

Spacelike mean curvature one surfaces in de Sitter 3-space

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The first author studied spacelike constant mean curvature one (CMC-1) surfaces in the de Sitter 3-space S_1^3 when the surfaces have no singularities except within some compact subsets and are of finite total curvature on the complement of this compact subset. However, there are many CMC-1 surfaces whose singular sets are not compact. In fact, such examples have already appeared in the construction of trinoids given by Lee and the last author via hypergeometric functions.

In this paper, we improve the Osserman-type inequality given by the first author. Moreover, we shall develop a fundamental framework that allows the singular set to be non-compact, and then will use it to investigate the global behavior of CMC-1 surfaces.

0. Introduction

A holomorphic map $F : M^2 \rightarrow \mathrm{SL}_2 \mathbf{C}$ of a Riemann surface M^2 into the complex Lie group $\mathrm{SL}_2 \mathbf{C}$ is called *null* if $\det(dF/dz)$ vanishes identically, where z is a local complex coordinate of M^2 . We consider two projections, one into the hyperbolic 3-space

$$\pi_{\mathrm{H}} : \mathrm{SL}_2 \mathbf{C} \longrightarrow H^3 = \mathrm{SL}_2 \mathbf{C} / \mathrm{SU}_2$$

and the other into the de Sitter 3-space

$$\pi_{\mathrm{S}} : \mathrm{SL}_2 \mathbf{C} \longrightarrow S_1^3 = \mathrm{SL}_2 \mathbf{C} / \mathrm{SU}_{1,1},$$

where the definition of $\mathrm{SU}_{1,1}$ is in Appendix B. It is well known that the projection of a holomorphic null immersion into H^3 by π_{H} gives a conformal constant mean curvature one (CMC-1) immersion (see [3, 6, 21]). Moreover, conformal CMC-1 immersions are always given locally in such a manner.

On the other hand, spacelike CMC-1 surfaces given by the projection of holomorphic null immersions into S_1^3 by π_{S} can have singularities, and

are called *CMC-1 faces*. We work with this class of surfaces that is larger than the class of CMC-1 immersions. In fact, the class of CMC-1 immersions is too small, since *there is only one, up to congruency, complete spacelike CMC-1 immersion* [2, 17], which we call an S_1^3 -horosphere. (We also give a simple proof of this here. See Remark 1.15)

The relationship between CMC-1 surfaces in H^3 and CMC-1 faces in S_1^3 is analogous to that between minimal surfaces in Euclidean 3-space \mathbf{R}^3 and spacelike maximal surfaces with singularities in the Lorentz-Minkowski 3-space \mathbf{R}_1^3 (called *maxfaces* [24]). Note that maximal surfaces also admit a Weierstrass-type representation formula ([11]). As in the case of maxfaces (see [24]), the first author [7] investigated the global behavior of CMC-1 faces in S_1^3 , in particular proving an Osserman-type inequality for complete CMC-1 faces of finite type whose ends are all elliptic, where a complete end of a CMC-1 face is called *elliptic*, *parabolic* or *hyperbolic* if the monodromy matrix of the holomorphic lift $F : M^2 \rightarrow \mathrm{SL}_2 \mathbf{C}$ is elliptic, parabolic or hyperbolic, respectively (see Section 1). One of our main results is the following, which implies that the ellipticity or parabolicity of ends follows from completeness:

Theorem 0.1. *A complete end of a CMC-1 face in S_1^3 is never hyperbolic, so must be either elliptic or parabolic. Moreover, the total curvature over a neighborhood of such an end is finite.*

We remark that there exist incomplete elliptic and parabolic ends.

It is remarkable that just completeness of an end is sufficient to conclude that it has finite total curvature. This is certainly not the case for CMC-1 surfaces in H^3 nor for minimal surfaces in \mathbf{R}^3 , but is similar to the case of maximal surfaces in \mathbf{R}_1^3 [24]. Although the asymptotic behavior of regular elliptic CMC-1 ends in S_1^3 is investigated in [7], there do also exist complete parabolic ends, and to describe them, a much deeper analysis is needed, which we will conduct in this article.

As an application of Theorem 0.1, we prove the following Osserman-type inequality, which improves the result of [7] by removing the assumptions of finite type and ellipticity of ends:

Theorem 0.2. *Suppose a CMC-1 face $f : M^2 \rightarrow S_1^3$ is complete. Then there exist a compact Riemann surface \overline{M}^2 and a finite number of points $p_1, \dots, p_n \in \overline{M}^2$ such that M^2 is biholomorphic to $\overline{M}^2 \setminus \{p_1, \dots, p_n\}$, and*

$$(*) \quad 2 \deg(G) \geq -\chi(\overline{M}^2) + 2n,$$

where G is the hyperbolic Gauss map of f and $\chi(\overline{M}^2)$ is the Euler characteristic of \overline{M}^2 . Furthermore, equality holds if and only if each end is regular and properly embedded.

CMC-1 trinoids in S_1^3 were constructed by Lee and the last author using hypergeometric functions [15], and those trinoids with elliptic ends are complete in the sense of [7], and attain equality in (*). However, those having other types of ends are not complete, as their singular sets are not compact. For this reason, our goal is not only to prove the above two theorems, but also to extend the framework for CMC-1 surfaces to include a larger class of surfaces, relaxing the immersedness and completeness conditions. If M^2 is of finite topology, i.e., if M is diffeomorphic to a compact Riemann surface \overline{M}^2 with finitely many punctures p_1, \dots, p_n , and if a CMC-1 face $f : M^2 \rightarrow S_1^3$ is weakly complete, whose precise definition will be given in Section 1, we say that f is a weakly complete CMC-1 face of finite topology. We shall develop the framework under this more general notion, which includes all the trinoids in [15].

In Section 1, we recall definitions and basic results. In Section 2, we investigate the monodromy of the hyperbolic metrics on a punctured disk around an end. As an application, we prove Theorem 0.1 in Section 3. In Section 4, we give a geometric interpretation of the hyperbolic Gauss map. In Section 5, we investigate the asymptotic behavior of regular parabolic ends, and prove Theorem 0.2. In Appendix A, we prove meromorphicity of the Hopf differential for complete CMC-1 faces. In Appendix B, we explain the conjugacy classes of $SU_{1,1}$.

Generic singularities of CMC-1 faces are classified in [9]. A CMC-1 face is called *embedded* (in the wider sense) if it is embedded outside of some compact set of S_1^3 . Examples of complete embedded CMC-1 faces are given in [8] as deformations of maxfaces in the Lorentz-Minkowski 3-space \mathbf{R}_1^3 .

1. Preliminaries

The representation formula

Let \mathbf{R}_1^4 be the Lorentz-Minkowski space of dimension 4, with the Lorentz metric

$$\langle (x_0, x_1, x_2, x_3), (y_0, y_1, y_2, y_3) \rangle = -x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3.$$

Then the de Sitter 3-space is

$$S_1^3 = \{(x_0, x_1, x_2, x_3) \in \mathbf{R}_1^4; -x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\}$$

with metric induced from \mathbf{R}_1^4 , which is a simply connected Lorentzian 3-manifold with constant sectional curvature 1. We identify \mathbf{R}_1^4 with the set of 2×2 Hermitian matrices $\text{Herm}(2) = \{X^* = X\}$ ($X^* := {}^t\bar{X}$) by

$$(1.1) \quad X = (x_0, x_1, x_2, x_3) \leftrightarrow X = \sum_{k=0}^3 x_k e_k = \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix},$$

where

$$(1.2) \quad e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $i = \sqrt{-1}$. Then S_1^3 is

$$S_1^3 = \{X; X^* = X, \det X = -1\} = \{Ae_3A^*; A \in \text{SL}_2 \mathbf{C}\}$$

with the metric

$$\langle X, Y \rangle = -\frac{1}{2} \text{trace}(Xe_2({}^tY)e_2), \quad \langle X, X \rangle = -\det X.$$

The projection $\pi_S: \text{SL}_2 \mathbf{C} \rightarrow S_1^3$ mentioned in the introduction is written explicitly as $\pi_S(A) = Ae_3A^*$. Note that the hyperbolic 3-space H^3 is given by $H^3 = \{AA^*; A \in \text{SL}_2 \mathbf{C}\}$ and the projection is $\pi_H(A) = AA^*$.

An immersion into S_1^3 is called *spacelike* if the induced metric on the immersed surface is positive definite. The complex Lie group $\text{SL}_2 \mathbf{C}$ acts isometrically on $\text{Herm}(2) = \mathbf{R}_1^4$, as well as S_1^3 , by

$$(1.3) \quad \text{Herm}(2) \ni X \mapsto AXA^*, \quad A \in \text{SL}_2 \mathbf{C}.$$

In fact, $\text{PSL}_2 \mathbf{C} = \text{SL}_2 \mathbf{C} / \{\pm \text{id}\}$ is isomorphic to the identity component $\text{SO}_{3,1}^+$ of the isometry group $\text{O}_{3,1}$ of S_1^3 . Note that each element of $\text{SO}_{3,1}^+$ corresponds to an orientation preserving and orthochronous (i.e., time orientation preserving) isometry. The group of orientation preserving isometries of S_1^3 is generated by $\text{PSL}_2 \mathbf{C}$ and the map

$$(1.4) \quad S_1^3 \ni X \mapsto -X \in S_1^3.$$

Aiyama-Akutagawa [1] gave a Weierstrass-type representation formula in terms of holomorphic data for spacelike CMC-1 immersions in S_1^3 . The

first author [7] extended the notion of CMC-1 surfaces as follows, like as for the case of maximal surfaces in the Lorentz-Minkowski space [24].

Definition 1.1 [7]. Let M^2 be a 2-manifold. A C^∞ -map $f: M^2 \rightarrow S_1^3$ is called a *CMC-1 face* if

- (1) there exists an open dense subset $W \subset M^2$ such that $f|_W$ is a spacelike CMC-1 immersion.
- (2) For any singular point (that is, a point where the induced metric degenerates) p , there exists a C^1 -differentiable function $\lambda: U \cap W \rightarrow (0, \infty)$, defined on the intersection of neighborhood U of p with W , such that λds^2 extends to a C^1 -differentiable Riemannian metric on U , where ds^2 is the first fundamental form, i.e., the pull-back of the metric of S_1^3 by f .
- (3) $df(p) \neq 0$ for any $p \in M^2$.

Remark 1.1. Although the original definition of CMC-1 faces in [7] assumed the orientability of the source manifold, our definition here does not. However, this difference is not of an essential nature. In fact, for any CMC-1 face $f: M^2 \rightarrow S_1^3$, M^2 is automatically orientable. (See [12].)

Remark 1.2. A C^∞ -map $f: M^2 \rightarrow S_1^3$ is called a *frontal* if f lifts to a C^∞ -map $L_f: M^2 \rightarrow P(T^*S_1^3)$ such that $dL_f(TM^2)$ lies in the canonical contact hyperplane field on $P(T^*S_1^3)$. Moreover, f is called a *wave front* or a *front* if L_f is an immersion, that is, $L_f(M^2)$ is a Legendrian submanifold. If a frontal L_f can lift up to a smooth map into $T^*S_1^3$, f is called *co-orientable*, and otherwise it is called *non-co-orientable*. Wave fronts are a canonical class for investigating flat surfaces in the hyperbolic 3-space H^3 . In fact, like for CMC-1 faces (see Theorem 0.2 in the introduction), an Osserman-type inequality holds for flat fronts in H^3 (see [14]). Although our CMC-1 faces belong to a special class of horospherical linear Weingarten surfaces (cf. [12]), they may not be (wave) fronts in general, but are co-orientable frontals. In particular, there is a globally defined non-vanishing normal vector field ν on the whole of M^2 for a given CMC-1 face $f: M^2 \rightarrow S_1^3$. It should be remarked that the limiting tangent plane at each singular point contains a lightlike direction, that is, a CMC-1 face is not spacelike on the singular set.

An oriented 2-manifold M^2 on which a CMC-1 face $f: M^2 \rightarrow S_1^3$ is defined always has a complex structure (see [7]). Since CMC-1 faces are all orientable and co-orientable (cf. [12]), from now on, we will treat M^2 as

a Riemann surface, and we can assume the existence of a globally defined non-vanishing normal vector field. The representation formula in [1] can be extended for CMC-1 faces as follows:

Theorem 1.1 ([7], **Theorem 1.9**). *Let \widetilde{M}^2 be a simply connected Riemann surface. Let g be a meromorphic function and ω a holomorphic 1-form on \widetilde{M}^2 such that*

$$(1.5) \quad d\hat{s}^2 = (1 + |g|^2)^2 |\omega|^2$$

is a Riemannian metric on \widetilde{M}^2 and $|g|$ is not identically 1. Take a holomorphic immersion $F = (F_{jk}) : \widetilde{M}^2 \rightarrow \text{SL}_2 \mathbf{C}$ satisfying

$$(1.6) \quad F^{-1}dF = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \omega.$$

Then $f : \widetilde{M}^2 \rightarrow S_1^3$ defined by

$$(1.7) \quad f = \pi_S \circ F := F e_3 F^*$$

is a CMC-1 face. The induced metric ds^2 on \widetilde{M}^2 , the second fundamental form II , and the Hopf differential Q of f are given as follows:

$$(1.8) \quad ds^2 = (1 - |g|^2)^2 |\omega|^2, \quad II = Q + \overline{Q} + ds^2, \quad Q = \omega dg.$$

The singularities of the CMC-1 face occur at points where $|g| = 1$.

Conversely, for any CMC-1 face $f : \widetilde{M}^2 \rightarrow S_1^3$, there exist a meromorphic function g (with $|g|$ not identically 1) and a holomorphic 1-form ω on \widetilde{M}^2 so that $d\hat{s}^2$ is a Riemannian metric on \widetilde{M}^2 and (1.7) holds, where $F : \widetilde{M}^2 \rightarrow \text{SL}_2 \mathbf{C}$ is an immersion satisfying (1.6).

Remark 1.3. By definition, CMC-1 faces have dense regular sets. However, the projection of null holomorphic immersions might not have dense regular sets, in general. Such an example has been given in ([7], Remark 1.8). Fortunately, we can explicitly classify such degenerate examples, as follows: Let M^2 be a connected Riemann surface and $F : M^2 \rightarrow \text{SL}_2 \mathbf{C}$ be a null immersion. We assume that the set of singular points of the corresponding map

$$f = F e_3 F^* : M^2 \longrightarrow S_1^3$$

has an interior point. Then the secondary Gauss map g is constant on M^2 and $|g| = 1$. Without loss of generality, we may assume $g = 1$. Since F is

an immersion, $(1 + |g|^2)|\omega|^2$ is positive definite. Then $\omega \neq 0$ everywhere. Hence for each $p \in M^2$, one can take a complex coordinate z such that $\omega = dz$ near p . Then F is a solution of

$$F^{-1}dF = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} dz.$$

Without loss of generality, we may assume that $F(0) = \begin{pmatrix} 1/2 & 1/2 \\ -1 & 1 \end{pmatrix}$. Then we have

$$F = \begin{pmatrix} z + 1/2 & -z + 1/2 \\ -1 & 1 \end{pmatrix},$$

and the corresponding map f is computed as

$$f = Fe_3F^* = \begin{pmatrix} 2 \operatorname{Re} z & -1 \\ -1 & 0 \end{pmatrix},$$

whose image is a lightlike line in S_1^3 . Thus, we have shown that the image of any degenerate CMC-1 surface is a part of a lightlike line.

Remark 1.4. Theorem 1.1 is an analogue of the Bryant representation for CMC-1 surfaces in H^3 , which explains why CMC-1 surfaces in both H^3 and S_1^3 are characterized by the projections $\pi_H \circ F$ and $\pi_S \circ F$. The CMC-1 surfaces in H^3 and S_1^3 are both typical examples in the class of linear Weingarten surfaces. A Bryant-type representation formula for linear Weingarten surfaces was recently given by Gálvez *et al.* [10].

Remark 1.5. Following the terminology of [21], g is called a *secondary* Gauss map of f . The pair (g, ω) is called *Weierstrass data* of f , and F is called a *holomorphic null lift* of f .

The holomorphic 2-differential Q as in (1.8) is called the *Hopf differential* of f . In analogy with the theory of CMC-1 surfaces in H^3 , the meromorphic function

$$(1.9) \quad G := \frac{dF_{11}}{dF_{21}} = \frac{dF_{12}}{dF_{22}}$$

is called the *hyperbolic Gauss map*. A geometric meaning for the hyperbolic Gauss map is given in Section 4.

Remark 1.6. Corresponding to Theorem 1.1, a Weierstrass-type representation formula is known for spacelike maximal surfaces in \mathbf{R}_1^3 [11]. In fact, the Weierstrass data (g, ω) as in Theorem 1.1 defines null curves in \mathbf{C}^3 by

$$F_0(z) := \int_{z_0}^z (-2g, 1 + g^2, i(1 - g^2))\omega.$$

Any maxface (see [24] for the definition) is locally obtained as the real part of some F_0 . Moreover, their first fundamental forms and Hopf differentials are given by (1.8). The meromorphic function g can be identified with the Lorentzian Gauss map.

Remark 1.7. Let G, g be meromorphic functions on a Riemann surface. Set

$$(1.10) \quad F = \begin{pmatrix} G \frac{da}{dG} - a & G \frac{db}{dG} - b \\ \frac{da}{dG} & \frac{db}{dG} \end{pmatrix}, \quad a = \sqrt{\frac{dG}{dg}}, \quad b = -ga.$$

Then F is a meromorphic null map with hyperbolic and secondary Gauss maps G and g respectively. Formula (1.10) is called Small’s formula [13, 19].

Remark 1.8. The holomorphic null lift F of a CMC-1 face f is unique up to right-multiplication by matrices in $SU_{1,1}$, that is, for each $A \in SU_{1,1}$, the projection of FA^{-1} is also f . Under the transformation $F \mapsto FA^{-1}$, the secondary Gauss map g changes by a Möbius transformation:

$$(1.11) \quad g \mapsto A \star g := \frac{A_{11}g + A_{12}}{A_{21}g + A_{22}}, \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

The conditions $|g| = 1$, $|g| > 1$, $|g| < 1$ are invariant under this transformation.

In particular, let $f: M^2 \rightarrow S_1^3$ be a CMC-1 face of a (not necessarily simply connected) Riemann surface M^2 . Then the holomorphic null lift F is defined only on the universal cover \widetilde{M}^2 of M^2 . Take a deck transformation $\tau \in \pi_1(M^2)$ in \widetilde{M}^2 . Since $\pi_S \circ F = \pi_S \circ F \circ \tau$, there exists a $\tilde{\rho}(\tau) \in SU_{1,1}$ such that

$$(1.12) \quad F \circ \tau = F\tilde{\rho}(\tau).$$

The induced homomorphism $\tilde{\rho}: \pi_1(M^2) \rightarrow SU_{1,1}$ is called the *monodromy representation*, which induces a $PSU_{1,1}$ -representation

$$\rho: \pi_1(M^2) \rightarrow PSU_{1,1} = SU_{1,1} / \{\pm 1\}$$

satisfying

$$(1.13) \quad g \circ \tau^{-1} = \rho(\tau) \star g.$$

Remark 1.9. The action $F \mapsto BF, B \in \text{SL}_2 \mathbf{C}$, induces a rigid motion $f \mapsto BfB^*$ in S_1^3 , and the isometric motion $f \mapsto -f$ as in (1.4) corresponds to

$$(1.14) \quad F \mapsto F^\natural = F \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = iFe_1.$$

The secondary Gauss map of F^\natural is $1/g$.

Remark 1.10. Let K_{ds^2} be the Gaussian curvature of ds^2 on the set of regular points of f . Then

$$(1.15) \quad d\sigma^2 := K_{ds^2} ds^2 = \frac{4|dg|^2}{(1-|g|^2)^2}$$

is a pseudometric of constant curvature -1 , which degenerates at isolated umbilic points. We have

$$(1.16) \quad d\sigma^2 \cdot ds^2 = 4|Q|^2.$$

Remark 1.11. The metric

$$(1.17) \quad ds_{\#}^2 := \text{trace}(\alpha_{\#})(\alpha_{\#})^* = (1 + |G|^2)^2 \left| \frac{Q}{dG} \right|^2 \quad (\alpha_{\#} := (F^{-1})^{-1}d(F^{-1}))$$

is induced from the canonical Hermitian metric of $\text{SL}_2 \mathbf{C}$ via $F^{-1}: \widetilde{M}^2 \rightarrow \text{SL}_2 \mathbf{C}$. When the CMC-1 face is defined on M^2 , G and Q are as well, so $ds_{\#}^2$ is well defined on M^2 , and is called the *lift metric*. It is nothing but the *dual metric* of the CMC-1 surface $\pi_H \circ F$ in H^3 , see [23].

Completeness

We now define two different notions of completeness for CMC-1 faces as follows:

Definition 1.2. We say a CMC-1 face $f: M^2 \rightarrow S_1^3$ is *complete* if there exists a symmetric 2-tensor field T which vanishes outside a compact subset $C \subset M^2$ such that the sum $T + ds^2$ is a complete Riemannian metric on M^2 .

See [7], with similar definitions in [14] for flat fronts in H^3 and in [24] for maxfaces.

Definition 1.3. We say that f is *weakly complete* if it is congruent to an S_1^3 -horosphere or if the lift metric (1.17) is a complete Riemannian metric on M^2 .

Here, the S_1^3 -horosphere is the totally umbilic CMC-1 surface, which is also the only complete CMC-1 immersed surface (see Remark 1.15). It has the Weierstrass data $g = c = \text{constant}$ ($|c| \neq 1$) and $\omega = dz$. The metric $ds_{\#}^2$ of an S_1^3 -horosphere cannot be defined by (1.17) as G is constant and Q is identically 0.

Definition 1.4. We say that f is of *finite type* if there exists a compact set C of M^2 such that the first fundamental form ds^2 is positive definite and has finite total (absolute) curvature on $M^2 \setminus C$.

Let $f: M^2 \rightarrow S_1^3$ be a CMC-1 face of finite topology, that is, M^2 is diffeomorphic to a compact Riemann surface \overline{M}^2 with a finite number of points $\{p_1, \dots, p_n\} \subset \overline{M}^2$ excluded. We can take a punctured neighborhood Δ_j^* of p_j which is biholomorphic to either the punctured unit disk $\Delta^* = \{z \in \mathbf{C}; 0 < |z| < 1\}$ or an annular domain, and p_j is called a *puncture-type end* or an *annular end*, respectively.

Proposition 1.1. *Let $f: M^2 \rightarrow S_1^3$ be a CMC-1 face. If f is complete, then*

- (1) *the singular set of f is compact,*
- (2) *f is weakly complete,*
- (3) *M^2 has finite topology and each end is of puncture-type.*

Proof. (1) is obvious. If f is totally umbilic, it is congruent to an S_1^3 -horosphere and the assertion is obvious. So we assume the Hopf differential Q does not vanish identically. Since the Gaussian curvature of f is non-negative, completeness implies (3) by the appendix of [24]. So we shall now prove that completeness implies weak completeness: fix an end p_j of f . By an appropriate choice of a coordinate z , the restriction of f to a neighborhood of p_j is $f_j: \Delta^* \rightarrow S_1^3$. We denote by $d\hat{s}^2$ the induced metric of the corresponding CMC-1 surface $\hat{f}_j = FF^*: \tilde{\Delta}^* \rightarrow H^3$ into hyperbolic 3-space (see (1.5), where $\tilde{\Delta}^*$ is the universal cover of Δ^*). Take a path $\gamma: [0, 1) \rightarrow \Delta^*$ such that $\gamma(t) \rightarrow 0$ as $t \rightarrow 1$. Then by (1.5) and (1.8), $d\hat{s}^2 \geq ds^2$ holds, and

hence completeness of f implies that each lift $\tilde{\gamma}: [0, 1) \rightarrow \tilde{\Delta}^*$ of γ has infinite length with respect to $d\tilde{s}^2$. Here, $d\tilde{s}^2$ and $ds_{\#}^2 = (1 + |G|^2)^2 |Q/dG|^2$ are the pull-backs of the canonical Hermitian metric of $SL_2 \mathbf{C}$ by F and F^{-1} , respectively. Yu [26] showed that completeness of these two metrics are equivalent. Hence, $\tilde{\gamma}$ has infinite length with respect to the metric $ds_{\#}^2$. Since $ds_{\#}^2$ is well defined on Δ^* , γ also has infinite length with respect to $ds_{\#}^2$, that is, the metric $ds_{\#}^2$ on Δ^* is complete at 0. Thus, f_j is a weakly complete end. \square

For further properties of complete ends, see Theorems 3.1 and 3.2.

Remark 1.12. The converse of Proposition 1.1 is true (see [25]). Our definition of weak completeness of CMC-1 faces is somewhat more technical than that of maxfaces [24], but it is the correctly corresponding concept in S_1^3 : for data (g, ω) , weak completeness of the associated maxface in \mathbf{R}_1^3 is equivalent to that of the CMC-1 face in S_1^3 .

Remark 1.13. The CMC-1 trinoids in S_1^3 constructed in [15] are all weakly complete (sometimes complete as well) and all ends are g -regular, see Section 3.

Remark 1.14. The Hopf differential Q of a complete CMC-1 face $f: M^2 \rightarrow S_1^3$ is meromorphic on its compactification \overline{M}^2 , even without assuming that all ends of f are regular. See Appendix A. It should be remarked that for CMC-1 surfaces in the hyperbolic 3-space, finiteness of total curvature is needed to show the meromorphicity of Q (see [3]).

Monodromy of ends of CMC-1 faces

For any real number t , we set

$$\begin{aligned}
 \Lambda_e(t) &:= \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}, \\
 \Lambda_p(t) &:= \begin{pmatrix} 1 + it & -it \\ it & 1 - it \end{pmatrix}, \\
 \Lambda_h(t) &:= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}.
 \end{aligned}
 \tag{1.18}$$

A matrix in $SU_{1,1}$ is called

- (1) *elliptic* if it is conjugate to $\Lambda_e(t)$ ($t \in (-\pi, \pi]$) in $SU_{1,1}$,
- (2) *parabolic* if it is conjugate to $\pm\Lambda_p(t)$ ($t \in \mathbf{R} \setminus \{0\}$) in $SU_{1,1}$, and
- (3) *hyperbolic* if it is conjugate to $\pm\Lambda_h(t)$ ($t > 0$) in $SU_{1,1}$.

Any matrix in $SU_{1,1}$ is of one of these three types, see Appendix B. Note that the parabolic matrices $\Lambda_p(t_1)$ and $\Lambda_p(t_2)$ are conjugate in $SU_{1,1}$ if and only if $t_1 t_2 > 0$. Although the set of conjugate classes of parabolic matrices is fully represented by $\{\pm \Lambda_p(\pm 1)\}$, we may use various values of t in this paper for the sake of simplicity.

Let $f: M^2 \rightarrow S_1^3$ be a weakly complete CMC-1 face of finite topology, where M^2 is diffeomorphic to a compact Riemann surface \bar{M}^2 with finitely many punctures $\{p_1, \dots, p_n\}$. Any puncture p_j , or occasionally a small neighborhood U_j of p_j , is called an *end* of f .

An end is called *elliptic*, *parabolic* or *hyperbolic* when the monodromy matrix $\tilde{\rho}(\tau) \in SU_{1,1}$ is elliptic, parabolic or hyperbolic, respectively, where $\tilde{\rho}$ is as in Remark 1.8 and $\tau \in \pi_1(M^2)$ is the deck transformation corresponding to the counterclockwise loop about p_j .

The Schwarzian derivative

Let (U, z) be a local complex coordinate of a Riemann surface M^2 , and $h(z)$ a meromorphic function on U . Then

$$S_z(h) := \left(\frac{h''}{h'}\right)' - \frac{1}{2} \left(\frac{h''}{h'}\right)^2 \quad \left(' = \frac{d}{dz}\right)$$

is the *Schwarzian derivative* of h with respect to the coordinate z .

If $h(z) = a + b(z-p)^m + o((z-p)^m)$ at $z = p \in U$ ($b \neq 0$), where $o((z-p)^m)$ denotes higher order terms, then the positive integer m is called the (*ramification*) *order* of $h(z)$, and we have

$$(1.19) \quad S_z(h) = \frac{1}{(z-p)^2} \left(\frac{1-m^2}{2} + o(1) \right).$$

We write $S(h) = S_z(h) dz^2$, which we also call the *Schwarzian derivative*. The Schwarzian derivative depends on the choice of local coordinates, but the difference does not, that is, $S(h_1) - S(h_2)$ is a well-defined holomorphic 2-differential.

The Schwarzian derivative is invariant under Möbius transformations: $S(h) = S(A \star h)$ holds for $A \in \mathrm{SL}_2 \mathbf{C}$, where \star denotes the Möbius transformation as in (1.11). Conversely, if $S_z(h) = S_z(g)$, there exists an $A \in \mathrm{SL}_2 \mathbf{C}$ such that $g = A \star h$.

Let $f: M^2 \rightarrow S_1^3$ be a CMC-1 face with the hyperbolic Gauss map G , a secondary Gauss map g and the Hopf differential Q . Then

$$(1.20) \quad S(g) - S(G) = 2Q.$$

Remark 1.15. Here we give a proof that the only complete CMC-1 immersion is the totally umbilic one, that is, the S_1^3 -horosphere, which is simpler than the original proofs in [2,17]. (The proof is essentially the same as for the case of maximal surfaces in R_1^3 given in [24, Remark 1.2].) Let $f: M^2 \rightarrow S_1^3$ be a complete CMC-1 immersion. Without loss of generality, we may assume that M^2 is both connected and simply connected. Then the Weierstrass data (g, ω) as in Theorem 1.1 is single valued on M^2 . Since f has no singular points, we may assume that $|g| < 1$ holds on M^2 . Since $(1 - |g|^2)^2 |\omega|^2 < |\omega|^2$, the metric $|\omega|^2$ is a complete flat metric on M^2 . Then the uniformization theorem yields that M^2 is bi-holomorphic to C , and g must be a constant function, which implies that the image of f must be totally umbilic.

2. Monodromy of punctured hyperbolic metrics

By Remark 1.8, the monodromy of a holomorphic null immersion F is elliptic, parabolic or hyperbolic if and only if the monodromy of its secondary Gauss map g is elliptic, parabolic or hyperbolic, respectively. In this section, in an abstract setting, we give results needed for investigating the behavior of g at a puncture-type end, in terms of the monodromy of g .

Lifts of $PSU_{1,1}$ -projective connections on a punctured disk

Let

$$\Delta^* = \Delta \setminus \{0\}, \quad \text{where } \Delta := \{z \in C; |z| < 1\},$$

be the punctured unit disk and $P = p(z)dz^2$ a holomorphic 2-differential on Δ^* . Then there exists a holomorphic developing map $g_P: \tilde{\Delta}^* \rightarrow C \cup \{\infty\}$ such that $S(g_P) = P$, where $\tilde{\Delta}^*$ is the universal cover of Δ^* . For any other holomorphic function h such that $S(h) = P$, there exists an $A \in SL_2 C$ so that $A \star g_P = h$. Thus there exists a matrix $T \in PSL_2 C$ such that

$$(2.1) \quad g_P \circ \tau^{-1} = T \star g_P,$$

where τ is the generator of $\pi_1(\Delta^*)$ corresponding to a counterclockwise loop about the origin. We call T the monodromy matrix of g_P . If there exists a g_P so that $T \in PSU_{1,1}$, P is called a $PSU_{1,1}$ -projective connection on Δ^* and

g_P is called a $\text{PSU}_{1,1}$ -lift of P . A $\text{PSU}_{1,1}$ -projective connection on Δ^* has a removable singularity, a pole or an essential singularity at 0, and is said to have a *regular singularity* at 0 if it has at most a pole of order 2 at 0. (The general definition of projective connections is given in [20, 22]. There exist holomorphic 2-differentials on Δ^* which are not $\text{PSU}_{1,1}$ -projective connections.) When $T \in \text{PSU}_{1,1}$, it is conjugate to one of the matrices in (1.18). The $\text{PSU}_{1,1}$ -projective connection P is then called *elliptic*, *parabolic* or *hyperbolic* when T is elliptic, parabolic or hyperbolic, respectively. This terminology is independent of the choice of g_P .

By property (1.13), the Schwarzian derivative $S(g)$ of the secondary Gauss map g of a CMC-1 face is an example of a $\text{PSU}_{1,1}$ -projective connection.

Note that a $\text{PSU}_{1,1}$ -lift g_P has the $\text{PSU}_{1,1}$ ambiguity $g_P \mapsto A \star g_P$ for $A \in \text{PSU}_{1,1}$. The property that $|g_P| > 1$ (resp. $|g_P| < 1$) is independent of this ambiguity.

Remark 2.1. Let g_P be a $\text{PSU}_{1,1}$ -lift of a $\text{PSU}_{1,1}$ -projective connection P . Then

$$\frac{1}{g_P} = D \star g_P \quad \left(D := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right)$$

is also a $\text{PSU}_{1,1}$ -lift of P , because $DAD^{-1} \in \text{PSU}_{1,1}$ for any $A \in \text{PSU}_{1,1}$. However, $D \notin \text{PSU}_{1,1}$, and one can show that there is no matrix $B \in \text{PSU}_{1,1}$ such that $1/g_P = B \star g_P$, that is, $1/g_P$ is not $\text{PSU}_{1,1}$ -equivalent to g_P .

In the rest of this article, as well as in the following proposition, we use

$$(2.2) \quad R := \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix},$$

which is motivated by an isomorphism between $\text{SL}_2 \mathbf{R}$ and $\text{SU}_{1,1}$. See Appendix B.

Proposition 2.1. *Let P be a $\text{PSU}_{1,1}$ -projective connection on Δ^* . Then the following assertions hold:*

- (1) *Suppose that P is elliptic. Then,*
 - (i) *there exist a real number μ and a single-valued meromorphic function $h(z)$ on Δ^* such that*

$$g(z) := z^\mu h(z)$$

is a $\text{PSU}_{1,1}$ -lift of P .

- (ii) P has a regular singularity at $z = 0$ if and only if $h(z)$ has at most a pole at $z = 0$.
- (2) Suppose that P is parabolic and take an arbitrary positive number t . Then,
 - (i) for each $\varepsilon \in \{-1, 1\}$, there exists a single-valued meromorphic function $h(z)$ on Δ^* such that

$$g(z) := R^{-1} \star \left(h(z) - \frac{\varepsilon t}{\pi i} \log z \right)$$

is a $\text{PSU}_{1,1}$ -lift of P .

- (ii) The function $h(z)$ has at most a pole at $z = 0$ if and only if P has a pole of order exactly 2 at $z = 0$.
- (iii) $h(z)$ is holomorphic at $z = 0$ if and only if $P - dz^2/(2z^2)$ has at most a pole of order 1 at $z = 0$.
- (iv) When $h(z)$ is holomorphic at $z = 0$, $|g(z)| > 1$ (resp. $|g(z)| < 1$) holds for sufficiently small $|z|$ if and only if $\varepsilon = +1$ (resp. $\varepsilon = -1$).
- (3) Suppose that P is hyperbolic. Then,
 - (i) there exist a positive number μ and a single-valued meromorphic function $h(z)$ on Δ^* such that

$$g(z) := R^{-1} \star (z^{i\mu} h(z))$$

is a $\text{PSU}_{1,1}$ -lift of P .

- (ii) $h(z)$ has at most a pole at $z = 0$ if and only if P has a pole of order exactly 2 at $z = 0$.

Remark 2.2. In the statements of Proposition 2.1, the function z^μ ($\mu \in \mathbf{C}$) is defined by

$$z^\mu := \exp(\mu \log z),$$

where $\log z$ is considered as a function defined on the universal cover $\tilde{\Delta}^*$ of Δ^* .

To prove this, we consider the following ordinary differential equation

$$(2.3) \quad X'' + \frac{1}{2}p(z)X = 0 \quad \left(' = \frac{d}{dz}, P = p(z) dz^2 \right).$$

If we assume $P(z)$ has a regular singularity at $z = 0$, then $p(z) = \alpha z^{-2}(1 + o(1))$ for some $\alpha \in \mathbf{C}$ and (2.3) has the fundamental system of solutions

$$(2.4) \quad \begin{aligned} X_1(z) &= z^{\mu_1} \xi_1(z) \\ X_2(z) &= z^{\mu_2} \xi_2(z) + k \log z X_1 \end{aligned} \quad (\operatorname{Re} \mu_1 \geq \operatorname{Re} \mu_2),$$

where $\xi_j(z)$ ($j = 1, 2$) are holomorphic functions on $\Delta = \{|z| < 1\}$ such that $\xi_j(0) \neq 0$ ($j = 1, 2$). The constant $k \in \mathbf{C}$ is called the *log-term coefficient* and μ_1, μ_2 are the solutions of the indicial equation

$$(2.5) \quad t(t - 1) + \frac{\alpha}{2} = 0.$$

If $\mu_1 - \mu_2 \notin \mathbf{Z}$, then k vanishes. (See [5] or the appendix of [18].) The following lemma is easy to show:

Lemma 2.1. *In the above setting, $S(g_0) = P$ if $g_0 := X_2/X_1$.*

Proof of Proposition 2.1. Take the matrix T as in (2.1).

We first prove the elliptic case. Since P is elliptic, there exist a $t \in \mathbf{R}$ and an $A \in \operatorname{SU}_{1,1}$ such that $ATA^{-1} = \Lambda_e(t)$. So $(A \star g) \circ \tau^{-1} = e^{2it}(A \star g)$, and $h(z) := z^{t/\pi}(A \star g(z))$ is single-valued on Δ^* , proving the first part of (1). If the origin 0 is at most a pole of h , a direct calculation shows that P has a regular singularity. To show the converse, we set $g_0 := X_2/X_1$, with $\{X_1, X_2\}$ as in (2.4). Then by Lemma 2.1, we have $S(g_0) = P$. The monodromy matrix $\pm T_0$ of g_0 is conjugate to

$$(2.6) \quad \begin{cases} \begin{pmatrix} e^{\pi i(\mu_1 - \mu_2)} & 0 \\ 0 & e^{\pi i(\mu_2 - \mu_1)} \end{pmatrix} & (\text{if } k = 0), \\ \begin{pmatrix} 1 & -2\pi i k \\ 0 & 1 \end{pmatrix} & (\text{if } k \neq 0). \end{cases}$$

Since P is elliptic, the log-term coefficient $k = 0$ and $\mu_2 - \mu_1 \in \mathbf{R}$. Thus

$$g_0(z) = z^\mu \frac{\xi_2(z)}{\xi_1(z)} \quad (\mu := \mu_2 - \mu_1).$$

Since $S(A \star g) = S(g_0)$, there exists a $B \in \operatorname{SL}_2 \mathbf{C}$ so that $A \star g = B \star g_0$. Then

$$\Lambda_e(t) \star (A \star g) = (A \star g) \circ \tau^{-1} = B \star (g_0 \circ \tau^{-1}) = B \Lambda_e(-\pi \mu) B^{-1} \star (A \star g),$$

so $\Lambda_e(t) = \pm B\Lambda_e(-\pi\mu)B^{-1}$. If $t \equiv 0 \pmod{\pi}$, then $A \star g$ is meromorphic, proving (1). Otherwise,

$$B = \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix}$$

for some $c \in \mathbf{C} \setminus \{0\}$, and (1) (ii) in the proposition follows from

$$A \star g(z) = c^2 z^\mu \frac{\xi_2(z)}{\xi_1(z)} \quad \text{or} \quad -c^2 z^{-\mu} \frac{\xi_1(z)}{\xi_2(z)},$$

respectively.

Next, we assume P is parabolic and take a positive number t and $\varepsilon \in \{-1, 1\}$. Then by Theorem B.1 and Remark B.2 in Appendix B, there exists a matrix $A \in \text{SU}_{1,1}$ such that ATA^{-1} is one of $\Lambda_p(\varepsilon t)$, $-\Lambda_p(\varepsilon t)$, $\Lambda_p(-\varepsilon t)$, $-\Lambda_p(-\varepsilon t)$. Note that $\Lambda_p(\varepsilon t)$ and $\Lambda_p(-\varepsilon t)$ are not conjugate in $\text{PSU}_{1,1}$. Replacing g with $1/g$ if $ATA^{-1} = \pm\Lambda_p(-\varepsilon t)$ (see Remark 2.1), we can choose a $\text{PSU}_{1,1}$ -lift g such that

$$ATA^{-1} = \pm\Lambda_p(\varepsilon t).$$

Then, $(A \star g) \circ \tau^{-1} = \Lambda_p(\varepsilon t) \star (A \star g)$. Here the \pm -ambiguity of ATA^{-1} does not affect the \star -action. Thus,

$$((RA) \star g) \circ \tau^{-1} = (RA) \star g + 2\varepsilon t, \quad \text{since} \quad R\Lambda_p(\varepsilon t) = \begin{pmatrix} 1 & 2\varepsilon t \\ 0 & 1 \end{pmatrix} R.$$

Hence $h(z) := (RA) \star g + (\varepsilon t/(\pi i)) \log z$ is a single-valued meromorphic function on Δ^* , proving the first part of (2) in the proposition. If $h(z)$ has at most a pole at $z = 0$, then a direct computation shows that P has a pole of order exactly 2. Therefore, it suffices to show that $h(z)$ has at most a pole at $z = 0$ when P has a regular singularity. We now show this:

We set $g_0 := X_2/X_1$. Since P is parabolic, (2.6) yields that the log-term coefficient $k \neq 0$ and $\mu := \mu_2 - \mu_1 \in \mathbf{Z}$ is non-positive. Hence

$$g_0 = z^\mu \frac{\xi_2(z)}{\xi_1(z)} + k \log z.$$

Here $z^\mu \xi_2(z)/\xi_1(z)$ is single-valued on Δ^* and has at most a pole at $z = 0$. Take a matrix $B \in \text{SL}_2 \mathbf{C}$ such that $(RA) \star g = B \star g_0$. Then

$$\begin{aligned} \begin{pmatrix} 1 & 2\varepsilon t \\ 0 & 1 \end{pmatrix} \star (RA \star g) &= (RA \star g) \circ \tau^{-1} = B \star (g_0 \circ \tau^{-1}) \\ &= B \begin{pmatrix} 1 & -2\pi i k \\ 0 & 1 \end{pmatrix} \star g_0 = B \begin{pmatrix} 1 & -2\pi i k \\ 0 & 1 \end{pmatrix} B^{-1} \star (RA \star g). \end{aligned}$$

Replacing X_2 by $(i\varepsilon t/(\pi k))X_2$ and renaming $(i\varepsilon t/(\pi k))\xi_2$ to ξ_2 , $-2\pi ik$ becomes $2\varepsilon t$, and we have

$$\begin{pmatrix} 1 & 2\varepsilon t \\ 0 & 1 \end{pmatrix} = B \begin{pmatrix} 1 & 2\varepsilon t \\ 0 & 1 \end{pmatrix} B^{-1},$$

and there is no \pm -ambiguity in the above equation, as the eigenvalues of the left-hand matrix must have the same sign as those of the right-hand matrix. Thus we can choose

$$B = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$$

for some $c \in \mathbf{C}$, which proves the second part of (2) in the Proposition 2.1. It is easy to see that $h(z)$ is holomorphic at $z = 0$ if and only if $\mu_2 = \mu_1$, that is, α in (2.5) is $1/2$, which proves the third part of (2). Assume h is holomorphic on Δ . Since the Möbius transformation $z \mapsto R \star z$ maps the disk Δ onto the upper-half plane $\{z; \text{Im } z > 0\}$, the condition $|g| > 1$ (equivalently $|A \star g| > 1$) is equivalent to $\text{Im}(RA \star g) < 0$ for all $A \in \text{SU}_{1,1}$. And since $|h|$ is bounded, this is equivalent to $\varepsilon > 0$. Thus we obtain the last part of (2).

Next, we assume P is hyperbolic. By Theorem B.1 in Appendix B, there are a matrix $A \in \text{SU}_{1,1}$ and $t > 0$ such that $ATA^{-1} = \Lambda_h(t)$ or $-\Lambda_h(t)$. Then $(A \star g) \circ \tau^{-1} = \Lambda_h(t) \star (A \star g)$, which implies $((RA) \star g) \circ \tau^{-1} = e^{2t}(RA) \star g$. So $h(z) := z^{-it/\pi}(RA) \star g$ is a single-valued meromorphic function in Δ^* . This proves the first part of (3). To prove the second part of (3), analogous to the parabolic case, we only need to prove one direction. Suppose that P has a regular singularity. We set $g_0 := X_2/X_1$. Since P is hyperbolic, (2.6) yields that $k = 0$ and $\mu_2 - \mu_1 \in i\mathbf{R}$. Thus

$$g_0(z) = z^{i\mu} \frac{\xi_2(z)}{\xi_1(z)}, \quad \mu := i(\mu_1 - \mu_2).$$

Exchanging X_1 and X_2 if necessary, we may assume $\mu > 0$ without loss of generality. Take a $B \in \text{SL}_2 \mathbf{C}$ such that $RA \star g = B \star g_0$. Then we have

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \star (RA \star g) = B \star (g_0 \circ \tau^{-1}) = B \begin{pmatrix} e^{\pi\mu} & 0 \\ 0 & e^{-\pi\mu} \end{pmatrix} B^{-1} \star (RA \star g),$$

so

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} = B \begin{pmatrix} e^{\pi\mu} & 0 \\ 0 & e^{-\pi\mu} \end{pmatrix} B^{-1},$$

that is, $t = \pm\pi\mu$. As we have assumed that $t > 0$ and $\mu > 0$, we have $t = \pi\mu$, and then B must be diagonal. Hence

$$RA \star g(z) = c^2 z^{it/\pi} \frac{\xi_2(z)}{\xi_1(z)}, \quad B = \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}$$

proving the assertion. □

Monodromy of punctured hyperbolic metrics

We consider a conformal metric $d\sigma^2$ on Δ^* of constant Gaussian curvature -1 , called a *punctured hyperbolic metric*. Then there exists a meromorphic function $g : \tilde{\Delta}^* \rightarrow \mathbf{C} \cup \{\infty\} \setminus \{|z| = 1\}$ such that

$$(2.7) \quad d\sigma^2 = \frac{4|dg|^2}{(1 - |g|^2)^2},$$

which is called the *developing map* of $d\sigma^2$. Since $d\sigma^2$ is a well defined hyperbolic metric on Δ^* , either $|g| < 1$ or $|g| > 1$ holds on Δ^* .

We remark that, for a CMC-1 immersion $f : \Delta^* \rightarrow S_1^3$, the metric $d\sigma^2$ as in (1.15) is an example of a hyperbolic metric, and the secondary Gauss map is a developing map of it.

The developing map g is not unique, and the set of all developing maps of $d\sigma^2$ coincides with

$$\left\{ A \star g; A \in \text{SU}_{1,1} \right\} \cup \left\{ A \star \frac{1}{g} = A \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \star g; A \in \text{SU}_{1,1} \right\}.$$

Set

$$(2.8) \quad S(d\sigma^2) := S(g) = \left(w_{zz} - \frac{(w_z)^2}{2} \right) dz^2,$$

where $d\sigma^2 = e^w |dz|^2$, that is, $w := \log(4|g_z|^2/(1 - |g|^2)^2)$. We call the projective connection $S(d\sigma^2)$ the *Schwarzian derivative* of $d\sigma^2$. Since the metric $d\sigma^2$ is well defined on Δ^* , the developing map g is a $\text{PSU}_{1,1}$ -lift of the $\text{PSU}_{1,1}$ -projective connection $S(d\sigma^2)$.

If g is a developing map of $d\sigma^2 = K_{ds^2} ds^2$ in (1.15), then

$$g \circ \tau^{-1} = T \star g \quad \text{for some } T \in \text{PSU}_{1,1}.$$

If the matrix T is elliptic (resp. parabolic, hyperbolic), the metric $d\sigma^2$ is said to have *elliptic* (resp. *parabolic*, *hyperbolic*) monodromy.

Definition 2.1. We say that a hyperbolic punctured metric $d\sigma^2$ has a *regular singularity* at the origin if $S(d\sigma^2)$ has a regular singularity at the origin, that is, it has at most a pole of order 2.

Theorem 2.1. *Any conformal hyperbolic metric on Δ^* has a regular singularity at $z = 0$.*

Proof. Let g be a developing map of a conformal hyperbolic metric $d\sigma^2$ on Δ^* .

Suppose $d\sigma^2$ has elliptic monodromy. Since $d\sigma^2$ has no singular points on Δ^* , $|g| < 1$ or $|g| > 1$ holds on Δ^* . Since $1/g$ is also a developing map of $d\sigma^2$, we may assume that $|g| < 1$. By Proposition 2.1, there exists a real number μ such that $h(z) := z^{-\mu}g(z)$ is a single-valued function on Δ^* . Multiplying $h(z)$ by z^k ($k \in \mathbf{Z}$), we may assume that $-1 < \mu \leq 0$ without loss of generality. Thus

$$|h(z)| = |z|^{-\mu}|g(z)| < |z|^{-\mu} < 1,$$

and $h(z)$ has more than two exceptional values, so has at most a pole at $z = 0$, by the Great Picard theorem. Then by (1) (ii) in Proposition 2.1, $S(g)$ has a regular singularity at the origin.

Suppose $d\sigma^2$ has parabolic monodromy. Applying Proposition 2.1 for the $\text{PSU}_{1,1}$ -projective connection $S(d\sigma^2)$ with $\varepsilon = -1$ and $t = \pi$, we can take a $\text{PSU}_{1,1}$ -lift g such that

$$h = \hat{g} + i \log z \quad \left(\hat{g}(z) := R \star g(z) = \frac{1}{i} \frac{g(z) + 1}{g(z) - 1} \right)$$

is a single-valued meromorphic function on Δ^* , where R is the matrix in (2.2). Since $d\sigma^2$ has no singular points on Δ^* , $|g| > 1$ or $|g| < 1$ holds. In particular, because $z \mapsto R \star z$ maps the unit disk into the upper-half plane, we have $\text{Im } \hat{g} > 0$ (resp. $\text{Im } \hat{g} < 0$) if $|g| < 1$ (resp. $|g| > 1$). Here, it holds that

$$|z \exp(ih)| = |\exp(i\hat{g})| = \exp(-\text{Im } \hat{g}).$$

Thus,

$$\begin{aligned} |z \exp(ih)| &= \exp(-\text{Im } \hat{g}) < 1 \quad (\text{if } |g| < 1), \\ \left| \frac{1}{z} \exp(-ih) \right| &= \exp(\text{Im } \hat{g}) < 1 \quad (\text{if } |g| > 1). \end{aligned}$$

Thus by the Great Picard theorem, there exist an integer m and a holomorphic function $\varphi(z)$ with $\varphi(0) \neq 0$ such that $\exp(\pm ih(z)) = z^m \varphi(z)$, that is,

$$\pm ih(z) = m \log z + \log \varphi(z).$$

Since $h(z)$ is single valued, m must be 0. Therefore, $h(z)$ can be extended to be holomorphic at $z = 0$, and then by (2)(ii) of Proposition 2.1, the origin is a regular singularity of $S(d\sigma^2)$.

To prove the hyperbolic case, we need the following

Fact 2.1 (Montel’s theorem). *If a family of holomorphic functions $\{f_n\}_{n=1,2,3,\dots}$ defined on a domain $D(\subset \mathbf{C})$ have two exceptional values in common, then they are a normal family, that is, there is a subsequence $\{f_{n_j}\}_{j=1,2,3,\dots}$ such that either $\{f_{n_j}\}_{j=1,2,3,\dots}$ or $\{1/f_{n_j}\}_{j=1,2,3,\dots}$ converges uniformly on every compact set in D .*

Proof of Theorem 2.1, continued. The proof for the hyperbolic case is parallel to the proofs of Propositions 4 and 5 in [3].

Suppose $d\sigma^2$ has hyperbolic monodromy. Again, we may assume that $|g| < 1$ without loss of generality. By Proposition 2.1, again replacing g by $A \star g$ for some $A \in \text{SU}_{1,1}$ if necessary, there exists a positive real number μ such that $h(z) := z^{-i\mu}(R \star g(z))$ is a single-valued meromorphic function on Δ^* . The function

$$\hat{g}(z) := R \star g(z) = \frac{1}{i} \frac{g(z) + 1}{g(z) - 1}$$

has neither zeros nor poles in Δ^* , and $\text{Im } \hat{g} > 0$. We now define a set

$$\Omega := \left\{ z \in \mathbf{C} ; 0 < |z| < 1, |\arg z| < \frac{2\pi}{3} \right\}$$

and analytic functions ζ and f_n for $n = 1, 2, 3, \dots$ from Ω to \mathbf{C} by $\zeta(z) := \hat{g}(z^2)$ and $f_n(z) := \hat{g}(z^2/2^{2n})$. Then $\{f_n\}_{n=1}^\infty$ is a family of holomorphic functions on Ω . Since $\text{Im } \hat{g} > 0$, we have $\text{Im } f_n > 0$. Thus $\{f_n\}$ is a normal family by Montel’s theorem. Two possible cases arise.

Case 1. First we consider the case that a subsequence $\{f_n\}$ converges to a holomorphic function uniformly on any compact subset of Ω . Since

$$\begin{aligned} \Omega_l := & \left\{ z \in \mathbf{C} ; |z| = \frac{1}{2^l}, |\arg z| \leq \frac{3}{5}\pi \right\} \cup \left\{ z \in \mathbf{C} ; |z| = \frac{1}{2^{l+1}}, |\arg z| \leq \frac{3}{5}\pi \right\} \\ & \cup \left\{ z \in \mathbf{C} ; |\arg z| = \frac{3}{5}\pi, \frac{1}{2^{l+1}} \leq |z| \leq \frac{1}{2^l} \right\} \end{aligned}$$

for a positive integer $l \in \mathbf{Z}_+$ is a compact subset of Ω , there exist a positive number $M \in \mathbf{R}_+$ and a positive integer $n_0 \in \mathbf{Z}_+$ such that $|f_n(z)| < M$ holds

on Ω_1 for $n \geq n_0$. This implies that $|\zeta(z)| < M$ on Ω_{n+1} for $n \geq n_0$. Then by the maximum principle, we have

$$|\zeta(z)| < M \quad \text{on} \quad \left\{ z \in \mathbf{C}; \frac{1}{2^{n+1}} \leq |z| \leq \frac{1}{2^n}, |\arg z| \leq \frac{3}{5}\pi \right\}$$

for each $n > n_0 + 1$. Thus we have

$$|\hat{g}(z^2)| = |\zeta(z)| < M \quad \text{on} \quad \left\{ z \in \mathbf{C}; 0 < |z| \leq \frac{1}{2^{n_0+1}}, |\arg z| \leq \frac{3}{5}\pi \right\}.$$

On the other hand, since $e^{-\pi|\mu|} < |z^{-i\mu}| < e^{\pi|\mu|}$ for $|z| < 1$ and $|\arg z| < \pi$, the function $h(z)$ is bounded in a punctured neighborhood of $z = 0$ and has a removable singularity there.

Case 2. Next, we consider the case that a subsequence $\{1/f_n\}$ converges to a holomorphic function f . Then we can conclude that $1/h(z)$ is bounded on Δ^* . In this case $h(z)$ has at most a pole at the origin.

In both cases, $S(g) = S(R^{-1} \star (z^{i\mu} h(z)))$ has at most a pole of order two at $z = 0$. □

Remark 2.3. In Corollary 3.1, we shall show that in fact the monodromy of $d\sigma^2$ can never be hyperbolic.

3. Intrinsic behavior of regular ends

Let $f: M^2 \rightarrow S_1^3$ be a weakly complete CMC-1 face of finite topology, and let \overline{M}^2 be a compact Riemann surface such that M^2 is diffeomorphic to $\overline{M}^2 \setminus \{p_1, \dots, p_n\}$.

Definition 3.1. A puncture-type end p_j of f is called *regular* if the hyperbolic Gauss map G has at most a pole at p_j .

Definition 3.2. On the other hand, we say a puncture-type end p_j is *g-regular* if the Schwarzian derivative $S(g)$ of the secondary Gauss map g has at most a pole of order 2 at p_j , that is, the pseudometric $d\sigma^2$ in (1.15) has a regular singularity at p_j (cf. Definition 2.1).

When g is single-valued, g -regularity implies that g has at most a pole at the end. When the Hopf differential has at most a pole of order 2, regularity and g -regularity are equivalent, by (1.20).

Theorem 2.1 can now be stated in terms of CMC-1 faces as follows:

Lemma 3.1. *All ends of a complete CMC-1 face are g -regular.*

Proof. By Proposition 1.1, all ends are of puncture-type. So we can set $M^2 = \overline{M}^2 \setminus \{p_1, \dots, p_n\}$, where \overline{M}^2 is a compact Riemann surface. Let (g, ω) be a Weierstrass data for f . Since the singular set is compact, the metric $d\sigma^2$ as in (1.15) is a punctured hyperbolic metric in a punctured neighborhood of p_j . Then $d\sigma^2$ has a regular singularity by Theorem 2.1, and hence f is g -regular at p_j . \square

Definition 3.3. An elliptic end of a CMC-1 face is *integral* if the monodromy of the secondary Gauss map is the identity, and *non-integral* otherwise.

Lemma E1. *Let $f: \Delta^* \rightarrow S_1^3$ be a g -regular non-integral elliptic end. Then the singular set does not accumulate at the end 0.*

Proof. One can take the secondary Gauss map g to be $g(z) = z^\mu h(z)$ on a neighborhood of the end, where $\mu \in \mathbf{R} \setminus \mathbf{Z}$ and $h(z)$ is holomorphic at the end $z = 0$ with $h(0) \neq 0$. Since $\mu \neq 0$, the singular set $\{|g| = 1\}$ cannot accumulate at the origin. \square

On the other hand, an integral elliptic end might or might not be complete:

Example 3.1. For non-zero integers m and n with $|m| \neq |n|$, we set $g = 1 - z^m$ and $G = z^n$. Setting $Q = (S(g) - S(G))/2$ and $\omega = Q/dg$, we see that (1.5) gives a Riemannian metric on $\mathbf{C} \setminus \{0\}$. So using Small's formula (1.10), we have a CMC-1 face defined on $\mathbf{C} \setminus \{0\}$ with integral elliptic ends at $z = 0, \infty$. The singular set is

$$\{z \in \mathbf{C} \setminus \{0\}; |z|^{2m} - 2\operatorname{Re}(z^m) = 0\}$$

(see Figure 1 for the case $m = 3$). Thus the singular set accumulates at $z = 0$ but not at $z = \infty$. Thus $z = \infty$ is a complete integral elliptic end, but $z = 0$ is an incomplete integral elliptic end.

To state the behavior of an incomplete (integral) elliptic end, we introduce a notation: For an $m \in \mathbf{Z}_+$, an $\varepsilon \in (0, \pi/(2m))$ and a $\delta \in [0, \pi/m]$, we

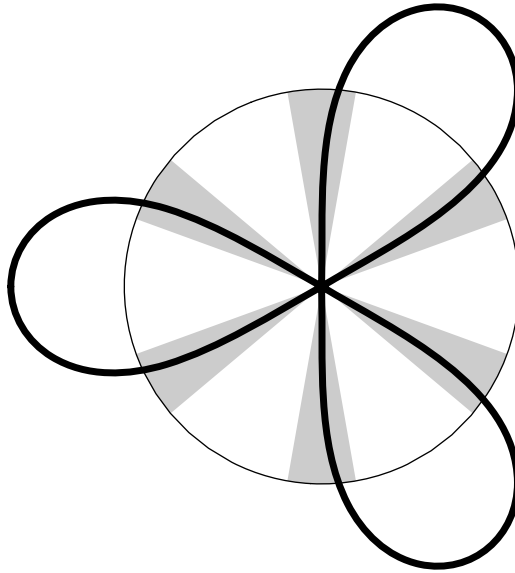


Figure 1: The thick curve is the singular set of Example 3.1 for the case $m = 3$. The shaded parts indicate the set $S(3, \varepsilon, \pi/6)$.

define the open subset (which is a union of sectors, see Figure 1)

$$S(m, \varepsilon, \delta) := \bigcup_{k=0}^{2m-1} \left\{ z \in \mathbf{C} \setminus \{0\}; \frac{k}{m}\pi + \delta - \varepsilon < \arg z < \frac{k}{m}\pi + \delta + \varepsilon \right\}.$$

Lemma E2. *Suppose $f : \Delta^* \rightarrow S_1^3$ is a g -regular integral elliptic end. If the singular set accumulates at the end, then there are an $m \in \mathbf{Z}_+$ and a $\delta \in \mathbf{R}$ such that, for any $\varepsilon > 0$, there exists an $r > 0$ so that the singular set of f in $\{z; 0 < |z| < r\}$ lies in $S(m, \varepsilon, \delta)$.*

Proof. The assertion of the lemma does not depend on a choice of the complex coordinate at the origin.

Since the singular set accumulates at 0, we have $|g(0)| = 1$. Then by Proposition 2.1(1), $g(z)$ is holomorphic at $z = 0$. Moreover, we may set $g(0) = 1$. Then $\varphi(z) := \log g(z)$ is well defined on a neighborhood of $z = 0$ and $\varphi(0) = 0$. Using the Weierstrass preparation theorem, we may further assume, without loss of generality, that $\varphi(z) = z^m$ for some positive integer m . Here, $|g(z)| = 1$ is equivalent to $\operatorname{Re} \varphi(z) = 0$. Thus, the singular set is expressed as $\{\cos m\theta = 0\}$, where $z = re^{i\theta}$. □

Definition 3.4. A parabolic end of a CMC-1 face is of the *first kind* if

$$S(d\sigma^2) - \frac{dz^2}{2z^2} = S(g) - \frac{dz^2}{2z^2} = S(G) + 2Q - \frac{dz^2}{2z^2}$$

has at most a pole of order 1. Otherwise, it is of the *second kind*.

Lemma P. *Let $f : \Delta^* \rightarrow S_1^3$ be a g -regular parabolic end. If the end is of the first kind, the singular set does not accumulate at the end. If the end is of the second kind, then the singular set does accumulate at the end. In this case, there exist an $m \in \mathbf{Z}_+$ and a δ ($\delta \in [0, \pi/m]$) such that, for all $\varepsilon > 0$, there exists an $r > 0$ so that the singular set of f in $\{z; 0 < |z| < r\}$ lies in $S(m, \varepsilon, \delta)$.*

Proof. Let g be the secondary Gauss map. Since the end is parabolic, the Schwarzian derivative $P := S(g)$ determines a $\text{PSU}_{1,1}$ -projective connection of parabolic monodromy. Then by (2) in Proposition 2.1 for $\varepsilon = -1$ and $t = \pi$, there exists a $\text{PSU}_{1,1}$ lift g_0 such that

$$h(z) = \hat{g}_0(z) + i \log z \quad \left(\hat{g}_0(z) = R \star g_0(z) = \frac{1}{i} \frac{g_0(z) + 1}{g_0(z) - 1} \right)$$

is a meromorphic function on Δ^* . Here, there exists a matrix $A \in \text{SU}_{1,1}$ such that $g = A \star g_0$ or $1/g = A \star g_0$ holds. Thus, by the $\text{SU}_{1,1}$ -ambiguity of the secondary Gauss map, we may assume $g = g_0$ or $1/g_0$. Moreover, replacing f with $-f$ if necessary (see Remark 1.9), we may assume $g = g_0$ without loss of generality.

Since the end is g -regular, (2)(ii) of Proposition 2.1 yields that $h(z)$ is meromorphic at $z = 0$. Thus, we can write

$$\hat{g}(z) = \hat{g}_0(z) = -i \log z + z^m \varphi(z) \quad (\varphi(0) \neq 0, m \in \mathbf{Z}),$$

where $\varphi(z)$ is a holomorphic function on a neighborhood of the origin. Then there exist an $a \in \mathbf{R} \setminus \{0\}$ and a $\gamma \in (-\pi, \pi)$ such that

$$\text{Im } \hat{g}(z) = -\log r + ar^m \sin(m\theta + \gamma) + o(r^{m+1}), \quad z = re^{i\theta}.$$

Here, the singular set $\{|g| = 1\}$ is written as $\{\text{Im } \hat{g} = 0\}$.

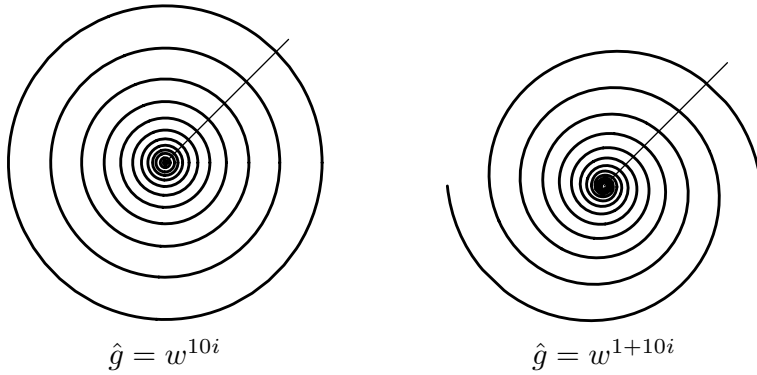


Figure 2: The thick curves indicate the singular sets of hyperbolic ends with secondary Gauss map $g = R^{-1} \star \hat{g}$, as discussed in Lemma H. The thin lines indicate rays in Δ^* emanating from the origin. Here we see that the singular sets intersect rays emanating from the origin infinitely many times.

If the end is of the first kind, then $m \geq 0$ by (2) (iii) in Proposition 2.1. Therefore, for each fixed θ , the right-hand side approaches ∞ as $r \rightarrow 0$, which implies that the singular points do not accumulate at the end.

If the end is of the second kind, then $m < 0$. Therefore, for each fixed θ , the right-hand side approaches ∞ if $a \sin(m\theta + \gamma) > 0$ and $-\infty$ if $a \sin(m\theta + \gamma) < 0$ as $r \rightarrow 0$, giving solutions of $\text{Im } \hat{g}(z) = 0$ for sufficiently small r near the lines $\sin(m\theta + \gamma) = 0$. This implies the second assertion. \square

Lemma H. *Let $f : \Delta^* \rightarrow S_1^3$ be a g -regular hyperbolic end. Then any ray in Δ^* emanating from the origin meets the singular set infinitely many times. (See Figure 2.)*

Remark 3.1. This intersection property does not depend on the choice of a complex coordinate for a punctured neighborhood of the end.

Proof of Lemma H. By Proposition 2.1 and an appropriate choice of g , if we set $\hat{g} = R \star g$, then there is a $\mu \in \mathbf{R} \setminus \{0\}$ such that

$$h(z) := z^{-i\mu} \hat{g}(z)$$

is a meromorphic function on Δ^* . Since f is g -regular, (3)(ii) of Proposition 2.1 implies h has at most a pole at the origin, and $S(g)$ has a pole of

order exactly 2 at $z = 0$. Thus we can rewrite

$$\hat{g}(z) = z^{m+i\mu}\varphi(z) \quad (\varphi(0) \neq 0, m \in \mathbf{Z}),$$

where $\varphi(z)$ is a single-valued holomorphic function on $\Delta = \Delta^* \cup \{0\}$. Now, we set

$$w = z \exp\left(\frac{\log \varphi}{m + i\mu}\right),$$

which gives a new coordinate w around the end, now at $w = 0$. Then $\hat{g}(w) = w^{m+i\mu}$. Since $g = (\hat{g} - i)/(\hat{g} + i)$, setting $w = re^{i\theta}$, the singular set is

$$\begin{aligned} \{w; |g(w)| = 1\} &= \{w; \text{Im}(\hat{g}(w)) = 0\} \\ &= \{(r, \theta) \in (0, 1) \times (-\pi, \pi); \mu \log r + m\theta \equiv 0 \pmod{\pi}\}, \end{aligned}$$

that is, $r = \exp((n\pi - m\theta)/\mu)$, $n \in \mathbf{Z}$, which is a log-spiral when $m \neq 0$. If $m = 0$, the singular set is a union of infinitely many disjoint circles. In any case, the singular set meets any ray based at $w = 0$ infinitely many times. (See Figure 2, left-hand side for the case $m = 0$ and right-hand side for the case $m \neq 0$.) □

Corollary 3.1. *The monodromy of a hyperbolic metric on Δ^* is either elliptic or parabolic. That is, hyperbolic monodromy never occurs.*

Proof. Suppose that a hyperbolic metric $d\sigma^2$ on Δ^* has hyperbolic monodromy. Let g be a developing map for $d\sigma^2$. The data $(g, \omega = dz)$ produces an F as in (1.6), and then the immersion $f = Fe_3F^* : \Delta^* \rightarrow S^3_1$ is without singularities, since $d\sigma^2$ is non-singular. By Theorem 2.1, f is g -regular at $z = 0$. Then by Lemma H, the singularities accumulate at the end, a contradiction. □

Lemmas E1, E2, P and H imply:

Corollary 3.2 (Characterization of hyperbolic ends). *A g -regular end $f : \Delta^* \rightarrow S^3_1$ of a CMC-1 face is hyperbolic if and only if every ray in Δ^* emanating from the origin meets the singular set infinitely many times.*

Completeness

We now give two theorems on complete CMC-1 faces.

Theorem 3.1. *Any complete end of a CMC-1 face is either g -regular elliptic or g -regular parabolic of the first kind.*

Proof. By Proposition 1.1, the end is of puncture-type. Moreover, Theorem 2.1 implies the end is g -regular. Thus the theorem follows from Corollary 3.1 and Lemma P. \square

Theorem 3.2. *Any complete CMC-1 face is of finite type. (The definition of finite type is given in Definition 1.4.)*

Proof. Let $f : M^2 \rightarrow S_1^3$ be a complete CMC-1 face. Then by Proposition 1.1, there is a compact Riemann surface \overline{M}^2 such that M^2 is biholomorphic to $\overline{M}^2 \setminus \{p_1, \dots, p_n\}$.

We fix any end p_j , and take a small coordinate neighborhood (U, z) with $z = 0$ at p_j . We may assume that there are no singular points on $U \setminus \{p_j\}$, and thus we may also assume that $|g| < 1$ on $U \setminus \{p_j\}$ for a secondary Gauss map g . We know from Theorem 3.1 that the end is a g -regular elliptic end or a g -regular parabolic end of the first kind.

First, we consider the elliptic case. Since $|g| < 1$, we may assume that there exist some $\mu (\geq 0)$ and a holomorphic function $h(z)$ on U with $h(0) \neq 0$ such that $g(z) = z^\mu h(z)$. If $|g(0)| = 1$, then $\mu = 0$ and

$$g(z) = e^{i\theta}(1 + az^m + o(z^m))$$

for some $\theta \in \mathbf{R}$, $a \in \mathbf{C} \setminus \{0\}$ and $m \in \mathbf{Z}_+$, which contradicts the fact that $|g| < 1$ on $U \setminus \{p_j\}$. Hence $|g(0)| < 1$. Therefore, there exist a neighborhood $\tilde{U} \subset U$ of p_j and an $\varepsilon > 0$ such that $|g|^2 < 1 - \varepsilon$ on \tilde{U} . So, on \tilde{U} ,

$$(3.1) \quad K_{ds^2} ds^2 = \frac{4|dg|^2}{(1 - |g|^2)^2} \leq \frac{4}{\varepsilon^2} \frac{4|dg|^2}{(1 + |g|^2)^2} = \frac{4}{\varepsilon^2} (-K_{d\hat{s}^2}) d\hat{s}^2,$$

where $d\hat{s}^2$ is the metric as in (1.5), which is defined on $\tilde{U} \setminus \{p_j\}$ because $g(z) = z^\mu h(z)$. Since p_j is a regular singularity of the punctured spherical metric

$$d\hat{\sigma}^2 := (-K_{d\hat{s}^2}) d\hat{s}^2 = \frac{4|dg|^2}{(1 + |g|^2)^2},$$

$d\hat{\sigma}^2$ has finite area, so ds^2 has finite total curvature on $\tilde{U} \setminus \{p_j\}$, by (3.1).

Next we consider the parabolic case. By Theorem 3.1, the end is parabolic of the first kind. Then without loss of generality, we may assume there exists a holomorphic function $h(z)$ on Δ such that (we set $t = \pi$ and replace $h(z)$)

by $ih(z)$ in the proof of Proposition 2.1)

$$(3.2) \quad \hat{g}(z) = i(h(z) \pm \log z), \quad \text{where} \quad \hat{g}(z) = R \star g(z) = \frac{1}{i} \frac{g(z) + 1}{g(z) - 1}.$$

If we set $k(z) := h(z) \pm \log z + 1$, we have $g = 1 - 2/k$, $g' = 2k'/k^2$ and

$$1 - |g(z)|^2 = \frac{4(\operatorname{Re} k(z) - 1)}{|k(z)|^2} = \frac{4(\operatorname{Re} h(z) \pm \operatorname{Re} \log z)}{|k(z)|^2}.$$

So we have

$$(3.3) \quad d\sigma^2 = \frac{4|dg|^2}{(1 - |g|^2)^2} = \frac{|h'(z) \pm (1/z)|^2 |dz|^2}{(\operatorname{Re} h(z) \pm \operatorname{Re} \log z)^2}.$$

We set $c := \sup_{z \in \Delta} |\operatorname{Re} h(z)|$ and $r = |z|$. Since $\log r \rightarrow -\infty$ as $z \rightarrow 0$, we may assume $-\log r > c$. Then

$$|\operatorname{Re} h(z) \pm \log r| \geq \left| |\operatorname{Re} h(z)| - |\log r| \right| \geq |c + \log r|, \quad \text{and}$$

$$(3.4) \quad d\sigma^2 \leq \frac{C^2}{r^2(c + \log r)^2} |dz|^2,$$

where $C = \sup_{z \in \Delta} |zh'(z) + 1|$. Since

$$\int_0^\varepsilon \frac{C^2 r \, dr}{r^2(c + \log r)^2} = -\frac{C^2}{c + \log \varepsilon} < \infty,$$

the area of a sufficiently small punctured neighborhood of $z = 0$ with respect to $d\sigma^2$ is finite, which proves the assertion. □

Theorem 0.1 in the introduction follows from Theorems 3.1 and 3.2.

4. The light-cone Gauss map and extrinsic behavior of ends

Let $\text{LC} = \{x \in \mathbf{R}_1^4; \langle x, x \rangle = 0\}$ be the light-cone of \mathbf{R}_1^4 , with future and past light cones

$$\text{LC}_\pm := \{x = (x_0, x_1, x_2, x_3) \in \text{LC}; \pm x_0 > 0\}.$$

The multiplicative group \mathbf{R}_+ of the positive real numbers acts on LC_\pm by scalar multiplication. The *ideal boundary* ∂S_1^3 of S_1^3 consists of two (future

and past) components

$$\partial_{\pm}S_1^3 := \text{LC}_{\pm}/\mathbf{R}_+,$$

each of which are identified with $\mathbf{C} \cup \{\infty\}$ by the projection

$$(4.1) \quad \pi: \partial_{\pm}S_1^3 \ni [(v_0, v_1, v_2, v_3)] \mapsto \frac{1}{v_0 - v_3}(v_1 + iv_2) \in \mathbf{C} \cup \{\infty\}.$$

The isometries of \mathbf{R}_1^4 induce Möbius transformations on $\mathbf{C} \cup \{\infty\}$. The boundary ∂S_1^3 is identified with the set of equivalence classes of oriented time-like geodesics in S_1^3 .

In particular, for a space-like immersion $f: M^2 \rightarrow S_1^3$ with the (time-like) unit normal vector field ν , the equivalence class $[f + \nu]$ determines a point in ∂S_1^3 for each $p \in M^2$. Hence we have the *light-cone Gauss map*

$$L = [f + \nu]: M^2 \rightarrow \partial S_1^3.$$

Let $f: M^2 \rightarrow S_1^3$ be a CMC-1 face, and $p \in M^2$ a regular point, that is, f is an immersion in a neighborhood of p . Under the identification of \mathbf{R}_1^4 and $\text{Herm}(2)$ as in (1.1), we can compute that the unit normal vector ν is

$$\nu = \frac{1}{|g|^2 - 1} F \begin{pmatrix} 1 + |g|^2 & 2g \\ 2\bar{g} & 1 + |g|^2 \end{pmatrix} F^*,$$

where F is the holomorphic lift of f and g is the secondary Gauss map. Hence

$$(4.2) \quad L = \left[\frac{2}{|g|^2 - 1} F \begin{pmatrix} |g|^2 & g \\ \bar{g} & 1 \end{pmatrix} F^* \right] = \text{sgn}(|g|^2 - 1) \left[F \begin{pmatrix} |g|^2 & g \\ \bar{g} & 1 \end{pmatrix} F^* \right],$$

where $\text{sgn}(|g|^2 - 1)$ is the sign of the function $|g|^2 - 1$. Thus, we have:

Proposition 4.1. *The light-cone Gauss map L of a CMC-1 face takes values in $\partial_+S_1^3$ (resp. $\partial_-S_1^3$) if $|g| > 1$ (resp. $|g| < 1$). Moreover, its projection $\pi \circ L$ is the hyperbolic Gauss map G as in (1.9), which extends to the singular set.*

Proof. By (4.2), the x_0 -component of $f + \nu$ is

$$\frac{1}{|g|^2 - 1} \text{trace} \left(F \begin{pmatrix} |g|^2 & g \\ \bar{g} & 1 \end{pmatrix} F^* \right) = \frac{1}{|g|^2 - 1} (|gF_{11} + F_{12}|^2 + |gF_{21} + F_{22}|^2),$$

where $F = (F_{ij})_{i,j=1,2}$. Here, $|gF_{11} + F_{12}|^2 + |gF_{21} + F_{22}|^2 > 0$ holds because $\det F = 1 \neq 0$, implying the first part of the proposition. By (4.2), (4.1), (1.6) and (1.9), we have

$$\pi \circ L = \pi \circ [f + \nu] = \frac{gF_{11} + F_{12}}{gF_{21} + F_{22}} = \frac{dF_{11}}{dF_{21}} = G,$$

and this completes the proof. □

Next, we give a criterion for when a complete regular end approaches $\partial_+ S_1^3$ or $\partial_- S_1^3$:

Proposition 4.2. *Let $f : \Delta^* \rightarrow S_1^3$ be a complete regular end at $z = 0$ and let g be a secondary Gauss map of f . Then the image of f converges to a point in $\partial_+ S_1^3$ (resp. $\partial_- S_1^3$) at the end if and only if $|g| < 1$ (resp. $|g| > 1$) near the end.*

Proof. We can change the holomorphic null lift F to F^\natural as in (1.14), so that f and its secondary Gauss map g change to $-f$ and $1/g$. The end of f approaches $\partial_\pm S_1^3$ if and only if $-f$ approaches $\partial_\mp S_1^3$, so it is sufficient to prove this result under the assumption $|g| < 1$ on Δ^* . By Theorem 3.1, the end is either elliptic or parabolic.

First we assume the end is elliptic. Replacing F by aFb^{-1} ($a \in \text{SL}_2 \mathbf{C}$, $b \in \text{SU}_{1,1}$) and using the Weierstrass preparation theorem if necessary, we may assume without loss of generality that the hyperbolic and secondary Gauss maps are

$$G(z) = z^m, \quad g(z) = z^\mu h(z) \quad \text{for some } m \in \mathbf{Z}_+, \mu \in \mathbf{R} \setminus \{0\},$$

where h is a holomorphic function on Δ with $h(0) \neq 0$. Here $\mu > 0$ because $|g| < 1$.

If $m \neq \mu$, Small's formula (1.10) implies that

$$F = \frac{1}{2\sqrt{m\mu}} \begin{pmatrix} -z^{\frac{m-\mu}{2}}(m+\mu)(1+o(1)) & z^{\frac{m+\mu}{2}}(m-\mu)(1+o(1)) \\ z^{\frac{-m-\mu}{2}}(m-\mu)(1+o(1)) & -z^{\frac{-m+\mu}{2}}(m+\mu)(1+o(1)) \end{pmatrix}.$$

Since m and μ are positive, the first component x_0 is

$$x_0 = \frac{1}{2} \text{trace}(Fe_3F^*) = \frac{(m-\mu)^2}{8m\mu} r^{-m-\mu}(1+o(1)) \rightarrow +\infty \quad (r \rightarrow 0),$$

where $z = re^{i\theta}$. Other components of $f = (x_0, x_1, x_2, x_3)$ are expressed as

$$\begin{aligned} x_1 + ix_2 &= e^{im\theta} r^{-\mu} \frac{\mu^2 - m^2}{4\mu m} (1 + o(1)), \\ x_3 &= -\frac{(m - \mu)^2}{8m\mu} r^{-m-\mu} (1 + o(1)). \end{aligned}$$

We now consider the stereographic projection given in [7]:

$$(4.3) \quad \Pi : \{(x_0, x_1, x_2, x_3) \in S_1^3; x_0 > 1\} \ni (x_0, x_1, x_2, x_3) \mapsto \frac{1}{1+x_0} (x_1, x_2, x_3) \in \left\{ (X_1, X_2, X_3) \in \mathbf{R}^3; \frac{1}{2} < \sum_{j=1}^3 (X_j)^2 < 1 \right\},$$

which is a diffeomorphism. Then $\Pi \circ f$ is expressed as

$$\Pi \circ f = (0, 0, -1) + o(1).$$

Thus, $\Pi \circ f$ approaches $(0, 0, -1) \in S^2 = \partial_+ S_1^3$.

When $\mu = m$, by (1.10) again, F_{11} , F_{12} and F_{22} are bounded on a neighborhood of 0, and these components can be extended to become holomorphic on a neighborhood of 0. If F_{21} is bounded, F must be holomorphic and then the induced metric is bounded, which contradicts the weak completeness of the end. Hence F_{21} has a pole at 0. So the x_0 -component of f is

$$x_0 = \frac{1}{2} |F_{21}|^2 + (\text{a bounded function}) \rightarrow +\infty \quad (z \rightarrow 0).$$

Moreover, since

$$\begin{aligned} x_3 &= -\frac{1}{2} |F_{21}|^2 + (\text{a bounded function}), \\ x_1 + ix_2 &= F_{11} \overline{F_{21}} - F_{12} \overline{F_{22}} = c \overline{F_{21}} + (\text{a bounded function}), \end{aligned}$$

where $c \in \mathbf{C} \setminus \{0\}$ is a constant, we have $\Pi \circ f \rightarrow (0, 0, -1)$ as $z \rightarrow 0$.

Next we assume the end is parabolic. Again we may set $G = z^m$, $m \in \mathbf{Z}_+$. Applying Proposition 2.1 for $t = 2m\pi$ and $\varepsilon = -1$, there exists a $\text{PSU}_{1,1}$ -lift g_0 of $S(g)$ such that $h(z) := R \star g_0(z) + 2mi \log z$ is a single-valued meromorphic function on Δ^* , and the secondary Gauss map g satisfies $g = A \star g_0$ or $1/g = A \star g_0$ for some $A \in \text{SU}_{1,1}$. By completeness, Lemma P implies that the end is of first kind. Hence by (2)(iii) of Proposition 2.1, $h(z)$ is holomorphic on a neighborhood of the origin. Moreover, by the assumption $|g| < 1$,

(2)(iv) of Proposition 2.1 yields that $g = A \star g_0$ for some $A \in \text{SU}_{1,1}$. Thus, without loss of generality, we may set

$$R \star g(z) = 2mi(k(z) - \log z),$$

here we set $h(z) = 2mik(z)$.

For a holomorphic null lift F of f with the secondary Gauss map g , set

$$\hat{F} = FB^{-1}, \quad \text{where } B := \left(\begin{pmatrix} 1 & 0 \\ 0 & 2i \end{pmatrix} R \right) = \begin{pmatrix} 1/2 & 1/2 \\ -1 & 1 \end{pmatrix} \in \text{SL}_2 \mathbf{C}.$$

Then \hat{F} is a holomorphic null immersion whose hyperbolic Gauss map \hat{G} and secondary Gauss map \hat{g} are given by

$$(4.4) \quad \begin{aligned} \hat{G} &= G = z^m, \\ \hat{g}(z) &= B \star g(z) = \begin{pmatrix} 1 & 0 \\ 0 & 2i \end{pmatrix} \star (R \star g(z)) = m(k(z) - \log z). \end{aligned}$$

So applying (1.10) for this (\hat{G}, \hat{g}) , the components of \hat{F} are written as

$$(4.5) \quad \begin{aligned} \hat{F}_{11}(z) &= -\frac{i}{2}z^{m/2}\varphi_1(z), & \hat{F}_{12}(z) &= -\frac{i}{2}z^{m/2}(m\varphi_1(z)\log z + \psi_1(z)), \\ \hat{F}_{21}(z) &= \frac{i}{2}z^{-m/2}\varphi_2(z), & \hat{F}_{22}(z) &= \frac{i}{2}z^{-m/2}(m\varphi_2(z)\log z + \psi_2(z)), \end{aligned}$$

where $\varphi_1, \varphi_2, \psi_1$ and ψ_2 are holomorphic functions defined on a neighborhood of the origin such that

$$\varphi_1(0) = \varphi_2(0) = 1.$$

Since $Be_3B^* = -e_1$, $f = Fe_3F^*$ satisfies

$$(4.6) \quad f = -\hat{F} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \hat{F}^* = - \begin{pmatrix} \hat{F}_{11}\overline{\hat{F}_{12}} + \hat{F}_{12}\overline{\hat{F}_{11}} & \hat{F}_{11}\overline{\hat{F}_{22}} + \hat{F}_{12}\overline{\hat{F}_{21}} \\ \hat{F}_{11}\overline{\hat{F}_{22}} + \hat{F}_{12}\overline{\hat{F}_{21}} & \hat{F}_{21}\overline{\hat{F}_{22}} + \hat{F}_{22}\overline{\hat{F}_{21}} \end{pmatrix}.$$

Hence the components of f are expressed as

$$(4.7) \quad \begin{aligned} x_0 &= \frac{m}{4}r^{-m}(\eta_1(u, v)\log r + \delta_1(u, v)), \\ x_3 &= \frac{m}{4}r^{-m}(-\eta_2(u, v)\log r + \delta_2(u, v)), \\ x_1 + ix_2 &= -\frac{m}{2}e^{im\theta}(\eta_3(u, v)\log r + \delta_3(u, v)), \end{aligned}$$

where $z = re^{i\theta} = u + iv$. Here, $\eta_j(u, v)$ ($j = 1, 2$) and $\delta_j(u, v)$ ($j = 1, 2$) (resp. $\eta_3(u, v)$ and $\delta_3(u, v)$) are real-valued (resp. complex-valued) differentiable functions defined on a neighborhood of the origin, such that $\eta_j(0, 0) = 1$ ($j = 1, 2, 3$).

Equations (4.7) yield that $x_0 \rightarrow +\infty$ and $\Pi \circ f \rightarrow (0, 0, -1)$ as $z \rightarrow 0$. \square

5. The Osserman-type inequality

Here we prove Theorem 0.2 stated in the introduction. First we prepare:

Lemma 5.1. *The Hopf differential of a CMC-1 face has a pole of order 2 at any complete regular parabolic end.*

Proof. Let $f : \Delta^* \rightarrow S_1^3$ be a complete regular parabolic end at $z = 0$. By Theorem 3.1, the end is of the first kind. Then

$$2Q + S_z(G) dz^2 = S_z(g) dz^2 = \frac{1}{z^2} \left(\frac{1}{2} + o(1) \right) dz^2.$$

Since G is meromorphic at $z = 0$, we may assume that $G = z^m \varphi(z)$, where m is a positive integer and $\varphi(z)$ is a holomorphic function on a neighborhood of 0 with $\varphi(0) \neq 0$. Applying (1.19) to $S_z(G)$, it follows that Q has a pole of order 2 at $z = 0$. \square

The next lemma improves a result in [7, Proposition 4.4]:

Lemma 5.2. *Let $f : \Delta^* \rightarrow S_1^3$ be a complete regular end at $z = 0$ of a CMC-1 face with Hopf differential Q and hyperbolic Gauss map G . Then the ramification order m of $G(z)$ at $z = 0$ satisfies*

$$(5.1) \quad m \geq \text{Ord}_{z=0}(Q) + 3.$$

(For the definition of the ramification order, see the subsection about the Schwarzian derivative in Section 1.) Here, $\text{Ord}_{z=0} Q$ denotes the order of Q at the origin, that is, $\text{Ord}_{z=0} Q = k$ if $Q = z^k \varphi(z) dz^2$, where $\varphi(z)$ is holomorphic at $z = 0$ and $\varphi(0) \neq 0$.

Proof. By Theorem 3.1, a complete end is either elliptic or parabolic of the first kind. The elliptic case has been proved in [7]. Assume that the end is parabolic. Then by Lemma 5.1, Q must have a pole of order 2 at $z = 0$, which proves the inequality since $m \geq 1$. \square

It should be remarked that the order of the metric $d\sigma_{\#}^2 = 4|dG|^2/(1 + |G|^2)^2$ at 0 is equal to $m - 1$, where m is the ramification order of G . Using Lemma 5.2 instead of [7, Proposition 4.4], the inequality in Theorem 0.2 is proved in the same way as [7, 23].

The condition for equality in (*) in Theorem 0.2 for elliptic ends was completely analyzed in [7]. So, it suffices to show the following theorem for parabolic ends. Note that $\text{Ord}_p(Q) = -2$ for complete regular parabolic ends, hence the equality in (5.1) holds if and only if G does not branch at p (see [7] for details).

Theorem 5.1. *A complete regular parabolic end of a CMC-1 face is properly embedded if and only if the hyperbolic Gauss map G does not branch at the end.*

Proof. Let $f : \Delta^* \rightarrow S_1^3$ be a complete regular parabolic end at $z = 0$. Taking $-f$ instead of f if necessary, we may assume that $|g| < 1$ in a neighborhood of the end, and that $G(z) = z^m, m \geq 1$ and $g(z) = R^{-1} \star (2mi(k(z) - \log z))$, as in the proof of Proposition 4.2. Then f is represented as in (4.5) and (4.6).

By Proposition 4.2, the image of f tends to a point in $\partial_+ S_1^3$. So we may assume that $x_0 > 1$ on Δ^* , and

$$\Pi \circ f : \Delta^* \ni z \longmapsto (X_1, X_2, X_3) \in \mathbf{R}^3$$

is well defined, where Π is the projection in (4.3).

Here, by (4.7),

$$U(z) := z^{-m} \frac{x_1 + ix_2}{1 + x_0} (= z^{-m}(X_1 + iX_2)) = -2 \frac{\eta_3 \log r + \delta_3}{\eta_1 \log r + \delta_1 + (4/m)r^m}.$$

Since η_1, η_3 and δ_1, δ_3 are differentiable functions defined on a neighborhood of 0, we have

$$(5.2) \quad \lim_{z \rightarrow 0} U(z) = -2 \neq 0, \quad \lim_{z \rightarrow 0} z \frac{\partial}{\partial z} U(z) = 0 \quad \text{and} \quad \lim_{z \rightarrow 0} z \frac{\partial}{\partial \bar{z}} U(z) = 0.$$

Now we suppose that the ramification order m of the hyperbolic Gauss map at $z = 0$ is 1, that is $m = 1$. As seen in the proof of Proposition 4.2, $\Pi \circ f$ converges to $(0, 0, -1)$. Then $X_1 + iX_2 = zU(z)$ and (5.2) yield that

$$\lim_{z \rightarrow 0} \frac{\partial}{\partial z} (X_1 + iX_2) \neq 0 \quad \text{and} \quad \lim_{z \rightarrow 0} \frac{\partial}{\partial \bar{z}} (X_1 + iX_2) = 0,$$

which implies that the correspondence $z \mapsto X_1 + iX_2$ is bijective near the origin, and the end is properly embedded.

Conversely, suppose that the end is properly embedded. We have already seen that $X_3 \rightarrow -1$ as $z \rightarrow 0$. Moreover $U(0) \neq 0$ implies that for any sufficiently small $\varepsilon > 0$, the image of the end $f(\{z; 0 < |z| < \varepsilon\})$ does not meet the X_3 -axis and is diffeomorphic to a cylinder. Then the image of $\Pi \circ f(\{z; |z| = \varepsilon\})$ by the orthogonal projection $(X_1, X_2, X_3) \mapsto X_1 + iX_2$ is an embedded closed curve with the winding number m with respect to the origin. So $m = 1$. □

Remark 5.1. In Proposition 4.4 of [7], the first author showed the equality condition in Theorem 0.2 for elliptic ends using the expression of the solution of the ordinary differential equation (1.6). Here we proved the equality condition in Theorem 0.2 for parabolic ends by using Small’s formula (1.10). It is also possible to prove the result in [7] more directly by using (1.10).

We give here three important examples:

Example 5.1 (An incomplete 3-noid not satisfying (*) in Theorem 0.2). We set $M^2 := \mathbb{C} \setminus \{0, 1\}$ and

$$G := z, \quad g := \frac{2z - 1}{2z(z - 1)} - \log \frac{z}{z - 1}.$$

Then (1.10) gives a CMC-1 face $f : M^2 \rightarrow S_1^3$ with hyperbolic and secondary Gauss maps G and g , and Hopf differential

$$Q = \frac{1}{2}(S(g) - S(G)) = -\frac{2dz^2}{z(z - 1)}.$$

Since the lift metric

$$ds_{\#}^2 = \frac{4(1 + |z|^2)^2}{|z(z - 1)|^2} |dz|^2$$

is complete on M^2 , f is weakly complete. The end $z = \infty$ is complete and elliptic, and $z = 0, 1$ are parabolic ends of the second kind. Hence $z = 0, 1$ are incomplete ends. Since $\deg(G) = 1$, f does not satisfy (*). This implies that completeness is an essential assumption in Theorem 0.2 in the introduction.

Example 5.2 (A 2-noid with complete parabolic ends satisfying the equality in (*)). We set

$$(5.3) \quad F(z) = \frac{i}{2\sqrt{2}} \begin{pmatrix} \sqrt{z} & 0 \\ 0 & \sqrt{z}^{-1} \end{pmatrix} \begin{pmatrix} 3 - \log z & -1 + \log z \\ 1 + \log z & -3 - \log z \end{pmatrix}.$$

Then $f = Fe_3F^* : \mathbf{C} \setminus \{0\} \rightarrow S_1^3$ has two parabolic regular ends. The hyperbolic Gauss map G , the secondary Gauss map g and the Hopf differential Q are computed as follows:

$$G = z, \quad g = \frac{\log z + 1}{\log z - 1}, \quad Q = \frac{dz^2}{4z^2}.$$

Since $\{z \in \mathbf{C}; |g(z)| = 1\} = \{z \in \mathbf{C}; |z| = 1\}$, the singular set is compact, and hence f is complete.

Any genus zero CMC-1 face with two parabolic regular ends and with degree 1 hyperbolic Gauss map is congruent to this f . We call this CMC-1 face the *parabolic catenoid*. On the other hand, the CMC-1 face with $G = z$, $g = z^\mu$ ($\mu \in \mathbf{R} \setminus \{0\}$) given in [7, Example 5.4] is called the *elliptic catenoid*.

Example 5.3 (A complete 4-noid with 4 integral elliptic ends satisfying the equality in (*)). Since $SL_2 \mathbf{C}$ can be identified with the complex hyperquadric Q^3 of \mathbf{C}^4 , the null (meromorphic) curves in $SL_2 \mathbf{C}$ can be identified with those in Q^3 . The null curve in $SL_2 \mathbf{C}$ with

$$G := \frac{3(z^3 + 2)}{4 - z}, \quad g := -\frac{z^3 - 12z^2 + 2}{3z}$$

belongs to the moduli space \mathcal{M}_4 in the classification list of null curves in Q^3 in Bryant [4], which has four integral elliptic ends at the roots of $1 + 6z^2 - z^3$ and $z = \infty$. Since G is of degree 3 and $\chi(\mathbf{C} \cup \{\infty\}) = 2$, the corresponding CMC-1 face attains equality in (*) of Theorem 0.2. (For the definition of an integral elliptic end, see Definition 3.3.)

Remark 5.2. We can deform an elliptic catenoid to a parabolic catenoid. Let f_μ be an elliptic catenoid with the hyperbolic Gauss map $G = z$ and the secondary Gauss map $g = z^\mu$, where $\mu > 0$. Then the hyperbolic metric corresponding to f_μ is

$$d\sigma_\mu^2 = \frac{4|dg|^2}{(1 - |g|^2)^2} = \frac{4\mu^2|z|^{2\mu-2}}{(1 - |z|^{2\mu})^2}|dz|^2.$$

It can be easily checked that

$$\lim_{\mu \rightarrow 0} d\sigma_\mu^2 = \frac{|dz|^2}{(r \log r)^2}, \quad \text{where } z = re^{i\theta},$$

which is the hyperbolic metric of a parabolic catenoid with

$$g(z) = R \star \log z = \frac{1 \log z + 1}{i \log z - 1},$$

see (3.3). On the other hand, by Small's formula (1.10) there exists a unique smooth 1-parameter family of CMC-1 faces \tilde{f}_μ ($\mu \geq 0$) with hyperbolic Gauss map $G = z$ and associated hyperbolic metric $d\sigma_\mu^2$. Then \tilde{f}_μ is congruent to f_μ , and \tilde{f}_0 gives a parabolic catenoid.

Remark 5.3. As a consequence of Remark 1.15, we know that there are no compact CMC-1 immersed surfaces in S_1^3 . Here we give an alternative proof of this: Let M^2 be a compact Riemann surface without boundary, and suppose there exists a compact CMC-1 face $f : M^2 \rightarrow S_1^3$ which has no singular points. Let F be a holomorphic lift of f . We may assume that $|g| < 1$ since there are no singular points. Then, by (1.6), we have

$$f_{z\bar{z}} = (1 - |g|^2)F \begin{pmatrix} g \\ 1 \end{pmatrix} (\bar{g} \quad 1) F^* |\hat{\omega}|^2,$$

where z is a local complex coordinate and $\omega = \hat{\omega} dz$. Thus, $\text{trace } f$ is a non-constant subharmonic function, which is a contradiction to the maximum principle.

This proof does not apply to compact CMC-1 faces, leading us to the following open problem:

Problem. Is there a compact CMC-1 face?

If such a CMC-1 face exists, the genus γ must be greater than or equal to 3, since equality in (*) in the introduction holds in this case and the degree of the hyperbolic Gauss map must be $\gamma - 1$.

Appendix A. Meromorphicity of the Hopf differential

In this appendix, we shall give a proof of the following

Theorem A.1. *Let \overline{M}^2 be a compact Riemann surface. Then the Hopf differential Q of a complete CMC-1 face*

$$f: \overline{M}^2 \setminus \{p_1, \dots, p_n\} \longrightarrow S_1^3$$

is meromorphic on \overline{M}^2 .

Proof. It is sufficient to show the meromorphicity of Q at a complete end $f: \Delta^* = \{z; 0 < |z| < 1\} \rightarrow S_1^3$ at the origin. We write the Hopf differential Q as

$$Q = \hat{Q} dz^2,$$

where \hat{Q} is a holomorphic function on Δ^* . By Theorem 3.1, a complete end $f: \Delta^* \rightarrow S_1^3$ is either a g -regular elliptic end or a g -regular parabolic end of the first kind. (The definition of g -regularity is given in Definition 3.2.)

First, we consider the case that f is elliptic. By g -regularity, the secondary Gauss map is written in the form

$$g = z^\mu h(z) \quad (h \text{ is a holomorphic function with } h(0) \neq 0),$$

where μ is a real number. Since $|g(0)| \neq 1$ by completeness, we may set $g(0) = 0$, or ∞ , because of the $SU_{1,1}$ -ambiguity of g . Moreover, replacing f by $-f$ if necessary, we may assume $\mu > 0$ without loss of generality. In this case, the corresponding hyperbolic metric $d\sigma^2$ is written as

$$d\sigma^2 = \left(\frac{2|z|^{\mu-1} |\mu h(z) + zh'(z)|}{|1 - |z|^{2\mu} |h(z)|^2} |dz| \right)^2, \quad \left(' = \frac{d}{dz} \right).$$

Since $|z|^\mu |h(z)|$ and $zh'(z)$ tend to 0 as $z \rightarrow 0$ and $h(z)$ is bounded near the origin, we have that

$$d\sigma \geq c|z|^{\mu-1} |dz| \geq c|z|^l |dz|$$

holds on a neighborhood of the origin, where l is the smallest integer such that $l \geq \mu - 1$ and c is a positive constant. Then, by (1.16), we have

$$ds = 2 \frac{|Q|}{d\sigma} \leq 2 \frac{|\hat{Q}|}{c|z|^l} |dz| = \frac{2}{c} \left| \frac{\hat{Q}}{z^l} dz \right|.$$

Since ds is complete at 0, we have meromorphicity of the one-form $z^{-l} \hat{Q} dz$ at the origin, because of [16, Lemma 9.6, page 83].

Next, we consider the case that f is parabolic. Since the end is g -regular parabolic of the first kind, one can choose the secondary Gauss map g as in (3.2):

$$g = R^{-1}\hat{g}, \quad \hat{g}(z) = i(h(z) \pm \log z),$$

where $h(z)$ is a holomorphic function on $\Delta := \Delta^* \cup \{0\}$. Hence $d\sigma^2$ is written as in (3.3):

$$d\sigma^2 = \left(\frac{|h'(z) \pm (1/z)|}{|\operatorname{Re} h(z) \pm \log |z||} |dz| \right)^2 \quad \left(' = \frac{d}{dz} \right).$$

Since $z \log |z| \rightarrow 0$ as $z \rightarrow 0$ and h is bounded on a neighborhood of the origin,

$$d\sigma = \frac{|1 \pm zh'(z)|}{|z \log |z|| \left| 1 \pm \frac{\operatorname{Re} h(z)}{\log |z|} \right|} |dz| \geq c|dz|$$

holds on a neighborhood of the origin, where c is a positive constant. Thus,

$$ds = 2 \frac{|Q|}{d\sigma} \leq \frac{2}{c} |\hat{Q} dz|.$$

Hence, by the same argument as in the elliptic case, we have meromorphicity of Q at the origin. □

Appendix B. Conjugacy classes of $SU_{1,1}$

The Lie group $SU_{1,1}$ is the set of matrices $S \in SL_2 \mathbf{C}$ satisfying $Se_3S^* = e_3$. Two matrices $A, B \in SU_{1,1}$ are called *conjugate in $SL_2 \mathbf{C}$* if there exists a matrix $P \in SL_2 \mathbf{C}$ such that $B = P^{-1}AP$, and are called *conjugate in $SU_{1,1}$* if $B = P^{-1}AP$ for some $P \in SU_{1,1}$.

As in (1.18) and (2.2), we set

$$\begin{aligned} \Lambda_e(t) &:= \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}, & \Lambda_p(t) &:= \begin{pmatrix} 1+it & -it \\ it & 1-it \end{pmatrix}, \\ \Lambda_h(t) &:= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} & \text{and } R &:= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \end{aligned}$$

for an arbitrary $t \in \mathbf{R}$.

Theorem B.1. *A matrix $A \in \text{SU}_{1,1}$ is conjugate in $\text{SU}_{1,1}$ to one of*

- 1) $\Lambda_e(s)$ ($s \in (-\pi, \pi]$),
- 2) $\pm\Lambda_p(t)$ or $\pm\Lambda_p(-t)$ ($t > 0$), or
- 3) $\pm\Lambda_h(t)$ ($t > 0$).

Remark B.1. Though the matrices $\Lambda_e(s)$ and $\Lambda_e(-s)$ are conjugate in $\text{SL}_2 \mathbf{C}$, they are not conjugate in $\text{SU}_{1,1}$ if $s \not\equiv 0 \pmod{2\pi}$. That is, for any elliptic matrix $A \in \text{SU}_{1,1}$, there exists a *unique* real number $t \in (-\pi, \pi]$ such that A and $\Lambda_e(s)$ are conjugate in $\text{SU}_{1,1}$.

Remark B.2. On the other hand, $\Lambda_p(t_1)$ and $\Lambda_p(t_2)$ ($t_1, t_2 \neq 0$) are conjugate in $\text{SU}_{1,1}$ if and only if $t_1 t_2 > 0$. In fact, if $t_1 \neq t_2$, $P\Lambda_p(t_1)P^{-1} = \Lambda_p(t_2)$ ($P \in \text{SU}_{1,1}$) holds if and only if

$$P = \pm \begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix}, \quad a = \cosh s + iu, \quad b = \sinh s + iu,$$

where $s = \log \sqrt{t_2/t_1} \in \mathbf{R}$ and $u \in \mathbf{R}$.

In particular, the sign of t in $\Lambda_p(t)$ is invariant under such a conjugation. Though $\Lambda_p(t)$ ($t \in \mathbf{R} \setminus \{0\}$) is conjugate with $\Lambda_p(1)$ or $\Lambda_p(-1)$, we choose various values of t in this paper for the sake of convenience.

Remark B.3. Since

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} \cosh t & -\sinh t \\ -\sinh t & \cosh t \end{pmatrix}$$

$\Lambda_h(t)$ and $\Lambda_h(-t)$ are conjugate in $\text{SU}_{1,1}$.

To show Theorem B.1, we use the following group isomorphism:

$$\rho: \text{SL}_2 \mathbf{R} \ni X \mapsto R^{-1}XR \in \text{SU}_{1,1}.$$

Note that $\Lambda_e(t)$, $\Lambda_p(t)$ and $\Lambda_h(t)$ are the images of

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \quad \begin{pmatrix} 1 & 2t \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

respectively, by ρ .

Lemma B.1. *Let A and B be 2×2 real matrices that are conjugate in $\text{SL}_2 \mathbf{C}$. Then A and either B or $e_3 B e_3$ are conjugate in $\text{SL}_2 \mathbf{R}$.*

Proof. By assumption, there is a $\tilde{P} \in \mathrm{SL}_2 \mathbf{C}$ with $A\tilde{P} = \tilde{P}B$. We set $\tilde{P} = U + iV$, for real matrices U, V . Then $AU = UB, AV = VB$ and

$$A(U + tV) = (U + tV)B \quad \text{for any } t \in \mathbf{R}.$$

If $\det(U + tV)$ vanishes identically for all $t \in \mathbf{R}$, holomorphicity of $\mathbf{C} \ni t \mapsto \det(U + tV)$ yields that $\det(U + iV) = \det \tilde{P} = 0$, a contradiction. Thus for some $t_0 \in \mathbf{R}$, $\det(U + t_0V) \neq 0$, and then $(U + t_0V)^{-1}A(U + t_0V) = B$. If $\det(U + t_0V) > 0$, we set $P = (U + t_0V)/\sqrt{\det(U + t_0V)} \in \mathrm{SL}_2 \mathbf{R}$, giving $P^{-1}AP = B$. If $\det(U + t_0V) < 0$, we set

$$P = \frac{(U + t_0V)e_3}{\sqrt{|\det(U + t_0V)|}} \in \mathrm{SL}_2 \mathbf{R},$$

giving $P^{-1}AP = e_3Be_3$. □

Proof of Theorem B.1. Let $A \in \mathrm{SU}_{1,1}$ and $\tilde{A} := \rho^{-1}(A) \in \mathrm{SL}_2 \mathbf{R}$.

If the eigenvalues of A are not real numbers, they are written as $\{e^{it}, e^{-it}\}$, where $t \in (-\pi, 0) \cup (0, \pi)$. In this case, \tilde{A} is conjugate in $\mathrm{SL}_2 \mathbf{C}$ to $B_e := R\Lambda_e(t)R^{-1}$. Hence by Lemma B.1, \tilde{A} is conjugate in $\mathrm{SL}_2 \mathbf{R}$ to B_e or $e_3B_e e_3$. Thus, $A = \rho(\tilde{A})$ is conjugate in $\mathrm{SU}_{1,1}$ to $\rho(B_e) = \Lambda_e(t)$ or $\rho(e_3B_e e_3) = \Lambda_e(-t)$.

If the eigenvalues of A are $\{\varepsilon, \varepsilon\}$ ($\varepsilon = \{-1, 1\}$) and $A \neq \varepsilon \mathrm{id}$, \tilde{A} is conjugate in $\mathrm{SL}_2 \mathbf{C}$ to $B_p := \varepsilon R\Lambda_p(t)R^{-1}$ for any $t \in \mathbf{R}_+$. Hence \tilde{A} is conjugate in $\mathrm{SL}_2 \mathbf{R}$ to either B_p or $e_3B_p e_3$. Thus, A is conjugate in $\mathrm{SU}_{1,1}$ to $\rho(B_p) = \varepsilon\Lambda_p(t)$ or $\rho(e_3B_p e_3) = \varepsilon\Lambda_p(-t)$. As mentioned in Remark B.2, $\Lambda_p(u)$ for $u \in \mathbf{R} \setminus \{0\}$ is conjugate in $\mathrm{SU}_{1,1}$ to $\Lambda_p(\mathrm{sgn} u) = \Lambda_p(\varepsilon)$.

If the eigenvalues of A are two distinct real numbers, they are represented as $\{\varepsilon e^t, \varepsilon e^{-t}\}$, where $t \in \mathbf{R}_+$ and $\varepsilon \in \{-1, 1\}$. Thus, \tilde{A} is conjugate in $\mathrm{SL}_2 \mathbf{C}$ to the diagonal matrix $B_h := \varepsilon R\Lambda_h(t)R^{-1}$. Hence, similarly to the first case, A is conjugate in $\mathrm{SU}_{1,1}$ to $\rho(B_h) = \varepsilon\Lambda_h(t)$. □

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