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Manifolds admitting both strongly irreducible and weakly reducible minimal genus Heegaard splittings

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We construct infinitely many manifolds admitting both strongly irreducible and weakly reducible minimal genus Heegaard splittings. Both closed manifolds and manifolds with boundary tori are constructed.

1. Introduction

The pioneering work of Casson and Gordon [1] shows that a minimal genus Heegaard splitting of an irreducible, non-Haken three-manifold is necessarily strongly irreducible; by contrast, Haken [2] showed that a minimal genus (indeed, any) Heegaard splitting of a composite three-manifold is necessarily reducible, and hence weakly reducible. The following question of Moriah [9] is therefore quite natural:

Question 1.1 ([9, Question 1.2]). Can a three-manifold M have both weakly reducible and strongly irreducible minimal genus Heegaard splittings?

We answer this question affirmatively:

Theorem 1.1. *There exist infinitely many closed, orientable three-manifolds of Heegaard genus 3, each admitting both strongly irreducible and weakly reducible minimal genus Heegaard splittings.*

Theorem 1.1 is proved in Section 3. In Remark 3.2 we offer a strategy to generalize Theorem 1.1 to construct examples of genus g, for each $g \geq 3$; it is easy to see that no such examples can exist if $g < 3$. In Section 4, we give examples of manifolds with one, two or three torus boundary components, each admitting both strongly irreducible and weakly reducible minimal genus Heegaard splittings. Moreover, for each manifold with two boundary components, we construct four minimal genus Heegaard surfaces, two weakly reducible, one separating the boundary components and one that does not, and similarly two strongly irreducible minimal genus Heegaard surfaces. For a precise statement, see Theorem 4.1.

In an effort to keep this article short, we refer the readers to Section 3 of [7] for definitions and background material. Unless otherwise stated, we follow the notations of that paper.

2. Preliminaries

2.1. Constructing strongly irreducible Heegaard splittings

In this section, we introduce a method for constructing strongly irreducible Heegaard splittings using two-bridge link exteriors; this is taken out of [6].

Definition 2.1.

(1) A *two-string tangle* $(B^3; t_1, t_2)$ is a pair of three-ball B^3 and two disjoint arcs t_1 and t_2 properly embedded in B^3.

(2) A tangle is called *two-string trivial tangle* if it is homeomorphic (as a triple) to $(D^2 \times [0,1]; \{p\} \times [0,1] : \{q\} \times [0,1])$, where D^2 is a two-disk and p and q are two distinct points in $\text{int}(D^2)$.

For $Y \subset X$ with $\dim X = \dim Y$, we denote the frontier of Y in X by $\text{Fr}_X(Y)$. Let $(B^3; t_1, t_2)$ be a two-string trivial tangle. Let $H = \text{cl}(B^3 \setminus (N(t_1) \cup N(t_2)))$ and $A_i = \text{Fr}_{B^3}(N(t_i))$, $i = 1, 2$. Note that H is a genus two handlebody, A_1 and A_2 are annuli in ∂H and the pair $\{A_1, A_2\}$ is primitive in H (see Figure 1), i.e., there exist pairwise disjoint meridian disks Δ_1, $\Delta_2 \subset H$ so that

(1) $\Delta_i \cap A_i$ is an essential arc in A_i $(i = 1, 2)$ and

(2) $\Delta_1 \cap A_2,\ \Delta_2 \cap A_1 = \emptyset$.

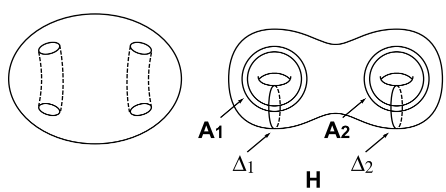

Figure 1: Exterior of a trivial tangle.

A link $L \subset S^3$ is called a two-bridge link, if it can be expressed as the union of two two-string trivial tangles; more precisely, if $(S^3; L) = (B; t_1, t_2) \cup (B'; t_1', t_2')$, where $(B; t_1, t_2)$ and $(B'; t_1', t_2')$ are two-string trivial tangles, $B \cap B' = \partial B = \partial B'$ and $L = (t_1 \cup t_1') \cup (t_2 \cup t_2')$. Note that in this paper by a two-bridge link we always mean a two-component link, and not a two-bridge knot.

Let $(H, A_1 \cup A_2)$ be as above and $(H', A_1' \cup A_2')$ be a copy of $(H, A_1 \cup A_2)$, $P = \text{cl}(\partial H \setminus (A_1 \cup A_2))$, and similarly, $P' = \text{cl}(\partial H' \setminus (A_1' \cup A_2'))$. Let L be a two-bridge link. Then we see from the above that there exists a homeomorphism $h : P \to P'$ such that $E(L)$, the exterior of L, is homeomorphic to $H \cup_h H'$ and $\partial E(L) = (A_1 \cup A_1') \cup (A_2 \cup A_2')$, so that ∂A_i and $\partial A_i'$ are meridian curves $(i = 1, 2)$. The image of $P = P'$ in $E(L)$ is called a bridge sphere.

Let N be a (possibly disconnected) orientable, irreducible, ∂-irreducible three-manifold such that ∂N consists of two tori T_1 and T_2 and each component of N has non-empty boundary (hence, N consists of at most two components). Suppose that there exists a three-dimensional sub-manifold $R \subset N$ such that

(1) each component of R is a handlebody and $\text{Fr}_N(R)$ is incompressible in N;

(2) $T_i \cap R$ $(i = 1, 2)$ consists of an annulus, say $\mathcal{A}_i$, such that
  (a) $\mathcal{A}_i$ is incompressible in N and
  (b) $\mathcal{A}_i$ is ∂-incompressible in R (i.e., there does not exist a disk properly embedded in R that intersects $\mathcal{A}_i$ in an essential arc);

(3) each component of $\text{cl}(N \setminus R) = R'$ is a handlebody such that $T_i \cap R'$ $(i = 1, 2)$ consists of an annulus, say $\mathcal{A}_i'$ satisfying
  (a) $\mathcal{A}_i'$ is incompressible in N and
  (b) $\mathcal{A}_i'$ is ∂-incompressible in R'.

With notations as above, let M be the three-manifold obtained from $E(L)$ and N by identifying their boundary by an orientation reversing homeomorphism $\partial N \to \partial(E(L))$ such that $\mathcal{A}_i$ ($\mathcal{A}_i'$ resp.) is mapped to A_i (A_i' resp.). Let $V = H \cup R \subset M$ and similarly $V' = H' \cup R' \subset M$. Since $A_1 \cup A_2$ is primitive in H, H is obtained by attaching a single one-handle to two solid tori, with A_1 a longitudinal annulus on one solid torus and A_2 a longitudinal annulus on the other. Hence, gluing H to R along $A_1 \cup A_2$ is equivalent to attaching a single one-handle to R, and similarly for H' and R'. We see that V (V' resp.) is a handlebody obtained from R (R' resp.) by attaching

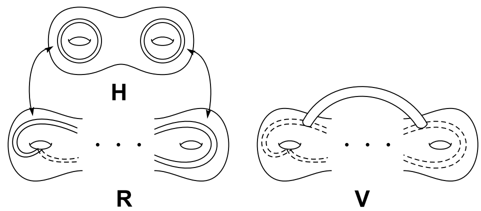

Figure 2: V is obtained from R by attaching a 1-handle.

a one-handle (see Figure 2), and therefore $V \cup V'$ is a Heegaard splitting of M. For this Heegaard splitting the following holds:

Proposition 2.1. *With notations as above, if L is not the trivial link or the Hopf link, then the Heegaard splitting $V \cup V'$ is strongly irreducible.*

Sketch of proof. Proposition 2.1 is identical to Proposition 3.1 of [6] and the proof can be found there. For the convenience of the readers, we sketch it here. Let $D \subset V$ and $D' \subset V'$ be a pair of meridian disks. Minimize the intersection of D with $A_1 \cup A_2$ and the intersection of D' with $A'_1 \cup A'_2$. By symmetry, we have the following three cases:

(1) $D \cap (A_1 \cup A_2) = \emptyset$ and $D' \cap (A'_1 \cup A'_2) = \emptyset$,

(2) $D \cap (A_1 \cup A_2) = \emptyset$ and $D' \cap (A'_1 \cup A'_2) \neq \emptyset$,

(3) $D \cap (A_1 \cup A_2) \neq \emptyset$ and $D' \cap (A'_1 \cup A'_2) \neq \emptyset$.

In the first case, D (resp. D') is the meridian disk of the tangle $(B; t_1, t_2)$ (resp. $(B'; t'_1, t'_2)$); since L is not the trivial link, D intersects D' more than twice. In the second case, D is the meridian disk of the tangles $(B; t_1, t_2)$. Consider an outermost disk on D', say δ'. Note that $\delta' \subset H'$. If the arc of δ' on A'_1 or A'_2 (say the latter) is inessential, we can surger D' along the disk component of $A'_2 \setminus \delta'$ to obtain a meridian disk of the tangle $(B'; t'_1, t'_2)$; the proof now is the same as the first case. Else, δ' gives a boundary compression for A'_1 or A'_2. Again, since L is not the trivial link, we see that $|D' \cap D| \geq |\delta' \cap D| > 1$.

In the third case, we consider outermost disks, δ on D, and δ' on D'. If the arc of δ on A_1 or A_2 is inessential, or the arc of δ' on A'_1 or A'_2 is inessential, then arguments similar to the above work. Suppose δ on A_1 or A_2 and δ' on A'_1 or A'_2 are essential. Since L is not the trivial link or the Hopf link, we see that $|D' \cap D| \geq |\delta' \cap \delta| \geq 1$. □

2.2. Spines of amalgamated Heegaard splittings.

A spine of a compression body C is a graph λ embedded in C so that $C \setminus (\lambda \cup \partial_- C)$ is homeomorphic to $\partial_+ C \times (-\infty, 0]$. Let $C \cup C'$ be a Heegaard splitting of a manifold M; a graph $\Gamma \subset M$ is a *spine for* C if there exists an ambient isotopy of M, so that the image of Γ after this isotopy is contained in C as a spine. *Simultaneous spines of* $C \cup C'$ are two disjointly embedded graphs Γ, $\Gamma' \subset M$, so that after an ambient isotopy of M, the image of Γ (Γ' resp.) is contained in C (C' resp.) as a spine.

For the definition of amalgamation of Heegaard splittings, see [10].

Proposition 2.2. *Let M_1 and M_2 be manifolds so that ∂M_1 and ∂M_2 are connected and homeomorphic. For $i = 1, 2$, let $H_i \cup C_i$ be Heegaard splittings of M_i, where H_i is a handlebody and C_i a compression body. Let μ_i (resp. λ_i) be a spine of H_i (resp. C_i). Let M be a manifold obtained by gluing M_1 and M_2 along their boundaries. Let $H \cup H'$ be the amalgamation of $H_1 \cup C_1$ and $H_2 \cup C_2$.*

Then there exist simultaneous spines of $H \cup H'$ so that $\mu_1 \cup \lambda_2$ is contained in a spine of H or H', and $\mu_2 \cup \lambda_1$ is contained in a spine of the other.

Proof. We denote the image of ∂M_i in M by F, the image of μ_i in M by μ_i and the image of λ_i in M by λ_i. By transversality, we assume as we may that $\lambda_1 \cap \lambda_2 = \emptyset$. The Heegaard surface that gives amalgamation of $H_1 \cup C_1$ and $H_2 \cup C_2$ is given by tubing F along λ_1 into M_1 and along λ_2 into M_2, see Figure 3 (this figure is based on Schultens' [10, Figure 3]). Note that the intersection of F and the amalgamated Heegaard surface is *not* transverse.

We may suppose that $\mu_1 \cup \lambda_2$ is contained in H and $\mu_2 \cup \lambda_1$ is contained in H'. By compressing H along the disks $\mathrm{cl}(\mathrm{int}(H) \cap F)$, we obtain two handlebodies. One handlebody is isotopic to H_1 and so we may take μ_1 as its spine. The other handlebody contains λ_2 and admits a deformation retract onto it; moreover, λ_2 intersects each disk of $\mathrm{cl}(\mathrm{int}(H) \cap F)$ at exactly one

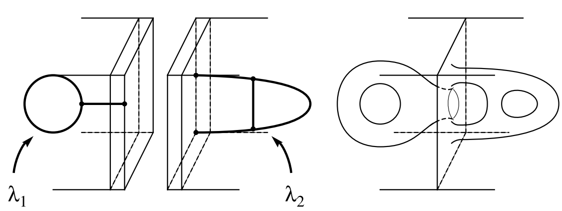

Figure 3: Amalgamation of Heegaard splittings.

point and has no other intersections with the boundary of this handlebody. Since the two handlebodies were obtained from H by compressing along the disks $\mathrm{cl}(\mathrm{int}(H) \cap F)$, it is easy to construct a spine for H by connecting λ_2 to μ_1. H' is treated similarly; the proposition follows. □

3. Proof of Theorem 1.1

We adopt the notations of Section 2.

Let 3_1 be the trefoil knot and 4_1 the figure eight knot. Let $L = l_1 \cup l_2$ be a hyperbolic two-bridge link. Denote $\partial N(l_i)$ by T_i $(i = 1, 2)$.

We note that there exists an essential annulus $\bar{A}$ in $E(3_1)$ such that the closures of the components of $E(3_1) \setminus \bar{A}$ are solid tori, say N_1 and N_1', where $\bar{A}$ wraps around N_1 longitudinally twice and around N_1' longitudinally three times. Hence, $N_1 \cap \partial E(3_1)$ and $N_1' \cap \partial E(3_1)$ are incompressible and boundary incompressible. On the other hand, we note that 4_1 is a genus 1 fibered knot. Hence we have the following: let $S \subset E(4_1)$ be a minimal genus Seifert surface for 4_1 (note that S is a once punctured torus). Let $N_2 = N(S)$ and $N_2' = \mathrm{cl}(E(4_1) \setminus N_2)$. Then N_2 (N_2' resp.) is homeomorphic to $S \times [0, 1]$, where $N_2 \cap \partial E(4_1)$ ($N_2' \cap \partial E(4_1)$ resp.) corresponds to $\partial S \times [0, 1]$. Note that $S \times [0, 1]$ is homeomorphic to a genus 2 handlebody, and $\partial S \times [0, 1]$ is incompressible and ∂-incompressible in $S \times [0, 1]$. Let P be a bridge sphere in $E(L)$. Then as in Section 2, P separates $E(L)$ into two genus 2 handlebodies, called H and H'. Finally, let M be a three-manifold obtained from $E(3_1) \cup E(4_1)$ and $E(L)$ by identifying their boundaries by a homeomorphism $h : (\partial E(3_1) \cup \partial E(4_1)) \to \partial E(L)(= T_1 \cup T_2)$, so that h satisfies the following conditions:

(1) $h(N_1 \cap \partial E(3_1)) = H \cap T_1$, hence $h(N_1' \cap \partial E(3_1)) = H' \cap T_1$.

(2) $h(N_2 \cap \partial E(4_1)) = H \cap T_2$, hence $h(N_2' \cap \partial E(4_1)) = H' \cap T_2$.

Note that the conditions of Proposition 2.1 are satisfied, and so we see that M admits a strongly irreducible genus 3 Heegaard splitting. Explicitly, the splitting surface is obtained from the bridge sphere P by attaching $\mathrm{Fr}_{E(4_1)} N_2$ (that is, two once-punctured tori) in $E(4_1)$ and $\bar{A}$ in $E(3_1)$. Denote this splitting by $V \cup_\Sigma V'$, where V and V' are the handlebodies of $N_1 \cup H \cup N_2$ and $N_1' \cup H' \cup N_2'$, respectively, and Σ is the splitting surface.

The decomposition $E(3_1) \cup E(L) \cup E(4_1)$ is the torus decomposition for M. In [4, Theorem], a complete list of Heegaard genus 2 three-manifolds admitting non-trivial torus decomposition is given. By consulting that list, we see that $g(M) > 2$. Above we constructed a strongly irreducible genus 3

Heegaard splitting for M. We conclude that $g(M) = 3$, and that M admits a strongly irreducible minimal genus Heegaard splitting.

We claim that the sub-manifold $E(3_1) \cup E(L)$ admits a genus 2 Heegaard splitting. Since A_1 is primitive in H and A_1' is primitive in H', $N_1 \cup H$ and $N_1' \cup H'$ are genus 2 handlebodies. Let $A = H \cap T_2$ and $A' = H' \cap T_2$. Let $C = \mathrm{cl}((N_1 \cup H) \setminus N(A, H))$ and $C' = (N_1' \cup H') \cup N(A, H)$. It is clear that C is a genus 2 handlebody. It is easy to see that A' is primitive in $N_1' \cup H'$, i.e., there is a meridian disk Δ' of $N_1' \cup H'$ such that $\Delta' \cap A'$ is an essential arc in A'. This implies that C' is a genus 2 compression body with $\partial_- C' = A \cup A' = T_2$. Denoting $\partial_+ C$ by Σ', we see that $C \cup_{\Sigma'} C'$ is a genus 2 Heegaard splitting of $E(3_1) \cup E(L)$.

Remark 3.1. For future reference, we note the following: Let α be a core curve of the solid torus N_1 and α' a core curve of the solid torus N_1'. By construction, α is contained in a spine of the handlebody C and α' is contained in a spine of the compression body C'. Similarly, the decomposition $M = \overline{C} \cup \overline{C}'$, where $\overline{C} = (N_1 \cup H) \cup N(A', H')$ and $\overline{C}' = \mathrm{cl}((N_1' \cup H') \setminus N(A', H'))$, gives another (possibly isotopic) genus 2 Heegaard splitting of $E(3_1) \cup E(L)$ so that α' is contained in a spine of the handlebody $\overline{C}'$ and α is contained in a spine of the compression body $\overline{C}$.

It is well known that $E(4_1)$ admits a genus 2 Heegaard splitting. By amalgamating a genus 2 Heegaard splitting for $E(4_1)$ with a genus 2 Heegaard splitting of $E(3_1) \cup E(L)$, we obtain a weakly reducible Heegaard splitting of M; by Schultens [10, Remark 2.7] (see also [7, Lemma 2.7] for a more general statement) this Heegaard splitting has genus 3. This establishes the existence of weakly reducible minimal genus Heegaard splittings of M.

This completes the proof of Theorem 1.1.

Remark 3.2. The following is a suggestion for a way to generalize the results of this paper. Fix $g \geq 3$. Let H (resp. H') be a genus $g - 1$ handlebody and $A_1, A_2 \subset \partial H$ (resp. $A_1', A_2' \subset \partial H'$) be two primitive annuli. Similar to the construction above, identify $\mathrm{cl}(\partial H \setminus (A_1 \cup A_2))$ with $\mathrm{cl}(\partial H' \setminus (A_1' \cup A_2'))$. To the resulting manifold, glue $E(3_1)$ and $E(4_1)$ in a way that $\partial H \setminus (A_1 \cup A_2)$ union two fibers of $E(4_1)$ union an essential annulus of $E(3_1)$ gives a genus g Heegaard splitting, say $V \cup V'$.

The curve complex distance of a Heegaard splitting was defined by Hempel [3] and was generalized by several authors to bridge decompositions; note that $H \cup H'$ is a genus $g - 3$, two-bridge decomposition (the link in question is the core of the attached solid tori when filling $H \cup H'$ along the slope defined by $H \cap H'$; see, for example, the proof of Proposition 2.2 of [8], where we defined generalized bridge decomposition in terms of a surface with boundary in the link exterior). It is reasonable to expect that if the distance of $H \cup H'$ is large, then $V \cup V'$ is strongly irreducible and minimal genus (Tomova's [11] should be useful here). Similar to the construction above, one obtains weakly reducible minimal genus Heegaard splittings by considering the decomposition $E(3_1) \cup H \cup H'$ and $E(4_1)$. This would give manifolds of genus g, for arbitrary $g \geq 3$, admitting both weakly reducible and strongly irreducible minimal genus Heegaard splittings.

4. Further examples: the bounded case

Throughout this section, let $M = E(3_1) \cup E(L) \cup E(4_1)$ be any of the manifolds constructed in the previous section. Let $V \cup_\Sigma V'$ be the strongly irreducible Heegaard splitting constructed there.

Let $\beta^* \subset E(4_1)$ be the simple, closed curve given in Figure 4. By Figure 4(a), β^* is contained in a once-punctured torus that is a fiber of the fibration of $E(4_1)$ over S^1. We may choose this fiber to be a component of $\Sigma \cap E(4_1)$.

Remark 4.1. We connect β^* to $\partial E(4_1)$ by an arc as in Figure 4(b). By using slideisotopy, we see that the exterior of a regular neighborhood of ($\partial E(4_1)$ together with the one-complex) is a genus 2 handlebody (see, for example, [5, Figures 5 and 6]). This shows that β^* is contained in a spine of a compression body (not handlebody) component of a genus 2 Heegaard splitting of $E(4_1)$.

Let α, α' be as in Remark 3.1, so that $\alpha \subset V$ and $\alpha' \subset V'$. Denote $\mathrm{cl}(M \setminus N(\alpha \cup \beta^* \cup \alpha'))$ by X. Denote the boundary components of X by $T_\alpha = \partial N(\alpha)$, $T_{\beta^*} = \partial N(\beta^*)$ and $T_{\alpha'} = \partial N(\alpha')$.

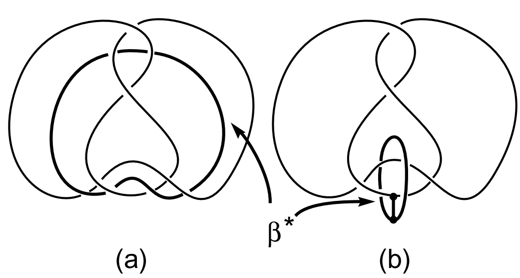

Figure 4: 4_1, β^*, and the arc connecting them.

Lemma 4.1. *X admits two genus three weakly reducible Heegaard surfaces, denoted by F_1 and F_2, so that*

(1) *F_1 separates $T_\alpha \cup T_{\beta^*}$ and $T_{\alpha'}$.*

(2) *F_2 separates T_α and $T_{\alpha'} \cup T_{\beta^*}$.*

Proof. By applying Proposition 2.2 to the Heegaard splitting $C \cup C'$ (recall Remark 3.1) and the genus 2 Heegaard splitting of $E(4_1)$ given in Remark 4.1 we obtain a genus 3 Heegaard splitting of M such that the Heegaard surface separates $\alpha \cup \beta^*$ and α', $\alpha \cup \beta^*$ is contained in a spine of one of the handlebodies and α' is contained in a spine of the other handlebody. This gives F_1.

Analogously, by applying Proposition 2.2 to the Heegaard splitting $\overline{C} \cup \overline{C}'$ (recall Remark 3.1) and the genus 2 Heegaard splitting of $E(4_1)$ given in Remark 4.1 we obtain F_2. □

Lemma 4.2. *$g(X) = 3$.*

Proof. Since M is obtained from X by Dehn filling, we have that $g(X) \geq g(M) = 3$. On the other hand, F_1 is a genus 3 Heegaard surface for X, showing that $g(X) \leq g(F_1) = 3$. □

Definition 4.1. Let C be a compression body and $\alpha_1, \dots, \alpha_n \subset C$ simple closed curves. We say that $\alpha_1, \dots, \alpha_n$ are *simultaneous cores*, if the following two conditions hold:

(1) There exist mutually disjoint annuli $A_1, \dots, A_n \subset C$ so that one component of ∂A_i is α_i and the other is on $\partial_+ C$.

(2) There exist mutually disjoint meridian disks $D_1, \dots, D_n \subset C$ so that α_i intersects D_i transversely at one point and for $i \neq j$, $\alpha_i \cap D_j = \emptyset$.

Remark 4.2. It is easy to see that $\alpha_1, \dots, \alpha_n \subset C$ are simultaneous cores if and only if $\text{cl}(C \setminus N(\cup_{i=1}^n \alpha_i))$ is a compression body.

Recall that $\beta^* \subset \Sigma \cap E(4_1)$. Let β (resp. β') be a curve obtained by pushing β^* slightly into V (resp. V').

Lemma 4.3. *The curves α, $\beta \subset V$ and α', $\beta' \subset V'$ are simultaneous cores.*

Proof. Recall the definition of the handlebody $H = V \cap E(L)$ given in Section 2, and let Δ_1 and Δ_2 be the meridian disks of H shown in Figure 1. Let

$\widehat{D}_\alpha$ be a meridian disk of the solid torus $N_1 = V \cap E(3_1)$ that intersects the annulus $\bar{A} = \Sigma \cap E(3_1)$ essentially. By attaching two copies of Δ_1 to $\widehat{D}_\alpha$, we obtain a meridian disk for V, denoted by D_α, that intersects α once and is disjoint from β.

Recall that $V \cap E(4_1)(= N_2)$ is homeomorphic to $S \times [0,1]$, where S is a once-punctured torus. We may suppose that β corresponds to a curve $\beta_S \times \{1/2\}$, where β_S is an essential curve on S. Let $\widehat{D}_\beta$ be a vertical disk in $V \cap E(4_1)$ that intersects β once, that is, $\widehat{D}_\beta$ corresponds to a disk of the form $\gamma \times [0,1]$, where γ is an arc properly embedded in S that intersects β_S transversely once. By attaching two copies of Δ_2 to $\widehat{D}_\beta$, we obtain a meridian disk for V, denoted by D_β, that intersects β once and is disjoint from α. It is easy to see that $D_\alpha \cap D_\beta = \emptyset$.

The annulus $A_\beta = \beta_S \times [1/2, 1]$ is embedded in V, with one boundary component β and the other on ∂V. Let $\widehat{A}_\alpha$ be an annulus embedded in N_1 with one boundary component α and the other on ∂N_1, which intersects the annulus $\bar{A}$ at three essential arcs. By attaching three copies of Δ_1 to $\widehat{A}_\alpha$, we obtain an annulus A_α embedded in V, with one boundary component α and the other on ∂V. By construction, $A_\alpha \cap A_\beta = \emptyset$.

Using D_α, D_β, A_α and A_β, we see that α and β are simultaneous cores. The curves α' and β' are treated similarly. □

Theorem 4.1. *For $i = 1, 2, 3$, there exists infinitely many manifolds M_i so that ∂M_i consists of exactly i tori, $g(M_i) = 3$, and each M_i admits both strongly irreducible and weakly reducible minimal genus Heegaard splittings.*

Moreover, each manifold M_2 admits four distinct minimal genus Heegaard surfaces, denoted as $F^{1,1}_{\mathrm{SI}}$, $F^{1,1}_{\mathrm{WR}}$, $F^{2,0}_{\mathrm{SI}}$, $F^{2,0}_{\mathrm{WR}}$, so that the following four conditions hold:

(1) *The Heegaard splittings given by $F^{1,1}_{\mathrm{SI}}$ and $F^{2,0}_{\mathrm{SI}}$ are strongly irreducible.*

(2) *The Heegaard splittings given by $F^{1,1}_{\mathrm{WR}}$ and $F^{2,0}_{\mathrm{WR}}$ are weakly reducible.*

(3) *$F^{1,1}_{\mathrm{SI}}$ and $F^{1,1}_{\mathrm{WR}}$ separate the two boundary components of M_2.*

(4) *$F^{2,0}_{\mathrm{SI}}$ and $F^{2,0}_{\mathrm{WR}}$ do not separate the boundary components of M_2.*

Before proving Theorem 4.1, we give the following definition.

Definition 4.2. Let Y_1 and Y_2 be manifolds so that Y_1 is obtained from Y_2 by Dehn filling (equivalently, Y_2 is obtained from Y_1 by removing an open regular neighborhood of a link in it). Note that $Y_2 \subset Y_1$.

Let $\Sigma_2 \subset Y_2$ be any Heegaard surface. Then Σ_2 is a Heegaard surface of Y_1. We say that $\Sigma_2 \subset Y_1$ is an *induced Heegaard surface* (or the *Heegaard surface induced by* Σ_2).

Let $\Sigma_1 \subset Y_1$ be a Heegaard surface. Suppose that $\Sigma_1 \subset Y_2$ and that Σ_1 is a Heegaard surface of Y_2. We say that $\Sigma_1 \subset Y_2$ is an *induced Heegaard surface* (or the *Heegaard surface induced by* Σ_1.)

The proof of the following lemma is easy and left to the readers:

Lemma 4.4. *Let Y_1 and Y_2 be as above. If a Heegaard surface Σ_2 of Y_2 is weakly reducible, then so is the induced Heegaard surface. On the other hand, if $\Sigma_1 \subset Y_1$ is a strongly irreducible Heegaard surface that induces a Heegaard surface for Y_2, then the induced Heegaard surface is strongly irreducible.*

Proof of Theorem 4.1. We deal with the cases $i = 1$, 2 and 3 in increasing order of difficulty.

For $i = 3$, let $M_3 = X$. Then by Lemmas 4.1 and 4.2, $g(X) = 3$ and X admits a weakly reducible minimal genus Heegaard splitting.

Note that β^* is isotopic to β; hence X is homeomorphic to $\mathrm{cl}(M \setminus N(\alpha \cup \alpha' \cup \beta))$. By Lemma 4.3 and Remark 4.2, $V \cup V'$ induces a genus 3 Heegaard splitting of $\mathrm{cl}(M \setminus N(\alpha \cup \alpha' \cup \beta))$. Since $V \cup V'$ is strongly irreducible, Lemma 4.4 shows that the induced Heegaard splitting is strongly irreducible. The case $i = 3$ follows.

For $i = 1$, let $M_1 = \mathrm{cl}(M \setminus N(\alpha))$. Then $g(M_1) \geq g(M) = 3$. Since X is obtained from M_1 by removing an open neighborhood of α' and β^*, $g(M_1) \leq g(X) = 3$. We see that $g(M_1) = 3$.

Note that M_1 is obtained by filling two boundary components of X. Hence the genus 3 weakly reducible Heegaard splittings for X given in Lemma 4.1 induces genus 3 weakly reducible Heegaard splittings for M_1.

By Lemma 4.3 and Remark 4.2, $V \cup V'$ induces a genus 3 Heegaard splitting for M_1. As above, the induced Heegaard splitting is strongly irreducible. The case $i = 1$ follows.

For $i = 2$, let $M_2 = \mathrm{cl}(M \setminus N(\alpha \cup \beta^*))$. Similar to M_1, it is easy to see that $g(M_2) = 3$.

By Lemma 4.4, each of the two genus 3 weakly reducible Heegaard splittings given in Lemma 4.1 induces a genus 3 weakly reducible Heegaard splitting on M_2, one not separating the components of ∂M_2 (corresponding to Lemma 4.1(1)), and the other separating them (corresponding to Lemma 4.1(2)). These are the surfaces $F^{2,0}_{\mathrm{WR}}$ and $F^{1,1}_{\mathrm{WR}}$ in the theorem.

Note that β (β' resp.) is isotopic to β^*; hence, M_2 is homeomorphic to $\mathrm{cl}(M \setminus N(\alpha \cup \beta))$ ($\mathrm{cl}(M \setminus N(\alpha \cup \beta'))$ resp.). By Lemma 4.3 and Remark 4.2, $V \cup V'$ induces a Heegaard splitting for $\mathrm{cl}(M \setminus N(\alpha \cup \beta))$ that does not separate the boundary components of $\mathrm{cl}(M \setminus N(\alpha \cup \beta))$. The corresponding Heegaard surface for M_2 is the surface $F_{\mathrm{SI}}^{2,0}$. Similarly, by Lemma 4.3 and Remark 4.2, $V \cup V'$ induces a Heegaard splitting for $\mathrm{cl}(M \setminus N(\alpha \cup \beta'))$ that separates the boundary components of $\mathrm{cl}(M \setminus N(\alpha \cup \beta'))$. The corresponding Heegaard surface for M_2 is the surface $F_{\mathrm{SI}}^{1,1}$. By Lemma 4.4, $F_{\mathrm{SI}}^{2,0}$ and $F_{\mathrm{SI}}^{1,1}$ are strongly irreducible. The case $i = 2$ follows. □

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