The Laplacian on hyperbolic 3-manifolds with Dehn surgery type singularities

FRANK PFÄFFLE AND HARTMUT WEISS

We study the spectrum of the Laplacian on hyperbolic 3-manifolds with Dehn surgery type singularities and its dependence on the generalized Dehn surgery coefficients.

1. Introduction

Let M be a complete non-compact hyperbolic 3-manifold of finite volume. The Laplacian considered as a symmetric densely defined operator

$$\Delta : C_0^\infty(M) \to L^2(M)$$

is essentially self-adjoint, cf. [1]. The essential spectrum of the unique self-adjoint extension of Δ consists of the interval $[1, \infty)$, cf. [2,3]. Let us further denote by $0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_k < 1$ the finitely many eigenvalues below the essential spectrum, cf. [4].

If $(M_i)_{i \in \mathbb{N}}$ is a sequence of compact hyperbolic 3-manifolds, which converges to M, say in the pointed Lipschitz topology, one can ask in what sense the spectrum of the limit manifold is related to the spectra of the approximators. Note that since M_i is a compact manifold, the spectrum of

the Laplacian on M_i is discrete for each $i \in \mathbb{N}$. Let us assume for simplicity that M has a single rank-2 cusp. The following results are known:

In [5], Colbois and Courtois show that the eigenvalues of M below the essential spectrum are the limits of eigenvalues of the M_i. More precisely, if $0 = \lambda_0^i < \lambda_1^i \leq \cdots \leq \lambda_{k(i)}^i < 1$ are the eigenvalues of M_i smaller than 1, then for i large enough one has $k(i) \geq k$ and further

$$(1.1) \qquad\qquad \lim_{i \to \infty} \lambda_j^i = \lambda_j$$

for $j = 0, \ldots, k$.

In [6], Chavel and Dodziuk show that the eigenvalues of the M_i accumulate in the interval $[1, \infty)$ as $i \to \infty$. Moreover, they determine the precise rate of clustering in terms of geometric data of the degenerating tube. Namely, with $\mathcal{N}_{\Delta_i, M_i}[1, 1 + x^2] = |\{\lambda \in \operatorname{spec} \Delta_i : 1 \leq \lambda \leq 1 + x^2\}|$ denoting the spectral counting function of the Laplacian on M_i, they obtain the estimate

$$(1.2) \qquad\qquad \mathcal{N}_{\Delta_i, M_i}[1, 1 + x^2] = \frac{x}{2\pi} \log\left(\frac{1}{l_i}\right) + O_x(1),$$

where l_i is the length of the shortest closed geodesic in M_i.

Let us mention that similar questions have been studied for the Laplacian on differential forms by Dodziuk and McGowan in [7] and for the Dirac operator on complex spinors by Bär in [8] and Pfäffle in [9].

From the point of view of the deformation theory of M, it is natural to bring a certain class of singular hyperbolic 3-manifolds into play: while due to Mostow–Prasad rigidity, cf. [10, 11], the hyperbolic structure on M may not be deformed through complete hyperbolic structures, there is actually a real two-dimensional deformation space of incomplete hyperbolic structures parameterized by the so-called generalized Dehn surgery coefficients. This is the essence of Thurston's Hyperbolic Dehn Surgery Theorem, cf. [12]. The existence of sequences M_i as above is, in fact, a consequence of that theorem.

These incomplete structures are said to have Dehn surgery type singularities, special cases include hyperbolic cone-manifold structures and in particular smooth hyperbolic structures on certain topological fillings.

The aim of this article is to study the basic spectral properties of the Laplacian on hyperbolic 3-manifolds with Dehn surgery type singularities and to prove analogues of the asymptotic statements in Equations (1.1) and (1.2) for this wider class of hyperbolic manifolds. The main results are Theorems 5.1 and 4.1. Note that due to Corollary 3.3, our results include the results of [5] and [6]. Moreover, we emphasize a "continuous" aspect of these

phenomena, namely all our estimates take place on the deformation space of structures (and do not make reference to specific sequences of manifolds).

2. Hyperbolic Dehn surgery

Let M be a complete hyperbolic 3-manifold of finite volume. As a consequence of the Margulis Lemma, cf. [15], M has only finitely many ends all of which are rank-2 cusps. More precisely, if for $\mu > 0$ we look at the thick–thin decomposition of M, i.e., $M = M_{(0,\mu)} \cup M_{[\mu,\infty)}$, where

$$M_{(0,\mu)} = \{p \in M : inj_p < \mu\}$$

is the μ-*thin* part of M, and

$$M_{[\mu,\infty)} = \{p \in M : inj_p \geq \mu\}$$

the μ-*thick* part, then the Margulis Lemma asserts the existence of a universal constant μ_0, such that the components of the μ-thin part of M have standard geometry for $\mu < \mu_0$: they are either *rank-2 cusps* or *smooth tubes*.

A *rank-2 cusp* is the quotient of a horoball in $\mathbb{H}^3$ by a rank-2 free abelian group of parabolic isometries, which we in the following denote by Γ_{cusp}. The intrinsic geometry of the boundary horosphere is that of flat $\mathbb{R}^2$, its principal curvatures are constantly 1. Let $T^2_{\text{cusp}} = \mathbb{R}^2 / \Gamma_{\text{cusp}}$ be the corresponding flat torus. Then the cusp based on T^2_{cusp} is given as the Riemannian manifold

$$((0,\infty) \times T^2, dt^2 + e^{-2t} g_{T^2}),$$

where $t \in (0,\infty)$ and g_{T^2} denotes the flat metric on T^2.

A *smooth tube* is the quotient of the distance tube of a geodesic γ in $\mathbb{H}^3$ by an infinite cyclic group of hyperbolic isometries. Let us denote its generator in the following by ϕ. The isometry ϕ is a screw-motion along γ, i.e., after orienting γ, we can associate the translation length $l > 0$ and the rotation angle $t \in [0, 2\pi)$ with ϕ. The boundary of the distance tube of radius r is intrinsically flat, its principal curvatures are given by $\coth(r)$ and $\tanh(r)$. Let Γ be the lattice in $\mathbb{R}^2$ generated by the vectors $(2\pi, 0)$ and (t, l) and let $T^2 = \mathbb{R}^2/\Gamma$. Then the tube of radius R based on T^2 is given as the

Riemannian manifold

$$((0, R) \times T^2, dr^2 + \sinh(r)^2 d\theta^2 + \cosh(r)^2 dz^2),$$

where $r \in (0, R)$ and $(\theta, z) \in \mathbb{R}^2$.

Let $\bar{M}$ denote the compact core of M obtained by removing the cusp components from the μ-thin part of M. The boundary components of $\bar{M}$ are horospherical tori, whose injectivity radii satisfy a universal lower bound. For simplicity we will assume in the following that M has a single cusp.

Let $\text{Def}(M)$ be the deformation space of (possibly incomplete) hyperbolic structures on M, for precise definitions see [16]. A hyperbolic structure is determined by a local diffeomorphism $\text{dev} : \tilde{M} \to \mathbb{H}^3$, the *developing map*, which is equivariant w.r.t. a group homomorphism $\text{hol} : \pi_1 M \to \text{SL}_2(\mathbb{C})$, the *holonomy representation*. The topology of C^∞-convergence on compact subsets of $\tilde{M}$ on the space of developing maps induces a topology on $\text{Def}(M)$. The map obtained by sending a hyperbolic structure to its holonomy representation induces a local homeomorphism $\text{Def}(M) \to X(\pi_1 M, \text{SL}_2(\mathbb{C}))$, where the latter is the space of group homomorphisms $\rho : \pi_1 M \to \text{SL}_2(\mathbb{C})$ considered up to conjugation in $\text{SL}_2(\mathbb{C})$, cf. [17], see also [16].

We fix generators $\mu, \lambda \in \pi_1 \partial \bar{M}$. Let ρ_0 be the holonomy of the complete structure and χ_0 the corresponding element in $X(\pi_1 M, \text{SL}_2(\mathbb{C}))$. For ρ a deformation of ρ_0 we consider the complex lengths $\mathcal{L}_\mu(\rho)$ and $\mathcal{L}_\lambda(\rho)$ of the isometries corresponding to $\rho(\mu)$ and $\rho(\lambda)$. Since the complex length is invariant under conjugation, but only determined up to addition of multiples of $2\pi i$ and multiplication by ± 1, $\mathcal{L}_\mu$ and $\mathcal{L}_\lambda$ may be considered as multi-valued functions on $X(\pi_1 \bar{M}, \text{SL}_2(\mathbb{C}))$.

In [18], it is shown that on a branched cover of a neighbourhood of χ_0 the functions $\mathcal{L}_\mu$ and $\mathcal{L}_\lambda$ can be defined as single-valued functions. More precisely, there exist open sets $0 \in U \subset \mathbb{C}$ and $\chi_0 \in V \subset X(\pi_1 \bar{M}, \text{SL}_2(\mathbb{C}))$, a branched cover $\pi : U \to V$ and a holomorphic map $f : U \to \mathbb{C}$ such that

$$\mathcal{L}_\mu(\pi(z)) = z$$

and

$$\mathcal{L}_\lambda(\pi(z)) = f(z).$$

Furthermore, for z small enough (w.l.o.g. for $z \in U$), the equation

(2.1) $$xz + yf(z) = 2\pi i$$

has a unique solution $(x, y) \in \mathbb{R}^2 \cup \{\infty\}$, the so-called *generalized Dehn surgery coefficients*. The map f satisfies $f(-u) = -f(u)$ (in particular $f(0) = 0$), such that the generalized Dehn surgery coefficients are determined by a character $\chi \in V$ up to sign. Hence, for a sufficiently small neighbourhood W of the complete structure in $\mathrm{Def}(M)$ there is a well-defined map

$$\mathrm{DS} : W \to (\mathbb{R}^2 \cup \{\infty\})/ \pm 1.$$

With these preparations we can state Thurston's Hyperbolic Dehn Surgery Theorem, cf. [12], see also [16, 18]:

Theorem 2.1 [12]. *There is a neighbourhood $W \subset \mathrm{Def}(M)$ of the complete structure such that the map*

$$\mathrm{DS} : W \to (\mathbb{R}^2 \cup \{\infty\})/ \pm 1$$

is a homeomorphism onto a neighbourhood W' of $\infty \in (\mathbb{R}^2 \cup \{\infty\})/ \pm 1$.

In the following, we give a geometric description of how these incomplete structures look like:

Let g denote the complete hyperbolic metric on M. Recall that $\partial \bar{M}$ equipped with g is a horospherical torus, i.e., intrinsically flat with principal curvatures 1. A structure that is close to the complete structure may be represented by a hyperbolic metric g' on $\bar{M}$, which is C^∞-close to g restricted to $\bar{M}$, such that $\partial \bar{M}$ equipped with g' becomes intrinsically flat with principal curvatures $\coth(R)$ and $\tanh(R)$ for $R > 0$ large. Then a *singular tube* of radius R is added, such that the principal curvature lines match.

A *singular tube* of radius R is the following obvious generalization of a smooth tube as above: let $\Gamma \subset \mathbb{R}^2$ be any lattice and consider the metric

$$(2.2) \qquad\qquad dr^2 + \sinh(r)^2 d\theta^2 + \cosh(r)^2 dz^2$$

on $(0, R) \times \mathbb{R}^2/\Gamma$, where $r \in (0, R)$ and $(\theta, z) \in \mathbb{R}^2$. We denote the tube of radius R based on the torus $T^2 = \mathbb{R}^2/\Gamma$ by $T^2_{(0,R)}$. The following cases occur:

1. $\Gamma \cap \{z = 0\} \neq \{0\}$: In this case, Γ is spanned by unique vectors $(\alpha, 0)$ with $\alpha > 0$ and (t, l) with $0 \leq t < \alpha$ and $l > 0$. Then $T^2_{(0,R)}$ is the smooth part of a cone tube with cone angle α, length l and twist t. In the special case $\alpha = 2\pi$ this is nothing but a smooth tube with the core geodesic removed. The principal curvature lines corresponding to $\coth(R)$ close up and are isotopic to a curve $p\mu + q\lambda$ with $\mu, \lambda \in \pi_1 \partial \bar{M}$

as above and p, q coprime integers. The generalized Dehn surgery coefficients of the structure are given by $(x, y) = \frac{2\pi}{\alpha}(p, q)$.

2. $\Gamma \cap \{z = 0\} = \{0\}$: In this case, the metric completion is not a manifold. The generalized Dehn surgery coefficients of the structure are of the form (x, y) with x/y irrational.

We will generally say that a hyperbolic 3-manifold obtained in this way has Dehn surgery type singularities.

The precise shape of the deformed tube is determined by the deformed holonomy of the boundary torus, i.e., by the complex lengths $\mathcal{L}_\mu$ and $\mathcal{L}_\lambda$. These in turn are determined by the generalized Dehn surgery coefficients via Equation (2.1); however, this dependence is not explicit. Qualitatively we can say the following:

Lemma 2.1. *Let M^∞ be a complete hyperbolic 3-manifold of finite volume with a single cusp. Then:*

1. *For any $\varepsilon > 0$ there is a neighbourhood $W \subset \mathrm{Def}(M^\infty)$ of the complete structure such that $\mathrm{diam}\, T^2 < \varepsilon$ for any hyperbolic structure in W, where $T^2 = \mathbb{R}^2/\Gamma$ is the base of the corresponding tube.*

2. *For any neighbourhood $W \subset \mathrm{Def}(M^\infty)$ of the complete structure there are constants $C_1, C_2 > 0$ such that $C_1 \le e^{2R}\mathrm{area}\, T^2 \le C_2$ for any hyperbolic structure in W.*

Proof. Recall that the complex lengths $\mathcal{L}_\mu$ and $\mathcal{L}_\lambda$ are defined as single-valued functions z and $f(z)$ on a branched cover of a neighbourhood of the complete structure with f holomorphic and $f(0) = 0$. Hence Γ is spanned by arbitrarily short vectors for hyperbolic structures close enough to the complete one.

Let T_r^2 denote the torus $T^2 = \mathbb{R}^2/\Gamma$ equipped with the Riemannian metric $\sinh(r)^2 d\theta^2 + \cosh(r)^2 dz^2$. Then $\mathrm{area}\, T_r^2 = \sinh(r)\cosh(r)\mathrm{area}\, T^2$ and $\mathrm{area}\, T_R^2$ differs from the area of the horospherical torus $\partial \bar{M}^\infty$ by some bounded amount depending on the neighbourhood $W \subset \mathrm{Def}(M^\infty)$. Since $\sinh(r)\cosh(r) \sim e^{2r}$, the second claim follows. $\qquad\square$

Note that a cone tube based on T^2 with small diameter (and hence small area) may have small cone angle or not. In any case, it is easy to see that the length of the tube has to be small.

Further, for a cone tube one has $\mathrm{area}\, T^2 = l\alpha$, where α is the cone angle and l the length of the tube. In particular, if we restrict to smooth fillings,

i.e., $\alpha = 2\pi$, Lemma 2.1 gives us constants $C_1', C_2' > 0$ such that

$$C_1' \leq e^{2R}l \leq C_2' \Leftrightarrow R - C_1'' \leq \tfrac{1}{2}\log\frac{1}{l} \leq R + C_2'',$$

which is the estimate used in [6] and [8]. This suggests that area T^2 in our arguments should play the role of l in the arguments of [6] and [8].

Lemma 2.2. *Consider $c > 4$ and $\beta > 0$. Then there exists $\mu > 0$ below the Margulis constant μ_0 and a neighbourhood W of $M^\infty \in \mathrm{Def}(M^\infty)$ such that one has:*

1. *For any $M \in W$ the μ-thick part $M_{[\mu,\infty)}$ is $(1+\beta)$-quasi-isometric to $M^\infty_{[\mu,\infty)}$,*

2. *any $M \in W$ contains a singular tube $T^2_{(0,\rho+c]}$ for some $\rho > 0$ such that $T^2_{[\rho,\rho+c]} \subset M_{[\mu,\infty)}$.*

Proof. The proof is evident from the discussion above. $\square$

3. The spectrum of the Laplacian

Let M be a hyperbolic 3-manifold with Dehn surgery type singularities. The Laplacian Δ considered on $\mathrm{dom}\,\Delta = C_0^\infty(M)$ is a symmetric, densely defined operator in $L^2(M)$. Since M is incomplete, we cannot expect the Laplacian to be essentially selfadjoint on that domain; in fact, we will see that it never is. Nevertheless, since Δ is non-negative on $C_0^\infty(M)$, i.e., $\langle \Delta f, f \rangle_{L^2} \geq 0$ for all $f \in C_0^\infty(M)$, there is always a distinguished self-adjoint extension at hand, namely, the so-called Friedrichs extension of Δ.

In the following, we briefly review the construction of the Friedrichs extension of a semibounded symmetric operator. Recall that if $\mathcal{H}$ is a Hilbert space and q a quadratic form defined on a dense domain in $\mathcal{H}$ such that $q \geq c$ for some $c \in \mathbb{R}$ and q is closable with closure $\bar{q}$, then there exists a unique self-adjoint operator A with $\mathrm{dom}\,A \subset \mathrm{dom}\,\bar{q}$ and $\langle Af, g \rangle = q(f,g)$ for all $f \in \mathrm{dom}\,q \cap \mathrm{dom}\,A$ and $g \in \mathrm{dom}\,q$. Furthermore, A satisfies the same lower bound as q, i.e., $A \geq c$. The domain of A is given by

$$\mathrm{dom}\,A = \{f \in \mathrm{dom}\,\bar{q} : \exists\, h \in \mathcal{H} \text{ s.t. } \bar{q}(f,g) = \langle h, g \rangle\ \forall g \in \mathrm{dom}\,q\}$$

and then $Af = h$.

The spectral theorem for self-adjoint operators yields the well-known variational characterization of the eigenvalues of A below the essential spectrum:

Theorem 3.1. *For* $\lambda_k = \inf\limits_{\substack{V \subset \mathrm{dom}\, q \\ \dim(V)=k}} \sup\limits_{f \in V \setminus \{0\}} \frac{q(f,f)}{|f|^2}$ *one has*

1. *the sequence* (λ_k) *is non-decreasing, and* $\lambda_k \to \lambda_\infty \leq \infty$,

2. *the* $\lambda_n < \lambda_\infty$ *are precisely the eigenvalues of* A *below* λ_∞,

3. λ_∞ *is the bottom of the essential spectrum of* A.

Now if A_0 is a densely defined symmetric operator in $\mathcal{H}$ with $A_0 \geq c$, then $q(f,g) = \langle A_0 f, g \rangle$ for $f, g \in \mathrm{dom}\, A_0$ is closable and trivially $q \geq c$. One has $\mathrm{dom}\, A = \mathrm{dom}\, A_0^* \cap \mathrm{dom}\, \bar{q}$ and $Af = A_0^* f$ for the corresponding self-adjoint operator A, which is the so-called *Friedrichs extension* of A_0.

If M is a Riemannian manifold (without boundary) we may apply this construction to the Dirichlet energy $q(f) = \int_M |df|^2$ on $\mathrm{dom}\, q = C_0^\infty(M)$ to obtain the Friedrichs extension of Δ, in the following denoted by Δ_{Fr}. One has $\mathrm{dom}\, \bar{q} = \mathrm{dom}\, d_{\min}$ and $\mathrm{dom}\, \Delta_{\mathrm{Fr}} = \mathrm{dom}\, \Delta_{\max} \cap \mathrm{dom}\, d_{\min}$, where for a differential operator P acting on compactly supported smooth sections of some vector bundle (equipped with a Euclidean metric) we set

$$P_{\max} = (P^t)^* \quad \text{and} \quad P_{\min} = \bar{P}.$$

Here, P^t denotes the formal adjoint of P and $\bar{P} = P^{**}$ the closure of P. It is easy to see that

$$\mathrm{dom}\, P_{\max} = \{s \in L^2 : Ps \in L^2\}$$

and

$$\mathrm{dom}\, P_{\min} = \{s \in L^2 : \exists s_n \in C_0^\infty \text{ with } s_n \xrightarrow{L^2} s,\ s_n \xrightarrow{L^2} Ps\},$$

where P is applied to L^2-sections in a distributional sense.

For example, if M is the interior of a compact manifold with boundary $\hat{M}$, i.e., $M = \hat{M} \setminus \partial\hat{M}$, then $\mathrm{dom}\, \bar{q} = H_0^1(M)$ and the Friedrichs extension of Δ is the Dirichlet extension with

$$\mathrm{dom}\, \Delta_{\mathrm{Dir}} = H^2(M) \cap H_0^1(M).$$

If we apply the Friedrichs construction to $q(f) = \int_M |df|^2$ on $\mathrm{dom}\, q = C^\infty(\hat{M})$, then $\mathrm{dom}\, \bar{q} = H^1(M)$ and the corresponding self-adjoint operator

is the Neumann extension of Δ with

$$\operatorname{dom}\Delta_{\mathrm{Neu}} = \{f \in H^2(M) : \nu f \in H_0^1(M)\},$$

where ν is a smooth extension of a unit normal to the boundary.

Returning to M being a hyperbolic 3-manifold with Dehn surgery type singularities, let $d : C^\infty(M) \to \Omega^1(M)$ denote the exterior differential on functions and $\delta : \Omega^1(M) \to C^\infty(M)$ the divergence on 1-forms. As usual let $H^1(M) = \operatorname{dom} d_{\max}$ and $H_0^1(M) = \operatorname{dom} d_{\min}$. From the preceding discussion it is clear that $\Delta_{\mathrm{Fr}} = \delta_{\max} d_{\min}$.

Theorem 3.2 (L^2-Stokes). *Let M be a hyperbolic 3-manifold with Dehn surgery type singularities. Then for $f \in \operatorname{dom} d_{\max}$ and $w \in \operatorname{dom} \delta_{\max}$ one has*

$$\langle df, \omega \rangle_{L^2} = \langle f, \delta\omega \rangle_{L^2}.$$

Proof. We may w.l.o.g. assume that f and ω are smooth, i.e., $f \in C^\infty(M)$ with $\|f\|_{L^2} < \infty$ and $\|df\|_{L^2} < \infty$ and $\omega \in \Omega^1(M)$ with $\|w\|_{L^2} < \infty$ and $\|\delta\omega\|_{L^2} < \infty$. We may also replace the hyperbolic metric by the flat metric $dr^2 + r^2 d\theta^2 + dz^2$, since the L^2-Stokes property is unaffected by passing to a quasi-isometric metric. Now

$$\int_{M \setminus T_{(0,r)}^2} df \wedge *\omega = \int_{M \setminus T_{(0,r)}^2} f * \delta\omega \pm \int_{T_r^2} f * \omega,$$

where T_r^2 denotes the cross-section $\{r\} \times T^2 \subset T_{(0,R)}^2$. We wish to show that the boundary integral vanishes in the limit as $r \to 0$ (or at least for a sequence $r_n \to 0$ as $n \to \infty$). If we write $\omega = \phi_r dr + \phi_\theta d\theta + \phi_z dz$, then the resriction of $*\omega$ to T_r^2 equals $\phi_r r d\theta \wedge dz$ and

$$\left| \int_{T_r^2} f * \omega \right| \le \left(\int_{T^2} f^2 \, r d\theta \wedge dz \right)^{\frac{1}{2}} \cdot \left(\int_{T^2} \phi_r^2 \, r d\theta \wedge dz \right)^{\frac{1}{2}}.$$

If $\varphi \in L^1(0,1)$ then Lemma 1.2 in [19] shows that there exists a sequence $r_n \to 0$ such that $\varphi(r_n) = o(r_n^{-1} |\log r_n|^{-1})$. Applied to the second factor this yields a sequence $r_n \to 0$ such that

$$\left(\int_{T^2} \phi_r^2 \, r_n d\theta \wedge dz \right)^{\frac{1}{2}} = o(r_n^{-\frac{1}{2}} |\log r_n|^{-\frac{1}{2}}).$$

To achieve a better estimate for the first factor we use that also $\|df\|_{L^2} < \infty$, cf. Lemma 2.3 in [19]:

$$\left(\int_{T^2}\left(\int_r^1 \frac{\partial f}{\partial r}(s)ds\right)^2 r d\theta \wedge dz\right)^{\frac{1}{2}} \le r^{\frac{1}{2}}|\log r|^{\frac{1}{2}} \int_{T^2}\left(\int_r^1 \left(\frac{\partial f}{\partial r}(s)\right)^2 s ds\right) d\theta \wedge dz.$$

Now

$$\int_r^1 \frac{\partial f}{\partial r}(s,\theta,z)ds = f(1,\theta,z) - f(r,\theta,z)$$

such that

$$\left(\int_{T^2} f^2 \, r d\theta \wedge dz\right)^{\frac{1}{2}} = O(r^{\frac{1}{2}}|\log r|^{\frac{1}{2}}).$$

Altogether we obtain that $\lim_{n\to\infty}\int_{T^2_{r_n}} f * \omega = 0$. $\hfill\square$

This proof also shows the following Green's formula.

Corollary 3.1. *If we denote the singular tube in M again by $T^2_{(0,r)}$, then $-\partial_r$ is the exterior normal vector field of X along T^2_r, and one has for all $f \in C^\infty(M)$ with $\|f\|_{L^2} < \infty$, $\|df\|_{L^2} < \infty$ and $\|\Delta f\|_{L^2} < \infty$:*

$$\int_{T^2_{(0,r)}} |df|^2 - \int_{T^2_{(0,r)}} f \cdot \Delta f = \pm \int_{T^2_r} f \cdot \partial_r f,$$

where all integrals are taken with respect to the induced volume measures.

Corollary 3.2. *Let M be a hyperbolic 3-manifold with Dehn surgery type singularities. Then:*

1. *$d_{\max} = d_{\min}$, i.e., $H^1(M) = H^1_0(M)$.*

2. *$0 \in \operatorname{spec}\Delta_{\mathrm{Fr}}$.*

Proof. Theorem 3.2 shows that $d_{\max} = \delta^*_{\max}$, where $\delta^*_{\max}$ denotes the Hilbert space adjoint of $\delta_{\max}$. Since in general one has $\delta^*_{\max} = d_{\min}$, the first assertion follows. Since $\Delta_{\mathrm{Fr}} = \delta_{\max}d_{\min}$ it is enough to show that $1 \in H^1_0(M)$. Clearly, $1 \in H^1(M)$, so the second assertion follows from the first. $\hfill\square$

Corollary 3.3. *Let $\hat{M}$ be a compact hyperbolic 3-manifold and let $\hat{\Delta}$ denote the Laplacian on $\hat{M}$. If $M = \hat{M} \setminus \gamma$ for $\gamma \subset \hat{M}$ a closed geodesic, then Δ_{Fr} coincides with the unique self-adjoint extension of $\hat{\Delta}$, i.e., $\operatorname{dom}\Delta_{\mathrm{Fr}} = H^2(\hat{M})$. In particular, $\operatorname{spec}\Delta_{\mathrm{Fr}} = \operatorname{spec}\hat{\Delta}$.*

Proof. Clearly $H_0^1(M) \subset H^1(\hat{M}) \subset H^1(M)$, hence by Corollary 3.2 one has, in particular, $H_0^1(M) = H^1(\hat{M})$. Applying the Friedrichs construction to both form domains yields the result. $\qquad\square$

To investigate further properties of Δ, we use a separation of variables argument on the singular tube. A direct calculation shows that the Laplacian on $T^2_{(0,R)}$ is given by

$$\Delta = -\partial_r^2 - 2\coth(2r)\partial_r + L(r),$$

where

$$L(r) = -\frac{1}{\sinh(r)^2}\partial_\theta^2 - \frac{1}{\cosh(r)^2}\partial_z^2$$

is the Laplacian on the cross-section T_r^2 with the induced metric. The volume form on $T^2_{(0,R)}$ is given by $\sinh(r)\cosh(r)dr \wedge d\theta \wedge dz$, hence

$$L^2(T^2_{(0,R)}) \to L^2((0,R) \times T^2)),$$
$$f \mapsto \sinh(r)^{\frac{1}{2}}\cosh(r)^{\frac{1}{2}}f$$

is a unitary operator. Since

$$\sinh(r)^{\frac{1}{2}}\cosh(r)^{\frac{1}{2}}\left((-\partial_r^2 - 2\coth(2r)\partial_r)\left(\sinh(r)^{-\frac{1}{2}}\cosh(r)^{-\frac{1}{2}}f\right)\right)$$
$$= -\partial_r^2 f + \left(2 - \coth(2r)^2\right)f,$$

we obtain that the Laplacian on $T^2_{(0,R)}$ is unitarily equivalent to the operator

$$-\partial_r^2 + 2 - \coth(2r)^2 + L(r).$$

If $T^2 = \mathbb{R}^2/\Gamma$, let Λ be the dual lattice to Γ, i.e.,

$$\Lambda = \{\lambda \in \mathbb{R}^2 : \langle\lambda,\gamma\rangle \in \mathbb{Z} \,\forall\, \gamma \in \Gamma\}.$$

The dual lattice Λ is spanned by the vectors

$$\left(\frac{l}{\sinh(r)\cdot\mathrm{covol}(\Gamma)}, \frac{-t}{\cosh(r)\cdot\mathrm{covol}(\Gamma)}\right)$$

and

$$\left(\frac{-s}{\sinh(r)\cdot\mathrm{covol}(\Gamma)}, \frac{\alpha}{\cosh(r)\cdot\mathrm{covol}(\Gamma)}\right).$$

We decompose as orthogonal Hilbert sums

$$L^2(\mathbb{R}^2/\Gamma) = \bigoplus_{\lambda \in \Lambda} \langle \Psi_\lambda \rangle$$

and

$$L^2((0, R), L^2(\mathbb{R}^2/\Gamma)) = \bigoplus_{\lambda \in \Lambda} L^2(0, R) \otimes \langle \Psi_\lambda \rangle,$$

where $\Psi_\lambda(x) = e^{2\pi i \langle \lambda, x \rangle} / \sqrt{\text{covol}(\Gamma)}$ and $\langle \Psi_\lambda \rangle$ denotes the span of Ψ_λ in $L^2(\mathbb{R}^2/\Gamma)$. We have

$$\text{spec } L(r) = \left\{ (2\pi)^2 \left(\frac{\lambda_1^2}{\sinh(r)^2} + \frac{\lambda_2^2}{\cosh(r)^2} \right) : \lambda = (\lambda_1, \lambda_2) \in \Lambda \right\}$$

and therefore the action of the operator $-\partial_r^2 + 2 - \coth(2r)^2 + L(r)$ on its domain intersected with $L^2(0, R) \otimes \langle \Psi_\lambda \rangle$ is given by the Schrödinger operator $P_{V_\lambda} = -\partial_r^2 + V_\lambda$ with potential

$$V_\lambda(r) = 2 - \coth(2r)^2 + (2\pi)^2 \left(\frac{\lambda_1^2}{\sinh(r)^2} + \frac{\lambda_2^2}{\cosh(r)^2} \right).$$

Let in the following $P_V = -\partial_r^2 + V(r)$ be a Schrödinger operator on the interval $(0, R)$ with continuous potential V. It has been shown by Sears in [13] that if

$$V(r) \geq \frac{3}{4r^2} + A$$

for some constant $A \in \mathbb{R}$, then the limit-point case holds at 0 (using Weyl's classical terminology), whereas the limit-circle case holds at 0 if

$$|V(r)| < \frac{\delta}{r^2} + A$$

for $0 \leq \delta < 3/4$ and some constant $A \in \mathbb{R}$.

The following lemma asserts that it is indeed necessary to specify a self-adjoint extension for the Laplacian on a hyperbolic 3-manifold with Dehn surgery type singularities.

Lemma 3.1. *Let M be a hyperbolic 3-manifold with Dehn surgery type singularities. Then Δ is not essentially self-adjoint.*

Proof. Since $\Delta_{\max}$ (resp. $\Delta_{\min}$) is unitarily equivalent to $\oplus_{\lambda \in \Lambda}(P_{V_\lambda})_{\max}$ (resp. to $\oplus_{\lambda \in \Lambda}(P_{V_\lambda})_{\min}$) on $T^2_{(0,R)}$, it is enough to exhibit at least one $\lambda \in \Lambda$ such that P_{V_λ} is in the limit-circle case at 0. Since $\lim_{r \to 0} r^2 V_0(r) = -\frac{1}{4}$ we obtain by Theorem 2 in [13] that P_{V_0} is indeed in the limit-circle case at 0. Note that there might actually be infinitely many $\lambda \in \Lambda$ such that P_{V_λ} is in the limit-circle case at 0. $\square$

It is again a result of Sears, cf. [14], that any self-adjoint extension of P_V on $(0, R)$ has discrete spectrum if

$$V(r) \geq -\frac{1}{4r^2} + A$$

for some constant $A \in \mathbb{R}$. Note that under the same condition P_V is semi-bounded by Hardy's inequality.

Lemma 3.2. *Let M be a hyperbolic 3-manifold with Dehn surgery type singularities. Then $\operatorname{spec} \Delta_{\mathrm{Fr}}$ is discrete.*

Proof. Since the essential spectrum is unaffected by removing a compact submanifold with boundary, cf. Proposition 1 in [8], it is sufficient to show that $\operatorname{spec} \Delta_{\mathrm{Fr}}$ is discrete on $T^2_{(0,R)}$.

We use that Δ_{Fr} is unitarily equivalent to $\oplus_{\lambda \in \Lambda}(P_{V_\lambda})_{\mathrm{Fr}}$ on $T^2_{(0,R)}$. From Theorem 1 in [14] we obtain that for $\lambda \in \Lambda$ any self-adjoint extension of P_{V_λ} in $L^2(0, R)$ has discrete spectrum, hence in particular the Friedrichs extension. Observe that P_{V_0} is a non-negative operator, i.e., $\langle P_{V_0} f, f \rangle_{L^2(0,R)} \geq 0$ for all $f \in C_0^\infty(0, R)$, and that $P_{V_\lambda} - P_{V_0} \geq C|\lambda|^2$ for all $\lambda \in \Lambda$ and some constant $C = C(R) > 0$. We may estimate

$$\langle P_{V_\lambda} f, f \rangle_{L^2(0,R)} = \langle P_{V_0} f, f \rangle_{L^2(0,R)} + \langle P_{V_\lambda} - P_{V_0} f, f \rangle_{L^2(0,R)}$$
$$\geq C|\lambda|^2 \|f\|^2_{L^2(0,R)}$$

for $f \in C_0^\infty(0, R)$. For $l > 0$, we obtain that $\operatorname{spec}(P_{V_\lambda})_{\mathrm{Fr}} \cap [0, l] \neq \emptyset$ only for finitely many $\lambda \in \Lambda$, hence that $\operatorname{spec} \Delta_{\mathrm{Fr}}$ is discrete on $T^2_{(0,R)}$. $\square$

We finish this section with another application of the variational characterization of eigenvalues.

Lemma 3.3. *Let X be a compact manifold with boundary, let $\beta > 0$ and let g_1 and g_2 be two Riemannian metrics on X being $(1 + \beta)$-quasi isometric:*

$$\frac{1}{(1 + \beta)^2} \, g_2 \leq g_1 \leq (1 + \beta)^2 \, g_2.$$

For $j = 1, 2$ let $0 < \lambda_1^j \leq \ldots \lambda_i^j \leq \ldots$ denote the eigenvalues of the Dirichlet problem on (X, g_j), resp. $0 = \lambda_0^j < \lambda_1^j \leq \ldots \lambda_i^j \leq \ldots$ the eigenvalues of the Neumann problem. Then for $i \geq 1$ one has

$$\frac{1}{(1 + \beta)^2} \, \lambda_i^1 \leq \lambda_i^2 \leq (1 + \beta)^2 \, \lambda_i^1.$$

Proof. Consider $f \in C^\infty(X)$. For the Rayleigh quotients one observes

$$\frac{1}{(1 + \beta)^2} \cdot \frac{\|df\|_{L^2(X, g_1)}^2}{\|f\|_{L^2(X, g_1)}^2} \leq \frac{\|df\|_{L^2(X, g_2)}^2}{\|f\|_{L^2(X, g_2)}^2} \leq (1 + \beta)^2 \cdot \frac{\|df\|_{L^2(X, g_1)}^2}{\|f\|_{L^2(X, g_1)}^2},$$

and Theorem 3.1 yields the claim for Neumann boundary conditions. Using the above estimate on the space of smooth functions compactly supported in the interior of X, then again Theorem 3.1 yields the claim for Dirichlet boundary conditions. $\square$

4. Clustering

Let M^∞ be a hyperbolic 3-manifold of finite volume with a single cusp in the following. For $M \in \mathrm{Def}(M^\infty)$ we fix the Friedrichs extension Δ_{Fr} for the Laplacian on M.

For a fixed interval $I \subset \mathbb{R}$ we view the spectral counting function $\mathcal{N}_{\Delta, M} I$ as a function on $\mathrm{Def}(M^\infty)$. Note that for $\mu > 0$ fixed the tube radius R may also be considered as a function on $\mathrm{Def}(M^\infty)$, cf. Section 2.

Theorem 4.1. *The spectral counting function on $\mathrm{Def}(M^\infty)$ satisfies*

$$\mathcal{N}_{\Delta, M}[1, 1 + x^2] = \frac{x}{\pi} R + O_x(1)$$

for $x > 0$. Here $O_x(1)$ denotes a function on $\mathrm{Def}(M^\infty)$ which is bounded in a neighbourhood of M^∞. (This neighbourhood and the bounds may depend on $x > 0$.)

We may reformulate this in terms of the tube shape, i.e., the geometry of the base of the singular tube:

Corollary 4.1. *The spectral counting function on* $\mathrm{Def}(M^\infty)$ *satisfies*

$$\mathcal{N}_{\Delta,M}[1, 1 + x^2] = \frac{x}{2\pi} \left(\frac{1}{\mathrm{area}\, T^2} \right) + O_x(1)$$

for $x > 0$.

Proof. The assertion follows from Theorem 4.1 using Lemma 2.1. □

We will bound the spectral counting function from above and from below, i.e., Theorem 4.1 will follow from Lemmas 4.2 and 4.1:

Lemma 4.1. *The spectral counting function on* $\mathrm{Def}(M^\infty)$ *satisfies*

$$\mathcal{N}_{\Delta,M}[1, 1 + x^2] \leq \frac{x}{\pi} R + O_x(1)$$

for $x > 0$. *Furthermore*, $\mathcal{N}_{\Delta,M}[0, 1] = O(1)$.

Proof. Let $x \geq 0$. We claim that

$$\mathcal{N}_{\Delta,M}[0, 1 + x^2] \leq \frac{x}{\pi} R + O_x(1),$$

from which the assertions trivially follow. To prove this, we decompose

$$M = M_{[\mu,\infty)} \cup T^2_{(0,R]},$$

such that by the domain decomposition principle, cf. [8, Prop. 3], we obtain

$$\mathcal{N}^{\mathrm{Fr}}_{\Delta,M}[0, 1 + x^2] \leq \mathcal{N}^{\mathrm{nat}}_{\Delta,M_{[\mu,\infty)}}[0, 1 + x^2] + \mathcal{N}^{\mathrm{nat}}_{\Delta,T^2_{(0,R]}}[0, 1 + x^2].$$

Here, we choose *natural* boundary conditions for the Laplacian on the various pieces, which are manifolds with boundary (compact or not), cf. [8]. More precisely, for a manifold with boundary N (compact or not) we are considering the self-adjoint extension of Δ obtained by applying the Friedrichs construction to the form domain $C_0^\infty(N)$, i.e., compactly supported smooth functions on N, whose support is allowed to hit ∂N. If N is in fact compact, this amounts to choosing Neumann conditions on ∂N.

We claim:

1. $\mathcal{N}^{\mathrm{nat}}_{\Delta,M_{[\mu,\infty)}}[0, 1 + x^2] = O_x(1)$.
2. $\mathcal{N}^{\mathrm{nat}}_{\Delta,T^2_{(0,R]}}[0, 1 + x^2] \leq \frac{x}{\pi} R + O_x(1)$.

The first claim follows from the fact that the metrics on the thick parts are uniformly quasi-isometric together with Lemma 3.3.

To prove the second claim, we will first show that only the 0-mode contributes to clustering. Towards that end, we choose $c = c(x) > 0$ and a neighbourhood W of M^∞ in $\mathrm{Def}(M^\infty)$ such that $V_\lambda - V_0 > 1 + x^2$ on the interval $(0, R - c]$ for all $M \in W$ and for all $\lambda \neq 0$. This is possible since there exists a constant $C > 0$ independent of $\lambda \neq 0$ such that for $r \in (0, R]$

$$V_\lambda(r) - V_0(r) \geq Ce^{2(R-r)}(V_\lambda(R) - V_0(R))$$

and hence for $r \in (0, R - c]$

$$V_\lambda(r) - V_0(r) \geq Ce^{2c}(V_\lambda(R) - V_0(R)).$$

Now, $V_\lambda(R) - V_0(R)$ is bounded from below by the first positive eigenvalue of $\partial T^2_{(0,R)} = \partial M_{[\mu,\infty)}$, which in turn is bounded from below by some positive constant, since the induced metrics on $\partial M_{[\mu,\infty)}$ are uniformly quasi-isometric in a neighbourhood of M^∞ in $\mathrm{Def}(M^\infty)$. We choose $c > 0$ accordingly and we may further decompose

$$T^2_{(0,R]} = T^2_{(0,R-c]} \cup T^2_{[R-c,R]}$$

to obtain

$$\mathcal{N}^{\mathrm{nat}}_{\Delta, T^2_{(0,R]}}[0, 1 + x^2] \leq \mathcal{N}^{\mathrm{nat}}_{\Delta, T^2_{(0,R-c]}}[0, 1 + x^2] + \mathcal{N}^{\mathrm{nat}}_{\Delta, T^2_{[R-c,R]}}[0, 1 + x^2].$$

Since the metrics on the regions $T^2_{[R-c,R]}$ are uniformly quasi-isometric in a neighbourhood of M^∞ in $\mathrm{Def}(M^\infty)$, we find

$$\mathcal{N}^{\mathrm{nat}}_{\Delta, T^2_{[R-c,R]}}[0, 1 + x^2] = O_x(1).$$

Clearly

$$\mathcal{N}^{\mathrm{nat}}_{P_{V_\lambda}, (0, R-c]}[0, 1 + x^2] = 0$$

since $V_\lambda - V_0 > 1 + x^2$ on $(0, R - c]$ for $\lambda \neq 0$ and $(P_{V_0})_{\mathrm{nat}} \geq 0$, cf. the proof of Lemma 3.2.

To estimate the contribution of the 0-mode, let $r_0 > 0$ be the unique positive zero of $V_0(r) = 2 - \coth(2r)^2$. We may assume that $R - c > r_0$. Using

the fact that $V_0 \geq 0$ for $r \geq r_0$ we obtain

$$
\begin{aligned}
\mathcal{N}^{\mathrm{nat}}_{P_{V_0},(0,R-c]}[0, 1+x^2] &\leq \mathcal{N}^{\mathrm{nat}}_{P_{V_0},(0,r_0]}[0, 1+x^2] + \mathcal{N}^{\mathrm{nat}}_{P_{V_0},[r_0,R-c]}[0, 1+x^2] \\
&\leq \mathcal{N}^{\mathrm{Neu}}_{P_{V_0},[r_0,R-c]}[0, 1+x^2] + O_x(1) \\
&\leq \mathcal{N}^{\mathrm{Neu}}_{-\partial_r^2,[r_0,R-c]}[0, 1+x^2] + O_x(1) = \frac{x}{\pi}R + O_x(1).
\end{aligned}
$$

This finishes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Lemma 4.2. *The spectral counting function on* $\mathrm{Def}(M^\infty)$ *satisfies*

$$
\mathcal{N}_{\Delta,M}[1, 1+x^2] \geq \frac{x}{\pi}R + O(1)
$$

for $x > 0$.

Proof. Note that $V_0(r) \leq 1$ for $r > 0$ and that $(P_{V_0})_{\mathrm{Fr}} \geq 0$, cf. the proof of Lemma 3.2. By domain monotonicity, cf. [8, Prop. 2], and the variational characterization of eigenvalues we get

$$
\begin{aligned}
\mathcal{N}^{\mathrm{Fr}}_{\Delta,M}[0, 1+x^2] &\geq \mathcal{N}^{\mathrm{Fr}}_{\Delta,T^2_{(0,R)}}[0, 1+x^2] \\
&\geq \mathcal{N}^{\mathrm{Fr}}_{P_{V_0},(0,R)}[0, 1+x^2] \\
&\geq \mathcal{N}^{\mathrm{Fr}}_{-\partial_r^2+1,(0,R)}[0, 1+x^2] \\
&= \mathcal{N}^{\mathrm{Dir}}_{-\partial_r^2,(0,R)}[0, x^2] = \frac{x}{\pi}R + O(1).
\end{aligned}
$$

Now by Lemma 4.1 we have $\mathcal{N}_{\Delta,M}[0, 1] = O(1)$, hence we obtain, as claimed, that $\mathcal{N}_{\Delta,M}[1, 1+x^2] \geq \frac{x}{\pi}R + O(1)$. $\qquad\qquad\qquad\qquad$ $\square$

5. Convergence of the small eigenvalues

Let M^∞ be a hyperbolic 3-manifold of finite volume with a single cusp in the following. Let $0 = \lambda_0^\infty < \lambda_1^\infty \leq \ldots \leq \lambda_{k^\infty}^\infty < 1$ denote the eigenvalues of the Laplacian on M^∞ below the essential spectrum. We fix some $\Lambda < 0$ with $\lambda_{k^\infty}^\infty < \Lambda < 1$. For $M \in \mathrm{Def}(M^\infty)$ let $0 = \lambda_0 < \lambda_1 \leq \ldots \leq \lambda_k \leq \ldots \to \infty$ denote the eigenvalues of Δ_{Fr} on M, and let $k(\Lambda)$ denote the number of positive eigenvalues strictly below Λ, which means $\lambda_{k(\Lambda)} < \Lambda \leq \lambda_{k(\Lambda)+1}$.

Theorem 5.1. *For each* $\varepsilon > 0$ *there exists a neighbourhood* W *of* M^∞ *in* $\mathrm{Def}(M^\infty)$ *such that for* $M \in W$ *one has:*

  1. $k(\Lambda) = k^\infty$,
  2. $|\lambda_i - \lambda_i^\infty| < \varepsilon$ *for* $i = 1, \ldots, k^\infty$.

We may reformulate this in terms of generalized Dehn surgery coefficients:

Corollary 5.1. *For each $\varepsilon > 0$ there exists a neighbourhood W' of ∞ in $(\mathbb{R}^2 \cup \{\infty\})/\pm 1$ such that for $(x, y) \in W'$ one has for $M = M^{(x,y)}$:*

1. $k(\Lambda) = k^\infty$,

2. $|\lambda_i - \lambda_i^\infty| < \varepsilon$ for $i = 1, \ldots, k^\infty$.

Proof. The assertion follows from Theorem 5.1 together with Theorem 2.1. □

In order to prove Theorem 5.1 we will extend Colbois' and Courtois' method of *parties fuyantes* (see [5]) to hyperbolic manifolds with Dehn surgery type singularities. We need some preliminary considerations.

If $\mu > 0$ is small enough the μ-thin part of M^∞ consists of a rank-2 cusp: $M^\infty_{(0,\mu)}$ can isometrically be identified with $(0, \infty) \times T^2$ carrying the Riemannian metric $dt^2 + e^{-2t} g_{T^2}$, and $M^\infty_{[\mu,\infty)}$ is a smooth compact manifold with boundary. Let $0 < \lambda_0^\infty(\mu) < \lambda_1^\infty(\mu) \leq \ldots \leq \lambda_i^\infty(\mu) \leq \ldots \to \infty$ be the eigenvalues of the Dirichlet problem on $M^\infty_{[\mu,\infty)}$, and denote the index of the last eigenvalue strictly below 1 by k_μ^∞. Then the same proof as the one of (1.4) in [5] shows:

Lemma 5.1. *For each $\varepsilon > 0$ with $\lambda_{k^\infty}^\infty + \varepsilon < 1$ there is a $\mu_\varepsilon > 0$ such that for all $\mu < \mu_\varepsilon$ one has*

1. $k_\mu^\infty = k^\infty$,

2. $\lambda_i^\infty \leq \lambda_i^\infty(\mu) < \lambda_i^\infty + \varepsilon$ for $i = 0, \ldots, k^\infty$.

And, by domain monotonicity, one also has $\lambda_{k^\infty+1}^\infty(\mu) \geq 1$.

Next, we note that elliptic regularity is a local statement, hence it also holds on manifolds with Dehn surgery type singularities. We get that any eigenfunction $f \in L^2(M)$ of Δ_{Fr} is smooth.

For the moment let us fix $\rho > 0$ and $c > 4$ and let us consider a hyperbolic 3-manifold M containing a singular tube $T^2_{(0,\rho+c)}$ isometric to $(0, \rho + c) \times T^2$ with the metric given in (2.2). For $0 \leq a < b \leq \rho + c$ we denote the domain that corresponds to $(a, b] \times T^2$ by $T^2_{(a,b]}$ and for $0 < r \leq \rho + c$ we denote the hypersurface that corresponds to $\{r\} \times T^2$ by T^2_r.

Lemma 5.2. *For any $\Lambda > 0$ and for any $f \in C^\infty(M)$ with $\|f\|_{L^2(M)}^2 = 1$ and $\|df\|_{L^2(M)}^2 < \Lambda$ there exists an $r \in [\rho + 2, \rho + c]$ such that*

1. $\|f\|_{H^1(T_{(r-1,r]}^2)}^2 < \frac{2\cdot(1+\Lambda)}{c-4}$ *and*

2. $\int_{T_r^2} \left\{ |f|^2 + |df|^2 \right\} < \frac{2\cdot(1+\Lambda)}{c-4}.$

Proof. In fact, the proof is the same as the one of [5, Lemma 2.4]: One defines $F = |f|^2 + |df|^2$ and gets

$$1 + \Lambda > \int_M F \geq \int_{T_{(\rho+2,\rho+c]}^2} F \geq \sum_{k=1}^{[\frac{c}{2}]-1} \int_{T_{(\rho+2k,\rho+2k+2]}^2} F.$$

Hence, there is a $k \in \left\{1, \ldots, [\frac{c}{2}] - 1\right\}$ with

$$\frac{1+\Lambda}{[\frac{c}{2}]-1} > \int_{T_{(\rho+2k,\rho+2k+2]}^2} F = \int_{\rho+2k}^{\rho+2k+2} dr \int_{T_r^2} F.$$

Furthermore, there is an $r \in [\rho + 2k + 1, \rho + 2k + 2] \subset [\rho + 2, \rho + c]$ with $\int_{T_r^2} F < \frac{1+\Lambda}{[\frac{c}{2}]-1} \leq \frac{2\cdot(1+\Lambda)}{c-4}$ and we are done. □

Corollary 5.2. *For $\Lambda > 0$, $f \in C^\infty(M)$ and $r \in [\rho + 2, \rho + c]$ as in Lemma 5.2 with $\|\Delta f\|_{L^2} < \infty$ one has*

$$(5.1) \qquad \left| \int_{T_{(0,r]}^2} |df|^2 - \int_{T_{(0,r]}^2} f \cdot \Delta f \right| < \frac{2 \cdot (1 + \Lambda)}{c - 4}.$$

Proof. We use the Green's formula in Corollary 3.1 to obtain

$$\left| \int_{T_{(0,r]}^2} |df|^2 - \int_{T_{(0,r]}^2} f \cdot \Delta f \right| = \left| \int_{T_r^2} f \cdot \partial_r f \right| \leq \sqrt{\int_{T_r^2} f^2} \cdot \sqrt{\int_{T_r^2} (\partial_r f)^2}.$$

Then Lemma 5.2 finishes the proof. □

Lemma 5.3. *Let $\Omega = T_{(0,R)}^2$ be a singular tube of radius R. Consider the Friedrichs extension Δ_{Fr} of the Laplace operator acting on $C_0^\infty(\Omega)$. For $h \in \mathrm{dom}(\Delta_{\mathrm{Fr}})$ one has*

$$\|h\|_{L^2(\Omega)} \leq \|dh\|_{L^2(\Omega)}.$$

Proof. For $h \in C_0^\infty (\Omega)$ one gets

$$(5.2) \qquad \int_\Omega \left| \partial_r (h^2) \right| = \int_\Omega |2h \cdot \partial_r h| \leq 2\|h\|_{L^2(\Omega)} \cdot \|dh\|_{L^2(\Omega)}.$$

As h is compactly supported we may integrate by parts and obtain

$$\|h\|_{L^2(\Omega)}^2 = \int_{T^2} d\theta \, dz \int_0^R h^2 \cdot \frac{d}{dr} \left(\frac{1}{2} \sinh^2(r) \right) dr$$

$$= - \int_{T^2} d\theta \, dz \int_0^R \frac{1}{2} \sinh^2(r) \cdot \partial_r \left(h^2 \right) dr$$

$$= -\tfrac{1}{2} \int_\Omega \tanh(r) \partial_r \left(h^2 \right) \leq \tfrac{1}{2} \int_\Omega \left| \partial_r (h^2) \right| \leq \|h\|_{L^2(\Omega)} \cdot \|dh\|_{L^2(\Omega)}$$

by (5.2). For the Rayleigh quotient this means

$$\frac{\|dh\|_{L^2(\Omega)}^2}{\|h\|_{L^2(\Omega)}^2} \geq 1.$$

By the construction of Friedrichs extensions this lower bound holds for any $h \in \operatorname{dom}(\Delta_{\mathrm{Fr}})$, and we are done. $\qquad \square$

Corollary 5.3. *Let M be a hyperbolic 3-manifold containing a singular tube $T_{(0,\rho+c]}^2$, where $\rho > 0$ and $c > 4$. Then for any $f \in \operatorname{dom}(\Delta_{\mathrm{Fr}})$ and any $r \in [\rho + 2, \rho + c]$ one obtains*

$$(5.3) \qquad \int_{T_{(0,r]}^2} |f|^2 - \int_{T_{(0,r]}^2} |df|^2 \leq 3 \cdot \|f\|_{L^2(T_{(r-1,r]}^2)}^2.$$

Proof. We choose a cut-off function $u \in C^\infty(M)$ with $\operatorname{supp}(u) \subset T_{(0,r)}^2$, $0 \leq u \leq 1$, $|\operatorname{grad}(u)| \leq 2$ and $u|_{T_{(0,r-1]}^2} \equiv 1$ and consider $h = u \cdot f$. Then h is in the domain of the Friedrichs extension of the Laplacian acting on $C_0^\infty(T_{(0,r)}^2)$, and by Lemma 5.3 we obtain

$$\int_{T_{(0,r]}^2} |f|^2 - \|f\|_{L^2(T_{(r-1,r)}^2)}^2 \leq \int_{T_{(0,r]}^2} |h|^2 \leq \int_{T_{(0,r]}^2} |dh|^2$$

$$\leq \int_{T_{(0,r]}^2} |df|^2 + 2 \cdot \|f\|_{L^2(T_{(r-1,r]}^2)}^2,$$

which finishes the proof. $\qquad \square$

Now, let $f \in C^\infty(M)$ be a normalized eigenfunction of Δ_{Fr} corresponding to an eigenvalue $\lambda < \Lambda < 1$, which means $\Delta_{\mathrm{Fr}}f = \lambda f$ and $\|f\|_{L^2(M)} = 1$. We take $r \in [\rho + 2, \rho + c]$ as in Lemma 5.2, and we abbreviate

$$A := \int_{T^2_{(0,r]}} |df|^2, \quad B := \int_{T^2_{(0,r]}} |f|^2, \quad \eta := \frac{2 \cdot (1 + \Lambda)}{c - 4}.$$

One gets $|A - \lambda \cdot B| < \eta$ by (5.1) and $B - A < 3\eta$ by (5.3) and Assertion 1 of Lemma 5.2. Together this yields $B - \lambda \cdot B < A + 3\eta - \lambda B < 4\eta$ and hence $B < \frac{4\eta}{1-\lambda} < \frac{4\eta}{1-\Lambda}$. Using $A + B \le (1 + \Lambda)B + \eta$ we conclude:

Proposition 5.1. *Let $f \in C^\infty(M)$ be an eigenfunction of Δ_{Fr} corresponding to an eigenvalue $\lambda < \Lambda < 1$ with $\|f\|_{L^2} = 1$. Then there is an $r \in [\rho + 2, \rho + c]$ such that*

$$\|f\|^2_{H^1(T^2_{(0,r]})} < \frac{40}{1 - \Lambda} \cdot \frac{1}{c - 4}.$$

Finally, we have all ingredients to prove Theorem 5.1.

Proof of Theorem 5.1. Let $\varepsilon > 0$. Without loss of generality we may assume that $\lambda^\infty_{k^\infty} + \varepsilon < \Lambda$ and $\Lambda + \varepsilon < 1$. Applying Lemma 5.1 for $\varepsilon/2$ we obtain $\mu_{\frac{\varepsilon}{2}} > 0$ such that for any $\mu < \mu_{\frac{\varepsilon}{2}}$ one has $k^\infty_\mu = k^\infty$ and

$$(5.4) \qquad \lambda^\infty_i \le \lambda^\infty_i(\mu) \le \lambda^\infty_i + \frac{\varepsilon}{2} \quad \text{for } i = 0, \cdots, k^\infty, \text{ and}$$

$$(5.5) \qquad 1 \le \lambda^\infty_{k^\infty + 1}(\mu).$$

We choose $\mu \in (0, \mu_{\frac{\varepsilon}{2}})$.

For $M \in \mathrm{Def}(M_\infty)$ we denote the Dirichlet-eigenvalues of $M_{[\mu,\infty)}$ by $\lambda_i(\mu)$. By Lemma 3.3 there exists $\beta > 0$ such that if the μ-thick parts $M_{[\mu,\infty)}$ and $M^\infty_{[\mu,\infty)}$ are $(1 + \beta)$-quasi-isometric, then one has

$$(5.6) \qquad \lambda^\infty_i(\mu) - \frac{\varepsilon}{2} \le \lambda_i(\mu) \le \lambda^\infty_i(\mu) + \frac{\varepsilon}{2} \quad \text{for } i = 0, \ldots, k^\infty, \text{ and}$$

$$(5.7) \qquad \lambda_{k^\infty + 1}(\mu) > \Lambda + \frac{\varepsilon}{2}.$$

From (5.4) and (5.6) we obtain for $i = 0, \ldots, k^\infty$

$$(5.8) \qquad \lambda^\infty_i - \frac{\varepsilon}{2} \le \lambda_i(\mu) \le \lambda^\infty_i + \varepsilon.$$

Domain monotonicity applied to $M_{[\mu,\infty)} \subset M$ gives

$$(5.9) \qquad \lambda_i \le \lambda_i(\mu) \le \lambda^\infty_i + \varepsilon < \Lambda$$

for $i = 0, \ldots, k^\infty$, and therefore $k(\Lambda) \geq k^\infty$.

For $\lambda_0, \ldots, \lambda_{k^\infty}$ we take L^2-orthonormal eigenfunctions $f_0, \ldots, f_{k^\infty}$. Now for $i \leq k^\infty$ Proposition 5.1 applies and we find $r_i \in [\rho + 2, \rho + c)$ with

$$\|f_i\|^2_{H^1(T^2_{(0,r_i]})} < \frac{40}{1 - \Lambda} \cdot \frac{1}{c - 4}.$$

We choose $\varphi_i \in C^\infty(M)$ with $0 \leq \varphi_i \leq 1$, $\varphi_i\big|_{T^2_{(0,\rho]}} \equiv 0$, $\varphi_i\big|_{M \setminus T^2_{(0,r_i]}} \equiv 1$ and $|\operatorname{grad} \varphi_i| \leq 2$ and obtain

$$(5.10) \qquad \|f_i - \varphi_i \cdot f_i\|^2_{H^1(M)} \leq 3 \cdot \|f_i\|^2_{H^1(T^2_{(0,r_i]})} < \frac{120}{1 - \Lambda} \cdot \frac{1}{c - 4}.$$

Now $g_i = \varphi_i \cdot f_i$ are smooth functions with compact support inside the interior of $M_{[\mu,\infty)}$. By (5.10) we get that for c large enough the functions g_i are linearly independent and their Rayleigh quotients are arbitrarily close to those of the f_i (the difference being uniformly controlled by c). Hence, by choosing c larger than some constant only depending on the number k_∞, we can achieve $\lambda_i(\mu) \leq \lambda_i + \frac{\varepsilon}{2}$ for $i = 0, \ldots, k^\infty$. Together with (5.8) and (5.9) this means

$$\lambda_i^\infty - \varepsilon \leq \lambda_i \leq \lambda_i^\infty + \varepsilon \quad \text{for } i = 0, \ldots, k^\infty.$$

If we suppose $\lambda_{k^\infty + 1} < \Lambda$, the same argument as above (possibly involving a new choice of c) yields $\lambda_{k^\infty + 1}(\mu) < \Lambda + \frac{\varepsilon}{2}$ which is a contradiction to (5.7). Therefore $k(\Lambda) \leq k^\infty$.

For these chosen μ, β and c, we take the neighbourhood W of $M^\infty \in \operatorname{Def}(M^\infty)$, which is given in Lemma 2.2, and by the preceding arguments it is clear that W is the desired neighbourhood in Theorem 5.1. $\qquad \square$

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INSTITUT FÜR MATHEMATIK DER UNIVERSITÄT POTSDAM
AM NEUEN PALAIS 10
14469 POTSDAM
GERMANY
E-mail address: `pfaeffle@math.uni-potsdam.de`

MATHEMATISCHES INSTITUT DER LMU MÜNCHEN
THERESIENSTR. 39
80333 MÜNCHEN
GERMANY
E-mail address: `weiss@mathematik.uni-muenchen.de`