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Heat trace asymptotics with singular weight functions

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We study the weighted heat trace asymptotics of an operator of Laplace type with Dirichlet boundary conditions where the weight function exhibits radial blowup. We give formulas for the first few terms in the expansion in terms of geometrical data.

1. Introduction

1.1. Motivation

The asymptotic analysis of the heat trace provides a natural link between the spectrum of Laplace-type operators $\mathcal{D}$ acting on functions on a m dimensional Riemannian manifold M and the underlying geometry of M. For small time t it links the distribution of the large energy part of the spectrum of $\mathcal{D}$ to local geometric invariants of M and its boundary which show up in its asymptotic expansion. These invariants play an important in many physical phenomena, e.g., in quantum statistical mechanics when taking the large volume limit or in the Casimir effect [9]. Typically the coefficient of the leading $t^{-m/2}$ term in the heat trace expansion for small t is determined by the interior (volume) of M. In many situations a detailed study of the boundary behaviour of the heat kernel associated with $\partial_t + \mathcal{D}$ is desirable. One way of obtaining this information is putting a weight in the evaluation of the heat trace. In the setting of the heat content of M this corresponds to giving M a non-uniform specific heat. It is well known that the diagonal element at $(x, x; t)$ of the Dirichlet heat kernel associated to $\mathrm{e}^{-t\mathcal{D}}$ vanishes like r^2 where $r = r(x)$ is the distance to the boundary. This allows the weights to diverge like $r^{-\alpha}$, where $\mathrm{Re}(\alpha) < 3$. We will show, using pseudo differential calculus, that a modified asymptotic series still exists in this case. For example, if $1 < \alpha < 3$ the leading behaviour of the heat trace is $t^{(1-m-\alpha)/2}$ with a coefficient determined by an integral over the boundary of M.

1.2. The heat equation

We adopt the *Einstein convention* and sum over repeated indices. Let M be a compact smooth Riemannian manifold of dimension m and with smooth boundary ∂M. Let V be a smooth vector bundle over M and let $\mathcal{D}$ be an operator of Laplace type on the space $C^\infty(V)$ of smooth sections to V. This means that the leading symbol of $\mathcal{D}$ is given by the metric tensor or, equivalently, that we may express in any system of local coordinates $x = (x^1, \ldots, x^m)$ and relative to any local frame for V the operator $\mathcal{D}$ in the form:

$$\mathcal{D} = -\left\{g^{\mu\nu}\,\mathrm{Id}\,\partial_{x_\mu}\partial_{x_\nu} + A_1^\nu\partial_{x_\nu} + A_0\right\}. \tag{1.1}$$

In Equation (1.1), let $1 \le \mu, \nu \le m$, let A_1^ν and A_0 be smooth endomorphisms (matrices), and let $g^{\mu\nu}$ be the inverse of the metric $g_{\mu\nu} := g(\partial_{x_\mu}, \partial_{x_\nu})$. Note that the Riemannian measure dx on M is given by

$$dx := g dx^1 \ldots dx^m, \quad \text{where} \quad g := \sqrt{\det(g_{\mu\nu})}.$$

Thus, for example, the scalar Laplacian $\Delta_M := \delta d$ is of Laplace type since

$$\Delta_M = -\left(g^{\mu\nu}\partial_{x_\nu}\partial_{x_\mu} + g^{-1}\partial_{x_\nu}\left\{g g^{\mu\nu}\right\}\partial_{x_\mu}\right). \tag{1.2}$$

We shall use the *Dirichlet realization* of the operator $\mathcal{D}$. For $t > 0$ and for $\phi \in L^2(V)$, the heat equation

$$(\partial_t + \mathcal{D})u(x;t) = 0, \quad u(\cdot;t)|_{\partial M} = 0, \quad \lim_{t\downarrow 0} u(\cdot;t) = \phi(\cdot) \text{ in } L^2(V)$$

has a solution $u = \mathrm{e}^{-t\mathcal{D}}\phi$ which is smooth in $(x;t)$. The operator $\mathrm{e}^{-t\mathcal{D}}$ has a kernel $p_{\mathcal{D}}(x,\tilde{x};t)$ which is smooth in $(x,\tilde{x};t)$ such that

$$u(x;t) = \int_M p_{\mathcal{D}}(x,\tilde{x};t)\phi(\tilde{x})d\tilde{x}.$$

In the case of the scalar Laplacian Δ_M, there is a complete orthonormal basis $\{\phi_i\}$ for $L^2(M)$ where the $\phi_i \in C^\infty(M)$ satisfy $\phi_i|_{\partial M} = 0$ and $\Delta_M\phi_i = \lambda_i\phi_i$. The corresponding Dirichlet heat kernel $p_M := p_{\Delta_M}$ is given in terms of the *spectral resolution* $\{\phi_i, \lambda_i\}$ via

$$p_M(x,\tilde{x};t) = \sum_i \mathrm{e}^{-t\lambda_i}\phi_i(x)\bar{\phi}_i(\tilde{x}).$$

1.3. Heat trace asymptotics in the smooth setting

We use the geodesic flow defined by the unit inward normal vector field to define a diffeomorphism for some $\varepsilon > 0$ between the collar $\mathcal{C}_\varepsilon := \partial M \times [0, \varepsilon]$ and a neighbourhood of the boundary in M which identifies $\partial M \times \{0\}$ with ∂M; the curves $r \to (y_0, r)$ for $r \in [0, \varepsilon]$ are then unit speed geodesics perpendicular to the boundary and r is the geodesic distance to the boundary. Let $F \in C^\infty(M)$ be an auxiliary weight function which is used for localization. On $\mathcal{C}_\varepsilon$, expand F in a Taylor series

$$F(y, r) \sim \sum_{i=0}^{\infty} F_i(y) r^i, \quad \text{where} \quad F_i = \tfrac{1}{i!} (\partial_r)^i F\big|_{r=0}.$$

Henceforth we shall let Tr denote the fibre trace and Tr_{L^2} denote the global L^2 trace. We then have:

$$\mathrm{Tr}_{L^2}(F \mathrm{e}^{-t\mathcal{D}}) = \int_M F(x) \,\mathrm{Tr}\{p_{\mathcal{D}}(x, x; t)\} dx. \tag{1.3}$$

We note for future reference that on the diagonal, the heat kernel $p_{\mathbb{R}^m}(x, x; t)$ for $\mathbb{R}^m$ and the heat kernel $p_H(x, x; t)$ on the half space $H := \{x : x_1 > 0\}$ of the scalar Laplacian are given by

$$p_{\mathbb{R}^m}(x, x; t) = (4\pi t)^{-m/2} \quad \text{and} \quad p_H(x, x; t) = (4\pi t)^{-m/2}(1 - \mathrm{e}^{-r^2/t}). \tag{1.4}$$

Let dy be the Riemannian measure on the boundary. To simplify future expressions, we set

$$\mathcal{I}\{F\} = \int_M F dx \quad \text{and} \quad \mathcal{I}^{bd}\{F\} = \int_{\partial M} F dy.$$

We will also use the notation $\mathcal{I}\{F d\nu\}[U]$ when it is necessary to specify the domain of integration U and/or the measure $d\nu$.

Theorem 1.1. *Let M be a compact smooth Riemannian manifold. Let $\mathcal{D}$ be the Dirichlet realization of an operator of Laplace type. Let $F \in C^\infty(M)$.*

(1) *There is a complete asymptotic expansion as $t \downarrow 0$ of the form:*

$$\mathrm{Tr}_{L^2}(F \mathrm{e}^{-t\mathcal{D}}) \sim t^{-m/2} \sum_{n=0}^{\infty} t^n a_n(F, \mathcal{D}) + t^{-(m-1)/2} \sum_{\ell=0}^{\infty} t^{\ell/2} a_\ell^{bd}(F, \mathcal{D}).$$

(2) *There are local invariants* $a_n = a_n(x, \mathcal{D})$ *defined on* M *and there are local invariants* $a_{\ell,i}^{bd} = a_{\ell,i}^{bd}(y, \mathcal{D})$ *defined on* ∂M *for* $0 \le i \le \ell$ *so that*

$$a_n(F, \mathcal{D}) = \mathcal{I}\{F a_n\} \quad \textit{and} \quad a_\ell^{bd}(F, \mathcal{D}) = \sum_{i=0}^{\ell} \mathcal{I}^{bd}\{F_i a_{\ell,i}^{bd}\}.$$

We refer to [8,13] for a proof of Theorem 1.1 where more general results are obtained in the context of elliptic operator theory and elliptic boundary conditions. We shall illustrate these formulas in Theorem 1.2 below. We add a caution that the notation we have chosen differs slightly from what is employed elsewhere.

1.4. The Bochner Laplacian

Before discussing the formulas in Theorem 1.1 in further detail, we must introduce the formalism of the Bochner Laplacian which will permit us to work in a tensorial and coordinate free fashion. If ∇ is a connection on V, then we use ∇ and the Levi–Civita connection defined by the metric to covariantly differentiate tensors of all types. Let ';' denote the components of multiple covariant differentiation — in particular, $\phi_{;\mu\nu}$ are the components of $\nabla^2\phi$. If E is an auxiliary endomorphism of V, we define the associated *modified Bochner Laplacian* by setting

$$\mathcal{D}(g, \nabla, E)\phi := -g^{\mu\nu}\phi_{;\nu\mu} - E\phi. \tag{1.5}$$

Let $\Gamma_{\mu\nu\sigma}$ and $\Gamma_{\mu\nu}{}^{\sigma}$ be the Christoffel symbols of the Levi–Civita connection on M. We have, adopting the notation of Equations (1.1) and (1.5), the following (see [7]):

Lemma 1.1. *If* $\mathcal{D}$ *is an operator of Laplace type, then there exists a unique connection* ∇ *on* V *and a unique endomorphism* E *on* V *so that* $\mathcal{D} = \mathcal{D}(g, \nabla, E)$. *The associated connection* 1*-form* ω *of* $\nabla = \nabla(\mathcal{D})$ *and the associated endomorphism* $E = E(\mathcal{D})$ *are given by:*

(1) $\omega_\mu = \frac{1}{2}(g_{\mu\nu}A_1^\nu + g^{\sigma\varepsilon}\Gamma_{\sigma\varepsilon\mu}\,\mathrm{Id})$.

(2) $E = A_0 - g^{\mu\nu}(\partial_{x_\nu}\omega_\mu + \omega_\mu\omega_\nu - \omega_\sigma\Gamma_{\mu\nu}{}^{\sigma})$.

1.5. Formulae for the heat trace asymptotics in the smooth setting

Let indices i, j, k, l range from 1 to m and index a local orthonormal frame $\{e_1, \dots, e_m\}$ for the tangent bundle of M. Let R_{ijkl} be the components of the Riemann curvature tensor; our sign convention is chosen so that $R_{1221} = +1$ on the sphere of radius 1 in $\mathbb{R}^3$. On the collar $\mathcal{C}_\varepsilon$, we normalize the choice of the local frame by requiring that $e_m = \partial_r$ is the inward unit geodesic normal. We let indices a, b, c, d range from 1 through $m-1$ and index the restricted orthonormal frame $\{e_1, \dots, e_{m-1}\}$ for the tangent bundle of the boundary. Let $L_{ab} := g(\nabla_{e_a} e_b, e_m)$ be the components of the second fundamental form. The scalar invariant L_{aa} is the unnormalized mean curvature (i.e. the geodesic curvature) and will play a central role in our investigation. One has the following formulae; note that $F_2 = \frac{1}{2} F_{;mm}$:

Theorem 1.2. *Let M be a compact smooth Riemannian manifold. Let $\mathcal{D}$ be the Dirichlet realization of an operator of Laplace type. Let $F \in C^\infty(M)$.*

(1) $a_0(F, \mathcal{D}) = (4\pi)^{-m/2} \mathcal{I}\{\mathrm{Tr}(F \,\mathrm{Id})\}$.

(2) $a_1(F, \mathcal{D}) = \frac{1}{6}(4\pi)^{-m/2} \mathcal{I}\{\mathrm{Tr}(6FE + FR_{ijji} \,\mathrm{Id})\}$.

(3) $a_0^{bd}(F, \mathcal{D}) = -\frac{1}{4}(4\pi)^{-(m-1)/2} \mathcal{I}^{bd}\{\mathrm{Tr}(F_0 \,\mathrm{Id})\}$.

(4) $a_1^{bd}(F, \mathcal{D}) = \frac{1}{6}(4\pi)^{-m/2} \mathcal{I}^{bd}\{\mathrm{Tr}(2F_0 L_{aa} \,\mathrm{Id} - 3F_1 \,\mathrm{Id})\}$.

(5) $$a_2^{bd}(F, \mathcal{D}) = -\tfrac{1}{384}(4\pi)^{-(m-1)/2} \mathcal{I}^{bd}\{\mathrm{Tr}(F_0(96E + [16R_{ijji} - 8R_{amma} \\ +7L_{aa}L_{bb} - 10L_{ab}L_{ab}] \,\mathrm{Id}) - 30F_1 L_{aa} \,\mathrm{Id} + 48F_2 \,\mathrm{Id})\}.$$

Formulas for the invariants $a_n(F, \mathcal{D})$ and $a_\ell^{bd}(F, \mathcal{D})$ are known for $n, \ell = 2, 3, 4, 5$. We refer to [9] for further details as the literature is vast and beyond the scope of the present paper to survey.

1.6. Singular weight functions

Fix $\alpha \in \mathbb{C}$. Let F be a smooth function on the interior of M such that $Fr^\alpha \in C^\infty(\mathcal{C}_\varepsilon)$; the parameter α controls the growth (if $\mathrm{Re}(\alpha) > 0$) or decay (if $\mathrm{Re}(\alpha) < 0$) of F near the boundary, assuming that Fr^α does not vanish identically on the boundary. We may expand $F|_{\mathcal{C}_\varepsilon}$ in a modified Taylor series:

$$F(y,r) \sim \sum_{i=0}^{\infty} F_i(y) r^{i-\alpha}, \quad \text{where} \quad F_i(y) = \tfrac{1}{i!}(\partial_r)^i (r^\alpha F)\big|_{r=0}.$$

1.7. Geometry near the boundary

The Riemannian measure is in general not product near the boundary, i.e., $dx \neq drdy$, and this plays an important role in our development. Let indices σ, ϱ range from 1 to $m-1$ and index the coordinate frame $\{\partial_{y_1}, \dots, \partial_{y_{m-1}}\}$ for the tangent bundle of the boundary. One may express the metric on the collar $\mathcal{C}_\varepsilon$ in the form:

$$ds_M^2 = g_{\sigma\varrho}(y,r) dy^\sigma \circ dy^\varrho + dr^2.$$

Fix $y_0 \in \partial M$ and choose the local coordinates so that $g_{\sigma\varrho}(y_0, 0) = \delta_{\sigma\varrho}$. Then we have that:

(1.6)
$$\begin{aligned} L_{\sigma\varrho} &= g(\partial_r, \nabla_{\partial_{x_\sigma}} \partial_{x_\varrho}) = \Gamma_{\sigma\varrho}{}^m = -\tfrac{1}{2}\partial_r g_{\sigma\varrho}, \\ g_M(y_0, r) &= \sqrt{\det\{\mathrm{Id} + \partial_r g_{\sigma\varrho}(y_0,0)\cdot r + O(r^2)\}} = 1 - rL_{aa}(y_0) + O(r^2), \\ dx &= (1 - rL_{aa}) drdy + O(r^2). \end{aligned}$$

Example 1.1. Let $x_1 = \zeta\cos\theta$ and $x_2 = \zeta\sin\theta$ be the usual polar coordinates on the unit disk in $\mathbb{R}^2$. One then has that $ds^2 = d\zeta^2 + \zeta^2 d\theta^2$ so $dx = \zeta d\theta d\zeta$. The geodesic distance to the boundary circle is given by $r = 1-\zeta$; thus $g_{\theta\theta} = (1-r)^2$ and $L_{aa} = 1$ so $dx = (1-r)drdy$.

1.8. Regularization

Before discussing the asymptotic expansion of the heat trace in the singular case, we must first discuss regularization; an analogous regularization was required when discussing the heat content for singular initial temperatures in [4]. Let H be smooth on the interior of M with $Hr^\alpha \in C^\infty(\mathcal{C}_\varepsilon)$. Then

$$\begin{aligned} dx &= (1 - rL_{aa}) drdy + O(r^2), \\ Hdx &= \{H_0 r^{-\alpha} + (H_1 - H_0 L_{aa}) r^{1-\alpha}\} drdy + O(r^{2-\alpha} drdy). \end{aligned}$$

For $\mathrm{Re}(\alpha) < 3$, define

$$\begin{aligned}\mathcal{I}_{\mathrm{Reg}}\{H\} := {} & \mathcal{I}\{H dx\}[M - \mathcal{C}_\varepsilon] \\ & + \mathcal{I}\left\{H dx - \left[H_0 r^{-\alpha} + (H_1 - H_0 L_{aa}) r^{1-\alpha}\right] dr dy\right\}[\mathcal{C}_\varepsilon] \\ & + \mathcal{I}^{bd}\{H_0\} \times \begin{cases} \dfrac{\varepsilon^{1-\alpha}}{1-\alpha} & \text{if } \alpha \neq 1, \\ \ln(\varepsilon) & \text{if } \alpha = 1. \end{cases} \\ & + \mathcal{I}^{bd}\{H_1 - H_0 L_{aa}\} \times \begin{cases} \dfrac{\varepsilon^{2-\alpha}}{2-\alpha} & \text{if } \alpha \neq 2, \\ \ln(\varepsilon) & \text{if } \alpha = 2. \end{cases}\end{aligned} \tag{1.7}$$

This is independent of the parameter ε and agrees with $\mathcal{I}\{H\}$ if $\mathrm{Re}(\alpha) < 1$. Because the integrand over $\mathcal{C}_\varepsilon$ is $O(r^{2-\mathrm{Re}(\alpha)})$ and $\mathrm{Re}(2-\alpha) > -1$, $\mathcal{I}_{\mathrm{Reg}}$ is well defined.

The regularization $\mathcal{I}_{\mathrm{Reg}}$ is a meromorphic function of α with simple poles at $\alpha = 1, 2$. At these exceptional values, $\mathcal{I}_{\mathrm{Reg}}$ is defined as the constant term in the appropriate Laurent expansion, thus dropping the pole. We shall apply this regularization to functions of the form $H(x) = F(x) a_n(x, \mathcal{D})$.

1.9. Heat trace asymptotics in the singular setting

The Dirichlet heat kernel satisfies $p_{\mathcal{D}}(x, (\tilde{y}, \tilde{r}), t)|_{\tilde{r}=0} = 0$. Since $p_{\mathcal{D}}$ is smooth for $t > 0$ and $\mathcal{C}_\varepsilon$ is compact, we may use the Taylor series expansion of $p_{\mathcal{D}}$ to derive the estimate:

$$|p_{\mathcal{D}}(x, (\tilde{y}, \tilde{r}); t)| \leq C(t)\tilde{r} \quad \text{on} \quad \mathcal{C}_\varepsilon.$$

A similar estimate holds for $|p_{\mathcal{D}}((y, r), \tilde{x}; t)|$. We set $\tilde{x} = (y, r)$ to derive the estimate on the diagonal:

$$|p_{\mathcal{D}}((y, r), (y, r); t)| \leq C(t) r^2 \quad \text{on} \quad \mathcal{C}_\varepsilon. \tag{1.8}$$

Thus if $\mathrm{Re}(\alpha) < 3$, then Equation (1.3) shows that $\mathrm{Tr}_{L^2}(F e^{-t\mathcal{D}})$ is convergent.

We shall begin our investigation in Section 2 by establishing the following result by a direct computation as this motivates our entire investigation; note that only limited smoothness is required of the boundary in this result. It has the further advantage of confirming by completely different means some of the constants that will be computed again in Sections 4 and 5. Let C here and elsewhere denote Euler's constant.

Theorem 1.3. *Let $M \subset \mathbb{R}^2$ be an open, bounded, and connected planar set with C^2 boundary. Let $0 < \varepsilon_0 < \varepsilon$. Set $F(x) := F_0(y) r^{-\alpha} \chi(r)$ where $\chi \in C^\infty(\mathcal{C}_\varepsilon)$ satisfies:*

$$\chi(r) = \begin{cases} 1 & \text{if } 0 \le r \le \varepsilon_0, \\ 0 & \text{if } \varepsilon \le r. \end{cases}$$

(1) *If $0 < \alpha < 1$ and $t \downarrow 0$, then:*

$$\begin{aligned}\mathrm{Tr}_{L^2}(F\mathrm{e}^{-t\Delta_M}) = \frac{1}{4\pi t}\{&\mathcal{I}\{F\} - \frac{1}{2}\Gamma\left(\frac{1-\alpha}{2}\right) t^{(1-\alpha)/2} \mathcal{I}^{bd}\{F_0\} \\ &+ \frac{4-\alpha}{4(3-\alpha)}\Gamma\left(\frac{2-\alpha}{2}\right) t^{1-\alpha/2}\mathcal{I}^{bd}\{F_0 L_{aa}\}\} + O(1).\end{aligned}$$

(2) *If $\alpha = 1$ and $t \downarrow 0$, then:*

$$\begin{aligned}\mathrm{Tr}_{L^2}\left(F\mathrm{e}^{-t\Delta_M}\right) = &\frac{1}{4\pi t}\left\{\mathcal{I}_{\mathrm{Reg}}\{F\} - \tfrac{1}{2}\ln(t)\cdot\mathcal{I}^{bd}\{F_0\} + \tfrac{C}{2}\mathcal{I}^{bd}\{F_0\}\right\} \\ &+ O(t^{-1/2}).\end{aligned}$$

This result extends to a very general setting. The following Theorem generalizes Theorem 1.1 to the singular setting where, in contrast to Theorem 1.3, we assume the boundary is C^∞. We also refer to [4] for further details where an analogous result was proved for the heat content asymptotics. In Section 5, we will use the pseudo-differential calculus to show that:

Theorem 1.4. *Let M be a compact smooth Riemannian manifold. Let $\mathcal{D}$ be the Dirichlet realization of an operator of Laplace type. Let $a_n = a_n(x, \mathcal{D})$ be the interior local heat trace asymptotics of Theorem 1.1. Fix $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) < 3$. Let F be a smooth function on the interior of M such that $Fr^\alpha \in C^\infty(\mathcal{C}_\varepsilon)$.*

(1) *If $\alpha \ne 1, 2$, there is a complete asymptotic expansion as $t \downarrow 0$ of the form:*

$$\mathrm{Tr}_{L^2}(F\mathrm{e}^{-t\mathcal{D}}) \sim t^{-m/2}\sum_{n=0}^{\infty} t^n \mathcal{I}_{\mathrm{Reg}}\{Fa_n\} + t^{-(m-1)/2}\sum_{\ell=0}^{\infty} t^{(\ell-\alpha)/2} a^{bd}_{\ell,\alpha}(F, \mathcal{D}).$$

(2) *If $\alpha = 1, 2$, there is a complete asymptotic expansion as $t \downarrow 0$ of the form:*

$$\mathrm{Tr}_{L^2}(F\mathrm{e}^{-t\mathcal{D}}) \sim t^{-m/2} \sum_{n=0}^{\infty} t^n \mathcal{I}_{\mathrm{Reg}}\{Fa_n\} + t^{-(m-1)/2} \sum_{\ell=0}^{\infty} t^{(\ell-\alpha)/2} a^{bd}_{\ell,\alpha}(F, \mathcal{D})$$
$$+ \ t^{-m/2} \ln(t) \sum_{k=0}^{\infty} t^{k/2} \tilde{a}^{bd}_{k,\alpha}(F, \mathcal{D}).$$

(3) *There exist local invariants $a^{bd}_{\ell,\alpha,i} = a^{bd}_{\ell,\alpha,i}(y, \mathcal{D})$ on ∂M, which are holomorphic in α for $\alpha \neq 1, 2$, so that*

$$a^{bd}_{\ell,\alpha}(F, \mathcal{D}) = \sum_{i=0}^{\ell} \mathcal{I}^{bd}\{F_i a^{bd}_{\ell,\alpha,i}\}.$$

The invariants $a^{bd}_{\ell,z,i}$ have simple poles at $z = 1, 2$ and

$$a^{bd}_{\ell,\alpha,i} = \left\{ a^{bd}_{\ell,z,i} - \frac{1}{z-\alpha} \mathrm{Res}_{z=\alpha}\, a^{bd}_{\ell,z,i} \right\}\Bigg|_{z=\alpha} \quad \textit{if} \quad \alpha = 1, 2.$$

(4) *The $\ln(t)$ coefficients in Assertion (2) are given by*

$$\tilde{a}^{bd}_{k,\alpha}(F, \mathcal{D}) = \begin{cases} -\frac{1}{2}\mathcal{I}^{bd}\{(Fa_n)_0\} & \textit{if } k = 2n \textit{ and } \alpha = 1, \\ -\frac{1}{2}\mathcal{I}^{bd}\{(Fa_n)_1 - (Fa_n)_0 L_{aa}\} & \textit{if } k = 2n \textit{ and } \alpha = 2, \\ 0 & \textit{if } k = 2n+1. \end{cases}$$

Throughout this paper, let

$$\kappa_\alpha := \tfrac{1}{2}\Gamma\left(\tfrac{1-\alpha}{2}\right).$$

The boundary invariants for $\alpha \neq 1, 2$ and for $\ell = 0, 1, 2$ are given by:

Theorem 1.5. *If $\alpha \neq 1, 2$, then one has:*

(1) $a^{bd}_{0,\alpha}(F, \mathcal{D}) = \kappa_\alpha (4\pi)^{-m/2} \mathcal{I}^{bd}\{\mathrm{Tr}(-F_0\,\mathrm{Id})\}$.

(2) $a^{bd}_{1,\alpha}(F, \mathcal{D}) = \kappa_{\alpha-1} (4\pi)^{-m/2} \mathcal{I}^{bd}\{\mathrm{Tr}(-F_1\,\mathrm{Id} + \frac{\alpha-4}{2(\alpha-3)} F_0 L_{aa}\,\mathrm{Id})\}$.

(3) $a^{bd}_{2,\alpha}(F, \mathcal{D}) = \kappa_{\alpha-2} (4\pi)^{-m/2} \mathcal{I}^{bd}\{\mathrm{Tr}(-F_2\,\mathrm{Id} + \frac{\alpha-5}{2(\alpha-4)} F_1 L_{aa}\,\mathrm{Id}$
$+\frac{1}{6} F_0 R_{amma}\,\mathrm{Id} - \frac{\alpha-7}{8(\alpha-6)} F_0 L_{aa} L_{bb}\,\mathrm{Id} + \frac{\alpha-5}{4(\alpha-6)} F_0 L_{ab} L_{ab}\,\mathrm{Id}$
$-\frac{1}{3(1-\alpha)} F_0 R_{ijji}\,\mathrm{Id} - \frac{2}{1-\alpha} F_0 E)\}$.

We remark that we recover Theorem 1.2 by setting $\alpha = 0$ in Theorem 1.5. We omit details as the calculation is entirely elementary. The boundary invariants for $\alpha = 1, 2$, and for $\ell = 0, 1, 2$ are given by:

Theorem 1.6.

(1) *When* $\alpha = 1$,
 (a) $a^{bd}_{0,1}(F, \mathcal{D}) = (4\pi)^{-m/2}\mathcal{I}^{bd}\{\mathrm{Tr}(\frac{C}{2}F_0 \,\mathrm{Id})\}$.
 (b) $a^{bd}_{1,1}(F, \mathcal{D}) = (4\pi)^{-m/2}\frac{\sqrt{\pi}}{2}\mathcal{I}^{bd}\{\mathrm{Tr}(-F_1 \,\mathrm{Id} + \frac{3}{4}F_0 L_{aa} \,\mathrm{Id})\}$.
 (c) $a^{bd}_{2,1}(F, \mathcal{D}) = (4\pi)^{-m/2}\mathcal{I}^{bd}\left\{\mathrm{Tr}\left(-\frac{1}{2}F_2 \,\mathrm{Id} + \frac{1}{3}F_1 L_{aa} \,\mathrm{Id} + \frac{1}{12}R_{amma} \,\mathrm{Id} - \frac{3}{40}L_{aa}L_{bb} \,\mathrm{Id} + \frac{1}{10}L_{ab}L_{ab} \,\mathrm{Id} + \frac{C}{12}R_{ijji} \,\mathrm{Id} + \frac{C}{2}E\right)\right\}$.

(2) *When* $\alpha = 2$,
 (a) $a^{bd}_{0,2}(F, \mathcal{D}) = (4\pi)^{-m/2}\sqrt{\pi}\mathcal{I}^{bd}\{\mathrm{Tr}(F_0 \,\mathrm{Id})\}$.
 (b) $a^{bd}_{1,2}(F, \mathcal{D}) = (4\pi)^{-m/2}\mathcal{I}^{bd}\{\mathrm{Tr}(\frac{C}{2}F_1 \,\mathrm{Id} - [\frac{C}{2} + \frac{1}{2}]F_0 L_{aa} \,\mathrm{Id})\}$.
 (c) $a^{bd}_{2,2}(F, \mathcal{D}) = (4\pi)^{-m/2}\sqrt{\pi}\mathcal{I}^{bd}\{\mathrm{Tr}(-\frac{1}{2}F_2 \,\mathrm{Id} + \frac{3}{8}F_1 L_{aa} \,\mathrm{Id} + F_0(\frac{1}{12}R_{amma} - \frac{5}{64}L_{aa}L_{bb} + \frac{3}{32}L_{ab}L_{ab} + \frac{1}{6}R_{ijji}) \,\mathrm{Id} + F_0 E)\}$.

Here is a brief guide to the remainder of this paper. In Section 2, we will make a special case calculation to establish Theorem 1.5. A probabilistic estimate of Lang [10] and of Lerche and Siegmund [11] plays a central role. In Section 3, we shall use dimensional analysis (scaling arguments) and various functorial properties to study the heat trace invariants. We will derive Theorem 1.4 (4) from the asymptotic series in Theorem 1.4 (3); another derivation will be given subsequently in Section 5. We shall examine the general form of the invariants and establish the following result.

Lemma 1.2.

(1) *There exist universal constants* $\{\bar{\kappa}_\alpha, \kappa^1_\alpha, \kappa^3_\alpha, \kappa^4_\alpha, \kappa^5_\alpha\}$ *so that:*
 (a) $a^{bd}_{0,\alpha}(F, \mathcal{D}) = (4\pi)^{-m/2}\mathcal{I}^{bd}\{\mathrm{Tr}(-\bar{\kappa}_\alpha F_0 \,\mathrm{Id})\}$.
 (b) $a^{bd}_{1,\alpha}(F, \mathcal{D}) = (4\pi)^{-m/2}\mathcal{I}^{bd}\{\mathrm{Tr}(-\bar{\kappa}_{\alpha-1}F_1 \,\mathrm{Id} + \kappa^1_\alpha F_0 L_{aa} \,\mathrm{Id})\}$.
 (c) $a^{bd}_{2,\alpha}(F, \mathcal{D}) = (4\pi)^{-m/2}\mathcal{I}^{bd}\{\mathrm{Tr}(-\bar{\kappa}_{\alpha-2}F_2 \,\mathrm{Id} + \kappa^1_{\alpha-1}F_1 L_{aa} \,\mathrm{Id} + F_0[\kappa^3_\alpha R_{amma} + \kappa^4_\alpha L_{aa}L_{bb} + \kappa^5_\alpha L_{ab}L_{ab}] \,\mathrm{Id} - \bar{\kappa}_\alpha F_0[E + \frac{1}{6}R_{ijji} \,\mathrm{Id}])\}$.

(2) *If* $\alpha \neq 1, 2$, *then* $\bar{\kappa}_\alpha = \kappa_\alpha$ *and* $\kappa^1_\alpha = \frac{1}{2}\Gamma\left(\frac{2-\alpha}{2}\right)\frac{\alpha-4}{2(\alpha-3)}$.

(3) $\bar{\kappa}_1 = \frac{C}{2}$.

In Section 4, we evaluate the remaining universal coefficients of Lemma 1.2 using the calculus of pseudo-differential operators and complete the proof of Theorem 1.5 by showing:

Lemma 1.3. *Adopt the notation of Lemma 1.2. If $\alpha \neq 1, 2$, then:*

$$\kappa_\alpha^3 = -\tfrac{\alpha-1}{24}\Gamma\left(\tfrac{1-\alpha}{2}\right), \quad \kappa_\alpha^4 = \tfrac{7-8\alpha+\alpha^2}{32(\alpha-6)}\Gamma\left(\tfrac{1-\alpha}{2}\right), \quad \kappa_\alpha^5 = \tfrac{6\alpha-5-\alpha^2}{16(\alpha-6)}\Gamma\left(\tfrac{1-\alpha}{2}\right).$$

We conclude the paper in Section 5 by using the pseudo-differential calculus to establish Theorem 1.4. We have postponed the proof of Theorem 1.4 until this point as much of the needed notation will be established in Section 4. We will also complete the proof of Theorem 1.6.

We have chosen to use special case calculations, the functorial method and the pseudo-differential calculus as our purpose in this paper is at least in part expository and we wish to illustrate the interplay amongst these methods. In a subsequent paper, we shall perform a similar analysis for other elliptic boundary conditions (Robin, transfer, transmittal, etc.); it will be necessary to restrict to $\mathrm{Re}(\alpha) < 1$ to ensure convergence, and regularization will not be required in that analysis.

2. Computations in $\mathbb{R}^2$

This section is devoted to the proof of Theorem 1.3, and we shall adopt the notation of that theorem throughout. As we shall be dealing with different weights, we drop the notation $\mathcal{I}$ and return to ordinary integrals in this section to perform a special case calculation in flat space. One has the following estimate of Lang [10] and of Lerche and Siegmund [11] that adjusts the formula of Equation (1.4) for the heat kernel on a half space to take into account the curvature of the boundary of M (for related results see also [3, 12]):

Lemma 2.1. *Adopt the notation of Theorem 1.3. Let $x = (y, r) \in \mathcal{C}_\varepsilon$. As $t \downarrow 0$,*

$$p_M(x, x; t) = \frac{1}{4\pi t}\left\{1 - \mathrm{e}^{-r^2/t} - L_{aa}(y) r^2 t^{-1/2} \int_{rt^{-1/2}}^{\infty} \mathrm{e}^{-\eta^2} d\eta\right\} + O(1).$$

Proof of Theorem 1.3 (1). Parametrize the boundary of M by arclength. There is no higher order correction in $\mathbb{R}^2$ and the $O(r^2)$ term in Equation (1.6) vanishes. Thus on the collar $\mathcal{C}_\varepsilon$, we have

$$dx = (1 - L_{aa}(y) r) dr dy.$$

Following Equation (1.4), we let $p_H(x,x;t)=(4\pi t)^{-1}(1-\mathrm{e}^{-r^2/t})$ be the Dirichlet heat kernel in the half space $r\geq 0$ on the diagonal. We take $0<\alpha<1$ and express:

$$\begin{aligned}
&\mathrm{Tr}_{L^2}(F\mathrm{e}^{-t\Delta_M})\\
&\quad=\int_{\partial M}\int_0^{\varepsilon}F_0(y)r^{-\alpha}\chi(r)(1-L_{aa}(y)r)p_M((y,r),(y,r);t)drdy\\
&\quad=\int_{\partial M}\int_0^{\infty}F_0(y)r^{-\alpha}\chi(r)(1-L_{aa}(y)r)p_M((y,r),(y,r);t)drdy\\
&\quad=D_1+D_2+D_3+D_4+D_5
\end{aligned}$$

where, motivated by Lemma 2.1, we have:

$$\begin{aligned}
D_1&:=\frac{1}{4\pi t}\int_M F(x)dx,\\
D_2&:=-\frac{1}{4\pi t}\int_{\partial M}\int_0^{\infty}F_0(y)r^{-\alpha}\mathrm{e}^{-r^2/t}drdy,\\
D_3&:=\frac{1}{4\pi t}\int_{\partial M}\int_0^{\infty}F_0(y)r^{-\alpha}(1-\chi(r))(1-L_{aa}(y)r)\mathrm{e}^{-r^2/t}drdy,\\
D_4&:=\frac{1}{4\pi t}\int_{\partial M}\int_0^{\infty}F_0(y)L_{aa}(y)r^{1-\alpha}\mathrm{e}^{-r^2/t}drdy,\\
D_5&:=\int_{\partial M}\int_0^{\infty}F_0(y)r^{-\alpha}\chi(r)(1-L_{aa}(y)r)\{(p_M-p_H)((y,r),(y,r);t)\}drdy.
\end{aligned}$$

A straightforward computation yields:

$$\begin{aligned}
D_2&=-\frac{1}{4\pi}\cdot\frac{1}{2}\Gamma\left(\frac{1-\alpha}{2}\right)t^{-(1+\alpha)/2}\int_{\partial M}F_0(y)dy,\\
D_3&=O\left(\mathrm{e}^{-\varepsilon_0^2/(2t)}\right),\\
D_4&=\frac{1}{4\pi}\cdot\frac{1}{2}\Gamma\left(\frac{2-\alpha}{2}\right)t^{-\alpha/2}\int_{\partial M}F_0(y)L_{aa}(y)dy.
\end{aligned}$$

We use Lemma 2.1 to compute D_5. Since $F_0(y)r^{-\alpha}\chi(r)$ is integrable on $\mathcal{C}_\varepsilon$, we have the $O(1)$ in Lemma 2.1 remains $O(1)$ as $t\downarrow 0$. Hence for $0<\alpha<1$,

$$\begin{aligned}
D_5=&-\frac{1}{4\pi t}\int_{\partial M}\int_0^{\infty}\int_{rt^{-1/2}}^{\infty}F_0(y)L_{aa}(y)r^{2-\alpha}t^{-1/2}\mathrm{e}^{-\eta^2}d\eta drdy\\
&+\frac{1}{4\pi t}\int_{\partial M}\int_0^{\infty}\int_{rt^{-1/2}}^{\infty}F_0(y)L_{aa}(y)L_{bb}(y)r^{3-\alpha}t^{-1/2}\mathrm{e}^{-\eta^2}d\eta drdy+O(1)
\end{aligned}$$

$$= \frac{1}{4\pi} \cdot \frac{2-\alpha}{4(\alpha-3)} \Gamma\left(\frac{2-\alpha}{2}\right) t^{-\alpha/2} \int_{\partial M} F_0(y) L_{aa}(y) dy + O(t^{(1-\alpha)/2}) + O(1)$$
$$= \frac{1}{4\pi} \cdot \frac{2-\alpha}{4(\alpha-3)} \Gamma\left(\frac{2-\alpha}{2}\right) t^{-\alpha/2} \int_{\partial M} F_0(y) L_{aa}(y) dy + O(1).$$

We collect terms to complete the proof of Theorem 1.3 (1). □

Remark 2.1. We have chosen to study the region $0 < \alpha < 1$. The reason for this is that F has to be integrable in order to control the $O(1)$ remainder in Lemma 2.1. If one wishes to obtain just the leading asymptotic behaviour of $Tr_{L^2}(F\mathrm{e}^{-t\Delta_M})$, then probabilistic estimates for

$$R_M(x;t) := (p_M - p_H)((y,r),(y,r);t) \tag{2.1}$$

along the lines of [1], and analogous to [2], could be used to show that for $1 \le \alpha < 3$,

$$\int_{\partial M} \int_0^{\varepsilon} \chi(r) r^{-\alpha} R(x;t) dr dy = O(t^{-(m-2+\alpha)/2}), \quad t \downarrow 0. \tag{2.2}$$

Proof of Theorem 1.3 (2). Recall from Equation (1.4) the formula for the heat kernel p_H on the diagonal for the half space. We decompose

$$\mathrm{Tr}_{L^2}(F\mathrm{e}^{-t\Delta_M}) = E_1 + E_2 + E_3$$

where we have:

$$E_3 := \int_{\partial M} \int_0^{\infty} F_0(y) r^{-1} \chi(r)(1 - L_{aa}(y) r)(p_M - p_H)((y,r),(y,r);t)) dr dy,$$
$$E_2 := -\frac{1}{4\pi t} \int_{\partial M} \int_0^{\infty} F_0(y) L_{aa}(y) \chi(r)(1 - \mathrm{e}^{-r^2/t}) dr dy,$$
$$E_1 := \frac{1}{4\pi t} \int_{\partial M} \int_0^{\infty} F_0(y) r^{-1} \chi(r)(1 - \mathrm{e}^{-r^2/t}) dr dy.$$

We apply Equations (2.1) and (2.2) with $m = 2$ and with $\alpha = 1$ to see that $E_3 = O(t^{-1/2})$. Furthermore, $E_2 = O(t^{-1/2})$. The leading term is provided by E_1. We integrate by parts to see that:

$$\begin{aligned} E_1 &= -\frac{1}{4\pi t}\int_{\partial M} F_0(y)\int_0^\infty \ln(r)\left\{\chi'(r)(1-\mathrm{e}^{-r^2/t})+\chi(r)\frac{2r}{t}\mathrm{e}^{-r^2/t}\right\}drdy \\ &= -\frac{1}{4\pi t}\int_{\partial M} F_0(y)\left\{\int_0^\infty \ln(r)\chi'(r)dr+\int_0^\infty \ln(r)\frac{2r}{t}\mathrm{e}^{-r^2/t}dr\right\}dy \\ &\quad + O(\mathrm{e}^{-\varepsilon_0^2/(2t)}) \\ &= \frac{1}{4\pi t}\cdot\frac{1}{2}\int_{\partial M} F_0(y)\left\{\ln\left(\tfrac{\varepsilon_0^2}{t}\right)+C+2\int_{\varepsilon_0}^{\varepsilon}\frac{\chi(r)}{r}dr\right\}+O(\mathrm{e}^{-\varepsilon_0^2/(2t)}). \end{aligned}$$

This completes the proof of Theorem 1.3. □

3. The functorial method

We adopt the notation of Theorem 1.4 throughout this section. We begin our study with the following:

Lemma 3.1. *There exist constants $\varepsilon^\nu_{\ell,\alpha}$ so that*

(1) $a^{bd}_{0,\alpha}(F,\mathcal{D}) = (4\pi)^{-m/2}\mathcal{I}^{bd}\{\mathrm{Tr}(\varepsilon^0_{0,\alpha}F_0\,\mathrm{Id})\}$.

(2) $a^{bd}_{1,\alpha}(F,\mathcal{D}) = (4\pi)^{-m/2}\mathcal{I}^{bd}\{\mathrm{Tr}(\varepsilon^0_{1,\alpha}F_1\,\mathrm{Id}+\varepsilon^1_{1,\alpha}F_0L_{aa}\,\mathrm{Id})\}$.

(3) $a^{bd}_{2,\alpha}(F,\mathcal{D}) = (4\pi)^{-m/2}\mathcal{I}^{bd}\{\mathrm{Tr}(\varepsilon^0_{2,\alpha}F_2\,\mathrm{Id}+\varepsilon^1_{2,\alpha}F_1L_{aa}\,\mathrm{Id}$
$+F_0[\varepsilon^2_{2,\alpha}R_{ijji}+\varepsilon^3_{2,\alpha}R_{amma}+\varepsilon^4_{2,\alpha}L_{aa}L_{bb}+\varepsilon^5_{2,\alpha}L_{ab}L_{ab}]\,\mathrm{Id}$
$+\varepsilon^6_{2,\alpha}F_0E)\}$.

Proof. We apply dimensional analysis — we shall suppose that $\alpha\notin\mathbb{Z}$ for the moment. Let $c>0$ define a rescaling $g_c := c^2g$. We then have

$$\begin{array}{lll} dx_c = c^m dx, & dy_c = c^{m-1}dy, & \mathcal{D}_c = c^{-2}\mathcal{D}, \\ r_c := cr, & \partial_{r_c} = c^{-1}\partial_r, & F_{i,c} = c^{\alpha-i}F_i, \\ \mathcal{I}_{\mathrm{Reg,c}} = c^m\mathcal{I}_{\mathrm{Reg}}, & \mathcal{I}^{bd}_c = c^{m-1}\mathcal{I}^{bd}. & \end{array}$$

Let $a_{n,c} := a_n(x,\mathcal{D}_c)$, $a_{\ell,\alpha,c} := a_{\ell,\alpha}(y,\mathcal{D}_c)$ and $a_{\ell,\alpha,i,c} := a_{\ell,\alpha,i}(y,\mathcal{D}_c)$ denote the local heat trace invariants defined by $\mathcal{D}_c$ on M and on ∂M, respectively. It is immediate that

$$\mathrm{Tr}_{L^2}(F\mathrm{e}^{-(tc^{-2})\mathcal{D}}) = \mathrm{Tr}_{L^2}(F\mathrm{e}^{-t\mathcal{D}_c}). \tag{3.1}$$

We expand both sides of Equation (3.1) in an asymptotic expansion:

$$\begin{aligned}
&t^{-m/2}c^m\sum_{n=0}^{\infty}(c^{-2}t)^n\mathcal{I}_{\mathrm{Reg}}\{Fa_n\}\\
&\quad+(c^{-2}t)^{-(m-1)/2}\sum_{\ell=0}^{\infty}c^{\alpha-\ell}t^{(\ell-\alpha)/2}\sum_{i=0}^{\ell}\mathcal{I}^{bd}\{F_ia_{\ell,\alpha,i}\}\\
&\sim t^{-m/2}\sum_{n=0}^{\infty}t^nc^m\mathcal{I}_{\mathrm{Reg}}\{Fa_{n,c}\}\\
&\quad+t^{-(m-1)/2}\sum_{\ell=0}^{\infty}t^{(\ell-\alpha)/2}\sum_{i=0}^{\ell}c^{m-1}c^{\alpha-i}\mathcal{I}^{bd}\{F_ia_{\ell,\alpha,i,c}\}.
\end{aligned}$$

Since $\alpha\notin\mathbb{Z}$, the interior and the boundary terms decouple. We equate terms in the asymptotic expansions to see that

$$a_{n,c}=c^{-2n}a_n\quad\text{and}\quad a_{\ell,\alpha,i,c}=c^{i-\ell}a_{\ell,\alpha,i}.$$

Examining relations of this kind is straightforward – they mean that the local formula $a_n(x,\mathcal{D})$ is homogeneous of weighted degree $2n$ in the jets of the derivatives of the symbol of $\mathcal{D}$ and that the local formula $a_{\ell,\alpha,i}(y,\mathcal{D})$ is homogeneous of weighted degree $\ell-i$ in the jets of the derivatives of the symbol of $\mathcal{D}$. One may use Weyl's theory of invariants to express a spanning set for the invariants which arise in this context and complete the proof of Lemma 3.1 for $\alpha\notin\mathbb{Z}$. We use analytic continuation to establish Lemma 3.1 when $\alpha=0,-1,-2,\dots$ as well. We refer to [7] for further details concerning this sort of dimensional analysis.

If $\alpha=1$, then the argument is rather different. Let ε_c be the width of the collar $\mathcal{C}$ with respect to the rescaled metric. The regularizing term in Equation (1.7) does not simply rescale. Rather we have:

$$\begin{aligned}
\ln(\varepsilon_c)\mathcal{I}^{bd}_c\{(Fa_{n,c})_{0,c}\}&=\ln(c\varepsilon)c^{\alpha}c^{-2n}c^{m-1}\mathcal{I}^{bd}\{(Fa_n)_0\}\\
&=c^{m-2n}\{\ln(c)+\ln(\varepsilon)\}\mathcal{I}^{bd}\{(Fa_n)_0\}.
\end{aligned}$$

This yields the modified relation:

$$\mathcal{I}_{\mathrm{Reg,c}}\{Fa_{n,c}\}=c^{m-2n}\mathcal{I}_{\mathrm{Reg}}\{Fa_n\}+\ln(c)c^{m-2n}\mathcal{I}^{bd}\{(Fa_n)_0\}.$$

A similar argument for $\alpha=2$ shows that:

$$\mathcal{I}_{\mathrm{Reg,c}}\{Fa_{n,c}\}=c^{m-2n}\mathcal{I}_{\mathrm{Reg}}\{Fa_n\}+\ln(c)c^{m-2n}\mathcal{I}^{bd}\{(Fa_n)_1-(Fa_n)_0L_{aa}\}.$$

When we use Equation (3.1) to equate coefficients in the asymptotic series, we have

$$a^{bd}_{\ell,\alpha}(F,\mathcal{D}_c)=c^{\ell}a^{bd}_{\ell,\alpha}(F,\mathcal{D}),$$

which completes the proof of Lemma 3.1 in these exceptional cases. We compare the terms involving $\ln(c)$ to obtain additional relations. The left-hand side in the following equation arises from $\ln(c^{-2}t)$ and the right-hand side arises from $\mathcal{I}_{reg,c}$ when we apply Equation (3.1); if $k=2n$, then

$$\begin{aligned}&-2\ln(c)\tilde{a}^{bd}_{k,1}(F,\mathcal{D})=\ln(c)\mathcal{I}^{bd}\{(Fa_n)_0\}\\&-2\ln(c)\tilde{a}^{bd}_{k,2}(F,\mathcal{D})=\ln(c)\mathcal{I}^{bd}\{(Fa_n)_1-(Fa_n)_0L_{aa}\}.\end{aligned}$$

There are no corresponding terms if k is odd and thus $\tilde{a}_{k,1}=0$ and $\tilde{a}_{k,2}=0$ if k is odd. This establishes Theorem 1.4 (4). An alternate proof is given in Section 5. □

We now use the functorial method to establish the following result:

Lemma 3.2.

(1) $\varepsilon^2_{2,\alpha}=\frac{1}{6}\varepsilon^0_{0,\alpha}$ *and* $\varepsilon^6_{2,\alpha}=\varepsilon^0_{0,\alpha}$.

(2) *The constants* $\varepsilon^{\mu}_{\ell,\alpha}$ *of Lemma 3.1 are dimension free.*

(3) $\varepsilon^0_{1,\alpha}=\varepsilon^0_{0,\alpha-1}$, $\varepsilon^0_{2,\alpha}=\varepsilon^0_{0,\alpha-2}$ *and* $\varepsilon^1_{2,\alpha}=\varepsilon^1_{1,\alpha-1}$.

Proof. Suppose that $M=M_1\times M_2$, that $g_M=g_{M_1}+g_{M_2}$, that $\mathcal{D}_M=\mathcal{D}_{M_1}+\mathcal{D}_{M_2}$ and that $F_M=F_1F_2$ where F_i are defined on M_i. We suppose that M_1 is a closed manifold and thus $\partial M=M_1\times\partial M_2$. We then have:

$$\begin{aligned}&\mathrm{e}^{-t\mathcal{D}_M}=\mathrm{e}^{-t\mathcal{D}_{M_1}}\mathrm{e}^{-t\mathcal{D}_{M_2}},\\&\mathrm{Tr}_{L^2}(F_M\mathrm{e}^{-t\mathcal{D}_M})=\mathrm{Tr}_{L^2}(F_1\mathrm{e}^{-t\mathcal{D}_{M_1}})\cdot\mathrm{Tr}_{L^2}(F_2\mathrm{e}^{-t\mathcal{D}_{M_2}}).\end{aligned}$$

Equating asymptotic series yields

$$a^{bd}_{\ell,\alpha}(F_M,\mathcal{D}_M)=\sum_{2k+j=\ell}a_k(F_1,\mathcal{D}_{M_1})a^{bd}_{j,\alpha}(F_2,\mathcal{D}_{M_2}),$$

and hence a corresponding decomposition of the local formulas:

$$a^{bd}_{\ell,\alpha,i}(y,\mathcal{D}_M)=\sum_{2k+j=\ell}a_k(x_1,\mathcal{D}_{M_1})a^{bd}_{j,\alpha,i}(y_2,\mathcal{D}_{M_2})\quad\text{for}\quad y=(x_1,y_2).\tag{3.2}$$

Assertion (1) now follows from Theorem 1.2 (2), from Lemma 3.1, and from Equation (3.2); the multiplicative constants $(4\pi)^{-m/2}$ play no role. If we take

$M_1 = S^1$ and $\mathcal{D}_{M_1} = -\partial_\theta^2$, then the structures are flat. Thus $a_0(x,\mathcal{D}_{M_1}) = 1/\sqrt{4\pi}$ and $a_k(x,\mathcal{D}_{M_1}) = 0$ for $k \geq 1$. Thus Equation (3.2) yields in this special case the following identity from which Assertion (2) follows after taking into account the multiplicative constants $(4\pi)^{-m/2}$:

$$a^{bd}_{\ell,\alpha,i}(y,\mathcal{D}_M) = \tfrac{1}{\sqrt{4\pi}} a^{bd}_{\ell,\alpha,i}(y_2,\mathcal{D}_{M_2}).$$

We prove Assertion (3) by index shifting. Let $\chi(r)$ be a smooth function so that $\chi \equiv 0$ near $r=\varepsilon$. Let $F(y,r) = F_0(y)\chi(r)r^{-\alpha_1}$ for $\mathrm{Re}(\alpha_1) < 3$. We apply Theorem 1.4 with $\alpha = \alpha_1$ and with $\alpha = \alpha_1 - 1$ to see that:

$$a^{bd}_{\ell,\alpha,j}(y,\mathcal{D}) = a^{bd}_{\ell-1,\alpha-1,j-1}(y,\mathcal{D}) \quad \text{for} \quad j \geq 1.$$

Assertion (3) now follows. □

Proof of Lemma 1.2. Assertion (1) of Lemma 1.2 follows from Lemma 3.1 and Lemma 3.2 by a suitable relabelling of the coefficients. Assertions (2) and (3) follow from Theorem 1.3. □

Proof of Theorem 1.6. We now derive Theorem 1.6 from Theorem 1.5 using Assertion (3) of Theorem 1.4. Certain of the coefficients are regular so the computation is elementary; we do not need to drop the pole. We simply set $\alpha = 1$ to compute $a^{bd}_{1,1}$ and $\alpha = 2$ to compute $a^{bd}_{0,2}$ and $a^{bd}_{2,2}$:

$$\begin{aligned}
a^{bd}_{0,2}(F,\mathcal{D}) &= \kappa_2(4\pi)^{-m/2}\mathcal{I}^{bd}\{\mathrm{Tr}(-F_0\,\mathrm{Id})\}\\
&= \sqrt{\pi}(4\pi)^{-m/2}\mathcal{I}^{bd}\{\mathrm{Tr}(F_0\,\mathrm{Id})\},\\
a^{bd}_{1,1}(F,\mathcal{D}) &= \kappa_0(4\pi)^{-m/2}\mathcal{I}^{bd}\{\mathrm{Tr}(-F_1\,\mathrm{Id} + \tfrac{3}{4}F_0L_{aa}\,\mathrm{Id})\}\\
&= \tfrac{\sqrt{\pi}}{8}(4\pi)^{-m/2}\mathcal{I}^{bd}\{\mathrm{Tr}(-4F_1\,\mathrm{Id} + 3F_0L_{aa}\,\mathrm{Id})\},\\
a^{bd}_{2,2}(F,\mathcal{D}) &= \kappa_0(4\pi)^{-m/2}\mathcal{I}^{bd}\{\mathrm{Tr}(-F_2\,\mathrm{Id} + \tfrac{\alpha-5}{2(\alpha-4)}F_1L_{aa}\,\mathrm{Id}\\
&\qquad + \tfrac{1}{6}F_0R_{amma}\,\mathrm{Id} - \tfrac{\alpha-7}{8(\alpha-6)}F_0L_{aa}L_{bb}\,\mathrm{Id} + \tfrac{\alpha-5}{4(\alpha-6)}F_0L_{ab}L_{ab}\,\mathrm{Id}\\
&\qquad - \tfrac{1}{3(1-\alpha)}F_0R_{ijji}\,\mathrm{Id} - \tfrac{2}{1-\alpha}F_0E)\}|_{\alpha=2}\\
&= \sqrt{\pi}(4\pi)^{-m/2}\mathcal{I}^{bd}\{\mathrm{Tr}(-\tfrac{1}{2}F_2\,\mathrm{Id} + \tfrac{3}{8}F_1L_{aa}\,\mathrm{Id} + \tfrac{1}{12}F_0R_{amma}\,\mathrm{Id}\\
&\qquad - \tfrac{5}{64}F_0L_{aa}L_{bb}\,\mathrm{Id} + \tfrac{3}{32}F_0L_{ab}L_{ab}\,\mathrm{Id} + \tfrac{1}{6}F_0R_{ijji}\,\mathrm{Id} + F_0E)\}.
\end{aligned}$$

We compute the remaining coefficients as follows. Let $c(a_{i,\alpha}^{bd}, A)$ be the coefficient of the monomial $\mathcal{I}^{bd}\{\mathrm{Tr}(A)\}$ in $(4\pi)^{m/2}a_{i,\alpha}^{bd}(F,\mathcal{D})$. It follows by Theorem 1.3 that

$$a_{0,1}^{bd}(F,\mathcal{D}) = \tfrac{C}{2}\cdot(4\pi)^{-m/2}\mathcal{I}^{bd}\{\mathrm{Tr}(F_0\,\mathrm{Id})\}.$$

We may expand

$$\kappa_{\alpha-1}\tfrac{\alpha-4}{2(\alpha-3)} = \kappa_{\alpha-1}\tfrac{2(\alpha-3)+(2-\alpha)}{2(\alpha-3)} = \kappa_{\alpha-1} + \Gamma\left(\tfrac{4-\alpha}{2}\right)\tfrac{1}{2(\alpha-3)}.$$

It now follows that

$$\begin{aligned}
&c(a_{1,2}^{bd}, F_0L_{aa}\,\mathrm{Id}) = -\tfrac{C}{2}-\tfrac{1}{2},\\
&a_{1,2}^{bd}(F,\mathcal{D}) = (4\pi)^{-m/2}\mathcal{I}^{bd}\{\mathrm{Tr}(\tfrac{C}{2}F_1\,\mathrm{Id}-(\tfrac{C}{2}+\tfrac{1}{2})F_0L_{aa}\,\mathrm{Id})\}.
\end{aligned}$$

Many of the terms in $a_{2,1}^{bd}(F,\mathcal{D})$ are in fact regular at $\alpha=1$. We have:

$$\begin{aligned}
&c(a_{2,1}^{bd}, F_2\,\mathrm{Id}) = -\tfrac{1}{2}, && c(a_{2,1}^{bd}, F_1L_{aa}\,\mathrm{Id}) = \tfrac{1}{3},\\
&c(a_{2,1}^{bd}, R_{amma}\,\mathrm{Id}) = \tfrac{1}{12}, && c(a_{2,1}^{bd}, L_{aa}L_{bb}\,\mathrm{Id}) = -\tfrac{3}{40},\\
&c(a_{2,1}^{bd}, L_{ab}L_{ab}\,\mathrm{Id}) = \tfrac{1}{10}.
\end{aligned}$$

The terms involving R_{ijji} and E can be written in the form $-\kappa_\alpha F_0(\frac{1}{6}R_{ijji}\,\mathrm{Id}$ $+E)$. Thus we may use the regularization of $-F_0\,\mathrm{Id}$ in $a_{0,1}^{bd}$ which was already computed to see

$$\begin{aligned}
a_{2,1}^{bd}(F,\mathcal{D}) = (4\pi)^{-m/2}\mathcal{I}^{bd}\Big\{\mathrm{Tr}\Big(&-\tfrac{1}{2}F_2\,\mathrm{Id}+\tfrac{1}{3}F_1L_{aa}\,\mathrm{Id}+\tfrac{1}{12}R_{amma}\,\mathrm{Id}\\
&-\tfrac{3}{40}L_{aa}L_{bb}\,\mathrm{Id}+\tfrac{1}{10}L_{ab}L_{ab}\,\mathrm{Id}+\tfrac{C}{12}R_{ijji}\,\mathrm{Id}+\tfrac{C}{2}E\Big)\Big\}.
\end{aligned}$$

This completes the derivation of Theorem 1.6 from Theorem 1.5. □

4. The pseudo-differential calculus

We adopt the following notational conventions. Let $\vec{\alpha}=(\alpha_1,\ldots,\alpha_m)$ be a multi-index. We set

$$\begin{aligned}
&|\vec{\alpha}| = \alpha_1+\ldots+\alpha_m, && \vec{\alpha}! = \alpha_1!\times\ldots\times\alpha_m!,\\
&x^{\vec{\alpha}} = x_1^{\alpha_1}\times\ldots\times x_m^{\alpha_m}, && d_x^{\vec{\alpha}} = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\times\ldots\times\left(\frac{\partial}{\partial x_m}\right)^{\alpha_m},\\
&D_{\vec{\alpha}}^x = (-\sqrt{-1})^{|\vec{\alpha}|}d_x^{\vec{\alpha}}.
\end{aligned}$$

We apologize in advance for the slight notational confusion involved with using α to control the growth of F and also to using $\vec{\alpha}$ as a multi-index.

We begin our discussion by reviewing the standard pseudo-differential computation of the resolvent on a closed manifold without boundary and refer to [5–8,13] for further details. Let $\mathcal{D}$ be an operator of Laplace type. We want to construct the resolvent of $\mathcal{D}-\lambda$ for large λ where we use Equation (1.1) to express:

$$\mathcal{D} = \sum_{|\vec{\alpha}|\leq 2} a_{\vec{\alpha}}(x) D^x_{\vec{\alpha}}.$$

For the symbol $\sigma(\mathcal{D})(x,\xi)$ of $\mathcal{D}$ this means

$$\sigma(\mathcal{D})(x,\xi) = \sum_{|\vec{\alpha}|\leq 2} a_{\vec{\alpha}}(x)\xi^{\vec{\alpha}};$$

note that for the scalar Laplacian Δ_M the zeroth term vanishes so $a_{\vec{0}} = 0$ in this setting. In the evaluation of the heat equation asymptotics homogeneity properties of symbols are relevant and it turns out that collecting terms according to

$$\begin{aligned} a_2(x,\xi,\lambda) &= -\lambda + \sum_{|\vec{\alpha}|=2} a_{\vec{\alpha}}(x)\xi^{\vec{\alpha}}, \\ a_j(x,\xi,\lambda) &= \sum_{|\vec{\alpha}|=j} a_{\vec{\alpha}}(x)\xi^{\vec{\alpha}}, \quad j=0,1 \end{aligned}$$

is fruitful. As a result, the symbol $\sigma(\mathcal{D}-\lambda)(x,\xi,\lambda)$ can be written as

$$\sigma(\mathcal{D}-\lambda)(x,\xi,\lambda) = \sum_{j=0}^{2} a_j(x,\xi,\lambda).$$

For the symbol of the resolvent of $\mathcal{D}-\lambda$ we make the Ansatz

$$\sigma((\mathcal{D}-\lambda)^{-1})(x,\xi,\lambda) \sim \sum_{l=0}^{\infty} q_{-2-l}(x,\xi,\lambda). \tag{4.1}$$

In view of the formula for the symbol of a product we see that q_{-2-l} is determined algebraically by

$$1 = a_2(x,\xi,\lambda) q_{-2}(x,\xi,\lambda), \tag{4.2}$$

$$0 = \sum_{\substack{\vec{\alpha},j,l\leq k \\ k=2+l+|\vec{\alpha}|-j}} \frac{1}{\vec{\alpha}!}[d_\xi^{\vec{\alpha}} a_j(x,\xi,\lambda)]\ [D^x_{\vec{\alpha}} q_{-2-l}(x,\xi,\lambda)] \quad \text{for } k\geq 1. \tag{4.3}$$

These symbols will play a crucial role in the proof of Theorem 1.4 in Section 5.

We now specialize to the case where $\mathcal{D} = \Delta_M$ is the scalar Laplacian. For our present considerations we need q_{-2}, q_{-3}, q_{-4}. For simplicity we skip the arguments in the following summary of results — we use in an essential fashion the fact that Δ_M is scalar. We also use the convention that repeated indices are summed over. We find (all greek indices will range over $\{1, 2, \ldots, m\}$):

$$\begin{aligned}
q_{-2} &= a_2^{-1},\\
q_{-3} &= -a_2^{-1}\left[a_1 q_{-2} + (D_\xi^\nu a_2)(\sqrt{-1}D_\nu^x q_{-2})\right],\\
q_{-4} &= -a_2^{-1}\Big[a_0 q_{-2} + a_1 q_{-3} + (D_\xi^\nu a_1)(\sqrt{-1}D_\nu^x q_{-2})\\
&\qquad + (D_\xi^\nu a_2)(\sqrt{-1}D_\nu^x q_{-3}) - \tfrac{1}{2}(D_\xi^{\nu\mu} a_2)(D_{\nu\mu}^x q_{-2})\Big].
\end{aligned}$$

For later use it will be advantageous to express the results in terms of q_{-2}^n. We then have

$$\begin{aligned}
q_{-3} &= -a_1 q_{-2}^2 + c_{-3,3} q_{-2}^3,\\
q_{-4} &= -a_0 q_{-2}^2 + c_{-4,3} q_{-2}^3 + c_{-4,4} q_{-2}^4 + c_{-4,5} q_{-2}^5,
\end{aligned}$$

where

$$\begin{aligned}
c_{-3,3} &= -\sqrt{-1}\,(\partial_\xi^\nu a_2)(\partial_\nu^x a_2),\\
c_{-4,3} &= a_1^2 - \sqrt{-1}\,(\partial_\xi^\nu a_1)(\partial_\nu^x a_2) - \sqrt{-1}\,(\partial_\xi^\nu a_2)(\partial_\nu^x a_1) - \tfrac{1}{2}(\partial_\xi^{\nu\mu} a_2)(\partial_{\nu\mu}^x a_2),\\
c_{-4,4} &= -3a_1 c_{-3,3} + \sqrt{-1}\,(\partial_\xi^\nu a_2)(\partial_\nu^x c_{-3,3}) + (\partial_\xi^{\nu\mu} a_2)(\partial_\nu^x a_2)(\partial_\mu^x a_2),\\
c_{-4,5} &= 3c_{-3,3}^2.
\end{aligned}$$

The relevant operator for our considerations is

$$\Delta_M - \lambda = -g^{\mu\nu}\partial_\mu\partial_\nu + b^\mu\partial_\mu - \lambda,$$

where we have changed notation slightly from that used previously. For the symbols this gives

$$\begin{aligned}
a_2(x,\xi,\lambda) &= g^{\mu\nu}\xi_\mu\xi_\nu - \lambda \equiv |\xi|^2 - \lambda,\\
a_1(x,\xi,\lambda) &= \sqrt{-1}b^\mu\xi_\mu \quad \text{and} \quad a_0(x,\xi,\lambda) = 0.
\end{aligned}$$

To state results for q_{-2}, q_{-3} and q_{-4} for this operator D we will as usual raise and lower indices using the inverse metric and the metric. Furthermore,

',' denotes partial differentiation. One computes easily that:

$$
\begin{aligned}
q_{-2}(x,\xi,\lambda) &= \tfrac{1}{|\xi|^2-\lambda},\\
q_{-3}(x,\xi,\lambda) &= -\tfrac{1}{(|\xi|^2-\lambda)^2}\sqrt{-1}b^\mu\xi_\mu - \tfrac{1}{(|\xi|^2-\lambda)^3}2\sqrt{-1}g^{\sigma\gamma}_{,\nu}\xi^\nu\xi_\sigma\xi_\gamma,\\
q_{-4}(x,\xi,\lambda) &= \tfrac{1}{(|\xi|^2-\lambda)^3}\left\{-b^\mu b^\nu\xi_\mu\xi_\nu + b^\nu g^{\sigma\beta}_{,\nu}\xi_\sigma\xi_\beta + 2b^\sigma_{,\nu}g^{\nu\beta}\xi_\beta\xi_\sigma - g^{\sigma\beta}_{,\nu\mu}g^{\nu\mu}\xi_\sigma\xi_\beta\right\}\\
&\quad + \tfrac{1}{(|\xi|^2-\lambda)^4}\left\{-6b^\mu g^{\sigma\gamma}_{,\nu}g^{\nu\beta}\xi_\mu\xi_\beta\xi_\sigma\xi_\gamma + 4g^{\sigma\gamma}_{,\beta\nu}g^{\beta\mu}g^{\nu\delta}\xi_\mu\xi_\sigma\xi_\gamma\xi_\delta\right.\\
&\qquad \left. +4g^{\sigma\gamma}_{,\beta}g^{\beta\mu}_{,\nu}g^{\nu\delta}\xi_\mu\xi_\sigma\xi_\gamma\xi_\delta + 2g^{\sigma\beta}_{,\nu}g^{\gamma\delta}_{,\mu}g^{\nu\mu}\xi_\sigma\xi_\beta\xi_\delta\xi_\gamma\right\}\\
&\quad + \tfrac{1}{(|\xi|^2-\lambda)^5}\left\{-12g^{\sigma\gamma}_{,\nu}g^{\nu\beta}g^{\delta\tau}_{,\mu}g^{\mu\rho}\xi_\beta\xi_\sigma\xi_\gamma\xi_\rho\xi_\delta\xi_\tau\right\}.
\end{aligned}
$$

If the manifold has a boundary the expansion (4.1) has to be augmented by a boundary correction. To formulate the conditions to be satisfied by the boundary correction we expand about $r = 0$. We adopt the notation established in Section 1.7 and expand

$$ds^2_M = g_{\sigma\varrho}(y,r)dy^\sigma \circ dy^\varrho + dr^2 \quad \text{on} \quad \mathcal{C}_\varepsilon.$$

The coordinate y locally parameterizes the boundary, and r is the geodesic distance to the boundary, so $x = (y,r)$. A tilde above any quantity will indicate that it is to be evaluated at the boundary, that is at $r = 0$. Furthermore, we use $\xi = (\omega,\tau)$.

We find

$$\Delta_M - \lambda = \sum_{k=0}^\infty \frac{1}{k!}r^k \sum_{|\vec\alpha|\le 2} \frac{\partial^k}{\partial r^k} a_{\vec\alpha}(y,r)\bigg|_{r=0} D^{\vec\alpha}_{y,r}$$

with the notation

$$D^{\vec\alpha}_{y,r} = \left(\prod_{i=1}^{m-1} D^{\alpha_i}_{y_i}\right) D^{\alpha_m}_r.$$

Introducing

$$a_j(y,r,\omega,D_r,\lambda) = \begin{cases} \sum_{|\vec\alpha|=j} a_{\vec\alpha}(y,r)\left(\prod_{i=1}^{m-1}\omega_i^{\alpha_i}\right) D^{\alpha_m}_r & \text{for} \quad j=0,1,\\ \sum_{|\vec\alpha|=2} a_{\vec\alpha}(y,r)\left(\prod_{i=1}^{m-1}\omega_i^{\alpha_i}\right) D^{\alpha_m}_r - \lambda & \text{for} \quad j=2,\end{cases}$$

we define the partial symbol

$$\sigma'(\Delta_M - \lambda) = \sum_{k=0}^\infty \frac{1}{k!} r^k \sum_{j=0}^2 \frac{\partial^k}{\partial r^k} a_j(y,r,\omega,D_r,\lambda)\bigg|_{r=0}.$$

As it turns out, the symbols

$$a^{(j)}(y,r,\omega,D_r,\lambda)=\sum_{l=0}^{2}\sum_{\substack{k=0\\ l-k=j}}^{\infty}\frac{1}{k!}r^k\frac{\partial^k}{\partial r^k}a_l(y,r,\omega,D_r,\lambda)\Big|_{r=0}$$

have suitable homogeneity properties and using these symbols we write

$$\sigma'(\Delta_M-\lambda)=\sum_{j=-\infty}^{2}a^{(j)}(y,r,\omega,D_r,\lambda).$$

We write the symbol of the resolvent as

$$\begin{aligned}&\sigma((\Delta_M-\lambda)^{-1})(y,r,\omega,\tau,\lambda)\\ &\quad=\sum_{j=0}^{\infty}q_{-2-j}(y,r,\omega,\tau,\lambda)-\mathrm{e}^{-\sqrt{-1}\tau r}\sum_{j=0}^{\infty}h_{-2-j}(y,r,\omega,\tau,\lambda),\end{aligned}\tag{4.4}$$

where the second term is the boundary correction. The factor $\mathrm{e}^{-\sqrt{-1}\tau r}$ appears because the operator constructed from these terms is the $Op'(h)$ in [13], and $Op'(h)=Op(h\mathrm{e}^{-\sqrt{-1}\tau r})$. This shows

$$\sigma'(\Delta_M-\lambda)\circ\sum_{j=0}^{\infty}h_{-2-j}(y,r,\omega,\tau,\lambda)=0.$$

Here $\circ$ denotes the symbol product on $\mathbb{R}^{m-1}$. Analogously to Equations (4.2) and (4.3) this equation leads to the differential equations

$$\begin{aligned}0&=a^{(2)}(y,r,\omega,D_r,\lambda)h_{-2}(y,r,\omega,\tau,\lambda),\\ 0&=a^{(2)}(y,r,\omega,D_r,\lambda)h_{-2-j}(y,r,\omega,\tau,\lambda)\\ &\quad+\sum_{\substack{\vec\alpha,k,l<j\\ j=l+2+|\vec\alpha|-k}}\frac{1}{\vec\alpha!}\left[D_\omega^{\vec\alpha}a^{(k)}(y,r,\omega,D_r,\lambda)\right]\left[(\sqrt{-1}D^y)_{\vec\alpha}h_{-2-l}(y,r,\omega,\tau,\lambda)\right].\end{aligned}$$

For the present considerations we need h_{-2-j} for $j=0,1,2$, and we have more explicitly (repeated letters $a,b,c,\dots$ run over tangential coordinates $\{1,2,\dots,m-1\}$)

$$\begin{aligned}0&=a^{(2)}(y,r,\omega,D_r,\lambda)h_{-2}(y,r,\omega,\tau,\lambda),\\ 0&=a^{(2)}(y,r,\omega,D_r,\lambda)h_{-3}(y,r,\omega,\tau,\lambda)+a^{(1)}(y,r,\omega,D_r,\lambda)h_{-2}(y,r,\omega,\tau,\lambda)\\ &\quad+\left[D_\omega^b a^{(2)}(y,r,\omega,D_r,\lambda)\right]\left[(\sqrt{-1}D^y)_b h_{-2}(y,r,\omega,\tau,\lambda)\right],\end{aligned}$$

$$
\begin{aligned}
0 &= a^{(2)}(y,r,\omega,D_r,\lambda)h_{-4}(y,r,\omega,\tau,\lambda) + a^{(0)}(y,r,\omega,D_r,\lambda)h_{-2}(y,r,\omega,\tau,\lambda) \\
&\quad + \left[D_\omega^b a^{(1)}(y,r,\omega,D_r,\lambda)\right]\left[(\sqrt{-1}D^y)_b h_{-2}(y,r,\omega,\tau,\lambda)\right] \\
&\quad + \tfrac{1}{2}\left[D_\omega^{bc} a^{(2)}(y,r,\omega,D_r,\lambda)\right]\left[(\sqrt{-1}D^y)_{bc} h_{-2}(y,r,\omega,\tau,\lambda)\right] \\
&\quad + a^{(1)}(y,r,\omega,D_r,\lambda)h_{-3}(y,r,\omega,\tau,\lambda) \\
&\quad + \left[D_\omega^b a^{(2)}(y,r,\omega,D_r,\lambda)\right]\left[(\sqrt{-1}D^y)_b h_{-3}(y,r,\omega,\tau,\lambda)\right].
\end{aligned}
$$

The relevant equations for $a^{(i)}(y,r,\omega,D_r,\lambda)$, $i=0,1,2$ are

$$
\begin{aligned}
a^{(2)}(y,r,\omega,D_r,\lambda) &= a_2(y,r,\omega,D_r,\lambda)|_{r=0} \\
&= \tilde{g}^{ab}\omega_a\omega_b + D_r^2 - \lambda, \\
a^{(1)}(y,r,\omega,D_r,\lambda) &= r(\partial_r a_2(y,r,\omega,D_r,\lambda))|_{r=0} + a_1(y,r,\omega,D_r,\lambda)|_{r=0} \\
&= r\tilde{g}^{ab}_{,r}\omega_a\omega_b + \sqrt{-1}\tilde{b}^a\omega_a + \sqrt{-1}\tilde{b}^r D_r, \\
a^{(0)}(y,r,\omega,D_r,\lambda) &= \tfrac{1}{2}r^2(\partial_r^2 a_2(y,r,\omega,D_r,\lambda))|_{r=0} + r(\partial_r a_1(y,r,\omega,D_r,\lambda))|_{r=0} \\
&\quad + a_0(y,r,\omega,D_r,\lambda)|_{r=0} \\
&= \tfrac{1}{2}r^2\tilde{g}^{ab}_{,rr}\omega_a\omega_b + r\sqrt{-1}\,\tilde{b}^a_{,r}\omega_a + r\sqrt{-1}\,\tilde{b}^r_{,r}D_r + \tilde{c}.
\end{aligned}
$$

The differential equations have to be augmented by a growth condition

$$
h_{-2-j}(y,r,\omega,\tau,\lambda) \to 0 \qquad \text{as} \quad r\to\infty, \tag{4.5}
$$

and an initial condition corresponding to the Dirichlet boundary condition

$$
h_{-2-j}(y,r,\omega,\tau,\lambda)|_{r=0} = q_{-2-j}(y,r,\omega,\tau,\lambda)|_{r=0}. \tag{4.6}
$$

Once the symbols h_{-2-j} have been determined, their contribution to the asymptotics of the trace of the heat kernel follows from multiple integration. As before, we suppose $r^\alpha F \in C^\infty(\mathcal{C}_\varepsilon)$. The contribution reads

$$
\sum_{l=0}^{\infty} t^{\frac{1-\alpha-m}{2}} t^{\frac{l}{2}} \int_{\partial M} \eta_{\frac{l}{2}}(y,F,\Delta_M)dy
$$

with

$$
\begin{aligned}
\eta_{\frac{l}{2}}(y,F,\Delta_M) &= \frac{1}{(2\pi)^{m+1}} \sum_{j+k=l} \int_{\mathbb{R}^{m-1}} d\omega \int_{-\infty}^{\infty} ds \int_0^\infty d\bar{r} e^{\sqrt{-1}\,s} \\
&\quad \times \left(-\int_\gamma d\tau e^{-\sqrt{-1}\,\tau\bar{r}}\right) h_{-2-j}(y,\bar{r},\omega,\tau,-\sqrt{-1}\,s)\bar{r}^{k-\alpha}F_k(y),
\end{aligned} \tag{4.7}
$$

where γ is anticlockwise enclosing the poles of h_{-2-j} in the lower half-plane. The integral with respect to s is the contour integral transforming the resolvent to the heat kernel; see Section 5. Note that from (4.4) the contribution to the heat kernel is **minus** the above.

As will become clear in the following, with $\Lambda = \sqrt{|\omega|^2 + \sqrt{-1}\, s}$, we need integrals of the type

$$\begin{aligned} T^{kljn}_{ab\ldots} \equiv & \int_{\mathbb{R}^{m-1}} d\omega \int_{-\infty}^{\infty} ds \int_0^\infty d\bar{r} \mathrm{e}^{\sqrt{-1}\, s} \left(- \int_\gamma d\tau \mathrm{e}^{-\sqrt{-1}\, \tau \bar{r}} \right) \\ & \times \frac{\tau^k \bar{r}^{l-\alpha} \omega_a \omega_b \ldots}{\Lambda^j (\tau^2 + \Lambda^2)^n} \mathrm{e}^{-\bar{r}\Lambda}. \end{aligned}$$

The τ integration can be done using

$$\int_\gamma d\tau \mathrm{e}^{-\sqrt{-1}\, \tau \bar{r}} \frac{\tau^k}{(\tau^2 + \Lambda^2)^l} = \frac{(\sqrt{-1})^k (-1)^{l+k} \pi}{(l-1)!} \left(\frac{1}{2\Lambda} \frac{d}{d\Lambda} \right)^{l-1} \left[\Lambda^{k-1} \mathrm{e}^{-\bar{r}\Lambda} \right].$$

So

$$\begin{aligned} T^{kljn}_{ab\ldots} = & \tfrac{(\sqrt{-1})^k (-1)^{n+k+1} \pi}{(n-1)!} \int_{\mathbb{R}^{m-1}} d\omega \int_{-\infty}^{\infty} ds \\ & \times \int_0^\infty d\bar{r} \mathrm{e}^{\sqrt{-1}\, s} \bar{r}^{l-\alpha} \tfrac{\omega_a \omega_b \ldots}{\Lambda^j} \mathrm{e}^{-\bar{r}\Lambda} \left(\tfrac{1}{2\Lambda} \tfrac{d}{d\Lambda} \right)^{n-1} \left[\Lambda^{k-1} \mathrm{e}^{-\bar{r}\Lambda} \right]. \end{aligned}$$

Performing the Λ-differentiation, different $\bar{r}$-dependent functions would occur. It is therefore desirable to first perform the $\bar{r}$-integration before performing the Λ-derivatives explicitly. This is achieved by noting that ($z = \Lambda$ has to be put after the Λ differentiation has been performed)

$$\begin{aligned} T^{kljn}_{ab\ldots} = & \frac{(\sqrt{-1})^k (-1)^{n+k+1} \pi}{(n-1)!} \int_{\mathbb{R}^{m-1}} d\omega \int_{-\infty}^{\infty} ds \mathrm{e}^{\sqrt{-1}\, s} \frac{\omega_a \omega_b \ldots}{\Lambda^j} \\ & \times \left(\frac{1}{2\Lambda} \frac{d}{d\Lambda} \right)^{n-1} \Lambda^{k-1} \int_0^\infty d\bar{r} \bar{r}^{l-\alpha} \mathrm{e}^{-\bar{r}(\Lambda + z)} \Big|_{z=\Lambda} \\ = & \frac{(\sqrt{-1})^k (-1)^{n+k+1} \pi}{(n-1)!} \Gamma(l+1-\alpha) \int_{\mathbb{R}^{m-1}} d\omega \int_{-\infty}^{\infty} ds \mathrm{e}^{\sqrt{-1}\, s} \frac{\omega_a \omega_b \ldots}{\Lambda^j} \\ & \left(\frac{1}{2\Lambda} \frac{d}{d\Lambda} \right)^{n-1} \frac{\Lambda^{k-1}}{(\Lambda + z)^{l+1-\alpha}} \Bigg|_{z=\Lambda}. \end{aligned}$$

We can proceed in general by introducing numerical multipliers c_{nkl} according to

$$\left(\frac{1}{2\Lambda}\frac{d}{d\Lambda}\right)^{n-1} \frac{\Lambda^{k-1}}{(\Lambda+z)^{l+1-\alpha}}\bigg|_{z=\Lambda} = c_{nkl}\frac{1}{\Lambda^{l+2n-k-\alpha}}.$$

The s-integration is then performed using

$$\int_{-\infty}^{\infty} ds \frac{\mathrm{e}^{\sqrt{-1}\,s}}{(|\omega|^2+\sqrt{-1}\,s)^{\beta}} = \frac{2\pi}{\Gamma(\beta)}\mathrm{e}^{-|\omega|^2}.$$

The final ω-integrations follow from

$$C(y) \equiv \int_{\mathbb{R}^{m-1}} d\omega \mathrm{e}^{-\tilde{g}^{ab}\omega_a\omega_b+\sqrt{-1}\,y^a\omega_a} = \pi^{\frac{m-1}{2}}\sqrt{\tilde{g}}\mathrm{e}^{-\frac{\tilde{g}_{ab}y^ay^b}{4}},$$

by observing that

$$\int_{\mathbb{R}^{m-1}} d\omega\ \omega_{a_1}\omega_{a_2}\dots\omega_{a_r}\mathrm{e}^{-\tilde{g}^{ab}\omega_a\omega_b} = \left(\frac{1}{\sqrt{-1}}\right)^r \frac{\partial}{\partial y^{a_1}}\cdots\frac{\partial}{\partial y^{a_r}}C(y)\bigg|_{y=0}.$$

In particular

$$\begin{aligned}
\int_{\mathbb{R}^{m-1}} d\omega\, \mathrm{e}^{-|\omega|^2} &= \pi^{\frac{m-1}{2}}\sqrt{\tilde{g}},\\
\int_{\mathbb{R}^{m-1}} d\omega\, \omega_a\omega_b\mathrm{e}^{-|\omega|^2} &= \tfrac{1}{2}\pi^{\frac{m-1}{2}}\sqrt{\tilde{g}}\tilde{g}_{ab},\\
\int_{\mathbb{R}^{m-1}} d\omega\, \omega_a\omega_b\omega_c\omega_d\mathrm{e}^{-|\omega|^2} &= \tfrac{1}{4}\pi^{\frac{m-1}{2}}\sqrt{\tilde{g}}\left(\tilde{g}_{ab}\tilde{g}_{cd}+\tilde{g}_{ac}\tilde{g}_{bd}+\tilde{g}_{ad}\tilde{g}_{bc}\right).
\end{aligned}$$

Introducing the numerical multipliers d_{kljn} according to

$$d_{kljn} = \frac{2(\sqrt{-1})^k(-1)^{n+k+1}\pi^2\Gamma(l+1-\alpha)c_{nkl}}{(n-1)!\Gamma\left(\frac{j+l-k-\alpha}{2}+n\right)},$$

we obtain the compact-looking answers

$$T^{kljn}_{ab\dots} = d_{kljn}\int_{\mathbb{R}^{m-1}} d\omega\ \omega_a\omega_b\dots\mathrm{e}^{-|\omega|^2},$$

where the last ω-integration is performed with the above results.

Note that the numerical multipliers d_{kljn} are easily determined using an algebraic computer program. Therefore, all appearing integrals can be very easily obtained.

Let us apply this formalism explicitly to the leading orders, and we start with $h_{-2}(y,r,\omega,\tau,\lambda)$. The relevant differential equation reads

$$(\partial_r^2-\Lambda^2)h_{-2}(y,r,\omega,\tau,\lambda)=0,$$

which has the general solution

$$h_{-2}(y,r,\omega,\tau,\lambda)=A\mathrm{e}^{-r\Lambda}+B\mathrm{e}^{r\Lambda}.$$

The asymptotic condition (4.5) on the symbol as $r\to\infty$ imposes $B=0$. The initial condition $h_{-2}|_{r=0}=q_{-2}|_{r=0}$ gives

$$A=\frac{1}{\tau^2+\Lambda^2}.$$

Putting the information together we have obtained

$$h_{-2}(y,r,\omega,\tau,\lambda)=\frac{1}{\tau^2+\Lambda^2}\mathrm{e}^{-r\Lambda}.$$

Performing the relevant integrals, with the notation

$$\int dI=\int_{\mathbb{R}^{m-1}}d\omega\int_{-\infty}^{\infty}ds\int_0^{\infty}d\bar{r}\mathrm{e}^{\sqrt{-1}\,s}\left(-\int_\gamma d\tau\mathrm{e}^{-\sqrt{-1}\,\tau\bar{r}}\right)\bar{r}^{-\alpha},$$

produces

$$\begin{aligned}\int dI h_{-2}(y,\bar{r},\omega,\tau,-\sqrt{-1}\,s)&=d_{0001}\pi^{\frac{m-1}{2}}\sqrt{\tilde{g}}=\frac{2^\alpha\pi^2\Gamma(1-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)}\pi^{\frac{m-1}{2}}\sqrt{\tilde{g}}\\&=\pi\Gamma\left(\frac{1-\alpha}{2}\right)\pi^{m/2}\sqrt{\tilde{g}}.\end{aligned}$$

Taking into account the prefactor in (4.7), this confirms the value of $\bar{\kappa}_\alpha$ in Lemma 1.2.

In the next order we obtain

$$(\partial_r^2-\Lambda^2)h_{-3}(y,r,\omega,\tau,\lambda)=(E+U_1)\mathrm{e}^{-r\lambda}+(F+U_2)r\mathrm{e}^{-r\Lambda},$$

where

$$\begin{aligned}E&=-\frac{\tilde{b}_r\Lambda}{\tau^2+\Lambda^2},\quad F=\frac{\tilde{g}^{ab}_{,r}\omega_a\omega_b}{\tau^2+\Lambda^2},\\U_1(\omega)&=\frac{\sqrt{-1}\,\tilde{b}^a\omega_a}{\tau^2+\Lambda^2}+\frac{2\sqrt{-1}\,\tilde{g}^{ac}_{,b}\omega^b\omega_a\omega_c}{(\tau^2+\Lambda^2)^2},\quad U_2(\omega)=\frac{\sqrt{-1}\,\tilde{g}^{ac}_{,b}\omega^b\omega_a\omega_c}{(\tau^2+\Lambda^2)\Lambda}.\end{aligned}$$

Note, for later arguments, that $U_1(\omega)$ and $U_2(\omega)$ are odd functions in ω. Furthermore, for the scalar Laplacian at hand $b^a = g^{bc}\Gamma_{bc}{}^a$; thus they contain only *tangential* derivatives of the metric.

Using for example the annihilator method, we write down the general form of the solution to this differential equation as

$$h_{-3}(y,r,\omega,\tau,\lambda) = c_1 e^{-r\Lambda} + c_2 r e^{-r\Lambda} + c_3 r^2 e^{-r\Lambda} + c_4 e^{r\Lambda}.$$

From the asymptotic condition (4.5) we conclude $c_4 = 0$. From the initial condition given in Equation (4.6) we obtain

$$c_1 = -\frac{\sqrt{-1}\,\tilde{b}^a\omega}{(\tau^2+\Lambda^2)^2} - \frac{\sqrt{-1}\,\tilde{b}^r\tau}{(\tau^2+\Lambda^2)^2} - \frac{2\sqrt{-1}\,\tilde{g}^{ab}_{,c}\omega^c\omega_a\omega_b}{(\tau^2+\Lambda^2)^3} - \frac{2\sqrt{-1}\,\tilde{g}^{ab}_{,r}\tau\omega_a\omega_b}{(\tau^2+\Lambda^2)^3}.$$

From the differential equation we derive

$$\begin{aligned} c_2 &= -\tfrac{1}{4\Lambda^2}(F+U_2) - \tfrac{1}{2\Lambda}(E+U_1),\\ c_3 &= -\tfrac{1}{4\Lambda}(F+U_2). \end{aligned}$$

Collecting the available information, we see

$$h_{-3}(y,r,\omega,\tau,\lambda) = D e^{-r\Lambda} + B r e^{-r\Lambda} + C r^2 e^{-r\Lambda} + O(\omega),$$

with

$$\begin{aligned} D &= -\frac{\sqrt{-1}\,\tilde{b}^r\tau}{(\tau^2+\Lambda^2)^2} - \frac{2\sqrt{-1}\,\tilde{g}^{ab}_{,r}\tau\omega_a\omega_b}{(\tau^2+\Lambda^2)^3},\\ B &= -\frac{\tilde{g}^{ab}_{,r}\omega_a\omega_b}{4\Lambda^2(\tau^2+\Lambda^2)} + \frac{\tilde{b}^r}{2(\tau^2+\Lambda^2)},\\ C &= -\frac{\tilde{g}^{ab}_{,r}\omega_a\omega_b}{4\Lambda(\tau^2+\Lambda^2)}, \end{aligned}$$

and where $O(\omega)$ is an odd function in ω. Furthermore, $O(\omega)$ contains only tangential derivatives of the metric. We next perform the multiple integrals; note, odd functions in ω do not contribute. We obtain

$$\begin{aligned} &\int dI h_{-3}(y,\bar{r},\omega,\tau,-\sqrt{-1}\,s)\\ &= \pi^{\frac{m-1}{2}}\sqrt{\tilde{g}}\tilde{g}^{ab}_{,r}\tilde{g}_{ab}\left\{-\tfrac{\sqrt{-1}}{2}d_{1002} + \tfrac{1}{4}d_{0101} - \sqrt{-1}\,d_{1003} - \tfrac{1}{8}d_{0121} - \tfrac{1}{8}d_{0211}\right\}\\ &= \frac{\pi(\alpha-4)}{4(3-\alpha)}\Gamma\left(\frac{2-\alpha}{2}\right)\pi^{m/2}\sqrt{\tilde{g}}\tilde{g}^{ab}_{,r}\tilde{g}_{ab}. \end{aligned}$$

This confirms the value of κ_α^1 in Lemma 1.2 after taking into account the prefactor in (4.7) and the fact that $\tilde{g}^{ab}_{,r}\tilde{g}_{ab} = -\tilde{g}^{ab}\tilde{g}_{ab,r} = 2g^{ab}L_{ab}$.

Up to this point the calculation can be considered a warm up for the next order. We would like to determine the universal coefficients of the geometric invariants R_{amma}, $L_{aa}L_{bb}$ and $L_{ab}L_{ab}$. In terms of the metric the last two are determined by

$$L_{ab} = -\tfrac{1}{2}\tilde{g}_{ab,r}.$$

Using the Christoffel symbols

$$\Gamma_{jk}{}^{i} = \tfrac{1}{2}g^{il}\left(g_{lj,k} + g_{kl,j} - g_{jk,l}\right),$$

and taking into account that with our sign convention the scalar curvature is given by the contraction $g^{jk}R_{ijk}{}^{i}$, we may expand the Riemann curvature tensor in the form:

$$R_{ijk}{}^{l} = \Gamma_{jk}{}^{l}{}_{,i} - \Gamma_{ik}{}^{l}{}_{,j} + \Gamma_{in}{}^{l}\Gamma_{jk}{}^{n} - \Gamma_{jn}{}^{l}\Gamma_{ik}{}^{n}.$$

The normal projection of the Riemann curvature tensor reads

$$\begin{aligned}\tilde{R}_{amma} &= -\tfrac{1}{2}\tilde{g}^{ac}_{,r}\tilde{g}_{ac,r} - \tfrac{1}{2}\tilde{g}^{ac}\tilde{g}_{ac,rr} - \tfrac{1}{4}\tilde{g}^{bc}_{,r}\tilde{g}^{ad}_{,r}\tilde{g}_{ca}\tilde{g}_{bd}\\ &= \tfrac{1}{4}\tilde{g}^{ab}\tilde{g}^{cd}\tilde{g}_{ac,r}\tilde{g}_{bd,r} - \tfrac{1}{2}\tilde{g}^{ac}\tilde{g}_{ac,rr}.\end{aligned}$$

The above results suggest a strategy for the calculation. It suffices to consider the special case where the metric is independent of y. As a consequence, our answer will have the form

$$(4.8) \qquad (4\pi)^{-m/2}\left\{A\tilde{g}^{ac}\tilde{g}_{ac,rr} + B\tilde{g}^{ab}\tilde{g}^{cd}\tilde{g}_{ac,r}\tilde{g}_{bd,r} + C\tilde{g}^{ab}\tilde{g}^{cd}\tilde{g}_{ab,r}\tilde{g}_{cd,r}\right\}.$$

This has to be compared with the terms in $a^{bd}_{2,\alpha}(F,\Delta_M)$ that possibly contribute to these geometric invariants. In detail one can show these terms are (mod terms with tangential derivatives of the metric)

$$\begin{aligned}&-\tfrac{1}{6}\kappa_\alpha\tilde{R} + \kappa_\alpha^3\tilde{R}_{amma} + \kappa_\alpha^4 L_{aa}L_{bb} + \kappa_\alpha^5 L_{ab}L_{ab}\\ &\quad= \tilde{g}^{ac}\tilde{g}_{ac,rr}\left(\tfrac{1}{6}\kappa_\alpha - \tfrac{1}{2}\kappa_\alpha^3\right) + \tilde{g}^{ab}\tilde{g}^{cd}\tilde{g}_{ab,r}\tilde{g}_{cd,r}\left(\tfrac{1}{24}\kappa_\alpha + \tfrac{1}{4}\kappa_\alpha^4\right)\\ &\qquad + \tilde{g}^{ab}\tilde{g}^{cd}\tilde{g}_{ac,r}\tilde{g}_{bd,r}\left(-\tfrac{1}{8}\kappa_\alpha + \tfrac{1}{4}\kappa_\alpha^3 + \tfrac{1}{4}\kappa_\alpha^5\right).\end{aligned}$$

So once we know A, B, C, we can deduce

(4.9)
$$\kappa_\alpha^3 = -2\left(A - \tfrac{1}{6}\kappa_\alpha\right), \quad \kappa_\alpha^4 = 4\left(C - \tfrac{1}{24}\kappa_\alpha\right), \quad \kappa_\alpha^5 = 4\left(B + \tfrac{1}{8}\kappa_\alpha - \tfrac{1}{4}\kappa_\alpha^3\right).$$

In summary, when writing down the differential equation for $h_{-4}(y, r, \omega, \tau, \lambda)$, we can neglect all terms that are odd in ω as well as all terms that contain tangential derivatives of the metric. We obtain (up to irrelevant terms)

$$(\partial_r^2 - \Lambda^2)h_{-4}(y, r, \omega, \tau, \lambda) = A\mathrm{e}^{-r\Lambda} + Br\mathrm{e}^{-r\Lambda} + Cr^2\mathrm{e}^{-r\Lambda} + Dr^3\mathrm{e}^{-r\Lambda},$$

where

$$\begin{aligned}
A &= \tfrac{\sqrt{-1}\,\tilde{b}^r\tilde{b}^r\Lambda\tau}{(\tau^2+\Lambda^2)^2} + \tfrac{2\sqrt{-1}\,\tilde{b}^r\tilde{g}^{ab}_{,r}\Lambda\tau\omega_a\omega_b}{(\tau^2+\Lambda^2)^3} - \tfrac{\tilde{b}^r\tilde{g}^{ab}_{,r}\omega_a\omega_b}{4\Lambda^2(\tau^2+\Lambda^2)} + \tfrac{\tilde{b}^r\tilde{b}^r}{2(\tau^2+\Lambda^2)},\\
B &= -\tfrac{\tilde{b}^r_{,r}\Lambda}{\tau^2+\Lambda^2} - \tfrac{\sqrt{-1}\,\tilde{g}^{ab}_{,r}\tilde{b}^r\tau\omega_a\omega_b}{(\tau^2+\Lambda^2)^2} - \tfrac{2\sqrt{-1}\,\tilde{g}^{ab}_{,r}\tilde{g}^{cd}_{,r}\tau\omega_a\omega_b\omega_c\omega_d}{(\tau^2+\Lambda^2)^3} - \tfrac{\tilde{g}^{ab}_{,r}\tilde{b}^r\omega_a\omega_b}{4\Lambda(\tau^2+\Lambda^2)} - \tfrac{\tilde{b}^r\tilde{b}^r\Lambda}{2(\tau^2+\Lambda^2)},\\
C &= \tfrac{\tilde{g}^{ab}_{,rr}\omega_a\omega_b}{2(\tau^2+\Lambda^2)} - \tfrac{\tilde{g}^{ab}_{,r}\tilde{g}^{cd}_{,r}\omega_a\omega_b\omega_c\omega_d}{4\Lambda^2(\tau^2+\Lambda^2)} + \tfrac{3\tilde{g}^{ab}_{,r}\tilde{b}^r\omega_a\omega_b}{4(\tau^2+\Lambda^2)},\\
D &= -\tfrac{\tilde{g}^{ab}_{,r}\tilde{g}^{cd}_{,r}\omega_a\omega_b\omega_c\omega_d}{4\Lambda(\tau^2+\Lambda^2)}.
\end{aligned}$$

So the solution has the form, taking into account the asymptotic behaviour (4.5),

$$h_{-4}(y, r, \omega, \tau, \lambda) = \tilde{\alpha}\mathrm{e}^{-r\Lambda} + \beta r\mathrm{e}^{-r\Lambda} + \gamma r^2\mathrm{e}^{-r\Lambda} + \delta r^3\mathrm{e}^{-r\Lambda} + \epsilon r^4\mathrm{e}^{-r\Lambda}.$$

From the initial condition $\tilde{\alpha} = q_{-4}(y, r, \omega, \tau, \lambda)|_{r=0}$ we obtain, up to irrelevant terms,

$$\begin{aligned}
\tilde{\alpha} &= \tfrac{1}{(\tau^2+\Lambda^2)^3}\left\{-\tilde{b}^r\tilde{b}^r\tau^2 + \tilde{b}^r\tilde{g}^{ab}_{,r}\omega_a\omega_b + 2\tilde{b}^r_{,r}\tau^2 - \tilde{g}^{ab}_{,rr}\omega_a\omega_b\right\}\\
&\quad + \tfrac{1}{(\tau^2+\Lambda^2)^4}\left\{-6\tilde{b}^r\tilde{g}^{ab}_{,r}\tau^2\omega_a\omega_b + 4\tilde{g}^{ab}_{,rr}\tau^2\omega_a\omega_b + 2\tilde{g}^{ab}_{,r}\tilde{g}^{cd}_{,r}\omega_a\omega_b\omega_c\omega_d\right\}\\
&\quad + \tfrac{1}{(\tau^2+\Lambda^2)^5}\left\{-12\tilde{g}^{ab}_{,r}\tilde{g}^{cd}_{,r}\omega_a\omega_b\omega_c\omega_d\tau^2\right\}.
\end{aligned}$$

From the differential equation we obtain the conditions

$$\begin{aligned}
A &= -2\Lambda\beta + 2\gamma, \quad B = -4\Lambda\gamma + 6\delta,\\
C &= -6\Lambda\delta + 12\epsilon, \quad D = -8\epsilon\Lambda.
\end{aligned}$$

This determines the numerical multipliers β, γ, δ and ϵ to be

$$\begin{aligned}
\beta &= -\tfrac{3}{8}\tfrac{D}{\Lambda^4} - \tfrac{1}{4}\tfrac{C}{\Lambda^3} - \tfrac{1}{4}\tfrac{B}{\Lambda^2} - \tfrac{1}{2}\tfrac{A}{\Lambda},\\
\gamma &= -\tfrac{3}{8}\tfrac{D}{\Lambda^3} - \tfrac{1}{4}\tfrac{C}{\Lambda^2} - \tfrac{1}{4}\tfrac{B}{\Lambda},\\
\delta &= -\tfrac{1}{4}\tfrac{D}{\Lambda^2} - \tfrac{1}{6}\tfrac{C}{\Lambda},\\
\epsilon &= -\tfrac{1}{8}\tfrac{D}{\Lambda}.
\end{aligned}$$

For the Laplacian on the manifold M we have

$$\begin{aligned}
\tilde b^r &= -\tfrac{1}{2}\tilde g^{ab}\tilde g_{ab,r},\\
\tilde b^r\tilde b^r &= \tfrac{1}{4}\tilde g^{ab}\tilde g^{cd}\tilde g_{ab,r}\tilde g_{cd,r},\\
\tilde b^r\tilde g_{ab}\tilde g^{ab}_{,r} &= \tfrac{1}{2}\tilde g^{ab}\tilde g^{cd}\tilde g_{ab,r}\tilde g_{cd,r},\\
\tilde b^r_{,r} &= \tfrac{1}{2}\tilde g^{ac}\tilde g^{bd}\tilde g_{cd,r}\tilde g_{ab,r} - \tfrac{1}{2}\tilde g^{ab}\tilde g_{ab,rr},\\
\tilde g_{ab}\tilde g^{ab}_{,rr} &= 2\tilde g^{ac}\tilde g^{bd}\tilde g_{ab,r}\tilde g_{cd,r} - \tilde g^{ab}\tilde g_{ab,rr},\\
\tilde g^{ab}_{,r}\tilde g^{cd}_{,r}\left(\tilde g_{ab}\tilde g_{cd}+\tilde g_{ac}\tilde g_{bd}+\tilde g_{ad}\tilde g_{bc}\right) &= \tilde g^{ab}\tilde g^{cd}\tilde g_{ab,r}\tilde g_{cd,r} + 2\tilde g^{ab}\tilde g^{cd}\tilde g_{ac,r}\tilde g_{bd,r}.
\end{aligned}$$

Performing the integrations we obtain the contributions (modulo $\pi^{(m-1)/2}$ $\sqrt{\tilde g}$)

$$\begin{aligned}
\tilde\alpha_I = {}& \tilde g^{ab}\tilde g^{cd}\tilde g_{ab,r}\tilde g_{cd,r}\left[-\tfrac{1}{4}d_{2003}+\tfrac{1}{4}d_{0003}-\tfrac{3}{2}d_{2004}+\tfrac{1}{2}d_{0004}-3d_{2005}\right]\\
&+\tilde g^{ab}\tilde g^{cd}\tilde g_{ac,r}\tilde g_{bd,r}\left[d_{2003}-d_{0003}+4d_{2004}+d_{0004}-6d_{2005}\right]\\
&+\tilde g^{ab}\tilde g_{ab,rr}\left[-d_{2003}+\tfrac{1}{2}d_{0003}-2d_{2004}\right],\\
\beta_I = {}& \tilde g^{ab}\tilde g^{cd}\tilde g_{ab,r}\tilde g_{cd,r}\left[\tfrac{3}{128}d_{0151}+\tfrac{1}{64}d_{0151}+\tfrac{\sqrt{-1}}{8}d_{1123}+\tfrac{\sqrt{-1}}{16}d_{1122}-\tfrac{\sqrt{-1}}{4}d_{1103}\right.\\
&\left.-\tfrac{\sqrt{-1}}{8}d_{1102}-\tfrac{1}{32}d_{0111}\right]\\
&+\tilde g^{ab}\tilde g^{cd}\tilde g_{ac,r}\tilde g_{bd,r}\left[\tfrac{3}{64}d_{0151}+\tfrac{1}{32}d_{0151}+\tfrac{\sqrt{-1}}{4}d_{1123}-\tfrac{1}{8}d_{0131}+\tfrac{1}{8}d_{0111}\right]\\
&+\tilde g^{ab}\tilde g_{ab,rr}\left[\tfrac{1}{16}d_{0131}-\tfrac{1}{8}d_{0111}\right],\\
\gamma_I = {}& \tilde g^{ab}\tilde g^{cd}\tilde g_{ab,r}\tilde g_{cd,r}\left[\tfrac{3}{128}d_{0241}+\tfrac{1}{64}d_{0241}+\tfrac{\sqrt{-1}}{8}d_{1213}+\tfrac{1}{32}d_{0201}-\tfrac{3}{64}d_{0221}\right.\\
&\left.+\tfrac{\sqrt{-1}}{16}d_{1212}+\tfrac{1}{64}d_{0221}\right]\\
&+\tilde g^{ab}\tilde g^{cd}\tilde g_{ac,r}\tilde g_{bd,r}\left[\tfrac{3}{64}d_{0241}+\tfrac{1}{32}d_{0241}+\tfrac{\sqrt{-1}}{4}d_{1213}-\tfrac{1}{8}d_{0221}+\tfrac{1}{8}d_{0201}\right]\\
&+\tilde g^{ab}\tilde g_{ab,rr}\left[\tfrac{1}{16}d_{0221}-\tfrac{1}{8}d_{0201}\right]\\
\delta_I = {}& \tilde g^{ab}\tilde g^{cd}\tilde g_{ab,r}\tilde g_{cd,r}\left[\tfrac{5}{192}d_{0331}-\tfrac{1}{32}d_{0311}\right]\\
&+\tilde g^{ab}\tilde g^{cd}\tilde g_{ac,r}\tilde g_{bd,r}\left[\tfrac{5}{96}d_{0331}-\tfrac{1}{12}d_{0311}\right]\\
&+\tilde g^{ab}\tilde g_{ab,rr}\left[\tfrac{1}{24}d_{0311}\right],\\
\epsilon_I = {}& \tilde g^{ab}\tilde g^{cd}\tilde g_{ab,r}\tilde g_{cd,r}\left[\tfrac{1}{128}d_{0421}\right]+\tilde g^{ab}\tilde g^{cd}\tilde g_{ac,r}\tilde g_{bd,r}\left[\tfrac{1}{64}d_{0421}\right].
\end{aligned}$$

Adding up all terms and simplifying using the functional equation and the doubling formula for the Γ-function, the contribution to the heat kernel coefficient reads

$$(4\pi)^{-m/2}\left\{\tilde{g}^{ab}\tilde{g}^{cd}\tilde{g}_{ab,r}\tilde{g}_{cd,r}\tfrac{3\alpha^2-16\alpha-27}{384(\alpha-6)}\Gamma\left(\tfrac{1-\alpha}{2}\right)\right.$$
$$\left.+\tilde{g}^{ab}\tilde{g}^{cd}\tilde{g}_{ac,r}\tilde{g}_{bd,r}\tfrac{5(9+4\alpha-\alpha^2)}{192(\alpha-6)}\Gamma\left(\tfrac{1-\alpha}{2}\right)+\tilde{g}^{ab}\tilde{g}_{ab,rr}\tfrac{\alpha+3}{48}\Gamma\left(\tfrac{1-\alpha}{2}\right)\right\}.$$

This allows us to read off A, B and C from Equations (4.8) and (4.9) and to conclude:

$$\kappa_\alpha^3=-\tfrac{1}{12}(\alpha-1)\kappa_\alpha,\quad \kappa_\alpha^4=\tfrac{7-8\alpha+\alpha^2}{16(\alpha-6)}\kappa_\alpha,\quad \kappa_\alpha^5=\tfrac{6\alpha-5-\alpha^2}{8(\alpha-6)}\kappa_\alpha.$$

This completes the proof of Lemma 1.3. □

5. The proof of Theorem 1.4

We adopt the notation of Theorem 1.4. We may suppose that the weighting function F is supported in a boundary coordinate neighbourhood U and that F has the form $F=r^{-\alpha}G$ with $G\in C^\infty_{\mathrm{comp}}(U)$. For appropriate functions q and q^{bd} set

$$\begin{aligned}
&Op(q)f(y,r):=\int\int e^{\sqrt{-1}y\cdot\omega+\sqrt{-1}r\tau}q(y,r,\omega,\tau,\lambda)\hat{f}(\omega,\tau)\bar{d}\omega\bar{d}\tau,\\
&Op'(q^{bd})f(y,r):=\int\int e^{\sqrt{-1}y\cdot\omega}q^{bd}(y,r,\omega,\tau,\lambda)\hat{f}(\omega,\tau)\bar{d}\omega\bar{d}\tau,\\
&\bar{d}\omega:=(2\pi)^{1-m}d\omega\quad\text{and}\quad\bar{d}\tau:=(2\pi)^{-1}d\tau.
\end{aligned}$$

Thus one has that $Op'(q^{bd})=Op(e^{-\sqrt{-1}r\tau}q^{bd})$. In local coordinates (y,r), we follow the construction in [13] to define the standard parametrix for $(\mathcal{D}-\lambda)^{-1}$:

$$Q_N(\lambda)=\sum_{n=0}^{N}\left\{Op(q_{-2-n}(\lambda))-Op'(q^{bd}_{-2-n}(\lambda))\right\}.$$

The q_j and q_j^{bd} are discussed in further detail in Section 4; we refer in particular to Equations (4.1)–(4.3). Note that the q_j^{bd} here is the h_j of Section 4.

We pass to a parametrix H_N for the operator $e^{-t\mathcal{D}}$ by performing a contour integration:

$$H_N(\mathcal{D},t):=\frac{1}{2\pi\sqrt{-1}}\int_\Gamma e^{-t\lambda}Q_N(\lambda)d\lambda=\sum_{n=0}^{N}\{Op(e_{-2-n})-Op'(e^{bd}_{-2-n})\};$$

in this expression, e_j and e_j^{bd} are the corresponding contour integrals of q_j and q_j^{bd}. For another construction of the heat parametrix see [8].

Take ϕ and ψ with compact support in U, with $\psi \equiv 1$ in a neighbourhood of the support of ϕ. Then

$$R_N = \phi \left[\mathrm{e}^{-t\mathcal{D}} - H_N\right] \psi$$

has a kernel $k(y, r, y', r'; t)$ which, for sufficiently large N, is C^2, $O(t^J)$ for large J (depending on N), and vanishes when $r = 0$, although not for $r' = 0$. To achieve vanishing in both variables, we consider

$$\begin{aligned} &\phi H_N(\mathcal{D}, t/2)\psi^2 H_N(\mathcal{D}^*, t/2)^*\phi - \phi\mathrm{e}^{-t\mathcal{D}/2}\psi^2(\mathrm{e}^{-t\mathcal{D}^*/2})^*\phi \\ &\quad = -R_N\psi H_N(\mathcal{D}^*, t/2)^*\phi - \phi H_N(\mathcal{D}, t/2)\psi R_N^* - R_N R_N^*. \end{aligned} \tag{5.1}$$

Here we compute the adjoints with respect to the measure $dydr$. The remainder given by Equation (5.1) has a kernel which is C^2, with derivatives $O(t^J)$ for large J, and vanishes when either $r = 0$ or $r' = 0$. By the pseudo-local properties of H_N and $\mathrm{e}^{-t\mathcal{D}}$, the same is true of

$$\phi H_N(\mathcal{D}, t/2) H_N(\mathcal{D}^*, t/2)^*\phi - \phi\,\mathrm{e}^{-t\mathcal{D}}\phi.$$

Hence the kernel of this operator is $O(r^2 t^J)$ for large J and we can deduce an expansion for $\mathrm{Tr}_{L^2}(F\mathrm{e}^{-t\mathcal{D}})$ from an expansion of $\mathrm{Tr}_{L^2}(FH_N H_N^*\phi)$, taking $\phi \equiv 1$ on the support of F. The error will be holomorphic in α for $\mathrm{Re}(\alpha) < 3$. Since $\mathrm{Tr}_{L^2}(F\mathrm{e}^{-t\mathcal{D}})$ is also holomorphic for $\mathrm{Re}(\alpha) < 3$, we may compute our expansion for $\mathrm{Re}(\alpha) < 1$ and continue analytically.

From the expansion of q_j in powers of $(|\omega|^2 + \tau^2 - \lambda)^{-1}$ (See Section 4) we obtain the estimates

$$|e_{-2-k}| \le Ct^{k/2}\mathrm{e}^{-ct(|\omega|^2+\tau^2)} \tag{5.2}$$

$$|e_j^{bd}| \le C(|\omega|^2 + \tau^2 + 1)^{-1}\mathrm{e}^{-|\omega|^2 t/2} \tag{5.3}$$

for suitably chosen constants C and c. Set $\xi = (\omega, \tau)$. The q_j and the q_j^{bd} have an appropriate homogeneity property [13]. This homogeneity yields that

$$e_j(y, r, s\xi, t/s^2) = s^{2+j} e_j(y, r, \xi, t), \quad \text{and} \tag{5.4}$$

$$e_j^{bd}(y, r/s, s\xi, t/s^2) = s^{2+j} e_j^{bd}(y, r, \xi, t). \tag{5.5}$$

Furthermore, the kernel k_1 of $Op(e_j)Op(e_k^*)^*$ and the kernel k_2 of $Op(e_j)Op'(e_k^{bd*})^*$ on the diagonal are given, respectively, by:

$$\begin{aligned} k_1(y,r,y,r;t) &= \int e_j(y,r,\xi,t/2)e_k(y,r,\xi,t/2)\bar{d}\xi \\ &= (t/2)^{-(m+j+k+4)/2}\int e_j(y,r,\xi,1)e_k(y,r,\xi,1)\bar{d}\xi, \end{aligned} \tag{5.6}$$

(5.7)

$$k_2(y,r,y,r;t) = (t/2)^{-(m+j+k+4)/2}\int e_j(y,r,\xi,1)e_k^{bd}(y,r(2/t)^{1/2},\xi,1)\bar{d}\xi.$$

There are similar formulas for the kernel of $Op'(e_j^{bd})Op(e_k^*)^*$ and for the kernel of $Op'(e_j^{bd})Op'(e_k^{bd*})^*$ on the diagonal.

We integrate $F(y,r)k_i(y,r,y,r;t)$. An expansion of Fe_je_k in powers $r^{\ell-\alpha}$ gives terms $t^{-(m+j+k+4)/2}$ times a meromorphic function with simple poles at $\alpha = 1,2,\ldots$. For other α, this gives the interior terms in Theorem 1.4; the terms with $j+k$ odd vanish because of the parity of e_j and e_k in ξ. For the terms in Equation (5.7), note that from Equations (5.3) and (5.5), e_k^{bd} decays exponentially as $r\to\infty$ so that we may integrate in r from 0 to ∞. An expansion of Fe_j in powers of r and a change of variable $r/\sqrt{t}\to r$ gives boundary terms of the form in Theorem 1.4 for $\alpha\neq 1,2$.

This proves Theorem 1.4 for $\mathrm{Re}(\alpha)<1$ and the rest follows by analytic continuation. Since for each t, the expansion is holomorphic in α for $\mathrm{Re}(\alpha)<3$, the residues arising from the interior integrals at $\alpha=1,2$ must be cancelled by residues from the boundary terms. And, since the various powers of t are linearly independent, the residue of each interior coefficient must be cancelled by the residue from the corresponding boundary coefficient. The residue from $t^n\mathcal{I}_{\mathrm{Reg}}(Fa_n) = t^n\mathcal{I}_{\mathrm{Reg}}(r^{-\alpha}Ga_n)$ is

$$\begin{cases} -t^n\mathcal{I}^{bd}\{(Fa_n)_0\} & \text{at } \alpha=1, \\ -t^n\mathcal{I}^{bd}\{(Fa_n)_1+(Fa_n)_0L_{aa}\} & \text{at } \alpha=2. \end{cases} \tag{5.8}$$

The corresponding boundary term when $\alpha=1$ is $t^{n-(\alpha-1)/2}a_{2n,\alpha}^{bd}(F,\mathcal{D})$. Let R be the residue of $a_{2n,\alpha}^{bd}(F,\mathcal{D})$ at $\alpha=1$. Then

$$t^{n-(\alpha-1)/2}a_{2n,\alpha}^{bd}(F,\mathcal{D}) = \frac{t^nR}{\alpha-1} - \frac{1}{2}t^n\ln(t)R + (\mathrm{const})t^n + O(\alpha-1)$$

Assertion (2) now follows. Since t^nR must cancel the residue in Equation (5.8), we get the $\ln(t)$ coefficient in Assertion (4) for $\alpha=1$. The proof of the rest of Assertion (4) and also of Assertion (3) follows in a similar fashion. $\square$

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