Sharp logarithmic Sobolev inequalities on gradient solitons and applications

José A. Carrillo and Lei Ni

We show that gradient shrinking, expanding or steady Ricci solitons have potentials leading to suitable reference probability measures on the manifold. For shrinking solitons, as well as expanding solitons with nonnegative Ricci curvature, these reference measures satisfy sharp logarithmic Sobolev inequalities with lower bounds characterized by the geometry of the manifold. The geometric invariant appearing in the sharp lower bound is shown to be nonnegative. We also characterize the expanders when such invariant is zero. In the proof, various useful volume growth estimates are also established for gradient shrinking and expanding solitons. In particular, we prove that the *asymptotic volume ratio* of any gradient shrinking soliton with nonnegative Ricci curvature must be zero.

1. Introduction

A complete Riemannian manifold (M, g) is called a gradient shrinking soliton (shrinker) if there exists a (smooth) function f, such that its Hessian f_{ij} satisfies

$$(1.1) \qquad R_{ij} + f_{ij} - \frac{1}{2}g_{ij} = 0.$$

Here R_{ij} denotes the Ricci curvature. As shown in Theorem 4.1 of [11], associated to the metric and the *potential function* f, there exists a family of metrics $g(\eta)$, a solution to Ricci flow

$$\frac{\partial}{\partial \eta}g(\eta) = -2\operatorname{Ric}(g(\eta)),$$

with the property that $g(0) = g$, the original metric and a family of diffeomorphisms $\phi(\eta)$, which is generated by the vector field $X = \frac{1}{\tau}\nabla f$, such that $\phi(0) = \mathrm{id}$ and $g(\eta) = \tau(\eta)\phi^*(\eta)g$ with $\tau(\eta) = 1 - \eta$, as well as $f(x, \eta) = \phi^*(\eta)f(x)$. Namely, there exists a self-similar (shrinking) family of metrics which is a solution to the Ricci flow. The metric $g(\eta)$ and $f(\eta)$, sometimes

also written as g^τ and f^τ, or simply g and f when the meaning is clear, satisfy that

$$(1.2) \qquad\qquad R_{ij} + f_{ij} - \frac{1}{2\tau} g_{ij} = 0.$$

We shall denote by $S(x)$ the scalar curvature and by $d\Gamma_\tau$ the volume element of g^τ.

Gradient shrinking solitons arise as the singularity models of Ricci flow. The more interesting cases are the noncompact ones. Trivial examples include the Euclidean space $\mathbb{R}^n$ and the cylinders $\mathbb{S}^k \times \mathbb{R}^{n-k}$ for $k \geq 2$. Nontrivial noncompact examples can be found in, for example, [18]. There is also a more recent construction of solitons with symmetry in [15]. The main result of this paper is the following theorem, which generalizes the sharp logarithmic Sobolev inequality (LSI) of the Euclidean space $\mathbb{R}^n$ [20]. (This result on $\mathbb{R}^n$ was referred as Stam–Gross LSI in [37, 38], where one can also find detailed historic accounts and more complete references.)

Theorem 1.1. *Assume that (M, g, f) is a gradient shrinking soliton, then:*

(i) *The potential e^{-f} is integrable on M and it can be normalized as*

$$(1.3) \qquad\qquad \frac{1}{(4\pi\tau)^{n/2}} \int_M e^{-f} d\Gamma_\tau = 1.$$

(ii) *LSI inequality: There exists a geometric invariant μ_s, under isometries, which depends only on the value of f and S at the minimum point of f and is independent of the parameter τ, such that*

$$\int_M \left\{ 4\tau |\nabla \sqrt{\rho}|_\tau^2 + \rho S(\cdot, \tau) - \rho \ln \rho - n\rho - \frac{n}{2} \ln(4\pi\tau)\rho \right\} d\Gamma_\tau \geq -\mu_s,$$

for any $\tau > 0$ and any nonnegative compactly supported smooth function ρ with unit integral on M. Moreover, the geometric invariant μ_s in the above inequality is sharp.

(iii) *If $|\operatorname{Rm}| \leq A$ for some $A > 0$, then $\mu_s \geq 0$.*

We refer to Section 2 for notations and the definition of the invariant μ_s. Let us observe that if we write $\rho = \frac{e^{-\psi}}{(4\pi\tau)^{n/2}}$, since wherever $\rho \to 0$ (hence $\psi \to -\infty$) we have $\rho\psi \to 0$, it is convenient and sensible to phrase the

LSI as

$$(1.4) \qquad \int_M \left\{ \tau \left[|\nabla \psi|_\tau^2 + S(\cdot, \tau) \right] + \psi - n \right\} \rho \, d\Gamma_\tau \geq -\mu_s.$$

Note also that for the Gaussian soliton, namely $(M, g, f) = (\mathbb{R}^n, g_{\text{can}}, \frac{1}{4}|x|^2)$, we get $\mu_s = 0$. The proof of the inequality uses the Bakry–Emery [2] criterion for LSI's, as obtained from the so-called HWI inequalities derived by Villani and coauthors in different settings [38], see Section 3 and references therein. Here, the main difficulty resides in showing the necessary bounds on the potential to normalize it as in (1.3) and thus, being able to apply these inequalities, which is done in Section 2. In Section 4 we also supply an alternate proof via the Fokker–Planck dynamics. We should point out that Perelman [34, Remark 3.2] has observed that when an LSI holds on a shrinking solitons, the sharp form can be justified using his entropy formula. This mainly applies to compact shrinkers since for the noncompact case, even a weak form of LSI is not known. For the compact shrinkers, our approach supplies a different argument. An immediate consequence of the theorem is the strong noncollapsing of the gradient shrinking solitons (in the scale proportional to $\sqrt{\tau}$). In the case that M has bounded nonnegative Ricci curvature the sharp LSI of Theorem 1.1 implies LSIs for all scales, not necessarily with sharp constants though, hence the noncollapsing at all scales. *Namely for any gradient shrinking soliton* (M, g, f), *if* $|\text{Ric}| \leq A$ *for some* A, *then there exists a* $\kappa > 0$ *such that for any ball* $B(x_0, r)$ *with the scalar curvature* $S(y) \leq \frac{1}{r^2}$ *for* $y \in B(x_0, r)$, *then the volume of* $B(x_0, s)$, $V(x_0, s) \geq \kappa s^n$ *for any* $s \leq r$. Related to this, there exists an earlier work of Naber [27]. Let us finally comment that the sharpness in the third statement of our main theorem means that taking the density ρ to be the normalized potential in the first statement, then the minimum μ_s is attained, see Section 4. We also remark that after establishing the invariance of μ_s (in terms of τ) in Section 2, the proof of LSI for $g(\tau)$ is equivalent to proving it for one fixed metric $g(1)$. This is what we shall do in Sections 3 and 4.

In our analysis of the gradient shrinking solitons, we also prove the following result.

Corollary 1.1. *Any nonflat gradient shrinking soliton with nonnegative Ricci curvature must have zero asymptotic volume ratio.*

This is done in Section 2. This result, in the case of gradient shrinking solitons, generalizes a previous result of Perelman [34] on ancient solutions with bounded nonnegative curvature operator. The result of Perelman [34, Proposition 11.4] draws the same conclusion for any ancient solutions with

bounded nonnegative curvature operator. Let us remark that Perelman also assumes the noncollapsing condition, which however is not needed in the proof. Gradient shrinking solitons are special ancient solutions. However our result is more general than that of Perelman for the following reasons: it cannot be derived from Perelman's since we assume nothing on the curvature operator only Ricci curvature must be nonnegative. On the other hand, the result of Perelman can be derived from the above result on gradient steady solitons by taking limits to the *asymptotic solitons*.

Gradient steady/expanding solitons (expanders) arise also in the singularity analysis of Ricci flow [21]. A gradient steady soliton is a triple (M, g, f) satisfying $R_{ij} = f_{ij}$ and a gradient expanding soliton is a triple (M, g, f) satisfying $R_{ij} + \frac{1}{2}g_{ij} = f_{ij}$. The technique employed here yields similar sharp geometric inequalities for gradient steady/expanding solitons as well. The following is the corresponding result for the gradient expanding solitons.

Theorem 1.2. *Assume that (M, g, f) is a gradient expanding soliton with* Ric ≥ 0. *Then for any nonnegative* $\rho(x) = \frac{e^{-\psi(x)}}{(4\pi)^{n/2}}$ *with* $\int_M \rho(x) \, d\Gamma(x) = 1$,

$$(1.5) \qquad \int_M \left(|\nabla \psi|^2 - 3S + \psi - n \right) \rho \, d\Gamma \geq -\mu_e.$$

Here μ_e is a geometric constant depending only on the value of f and S at the minimum point of f. The inequality is sharp for such μ_e. Moreover $\mu_e \geq 0$ with equality if and only if (M, g) is isometric to $\mathbb{R}^n$.

This is done in Section 5. As before the expression still makes sense even over the points where ρ vanishes. An equivalent expression of the integrand appeared in a recent interesting preprint of Cao and Hamilton [3] on pointwise differential estimates of Li–Yau–Hamilton type.

For expanding solitons, we also obtain a volume estimate, which generalizes a recent result of Hamilton [22], see also [11], asserting that *the asymptotic volume ratio of gradient expanding solitons with bounded positive Ricci curvature must be positive*. The following is one of our statements.

Corollary 1.2. *Assume that (M, g, f) is a gradient expanding soliton with $S(x) \geq -\beta$ for some constant $\beta \geq 0$. Then for any $o \in M$ and $r \geq r_0$*

$$V(o, r) \geq V(o, r_0) \left(\frac{r + a}{r_0 + a} \right)^{n - 2\beta}$$

with $a = 2\sqrt{f(o) + \mu_e + \beta}$.

The result of Hamilton mentioned previously follows from the above statement applied to the case $\beta = 0$. For general β, the growth rate in our estimate is sharp as shown by examples. This is proved in Section 5. A similar/independent result can also be found in a recent preprint [8].

For gradient steady solitons, since one cannot expect that the LSI to hold in general in view of the existence of Hamilton's "cigar" manifold, we obtain a sharp weighted Poincaré inequality instead. The proof is relatively easy, without appealing the above-mentioned theory involving the Bakry–Emery criterion, and is done in Section 6.

The part $\mu_s, \mu_e \geq 0$ of the main theorems is finally proved in Section 7. This is motivated by the Zamolodchikov's [41] c-theorem of the renormalization group flow. In view of the entropy monotonicity formula of Perelman, its connection with the LSI, as well as the fact that gradient shrinking solitons arise as the singularity models (at least for the cases that the blow-up has nonnegative curvature operator), this result can be viewed as an analog of Zamolodchikov's c-theorem for the renormalization group flow. The proof makes use a Li–Yau–Hamilton type inequality of Perelman [34] and the entropy formula/monotonicity for the linear heat equation of [29].

By the work of Dolbeault and Del Pino [16], also Toscani and the first author [7], the sharp form of Sobolev–Gagliardo–Nirenberg inequalities on $\mathbb{R}^n$ are related to the nonlinear Fokker–Planck dynamics of porous medium/fast diffusion type equations. It is interesting to find out if such relation can lead to sharp inequalities on solitons along a similar line of argument of this paper.

2. Geometric estimates on gradient shrinking solitons

We shall follow the notations of the introduction being our first objective to show the integrability on the potential function for solitons allowing for normalization (1.3). The following equations are simple consequences of the soliton equation (1.2):

$$(2.1) \qquad S + \Delta f - \frac{n}{2\tau} = 0,$$

$$(2.2) \qquad S + |\nabla f|^2 - \frac{f}{\tau} = \frac{\mu_s(\tau)}{\tau},$$

where $\mu_s(\tau)$ is a constant that will be eventually chosen by the normalization of the potential as in (1.3). Here S is the scalar curvature. See, for example [21] or [11], for a proof. Equations (2.1) and (2.2) imply that

$$(2.3) \qquad 2\Delta f - |\nabla f|^2 + S + \frac{f - n}{\tau} = -\frac{\mu_s(\tau)}{\tau}.$$

The lemma below implies that the integral involved in normalization (1.3), as well as other integrals involved later in the proof of Theorem 1.1, are finite.

Lemma 2.1. *Let $r(x)$ be the distance function to a fixed point $o \in M$ with respect to $g(\eta)$ metric. Then there exist constants C_1 and C_2 such that*

$$(2.4) \qquad f(x) \geq \frac{1}{4\tau}(r(x) - C_1)^2$$

and

$$(2.5) \qquad f(x) \leq \frac{1}{4\tau}(r(x) + C_1)^2, \qquad |\nabla f|(x) \leq \frac{1}{2\tau}(r(x) + C_1)$$

for $r(x) \geq C_2$.

Proof. First, we observe that $S \geq 0$ by a gradient estimate argument of Chen [10] (see also the appendix of [40]). Estimate (2.4) then follows verbatim from [17, pp. 655–656].

Now (2.2) and $S \geq 0$ imply that

$$(2.6) \qquad |\nabla f| \leq \sqrt{\frac{f + \mu_s(\tau)}{\tau}}.$$

The first estimate in (2.5) follows easily from this by integrating $f + \mu_s(\tau)$ along minimizing geodesics from o, see also the proof of Proposition 5.1. The second estimate in (2.5) follows from the first one via (2.6). $\square$

Corollary 2.1. *Let (M, g, f) be a gradient shrinking soliton. Then the minimum of f can be achieved somewhere, say $o \in M$. Hence $f(o)$ and $S(o)$ are fixed for different η and the constant $\mu_s(\tau)$ in (2.2) is a constant independent of τ. Therefore $\mu_s \doteq \mu_s(\tau)$ is an invariant of the soliton. Moreover,*

$$(2.7) \qquad \int_M \left(|\Delta f| + |\nabla f|^2 + |f| + S \right) e^{-f} \, d\Gamma_\tau < \infty.$$

Proof. The first part of the corollary is evident since o is the fixed point of $\phi(\eta)$. The change of S (from the shrinking) is compensated by the factor τ. For the second part, observe that a result of [39, p. 7] implies

$$e^{-f} J(r, \theta) \leq A_1 e^{a_2 r - \frac{1}{2\tau} r^2}$$

for some positive constants A_1, a_2 independent of r. Here $J(r, \theta)$ is the area element of the sphere $\partial B_o(r)$. Namely, $\text{Area}(\partial B_o(r)) = \int_{\mathbb{S}^{n-1}} J(r, \theta) \, d\theta$.

Notice that (2.1) and (2.2), along with (2.4) and (2.5) effectively imply that

$$(2.8) \qquad 0 \leq S \quad \leq \quad \frac{1}{4\tau^2}(r(x) + C_1)^2,$$

$$(2.9) \qquad -\frac{n}{2} \leq -\Delta f \quad \leq \quad \frac{1}{4\tau^2}(r(x) + C_1)^2.$$

Since $\int_M (\cdot)\, d\Gamma_\tau = \int_0^\infty \int_{\mathbb{S}^{n-1}} (\cdot) J(r,\theta)\, d\theta\, dr$, the finiteness of the integral in (2.7) follows easily. $\qquad\qquad\square$

Note that by Theorem 4.1 of [11], $\frac{e^{-f}}{(4\pi\tau)^{n/2}}$ satisfies the conjugate heat equation

$$\left(\frac{\partial}{\partial\tau} - \Delta + S \right) \left(\frac{e^{-f}}{(4\pi\tau)^{n/2}} \right) = 0.$$

Hence the total mass of e^{-f}, namely normalization (1.3) is preserved along the evolution. In other words, if

$$\int_M \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}}\, d\Gamma_\tau = 1$$

holds at $\tau = 1$ (which corresponds to $\eta = 0$), it holds for all $\tau > 0$. Also note that $\mu_s(M, g) = \mu_s(M', g')$ if (M, g) is isometric to (M', g') by the virtue of [27, Lemma 1.2]. Equivalently, the invariant $\mu_s(M, g)$ is independent of the choice of the potential function f since the difference of two potential functions is either a constant or a linear function, since they have the same Hessian. In the first case, the normalization makes the constant zero. For the second case, namely the difference of the two potential functions is a linear function, then the manifold M splits off a line. Some simple calculation also shows that the normalization would make the constants μ_s in (2.2) identical for the two different potential functions. In fact, if the difference of two potential functions $h \doteq f_1 - f_2$ (assuming $\tau = 1$ without the loss of the generality) is a linear function of $\mathbb{R}$ and $M = \mathbb{R} \times M_1$, using the soliton equation (2.1) one can write $f_k(x, y) = \frac{1}{4}x^2 + b_k x + c_k(y)$ for $k = 1, 2$. Here we denote the coordinate of $\mathbb{R}$ by x and the coordinate of M' by y. Since $h(x) = ax + b$ for constants a and b. Hence we have $c_1(y) - c_2(y) = c$. Now if $\int_M e^{-f_1} = \int_M e^{-f_2}$, by simple direct calculation we have that $b_1^2 = c + b_2^2$. Direct calculation shows that

$$(|\nabla f_1|^2 + S - f_1) - (|\nabla f_2|^2 + S - f_2) = b_1^2 - b_2^2 - (c_1(y) - c_2(y)) = 0.$$

The next result concerns the behavior of the volume $V(o, r)$ of balls $B(o, r)$, especially as $r \to \infty$. We start with the easier case of the Ricci curvature being bounded.

Corollary 2.2. *Let (M, g) be a nonflat gradient shrinking soliton with $\mathrm{Ric} \geq 0$. Assume further that the scalar curvature $S(x) \leq C_1$ for some $C_1 > 0$. Then, there exists a constant $\delta = \delta(M, f) > 0$ with the property that for any $o \in M$, there exists $a = a(M, f, C_1)$ such that for any $r \geq r_0 \geq a$*

$$(2.10) \qquad\qquad V(o, r) \leq V(o, r_0) \left(\frac{r - a}{r_0 - a} \right)^{n - \delta}.$$

Proof. Without the loss of generality we may assume that $\tau = 1$. By [30, Proposition 1.1], we have $\delta = \delta(M, f) > 0$ such that $S \geq \frac{\delta}{2}$. On the other hand, by Perelman [34, Section 8], see also the proof of [30, Proposition 1.1], for any minimizing geodesic joining o to $x = \gamma(s_0)$ with $s_0 \geq 2$ and $r_0 > 0$ such that $s_0 - r_0 \geq 1$, we have the estimate

$$(2.11) \qquad\qquad \int_0^{s_0 - r_0} \mathrm{Ric}(\gamma', \gamma') \, ds \leq C_4(M) + \frac{n - 1}{r_0}.$$

This implies, again by the argument in the proof of [30, Proposition 1], that

$$\frac{\partial f}{\partial r}(x) \geq \frac{r(x)}{2} - C_6(M, f, o).$$

Now integration by parts on equation (2.1) over $B(o, r)$ yields that

$$\begin{aligned}
\frac{n - \delta}{2} V(o, r) &\geq \int_{B(o,r)} \left(\frac{n}{2} - S \right) d\Gamma \\
&= \int_{\partial B(o,r)} \frac{\partial f}{\partial r} \, dA \\
&\geq A(o, r) \left(\frac{r}{2} - C_6 \right).
\end{aligned}$$

Here $A(o, r)$ is the surface area of $\partial B(o, r)$. The result follows from integrating the above estimate on $[r_0, r]$. $\qquad\square$

Remark 2.1. Being Corollary 2.2 proved under no restriction on the boundedness of the Ricci curvature, it might be used to prove [34, Proposition 11.4]. This result concludes that any nonflat ancient solution of Ricci flow with bounded nonnegative curvature operator must have the asymptotic

volume ratio $\lim_{r\to\infty} \frac{V(o,r)}{r^n} = 0$. In fact one can derive [34, Proposition 11.4] by contradiction: Assume the claim is false, one obtains an asymptotic soliton by [34, Proposition 11.2] as a blow-down limit, which is nonflat and that has the maximum volume growth. This is a contradiction with estimate (2.10).

With some extra effort, we can indeed prove such desired volume estimate without assuming the Ricci curvature upper bound.

Proposition 2.1. *Let (M, g, f) be a nonflat gradient shrinking soliton with $\mathrm{Ric} \geq 0$. Then*

$$\lim_{r\to\infty} \frac{V(o,r)}{r^n} = 0.$$

Here $V(o,r)$ is the volume of $B(o,r)$.

Proof. We can reduce ourselves to the case $\tau = 1$ without loss of generality. For simplicity, after translation we may assume that the potential function f satisfies $|\nabla f|^2 - S = f$. It is also more convenient to work with sub-level sets of f. Let us consider the sets

$$F_r \doteqdot \{x \in M \mid 2\sqrt{f(x)} \leq r\}$$

and $\widetilde{V}(r) = \mathrm{Vol}(F_r)$. Assume that the conclusion is not true, then $\liminf_{r\to\infty} r^{-n} V(o,r) \geq \eta > 0$ for some $\eta > 0$. This clearly implies that $\liminf_{r\to\infty} r^{-n} \widetilde{V}(r) \geq \eta' > 0$.

On the other hand, following [5, 26] we consider the function

$$\chi(r) = \int_{F_r} S \, d\Gamma.$$

Using $S \geq \delta > 0$ for some $\delta > 0$, which is ensured by Ni [30, Proposition 1.1], we conclude that for any small $\epsilon > 0$

$$
\begin{aligned}
(n - \delta')\widetilde{V}(r) + 2(1 - \epsilon)\chi(r) &= \int_{F_r} 2\left(\frac{n}{2} - \epsilon\delta + (1 - \epsilon)S\right) d\Gamma \\
&\geq \int_{F_r} 2(\frac{n}{2} - S) \, d\Gamma \\
&= \int_{F_r} 2\Delta f \, d\Gamma \\
&= r\widetilde{V}'(r) - \frac{4\chi'(r)}{r}.
\end{aligned}
$$

Here $\delta' = 2\epsilon\delta$ and in the last line we used the computation in equation (5) of [26]. Integrating the above estimate as in [5, 26], we arrive at

$$\frac{\widetilde{V}(r)}{r^{n-\delta'}} - \frac{\widetilde{V}(r_0)}{r_0^{n-\delta'}} \leq \frac{4\chi(r)}{r^{n-\delta'+2}}$$

for $r \geq r_0 \geq 8\sqrt{n+2}$. Now using that $2\chi(r) \leq n\widetilde{V}(r)$, we deduce that the right-hand side above tends to zero as $r \to \infty$. This induces that $\limsup_{r\to\infty} r^{-n}\widetilde{V}(r) = 0$, which is a contradiction. $\qquad\square$

We should remark that there exists a proof to Perelman's result by Hamilton via his singularity analysis of ancient solutions. The interested reader can find the details of Hamilton's argument in [11]. It is interesting to find out whether or not Proposition 2.1 can be generalized to ancient solutions with nonnegative Ricci curvature.

3. Optimal transport and LSIs

In this section, we will work with Riemannian manifolds (M, g) endowed with a reference probability measure $\mathrm{e}^{-V} d\Gamma$ where the potential $V \in C^2(M)$ verifies a curvature-dimension bound of the type $C(K, \infty)$ with $K \in \mathbb{R}$, i.e.,

$$R_{ij} + V_{ij} \geq Kg_{ij}.$$

Here $d\Gamma$ is the volume measure associated to (M, g). This section is devoted to collect several results presented in the literature [38]. Let us assume that the reference measure is normalized by

$$\int_M \mathrm{e}^{-V} d\Gamma = 1.$$

For any positive function ρ with $\int_M \rho \, d\Gamma = 1$ let $\xi = \log \rho + V$.

Let us define *the Boltzmann relative entropy functional*, called also Nash entropy, as

$$H_V(\rho) \doteq \int_M \rho\xi \, d\Gamma$$

and define the *relative Fisher information* as

$$I_V(\rho) \doteq \int_M |\nabla\xi|^2 \rho \, d\Gamma.$$

Related to these functionals, there is another quantity that is involved in these inequalities: *the Euclidean Wasserstein distance* between any two probability measures ν_0, ν_1 on the manifold M, i.e.,

$$(3.1) \qquad W_2(\nu_0, \nu_1) \doteq \inf \left\{ \int_{M \times M} r^2(x, y) \, d\theta(x, y); \ \theta \in \Theta(\nu_0, \nu_1) \right\}^{1/2},$$

where $\Theta(\nu_0, \nu_1)$ is the set of probability measures on $M \times M$ having marginals ν_0 and ν_1, $r(x, y)$ is the Riemannian distance between x and y. This distance is well defined for probability measures ν_0 and ν_1 with second moment bounded, $\mathcal{P}_2(M)$, and metrizes the weak convergence of measures in the sense of [38, Definition 6.7, Theorem 6.8]. The expression "second moment bounded" refers to the fact that the squared distance function $r^2(x)$ is integrable against the measures ν_0 and ν_1. It is worth to mention that the curvature-dimension bound $C(K, \infty)$ with $K > 0$ implies that the second moment of the reference measure (actually, all moments) e^{-V} is bounded, see [38, Theorem 18.11].

Recently, several authors [24, 35] based on early works [14, 25], see [38, Chapter 17] for a whole account of the history, have characterized curvature-dimension bounds in terms of the displacement convexity of the Boltzmann relative entropy functional. The notion of displacement convexity refers to convexity along paths of minimal transport distance W_2 in the set of probability measures $\mathcal{P}_2(M)$. An expression of the convexity of these functionals is the so-called HWI inequalities, named in this way since they involved three functionals $H_V(\rho)$, $I_V(\rho)$ and W_2. In the following, we will work with measures absolutely continuous against volume measure and we identify the measures with their densities for notational convenience. The main results we need are the following:

Theorem 3.1 ([38, Corollary 20.13]; [2]). *Let M be a Riemannian manifold equipped with a reference measure $\mathrm{e}^{-V} d\Gamma$ where the potential $V \in C^2(M)$ verifies a curvature-dimension bound of the type $C(K, \infty)$ with $K \in \mathbb{R}$. Then, for any given $\nu \in \mathcal{P}_2(M)$ absolutely continuous with respect to volume measure $d\Gamma$ with density ρ, it holds the HWI inequality:*

$$H_V(\rho) \leq W_2(\rho, \mathrm{e}^{-V}) \sqrt{I_V(\rho)} - \frac{K}{2} W_2(\rho, \mathrm{e}^{-V})^2.$$

As a consequence, we know that whenever $K > 0$, the LSI is follows:

$$H_V(\rho) \leq \frac{1}{2K} I_V(\rho).$$

The HWI inequalities were originally introduced in [33] and used in other models in nonlinear PDEs in [6]. Later, they were generalized to compact manifolds in [24] and in this generality in [38]. To see that the LSI inequality follows from the HWI inequality, it suffices to consider the right-hand side of the HWI inequality as a function of W_2 and maximize that function.

Consider the positive solution ρ to the Fokker–Planck equation

$$(3.2) \qquad \frac{\partial \rho}{\partial t} - \operatorname{div}(\rho \nabla(\log \rho + V)) = 0$$

with initial value $\rho(0)$. Recall that $\xi = \log \rho + V$. It is easy to see that

$$\left(\frac{\partial}{\partial t} - \Delta\right)\xi = \langle \nabla \xi, \nabla \log \rho \rangle.$$

We have immediately the following dissipation of the Boltzmann relative entropy functional,

$$(3.3) \qquad \frac{d}{dt} H_V(\rho(t)) = -\int_M |\nabla \xi|^2 \rho \, d\Gamma = -I_V(\rho(t)),$$

where computations are made for smooth, fast-decaying at infinity for non-compact manifolds, solutions on M. This computation shows us that these two quantities, the relative Boltzmann entropy $H_V(\rho)$ and the relative Fisher information $I_V(\rho)$, are intimately related at least for solutions of (3.2). However, as it was discovered in the case of $\mathbb{R}^n$, and in the case of a manifold in [1, 2] for linear diffusions or in [7, 16, 32] for nonlinear diffusions, this relation is really through functional inequalities, see also [33].

Let us remark that some proofs of the LSI inequality use the Fokker–Planck dynamics (3.2), called the Bakry–Emery strategy, but the referred functional proof through the HWI inequalities allows us to overcome discussions on integrability issues and the decay at infinity for noncompact manifolds of solutions to (3.2). In fact, a direct application of the LSI on (3.3) gives the exponential decay of the Boltzmann relative entropy functional for solutions of (3.2) with initial density in $\mathcal{P}_2(M)$ in case $C(K, \infty)$ with $K > 0$ holds, i.e., given a solution $\rho(t)$ of (3.2) then

$$H_V(\rho(t)) \leq H_V(\rho(0)) \, \mathrm{e}^{-2Kt} \qquad \text{for all } t \geq 0.$$

4. Main result and applications

Now, let us come back to the precise situation we have, the case of a shrinking soliton, and prove the main Theorem 1.1. We shall prove it for $g(1)$,

namely $\tau = 1$. Let us define the potential $V = f + \frac{n}{2}\log(4\pi)$ for the fixed time slice of the shrinking soliton (M, g) at time $\tau = 1$. Lemma 2.1 and Corollary 2.1 imply that e^{-V} is a well-defined probability measure. Moreover, we deduce from the soliton definition (1.2) that this reference measure verifies the $C(\frac{1}{2}, \infty)$ condition. Therefore, Theorem 3.1 implies that for any probability density of the form

$$\rho(x) = \frac{e^{-\psi(x)}}{(4\pi)^{\frac{n}{2}}}$$

with second moment bounded, we get the LSI

$$H_V(\rho) \leq I_V(\rho).$$

Using now the soliton equation (2.3), we deduce

$$\begin{aligned}
I_V(\rho) &= \int_M (|\nabla\psi|^2\rho + 2\langle\nabla f, \nabla\rho\rangle + |\nabla f|^2\rho)\, d\Gamma \\
&= \int_M [|\nabla\psi|^2\rho + (-2\Delta f + |\nabla f|^2)\rho]\, d\Gamma \\
&= \int_M [|\nabla\psi|^2 + S + f + \mu_s - n]\rho\, d\Gamma.
\end{aligned}$$

Thus, the LSI inequality is equivalent to

(4.1) $$\int_M \left[|\nabla\psi|^2 + S + \psi - n\right]\rho\, d\Gamma \geq -\mu_s,$$

for all densities ρ with bounded second moment for the shrinking soliton, with μ_s characterized by Corollary 2.1.

Nevertheless, let us remind the reader that assuming all integrability and behavior at the (spatial) infinity are met for all integration by parts below, we can obtain the evolution of the relative Fisher information (see also [1, 2, 37, 38] for these computations) and obtain a direct proof of LSI via the Fokker–Planck dynamics started last section without appealing Theorem 3.1. Given $\rho(x)$ consider the solution $\rho(x, t)$ to the Fokker–Planck equation (3.2) with initial value $\rho(x, 0) = \rho(x)$. Note that the Laplacian and covariant differentiation are with respect to a fixed Riemannian metric, namely $g(1)$ (hence the parameter t has nothing to do with the Ricci flow). To take the

time derivative of $I_V(\rho(t))$ note the Bochner-type formula

$$(4.2) \qquad \left(\frac{\partial}{\partial t} - \Delta\right)|\nabla\xi|^2 = -2\xi_{ij}^2 + 2\langle\nabla(\langle\nabla\xi, \nabla\log\rho\rangle), \nabla\xi\rangle - 2R_{ij}\xi_i\xi_j.$$

Using the above formula, we arrive at

$$\frac{d}{dt}I_V(\rho(t)) = \int_M (\Delta|\nabla\xi|^2)\rho + |\nabla\xi|^2 \operatorname{div}(\nabla\rho + \rho\nabla V)\, d\Gamma$$
$$+ \int_M \left(-2\xi_{ij}^2 + 2\langle\nabla(\langle\nabla\xi, \nabla\log\rho\rangle), \nabla\xi\rangle - 2R_{ij}\xi_i\xi_j\right)\rho\, d\Gamma.$$

Since

$$\int_M \langle\nabla(\langle\nabla\xi, \nabla\log\rho\rangle), \nabla\xi\rangle\rho\, d\Gamma = \int_M \langle\nabla(|\nabla\xi|^2 - \langle\nabla V, \nabla\xi\rangle), \nabla\xi\rangle\rho\, d\Gamma$$
$$= \int_M \langle\langle\nabla|\nabla\xi|^2, \nabla\rho\rangle + \langle\nabla|\nabla\xi|^2, \nabla V\rangle\rho\, d\Gamma$$
$$- \int_M \langle\nabla\langle\nabla V, \nabla\xi\rangle, \nabla\xi\rangle\rho\, d\Gamma$$

we arrive at

$$\frac{d}{dt}I_V(\rho(t)) = \int_M (-2\xi_{ij}^2 - 2R_{ij}\xi_i\xi_j)\rho\, d\Gamma$$
$$+ \int_M \langle\nabla|\nabla\xi|^2, \nabla V\rangle\rho - 2\langle\nabla\langle\nabla V, \nabla\xi\rangle, \nabla\xi\rangle\rho\, d\Gamma$$
$$(4.3) \qquad\qquad = \int_M (-2\xi_{ij}^2 - 2(R_{ij} + V_{ij})\xi_i\xi_j)\rho\, d\Gamma.$$

As a consequence, due to the curvature-dimension bound $C(K, \infty)$, we have

$$\frac{d}{dt}I_V(\rho(t)) \le -\tau \int_M |\nabla\xi|^2\rho\, d\Gamma,$$

and thus

$$I_V(\rho(t)) \le I_V(\rho(0))\, e^{-\tau t} \qquad \text{for all } t \ge 0.$$

Integrating the above estimate on $[0, \infty)$ and noting that $\lim_{t\to\infty} H_V(\rho(t)) = 0$, we deduce that $H_V(\rho(0)) \le I_V(\rho(0))$. As above, this proves (1.4) for $\tau = 1$. The general version of (1.4) follows from scaling.

Now recall Perelman's entropy functional

$$\mathcal{W}(g^\tau, u, \tau) \doteq \int_M [\tau(|\nabla\psi|^2 + S) + \psi - n]u\, d\Gamma_\tau$$

is defined for $u = \frac{e^{-\psi}}{(4\pi\tau)^{n/2}}$ with $\int_M u\, d\Gamma_\tau = 1$. Theorem 1.1 implies that for (M, g^τ), $W(g^\tau, u, \tau) \geq -\mu_s$. Namely Perelman's μ-invariant

$$\mu(g^\tau, \tau) \doteq \inf_{\int_M u = 1} \mathcal{W}(g^\tau, u, \tau)$$

is bounded from below by $-\mu_s$. From (2.3) it is easy to see that

$$\tau(2\Delta f - |\nabla f|^2 + S) + f - n = -\mu_s.$$

Hence $u = \frac{e^{-f}}{(4\pi\tau)^{n/2}}$ is the minimizer for Perelman's $\mu(g, \tau)$, cf. [34, Remark 3.2]. This shows that the inequality of Theorem 1.1 is sharp. Summarizing, we have proved the following result.

Corollary 4.1. *Let (M, g, f) be a gradient shrinking soliton satisfying (1.1). Then*

$$\mu(g, 1) = -\mu_s.$$

Remark 4.1. When $f = $ constant, (M, g) is a Einstein manifold with $\mathrm{Ric}_M = \frac{1}{2}g_M$. In this case we obtain a log-Sobolev inequality for $S = \frac{n}{2}$ and

$$\mu_s = \frac{n}{2} - \log(V(M)) + \frac{n}{2}\log(4\pi),$$

where $V(M)$ is the volume of (M, g_M). The μ-invariant was computed in [4] for many examples of four manifolds.

When $M = \mathbb{R}^n$ with $f = \frac{1}{4}|x|^2$, direct calculation shows that $\mu_s = 0$. Hence the classical LSI of Stam–Gross is a special case.

Recall here that a solution of Ricci flow is called κ noncollapsed, if for any (x_0, t_0) and $r \geq 0$, such that on $P(x_0, t_0, r) = B_{g(t_0)}(x_0, r) \times [t_0 - r^2, t_0]$, $|Rm|(x, t) \leq r^{-2}$, then $V_{g(t_0)}(x_0, r) \geq \kappa r^n$. Here $V_{g(t_0)}(x_0, r)$ is the volume of $B_{g(t_0)}(x_0, r)$ with respect to $g(t_0)$. Perelman [34, Theorem 1.1] implies the following volume noncollapsing result for gradient shrinking solitons.

Corollary 4.2. *Let (M, g, f) be a gradient shrinking soliton satisfying (1.1). Then there exists a $\kappa = \kappa(\mu_s) > 0$ such that if in a ball $B(x_0, 1)$, $|\mathrm{Ric}| \leq 1$, then $V(x_0, 1) \geq \kappa$. In particular, if the Ricci curvature is bounded on M which is noncompact, then M has at least linear volume growth.*

Proof. Follows from Theorem 1.1 and Section 4 of [34]. See also [12, 36]. We thank Zhenlei Zhang for pointing out to us the consequence of the linear volume growth after the first version of this paper in June 2008. $\square$

In [27] there is a related result asserting the κ-noncollapsing of gradient shrinking solitons with bounded curvature, in the sense defined right above the corollary. The conclusion in the above corollary appears stronger since it only requires global lower bound on the scalar curvature and the local bound of the Ricci curvature over the ball, for a fixed time-slice only.

When $\mathrm{Ric}(M, g_{\tau=1}) \geq 0$ and is bounded, one can derive the LSI for all scales. This is done in the following two propositions.

Proposition 4.1 (Scale > 1). *Let (M, g) be a gradient shrinking soliton satisfying (1.1). Assume that $\mathrm{Ric} \geq 0$. Then, there exists positive $\delta = \delta(M) < 1$ such that for any $\sigma > 1$,*

$$\int_M [\sigma(|\nabla\tilde{\psi}|^2 + S) + \tilde{\psi} - n]\frac{e^{-\tilde{\psi}}}{(4\pi\sigma)^{\frac{n}{2}}}\, d\Gamma \geq -\mu_s + \frac{n}{2} - \delta - \frac{n}{2}\log\left(\frac{n}{2\delta}\right)$$

for any $\tilde{\psi}$ satisfying that $\int_M e^{-\tilde{\psi}}/(4\pi\sigma)^{\frac{n}{2}}\, d\Gamma = 1$.

Proof. Clearly, only the nonflat case worths the proof (since the flat one is isometric to $\mathbb{R}^n$). By Ni [30, Proposition 1.1], for a nonflat gradient shrinking soliton, there exists $\delta = \delta(M, f) > 0$ such that $S(x) \geq \delta$ for any $x \in M$. Let $\psi = \tilde{\psi} + \frac{n}{2}\log\sigma$. Then it is easy to see that

$$\int_M [\sigma(|\nabla\tilde{\psi}|^2 + S) + \tilde{\psi} - n]\frac{e^{-\tilde{\psi}}}{(4\pi\sigma)^{\frac{n}{2}}}\, d\Gamma$$
$$= \int_M (|\nabla\psi|^2 + S + \psi - n)\frac{e^{-\psi}}{(4\pi)^{\frac{n}{2}}}\, d\Gamma + (\sigma - 1)\int_M (|\nabla\psi|^2 + S)\frac{e^{-\psi}}{(4\pi)^{\frac{n}{2}}}\, d\Gamma$$
$$- \frac{n}{2}\log\sigma$$
$$\geq -\mu_s + \delta(\sigma - 1) - \frac{n}{2}\log\sigma,$$

where we have used Theorem 1.1 in the last estimate. Since $\delta(\sigma - 1) - \frac{n}{2}\log\sigma \geq \frac{n}{2} - \delta - \frac{n}{2}\log(\frac{n}{2\delta})$, the claimed result follows. $\square$

From the proof, the following corollary is evident, observing that $S \geq 0$ for shrinking solitons, which is clear from [10], see also the appendix of [40].

Corollary 4.3. *Let (M, g) be a gradient shrinking soliton satisfying* (1.1). *Then for any $\sigma > 1$,*

$$\int_M [\sigma(|\nabla\tilde\psi|^2 + S) + \tilde\psi - n]\frac{\mathrm{e}^{-\tilde\psi}}{(4\pi\sigma)^{\frac{n}{2}}}\, d\Gamma \geq -\mu_s - \frac{n}{2}\log\sigma.$$

Proposition 4.2 (Scale < 1). *Assume that $0 \leq \mathrm{Ric} \leq A$. Then for any $0 < \sigma \leq 1$,*

$$\int_M [\sigma(|\nabla\tilde\psi|^2 + S) + \tilde\psi - n]\frac{\mathrm{e}^{-\tilde\psi}}{(4\pi\sigma)^{\frac{n}{2}}}\, d\Gamma \geq -\mu_s - nA$$

for any $\tilde\psi$ satisfying that $\int_M \mathrm{e}^{-\tilde\psi}/(4\pi\sigma)^{\frac{n}{2}}\, d\Gamma = 1$.

Proof. Define

$$\mu_0(g, \sigma) \doteq \inf_{\int_M u_0 = 1} \int_M (\sigma|\nabla\tilde\psi|^2 + \tilde\psi - n)u_0\, d\Gamma$$

with $u_0 = \mathrm{e}^{-\tilde\psi}/(4\pi\sigma)^{\frac{n}{2}}$. Theorem 1.1 implies that $\mu_0(g, 1) \geq -\mu_s - nA$. Now for any u_0 which is compactly supported, let $u(x, t)$ be the heat equation solution with $u(x, 0) = u_0$. Then by the entropy monotonicity result in [29], for $\sigma \leq 1$,

$$\int_M (\sigma|\nabla\tilde\psi|^2 + \tilde\psi - n)\frac{\mathrm{e}^{-\tilde\psi}}{(4\pi\sigma)^{\frac{n}{2}}} \geq \int_M (|\nabla\varphi|^2 + \varphi - n)u(y, 1 - \sigma)\, d\Gamma(y)$$
$$\geq \mu_0(g, 1),$$

where $u(y, 1 - \sigma) = \mathrm{e}^{-\varphi(y)}/(4\pi(1 - \sigma))^{\frac{n}{2}}$. This implies the claimed result. $\qquad\square$

The above two propositions imply that Perelman's $\nu(g)$ invariant (see also Section 7 for the definition) satisfies $\nu(g) > -\infty$, hence the strong κ-noncollapsing result for gradient shrinking solitons with bounded and nonnegative Ricci curvature as in [34] (see also [12] and [36]). For the general case without assuming $\mathrm{Ric} \geq 0$, one can still obtain a logarithmic Sobolev for scales less than one, see Section 7.

5. Expanding solitons

Recall that (M, g) is called a gradient expanding soliton if there exists f such that

$$(5.1) \qquad R_{ij} + \frac{1}{2}g_{ij} = f_{ij}.$$

It is easy to show that

$$(5.2) \qquad \Delta f = S + \frac{n}{2},$$

$$(5.3) \qquad S + |\nabla f|^2 - f = \mu_e$$

for some constant μ_e. As before we will eventually choose μ_e by the normalizing condition $\int_M e^{-f}/(4\pi)^{n/2} \, d\Gamma = 1$. This will make μ_e a geometric invariant of (M, g).

Our first concern is about the behavior of the volume of balls $B(o, r)$ in M for any given $o \in M$. Along this direction, Hamilton [22] proved the following result:

Theorem 5.1. *Let (M, g) be a gradient expanding soliton has bounded nonnegative Ricci curvature. Then (M, g) has maximum volume growth. Namely*

$$\liminf_{r \to \infty} \frac{V(o, r)}{r^n} > 0.$$

For the exposition of this result please see [11, Proposition 9.46]. Let us remark that the assumption of uniform boundedness of the Ricci curvature is used in the proof to bound $\int_\gamma \text{Ric}(\gamma' \gamma')$ as in Section 2. Here, the limit always exists due to the Bishop–Gromov volume comparison. The limit of the quotient is called the *asymptotic volume ratio*. This compares sharply with Proposition 2.1 for gradient shrinking solitons. The result below is a generalization of the above result of Hamilton.

Proposition 5.1. *Let (M, g, f) be an gradient expanding soliton.*
(1) If $S(x) \geq 0$ for all $x \in M$, without assuming any curvature bound, then for any $o \in M$, $r \geq r_0$.

$$V(o, r) \geq V(o, r_0) \left(\frac{r + a}{r_0 + a} \right)^n$$

with $a = 2\sqrt{f(o) + \mu_e}$.

(2) *Assume that $S(x) \geq -\beta$ for some constant $\beta > 0$. Then for any $o \in M$ and $r \geq r_0$,*

$$V(o, r) \geq V(o, r_0) \left(\frac{r + a}{r_0 + a} \right)^{n - 2\beta}$$

with $a = 2\sqrt{f(o) + \mu_e + \beta}$.

Proof. In case (1), from the assumption and (5.3) we have that $f + \mu_e \geq 0$. Consider any minimizing geodesic $\gamma(s)$ from $o \in M$ a fixed point of M. Then (5.3) implies that for any s

$$\left| \frac{d}{ds} f(\gamma(s)) \right|^2 \leq f + \mu_e.$$

This implies, by the ODE comparison, that

$$(2\sqrt{f + \mu_e})(\gamma(s)) \leq s + a,$$

where $a = 2\sqrt{f(o) + \mu_e}$, which then implies that

(5.4) $$\left| \frac{\partial f}{\partial r} \right| (\gamma(s)) \leq \frac{s}{2} + \frac{a}{2}.$$

Now we integrate (5.2) on $B(o, r)$ and obtain that

$$\frac{n}{2} V(o, r) \leq \frac{n}{2} V(o, r) + \int_{B(o,r)} S \, d\Gamma$$
$$= \int_{B(o,r)} \Delta f \, d\Gamma$$
$$\leq \int_{\partial B(o,r)} \left| \frac{\partial f}{\partial r} \right| (y) \, dA(y).$$

Using (5.4) we conclude

$$\frac{n}{2} V(o, r) \leq A(o, r) \left(\frac{r}{2} + \frac{a}{2} \right).$$

The result follows by dividing the both sides of the above by $V(o, r)$ and then integrating the resulting estimate on the interval $[r_0, r]$. The proof for case (2) is similar. $\qquad \square$

Remark 5.1. The estimates in both cases have the sharp power. To see this consider $M = N^k \times \mathbb{R}^{n-k}$ where N is a compact Einstein manifold with $\text{Ric}_N = -\frac{1}{2} g_N$, $\mathbb{R}^{n-k}$ is the Gaussian expanding soliton.

Now we derive the LSI for the expanders. To make sure that the integral $\int_M e^{-f} \, d\Gamma$ is finite we have to make an assumption that there exists some $\epsilon > 0$,

(5.5) $$f_{ij} = \frac{1}{2} g_{ij} + R_{ij} \geq \epsilon g_{ij}.$$

Under this assumption, it is easy to see that

$$f(x) \geq \frac{\epsilon}{4} r^2(x) - C$$

for some $C = C(M, f)$. Since $R_{ij} \geq -\frac{1}{2} g_{ij}$, the volume $V(o, r) \leq \exp(A(r + 1))$ for some $A = A(n)$. This together with the lower estimate above ensures that the integral $\int_M e^{-f} \, d\Gamma$ is finite, see also [39]. Notice that under our assumption (5.5), as in the proof of Proposition 5.1,

$$f(x) \leq \left(\frac{r(x)}{2} + b \right)^2$$

for some $b = b(M, f)$. This ensures the finiteness of the integral

$$\int_M (|\nabla f|^2 + |\Delta f| + |S|) \frac{e^{-f}}{(4\pi)^{n/2}} \, d\Gamma.$$

Note that (5.2) and (5.3) implies that

(5.6) $$2\Delta f - |\nabla f|^2 - 3S + f - n = -\mu_e.$$

Integrating (5.6), we arrive at

$$\int_M (|\nabla f|^2 - 3S + f - n) \frac{e^{-f}}{(4\pi)^{n/2}} \, d\Gamma = -\mu_e.$$

It is clear that assumption (5.5) is trivially satisfied for the case that M has nonnegative Ricci curvature.

Assume in the rest of this section that $\mathrm{Ric} \geq 0$, let us define the potential $V = f - \frac{n}{2} \log(4\pi)$. Previous arguments imply that the reference measure e^{-V} is a well-defined probability measure. Moreover, we deduce from the soliton definition (5.1) and being $\mathrm{Ric} \geq 0$ that this reference measure verifies the $C(\frac{1}{2}, \infty)$ condition. Therefore, Theorem 3.1, together with a similar calculation as before, implies the following LSI inequality.

Theorem 5.2. *Assume that (M, g, f) is gradient expanding soliton with* Ric ≥ 0. *Then for any $\rho(x) = e^{-\psi(x)}/(4\pi)^{n/2}$ with $\int_M \rho(x) \, d\Gamma(x) = 1$, we have that*

$$(5.7) \qquad \int_M \left(|\nabla \psi|^2 - 3S + \psi - n \right) \rho \, d\Gamma \geq -\mu_e.$$

Here μ_e, as before, is a geometric invariant (in the sense of Section 2), which is the same for two isometric metrics. One can write in the dynamic form by considering the family of metrics $g(\tau)$ (in this case with $g(1)$ being the original metric, and $0 < \tau < \infty$) generated by the diffeomorphisms, as for the shrinking solitons case described in the introduction. Since it is the same inequality by re-scaling we omit its full statement. Note that in the left-hand side of (5.7) an equivalent integrand is

$$\tau(2\Delta\psi - |\nabla\psi|^2 - 3S) + \psi - n.$$

This expression also showed itself up in a differential Harnack or Li–Yau–Hamilton type calculation, in a recent preprint of Cao and Hamilton [3], where however the nonnegativity of the curvature operator is required. We think it interesting to explore the connections between the LSI and the Li–Yau–Hamilton type estimates for Ricci flow solutions. The discussion before Theorem 5.2 also yields the following useful result.

Proposition 5.2. *Let (M, g) be an expanding soliton as in Theorem 5.2. Then M is diffeomorphic to $\mathbb{R}^n$.*

Proof. First it is easy to see that M is of finite topological type. This follows from the observation that f is a proper function and has no critical point outside a compact subset [17], since for any $x \in M$ and $\gamma(s)$ a minimizing geodesic jointing $o \in M$, a fixed point, to x, with $f(\gamma(0)) = o$ and $f(\gamma(s_0)) = x$

$$f'(\gamma(s_0)) = f'(\gamma(0)) + \int_0^{s_0} f''(\gamma(s)) \, ds$$
$$\geq \frac{s_0}{2} + f'(\gamma(0)).$$

The conclusion follows from the uniqueness of the critical point along with the strict convexity of f. $\qquad\square$

6. Gradient steady solitons

Now we consider the gradient steady solitons. Recall that a gradient steady soliton (M, g) has a potential function f satisfying that

$$(6.1) \qquad\qquad\qquad\qquad R_{ij} = f_{ij}.$$

It was shown in [21] that

$$(6.2) \qquad\qquad\qquad\qquad |\nabla f|^2 + S = \lambda$$

for some λ. Similar as before there is a solution to Ricci flow $g(\tau)$ associated with the gradient steady soliton (M, g, f) [11]. We first need the following lemma to ensure the finiteness of $\int_M e^{-f} \, d\Gamma$ and other integrals later involved, under some geometric assumptions.

Lemma 6.1. *Let (M, g, f) be a gradient steady soliton. Assume that there exists a point $o \in M$ such that $S(o) = \max_M S$ and either $\mathrm{Ric}(x) > 0$ for all $x \in M$, or $\mathrm{Ric} \geq 0$ and*

$$\limsup_{x \to \infty} S(x) < \max_M S.$$

Then o is a minimum of f and there exists $\delta > 0$ and $C = C(M, f)$ so that

$$(6.3) \qquad\qquad\qquad\qquad f(x) \geq \delta r(x) - C.$$

Here $r(x)$ is the distance function to o. In particular, M is diffeomorphic to $\mathbb{R}^n$ in the case $\mathrm{Ric} > 0$ and of finite topological type in the case $\mathrm{Ric} \geq 0$.

Proof. For the first case, it was shown in [21, Theorem 20.1] that o is the unique minimum of f. Note that the argument there actually requires $\mathrm{Ric} > 0$ even though it was not stated; it is also necessary, as shown by easy examples. Note that for any geodesic $\gamma(s)$ from o, we have that

$$\frac{d^2}{ds^2}(f(\gamma(s)) = \mathrm{Ric}(\gamma', \gamma') > 0.$$

Hence we have for any $s_0 > 0$, $\frac{d}{ds}(f(\gamma(s_0))) > 0$. Then $f(\gamma(s)) \geq \frac{d}{ds}(f(\gamma(s_0)))(s - s_0) + f(\gamma(s_0))$, which implies the desired lower estimate.

For the second case, the assumption already excludes the Ricci flat situation. We first claim that under the assumption on the behavior of S at

the infinity, $S(o) = \lambda$. Suppose it is not true, then $\max_M S < \lambda$ and $|\nabla f|^2 \geq \lambda - \max_M S$. Let $\sigma(u)$ be an integral curve of ∇f passing o with $\sigma(0) = o$. Direct calculation shows that $\frac{d}{du}(|\nabla f|^2(\sigma(u))) = 2\operatorname{Ric}(\nabla f, \nabla f)(\sigma(u)) \geq 0$. This shows that $|\nabla f|^2(\sigma(u)) = |\nabla f|^2(\sigma(0))$ for $u \leq 0$ since $|\nabla f|^2$ has its minimum at o. Hence we conclude that $S(\sigma(u)) = \max_M S$ for all $u \leq 0$. However since $-f(\sigma(u)) = -f(\sigma(0)) + \int_u^0 |\nabla f|^2 \, du = -f(\sigma(0)) - u|\nabla f|^2(\sigma(0))$ $\to +\infty$ as $u \to -\infty$ we can conclude that $\sigma(u) \to \infty$. This is a contradiction with the assumption that $\limsup_{x\to\infty} S(x) < \max_M S$. Hence we have that $\lambda = \max_M S$, which implies $\nabla f = 0$ at o and

$$\liminf_{x\to\infty} |\nabla f|^2 \geq 2\eta^2 \doteq \lambda - \limsup_{x\to\infty} S(x) > 0.$$

By considering any minimizing geodesic $\gamma(s)$ emitting from o and the fact $\frac{d}{ds}(f(\gamma(0))) = 0$ and $\frac{d^2}{ds^2}(f(\gamma(s))) \geq 0$, it is clear that o is the minimal point of f and $\langle \nabla f, \nabla r \rangle(x) \geq 0$ for any $x \in M \setminus \{o\}$. Let R_0 be such that $|\nabla f|^2(x) \geq \eta^2$ for all $x \in M \setminus B(o, R_0)$. Consider again an integral curve $\sigma(u)$ passing x. Since $|\nabla f|$ is bounded and M is complete, the curve is defined for all $-\infty < u < +\infty$. Notice that $\sigma(u) \in B(o, r(x))$ for all $u \leq 0$ and

$$f(\sigma(0)) - f(\sigma(u)) = \int_u^0 |\nabla f|^2 \, du \geq (-u)\eta^2$$

as along as $\sigma(u) \in M \setminus B(o, R_0)$. From this we infer that there exist some u_0 such that $\sigma(u_0) \in B(o, R_0)$. On the other hand,

$$\begin{aligned} f(x) &= f(\sigma(u_0)) + \int_{u_0}^0 |\nabla f|^2 \, du \geq f(\sigma(u_0)) + \eta \int_{u_0}^0 |\sigma'(u)| \, du \\ &\geq f(\sigma(u_0)) + \eta d(x, \sigma(u_0)). \end{aligned}$$

This implies the desired lower estimate. The final conclusion follows easily from the above estimate on $|\nabla f|$ and the convexity of f. $\qquad \square$

Remark 6.1. If the sectional curvature of (M, g) is nonnegative, one can show that the claim of the lemma holds under the assumption that $S(o) = \max_M S$, as long as M does not admit any flat factor $\mathbb{R}^k$. The reason is the following. First if the claimed result fails, one can conclude that f_{ij} has an eigenvector corresponding to the zero eigenvalue somewhere. Note that for the associated Ricci flow, the function $f(x, \tau)$, defined as the pull back via the diffeomorphism generated by ∇f, satisfies the heat equation (cf. [11] for details). Then the result follows from the strong tensor maximum principle and a splitting theorem of noncompact manifolds proved in [28].

Assume that (M, g, f) is as in Lemma 6.1, then the point o in the lemma is a minimum point of f and λ in (6.2) is a geometric invariant, namely $\max_{x \in M} S(x)$. Also we have seen that both $|\nabla f|$ and $|\Delta f|$ are bounded. We normalize f so that $\int_M e^{-f} d\Gamma = 1$. Integration by parts gives the following weighted Poincaré inequality.

Proposition 6.1. *Let (M, g, f) be a gradient steady soliton as in Lemma 6.1. Then for any compact supported smooth function $u = e^{-\psi}$ with $\int_M u \, d\Gamma = 1$,*

$$\int_M (|\nabla \psi|^2 - 3S) u \, d\Gamma \geq -\lambda.$$

Proof. The proof follows from the following simple calculation:

$$\int_M (|\nabla \psi|^2 - 3S) u = \int_M (|\nabla \psi|^2 - 2\langle \nabla \psi, \nabla f \rangle + |\nabla f|^2 + 2\Delta f - |\nabla f|^2 - 3S) u$$

$$\geq \int_M (2\Delta f - |\nabla f|^2 - 3S) u = -\lambda,$$

for all normalized u. $\qquad \square$

This is a sharp inequality, at least it is so under the assumption of Lemma 6.1, since for this case the equality holds when $u = e^{-f}$. An equivalent form is that

$$\int_M (4|\nabla \varphi|^2 - 3S\varphi^2) \, d\Gamma \geq -\lambda \int_M \varphi^2 \, d\Gamma$$

for any $\varphi \in L^2(M)$. The weighted Poincaré inequality and its interplay with the geometry has recently been studied in [23].

7. An analog of the c-theorem — nonnegativity of μ_s and μ_e

For the renormalization group flow, there exists the so-called central charge $c(t)$ invariant [41] for the flow such that $c(t)$ *is monotone nonincreasing in t.* (Here in this section only, the parameter t is used as the time parameter for the renormalization group flow or the Ricci flow.) Moreover $c(t)$ *is always nonnegative.* For Ricci flow, there are Perelman's monotonic quantities such as the $\mathcal{W}(g, \sigma, f)$-entropy, defined as

$$\mathcal{W}(g, \sigma, \varphi) \doteq \int_M (\sigma(|\nabla \varphi|^2 + S) + \varphi - n) u \, d\Gamma$$

for any $u = e^{-\varphi}/(4\pi\sigma)^{\frac{n}{2}}$ with $\int_M u = 1$, and associated $\mu(g, \sigma) \doteq \inf_{\int_M u = 1} \mathcal{W}(g, \sigma, \varphi)$, $\nu(g) \doteq \inf_{\sigma > 0} \mu(g, \sigma)$ invariants, as well as the so-called reduced

volume $\mathcal{V}(t)$ (cf. [34] for the definition). The quantity $\mathcal{W}(g, \sigma, \varphi)$, $\mu(g, \sigma)$ and $\nu(g)$ may not be finite when M is not compact. Propositions 4.1 and 4.2 ensures that is the case for the shrinkers with bounded nonnegative Ricci curvature. Utilizing the sharp LSI's proved for the shrinkers and expanders we shall show in this section that the logarithmic Sobolev constants μ_s and μ_e are nonnegative, at least for the gradient shrinking/expanding solitons (with some mild assumptions on the Ricci curvature). In view of the monotonicity of the entropy, and the fact that the gradient shrinking solitons often arise at the singularity, one can view the monotonicity of the entropy together with the result proved here as an analog of the c-theorem. Namely, for the solution to the Ricci flow, one can view $-\mu(g, \tau)$ as the analog of the $c(t)$-invariant. Perelman's entropy formula concludes that it is monotone nonincreasing. Our result concludes that $-\mu(g, 1) = \mu_s$ and it is nonnegative. One should note that if the $\nu(g(t))$ invariant of Perelman [34, Section 3] is well-defined/finite, unfortunately this is not always the case, then at least for the compact manifolds, the $-\nu(g(t))$ would be nonnegative and nonincreasing along the Ricci flow. *However, $-\log \mathcal{V}(t)$ is a nonnegative quantity which is monotone nonincreasing in t.*

We shall show two results on the sign of the invariants μ_s and μ_e. The case of μ_e is an easy application of a rigidity result in [29].

Proposition 7.1. *Let (M, g, f) be a gradient expanding soliton with $\mathrm{Ric} \geq 0$. Then $\mu_e \geq 0$. If $\mu_e = 0$ then (M, g) must be isometric to $\mathbb{R}^n$.*

Proof. Assuming that $\mu_e \leq 0$, Theorem 5.2 then implies that

$$\int_M (|\nabla \psi|^2 + \psi - n) \rho \, d\Gamma \geq 0.$$

Then by Ni [29, Theorem 1.4], one can see a detailed account in [12, pp. 314–333], we can conclude that (M, g) is isometric to $\mathbb{R}^n$, on which $\mu_e = 0$. $\square$

Similar result holds for gradient shrinking solitons. For that we have to assume that the curvature tensor of (M, g) is bounded.

Theorem 7.1. *Let (M, g) be a gradient shrinking soliton with bounded curvature. Let f be the normalized potential function as before, then $\mu_s \geq 0$.*

Remark 7.1. After the appearance of our paper, Yokota [40] generalized the above result by assuming only the lower bound of Ricci curvature. The proof makes uses of Perelman's reduced volume.

Proof. We first prove the result under the extra assumption that Ric ≥ 0. Recall from the introduction that there is an associated solution $g(t)$ (with $-\infty < t < 0$, $t = \eta - 1$) to Ricci flow generated by pulling back the metric via the diffeomorphisms generated by the vector field ∇f. The original metric g corresponds to the one $g(-1)$ (meaning $t = -1$). Propositions 4.1 and 4.2 imply that $\mu(g(-1), \sigma)$ and $\nu(g(-1))$ are finite. Since $g(t)$ is just the re-scale of $g(-1)$, we know that for any $-\infty < t < 0$, $\mu(g(t), \sigma)$ and $\nu(g(t))$ are also finite. Now let $H(y, t; x, t_0)$ (with $t < t_0 < 0$) be the (minimal) positive fundamental solution to the conjugate heat equation:

$$\left(-\frac{\partial}{\partial t} - \Delta_y + S(y, t) \right) H(y, t; x, t_0) = 0$$

being the $\delta_x(y)$ at $t = t_0$. By a result of Perelman [34, Corollary 9.3], see also [9, 31], we know that

$$v_H(y, t) \doteq (t_0 - t)(2\Delta\varphi - |\nabla\varphi|^2 + S) + \varphi - n \leq 0$$

with $H(y, t; x, t_0) = \mathrm{e}^{-\varphi(y,t)}/(4\pi(t_0 - t))^{\frac{n}{2}}$. This implies in particular

$$\mu(g(-1), t_0 + 1) \leq \int_M v_H(y, -1) H(y, -1)\, d\Gamma_{g(-1)} \leq 0.$$

Here to ensure the inequality $v_H \leq 0$ the extra assumption that the curvature tensor of M is uniformly bounded is needed [9].

On the other hand Theorem 1.1 asserts that $\mu(g(-1), 1) \geq -\mu_s$. The result would follow if we show that $\mu(g(-1), t_0 + 1) \to \mu(g(-1), 1)$ as $t_0 \to 0$. For $t_{0,i} \to 0$, consider minimizers φ_i of $\mathcal{W}(g(-1), 1 + t_{0,i}, \varphi)$ (for simplicity we write $g(-1)$ back to g from now on). Let $\sigma_i = 1 + t_{0,i} \to 1$. We assume that $\frac{1}{2} \leq \sigma_i \leq 1$. By Proposition 4.2 and the above we have

$$0 \geq \mu(g, \sigma_i) \geq -\mu_s - nA.$$

Write $w_i = \mathrm{e}^{-\varphi_i/2}$. Then, essentially from definition, the $w_i \in W^{1,2}(M)$. The Euler–Lagrangian equation is

(7.1) $-4\sigma_i \Delta w_i + \sigma_i S w_i - n w_i - 2 w_i \log w_i = \mu(g, \sigma_i) w_i$

for $\int_M w_i^2 = (4\pi\sigma_i)^{\frac{n}{2}} \leq (4\pi)^{\frac{n}{2}}$. Integrating over M we have that

$$4\sigma_i \int_M |\nabla w_i|^2 = \mu(g, \sigma_i)(4\pi\sigma_i)^{\frac{n}{2}} + \int_M (w_i^2 \log w_i^2 + n w_i^2 - \sigma_i S w_i^2)$$

which implies

(7.2) $$4\sigma_i \int_M |\nabla w_i|^2 \leq \int_M w_i^2 \log w_i^2 + n(4\pi)^{\frac{n}{2}}.$$

On the other hand, writing $w_i^2/(4\pi\sigma_i)^{\frac{n}{2}} = e^{-\tilde{\psi}}/\pi^{\frac{n}{2}}$ and using that $\mathcal{W}(g, \frac{1}{4}, \tilde{\psi}) \geq \mu(g, \frac{1}{4})$,

$$\int_M |\nabla w_i|^2 \geq (4\pi\sigma_i)^{\frac{n}{2}}\mu(g, \frac{1}{4}) - \frac{nA}{4}(4\pi\sigma_i)^{\frac{n}{2}} + \int_M w_i^2 \log w_i^2 - \frac{n}{2}\log(4\pi\sigma_i).$$

Combining with (7.2), one can find $C = C(A, n)$ such that

$$\int_M |\nabla w_i|^2 \leq C(A, n)$$

which implies that $\|w_i\|_{W^{1,2}(M)}$ is uniformly bounded. It then implies that $w_i \to w_\infty$ in the dual norm of $W^{1,2}(M)$ and strongly in $L^2(M)$, for some $w_\infty \in W^{1,2}(M)$. Due to the bound $\mu(g, \sigma_i)$ we may also assume that $\mu(g, \sigma_i) \to \mu_\infty(g)$. Clearly $\mu_\infty(g) \leq 0$. It is evident that $\int_M w_\infty^2 = (4\pi)^{\frac{n}{2}}$. We shall show that on every compact subset K, after passing to subsequences, w_i converges to w_∞, say in C^0-fashion. This will imply that w_∞ satisfies the equation

$$-4\Delta w_\infty + S w_\infty - n w_\infty - 2 w_\infty \log w_\infty = \mu_\infty(g) w_\infty.$$

Integration by parts yields that

$$\int_M (4|\nabla w_\infty|^2 + S w_\infty^2 - 2 w_\infty^2 \log w_\infty - n w_\infty^2) = \mu_\infty(g)(4\pi)^{\frac{n}{2}}.$$

This implies that $\mu_\infty(g) \geq \mu(g, 1)$, which is enough to conclude that $\mu_s \geq 0$ since $0 \geq \mu_\infty(g) \geq \mu(g, 1) = -\mu_s$. The claim that $w_i \to w_\infty$ in C^0 norm can be proved using Sobolev embedding theorem (over compact region K), interior L^p-estimates, and the compactness of the Sobolev embedding. Since it is rather standard we leave the details to the interested reader. One can also find this in the forthcoming book [13].

Now we point out how one can modify the above argument to the general case. In fact in the proof above the assumption that $\mathrm{Ric} \geq 0$ is only used, via Proposition 4.2, to ensure that $\mu(g, \sigma)$ is uniformly bounded for $1 - \delta \leq \sigma < 1$, for some $\delta > 0$. This can be done for the case that $|\mathrm{Ric}| \leq A$ for some $A > 0$. We state this as a separate result below. $\qquad\square$

Proposition 7.2. *Assume that on a complete Riemannian manifold (M, g), $\mu(g, 1) > -\infty$ and $\mathrm{Ric} \geq -A$ and $S \leq B$ for some positive numbers A and B. Then for any $0 < \sigma < 1$,*

$$(7.3) \qquad \mu(g, \sigma) \geq \mu(g, 1) - nA\sigma - B - \left(\frac{A^2 n}{2} + An\right)(1 - \sigma).$$

Proof. As in Proposition 4.2, $\mu_0(g, 1) \geq \mu(g, 1) - B$. Let $u_0(x) = \mathrm{e}^{-\tilde{\psi}}/(4\pi\sigma)^{n/2}$ be a smooth function with compact support such that $\int_M u_0 = 1$. Similarly let $u(x, t) = \mathrm{e}^{-\varphi}/(4\pi\tau)^{n/2}$ be the solution to the heat equation with $u(x, 0) = u_0(x)$. Here $\tau(t) = \sigma + t$. We shall use the entropy formula from [29] to estimate

$$\mathcal{W}_0(0) \doteq \int_M (\sigma|\nabla\tilde{\psi}|^2 + \tilde{\psi} - n)u_0.$$

Let $F(t) = \int_M |\nabla\varphi|^2 u$. The entropy formula of [29] implies that the entropy

$$\mathcal{W}_0(t) \doteq \int_M (\tau|\nabla\varphi|^2 + \varphi - n)u$$

satisfies the estimate

$$\begin{aligned}
\frac{d}{dt}\mathcal{W}_0(t) &\leq -2\tau \int_M \left|\nabla_i\nabla_j\varphi - \frac{1}{2\tau}g_{ij}\right|^2 u + 2\tau A F(t) \\
&\leq -\frac{2\tau}{n} \int_M \left(\Delta\varphi - \frac{n}{2\tau}\right)^2 u + 2\tau A F(t) \\
&\leq -\frac{2\tau}{n} \left(\int_M (\Delta\varphi - \frac{n}{2\tau})u\right)^2 + 2\tau A F(t) \\
&= -\frac{2\tau}{n} \left(F(t) - \frac{n}{2\tau}\right)^2 + 2\tau A F(t).
\end{aligned}$$

Viewing the right-hand side above as a quadratic polynomial in $X = F(t) - \frac{n}{2\tau}$, by an elementary consideration we deduce that

$$\frac{d}{dt}\mathcal{W}_0(t) \leq \frac{A^2 n}{2} + nA$$

for $\tau \leq 1$. Hence

$$\mathcal{W}_0(0) \geq \mathcal{W}_0(1 - \sigma) - \left(\frac{A^2 n}{2} + An\right)(1 - \sigma).$$

This shows that

$$\mu_0(g, \sigma) \geq \mu_0(g, 1) - \left(\frac{A^2 n}{2} + An\right)(1 - \sigma).$$

Finally, we have that $\mu(g, \sigma) \geq \mu(g, 1) - nA\sigma - B - \left(\frac{A^2 n}{2} + An\right)(1 - \sigma)$. $\square$

When $f = \text{constant}$, (M, g) is a compact Einstein manifold with $\text{Ric}_M = \frac{1}{2}g_M$. The theorem concludes that

$$\mu_s = \frac{n}{2} - \log(V(M)) + \frac{n}{2}\log(4\pi) \geq 0,$$

where $V(M)$ is the volume of (M, g_M). Among all such manifolds the sphere $\mathbb{S}^n$ has the smallest μ_s. In this case μ_s is monotone nonincreasing in n and has the limit $\frac{1}{2}\log\frac{e}{2}$ as $n \to \infty$, at least for the case that n is even. In fact,

$$\mu_s(\mathbb{S}^{2k}) = \log\frac{e^k(2k-1)!}{(2(2k-1))^k(k-1)!}.$$

It is also easy to see that $\mu_s(\mathbb{R}^n) = 0$ and $\mu_s(M_1 \times M_2) = \mu_s(M_1) + \mu_s(M_2)$.

Remark 7.2. If $(M, g(t))$ is a solution to Ricci flow on compact manifold M over $[0, T)$. Then for any $0 \leq t_1 < T$, $\mu(g(t_1), T - t_1) \leq 0$ by an argument similar as (but easier than) the above. For the steady gradient soliton, it is clear that $\lambda \geq 0$ for any steady solitons with $S \geq 0$. We conjecture that if $\mu_s = 0$, then the shrinker has to be isometric to $\mathbb{R}^n$. In [40], this conjecture has been proved recently.

Corollary 7.1. *Let (M, g, f) be a gradient shrinking soliton as in Theorem 7.1. Then*

$$\int_M f\frac{e^{-f}}{(4\pi)^{n/2}} \leq \frac{n}{2}.$$

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ICREA and Departament de Matemàtiques
Universitat Autònoma de Barcelona
E-08193-Bellaterra
Spain
E-mail address: carrillo@mat.uab.es

Department of Mathematics
University of California at San Diego
La Jolla
CA 92093
USA
E-mail address: lni@math.ucsd.edu