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Examples of hypersurfaces flowing by curvature in a Riemannian manifold

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This paper gives some examples of hypersurfaces $\varphi_t(M^n)$ evolving in time with speed determined by functions of the normal curvatures in an $(n+1)$-dimensional hyperbolic manifold; we emphasize the case of flow by harmonic mean curvature. The examples converge to a totally geodesic submanifold of any dimension from 1 to n, and include cases which exist for infinite time. Convergence to a point was studied by Andrews, and only occurs in finite time. For dimension $n = 2$, the destiny of any harmonic mean curvature flow is strongly influenced by the genus of the surface M^2.

1. Background

Unless otherwise mentioned, all Riemannian manifolds in this article are connected and complete. Let M^n be a smooth, connected, orientable compact manifold of dimension $n \geq 2$, without boundary, and let (N^{n+1}, g^N) be a smooth connected Riemannian manifold. σ^N is any sectional curvature of N^{n+1}, $\mathscr{R}$ is the Riemann tensor of N^{n+1} and ∇^N is the Levi-Civita connection corresponding to g^N. For a hyperbolic manifold, $\sigma^N \equiv -1$. When an index, such as i, is repeated in one term of an expression, summation $1 \leq i \leq n$ is indicated.

Suppose $\varphi_0 : M^n \to N^{n+1}$ is a smooth immersion of an oriented manifold M^n into N^{n+1}; write $\vec{v}$ for the induced normal vector to $\varphi_0(M)$. The second fundamental form of M is a covariant tensor, which we represent at each point by a matrix A, where the entry $A_{ij} = h_{ij} = \langle \nabla^N_{\frac{\partial}{\partial x_i}} \vec{v}, \frac{\partial}{\partial x_j} \rangle_{g^N}$. The Weingarten tensor is given by the matrix $\mathscr{W}$, whose entry $\omega_i^k = h_{ij} g^{jk}$, and $\{g^{jk}\}$ is the pointwise inverse matrix of $\{g_{jk}\}$.

We seek a solution $\varphi : M^n \times [0, T) \to N^{n+1}$ to an equation

$$\frac{\partial}{\partial t}\varphi(x,t) = -f(\lambda(\mathscr{W}(x,t)))\vec{v}(x,t), \tag{1.1}$$
$$\varphi(x,0) = \varphi_0(x),$$

where $F(x,t) = f(\lambda(\mathscr{W}(x,t)))$ and f is a smooth symmetric function, where $\vec{v}(x,t)$ is the outward normal vector to $\varphi(M^n,t)$. $\mathscr{W}(x,t)$ is the Weingarten matrix of $\varphi(M^n,t)$ in N^{n+1} and $\lambda(\mathscr{W})$ is the set of eigenvalues $(\lambda_1,\dots,\lambda_n)$ of $\mathscr{W}$. Define $\varphi_t(x) = \varphi(x,t)$, then $(\lambda_1,\dots,\lambda_n)$ are the principal curvatures of the hypersurface $M_t \triangleq \varphi_t(M) \subset N$.

For example, (1.1) becomes mean curvature flow when $f(\lambda) = \sum_i \lambda_i$ (see [4,8]).

Consider the solution $\varphi : M^n \times [0,T) \to N^{n+1}$ of the following equations:

$$\text{(1.2)} \qquad \frac{\partial}{\partial t}\varphi(x,t) = -\left(\sum_i \lambda_i^{-1}\right)^{-1} \vec{v}(x,t),$$
$$\varphi(x,0) = \varphi_0(x).$$

Such a solution $\varphi(x,t)$ is harmonic mean curvature flow; $f(\lambda) = (\sum_i \lambda_i^{-1})^{-1}$ is the harmonic mean of the numbers $\lambda_1,\dots,\lambda_n$.

It has been noted that the mean curvature flow of hypersurfaces in a Riemannian $(n+1)$-dimensional manifold, $n \geq 2$, does not have all the desirable properties satisfied for $n = 1$ [3]. For some purposes, harmonic mean curvature flow (1.2) may be the preferred way to extend curve-shortening flow to $n \geq 2$.

Andrews proved the following theorem in [2]:

Theorem 1.1. *Let M^n and φ_0 be assumed as at the beginning of this paper, and that the Riemannian manifold (N^{n+1}, g^N) satisfies the following conditions:*

$$-K_1 \leq \sigma^N \leq K_2, \quad |\nabla^N R^N|_{g^N} \leq L$$

for some non-negative constants K_1, K_2 and L.

Assume every principal curvature λ_i of φ_0 satisfies the following condition:

$$\lambda_i > \sqrt{K_1}.$$

Then there exists a unique smooth solution to (1.2) *on a maximal time interval $[0,T)$, $T < \infty$, and the immersion φ_t converges uniformly to a round point p in N^{n+1} as t approaches T.*

Also, we have the following theorem, to appear in [7]:

Theorem 1.2. *Let M^n be a smooth, connected, orientable compact manifold of dimension $n \geq 2$, without boundary. Assume N^{n+1} is a non-positively* *curved, simply connected smooth manifold, and suppose* $\varphi_0 : M^n \to N^{n+1}$ *is a smooth immersion of* M^n. *Assume every principal curvature of* $\varphi_0(M)$ *is positive. Then there exists a unique smooth solution to* (1.2) *on a maximal time interval* $[0, T)$, $T < \infty$, *and the immersion* φ_t *converges uniformly to a round point* p *in* N^{n+1} *as* t *approaches* T.

In the rest of this paper, except for Section 6, and unless otherwise mentioned, we consider harmonic mean curvature flow and let $f(\lambda) = (\sum_i \lambda_i^{-1})^{-1}$. We provide two specific examples of harmonic mean curvature flow for infinite time: in Section 2, with dimension reduction in the limit and in Section 3, with the limit manifold of the same dimension as M. Note, these examples in Sections 2 and 3 provide barriers for harmonic mean curvature flow in Riemannian manifolds; further applications will be addressed in [7]. We discuss the limit behavior of the harmonic mean curvature flow at infinite time in section 4. Then we treat the special consequences of the Gauss–Bonnet theorem for two-dimensional surfaces in Section 5, and turn to examples of more general flows by functions of normal curvatures in Section 6.

2. The dimension-reduction example

In this section, we give an example where φ_t converges to φ_∞ in the C^∞ topology but the dimension of $M_\infty = \varphi_\infty(M)$ is less than the dimension of M_t, i.e., there is *dimension reduction*.

Theorem 2.1. *Let* N^3 *be a hyperbolic manifold containing an embedded closed geodesic* M_∞. *Then there is a flow* $\varphi_t : M^2 \to N^3$ *by harmonic mean curvature, where* M^2 *is a torus, which converges to* M_∞ *as* $t \to +\infty$. *The flow consists of immersions* φ_t, *which become embedded for* t *sufficiently large.*

For example, we may let the ambient manifold N be $H^3/\mathbb{Z}$, where H^3 is hyperbolic space, represented as the Poincaré half-space $(R^3)^+ = \{(x, y, z) | (x, y, z) \in \mathbb{R}^3, z > 0\}$ with the metric $g_{ij}^N = \frac{1}{z^2}\delta_{ij}$ ($\delta_{ij} = \delta_i^j$ = Kronecker delta), and the $\mathbb{Z}$ action $f : \mathbb{Z} \times H^3 \to H^3$ is defined as

$$f(k)(x, y, z) = 2^k(x, y, z).$$

Recall that $f(k)$ is an isometry of H^3 for each $k \in \mathbb{Z}$.

Now we let N be the quotient manifold of H^3 under the $\mathbb{Z}$-action, with fundamental domain $\{(x, y, z) | \, 1 \le \sqrt{x^2 + y^2 + z^2} \le 2\}$. Then $M_\infty =$ the positive z-axis, modulo $f(1)$, is a closed geodesic in N.

Proof. Let $\psi_0 : \mathbb{S}^1 \to N$ be an embedding as the given closed geodesic curve M_∞ in N. We choose a unit vector field $w(x)$ in $(T_x\psi_0)^\perp$. Then for $r > 0$, we define

$$\psi(x, \theta, r) = \psi_r(x, \theta) : \mathbb{S}^1 \times \mathbb{S}^1 \to N^3$$

by

$$\psi(x, \theta, r) = \psi_r(x, \theta) = \gamma(x, \theta, r),$$

where $\gamma(x, \theta, \cdot)$ is the unit-speed geodesic in N with $\gamma(x, \theta, 0) = \psi_0(x)$ and $\frac{d}{dr}\gamma(x, \theta, r) = \vec{N}(x, \theta)$ at $r = 0$. Here $\vec{N}(x, \theta)$ is the unit tangent vector in $T_{\psi_0(x)}N^3$, which is perpendicular to $T_x\psi_0$ and makes the angle θ with $w(x)$. Then $\psi_r(\mathbb{S}^1 \times \mathbb{S}^1)$ has two principal curvatures:

$$\lambda_1(r) \equiv \tanh r, \quad \lambda_2(r) \equiv \coth r.$$

In fact, for $i = 1, 2$, $\lambda_i(r)$ is the logarithmic derivative of the length of a Jacobi field, and hence satisfies the Ricatti equation $\lambda_i'(r) + (\lambda_i(r))^2 = 1$.

We have constructed a one-parameter family of immersions $\psi_r : M \to N$, $-\infty < r < \infty$, with two principal curvatures: $\lambda_1(r) \equiv \tanh r$ and $\lambda_2(r) \equiv \coth r$. It may be observed that ψ_r is an embedding for r sufficiently small.

Now consider the harmonic mean curvature flow $\varphi_t = \psi_{r(t)} : M \to N$, with initial conditions $\varphi_0 = \psi_{r_0}$, $r(0) = r_0$, where r_0 is some fixed positive constant. The speed must satisfy

$$\begin{aligned} \frac{\partial r}{\partial t} &= \left\langle \frac{\partial \gamma}{\partial r}\frac{\partial r}{\partial t}, \vec{v} \right\rangle = \left\langle \frac{\partial \gamma(x, r)}{\partial t}, \vec{v} \right\rangle \\ &= \left\langle \frac{\partial \psi(x, r)}{\partial t}, \vec{v} \right\rangle = \left\langle \frac{\partial \varphi(x, t)}{\partial t}, \vec{v} \right\rangle \\ &= \langle -F\vec{v}, \vec{v} \rangle = -F(\lambda_1, \lambda_2) \\ &= -\frac{1}{\lambda_1^{-1} + \lambda_2^{-1}} = -\frac{\sinh r \cosh r}{(\sinh r)^2 + (\cosh r)^2}. \end{aligned}$$

In the first equation, we use the fact $\frac{\partial \gamma}{\partial r} = \vec{v}$; in the third equation, we use the definition of ψ_r, where $\vec{v} = \vec{N}(x, \theta)$ is the outward normal vector of $\psi_r(M)$ at $(x, \theta) \in \mathbb{S}^1 \times \mathbb{S}^1$.

Solving, we find

$$r(t) = \frac{1}{2}\sinh^{-1}\left(e^{-t}\sinh 2r_0\right).$$

Note that $r(t) \to 0$ as $t \to \infty$. □

3. The no-dimension-reduction example

In this section, we give an example in which M_t converges to M_∞ in the C^∞ topology and the dimension of M_∞ is the same as the dimension of M_t, i.e., there is *no dimension reduction.*

Theorem 3.1. *There is a compact surface M^2 of genus 2, a hyperbolic manifold N^3 diffeomorphic to $M \times \mathbb{R}$, a totally geodesic embedding $\psi_0 : M \to N$ and a flow by harmonic mean curvature $\varphi_t : M \to N$ such that as $t \to +\infty$, $\varphi_t(M) \to \psi_0(M)$ smoothly.*

Proof. Let Ω be a regular geodesic octagon in the hyperbolic plane H^2, with angles $\pi/2$, and thus area 4π. Label the edges as

$$\beta_1, \alpha_1', -\beta_1', -\alpha_1, \beta_2, \alpha_2', -\beta_2', -\alpha_2,$$

in that order, where the signs indicate orientation. Let A_1 be the orientation-preserving isometry of H^2, which maps the oriented geodesic segments α_1 to α_1'; A_2 maps α_2 to α_2'; B_1 maps β_1 to β_1' and B_2 maps β_2 to β_2'. The group G of isometries of H^2 generated by A_1, A_2 and B_1 also includes B_2. G is isomorphic to the fundamental group of the compact surface of genus 2. (See [Katok [6], pp. 95–98] for the arithmetic properties of the group G.)

Let $\psi_0 : H^2 \to H^3$ be an embedding as a totally geodesic surface in H^3. The isometries in G extend in a well-known fashion to isometries of H^3, leaving the distance from $\psi_0(H^2)$ invariant.

Choose a unit normal vector field $\vec{N}$ to $\psi_0(H^2)$. Define $\psi(\cdot, r) : H^2 \to H^3$ by $\psi(x, r) = \psi_r(x) = \gamma(x, r)$ and $\psi(x, 0) = \psi_0(x)$, where $\gamma(x, \cdot)$ is the unit-speed geodesic in H^3 with $\gamma(x, 0) = x$ and $\frac{\partial}{\partial r}\gamma(x, 0) = \vec{N}(x)$.

Then $\psi_r(H^2)$ is totally umbilic, with normal curvatures $\lambda(r) \equiv \tanh r$. In fact, $\lambda(r)$ satisfies the Ricatti equation $\lambda'(r) + (\lambda(r))^2 = 1$, with the initial condition $\lambda(0) = 0$.

Now let the group G act by isometries on H^2 and on H^3. The quotient $H^2/G = M^2$ is a compact surface of genus 2, with fundamental domain Ω, and the quotient $H^3/G = N^3$ is a non-compact hyperbolic manifold diffeomorphic to $M \times \mathbb{R}$. The group G acting on N preserves each of the hypersurfaces $\psi_r(H^2)$. We have constructed a one-parameter family of totally umbilic embeddings $\psi_r : M \to N$, $-\infty < r < \infty$, with normal curvatures $\equiv \tanh r$.

Now consider the harmonic mean curvature flow $\varphi_t : M \to N$, with initial conditions $\varphi_0 = \psi_{r_0}$, where r_0 is some fixed positive constant. The speed must satisfy

$$\begin{aligned}\frac{\partial r}{\partial t} &= \left\langle \frac{\partial \gamma}{\partial r}\frac{\partial r}{\partial t}, \vec{v} \right\rangle = \left\langle \frac{\partial \gamma(x,r)}{\partial t}, \vec{v} \right\rangle = \left\langle \frac{\partial \psi(x,r)}{\partial t}, \vec{v} \right\rangle \\ &= \left\langle \frac{\partial \varphi(x,t)}{\partial t}, \vec{v} \right\rangle = \langle -F\,\vec{v}, \vec{v}\rangle = -F(\lambda_1, \lambda_2) \\ &= -\frac{1}{\lambda_1^{-1} + \lambda_2^{-1}} = -\frac{1}{2}\tanh r.\end{aligned}$$

In the first equation, we use the fact $\frac{\partial \gamma}{\partial r} = \vec{N}(x) = \vec{v}$. In the third equation, we use the definition of ψ_r, where $\vec{v}$ is the outward normal vector of ψ_r.

Solving, we find

$$r(t) = \sinh^{-1}(e^{-t/2}\sinh r_0).$$

Note that $r(t) \to 0$ as $t \to \infty$. □

4. The limit behavior of harmonic mean curvature flow at infinite time

In this section, we will give a sufficient condition where the harmonic mean curvature flow will exist forever and discuss the limit behavior. Let $\varphi_t : M \to N$ be an immersion of M^n into a hyperbolic manifold N^{n+1}.

Definition 4.1. We define the following notation:

$$\begin{aligned}&\dot{F}^{kl} = \frac{\partial F}{\partial h_{kl}}, \quad \ddot{F}^{kl,pq} = \frac{\partial^2 F}{\partial h_{kl}\partial h_{pq}}, \quad \dot{H}_k^i = \frac{\partial H}{\partial \omega_i^k}, \\ &\ddot{H}_{r,k}^{s,i} = \frac{\partial^2 H}{\partial \omega_i^k \partial \omega_s^r}, \quad \mathscr{R}_{ij} = \mathscr{R}_{i0j0},\end{aligned}$$

where 0 appearing as a tensor index represents the normal vector $\vec{v}$ of $\varphi(M)$ in N. For any $W : M \to \mathbb{R}$, we define:

$$\mathscr{L}(W) = \dot{F}^{kl}\nabla_k\nabla_l W.$$

Recall from Andrews [2] that $\mathscr{L}$ is elliptic as long as $\varphi_t(M)$ remains locally strictly convex.

Theorem 4.1. *If N^{n+1} is a hyperbolic manifold, $F(x) < \frac{1}{n}$ for any $x \in M$, then $\varphi_t(M)$ remains locally convex and $F(x,t) < \frac{1}{n}$ for any $x \in M$, $t \in$*

$[0, +\infty)$, $\lim_{t\to\infty} F(x,t) = 0$, *and the harmonic mean curvature flow exists for all* t *in* $[0, +\infty)$.

Proof. By Andrews [2], using a curvature coordinate system at one point, we have the following formula:

$$\begin{aligned} \frac{\partial F}{\partial t} &= \mathscr{L}(F) + F < \dot{F}, (\mathscr{W}^2) > +F < \dot{F}^{ij}, (\mathscr{R}_{ij}) > \\ &= \mathscr{L}(F) + \sum_i F \frac{\partial f}{\partial \lambda_i} (\lambda_i^2 + \mathscr{R}_{ii}) \\ &\le \mathscr{L}(F) + F^3 (n - \sum_i \lambda_i^{-2}) \le \mathscr{L}(F) + F^3 \left(n - \frac{1}{n} F^{-2} \right). \end{aligned} \tag{4.1}$$

Consider the ordinary differential equation (ODE)

$$\frac{\partial \tilde{F}}{\partial t} = \tilde{F}^3 \left(n - \frac{1}{n} \tilde{F}^{-2} \right),$$
$$\tilde{F}(0) = \max_{x\in M} F(x,0).$$

Solving the above ODE, we get $\widetilde{F}(t)^{-2} - n^2 = (\widetilde{F}(0)^{-2} - n^2)\mathrm{e}^{2t/n}$. Because $0 < \widetilde{F}(0) = \max_{x\in M^n} F(x,0) < \frac{1}{n}$, we get $\lim_{t\to\infty} \tilde{F}(t) = 0$.

By the maximum principle, $F(x,t) \le \tilde{F}(t) < \frac{1}{n}$, for all $x \in M$, $t \in [0, +\infty)$, and therefore $\lim_{t\to\infty} F(x,t) = 0$.

On the other hand, we have the following estimate by the above evolution equation of F:

$$\frac{\partial F}{\partial t} \ge \mathscr{L}(F) + F^3 \left(-\sum_i \lambda_i^{-2} \right) \ge \mathscr{L}(F) - F.$$

Now consider the ODE

$$\frac{\partial \widehat{F}}{\partial t} = -\widehat{F},$$
$$\widehat{F}(0) = \min_{x\in M} F(x,0).$$

Then by the maximum principle again, we get for all $x \in M, t \in [0, +\infty)$

$$F(x,t) \ge \widehat{F}(t) = \min_{x\in M} F(x,0)\,\mathrm{e}^{-t} > 0.$$

In particular, $\varphi_t(M)$ remains convex for all t.

Finally, we have the following estimate of H. By Andrews [2]

$$\begin{aligned}\frac{\partial}{\partial t}\omega_i^r &= \dot{F}^{kl}\nabla_k\nabla_l\omega_i^r + \ddot{F}^{kl,pq}(\nabla_i h_{kl})(\nabla_j h_{pq})g^{jr} \\ &\quad + \dot{F}^{kl}(h_{ml}\omega_k^m)\omega_i^r + \dot{F}^{st}\mathscr{R}_{st}h_{ij}g^{jr} + 2\dot{F}^{pm}g^{tr}\omega_m^q\mathscr{R}_{piqt} \\ &\quad - \dot{F}^{pq}(g^{tr}\omega_i^s\mathscr{R}_{psqt} + g^{ts}\omega_s^r\mathscr{R}_{piqt}) + \dot{F}^{pq}g^{tr}(\nabla_i\mathscr{R}_{tpq0} - \nabla_p\mathscr{R}_{qit0}).\end{aligned}$$

Now referring to the last five terms above, we define

$$\text{(I)} = \dot{H}_r^i\dot{F}^{kl}(h_{ml}\omega_k^m)\omega_i^r, \quad \text{(II)} = \dot{H}_r^i\dot{F}^{st}\mathscr{R}_{st}h_{ij}g^{jr},$$
$$\text{(III)} = 2\dot{H}_r^i\dot{F}^{pm}g^{tr}\omega_m^q\mathscr{R}_{piqt}, \quad \text{(IV)} = -\dot{H}_r^i(\dot{F}^{pq}g^{tr}\omega_i^s\mathscr{R}_{psqt} + \dot{F}^{pq}g^{ts}\omega_s^r\mathscr{R}_{piqt}),$$
$$\text{(V)} = \dot{H}_r^i\dot{F}^{pq}g^{tr}(\nabla_i\mathscr{R}_{tpq0} - \nabla_p\mathscr{R}_{qit0}),$$

then

$$\begin{aligned}\frac{\partial}{\partial t}H &= \dot{H}_r^i\left(\frac{\partial}{\partial t}\omega_i^r\right) \\ &= \dot{H}_r^i(\dot{F}^{kl}\nabla_k\nabla_l\,\omega_i^r) + \dot{H}_r^i\ddot{F}^{kl,pq}(\nabla_i h_{kl})(\nabla_j h_{pq})g^{jr} + (I) + \cdots + \text{(V)}.\end{aligned}$$

Note

$$\dot{F}^{kl}\nabla_k\nabla_l H = \dot{F}^{kl}\nabla_k(\dot{H}_r^i\nabla_l\omega_i^r) = \dot{F}^{kl}\ddot{H}_{r,\tilde{r}}^{i,\tilde{i}}(\nabla_k\omega_{\tilde{i}}^{\tilde{r}})(\nabla_l\omega_i^r) + \dot{F}^{kl}\dot{H}_r^i\nabla_k\nabla_l\omega_i^r.$$

Define

$$\text{(J)} = \dot{H}_r^i\ddot{F}^{kl,pq}(\nabla_i h_{kl})(\nabla_j h_{pq})g^{jr} - \dot{F}^{kl}\ddot{H}_{r,\tilde{r}}^{i,\tilde{i}}(\nabla_k\omega_{\tilde{i}}^{\tilde{r}})(\nabla_l\omega_i^r),$$

we get

$$\frac{\partial}{\partial t}H = \mathscr{L}(H) + (J) + (I) + \cdots + \text{(V)}.$$

It is straightforward to get

$$\text{(I)} + \text{(II)} = H[< \dot{F}, (\mathscr{W}^2) > + \dot{F}^{ij}\mathscr{R}_{i0j0}] \leq nF^2H \leq \frac{1}{n}H$$

and

$$\text{(V)} = \frac{\partial f}{\partial\lambda_i}(\nabla_j\mathscr{R}_{jii0} - \nabla_i\mathscr{R}_{ijj0}) = 0.$$

Choose a curvature coordinate system around one point; then we could do the following calculation:

$$\text{(J)} = \ddot{F}^{kl,pq}(\nabla_i h_{kl})(\nabla_i h_{pq}).$$

But, by Lemma 2.22 in [1], we know F is concave from the fact that f is concave. So, we get $(J) \leq 0$.

Now

$$
\begin{aligned}
(\mathrm{III}) + (\mathrm{IV}) &= 2\dot{H}^i_r \dot{F}^{pm} g^{tr} \omega^q_m \mathscr{R}_{piqt} - \dot{H}^i_r (\dot{F}^{pq} g^{tr} \omega^s_i \mathscr{R}_{psqt} + \dot{F}^{pq} g^{ts} \omega^r_s \mathscr{R}_{piqt}) \\
&= 2\delta^i_r \frac{\partial f}{\partial \lambda_p} \delta^m_p \delta^r_t \lambda_q \delta^m_q \mathscr{R}_{piqt} \\
&\quad - \delta^i_r \left(\frac{\partial f}{\partial \lambda_p} \delta^q_p \delta^r_t \lambda_i \delta^s_i \mathscr{R}_{psqt} + \frac{\partial f}{\partial \lambda_p} \delta^q_p \delta^s_t \lambda_s \delta^r_s \mathscr{R}_{piqt} \right) \\
&= 2\mathscr{R}_{prpr} \frac{\partial f}{\partial \lambda_p} (\lambda_p - \lambda_r) = 2 \sum_{p<r} \mathscr{R}_{prpr} \left(\frac{\partial f}{\partial \lambda_p} - \frac{\partial f}{\partial \lambda_r} \right) (\lambda_p - \lambda_r) \\
&= \left(\sum_k \lambda_k^{-1} \right)^{-2} \cdot \sum_{i,j} (-\mathscr{R}_{ijij}) \cdot (\lambda_i - \lambda_j)^2 (\lambda_i + \lambda_j) \cdot \lambda_i^{-2} \lambda_j^{-2} \\
&\leq \sum_{i,j} (\lambda_i + \lambda_j) \cdot \left(\frac{\lambda_i^{-1} - \lambda_j^{-1}}{\sum_k \lambda_k^{-1}} \right)^2 \leq \sum_{i,j} (\lambda_i + \lambda_j) = 2nH.
\end{aligned}
$$

We have the following inequality for H by the above estimates:

$$
\frac{\partial H}{\partial t} \leq \mathscr{L}(H) + \left(2n + \frac{1}{n} \right) H.
$$

Now consider the ODE

$$
\begin{aligned}
\frac{\partial \widehat{H}}{\partial t} &= \left(2n + \frac{1}{n} \right) \widehat{H}, \\
\widehat{H}(0) &= \max_{x \in M} H(x, 0).
\end{aligned}
$$

Then by the maximum principle again, we get for all $x \in M, t \in [0, +\infty)$:

$$
H(x,t) \leq \widehat{H}(t) = \max_{x \in M} H(x, 0)\, \mathrm{e}^{(2n + \frac{1}{n})t} < +\infty.
$$

This shows that the harmonic mean curvature flow exists on $[0, +\infty)$. □

In the rest of this section, we do not assume the ambient manifold N^{n+1} is a hyperbolic manifold.

Proposition 4.1. *Assume N^{n+1} is a smooth $n + 1 \geq 3$ dimensional manifold which is convex at infinity, the maximal existence time of the harmonic mean curvature flow $\varphi : M \times [0, T) \to N$ is $T = +\infty$, and as $t \to +\infty$, $M_t =$*

$\varphi(M,t)$ converges to a smooth n-dimensional submanifold M_∞ of N in the C^∞-topology; then

$$\max_{x\in M,\, t\in[0,+\infty)} \{|F(x,t)|, |\nabla F(x,t)|, |\nabla^2 F(x,t)|\} \le C,$$

where C is a constant depending on M_0, N^{n+1} and M_∞.

Proof. Straightforward from the assumptions. □

Proposition 4.2. *Assume N and $M_t \to M_\infty$ are as in the hypotheses of Proposition 4.1. Then*

$$\lim_{t\to\infty} \int_{M_t} F^2 \, d\mu_t = 0.$$

Proof. By Theorem 1.1 in [5], we have the formula $\frac{\partial}{\partial t}(\int_{M_t} d\mu_t) = -\int_{M_t} FH d\mu_t$. Because $\int_{M_t} d\mu_t \to \mu(M_\infty)$ as $t \to \infty$, we could find an ϵ-dense set $\{t_k\}_{k=1}^\infty$ for any positive constant $\epsilon > 0$ such that

$$\lim_{k\to\infty} t_k = \infty$$

and

$$\lim_{k\to\infty} \int_{M_{t_k}} FH \, d\mu_{t_k} = 0.$$

Then using the inequality $H \ge n^2 F$, we get $\lim_{k\to\infty} \int_{M_{t_k}} F^2 \, d\mu_{t_k} = 0$.

Now to get our conclusion we only need to show $\frac{\partial}{\partial t} \int_{M_t} F^2 d\mu_t$ is uniformly bounded. First, we know from Proposition 4.1 that $|F|$, $|\nabla F|$ and $|\nabla^2 F|$ are uniformly bounded. So, we have

$$\begin{aligned} \frac{\partial}{\partial t}\left(\int_{M_t} F^2 \, d\mu_t\right) &= \int 2FF_t + F^2(-FH) \, d\mu_t \\ &= \int 2F\left(\mathscr{L}(F) + \sum_{i=1}^n F\left(\frac{\partial f}{\partial \lambda_i}\right)(\lambda_i^2 + \mathscr{R}_{ii})\right) - F^3 H \, d\mu_t \end{aligned}$$

(where we use equation (4.1))

$$\begin{aligned}
&= \int 2nF^4 + 2F^4\left(\sum_{i=1}^{n}\lambda_i^{-2}\mathscr{R}_{ii}\right) + 2F\mathscr{L}(F) - F^3H\,d\mu_t \\
&\le \int 2F^4K_2\left(\sum_{i=1}^{n}\lambda_i^{-2}\right)\,d\mu_t + \int 2F\mathscr{L}(F)\,d\mu_t \\
&\le C\int F^2\,d\mu_t + 2\int F\mathscr{L}(F)\,d\mu_t,
\end{aligned}$$

where the first inequality uses the following facts:

M_t is always contained in some compact set of N^{n+1}, since N^{n+1} is convex at infinity, so its sectional curvature is bounded above by some constant K_2; and $HF^{-1} = (\sum_{i=1}^{n}\lambda_i)(\sum_{i=1}^{n}\lambda_i^{-1}) \ge n^2 \ge 2n$.

Next, since we know the volume of M_t is always non-increasing and $|F|$ is uniformly bounded, we get

$$C\int_{M_t} F^2\,d\mu_t \le C_1,$$

where C_1 is some constant depending only on M_0, N and M_∞.
Since $|\nabla^2 F|$ is uniformly bounded, we get

$$2\int F\mathscr{L}(F)\,d\mu_t \le 2n^2\int F|\nabla^2F|\,d\mu_t \le C_2,$$

where C_2 is some constant depending on M_0, N and M_∞.
By all the above, we get

$$\frac{\partial}{\partial t}\left(\int F^2\,d\mu_t\right) \le C_3,$$

where C_3 is another constant depending on M_0, N and M_∞.

Therefore,

$$\lim_{t\to\infty}\int_{M_t} F^2\,d\mu_t = 0. \qquad \square$$

Corollary 4.1. *Assume N and $M_t \to M_\infty$ are as assumed for Proposition 4.1. Then we have*

$$\lim_{t\to\infty}\left(\max_{x\in M} F(x,t)\right) = 0.$$

Proof. By Proposition 4.2, we have

$$0 = \lim_{t\to\infty} \int_{M_t} F^2 \, d\mu_t = \int_{M_\infty} \lim_{t\to\infty} F^2(x,t) \, d\mu_\infty,$$

so the corollary follows. □

By the above results, assume N and $M_t \to M_\infty$ are as in the hypotheses of Proposition 4.1, we know that $F \equiv 0$ on the limit surface M_∞, if M_∞ is the smooth limit of the harmonic mean curvature flow, which implies that $\det \mathscr{W} = 0$ on M_∞.

5. Classification of harmonic mean curvature flow on surfaces

In this section, we consider harmonic mean curvature flow for $n = 2$, where M^2 is an orientable surface, N^3 is a hyperbolic manifold and the harmonic mean $f(\lambda) = \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}$. As before, we assume that $\varphi_0(M)$ is locally strictly convex.

In the following, we always assume $F(x,0) < \frac{1}{2}$, i.e., $\lambda_1^{-1} + \lambda_2^{-1} > 2$, which will guarantee, that the harmonic mean curvature flow exists forever by Theorem 4.1. Note that, for example, $f(\lambda_1, \lambda_2) < \frac{1}{2}$ for the examples of Theorems 2.1 and 3.1, and that the horospheres have $f(\lambda_1, \lambda_2) \equiv \frac{1}{2}$.

We define $C_0 = 2\pi\chi(M_0) = \int_{M_t} (K-1) \, d\mu_t$, where the second equation is true for any M_t because of the Gauss–Bonnet theorem, where $\chi(M_0)$ is the Euler number of M_0; $K(x,t) = \lambda_1(x,t)\lambda_2(x,t)$, $\lambda_1(x,t)$ and $\lambda_2(x,t)$ are the principal curvatures at the point x on M_t in the ambient hyperbolic manifold N^3, and the Gauss equation, which implies the Gauss curvature $= K - 1$.

First, define $V(t) = \int_{M_t} 1 \, d\mu_t$, the area of M_t. Then using the formula

$$\frac{\partial}{\partial t} d\mu_t = -FH d\mu_t,$$

we get

$$\begin{aligned}\frac{d}{dt} V(t) &= \int_{M_t} \frac{\partial}{\partial t} \, d\mu_t = \int_{M_t} (-FH) \, d\mu_t = \int_{M_t} (-K) \, d\mu_t \\ &= -\int_{M_t} (K-1) \, d\mu_t - \int_{M_t} 1 \, d\mu_t = -C_0 - V(t).\end{aligned}$$

Solving the above ODE, we get

$$V(t) = (V(0) + C_0)\mathrm{e}^{-t} - C_0.$$

This shows that the area of M_t is determined by its genus and the area $V(0)$ of the initial surface M_0.

There are three cases: $C_0 < 0$ and $C_0 = 0$ and $C_0 > 0$, corresponding to the surfaces with genus $g > 1$ (case I) , $g = 1$ (case II) and $g = 0$ (case III), respectively.

(I) Let us first consider the case $C_0 = 2\pi\chi(M_0) < 0$. In this case, we have

$$\lim_{t\to\infty} V(t) = -C_0 > 0,$$

which means the limit surface has non-zero volume. We conjecture that in a hyperbolic manifold N^3, the limit surface will be the totally geodesic surface, if there is one in the homotopy class of M_0. This behavior is seen in Theorem 3.1.

(II) When $C_0 = 2\pi\chi(M_0) = 0$, we have

$$\lim_{t\to\infty} V(t) = -C_0 = 0,$$

which means the limit surface has zero volume. In fact, we could prove the following:

Proposition 5.1. *If N^3 is a hyperbolic manifold, $F(x,0) < \frac{1}{2}$ for all $x \in M$ and the genus of $M = 0$, then*

$$\lim_{t\to\infty} (\max_{x\in M_t} H(x,t)) = +\infty.$$

Proof. Because $\int_{M_t} (K-1)\, d\mu_t = C_0 = 0$, we have $\max_{x\in M_t} K(x,t) \geq 1$. We also have $\lim_{t\to\infty}(\max_{x\in M_t} F(x,t)) = 0$, using the assumption $F(x,0) < \frac{1}{2}$, by Theorem 4.1. Then for any $x \in M_t$, $t > 0$, we have the following:

$$K(x,t) = H(x,t)F(x,t) \leq F(x,t)(\max_{x\in M_t} H(x,t)).$$

Taking the maximum on the both sides of the above inequality, we have

$$1 \leq \max_{x\in M_t} K(x,t) \leq (\max_{x\in M_t} F(x,t))(\max_{x\in M_t} H(x,t)).$$

So

$$\max_{x\in M_t} H(x,t) \geq \frac{1}{\max_{x\in M_t} F(x,t)}.$$

Taking the limit on both sides, we get

$$\lim_{t\to\infty}(\max_{x\in M_t} H(x,t)) \geq \frac{1}{\lim_{t\to\infty}(\max_{x\in M_t} F(x,t))} = +\infty.$$

□

The above proposition means that there exists at least one blow-up point on the limit set; the example of Theorem 2.1 blows up at every point.

(III) Finally, when $C_0 = 2\pi\chi(M_0) > 0$, we have an interesting geometric result. In this case, because

$$V(t) = (V(0) + C_0)\mathrm{e}^{-t} - C_0,$$

there exists some T_0, $0 < T_0 < +\infty$, such that $V(T_0) = 0$. That means the harmonic mean curvature flow stops in finite time. But we have already proved that the flow will exist forever if $F < \frac{1}{2}$. So under the assumption $F < \frac{1}{2}$, this surface will not exist.

Remark 5.1. Observe that the non-existence of the initial surfaces in Case (III) above may also be proven by lifting the simply connected surface M_0 to the universal cover H^3 of N^3 and applying the comparison principle with shrinking spheres centered at a point: the sphere of radius r has $F = \frac{1}{2}\coth r > \frac{1}{2}$.

6. General geometric flows

In this section, we give examples for a general geometric flow (1.1) in a hyperbolic manifold N^{n+1}, which will exist forever or for a computable finite time, and converge to a given totally geodesic submanifold P^k of any codimension. In this section, we always assume the existence of a totally geodesic submanifold P^k in N^{n+1}.

First, by similar methods to those of Sections 2 and 3, we may prove a theorem for general dimensions and codimensions.

Theorem 6.1. *Assume P^k is a compact totally geodesic submanifold of the hyperbolic manifold N^{n+1}, where $1 \leq k \leq n$. Let M be diffeomorphic to the unit sphere bundle of the normal bundle $\perp P$ when $k < n$; we choose M to be one of the two connected components of the unit sphere bundle of the normal bundle $\perp P$ when $k = n$. Then, we have a flow by harmonic mean curvature $\varphi_t : M \to N$ such that as $t \to +\infty$, $\varphi_t(M) \to P$.*

Proof. We only sketch the proof. We find the second fundamental form matrix of $\psi_r(M)$ with respect to a basis of curvature directions in the following:

$$\mathscr{W} = \begin{pmatrix} I_k \tanh r & 0_{k\times(n-k)} \\ 0_{(n-k)\times k} & I_{n-k} \coth r \end{pmatrix}.$$

Then, we find

$$\frac{\partial r}{\partial t} = -F = -\frac{\tanh r}{k + (n-k)(\tanh r)^2}. \tag{6.1}$$

Solving this ODE, we get

$$(\sinh r(t))^k (\cosh r(t))^{n-k} = C\mathrm{e}^{-t},$$

where $C = (\sinh r_0)^k (\cosh r_0)^{n-k}$ is a fixed positive constant. This shows that $\varphi_t := \psi_{r(t)}$ is a solution of harmonic mean curvature flow.

Note that $r(t) \to 0$ as $t \to +\infty$. □

Now let M^n be diffeomorphic to (one connected component of) the unit sphere normal bundle of P^k in N^{n+1}, and let $\psi_r : M \to N$ define the hypersurface at distance $r > 0$ from P^k. We consider flow by an arbitrary symmetric function of the normal curvatures.

Theorem 6.2. *For the symmetric function $f(\lambda_1, \ldots, \lambda_n)$, define*

$$h(r) = f(\tanh r, \ldots, \coth r),$$

where $\tanh r$ *is repeated k times and* $\coth r$ *is repeated $n-k$ times. Choose $r_0 > 0$ and define*

$$T_0 = \int_0^{r_0} \frac{1}{h(r)} dr, \quad 0 < T_0 \le +\infty.$$

Then we may construct a flow

$$\frac{\partial}{\partial t}\varphi(\cdot, t) = f(\lambda(\mathscr{W}(x,t)))\, \vec{v}(x,t) \tag{6.2}$$

with initial condition $\varphi(\cdot, 0) = \psi_{r_0}$, which exists for time $0 \le t \le T_0 \le \infty$, and $\varphi(\cdot, t)$ converges to the totally geodesic k-dimensional submanifold P^k as $t \to T_0$.

Proof. The hypersurface defined by $\varphi(\cdot, t) := \psi_{r(t)}$ flows by (6.2) if

$$\begin{aligned}
\text{(6.3)} \qquad & \frac{\partial r}{\partial t} = -F(x,t) \equiv -h(r) \\
& \Longrightarrow \int_{r(0)}^{r(T_0)} \frac{1}{h(r)} dr = \int_0^{T_0} -1\, dt \\
& \Longrightarrow T_0 = \int_0^{r_0} \frac{1}{h(r)} dr.
\end{aligned}$$

The conclusion now follows from the proof of Theorem 6.1, replacing equation (6.1) with equation (6.3). □

Remark 6.1. Note that the flow (6.2) is parabolic if $\frac{\partial f}{\partial \lambda_i} > 0$ $(1 \le i \le n)$; parabolic for backwards time if $\frac{\partial f}{\partial \lambda_i} < 0$ $(1 \le i \le n)$ and is a first-order partial differential equation (PDE) if f is constant.

The following corollary is a generalization of both mean curvature flow ($m = 1$, $\ell = 0$) and of harmonic mean curvature flow ($m = n$, $\ell = n - 1$).

Corollary 6.1. *Assume P^k is a compact totally geodesic submanifold of N^{n+1}, where $1 \le k \le n$. Let M be diffeomorphic to the unit sphere bundle of the normal bundle $\perp P$ when $k < n$; M is one of the two components of the unit sphere bundle of $\perp P$ when $k = n$.*

For integers $0 \le m$, $\ell \le n$, let S_m and S_ℓ be the elementary symmetric functions of degree m, ℓ, respectively, of the principal curvatures $\lambda_1, \dots, \lambda_n$ of M_t. We have a flow by curvature function

$$F(x,t) = \frac{S_m(\lambda_1, \dots, \lambda_n)}{S_\ell(\lambda_1, \dots, \lambda_n)},$$

for time $0 \le t < \infty$, such that $\varphi(t) : M \to N$ and $\varphi_t(M) \to P$ as $t \to +\infty$; assuming that the integers m, ℓ satisfy $|m - (n-k)| < |\ell - (n-k)|$.

Remark 6.2. Theorem 6.2 may also be applied to prove a partial converse of Corollary 6.1: assuming P^k and N^{n+1} are as in Corollary 6.1, if the opposite condition $|m - (n-k)| \ge |\ell - (n-k)|$ holds, then the same construction yields a flow of hypersurfaces by the curvature function $F = \frac{S_m}{S_\ell}$, which converges to the totally geodesic submanifold P^k in finite time T_0.

Proof. In the following, we fix an arbitrary positive constant $r(0) = r_0$. First, we have

$$S_m = \sum_{\substack{p+q=m \\ 0\le p\le k \\ 0\le q\le n-k}}^{n} C_k^p (\tanh r)^p\, C_{n-k}^q (\coth r)^q = \sum C_k^p C_{n-k}^q (\coth r)^{q-p},$$

where C_k^p is the combinatorial coefficient $\frac{k!}{p!(k-p)!}$.

Since $\coth r \ge 1$, it is easy to see

$$S_m \sim \begin{cases} (\coth r)^m & \text{if } m \le n-k, \\ (\coth r)^{2(n-k)-m} & \text{if } m > n-k, \end{cases}$$

where the notation $S_m \sim (\coth r)^j$ means that there exist positive constants C_1 and C_2 such that $C_1(\coth r)^j \le S_m \le C_2(\coth r)^j$. Here C_1 and C_2 will depend only on m, n, k, ℓ and r_0.

Similarly, we have

$$S_\ell \sim \begin{cases} (\coth r)^\ell & \text{if } \ell \le n-k, \\ (\coth r)^{2(n-k)-\ell} & \text{if } \ell > n-k. \end{cases}$$

Therefore,

$$F = \frac{S_m}{S_\ell} \sim \begin{cases} (\coth r)^{m-\ell} & \text{if } m, \ell \le n-k, \\ (\coth r)^{\ell-m} & \text{if } m, \ell > n-k, \\ (\coth r)^{2(n-k)-m-\ell} & \text{if } \ell \le n-k < m, \\ (\coth r)^{m+\ell-2(n-k)} & \text{if } m \le n-k < \ell. \end{cases}$$

By Theorem 6.2, we obtain that the flow exists forever if and only if the power of $\coth r$ is negative in the asymptotic estimate for F above. That is, if and only if m and ℓ satisfy one of the following conditions:

$$\begin{cases} m < \ell & \text{if } m, \ell \le n-k, \\ \ell < m & \text{if } m, \ell > n-k, \\ 2(n-k) < m+\ell & \text{if } \ell \le n-k < m, \\ m+\ell < 2(n-k) & \text{if } m \le n-k < \ell. \end{cases}$$

It is straightforward to see the above inequalities are equivalent to the inequality $|m-(n-k)| < |\ell-(n-k)|$, which is our conclusion. □

Remark 6.3. In particular, the cases $k = n$, $m = 1$ and $\ell = 0$ are the first examples we are aware of in the literature of a locally convex compact hypersurface flowing by mean curvature and converging smoothly to a submanifold in infinite time. In addition, the cases $k = n - 1$, $m = 0$ and $\ell = 1$ give examples of (backwards parabolic) inverse mean curvature flow existing forever and converging to a totally geodesic hypersurface. After reversing time to obtain parabolicity, this example of $-\frac{1}{H}$ flow is properly divergent as $t \to \infty$.

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