ i=1,\dots,\ell, \quad \Phi\circ\tilde{\delta},\ \Phi\circ\tilde{\gamma}. \tag{6.13}$$

To show that $V_{n,k} \to X^{2\ell+1,0}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\dots,\frac{k}{n}}$ is orientable, it suffices to show that its restriction to each of the $(4\ell+2)$ loops in (6.13) is orientable, or equivalently:

Proposition 6.4. *Let $\ell \geq 1$ and let $n > 1$. The restriction of the vector bundle $W_{n,k} \to Z^{\ell,2}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\dots,\frac{k}{n}}$ to each of the following $(4\ell+2)$ loops is orientable:*

$$\tilde{\alpha}_i,\ \tilde{\beta}_i,\ \tilde{\alpha}_i',\ \tilde{\beta}_i',\ i=1,\dots,\ell, \quad \tilde{\delta},\ \tilde{\gamma}.$$

Proof. See Section 7.4. □

7. Orientability along loops

Our approach is similar to that in the proof of [9, Proposition 8.1.4]. Let $\gamma : S^1 \to Z^{\ell,i}_{\mathrm{YM}}(U(n))_{\frac{k}{n},\dots,\frac{k}{n}}$ be any of the loops in Propositions 6.1–6.4. We need to show that $E := \gamma^* W_{n,k}$ is an orientable real vector bundle over S^1. The fiber of E over $\theta \in S^1$ is given by $E_\theta = H^1(\tilde{\Sigma}, \mathcal{M}_\theta)^{\tau_\theta}$, where $\mathcal{M}_\theta$ is an S^1-family of holomorphic vector bundles over $\tilde{\Sigma}$. Our strategy is to use degeneration and normalization of the Riemann surface $\tilde{\Sigma}$ to show that $E \cong E_{\mathbb{C}} \oplus E_{\mathbb{R}}$, where $E_{\mathbb{C}}$ is a complex vector bundle over S^1, and $E_{\mathbb{R}}$ is a (possibly zero, possibly virtual) real vector bundle which we can describe very explicitly. The explicit description of $E_{\mathbb{R}}$ allows us to compute $w_1(E_{\mathbb{R}}) = 0$.

7.1. Degeneration of the Riemann surface

We degenerate the smooth Riemann surface $\tilde{\Sigma} = \Sigma_0^{2\ell+i-1}$ to a nodal Riemann surface C with three irreducible components C_+, C_0 and C_-, where $C_\pm \cong \Sigma_0^\ell$, $C_0 \cong \Sigma_i^0$ and C_0 intersects $C_\pm$ at a node $p_\pm$. More precisely, consider a family of Riemann surfaces $\tilde{\Sigma}_t$, where $t \in I = [0,1]$, such that

(i) $\tilde{\Sigma}_1 = \tilde{\Sigma}$, $\tilde{\Sigma}_0 = C$.

(ii) $\tilde{\Sigma}_t$ is smooth for $0 < t \le 1$.

(iii) There is a family of anti-holomorphic involutions $\tau_t : \tilde{\Sigma}_t \to \tilde{\Sigma}_t$, such that

$$\tau_0(C_\pm) = C_\mp, \quad \tau_0(p_\pm) = p_\mp, \quad \tau_0(C_0) = C_0, \quad \tau_1 = \tau.$$

The $i = 1$ and 2 cases are shown in figures 1 and 2, respectively. In figures 1 and 2, $\tau_t(q_\pm) = q_\mp$, $\tau_t(r_\pm) = r_\mp$, $\tau_t(\epsilon_\pm) = \epsilon_\mp$. Note that our loops start from $p_\pm$, so the loop α_1 (resp. α_1') contains the path from p_+ to q_+ (resp. from p_- to q_-) and its inverse; the loop β_1 (resp. β_1') contains the path from p_+

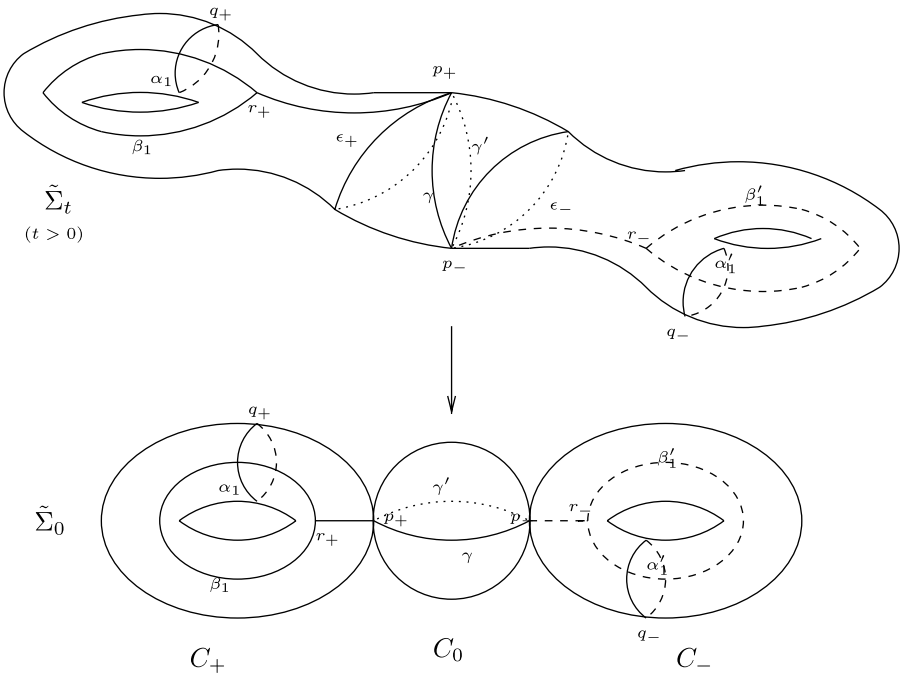

Figure 1: Degeneration of $\widetilde{\Sigma_1^\ell} = \Sigma_0^{2\ell}$.

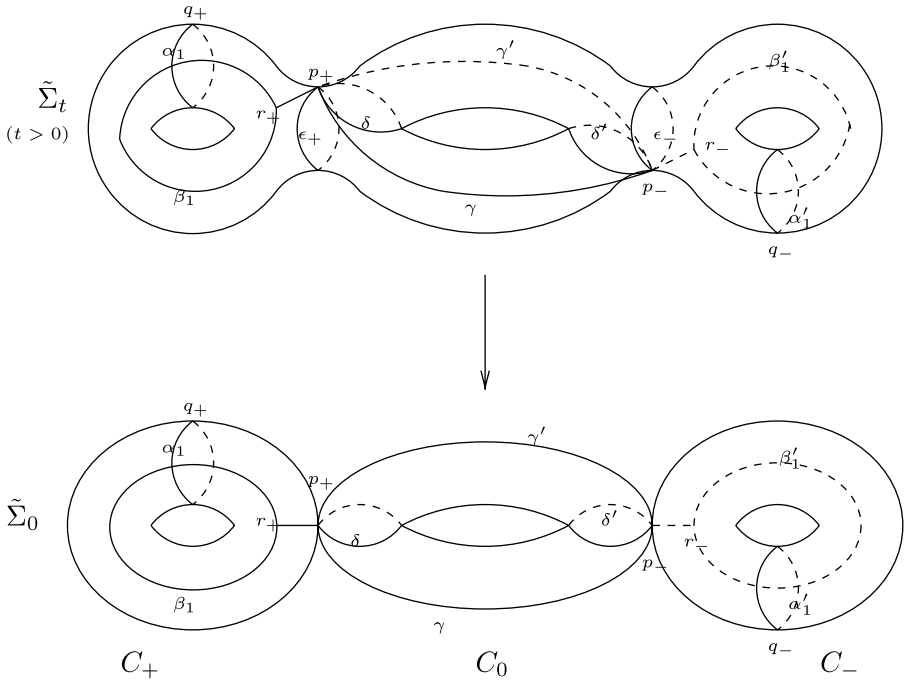

Figure 2: Degeneration of $\widetilde{\Sigma_2^\ell} = \Sigma_0^{2\ell+1}$.

to r_+ (resp. from p_- to r_-) and its inverse. In the degeneration $t \to 0$, the loop $\epsilon_\pm$ shrinks to the point $p_\pm$, respectively.

In Sections 7.3 and 7.4 we will degenerate the family $\mathcal{M}_\theta$ together with the base $\tilde{\Sigma}$ to obtain an S^1-family of vector bundles $\mathcal{M}_{\theta,0} \to \tilde{\Sigma}_0 = C$. Then we will reduce the orientability of $E \to S^1$ to the orientability of $E_0 \to S^1$ whose fiber over $\theta \in S^1$ is $H^1(C, \mathcal{M}_{\theta,0})^{\tau_{\theta,0}}$.

7.2. Normalization

Let C_+, C_0, C_- and C be defined as in Section 7.1. The normalization $\tilde{C}$ of C is a disconnected smooth Riemann surface which can be identified with the disjoint union of C_+, C_0 and C_-. There is a normalization map $\tilde{\nu} : \tilde{C} \to C$, identifying a point on $C_\pm$ to a point in C_0 (which becomes the node $p_\pm$). We will use the following lemma to study the orientability of $E_0 \to S^1$.

Lemma 7.1. *Let $\mathcal{M} \to C$ be a holomorphic vector bundle of rank r. Let $\mathcal{M}_+ \to C_+$, $\mathcal{M}_0 \to C_0$ and $\mathcal{M}_- \to C_-$ be the restrictions of $\mathcal{M}$ to C_+, C_0 and C_-, respectively. Then we have a long exact sequence of complex vector*

spaces:

$$\begin{aligned}(7.1)\qquad 0 &\to H^0(C,\mathcal{M}) \to H^0(C_+,\mathcal{M}_+)\oplus H^0(C_0,\mathcal{M}_0)\oplus H^0(C_-,\mathcal{M}_-)\\ &\to \mathbb{C}^r_{p_+}\oplus\mathbb{C}^r_{p_-}\to H^1(C,\mathcal{M})\\ &\to H^1(C_+,\mathcal{M}_+)\oplus H^1(C_0,\mathcal{M}_0)\oplus H^1(C_-,\mathcal{M}_-)\to 0.\end{aligned}$$

Proof. Let $\iota_+ : C_+ \hookrightarrow C$, $\iota_0 : C_0 \hookrightarrow C$ and $\iota_- : C_- \hookrightarrow C$ be inclusion maps, and let $\nu : \tilde{C} \to C$ be the normalization map. Then

$$\nu_*\mathcal{O}_{\tilde{C}} = \iota_{+*}\mathcal{O}_{C_+}\oplus\iota_{0*}\mathcal{O}_{C_0}\oplus\iota_{-*}\mathcal{O}_{C_-}.$$

We have a short exact sequence of sheaves on C (known as the normalization sequence):

$$(7.2)\qquad 0\to\mathcal{O}_C\to\iota_{+*}\mathcal{O}_{C_+}\oplus\iota_{0*}\mathcal{O}_{C_0}\oplus\iota_{-*}\mathcal{O}_{C-}\to\mathcal{O}_{p_+}\oplus\mathcal{O}_{p_-}\to 0.$$

(See, e.g., [15, p.81].) Twisting the normalization sequence (7.2) by $\mathcal{M}$, we obtain a short exact sequence of sheaves on C:

$$(7.3)\qquad 0\to\mathcal{M}\to\iota_{+*}\mathcal{M}_+\oplus\iota_{0*}\mathcal{M}_0\oplus\iota_{-*}\mathcal{M}_-\to\mathcal{O}_{p_+}^{\oplus r}\oplus\mathcal{O}_{p_-}^{\oplus r}\to 0.$$

The long exact sequence of cohomology groups associated to (7.3) is

$$\begin{aligned}(7.4)\qquad 0 &\to H^0(C,\mathcal{M})\to H^0(C,\iota_{+*}\mathcal{M}_+)\oplus H^0(C,\iota_{0*}\mathcal{M}_0)\oplus H^0(C,\iota_{-*}\mathcal{M}_-)\\ &\to\mathbb{C}^r_{p_+}\oplus\mathbb{C}^r_{p_-}\to H^1(C,\mathcal{M})\\ &\to H^1(C,\iota_{+*}\mathcal{M}_+)\oplus H^1(C,\iota_{0*}\mathcal{M}_0)\oplus H^1(C,\iota_{-*}\mathcal{M}_-)\to 0.\end{aligned}$$

For $k=0,1$, we have

$$H^k(C,\iota_{\pm*}\mathcal{M}_\pm)=H^k(C_\pm,\mathcal{M}_\pm),\quad H^k(C,\iota_{0*}\mathcal{M}_0)=H^k(C_0,\mathcal{M}_0).$$

So (7.4) is equivalent to (7.1). □

7.3. The $i=1$ case

Proof of Propositions 6.1 *and* 6.3. We will show that $\tilde{\alpha}_j^*W_{n,k}\to S^1$ is orientable, $j=1,\dots,\ell$. The other loops $\tilde{\beta}_j,\tilde{\alpha}'_j,\tilde{\beta}'_j$ are similar.

Note that $\tilde{\alpha}_j^* W_{n,k} = (\phi_T \circ \tilde{\alpha}_j)^* U_{n,k}^{\hat{\tau}}$. We have

$$\tilde{\alpha}_j(\theta) = \left(V_\theta, \mathrm{e}^{\sqrt{-1}\pi k/2n} I_n, V', \mathrm{e}^{\sqrt{-1}\pi k/2n} I_n, -2\pi\sqrt{-1}\frac{k}{n} I_n \right),$$

where $V_\theta, V' \in U(n)^{2\ell}$. Note that this is also true for the $n = 1$ case. The loop $\phi_T \circ \tilde{\alpha}_j : S^1 \to Z^{\ell,1}_{\mathrm{YM}}(U(n^2))_{-\frac{2k}{n},\ldots,-\frac{2k}{n}}$ is given by (see (6.6) for the definition of ϕ_T):

$$\phi_T \circ \tilde{\alpha}_j(\theta) = \left(\bar{V}_\theta \otimes V', T, V_\theta \otimes \bar{V}', T, 4\pi\sqrt{-1}\frac{k}{n} I_{n^2} \right). \tag{7.5}$$

The loop $\phi_T \circ \tilde{\alpha}_j(\theta)$ can be viewed as an S^1-family of polystable holomorphic vector bundles $\mathcal{M}_\theta$ of rank n^2, degree $-2kn$ over the Riemann surface $\tilde{\Sigma} \cong \Sigma_0^{2\ell}$. We now consider holomorphic vector bundles $\mathcal{M}_{\theta,t}$ of rank n^2, degree $-2kn$ over $\tilde{\Sigma}_t$ with the following properties:

(i) For $0 < t \leq 1$, the holonomies of $\mathcal{M}_{\theta,t}$ over $\tilde{\Sigma}_t$ are given by (7.5).

(ii) When $t = 0$, we have

$$\mathcal{M}_{\theta,0}\Big|_{C_\pm} = \mathcal{M}_\pm(\theta), \quad \mathcal{M}_{\theta,0}\Big|_{C_0} = \mathcal{M}_0(\theta),$$

where $\mathcal{M}_\pm(\theta)$ is a rank n^2, degree $-kn$ polystable holomorphic vector bundle over $C_\pm \cong \Sigma_0^\ell$ and $\mathcal{M}_0(\theta)$ is a rank n^2, degree 0 polystable holomorphic vector bundle over $C_0 \cong \mathbb{P}^1$.

(iii) The holonomies of $\mathcal{M}_+(\theta)$ starting from p_+ along $(\alpha_1, \beta_1, \ldots, \alpha_\ell, \beta_\ell)$ are given by

$$\left(\bar{V}_\theta \otimes V', 2\pi\sqrt{-1}\frac{k}{n} I_{n^2} \right) \in X^{\ell,0}_{\mathrm{YM}}(U(n^2))_{-\frac{k}{n},\ldots,-\frac{k}{n}},$$

and the holonomies of $\mathcal{M}_-(\theta)$ starting from p_- along $(\beta'_\ell, \alpha'_\ell, .., \beta'_1, \alpha'_1)$ are given by

$$\left(\mathfrak{r}(V_\theta \otimes \bar{V}'), 2\pi\sqrt{-1}\frac{k}{n} I_{n^2} \right) \in X^{\ell,0}_{\mathrm{YM}}(U(n^2))_{-\frac{k}{n},\ldots,-\frac{k}{n}}.$$

(iv) By (ii), $\mathcal{M}_0(\theta) \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus n^2}$, since the trivial bundle is the only degree zero polystable bundle on $\mathbb{P}^1$. Thus, $\mathcal{M}_0(\theta)$ is independent of θ and will be denoted by $\mathcal{M}_0$. The holonomy of $\mathcal{M}_0$ along the equator $\gamma\gamma'$ is $TT = I_{n^2}$ as expected.

(v) For all $(\theta,t)\in S^1\times I$, we have $\overline{\tau_t^*\mathcal{M}_{\theta,t}}=\mathcal{M}_{\theta,t}$, so there is a conjugate linear involution $\hat{\tau}_{\theta,t}$ on $H^*(\tilde{\Sigma}_t,\mathcal{M}_{\theta,t})$.

As (θ,t) varies, the real vector spaces

$$\{H^1(\tilde{\Sigma}_t,\mathcal{M}_{\theta,t})^{\hat{\tau}_{\theta,t}}\mid(\theta,t)\in S^1\times I\}$$

form a real vector bundle E over the cylinder $S^1\times I$. Let $i_t:S^1\to S^1\times I$ be the embedding $\theta\mapsto(\theta,t)$. Then

$$i_1^*E=(\phi_T\circ\tilde{\alpha}_j)^*U_{n,k}^{\hat{\tau}}=\tilde{\alpha}_j^*W_{n,k}.$$

The maps i_0 and i_1 homotopic, so i_0^*E and i_1^*E are isomorphic real vector bundles over S^1. Thus $\tilde{\alpha}_j^*W_{n,k}$ is orientable if and only if i_0^*E is an orientable real vector bundle over S^1.

By Lemma 7.1, we have the following long exact sequence:

(7.6)
$$\begin{aligned}0\to{}&H^0(C,\mathcal{M}_{\theta,0})\to H^0(C_+,\mathcal{M}_+(\theta))\oplus H^0(C_0,\mathcal{M}_0)\oplus H^0(C_-,\mathcal{M}_-(\theta))\\&\to\mathbb{C}_{p_+}^{n^2}\oplus\mathbb{C}_{p_-}^{n^2}\to H^1(C,\mathcal{M}_{\theta,0})\\&\to H^1(C_+,\mathcal{M}_+(\theta))\oplus H^1(C_0,\mathcal{M}_0)\oplus H^1(C_-,\mathcal{M}_-(\theta))\to 0,\end{aligned}$$

where $p_\pm$ is the node at which C_0 and $C_\pm$ intersect. We have

$$\deg\mathcal{M}_{\theta,0}=-2nk<0,\quad \deg\mathcal{M}_\pm(\theta)=-nk<0,$$

so

$$H^0(C,\mathcal{M}_{\theta,0})=H^0(C_+,\mathcal{M}_+(\theta))=H^0(C_-,\mathcal{M}_-(\theta))=0.$$

By (iv), $\mathcal{M}_0\cong\mathcal{O}_{\mathbb{P}^1}^{\oplus n^2}$, so $H^0(C_0,\mathcal{M}_0)\cong\mathbb{C}^{\oplus n^2}$ and $H^1(C_0,\mathcal{M}_0)=0$. The map $H^0(C_0,\mathcal{M}_0)\to\mathbb{C}_{p_\pm}^{n^2}$ is the evaluation map $s\mapsto s(p_\pm)$. Therefore (7.6) is reduced to

(7.7)
$$\begin{aligned}0\to\mathbb{C}^{n^2}\xrightarrow{j}{}&\mathbb{C}_{p_+}^{\oplus n^2}\oplus\mathbb{C}_{p_-}^{\oplus n^2}\to H^1(C,\mathcal{M}_{\theta,0})\\&\to H^1(C_+,\mathcal{M}_+(\theta))\oplus H^1(C_-,\mathcal{M}_-(\theta))\to 0,\end{aligned}$$

where $j(v)=(v,v)$. The involution $\hat{\tau}_{\theta,0}$ acts on the exact sequence (7.7) in the following way:

$$H^1(C,\mathcal{M}_{\theta,0})\overset{\hat{\tau}_{\theta,0}}{\longleftrightarrow}H^1(C,\mathcal{M}_{\theta,0}),\quad H^1(C_+,\mathcal{M}_+(\theta))\overset{\hat{\tau}_{\theta,0}}{\longleftrightarrow}H^1(C_-,\mathcal{M}_-(\theta)).$$

The involution on $\mathbb{C}^{\oplus n^2}$ and $\mathbb{C}_{p_+}^{\oplus n^2} \oplus \mathbb{C}_{p_-}^{\oplus n^2}$ is independent of θ:

$$\hat{\tau}_{\theta,0} : \mathbb{C}^{n^2} \to \mathbb{C}^{n^2}, \quad v \mapsto \bar{v},$$
$$\hat{\tau}_{\theta,0} : \mathbb{C}_{p_+}^{n^2} \oplus \mathbb{C}_{p_-}^{n^2} \to \mathbb{C}_{p_+}^{n^2} \oplus \mathbb{C}_{p_-}^{n^2}, \quad (v,w) \mapsto (\bar{w}, \bar{v}).$$

Thus, we have

$$0 \to \mathbb{R}^{n^2} \to H^1(C, \mathcal{M}_{\theta,0})^{\hat{\tau}_\theta,0} \to H^1(C_+, \mathcal{M}_+(\theta)) \to 0.$$

We conclude that $i_0^* E \cong E_{\mathbb{C}} \oplus E_{\mathbb{R}}$, where $E_{\mathbb{C}} \to S^1$ is a complex vector bundle whose fiber at $\theta \in S^1$ is $H^1(C_+, \mathcal{M}_+(\theta))$, and $E_{\mathbb{R}} \to S^1$ is a trivial real vector bundle of rank n^2. Therefore $i_0^* E$ is orientable. □

7.4. The $i = 2$ case

Proof of Propositions 6.2 *and* 6.4. We will study the orientability of the real vector bundle

$$\lambda^* W_{n,k} = (\phi_T \circ \lambda)^* U_{n,k}^{\hat{\tau}}$$

over S^1, where $\lambda : S^1 \to Z_{\mathrm{YM}}^{\ell,2}(U(n))_{\frac{k}{n},\dots,\frac{k}{n}}$ is one of the $4\ell+2$ loops. Note that λ is of the form

$$\lambda(\theta) = \left(V_\theta, \sqrt{-1}\mathrm{e}^{\frac{\pi\sqrt{-1}k}{2n}} g_\theta, c_\theta, V', -\sqrt{-1}\mathrm{e}^{\frac{\pi\sqrt{-1}k}{2n}} \bar{g}_\theta, I_n, -2\pi\sqrt{-1}\frac{k}{n} I_n \right),$$

where $V_\theta, V_\theta' \in U(n)^{2\ell}$ and g_θ, c_θ are diagonal matrices in $U(n)$. Note that this is also true when $n = 1$.

The loop $\phi_T \circ \lambda : S^1 \to Z_{\mathrm{YM}}^{\ell,1}(U(n^2))_{-\frac{2k}{n},\dots,-\frac{2k}{n}}$ is given by (see (6.7) for the definition of ϕ_T):

$$\phi_T \circ \lambda(\theta) = \Big(\bar{V}_\theta \otimes V', -\bar{g}_\theta \otimes \bar{g}_\theta, (\bar{c}_\theta \otimes I_n)\, T, V_\theta \otimes \bar{V}', -g_\theta \otimes g_\theta, (c_\theta \otimes I_n)\, T, 4\pi\sqrt{-1}\frac{k}{n} I_{n^2} \Big). \tag{7.8}$$

The loop $\phi_T \circ \lambda(\theta)$ can be viewed as an S^1-family of polystable holomorphic vector bundles $\mathcal{M}_\theta$ of rank n^2, degree $-2kn$ over the Riemann surface $\tilde{\Sigma} \cong \Sigma_0^{2\ell+1}$. We now consider holomorphic vector bundles $\mathcal{M}_{\theta,t}$ of rank n^2, degree $-2kn$ over $\tilde{\Sigma}_t$ with the following properties:

(i) For $0 < t \leq 1$, the holonomies of $\mathcal{M}_{\theta,t}$ are given by (7.8).

(ii) When $t = 0$, we have

$$\mathcal{M}_{\theta,0}\Big|_{C_\pm} = \mathcal{M}_\pm(\theta), \quad \mathcal{M}_{\theta,0}\Big|_{C_0} = \mathcal{M}_0(\theta),$$

where $\mathcal{M}_\pm(\theta)$ is a rank n^2, degree $-kn$ polystable holomorphic vector bundle over $C_\pm \cong \Sigma_0^\ell$ and $\mathcal{M}_0(\theta)$ is a rank n^2, degree 0 polystable holomorphic vector bundle over $C_0 \cong S^1 \times S^1$.

(iii) The holonomies of $\mathcal{M}_+(\theta)$ starting from p_+ along $(\alpha_1, \beta_1, \ldots, \alpha_\ell, \beta_\ell)$ are given by

$$(\bar{V}_\theta \otimes V', 2\pi\sqrt{-1}\frac{k}{n} I_{n^2}) \in X^{\ell,0}_{\mathrm{YM}}(U(n^2))_{-\frac{k}{n},\ldots,-\frac{k}{n}},$$

and the holonomies of $\mathcal{M}_-(\theta)$ along $(\beta'_\ell, \alpha'_\ell, .., \beta'_1, \alpha'_1)$ (starting from p_-) are given by

$$(\mathfrak{r}(V_\theta \otimes \bar{V}'), 2\pi\sqrt{-1}\frac{k}{n} I_{n^2}) \in X^{\ell,0}_{\mathrm{YM}}(U(n^2))_{-\frac{k}{n},\ldots,-\frac{k}{n}}.$$

(iv) The holonomies of $\mathcal{M}_0(\theta)$ along δ, γ, δ', γ' are given by

$$\lambda_0(\theta) = (-\bar{g}_\theta \otimes \bar{g}_\theta, (\bar{c}_\theta \otimes I_n)T, -g_\theta \otimes g_\theta, (c_\theta \otimes I_n)T) \in Z^{0,2}_{\mathrm{flat}}(U(n^2))^{\hat{\tau}}. \tag{7.9}$$

Therefore, the holomomies of $\mathcal{M}_0(\theta)$ along the loops δ^{-1}, $\gamma\gamma'$ are given by

$$\Phi^{0,2} \circ \lambda_0(\theta) = (-g_\theta \otimes g_\theta, \bar{c}_\theta \otimes c_\theta) \in X^{1,0}_{\mathrm{flat}}(U(n^2)).$$

(v) For all $(\theta, t) \in S^1 \times I$, we have $\overline{\tau_t^* \mathcal{M}_{\theta,t}} = \mathcal{M}_{\theta,t}$, so there is a conjugate linear involution $\hat{\tau}_{\theta,t}$ on $H^*(\tilde{\Sigma}_t, \mathcal{M}_{\theta,t})$.

As (θ, t) varies, the real vector spaces

$$\{H^1(\tilde{\Sigma}_t, \mathcal{M}_{\theta,t})^{\hat{\tau}_{\theta,t}} \mid (\theta, t) \in S^1 \times I\}$$

form a real vector bundle E over the cylinder $S^1 \times I$. Let $i_t : S^1 \to S^1 \times I$ be the embedding $\theta \mapsto (\theta, t)$. Then

$$i_1^* E = (\phi_T \circ \lambda)^* U^{\hat{\tau}}_{n,k} = \lambda^* W_{n,k}.$$

The maps i_0 and i_1 are homotopic, so $i_0^* E$ and $i_1^* E$ are isomorphic real vector bundles over S^1. Thus $\lambda^* W_{n,k}$ is orientable if and only if $i_0^* E$ is an orientable vector bundle over S^1.

By Lemma 7.1, we have the following long exact sequence:

(7.10)
$$\begin{aligned} 0 \to & H^0(C, \mathcal{M}_{\theta,0}) \to H^0(C_+, \mathcal{M}_+(\theta)) \oplus H^0(C_0, \mathcal{M}_0(\theta)) \oplus H^0(C_-, \mathcal{M}_-(\theta)) \\ & \to \mathbb{C}^{n^2}_{p_+} \oplus \mathbb{C}^{n^2}_{p_-} \to H^1(C, \mathcal{M}_{\theta,0}) \\ & \to H^1(C_+, \mathcal{M}_+(\theta)) \oplus H^1(C_0, \mathcal{M}_0(\theta)) \oplus H^1(C_-, \mathcal{M}_-(\theta)) \to 0, \end{aligned}$$

where $p_\pm$ is the node at which C_0 and $C_\pm$ intersect. We have

$$\deg \mathcal{M}_{\theta,0} = -2nk < 0, \quad \deg \mathcal{M}_\pm(\theta) = -nk < 0,$$

so

$$H^0(C, \mathcal{M}_{\theta,0}) = H^0(C_+, \mathcal{M}_+(\theta)) = H^0(C_-, \mathcal{M}_-(\theta)) = 0.$$

Note that the holonomies of $\mathcal{M}_0(\theta)$ are diagonal, so it is the direct sum of n^2 holomorphic line bundles of degree 0. Let $\mathcal{L}_{a,b}$ denote the degree 0 holomorphic line bundle whose holonomies along the loops δ^{-1}, $\gamma\gamma'$ are given by

$$(a, b) \in X^{1,0}_{\text{flat}}(U(1)) = U(1)^2.$$

Then

$$H^0(C_0, \mathcal{L}_{a,b}) = H^1(C_0, \mathcal{L}_{a,b}) = \begin{cases} \mathbb{C}, & (a, b) = (1, 1), \\ 0, & (a, b) \neq (1, 1). \end{cases}$$

Case 1. $\lambda = \tilde{\alpha}_i, \tilde{\beta}_i, \tilde{\alpha}'_i, \tilde{\beta}'_i$.

$$\begin{aligned} & g_\theta = c_\theta = I_n, \quad (-g_\theta \otimes g_\theta, \bar{c}_\theta \otimes c_\theta) = (-I_{n^2}, I_{n^2}), \\ & \mathcal{M}_0(\theta) = \mathcal{L}^{\oplus n^2}_{-1,1}, \quad H^0(C_0, \mathcal{M}_0(\theta)) = 0 = H^1(C_0, \mathcal{M}_0(\theta)). \end{aligned}$$

Case 2. $\lambda = \tilde{\delta}$.

$$\begin{aligned} & g_\theta = \bar{a}_\theta, \quad c_\theta = I_n, \quad (-g_\theta \otimes g_\theta, \bar{c}_\theta \otimes c_\theta) = (-\bar{a}_\theta \otimes \bar{a}_\theta, I_{n^2}), \\ & \mathcal{M}_0(\theta) = \mathcal{L}_{-\mathrm{e}^{-2\sqrt{-1}\theta},1} \oplus \mathcal{L}^{\oplus 2(n-1)}_{-\mathrm{e}^{-\sqrt{-1}\theta},1} \oplus \mathcal{L}^{\oplus (n-1)^2}_{-1,1}. \end{aligned}$$

Case 3. $\lambda = \tilde{\gamma}$.

$$\begin{aligned} & g_\theta = I_n, \quad c_\theta = a_\theta, \quad (-g_\theta \otimes g_\theta, \bar{c}_\theta \otimes c_\theta) = (-I_{n^2}, \bar{a}_\theta \otimes a_\theta), \\ & \mathcal{M}_0(\theta) = \mathcal{L}^{\oplus (n-1)}_{-1,\mathrm{e}^{-\sqrt{-1}\theta}} \oplus \mathcal{L}^{\oplus (n-1)}_{-1,\mathrm{e}^{\sqrt{-1}\theta}} \oplus \mathcal{L}^{\oplus (n^2-2n+2)}_{-1,1}, \\ & H^0(C_0, \mathcal{M}_0(\theta)) = 0 = H^1(C_0, \mathcal{M}_0(\theta)). \end{aligned}$$

In Cases 1 and 3, (7.10) is reduced to

(7.11)
$$0 \to \mathbb{C}^{n^2}_{p_+} \oplus \mathbb{C}^{n^2}_{p_-} \to H^1(C, \mathcal{M}_{\theta,0}) \to H^1(C_+, \mathcal{M}_+(\theta)) \oplus H^1(C_-, \mathcal{M}_-(\theta)) \to 0.$$

The involution $\hat{\tau}_{\theta,0}$ acts on the exact sequence (7.11) in the following way:

$$\hat{\tau}_{\theta,0} : \mathbb{C}^{n^2}_{p_+} \oplus \mathbb{C}^{n^2}_{p_-} \to \mathbb{C}^{n^2}_{p_+} \oplus \mathbb{C}^{n^2}_{p_-}, \quad (v, w) \mapsto (\bar{w}, \bar{v}) \qquad \text{(independent of } \theta\text{)},$$
$$H^1(C, \mathcal{M}_{\theta,0}) \overset{\hat{\tau}_{\theta,0}}{\longleftrightarrow} H^1(C, \mathcal{M}_{\theta,0}), \quad H^1(C_+, \mathcal{M}_+(\theta)) \overset{\hat{\tau}_{\theta,0}}{\longleftrightarrow} H^1(C_-, \mathcal{M}_-(\theta)).$$

Thus, we have

$$0 \to \mathbb{C}^{\oplus n^2} \to H^1(C, \mathcal{M}_{\theta,0})^{\hat{\tau}_{\theta,0}} \to H^1(C_+, \mathcal{M}_+(\theta)) \to 0.$$

Therefore $i_0^* E \cong E_{\mathbb{C}}$, where $E_{\mathbb{C}} \to S^1$ is a complex vector bundle whose fiber at $\theta \in S^1$ is $H^1(C_+, \mathcal{M}_+(\theta)) \oplus \mathbb{C}^{\oplus n^2}$. Therefore $i_0^* E$ is orientable.

In Case 2, (7.10) is reduced to

(7.12)
$$\begin{aligned} 0 \to & H^0(C_0, \mathcal{M}_0(\theta)) \to \mathbb{C}^{n^2}_{p_+} \oplus \mathbb{C}^{n^2}_{p_-} \to H^1(C, \mathcal{M}_{\theta,0}) \\ & \to H^1(C_+, \mathcal{M}_+(\theta)) \oplus H^1(C_0, \mathcal{M}_0(\theta)) \oplus H^1(C_-, \mathcal{M}_-(\theta)) \to 0. \end{aligned}$$

Taking fixed points of the involution $\hat{\tau}_{\theta,0}$ on (7.12) yields

(7.13)
$$\begin{aligned} 0 \to & H^0(C_0, \mathcal{M}_0(\theta))^{\hat{\tau}_{\theta,0}} \to \mathbb{C}^{n^2}_{p_+} \to H^1(C, \mathcal{M}_{\theta,0})^{\hat{\tau}_{\theta,0}} \\ & \to H^1(C_+, \mathcal{M}_+(\theta)) \oplus H^1(C_0, \mathcal{M}_0(\theta))^{\hat{\tau}_{\theta,0}} \to 0, \end{aligned}$$

where $\mathbb{C}^{n^2}_{p_+}$ and $H^1(C_+, \mathcal{M}_+(\theta))$ are complex vector spaces.

Recall from (7.9) that the holonomies of $\mathcal{M}_0(\theta)$ along δ, γ, δ', γ' are given by

$$(-a_\theta \otimes a_\theta, T, -\bar{a}_\theta \otimes \bar{a}_\theta, T).$$

Let $\tilde{\delta}_0 : S^1 \to Z^{0,2}_{\text{flat}}(U(n^2))^{\hat{\tau}}$ be the loop defined by

(7.14)
$$\tilde{\delta}_0(\theta) = (-a_\theta \otimes a_\theta, T, -\bar{a}_\theta \otimes \bar{a}_\theta, T),$$

then the loop $\tilde{\delta}_0$ can be viewed as the S^1-family of the bundles $\mathcal{M}_0(\theta)$. Let U_n^{vir} be defined as in Definition 6.2. Then the fiber of (the pull back bundle)

$E_0^{\mathrm{vir}} = \tilde{\delta}_0^* \left((U_n^{\mathrm{vir}})^{\hat{\tau}} \right)$ at $\theta \in S^1$ is the virtual vector space

$$H^1(C_0, \mathcal{M}_0(\theta))^{\hat{\tau}_{\theta,0}} - H^0(C_0, \mathcal{M}_0(\theta))^{\hat{\tau}_{\theta,0}}.$$

To show that $\tilde{\delta}^* W_{n,k} \to S^1$ is orientable, it remains to show that the virtual real vector bundle $E_0^{\mathrm{vir}} \to S^1$ is orientable. This is true by Lemma 7.2 below. □

Lemma 7.2. *E_0^{vir} is an orientable virtual real vector bundle over S^1.*

Proof. We use the notation in the above proof.
Step 1. Recall that T is an involution on $\mathbb{C}^n \otimes \mathbb{C}^n = \mathbb{C}^{n^2}$ defined by $T(u \otimes v) = v \otimes u$ for $u, v \in \mathbb{C}^n$. The eigenspaces of T are $E_{+1} = \mathrm{Sym}^2(\mathbb{C}^n)$ and $E_{-1} = \Lambda^2(\mathbb{C}^n)$. We have an orthogonal decomposition

$$\mathbb{C}^n \otimes \mathbb{C}^n = E_{+1} \oplus E_{-1}.$$

Let $I_{E_{\pm 1}} : E_{\pm 1} \to E_{\pm 1}$ be the identity map. Then

$$I_{n^2} = I_{E_{+1}} \oplus I_{E_{-1}}, \quad T = I_{E_{+1}} \oplus (-I_{E_{-1}}).$$

Define $\chi : [0,1] \to U(n^2)$ by

$$\chi(t) = I_{E_{+1}} \oplus (-e^{\sqrt{-1}\pi t} I_{E_{-1}}).$$

Then

$$\chi(0) = T, \quad \chi(1) = I_{n^2}, \quad \overline{\chi(t)} = \chi(t)^{-1} = \chi(-t).$$

Note that $a_\theta \otimes a_\theta$ is of the form $a_\theta \otimes a_\theta = A_\theta^+ \oplus A_\theta^-$, where A_θ^+ and A_θ^- are linear automorphisms on E_{+1} and E_{-1}, respectively. Therefore $a_\theta \otimes a_\theta$ and $\chi(t)$ commute for any $\theta \in S^1$ and $t \in [0,1]$.
Step 2. For any $t \in [0,1]$ and $\theta \in S^1$, define

$$\tilde{\delta}_t(\theta) = (-a_\theta \otimes a_\theta, \chi(t), -\bar{a}_\theta \otimes \bar{a}_\theta, \overline{\chi(t)}). \tag{7.15}$$

In particular, $\tilde{\delta}_0(\theta)$ is given by (7.14). By Step 1, the right-hand side of (7.15) lies in $Z^{0,2}_{\mathrm{flat}}(U(n^2))^{\hat{\tau}}$. So for each $t \in [0,1]$, (7.15) defines a loop $\tilde{\delta}_t : S^1 \to Z^{0,2}_{\mathrm{flat}}(U(n^2))^{\hat{\tau}}$. Let $E_1^{\mathrm{vir}} = \tilde{\delta}_1^* \left((U_n^{\mathrm{vir}})^{\hat{\tau}} \right) \to S^1$. The loops $\tilde{\delta}_0$ and $\tilde{\delta}_1$ are

homotopic, so E_0^{vir} is orientable if and only if E_1^{vir} is orientable.
Step 3. Let $\mathcal{D}(\theta)$ be the rank n^2, degree 0 polystable vector bundle over C_0 whose holonomies along $\delta, \gamma, \delta', \gamma'$ are given by

$$\tilde{\delta}_1(\theta) = (-a_\theta \otimes a_\theta, I_{n^2}, -\bar{a}_\theta \otimes \bar{a}_\theta, I_{n^2}). \tag{7.16}$$

The holonomies of $\mathcal{D}(\theta)$ along γ, γ' are I_{n^2}, so we may degenerate the torus C_0 to a rational nodal curve C' (see figure 3), and degenerate $\mathcal{D}(\theta)$ to $\mathcal{D}'(\theta)$.

More explicitly, we consider a family of Riemann surface R_s, where $s \in I = [0,1]$, such that

(i) $R_1 = C_0$, $R_0 = C'$.

(ii) R_s is a smooth torus for $0 < s \leq 1$.

(iii) There is a family of anti-holomorphic involution $\sigma_s : R_s \to R_s$, such that $\sigma_0(p) = p$, $\sigma_1 = \tau_0$, $\sigma_s(\delta) = \delta'$ and $\sigma_s(\gamma) = \gamma'$ if $s \neq 0$.

(iv) There is a normalization map $\nu : \mathbb{P}^1 \to C'$ such that $\nu(q_\pm) = p$ (see figure 3). There is an anti-holomorphic involution $\tilde{\sigma} : \mathbb{P}^1 \to \mathbb{P}^1$ such that

$$\nu \circ \tilde{\sigma} = \sigma_0 \circ \nu, \quad \tilde{\sigma}(q_\pm) = q_\mp.$$

In the degeneration $s \to 0$, the loop $\gamma\gamma'$ shrinks to a point p.

We consider a family of polystable holomorphic vector bundles $\mathcal{D}_{\theta,s}$ of rank n^2, degree 0 over R_s with the following properties:

(i) For $0 < s \leq 1$, the holonomies of $\mathcal{D}_{\theta,s}$ along $\delta, \gamma, \delta', \gamma'$ are given by (7.16), so we have $\overline{\sigma_s^* \mathcal{D}_{\theta,s}} = \mathcal{D}_{\theta,s}$.

(ii) When $s = 0$, we have $\mathcal{D}_{\theta,0} = \mathcal{D}'(\theta)$, and $\nu^* \mathcal{D}'(\theta) = \mathcal{O}_{\mathbb{P}^1}^{\oplus n^2}$. $\mathcal{D}'(\theta)$ is obtained by identifying $v \in \mathbb{C}_{q_+}^{n^2}$ with $(-a_\theta \otimes a_\theta)v \in \mathbb{C}_{q_-}^{n^2}$.

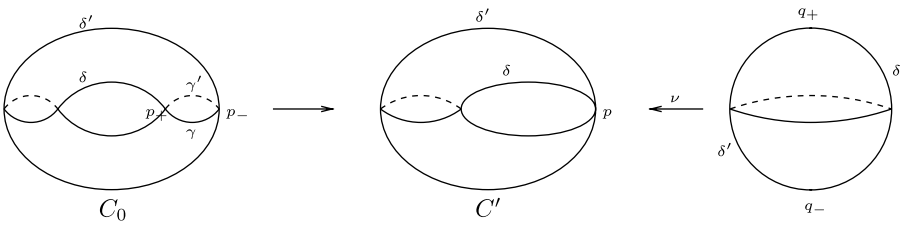

Figure 3: Degeneration of $C_0 = S^1 \times S^1$.

(iii) For all $(\theta, s) \in S^1 \times I$, there is a conjugate linear involution $\sigma_{\theta,s}$ on $H^*(R_s, \mathcal{D}_{\theta,s})$ such that

$$H^1(R_1, \mathcal{D}_{\theta,1})^{\sigma_{\theta,1}} - H^0(R_1, \mathcal{D}_{\theta,1})^{\sigma_{\theta,1}}$$

is the fiber of E_1^{vir} at $\theta \in S^1$.

(iv) The conjugate linear involution $\sigma_{\theta,0}$ on $H^0(R_0, \mathcal{D}_{\theta,0})$ induces a conjugate linear involution $\sigma_{\theta,0}$ on $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}^{\oplus n^2}) \cong \mathbb{C}^{n^2}$ (constant sections) that is given by $v \mapsto \bar{v}$.

As (θ, s) varies, the virtual real vector spaces

$$\tilde{E}_{\theta,s}^{\mathrm{vir}} = H^1(R_s, \mathcal{D}_{\theta,s})^{\sigma_{\theta,s}} - H^0(R_s, \mathcal{D}_{\theta,s})^{\sigma_{\theta,s}}$$

form a virtual real vector bundle $\tilde{E}^{\mathrm{vir}}$ over the cylinder $S^1 \times I$. Let $i_s : S^1 \to S^1 \times I$ be the embedding $\theta \mapsto (\theta, s)$. Then

$$i_0^* \tilde{E}^{\mathrm{vir}} = F^{\mathrm{vir}}, \quad i_1^* \tilde{E}^{\mathrm{vir}} = E_1^{\mathrm{vir}},$$

where the fiber of F^{vir} at $\theta \in S^1$ is

$$(F^{\mathrm{vir}})_\theta = H^1(C', \mathcal{D}'(\theta))^{\sigma_{\theta,0}} - H^0(C', \mathcal{D}'(\theta))^{\sigma_{\theta,0}}. \tag{7.17}$$

The maps i_0 and i_1 are homotopic, so E_1^{vir} is orientable if and only if F^{vir} is orientable.

Step 4. We have a long exact sequence

$$0 \to H^0(C', \mathcal{D}'(\theta)) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}^{\oplus n^2}) \cong \mathbb{C}^{n^2} \xrightarrow{f_\theta} \mathbb{C}_p^{n^2} \to H^1(C', \mathcal{D}'(\theta)) \to 0 \tag{7.18}$$

where $f_\theta(v) = v + (a_\theta \otimes a_\theta)v$ and we used $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}^{\oplus n^2}) = 0$. The involution $\sigma_{\theta,0}$ acts on (7.18) by

$$\begin{array}{ll} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}^{\oplus n^2}) \cong \mathbb{C}^{n^2} \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}^{\oplus n^2}), & v \mapsto \bar{v}, \\ \mathbb{C}_p^{n^2} \to \mathbb{C}_p^{n^2} \qquad\qquad\qquad\qquad\quad , & v \mapsto (a_\theta \otimes a_\theta)\bar{v}. \end{array}$$

Therefore

$$\begin{aligned} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}^{\oplus n^2})^{\sigma_{\theta,0}} &= \mathbb{R}^{n^2} \\ \left(\mathbb{C}_p^{n^2}\right)^{\sigma_{\theta,0}} &= \mathrm{e}^{\sqrt{-1}\theta}\mathbb{R} \oplus \mathrm{e}^{\sqrt{-1}\theta/2}\mathbb{R}^{2(n-1)} \oplus \mathbb{R}^{(n-1)^2}. \end{aligned}$$

So we have an exact sequence

$$0 \to H^0(C', \mathcal{D}'(\theta))^{\sigma_{\theta,0}} \to \mathbb{R}^{n^2} \xrightarrow{f_\theta} E_{\mathbb{R},\theta} \to H^1(C', \mathcal{D}'(\theta))^{\sigma_{\theta,0}} \to 0, \tag{7.19}$$

where

$$E_{\mathbb{R},\theta} = \mathrm{e}^{\sqrt{-1}\theta}\mathbb{R} \oplus \mathrm{e}^{\sqrt{-1}\theta/2}\mathbb{R}^{2(n-1)} \oplus \mathbb{R}^{(n-1)^2}.$$

To show that F^{vir} is orientable, it suffices to show that $E_{\mathbb{R},\theta}$ form an orientable real vector bundle $E_{\mathbb{R}} \to S^1$. Let L_0 and L_1 denote the trivial and nontrivial real line bundles over S^1, so that

$$w_1(L_j) = j \in \mathbb{Z}/2\mathbb{Z} = H^1(S^1, \mathbb{Z}/2\mathbb{Z}), \quad j = 0, 1.$$

Then

$$E_{\mathbb{R}} = L_0^{\oplus(n^2-2n+2)} \oplus L_1^{\oplus(2n-2)}, \quad w_1(E_{\mathbb{R}}) = 0 \in \mathbb{Z}/2\mathbb{Z} = H^1(S^1, \mathbb{Z}/2\mathbb{Z}).$$

Therefore $E_{\mathbb{R}} \to S^1$ is orientable. □

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