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3-manifolds in Euclidean space from a contact viewpoint

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We study the geometry of 3-manifolds generically embedded in $\mathbb{R}^n$ by means of the analysis of the singularities of the distance-squared and height functions on them. We describe the local structure of the discriminant (associated to the distribution of asymptotic directions), the ridges and the flat ridges.

1. Introduction

The study of the contacts of a submanifold of Euclidean space with objects, such as the hyperspheres and hyperplanes, that are invariant through the action of the Euclidean group provides a useful information on its extrinsic geometry, which leads to interesting global results [6, 28]. The main tool in this study is the analysis of the singularities of the distance squared and height functions on the submanifold. The generic singularities of the family of distance squared functions were initially studied by Porteous [22], who determined the relations between the singular set, the catastrophe map and the bifurcation set of this family with, respectively, the normal bundle, the normal exponential map and the focal set of the submanifold. He also introduced the concepts of ribs and ridges in connection with special contacts of the submanifold with its focal hyperspheres. These sets have a special interest from the viewpoint of applications in Image Analysis [3, 7, 9, 10]. A detailed study for surfaces in 3-space can be found in [23] and for surfaces in 4-space in [18].

On the other hand, the generic singularities of height functions on hypersurfaces were analyzed by Bruce [4] and Romero Fuster [24]. The corresponding study for surfaces in $\mathbb{R}^4$ and $\mathbb{R}^5$ can be, respectively, found in [14] and [17]. The concept of flat ridge of submanifolds with codimension 2 was introduced in [27] as the natural analogue of the ridges for the contacts with hyperplanes. In the case of a hypersurface, they can be seen as the intersection of the ridge and the parabolic sets. Other properties concerning submanifolds in n-space and their contacts with hyperplanes, in particular on the behavior of the binormal and asymptotic directions, can be found in [15, 16, 26].

The generic behavior of height functions on 3-manifolds in $\mathbb{R}^4$ was treated with detail in [20], where a duality relation with the singularities of projections of M onto hyperplanes was also described. An initial approach to the study of 3-manifolds in $\mathbb{R}^5$ can be found in [13]. Apart from these, there is not much information on the generic extrinsic geometry of 3-manifolds which, being richer and more complicated than that of surfaces, deserves a special attention. In particular, the ridges and flat ridges on 3-manifolds form surfaces with possible singularities. In this paper, we shall concentrate our attention in the study of 3-manifolds immersed in $\mathbb{R}^n$. The main tool for the study of a 3-manifold M embedded in codimension higher than one relies on the consideration of the normal Gauss map Γ on the canal hypersurface CM and the analysis of the generic singularities of the restrictions of the natural projection $\eta : CM \to M$ to different subsets of the singular set of Γ. Sections 2 and 3 contain some preliminaries on singularities and contacts. Section 4 is devoted to the canal hypersurface, its Gauss map and their connections with the height functions singularities. In Section 5 we describe the generic local behavior of the discriminant surface that separates regions with different number of asymptotic directions. The generic structure of the flat ridges is studied in Section 6. Finally, in Section 7, we use the fact that stereographic projection provides a link between the contacts of submanifolds of codimension k with hyperspheres in $\mathbb{R}^n$ and those of submanifolds of codimension $k+1$ with hyperplanes in $\mathbb{R}^{(n+1)}$ (see [25,29]) in order to obtain conclusions on the generic behavior of the ridges. We observe that the methods developed here for 3-manifolds can be naturally generalized to higher dimensions.

2. Contacts and singularities

Let X_i, Y_i $(i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. We say that the *contact of* X_1 and Y_1 at y_1 is of the same type as the *contact of* X_2 and Y_2 at y_2 if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n, y_1) \to (\mathbb{R}^n, y_2)$ such that $\Phi(X_1) = X_2$ and $\Phi(Y_1) = Y_2$. In this case we write $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$. Since this is a local concept, it is clear that $\mathbb{R}^n$ can be replaced by any manifold in this definition. Montaldi [19] gives the following characterization of the notion in terms of Mather's contact equivalence ($\mathcal{K}$-equivalence):

Theorem 2.1. *Let* M_i, N_i $(i = 1, 2)$ *be submanifolds of* $\mathbb{R}^n$ *with* $\dim M_1 = \dim M_2$ *and* $\dim N_1 = \dim N_2$. *Let* $f_i : (M_i, x_i) \to (\mathbb{R}^n, y_i)$ *be immersion germs and* $g_i : (\mathbb{R}^n, y_i) \to (\mathbb{R}^r, 0)$ *be submersion germs with* $(N_i, y_i) = (g_i^{-1}(0), y_i)$. *Then* $K(M_1, N_1; y_1) = K(M_2, N_2; y_2)$ *if and only if* $g_1 \circ f_1$ *and* $g_2 \circ f_2$ *are* $\mathcal{K}$*-equivalent.*

Therefore, given two submanifolds M and N of $\mathbb{R}^n$, with a common point p, an immersion germ $f : (M, x) \to (\mathbb{R}^n, p)$ and a submersion germ $g : (\mathbb{R}^n, p) \to (\mathbb{R}^r, 0)$, such that $N = g^{-1}(0)$, the contact of $M \equiv f(M)$ and N at p is completely determined by the $\mathcal{K}$-singularity type of the germ $(g \circ f, x)$ (see [8] for details on $\mathcal{K}$-equivalence).

When N is a hypersurface, we have $r = 1$, and the function germ $(g \circ f, x)$ has a degenerate singularity if and only if its Hessian, $\mathcal{H}(g \circ f)(x)$, is a degenerate quadratic form. In such a case, the tangent directions lying in the kernel of this quadratic form are called *contact directions* for M and N at p.

Since our study is of local character, we shall consider in what follows that the submanifold M is given by the image on an embedding $f : \mathbb{R}^m \to \mathbb{R}^n$.

We analyze here the following two families of functions:

(a) *Height functions* on M, given by

$$\begin{array}{rccl} \lambda(f) : & M \times S^{n-1} & \longrightarrow & \mathbb{R} \\ & (x, v) & \longmapsto & \langle f(x), v \rangle = f_v(x). \end{array}$$

The singularities of these functions describe the contacts of M with the hyperplanes of $\mathbb{R}^n$. We observe that a height function f_v has a singularity at $x \in M$ if and only if v is normal to M at x, then the singularity type of f_v at x determines the contact of M with the hyperplane orthogonal to v passing through x.

(b) *Distance squared functions* over M, defined as

$$\begin{array}{rccl} d : & M \times \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ & (x, a) & \longmapsto & d_a(x) = \|f(x) - a\|^2. \end{array}$$

This family measures the contacts of M with the hyperspheres of $\mathbb{R}^n$. In this case we have that $x \in M$ is a singular point of a function d_a if and only if the vector $a - f(x)$ lies in the normal subspace N_xM of M at x. The singularity type of d_a at x determines the contact of M with the hypersphere with center a passing through x.

It follows from the works of Looijenga [11], or Montaldi [19], that there is a residual subset $\mathcal{E}$ of embeddings of $\mathbb{R}^m$ into $\mathbb{R}^n$ with the Whitney C^∞-topology such that for any f belonging to it the corresponding families λ and d are generic families of functions on $\mathbb{R}^m$. For a detailed description of the term "generic family of functions" we refer to [11] or [30]. This means, in particular, that these families are topologically stable, and for $n \leq 5$, smoothly stable too. The singularities of the different functions in such a generic family may have codimension at most $n-1$ in the case of height functions and n in the case of distance squared functions. These are well known for small enough values of n. For instance, for $n \leq 6$, they are all simple singularities and correspond to the extended list of catastrophe germs determined by Arnold et al. [1]. A more complete classification, including all possible singularities up to codimension 14 can be found in [2].

It can be seen that the inverse $\varphi : \mathbb{R}^n \to S^n$ of the stereographic projection determines a $\mathcal{K}$-equivalence between the family of the distance squared functions on an m-manifold M immersed in $\mathbb{R}^n$ and the family of height functions over the m-manifold $\varphi(M) \subset S^n \subset \mathbb{R}^{n+1}$ (see [25] or [29]). Therefore it takes the singularities of a given type of distance squared functions of a k-codimension submanifold M of $\mathbb{R}^n$ into the singularities of the same type for height functions on the $(k+1)$-codimension submanifold $\varphi(M)$ of $\mathbb{R}^{n+1}$. It thus follows that the properties associated to the *round geometry* of submanifolds of $\mathbb{R}^n$ can be obtained, as a particular case, from those associated to the *flat geometry* of submanifolds of $\mathbb{R}^{n+1}$. We use here this fact and analyze first the behavior of height functions on 3-manifolds and then obtain, as a consequence, the corresponding properties related to the behavior of the distance squared functions on them.

3. Height functions, binormals and asymptotic directions

Consider a 3-manifold M given by the image of an embedding $f : M \to \mathbb{R}^{3+k}$ that lies in the residual subset $\mathcal{E}$ of $\mathrm{Emb}(M, \mathbb{R}^{3+k})$. As mentioned above, for $f \in \mathcal{E}$, the height functions family of f is a generic family of functions, which in particular implies that any height function f_v on M has only singularities of codimension less or equal to $k+2$. Moreover, those of corank one ($\mathrm{corank}(f_v) = \mathrm{corank}(\mathrm{Hess}(f_v))$) belong to the series $\{A_j\}_{j\geq 1}$, known as the cuspoids family. We recall that the $\mathcal{A}$-codimension of a singular germ of type A_j is $j-1$ (see [1]).

Denote by N_xM the normal space to M at x. A direction $v \in N_xM$ is a *degenerate direction* if x is singularity of f_v more degenerate than Morse, that is, a singularity of $\mathcal{A}$-codimension at least 1. In such a case, the kernel of the Hessian quadratic form, $\mathrm{Hess}(f_v)(x)$, associated to f_v at x contains non-zero vectors. Any direction $u \in \mathrm{Ker}(\mathrm{Hess}(f_v)(x))$ will be called *contact direction associated to v*.

A unit vector $v \in N_xM$ is said to be a *binormal direction* for M if and only if f_v has a singularity of type A_j, $j \geq k$ (so the $\mathcal{A}$-codimension of f_v is $\geq k-1$) at x. Binormal directions are a particular case of degenerate normal directions. We call them binormal by analogy in the case of curves in $\mathbb{R}^3$. The tangent hyperplane orthogonal to a binormal direction is said to be an *osculating hyperplane* of M at the considered point. If $v \in N_xM$ is a binormal vector, the tangent direction determined by the kernel of the Hessian quadratic form of f_v at x is said to be the *asymptotic direction* associated to the binormal v at x. The existence of binormal and asymptotic directions has been studied in [14] for the case of generic surfaces in 4-space, and in [15] for the generic submanifolds of codimension 2 in Euclidean space. We observe that in the last case the binormal and the degenerate directions coincide. An interesting feature of these is the relation between the convexity and the existence of binormal directions at a given point. Moreover, in a recent paper [21], Nuño Ballesteros and the second author show that a necessary and sufficient condition for the vanishing of the normal curvature at a point p of an m-submanifold of codimension 2 of Euclidean space is the existence of exactly m mutually orthogonal asymptotic directions at x.

The asymptotic directions were also characterized in [16] in terms of normal sections of M: Let v be a degenerate direction at a point x of M such that $\mathrm{corank}(\mathrm{Hess}(f_v)(x)) = 1$, and let θ be a tangent vector in the kernel of the quadratic form $\mathrm{Hess}(f_v)(x)$. We denote by γ_θ the normal section of the surface M in the tangent direction θ. That is, γ_θ is a curve in the $(k+1)$-space $V_\theta = \langle\theta\rangle \oplus N_xM$, obtained as the intersection of this $(k+1)$-space with M.

Proposition 3.1. *Let $x \in M$ and $v \in N_qM$ a degenerate direction for M at x. Let θ be a tangent direction in $\mathrm{Ker}(\mathrm{Hess}(f_v)(x))$. Then θ is an asymptotic direction corresponding to the binormal v if and only if v is the binormal direction at x for the curve γ_θ in the $(k+1)$-space V_θ.*

The binormal and asymptotic directions on generic surfaces in $\mathbb{R}^5$ were introduced in [17], where it was shown that there exist at least one and at most five at each point of such surfaces. The number of these directions is determined by the number of real roots of certain polynomials and jumps by two when crossing the discriminant set, which consists of closed regular curves made of points at which the considered polynomials admit multiple roots. The generic behavior of the asymptotic lines near the critical points and the discriminant is described in [26].

4. Flat geometry and canal hypersurfaces

Let M be an m-manifold immersed into $\mathbb{R}^{m+k}$ and let $\Lambda : M \times S^{m+k-1} \to S^{m+k-1} \times \mathbb{R}$ be the unfolding associated to the family λ. The singular set of the unfolding

$$\begin{aligned} \Lambda(f) : M \times S^{m+k-1} &\longrightarrow \mathbb{R} \times S^{m+k-1} \\ (x, v) &\longmapsto (f_v(x), v), \end{aligned}$$

associated to the family $\lambda(f)$ is given by

$$\Sigma\Lambda = \{(x, v) \in M \times S^{m+k-1} : \langle v, df(x)\rangle = 0\}.$$

This can be identified with an ϵ-tube around M,

$$C_\epsilon M = \{x + \epsilon v \in \mathbb{R}^{m+k} : v \perp T_x M\},$$

which for a small enough $\epsilon \in \mathbb{R}_+$ can be seen to be a hypersurface immersed in $\mathbb{R}^{m+k}$. This is also known as the *canal hypersurface* of M in $\mathbb{R}^{m+k}$. We denote it by CM and observe that the restriction of the natural projection $\pi : M \times S^{m+k-1} \to S^{m+k-1}$ to the submanifold $\Sigma\Lambda \equiv CM$ can be viewed as the normal Gauss map $\Gamma : CM \to S^{m+k-1}$ on the hypersurface CM. This map is also known as the *generalized normal Gauss map* of M. When M is a hypersurface ($k = 1$), we have that M and CM are locally diffeomorphic and hence Γ is locally equivalent to the normal Gauss map on M.

If we denote by $h_v : CM \to \mathbb{R}$ the height function in the direction v over CM and by I the $(k-1) \times (k-1)$-identity matrix, it is not difficult to check that, in appropriate coordinate systems

$$\mathrm{Hess}(h_v)(x, v) = \begin{bmatrix} \mathrm{Hess}(f_v)(x) & X \\ 0 & I \end{bmatrix} = D\Gamma(x, v).$$

The determinant of $D\Gamma(x, v)$ is the Gauss–Kronnecker curvature function $\mathcal{K}$ of CM at the point (x, v). The singular set $\Sigma\Gamma = \mathcal{K}^{-1}(0)$ is the *parabolic set* of CM. It follows from the above expression that $(x, v) \in \Sigma\Gamma$ if and only if (x, v) is a degenerate singularity of h_v, which is in turn equivalent to saying that x is a degenerate singularity of f_v.

We now recall the definition of the Thom–Boardman symbols for a map $H : X \to Y$ [8]: We say that H has a singularity of type S_r at $p \in X$ if $DH(p)$ drops rank by r; i.e., rank $DH(x) = \min(\dim X, \dim Y) - r$. We denote by $S_r(H)$ the singularities of type S_r in X. It is well known that for a generic map H, the subsets $S_r(H)$ are submanifolds of codimension $r^2 + er$ of X, where $e = |\dim X - \dim Y|$. In such a case, we can consider the restriction $H|_{S_r(H)} : S_r(H) \to Y$. Then we denote by $S_{r,s}(H)$ the set of points where this restriction drops rank s. Again, under appropriate genericity conditions on the jet extensions of H, these subsets are submanifolds of $S_r(H)$ and then it is possible to define inductively a nested family of submanifolds $S_{i_1,\cdots,i_q}(H)$ of X. The points of $S_{i_1,\cdots,i_q}(H)$ are said to be singularities of H with Thom–Boardman symbol $\Sigma^{i_1,\cdots,i_q}$.

As mentioned in Section 2, for any immersion f lying in the residual set $\mathcal{E}$ of immersions of the m-manifold M in $\mathbb{R}^{m+k}$, the germ of $\lambda(f)$ at any point (x, v) is a versal unfolding of the germ of f_v at x. It follows from standard results on stable families [12] that the subsets

$$S_r(\Gamma) = \{(x, v) \in CM : (x, v) \text{ is a singularity of corank } r \text{ of } \Gamma\}, r \geq 1$$

are submanifolds of CM, and satisfy that codim $S_1(\Gamma) = 1$ and $\cup_{i\geq 2} S_i(\Gamma)$ is a stratified subset of CM with codimension ≥ 3. We observe that, considered as a smooth map over equidimensional manifolds, Γ is stable over the points of S_1, but it is not stable over the points lying in $\cup_{i\geq 2} S_i(\Gamma)$. It follows from the above considerations that

$$\begin{aligned}(x, v) \in S_r\Gamma &\Leftrightarrow (x, v) \text{ is a singularity of corank } r \text{ of } h_v, r = 1, 2, \ldots\\ &\Leftrightarrow x \text{ is a singularity of corank } r \text{ of } f_v, r = 1, 2, \ldots.\end{aligned}$$

In order to simplify the notation, we shall denote $S_{1,1} = S_{1_2},\ S_{1,1,1} = S_{1_3}$ and so on. We have,

$$\begin{aligned}S_{1_k}(\Gamma) &= \{(x, v) \in CM : (x, v) \text{ is a singularity of corank 1 of } \Gamma|_{S_{1_{k-1}}(\Gamma)}\}\\ &= \{(x, v) \in CM : (x, v) \text{ is a singularity of type } A_{k+1} \text{ of } f_v\}.\end{aligned}$$

Observe that given $(x, v) \in S_{1_k}(\Gamma)$, there is a unique principal asymptotic direction $\pm u(x, v) \in T_{(x,v)}CM$, and thus we can write,

$$S_{1_k}(\Gamma) = \{(x, v) \in S_{1_{k-1}}(\Gamma) : u(x, v) \in T_{(x,v)} S_{1_{k-1}}(\Gamma)\}.$$

Then, since the normal direction to $S_1(\Gamma)$ in CM is given by grad $\mathcal{K}(x,v)$, we have that $(x,v) \in S_{1_2}(\Gamma)$ if and only if $\mathcal{K}_2(x,v) = \langle u(x,v), \text{grad}\, \mathcal{K}(x,v)\rangle = 0$. Analogously, given $(x,v) \in S_{1_2}(\Gamma)$, we have that $(x,v) \in S_{1_3}(\Gamma)$ if and only if $\mathcal{K}_3(x,v) = \langle u(x,v), \text{grad}\ \mathcal{K}_2\rangle = 0$, and so on. We inductively define in this way a set of functions $\mathcal{K}_j$ over $S_1(\Gamma)$ that depend on the derivatives of the immersion f at each point and satisfy that $(x,v) \in S_{1_r}(\Gamma)$ if and only if $\mathcal{K}_j(x,v) = 0, \forall\, j = 1,\dots,r$, where $\mathcal{K}_1(x,v) = \mathcal{K}(x,v)$. Observe that these functions are independent, for as j increases their coefficients involve higher order derivatives of the (generic) embedding f. We can thus view the r-codimensional submanifold $S_{1_r}(\Gamma)$ as the set of zeroes of the r implicit equations $\mathcal{K}_j(x,v) = 0,\ j = 1,\dots,r$ on CM. In particular, we have that $S_{1_{k-1}}(\Gamma)$ is an m-dimensional submanifold of CM.

Consider the natural projection $\eta : CM \to M$ and denote by η_j its restriction to $S_{1_j}(\Gamma)$. For $j = k-1$, we get the map $\eta_{k-1} : S_{1_{k-1}}(\Gamma) \to M$ between equidimensional manifolds. We can characterize the asymptotic directions of M in terms of the principal asymptotic directions of the hypersurface CM as follows.

Proposition 4.1. *Given $(x,v) \in S_{1_{k-1}}(\Gamma)$, the linear map*

$$D_{(x,v)}\eta_{k-1} : T_{(x,v)}S_{1_{(k-1)}}(\Gamma) \to T_xM$$

takes the unique principal asymptotic direction of CM at (x,v) to the asymptotic direction of M associated to v at x.

Proof. This follows easily from the equality

$$D\Gamma(x,v) = \begin{bmatrix} \text{Hess}(f_v)(x) & X \\ 0 & I \end{bmatrix}. \qquad \square$$

It follows that given a point $x \in M$, the asymptotic directions at x come from the images of the principal asymptotic directions at all the points of the fibre $\eta_{k-1}^{-1}(x)$ in CM. Then the total number of asymptotic (or binormal) directions at x is given by the cardinality of $\eta_{k-1}^{-1}(x)$. This may vary from one point to another in M. In fact, we define the *discriminant set* of M as $\Delta = \eta_{k-1}(\Sigma\eta_{k-1})$. Generically, this subset has codimension one in M and separates regions with different number of binormal/asymptotic directions. It has been shown in [15] that the maximum number of binormal directions at a point of an m-manifold immersed in $\mathbb{R}^{m+2}$ is m. It can be shown that for submanifolds immersed in higher codimensions, this number may increase, but it is always finite, so the map $\eta_{k-1} : S_{1_{(k-1)}}(\Gamma) \to M$ is finite-to-one.

In the following sections we study the generic behavior of the restrictions of the map η to the submanifolds S_{1_r}, $r \geq k-1$. For this purpose, we shall consider several algebraic subsets given by implicit equations over $S^{2+k} \times J^q(M, \mathbb{R}^{3+k})$. In order to apply the Thom's Transversality Theorem [8] in each case, we use the fact that these equations determine semialgebraic subsets in convenient jet spaces $J^q(M, \mathbb{R}^{3+k})$. This can be seen as follows: Consider the algebraic subset

$$\mathcal{W} = \{(v, j^q f(x)) \in S^{2+k} \times J^q(M, \mathbb{R}^{3+k}) : (v, x) \in S_{1_{(k-1)}}(\Gamma)\},$$

and let $\pi : \mathcal{W} \to J^q(M, \mathbb{R}^{3+k})$ be the natural projection. Then, given any algebraic subset S of codimension r (defined by r independent implicit equations) in $\mathcal{W}$, its image $\pi(S)$ has codimension $\geq r$ in $J^q(M, \mathbb{R}^{3+k})$. Moreover, provided π is finite-to-one, we have that $\operatorname{codim}(S) = \operatorname{codim} \pi(S))$. We now observe that since η_{k-1} is finite-to-one so must be π.

5. Generic structure of discriminant sets

We have characterized the 3-manifold $S_{1_{k-1}}(\Gamma)$ by $k-1$ implicit equations $\mathcal{K}_i(x, v) = 0$ on CM, with $\mathcal{K}_1(x, v) = \det D\Gamma(x, v)$ and

$$\mathcal{K}_i(x, v) = \langle u(x, v), \operatorname{grad} \mathcal{K}_{i-1}(x, v)\rangle, \quad 2 \leq i \leq k-1,$$

where $u(x, v)$ is the asymptotic principal direction of CM at (x, v). In order to describe the generic local structure of Δ in M we need to analyze the generic singularities of the map $\eta_{k-1} : S_{1_{k-1}}(\Gamma) \to M$. This is a map between 3-manifolds. We recall that the stable singularities between 3-manifolds may be one of the following types (see [8, p. 191]): S_1 (fold), S_{1_2} (cusp) and S_{1_3} (swallowtail).

Theorem 5.1. *For a generic embedding $f : M \to \mathbb{R}^{3+k}$, $k \geq 2$, the map $\eta_{k-1} : S_{1_{k-1}}(\Gamma) \to M$ is locally stable. That is, it may only have fold singularities on a surface, cusp singularities on a curve and isolated swallowtail points. Moreover, for $k = 2$, η_{k-1} does not have swallowtail points.*

Proof. (i) Given $\eta : CM \to M$, we observe that $\operatorname{Ker} D\eta(x, v)$ is the tangent space to the fiber $F_x = \eta^{-1}(x)$ of CM over x and has dimension $k-1$. Clearly $\operatorname{Ker} D\eta_{k-1}(x, v) = \operatorname{Ker} D\eta(x, v) \cap T_{(x,v)} S_{1_{k-1}}(\Gamma)$. Then we have:

(1) We have that $(x, v) \in S_1(\eta_{k-1})$ if and only if $\operatorname{Ker} D\eta(x, v) \cap T_{(x,v)} S_{1_{k-1}}(\Gamma)$ has dimension exactly 1. But this means that

$\dim(T_{(x,v)}S_{1_{k-1}}(\Gamma) + T_{(x,v)}F_x) = k+1$. Or equivalently, in terms of normal spaces (in CM), this implies

$$\dim(N_{(x,v)}S_{1_{k-1}}(\Gamma) + N_{(x,v)}F_x) = k+1.$$

We have seen that $S_{1_{k-1}}(\Gamma)$ is given by $\mathcal{K}_i(x,v) = 0$, $1 \leq i \leq k-1$, so $N_{(x,v)}S_{1_{k-1}}(\Gamma)$ is generated by the vectors $\operatorname{grad}\mathcal{K}_i$. On the other hand, $N_{(x,v)}F_x = T_xM$ is generated by f_{x_1}, f_{x_2} and f_{x_3}. If we consider the matrix $L = [f_{x_1}, f_{x_2}, f_{x_3}, \operatorname{grad}\ \mathcal{K}_1, \ldots, \operatorname{grad}\ \mathcal{K}_{k-1}]$, whose entries correspond to vectors in the $(2+k)$-dimensional space $T_{(x,v)}CM$, the above condition is equivalent to asking that $\det L = 0$. This determines an equation in terms of the derivatives of the embedding of order lesser or equal to $(k+1)$. In fact, this condition defines an algebraic subset, $\mathcal{S}_1$, of codimension 1 in the jet space $S^{2+k} \times J^{k+1}(M, \mathbb{R}^{3+k})$.

(2) We then have that $(x,v) \in S_{1_2}(\eta_{k-1})$ if and only if $\operatorname{Ker} D\eta(x,v) \cap T_{(x,v)}S_1(\eta_{k-1})$ has dimension exactly 1. As above, this is equivalent to,

$$\dim(N_{(x,v)}S_1(\eta_{k-1}) + T_xM) = k+1.$$

Now we observe that $S_1(\eta_{k-1})$ is given by the implicit equations,

$$\mathcal{F}_1(x,v) = 0 \text{ and } \mathcal{K}_i(x,v) = 0,$$

$1 \leq i \leq k-1$, where we denote $\mathcal{F}_1(x,v) = \det\ L(x,v)$, and hence the linear subspace $N_{(x,v)}S_1(\eta_{k-1})$ is generated by the vectors $\{\operatorname{grad}\ \mathcal{K}_i(x,v)\}_{i=1}^{k-1}$ and $\operatorname{grad}\ \mathcal{F}_1(x,v)$. Then the above condition is equivalent to asking that the $(k+3)\times(k+2)$-matrix

$$L_1 = [\operatorname{grad}\mathcal{K}_1(x,v), \ldots, \operatorname{grad}\ \mathcal{K}_{k-1}(x,v), \operatorname{grad}\ \mathcal{F}_1(x,v), f_{x_1}, f_{x_2}, f_{x_3}]$$

has rank $k+1$. Since we are assuming already that $(x,v) \in S_1(\eta_{k-1})$, we have that this only adds one equation to the above one. Therefore, this determines an algebraic subset, $\mathcal{S}_2$, of codimension 2 in the jet space $S^{2+k} \times J^{k+2}(M, \mathbb{R}^{3+k})$.

(3) We get analogously that $(x,v) \in S_{1_3}(\eta_{k-1})$ if and only if

$$\dim(N_{(x,v)}S_{1_2}(\eta_{k-1}) + T_xM) = k+1.$$

It can be seen, in a similar way than in (1) and (2) above, that this condition determines an algebraic subset, $\mathcal{S}_3$, of codimension 3 in the jet space $S^{2+k} \times J^{k+3}(M, \mathbb{R}^{3+k})$.

(4) Moreover, $S_{1_4}(\eta_{k-1}) = \emptyset$ because it is impossible $\operatorname{Ker} D\eta(x,v) \cap T_{(x,v)}S_{1_3}(\eta_{k-1})$ to have dimension 1 since $S_{1_3}(\eta_{k-1})$ has dimension zero.

(5) Finally, we observe that $(x,v) \in S_2(\eta_{k-1})$ if and only if $\operatorname{Ker} D\eta(x,v) \cap T_{(x,v)}S_{1_{k-1}}(\Gamma)$ has dimension exactly 2. In the case $k = 2$ we have that $\dim \operatorname{Ker} D\eta(x,v) = 1$. Hence we must have that $S_2(\eta_{k-1}) = \emptyset$. Suppose now that $k \geq 3$. By using analogous arguments as above, we have that $(x,v) \in S_2(\eta_{k-1})$ if and only if $\dim\,(T_{(x,v)}S_{1_{k-1}}(\Gamma) + T_{(x,v)}F_x) = (k-1)+3-2 = k$. But this is equivalent to asking that $\dim\,(N_{(x,v)}F_x + N_{(x,v)}S_{1_{k-1}}(\Gamma)) = k$. Which means that the rank of the subset of $(k+2)$ vectors $\{\operatorname{grad}\mathcal{K}_1(x,v), \ldots, \operatorname{grad}\mathcal{K}_{k-1}(x,v), f_{x_1}, f_{x_2}, f_{x_3}\}$ in $T_{(x,v)}$ CM must be k. This determines four independent conditions on the derivatives of order less or equal to $(k+1)$ of the embedding and thus defines an algebraic subset, $\mathcal{S}_4$, of codimension 4 in the jet space $S^{2+k} \times J^{k+1}(M, \mathbb{R}^{3+k})$.

It is now a straightforward consequence of the Thom's Transversality Theorem [8] that there is a residual subset $\mathcal{E}_1 \subset \mathcal{E}$ of $\operatorname{Emb}(M, \mathbb{R}^{3+k})$ for which the map $\eta_{k-1} : S_{1_{k-1}}(\Gamma) \to M$ only has fold singularities over a surface in M, cusp singularities along a curve and perhaps isolated swallowtail points.

We now prove that for $k = 2$ the subset $S_{1_3}(\eta_1)$ is empty. Suppose that there exists a point $(p, \bar{v}) \in S_{1_3}(\eta_1)$. By an appropriate change of coordinates we can put $p = 0$, $f_{x_i}(0) = 0$ and $\bar{v} = (0,1)$. We now look for all $(0,v)$ such that $(0,v) \in S_{1_3}(\eta_1)$. It follows from our construction that if $(0,v) \in S_1(\Gamma)$, then $(0,v)$ satisfies the implicit equation $\mathcal{K}_1(x,v) = 0$. Moreover, we can also take $v \in N_pM$, $v = (0,0,0,v_4,v_5)$ with $v_5 = 1$. Then $\bar{\mathcal{K}}_1(x,v_4) = \mathcal{K}_1(x,v_4,1) = \det\,(D\Gamma(x,v_4,1))$ is a non-homogeneous polynomial of degree 3 in the variable v_4, with null constant term because by hypothesis $(0,\bar{v})$ is a solution of $\mathcal{K}_1(0,v) = 0$.

We analyze now the remaining implicit equations, $\mathcal{F}_i(0,v_4) = 0$, $1 \leq i \leq 3$, that define $S_{1_3}(\eta_1)$. Consider the 4×5-matrix $L = [f_{x_1}, f_{x_2}, f_{x_3}, \operatorname{grad}\mathcal{K}_1]$, evaluated at $(0,v_4,1)$. Since we are assuming that $v_5 = 1$, then $\dfrac{\partial \mathcal{K}_1}{\partial v_5} = 0$ and the last column of L is null. Let $\bar{L}$ be the matrix obtained by elimination of the last column in L. This has corank 1 provided $\mathcal{F}_1(0,v_4) = \det(\bar{L}) = 0$. Note that in $(0,v_4,1)$, we have $\det(\bar{L}) = \det(I_3)\dfrac{\partial \mathcal{K}_1(0,v)}{\partial v_4}$, and hence $\mathcal{F}_1(0,v_4) = \dfrac{\partial \mathcal{K}_1(0,v)}{\partial v_4}$.

Consider the matrices $L_1 = [L, \operatorname{grad} \mathcal{F}_1]$, $L_2 = [L_1, \operatorname{grad} \mathcal{F}_2]$, from which we respectively define $\bar{L}_1, \bar{L}_2$ by elimination of the (null) last column. Then we have that $\bar{L}_2$ has corank 3 provided $\mathcal{F}_1(0, v_4) = \mathcal{F}_2(0, v_4) = \mathcal{F}_3(0, v_4) = 0$, where $\mathcal{F}_2(0, v_4) = \dfrac{\partial \mathcal{F}_1(0, v_4)}{\partial v_4}$ and $\mathcal{F}_3(0, v_4) = \dfrac{\partial \mathcal{F}_2(0, v_4)}{\partial v_4}$.

We can locally write, in a neighborhood of $(0, \bar{v})$,

$$\mathcal{K}_1(x, v_4, 1) = \bar{\mathcal{K}}_1(x, v_4) = \frac{\partial \bar{\mathcal{K}}_1(0,0)}{\partial v_4} v_4 + \frac{\partial^2 \bar{\mathcal{K}}_1(0,0)}{\partial v_4^2} v_4^2 + \frac{\partial^3 \bar{\mathcal{K}}_1(0,0)}{\partial v_4^3} v_4^3,$$

which has vanishing constant term, since $(0, \bar{v})$ is solution. Besides, the condition $(0, \bar{v}) \in S_{1_3}(\eta_1)$ means that

$$\mathcal{F}_1(0, v_4) = \mathcal{F}_2(0, v_4) = \mathcal{F}_3(0, v_4) = 0$$

and by the previous calculations these are equivalent to

$$\frac{\partial^{j-1} \bar{\mathcal{K}}_1(0,0)}{\partial v_4^{j-1}} = 0, \quad 2 \leq j \leq 4.$$

We can then conclude that for $(0, \bar{v}) \in S_{1_3}(\eta_1)$ the polynomial $\bar{\mathcal{K}}_1(0, v_4)$ is a null polynomial. But this implies that there must be infinite solutions $v \in N_p M$ for $p = 0$, which contradicts the fact that the number of binormals is always finite on $S_1(\Gamma)$. □

We show in Figure 1 all the possibilities for the generic local structure of the discriminant.

Remark 5.1. We observe that the proof that $S_{1_3}(\eta_1)$ is empty can be extended in a straightforward manner to prove that $S_{1_m}(\eta_1) = \emptyset$ for any m-manifold with codimension 2, that is $m \geq 2$ and $k = 2$. This means that the last possible stable singularity (see page 191 of [8]) for the projection η_1 between equidimensional manifolds, does not occur. For 3-manifolds, this fact can also be concluded from Theorem 1 in [13].

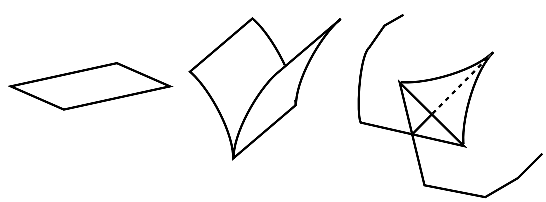

Figure 1: Generic local structure of Δ.

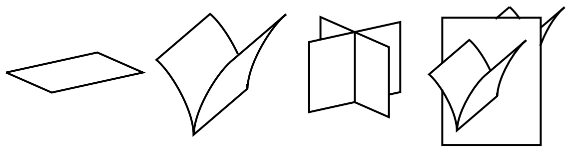

Figure 2: Generic structure of Δ for a 3-manifold in $\mathbb{R}^5$.

By using the multijet version of Thom's Transversality Theorem [8] we can also prove that the discriminant Δ is a surface with normal crossings. The self-intersections of Δ consist of curves of double points, isolated fold-cusp points and isolated triple points. Moreover, we have,

Corollary 5.1. *Given a 3-manifold generically embedded in $\mathbb{R}^5$, the discriminant set Δ has no triple points. Therefore it is a closed surface with possible singularities along cuspidal edges and closed curves of double points with isolated singularities at their intersections with the cuspidal edges.*

Proof. The discriminant set separates M into open regions with different number of asymptotic directions. This number jumps by two from one side of Δ to the other. Then if Δ had a triple point, by looking to its local structure in a neighborhood of such a point we would have that M should admit at least five asymptotic directions over some region. But this contradicts the fact that the maximum number of asymptotic directions on 3-manifolds immersed in codimension 2 is 3 [15]. $\square$

Figure 2 illustrates the generic structure of the discriminant from the multilocal viewpoint.

6. Flat ridges

We define the *flat ribs of order i* in a 3-manifold M embedded in $\mathbb{R}^{3+k}, k \geq 1$ as the subset $S_{1_i}(\Gamma), i \geq 2$, in CM. The *flat ridge of order i* in a 3-manifold M embedded in $\mathbb{R}^{3+k}, k \geq 1$ is the set of points that are singularities of type A_i, $i \geq k+1$, for some height function. In other words, the projection through η of the flat rib of order $i, i \geq k+1$. The *highest-order flat ridge points* are defined as the singularities of type A_i, $i \geq 3+k$, for some height function. These are, generically, isolated points.

We have the following characterization of flat ridges in terms of the normals sections of M.

Given a curve $\gamma : \mathbb{R} \to \mathbb{R}^{k+1}$, consider its Frenet-Serret frame $\{T, N_1, \dots, N_k\}$ and the corresponding curvature functions $\{\kappa_1, \dots, \kappa_k\}$. We say that a point $x = \gamma(t_0)$ is a *flattening* of γ, provided $\kappa_k(t_0) = 0$. A flattening $x = \gamma(t_0)$ is said to be *degenerate of order* r, $r > 0$, provided $\kappa_k(t_0) = \kappa_k'(t_0) = \cdots = \kappa_k^{(r)}(t_0) = 0$

Proposition 6.1. *Let $x \in M$ and $v \in N_xM$ a binormal direction. Let θ be its corresponding asymptotic direction and γ_θ the corresponding normal section of M. Then*

(a) $x = \gamma_\theta(0)$ is a flat ridge of M if and only if x is a flattening of γ_θ (as a curve in the $(k+1)$-space V_θ).

(b) $x = \gamma_\theta(0)$ is a flat ridge of order $k + r$ of M if and only if x is a degenerate flattening of order r of γ_θ.

Proof. Since θ is the contact direction associated to f_v at x, we have that x is a singularity of type A_j of $f_v|_{\gamma_\theta}$ if and only if it is a singularity of type A_j of the height function f_v over M. Then the fact that the point $x = \gamma_\theta(0)$ is a flattening of γ_θ if and only if it is a singularity of type $A_j, j \geq k+1$ for $f_v|_{\gamma_\theta}$ leads to the required results. □

We can view the flat ridges of order $k + r$ of M as the images by η of the submanifolds $S_{1_{k+r-1}}(\Gamma)$ of CM into M. In order to study their local structure we shall analyze the generic singularities of the maps $\eta_{k+r-1} : S_{1_{k+r-1}}(\Gamma) \to M$.

The $(3 - r)$-manifold $S_{1_{k+r-1}}(\Gamma)$ is characterized by $k + r - 1$ implicit equations on CM:

$$\mathcal{K}_1(x, v) = \det D_{(x,v)}\Gamma = 0,$$
$$\mathcal{K}_i(x, v) = \langle u(x, v), \operatorname{grad} \mathcal{K}_{i-1}(x, v)\rangle = 0, 2 \leq i \leq k + r - 1,$$

where u is the principal asymptotic direction at x associated to v.

Theorem 6.1. *For a generic immersion $f : M \to \mathbb{R}^{3+k}$, $k \geq 2$, the map $\eta_k : S_{1_k}(\Gamma) \to M$ is locally stable.*

Proof. The subset $S_{1_k}(\Gamma)$, given by $\mathcal{K}_i(x, v) = 0,\ 1 \leq i \leq k$, is a regular surface in the $(k + 2)$-dimensional manifold CM. Consider the map $\eta_k : S_{1_k}(\Gamma) \to M$. We have that $(x, v) \in S_1(\eta_k)$ if and only if $T_{(x,v)}S_{1_k}(\Gamma)$ is not transversal to $D\eta(x,v)$ in the $(2+k)$-dimensional space $T_{(x,v)}CM$. Or equivalently, in terms of normal spaces,

$$\dim(N_{(x,v)}S_{1_k}(\Gamma)+N_{(x,v)}F_x)=k+1,$$

where $N_{(x,v)}S_{1_k}(\Gamma)$ is generated by $\{\operatorname{grad}\mathcal{K}_i\}_{i=1}^k$, and $N_{(x,v)}F_x=T_xM$, is generated by $\{f_{x_1},f_{x_2},f_{x_3}\}$. So we must have

$$\operatorname{rank}\{\operatorname{grad}\mathcal{K}_1,\dots,\operatorname{grad}\mathcal{K}_k,f_{x_1},f_{x_2},f_{x_3}\}=k+1.$$

Consider the $(k+3)\times(k+2)$-matrix $B_1=[f_{x_1},f_{x_2},f_{x_3},\operatorname{grad}\mathcal{K}_1,\cdots,\operatorname{grad}\mathcal{K}_k]$, in the $(2+k)$-dimensional space $T_{(x,v)}CM$. Since the corank of B_1 must be 2, then there exists a $(k+1)\times(k+2)$-sub-matrix B such that $B_1=[B,l_1,l_2]$ where we get that $(x,v)\in S_1(\eta_k)$ if and only if $\mathcal{F}_1(x,v)=\det(B,l_1)=0$ and $\mathcal{F}_2(x,v)=\det(B,l_2)=0$. Analogously to the proofs of the previous section, using the Thom's transversality Theorem, these equations determine, generically, isolated points in $S_{1_k}(\Gamma)$. Since $S_1(\eta_k)$ has dimension zero then $\eta_k:S_1(\eta_k)\to M$ cannot drop rank and we conclude that $S_{1_j}(\eta_k)=\emptyset$ to $j\geq 2$.

We now see that $S_2(\eta_k)=\emptyset$ and thus $S_j(\eta_k)=\emptyset$, $\forall j\geq 2$. In fact, for $k=2$, CM has dimension 4, and $(x,v)\in S_2(\eta_2)$ if and only if

$$\dim(T_{(x,v)}S_{1_2}(\Gamma)\cap\operatorname{Ker}(D\eta(x,v)))=2.$$

But this is impossible because $\dim(\operatorname{Ker}(D\eta(x,v)))=1$.

For $k=3$, $(x,v)\in S_2(\eta_3)$ if and only if

$$\dim(T_{(x,v)}S_{1_3}(\Gamma)\cap\operatorname{Ker}(D\eta(x,v)))=2.$$

In other words, $T_{(x,v)}S_{1_3}(\Gamma)=\operatorname{Ker}(D\eta(x,v))$, or equivalently

$$\dim(N_{(x,v)}S_{1_3}(\Gamma)+T_xM)=3.$$

Then the 6×5-order matrix $L=[f_{x_1},f_{x_2},f_{x_3},\operatorname{grad}\mathcal{K}_1,\operatorname{grad}\mathcal{K}_2,\operatorname{grad}\mathcal{K}_3]$ must have corank 3. This gives rise to six independent equations on the derivatives of the embedding. Therefore we conclude that generically S_2 is empty.

This proof can be extended in a straightforward manner for any $k\geq 4$. □

Corollary 6.1. *The flat ridge set of a* 3*-manifold M generically embedded in $\mathbb{R}^{3+k}$, $k\geq 2$, is a surface with possible isolated cross-caps and transverse* *self-intersections. Inside this surface, we may have regular immersed curves corresponding to the $(k+2)$-order flat ridge and isolated highest order flat ridges.*

The following provides a characterization of the flat ridges of different orders in terms of the relative positions with respect to the asymptotic lines.

Proposition 6.1. *On a 3-manifold generically embedded in $\mathbb{R}^{3+k}$, the flat ridges of order $k+2$ are the points at which the flat ridge surface $\eta(S_{1_k}(\Gamma))$ is tangent to some asymptotic line and the highest order flat ridges are the points at which the flat ridge curve $\eta(S_{1_{k+1}}(\Gamma))$ is tangent to some asymptotic line.*

Proof. We observe that for a generic embedding $f : \mathbb{R}^3 \to \mathbb{R}^{3+k}$, the subset $S_{1_k}(\Gamma)$ is a surface in the $(k+2)$-manifold CM. Then it follows from the definition of Thom–Boardman singularities that a point $(x, v) \in S_{1_k}(\Gamma)$ lies in the curve $S_{1_{k+1}}(\Gamma)$ if and only if the principal asymptotic direction of M at (x, v) (which is the contact direction associated to the height function h_v on CM at (x, v)) is tangent to the surface $S_{1_k}(\Gamma)$. Analogously, a point $(x, v) \in S_{1_k}(\Gamma)$ lies in $S_{1_{k+2}}(\Gamma)$ if and only if the principal asymptotic direction of M at (x, v) is tangent to the curve $S_{1_{k+1}}(\Gamma)$. Then the result follows immediately by taking the corresponding images through $\eta : CM \to M$ into M. □

7. Round geometry on 3-manifolds

The generic singularities of the family d were initially studied by Porteous [22], who observed that the corresponding catastrophe manifold,

$$\Sigma(d) = \left\{ (g(x), a) \in M \times \mathbb{R}^n | \ \frac{\partial d_a}{\partial x} = 0 \right\}$$

coincides with the normal bundle NM of M in $\mathbb{R}^n$. The restriction of the projection $\pi : M \times \mathbb{R}^n \to \mathbb{R}^n$ to $\Sigma(d) = NM \subset M \times \mathbb{R}^n$, i.e., the *catastrophe map* associated to the family d, is the normal exponential map $\exp_N$ of M. The *bifurcation set*

$$\mathcal{B}(d) = \{ a \in \mathbb{R}^n | \ \exists \ x \in \mathbb{R}^{n-1} \text{ where } d_a \text{ has a degenerate singularity } \}$$

is made of all the centers of hyperspheres having contact of order at least 2 with M in the sense that the contact function-germ d_a at x has codimension at least 1, i.e., it is not a Morse function. This subset is classically known as *focal set* of M and the hyperspheres tangent to M whose centers lie in $\mathcal{B}(d)$ are called *focal hyperspheres* of M.

We remind that when M is a 3-manifold in $\mathbb{R}^4$ and $\Gamma : M \to S^3$ represents its normal Gauss map, then the eigenvectors of $D\Gamma(x)$ are the *principal directions* of curvature of M at the point x and the corresponding eigenvalues, $\{\kappa_i(x)\}_{i=1}^3$, are the *principal curvatures.* A curve all of whose tangents are principal directions is a *curvature line.* We shall say that a point $x \in M$ is *umbilic* if the three principal curvatures are equal at x and we call it *pre-umbilic* when two of them coincide. It can be seen that the principal directions are the contact directions corresponding to the distance squared functions on M, i.e., they are the contact directions of M with its focal hyperspheres at each point (see [15]). The pre-umbilics are singularities of corank two of distance-squared functions on M, the umbilics being those of maximal corank 3. We shall denote by $\mathcal{PU}$ the subset of pre-umbilics of M. For a generic 3-manifold M, the subset $M - \mathcal{PU}$ is an open and dense submanifold of M. Provided $x \in M - \mathcal{PU}$, we can find exactly three focal hyperspheres at it, whose centers are given by $a_i(x) = f(x) + r_i(x)\Gamma(x), i = 1, 2, 3,$ and whose radii are $r_i(x) = 1/\kappa_i(x), i = 1, 2, 3$. If some of the principal curvatures vanishes, so x is a *parabolic point* of M, then the corresponding focal hypersphere becomes a tangent hyperplane. This can be generalized to the case of a 3-manifold embedded with higher codimension by saying that a point is *pre-umbilic* when it is a singularity of corank 2 of some distance-squared function d_a. The point a is the center of a hypersphere with a special contact (of corank 2) with M, we call it *pre-umbilic center.* When the corank is equal to 3 we say that the point is an *umbilic* and the center a *umbilic center.*

The focal hyperspheres at a point x in a 3-manifold M embedded in $\mathbb{R}^{3+k}, k \geq 1$ are the tangent hyperspheres whose centers lie in the complement of $S_{1,0}(\exp_{\mathrm{N}})$ in the singular set of $\exp_{\mathrm{N}}$ (i.e., they lie in the *focal set* of M). They define distance-squared functions with a singularity of type A_3 or worse. The *rib of order i* of M is defined as the subset $S_{1_i}(\exp_N), i \geq 2$, of NM. This, together with the subset $S_{k\geq 2}(\exp_N)$ form the singular part of the focal set. The *ridge of order i* in a 3-manifold M embedded in $\mathbb{R}^{3+k}, k \geq 1$ is the set of points that are singularities of type A_i, $i \geq k+2$, for some distance-squared function. In other words, they are the projection through $\exp_N$ of the rib of order $i, i \geq k+2$. The *highest-order ridge points* are defined as the singularities of type A_i, $i \geq 4+k$, for some distance-squared function. These are, generically, isolated points.

We characterize next the ridge points in terms of the normal sections of the manifold at a given point.

We recall that a *vertex* of a curve $\alpha : \mathbb{R} \to \mathbb{R}^{1+k}$ is a point $t_0 \in \mathbb{R}$ for which there is a point $a \in \mathbb{R}^{1+k}$ such that the distance-squared function $d_a^\alpha : \mathbb{R} \to \mathbb{R}$, defined as $d_a^\alpha(t) = ||\alpha(t) - a||^2$, has a singularity of type $A_j, j \geq k+2$. The vertices of α are the points at which the focal hypersphere has higher contact with the curve [5].

Proposition 7.1. *Given a 3-manifold M immersed in $\mathbb{R}^{3+k}$, let $x \in M$ and $\theta \in T_xM$ be a principal direction at x. Denote by α_θ the normal section of M in the direction θ. Then α_θ is a curve in the $(k+1)$-dimensional subspace $V_\theta = \langle\theta\rangle \oplus N_xM \subset T_x\mathbb{R}^{3+k}$. Then we have that $x = \alpha_\theta(0)$ is a ridge of M if and only if x is a vertex of γ_θ (as a curve in the $(k+1)$-space V_θ).*

Proof. Observe that θ is the contact direction associated to the distance-squared function d_a at x, where a is the focal center corresponding to the principal direction θ at x. Then we have that x is a singularity of type A_j of $d_a|_{\alpha_\theta}$ if and only if it is a singularity of type A_j of the function d_a over M. Since the point $x = \alpha_\theta(0)$ is a vertex of α_θ if and only if it is a singularity of type $A_j, j \geq k+1$ for $d_a|_{\alpha_\theta}$ we obtain the required results. □

Consider now the inverse sterographic projection, $\varphi : \mathbb{R}^{2+k} \to S^{2+k}$. Given any 3-manifold M in $\mathbb{R}^{2+k}$ and a tangent hypersphere $S(a)$ at a point, centered at $a \in \mathbb{R}^{2+k}$, the map φ determines a diffeomorphism onto S^{2+k} (minus a point), that takes the pair $(M, S(a))$ onto a pair (M', S'), preserving their contact. We can consider M' as a 3-manifold in $\mathbb{R}^{3+k}$. Let H be the hyperplane determined by the hypersphere S' in $\mathbb{R}^{3+k}$. The contact functions of the pairs $(M, S(a))$ and (M', H) are $\mathcal{K}$-equivalent [25]. This implies that φ takes diffeomorphically the ridges of order r of M onto the flat ridges of order r of M' and the pre-inflections of M onto the pre-umbilics of M'. Consequently, we get the following results as corollaries of those obtained in the previous sections.

Corollary 7.1. *The ridge sets of a 3-manifold M generically embedded in $\mathbb{R}^{2+k}$, $k \geq 2$, form a surface with possible isolated cross-caps and transverse self-intersections. Inside this surface, we may have regular immersed curves corresponding to the $(k+2)$-order ridge set and isolated highest order ridge points.*

We also have the following characterization of the ridges of different orders in terms of the relative positions with respect to the curvature lines, which is a natural generalization of the corresponding property for surfaces in $\mathbb{R}^3$ (see [23]).

Proposition 7.2. *On a* 3*-manifold generically embedded in* $\mathbb{R}^{2+k}$*,* $k \geq 2$*, the ridges of order* $k+2$ *are the points at which the* $(k+1)$*-order ridge surface is tangent to some curvature line and the highest order ridges are the points at which the* $(k+2)$*-order ridge curve is tangent to some curvature line.*

We can summarize the above results as follows:

(a) *Generic* 3*-manifolds in* $\mathbb{R}^4$*:* The ridges form surfaces with isolated cross-caps and transverse self-intersections. Their intersection with the parabolic set are the flat ridges. These form regular curves.

(b) *Generic* 3*-manifolds in* $\mathbb{R}^5$*:* The flat ridges of order 3 form surfaces with isolated cross-caps and transverse self-intersections. The ridges of order 4 form surfaces with isolated cross-caps and transverse self-intersections. The corresponding rib points may go to infinity along regular curves, these are the flat ridges of order 4.

(c) *Generic* 3*-manifolds in* $\mathbb{R}^6$*:* The flat ridges (of order ≥ 4) form a surface F with isolated cross-caps and transverse self-intersections over which there are regular curves of flat ridges of order 4 and isolated ridges of order 6. The ridges of order 5 form surfaces with isolated cross-caps and transverse self-intersections that intersect F at the curves of flat ridges of order 4 (corresponding to rib points at infinity).

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