Gluing constructions for asymptotically hyperbolic manifolds with constant scalar curvature

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We show that asymptotically hyperbolic initial data satisfying smallness conditions in dimensions $n \geq 3$, or fast decay conditions in $n \geq 5$, or a genericity condition in $n \geq 9$, can be deformed, by a deformation that is supported arbitrarily far in the asymptotic region, to ones that are exactly Kottler ("Schwarzschild- adS") in the asymptotic region.

1. Introduction

One of the key problems in mathematical general relativity is the understanding of the space of solutions of the vacuum constraint equations. In this context, an important gluing method has been introduced by Corvino and Schoen [12, 13] for vacuum data with vanishing cosmological constant. The object of this paper is to present related gluing results when the cosmological constant Λ is negative. The question we address is the possibility of deforming an asymptotically hyperbolic Riemannian manifold of constant scalar curvature, and hence a time-symmetric vacuum initial data set, to one with a Kottler metric (sometimes known as Schwarzschild — anti-de Sitter metric) outside of a compact set. We establish deformation or extension theorems in dimensions $n \geq 3$ under a smallness condition for metrics sufficiently close to (generalized) Kottler metrics, or under smallness and parity conditions for metrics close to a standard hyperbolic metric, or assuming a rapid decay condition in dimensions $n \geq 5$.

More precisely, we consider n-dimensional manifolds containing asymptotic ends

$$(1.1) \qquad M_{\text{ext}} := (r_0, \infty) \times N,$$

where N is a compact manifold. We are interested in constant scalar curvature metrics that asymptote, as r goes to infinity, to a background metric b of the form[1]

$$(1.2) \qquad b = \frac{dr^2}{r^2 + k} + r^2 \widehat{b},$$

where $k \in \{0, \pm 1\}$, and where $\widehat{b}$ is a (r-independent) metric on N satisfying $\mathrm{Ric}(\widehat{b}) = k(n-2)\widehat{b}$. A family of examples is provided by the (generalized) Kottler metrics,

$$(1.3) \qquad b_m = \frac{dr^2}{r^2 + k - \frac{2m}{r^{n-2}}} + r^2 \widehat{b}.$$

Note that $b_0 = b$, with b as in (1.2).

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[1] The constant k in (1.2)–(1.3) is of course unrelated to the order of differentiability k used elsewhere, we hope that this will not confuse the reader.

For the purpose of the next theorem define the manifold M to be

$$M = (r_0, r_2] \times N,$$

and suppose that g is a constant negative scalar curvature metric on M close to b, or to b_m. There are two natural questions:

First, choose r_1 satisfying $r_0 < r_1 < r_2$, can one deform g, keeping the scalar curvature fixed, so that the resulting metric coincides with g on $(r_0, r_1] \times N$, and with b_m, for some m, near $\{r_2\} \times N$? In this case we set $M' = (r_0, r_1] \times N$, $M'' = [r_2, \infty) \times N$, and we refer to this case as the *deformation problem*.

Next, let $r_3 > r_2$, can one extend g to a new metric of constant scalar curvature on $(r_0, \infty) \times N$ so that the extended metric coincides with b_m, for some m, on $[r_3, \infty) \times N$? In this case we set $M' = M$, $M'' = [r_3, \infty) \times N$, and we refer to this case as the *extension problem*. It is shown in [8, Section 8.6] how to reduce this problem to the deformation one.

Our aim here is to show that those problems can always be solved when g is sufficiently close to b, except perhaps when $(N, \widehat{b})$ is a round sphere and $m = 0$, in which case we need to impose a restrictive condition: For $(r, q) \in M$ let $\psi(r, q) = (r, \phi(q))$, where ϕ is the antipodal map of the sphere. A metric g on M will be said to be parity symmetric if $\psi^* g = g$. At the end of Section 5 we prove:

Theorem 1.1. *Let $n \geq 3$, $\mathbb{N} \ni \ell > \lfloor \frac{n}{2} \rfloor + 4$, $\lambda \in (0, 1)$, $m \in \mathbb{R}$. If $(N, \widehat{b})$ is a round sphere and $m = 0$, we suppose moreover that g is parity symmetric. There exists $\varepsilon > 0$ such that if $\|g - b_m\|_{C^{\ell, \lambda}(M)} < \varepsilon$, then there exists on M_{ext} a $C^{\ell, \lambda}$ metric of constant negative scalar curvature that coincides with g on M', and which is a Kottler metric on M''. If g is smooth, then so is the solution of the deformation problem.*

We emphasize that g and b_m are only required to be close to each other on an "annulus" as above, and in fact b_m is not even defined throughout the original manifold.

We can also prove a result without smallness assumptions which, however, excludes dimensions three and four. Moreover, the decay rates are undesirably restrictive in dimensions five, six and seven; they are satisfactory, but not as weak as one would wish, in higher dimensions.

Let g be an asymptotically hyperbolic metric as defined in Section 2, and let $p_{(\mu)}^0$ be the momentum vector of g, obtained by passing to the limit as r goes to infinity of the integral (5.1) below over submanifolds $r = \mathrm{const}$. Let

$b_{p_{(\mu)}}$ denote a (generalized, boosted) Kottler metric with momentum vector $p_{(\mu)}$. Suppose that

$$(1.4) \qquad n \geq 5, \quad \alpha > \begin{cases} 8, & n = 5, 6, 7, \\ \dfrac{8+n}{2}, & n \geq 8. \end{cases}$$

(This can be compared with the conditions $\alpha > n/2$, $n \geq 3$, needed for $p^0_{(\mu)}$ to be well defined, or $\alpha = n$, which holds for Kottler metrics.) For $\alpha \leq n$ we assume that g has the following asymptotic behaviour:

$$(1.5) \qquad |g - b|_b + |\mathring{D}(g-b)|_b + \ldots + |\mathring{D}^{(k+2)}(g-b)|_b = O(\rho^\alpha);$$

where $\mathring{D}$ is the covariant derivative operator of b. For $\alpha > n$, if $(N, \widehat{b})$ is a round sphere, we assume that the momentum vector $p^0_{(\mu)}$ of g is *timelike*,[2] so that an associated (perhaps boosted) Kottler metric $b_{p^0_{(\mu)}}$ exists. Whether or not $(N, \widehat{b})$ is a round sphere, for $\alpha > n$ instead of (1.5) we suppose that

$$(1.6) \qquad |g - b_{p^0_{(\mu)}}|_b + |\mathring{D}(g - b_{p^0_{(\mu)}})|_b + \ldots + |\mathring{D}^{(k+2)}(g - b_{p^0_{(\mu)}})|_b = O(\rho^\alpha);$$

in fact, (1.6) is equivalent to (1.5) if $\alpha \leq n$.

Letting M_δ be as in (2.1), and $A_{\delta,4\delta}$ as in (2.2), in Section 4 we prove:

Theorem 1.2. *Let $n \geq 5$, $\mathbb{N} \ni \ell > \lfloor \frac{n}{2} \rfloor + 4$, $\lambda \in (0,1)$, and let $\alpha > 0$ satisfy (1.4). Let g be a $C^{\ell,\lambda}$ asymptotically hyperbolic metric with constant negative scalar curvature satisfying (1.6) with $k = \ell - 4$. We furthermore assume that (1.6) holds with $k = \ell - 2$ and $\alpha = 0$, and that the energy-momentum vector is timelike if $(N, \widehat{b})$ is a round sphere. There exists $\delta_0 > 0$ such that for all $0 < \delta \leq \delta_0$ the metric g can be deformed across an annulus $A_{\delta,4\delta}$ to a constant scalar curvature metric, of $C^{\ell,\lambda}$ differentiability class, which coincides with g on $M \setminus M_{4\delta}$ and with a Kottler metric $b_{p_{(\mu)}}$ on M_δ. The solution is smooth if g is.*

A key role in our analysis is played by the kernel of the operator P^*_g given by (2.6) below; it is known that this kernel is trivial for any open subset of M for generic metrics [4]. Deforming first the metric as in Section 6, and applying Theorem 1.2 to the new metric one concludes:

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[2] Both past and future pointing $p^0_{(\mu)}$ are allowed.

Corollary 1.1. *Let $n \geq 9$. Under the remaining hypotheses of Theorem 1.2, suppose instead of (1.4) that $\alpha > n/2$. If there are no neighbourhoods of the conformal boundary at infinity on which P_g^* has a kernel, then the conclusions of Theorem 1.2 hold.*

It might be helpful to the reader to recall briefly the Corvino–Schoen method, as adapted to our setting. We work on an end $(r_0, +\infty) \times N$, where we have a metric g asymptotic to a background metric b. We also have a d-parameter family of references metrics b_p, all asymptotic to b, all having the same, constant scalar curvature. The gluing is performed on an annulus $A_R = \{R < r < 4R\}$, with $R \gg 1$, in four steps:

Step 1. Do a scaling in order to work on a fixed annulus A_1.

Step 2. Establish a weighted estimate of the form $|P^*u|_{L^2} \geq c|u|_{H^2}$, where P^* is the adjoint of the linearized scalar curvature operator P and u is orthogonal to the d-dimensional kernel K of P^*. The constant c has to be uniform in the family of metrics under consideration, with controlled dependence upon R. In fact, in previous applications c was R–independent.

Step 3. By step 2, and up to weighting functions, the operator $L = PP^*$ is an isomorphism modulo projections onto $K^\perp$. By the inverse function theorem, for $R \gg 1$, the gluing of g with *any* b_p can be done modulo weighted L^2-projection onto $K^\perp$.

Step 4. Estimate the projection onto K and show that you can adjust the parameter p to obtain a solution.

So, the overall strategy is the same as in [8, 12, 13]. However, in our case essential new difficulties arise: the scaling transformation in the asymptotically flat case leads to a family of uniformly equivalent operators on a fixed annulus, while this is not the case anymore for negative Λ. To handle this we prove a sharp estimate on the family of operators that arise in our context; unfortunately, the estimate degenerates as the gluing annuli recede to infinity, as the sharp constant c in Step 2 above goes to zero. This results in the undesirable restrictions described above. A possible approach to improve this state of affairs could be to devise a method which, first, deforms any asymptotically hyperbolic metric to one for which our Theorem 1.2 (or some variation thereof) applies. Alternatively, a completely different method of approaching the problem is needed.

Our work has been largely motivated by Andersson *et al.* [1], to remove the sign condition on the mass aspect function imposed there. Our deformation produces a metric with a constant mass aspect without a priori

assuming such a sign in dimensions larger than eight; our result is, however, irrelevant for the main result in [1], which is only proved for $n \leq 7$.

2. Definitions, notations and conventions

Let $\overline{M}$ be a smooth, compact n-dimensional manifold with boundary ∂M. Let $M := \overline{M} \backslash \partial M$, a non-compact manifold without boundary. In our context, the boundary ∂M will play the role of a *conformal boundary at infinity* of M. We will choose a defining function ρ for ∂M, that is a non-negative smooth function on $\overline{M}$, vanishing precisely on ∂M, with $d\rho$ never vanishing there.

We will work near the infinity of M, so it is convenient to define, for small $\varepsilon > 0$, the manifold

$$(2.1) \qquad M_\varepsilon = \{ x \in M, \rho(x) < \varepsilon \}.$$

We also define for small $\varepsilon > \delta > 0$, the "annulus"

$$(2.2) \qquad A_{\delta,\varepsilon} := M_\varepsilon \backslash \overline{M_\delta}.$$

We continue by defining a class of background metrics of interest. For k equal to -1, 0 or 1, let $\widehat{b}$ be a metric on ∂M satisfying $\mathrm{Ric}(\widehat{b}) = k(n-2)\widehat{b}$. For ρ_0 such that $1 - k(\frac{\rho}{2})^2$ has no zeros on $(0, \rho_0)$, consider the metric

$$(2.3) \qquad b = \rho^{-2} \left(d\rho^2 + \frac{(4 - k\rho^2)^2}{16} \widehat{b} \right) =: \rho^{-2} \overline{b}$$

defined on $(0, \rho_0) \times \partial M$. Then b is Einstein, $\mathrm{Ric}(b) = -(n-1)b$, in particular it has constant scalar curvature $R(b) = -n(n-1)$, and in fact provides initial data for a static solution of the vacuum Einstein equations with a negative cosmological constant. These are of course identical to (1.2) (use $r = \rho^{-1}[1 - k(\frac{\rho}{2})^2]$). The basic example of such a background is the standard hyperbolic metric. In that case M is the unit ball of $\mathbb{R}^n$, with

$$(2.4) \qquad b = \omega^{-2}\delta,$$

δ is the Euclidean metric, $\omega(x) = \frac{1}{2}(1 - |x|_\delta^2)$.

A metric g will be called *asymptotically hyperbolic* if g tends to a background metric as in (2.3) when approaching ∂M. The precise decay rates will be indicated whenever needed. The terminology is motivated by the fact that the sectional curvatures of g tend to -1 as ρ approaches zero; cf.,

e.g., [17]. One should, however, keep in mind that b does not necessarily have constant sectional curvature in space–time dimension other than four. Moreover, metrics which asymptote to hyperbolic metrics in cuspidal ends do not necessarily belong to our class.

An important class of asymptotically hyperbolic metrics is given by the *(generalized) Kottler metrics* [14] (cf. [5]) as given by (1.3). In the coordinate system of (2.3) they read

$$b_m = \rho^{-2} \left\{ \left[1 - 2m\rho^n \left(1 - k \left(\frac{\rho}{2} \right)^2 \right)^{2-n} \left(1 + k \left(\frac{\rho}{2} \right)^2 \right)^{-2} \right]^{-1} d\rho^2 \right.$$

$$\left. + \left(1 - k \left(\frac{\rho}{2} \right)^2 \right)^2 \widehat{b} \right\}$$

$$\text{(2.5)} \quad = \rho^{-2} \left[\overline{b} + 2m\rho^n (1 + O(\rho^2)) d\rho^2 \right],$$

where, as before, $\widehat{b}$ is a fixed metric on the boundary at infinity N satisfying $\mathrm{Ric}(\widehat{b}) = k(n-2)\widehat{b}$. Those metrics satisfy $R(b_m) = -n(n-1)$, and again provide initial data for static Einstein metrics.

If $\widehat{b}$ is *not* the round metric on a sphere, the only energy-like Hamiltonian invariant of b_m is m, see, e.g., [6] and references therein. Otherwise b is the standard hyperbolic metric, and the energy-momentum vector of b_m, say $p_{(\mu)}$ (as defined in [10] or [22], see (4.18) with $r \to +\infty$)), is proportional to $(m, \vec{0})$. Under isometries of hyperbolic space, $p_{(\mu)}$ transforms as a Lorentz vector, and a metric with any timelike $p_{(\mu)}$ can be obtained by applying such an isometry to some b_m.

In this way we generate a family of metrics with any *timelike* $p_{(\mu)}$, as needed for the Brouwer fixed point argument when compensating for the cokernel below. (On the other hand, we are not aware of existence of such metrics with non-timelike non-zero $p_{(\mu)}$, whence the restriction of timelike-ness in our results when $(N, \widehat{b})$ is a round sphere.) We denote by $b_{p_{(\mu)}}$ the resulting metrics, and we will refer to them as Kottler metrics, or boosted Kottler metrics when ambiguities are likely to occur.

Recall that the linearized scalar curvature operator $P = P_g$ is

$$P_g h := DR(g)h = -\nabla^k \nabla_k (tr_g\ h) + \nabla^k \nabla^l h_{kl} - R^{kl} h_{kl},$$

so that its L^2 formal adjoint reads

$$\text{(2.6)} \quad P_g^* f = [DR(g)]^* f = -\nabla^k \nabla_k f g + \nabla \nabla f - f \, \mathrm{Ric}(g).$$

(We use the summation convention, indices are lowered with g_{ij} and raised with its inverse g^{ij}.) We note that

$$\operatorname{Tr} P_g^* f = (n-1)\nabla^*\nabla f - Rf.$$

Let

(2.7)
$$b_\delta := z^{-2}\left(dz^2 + \delta^{-2}\widehat{b}(\delta z)\right)$$

be a hyperbolic metric scaled up in ρ from $A_{\delta,4\delta}$ to

(2.8)
$$A \equiv A_{(1,4)} := (1,4) \times \partial M.$$

We have $\operatorname{Ric}(b_\delta) = -(n-1)b_\delta$, thus

(2.9)
$$P_{b_\delta}^* u := \nabla\nabla u + \left((n-1)u - \nabla^k\nabla_k u\right)b_\delta,$$

where ∇ is associated with the metric b_δ.

It is well known that the kernel of P^* has dimension at most $n+1$, see [12] for instance. For the hyperbolic metric b on the unit n-dimensional ball $B^n(1) \subset \mathbb{R}^n$, in the representation (2.4), the kernel of P_b^* is spanned by the following functions, which are the restrictions to the hyperboloid $\mathbb{H}^n$ of the coordinates functions in Minkowski $\mathbb{R}^{n,1}$:

(2.10)
$$V_{(0)} := \frac{1+|x|^2}{1-|x|^2} = \rho^{-1}\left(1 + \left(\frac{\rho}{2}\right)^2\right),$$

(2.11)
$$V_{(k)} := -\frac{2x^k}{1-|x|^2} = -\rho^{-1}\left(1 - \left(\frac{\rho}{2}\right)^2\right)\frac{x^k}{|x|},$$

with $\rho = 2(1-|x|)/(1+|x|)$. We can also rewrite (2.4) as

(2.12)
$$b = \rho^{-2}(d\rho^2 + \widehat{b}(\rho)),$$

with $\widehat{b}(\rho) = \left(1 - \left(\frac{\rho}{2}\right)^2\right)^2\widehat{b}(0)$, where $\widehat{b}(0)$ is the round unit metric on S^n. Setting $\rho = \delta z$, defining

(2.13)
$$V_{\delta,(\mu)}(z,\theta) = V_{(\mu)}(\delta z, \theta),$$

and letting δ tend to zero, the functions $\delta V_{\delta,(\mu)}$ tend to

(2.14)
$$u_{(0)} := z^{-1}, \quad u_{(k)} := -z^{-1}\frac{x^k}{|x|}.$$

For non-spherical boundary metrics $\widehat{b} \equiv \widehat{b}(0)$ we still write $\{V_{(\mu)}\}$ for any basis of Ker P_b^*, and then $V_{\delta,(\mu)}$ is defined by (2.13). By hypothesis the scalar curvature of $\widehat{b}$ is constant, so that we can invoke a theorem of Obata [20] (see also [15, Theorem 24]) to conclude that the only Riemannian manifold $(\widehat{M}, \widehat{b})$ of dimension $n-1$ with non-constant solutions v to the equation $DDv + \frac{D^*Dv}{n-1}\widehat{b} = 0$ is a round sphere. For metrics of the form (2.3) with $\widehat{b}$ different from a round sphere, this implies (cf. Appendix A) that dim Ker $P^* = 1$ and $V_{(0)} = V_{(0)}(\rho)$, with the $u_{(\mu)}$'s proportional to $u_{(0)} = z^{-1}$ (cf. (3.19) and (3.20)).

Definition 2.1. Let $k \in \mathbb{N}$, $C, \sigma \geq 0$. Let b be of the form (2.3), with $\widehat{b}$ an Einstein metric on ∂M with scalar curvature $(n-1)(n-2)\kappa$, $\kappa \in \{0, \pm 1\}$, and with $\rho \in (0, 2\rho_0]$. We will say that g is (C, k, σ)-*asymptotically hyperbolic* if we have

$$(2.15) \qquad |g - b|_b + |\nabla g|_b + \ldots + |\nabla^{(k)} g|_b \leq C\rho^\sigma,$$

where the norm and covariant derivatives are defined by b. For $\alpha \in (0,1)$ we will say that g is $(C, k+\alpha, \sigma)$-*asymptotically hyperbolic* if the derivatives of order k of $g - b$ further satisfy a weighted Hölder condition of order α, as in [16].

Let g be a Riemannian metric on M, recall that (M, g) is *conformally compact* if there exists on $\overline{M}$ a smooth defining function ρ for ∂M (that is $\rho \in C^\infty(\overline{M})$, $\rho > 0$ on M, $\rho = 0$ on ∂M and $d\rho$ nowhere vanishing on ∂M; the symbol ρ will be used throughout this work to denote such a function) such that $\overline{g} := \rho^2 g$ is a Riemannian metric on $\overline{M}$, we will denote by $\widehat{g}$ the metric induced on ∂M. The background metrics b considered above are conformally compact in this sense.

It is well know that, near infinity, for any sufficiently differentiable conformally compact metric g we may choose the defining function ρ to be the $\overline{g}$-distance to the boundary. Thus, if ε is small enough, M_ε can be identified with $(0, \varepsilon) \times \partial M$ equipped with the metric

$$(2.16) \qquad g = \rho^{-2}(d\rho^2 + \widehat{g}(\rho)) = \rho^{-2}(d\rho^2 + \widehat{g}_{AB}(\rho) d\theta^A d\theta^B),$$

where $\{\widehat{g}(\rho)\}_{\rho \in (0,\varepsilon)}$ is a family of smooth, uniformly equivalent, metrics on ∂M, with $\widehat{g}(0) = \widehat{g}$. However, the introduction of this system of coordinates might lead to a loss of up to two derivatives of the metric. This can be circumvented for (C, k, σ)-asymptotically hyperbolic metrics by introducing

a coordinate system as in [2, Appendix B] in which g takes the form

$$(2.17) \quad g = \rho^{-2}\left((1 + O(\rho^{k+\sigma}))d\rho^2 + \widehat{g}_{AB}(\rho)d\theta^A d\theta^B + O(\rho^{k+\sigma})_A d\rho d\theta^A\right),$$

with all metric coefficients of original differentiability class.

If g is (C, k, σ)-asymptotically hyperbolic with $k \geq 2$ and $\sigma > 0$, we have

$$(2.18)$$
$$P_g^* u := \nabla\nabla u - \nabla^k \nabla_k u\, g - u\operatorname{Ric}(g) = \nabla\nabla u + \left((n-1)u - \nabla^k \nabla_k u\right)g + O(\rho^\sigma)u,$$

where the covariant derivatives are related to g, and the $O(\rho^\sigma)$ term is bounded (in b-norm) together with its b-derivatives up to order $k - 2$, by ρ^σ times a constant depending on C and k.

3. A uniform estimate for P^*

Let y be the function on A defined by

$$(3.1) \qquad \begin{aligned} y: \quad (1,4) \times \partial M \quad &\longrightarrow \qquad \mathbb{R} \\ (z, \theta) \quad &\longmapsto \quad \frac{4}{3}\left(1 - \frac{z}{4}\right)(z - 1). \end{aligned}$$

We claim that:

Proposition 3.1. *Let $c_0, \sigma > 0$ and $s \geq 0$. There exist constants $c_1 = c_1(n, s, c_0, \sigma) > 0$ and $\delta_0 = \delta_0(n, s, c_0, \sigma) > 0$ such that for all $(c_0, 4, \sigma)$-asymptotically hyperbolic metrics g, for all $0 < \delta \leq \delta_0$, and for all u satisfying*

$$(3.2) \qquad \forall \mu = 0, \ldots, k, \quad \int_A e^{-s/y} u u_{(\mu)}\, d\mu_{b_\delta} = 0,$$

where $k = n$ if b is the standard hyperbolic metric, and $k = 0$ otherwise, we have

$$(3.3)$$
$$\int_A e^{-s/y} y^8 |P_{g_\delta}^* u|_{g_\delta}^2\, d\mu_{g_\delta} \geq c_1 \delta^4 \int_A e^{-s/y}(y^8|\nabla\nabla u|_{g_\delta}^2 + y^4|\nabla u|_{g_\delta}^2 + u^2)\, d\mu_{g_\delta},$$

provided that the right-hand side is finite. Similarly (3.3) holds (with perhaps a different constant c_1) if

$$(3.4) \qquad \forall \mu = 0, \ldots, k, \quad \int_A e^{-s/y} u V_{\delta,(\mu)}\, d\mu_{g_\delta} = 0,$$

or if in (3.2) the measure $z^{-n}\, dz\, d\mu_{\widehat{b}(0)}$ is used.

Remark 3.1. There is little doubt that the result remains valid for $(c_0, 2, \sigma)$ asymptotically hyperbolic metrics, or for those conformally compact metrics which are C^2 up-to-boundary after the conformal rescaling, by using coordinates as in (2.17). For simplicity of calculations we assume (2.16), since our main gluing results require $(c_0, 4, \sigma)$ asymptotically hyperbolic metrics anyway.

Remark 3.2. The power of δ in (3.3) cannot be improved, which can be seen by considering a function of the form $u(z, \theta) = v(\theta)/z$, with a non-trivial v of vanishing integral on ∂M, such that $DDv + \frac{D^*Dv}{n-1}\widehat{b}(0) \neq 0$, where D is the covariant derivative operator of $\widehat{b}(0)$, and such that v is $L^2(\partial M, \widehat{b}(0))$-orthogonal to the kernel of $P^*_{\widehat{b}(0)}$ (see (3.15) below).

Proof. In some of the calculations of this proof the reader might find it convenient to use the coordinate system of (2.16). Without loss of generality, we can assume that $\sigma \leq 1$. Let us define $d\nu_{g_\delta} = \delta^{n-1}d\mu_{g_\delta}$, and note that the measure $d\mu_{g_\delta}$ can be replaced by $d\nu_{g_\delta}$ in (3.2) to (3.4); e.g., (3.2) can be replaced by

$$(3.5) \qquad \forall \mu = 0, \ldots, k, \quad \int_A e^{-s/y} u u_{(\mu)} \, d\nu_{g_\delta} = 0.$$

Suppose that (3.3) with $d\mu_{g_\delta}$ there replaced by $d\nu_{g_\delta}$ does not hold, then there exist sequences $\delta_n \to 0$, $g^{(n)}$ and u_n satisfying (3.2) (respectively (3.4)) such that the right-hand side equals one, while the reverse inequality to (3.3) holds with c_1 replaced by $1/n$:

$$(3.6) \qquad \int_A e^{-s/y} y^8 |P^*_{\delta_n} u_n|^2_{g_n} \, d\nu_{g_n} \leq \frac{\delta_n^4}{n},$$

$$(3.7) \qquad \int_A e^{-s/y} (y^8 |\nabla\nabla u_n|^2_{g_n} + y^4 |\nabla u_n|^2_{g_n} + u_n^2) \, d\nu_{g_n} = 1,$$

where we have set

$$g_n := g^{(n)}_{\delta_n} \quad \text{and} \quad P^*_{\delta_n} \equiv P^*_{g_n}.$$

Let y be the function on A defined in (3.1). Using (2.18) to express $\nabla\nabla u_n$ in terms of $P^*_{g_n} u_n$ and u_n one obtains (cf. (3.22) below)

$$\int_A e^{-s/y} y^8 |\nabla\nabla u_n|^2_{g_n} \, d\nu_{g_n} \leq C \int_A e^{-s/y} \left(y^8 |P^*_{\delta_n}|^2_{g_n} + u_n^2 \right) d\nu_{g_n}$$

$$(3.8) \qquad\qquad\qquad \leq C \left(\frac{\delta_n^4}{n} + \int_A e^{-s/y} u_n^2 \right) d\nu_{g_n},$$

which together with (3.7) implies that there exists $c > 0$ such that

$$(3.9) \qquad \int_A e^{-s/y}(y^4|\nabla u_n|^2_{g_n} + u_n^2)\, d\nu_{g_n} \geq c.$$

Now,

$$(3.10) \qquad |\nabla u_n|^2_{g_n} = z^2(|\partial_z u_n|^2 + \delta_n^2|\partial_\theta u_n|^2_{\widehat{g}_n}),$$

where $|\cdot|_{\widehat{g}_n}$ denotes the norm of a tensor field on ∂M with respect to the metric

$$\widehat{g}_n(z) := \widehat{g}^{(n)}(\delta_n z).$$

Note that, decreasing the constant ρ_0 of Definition 2.1 if necessary, all the $\widehat{g}_n$'s are uniformly equivalent to $\widehat{b}(0)$. From (3.9) we obtain

$$(3.11) \qquad \int_A e^{-s/y}(y^4|\partial_z u_n|^2 + u_n^2)\, d\nu_{g_n} \geq c,$$

for some $c > 0$.

Clearly the trace of $P^*_{g_n} u$ satisfies an estimate of the form (3.6) (cf. Appendix A)

$$(3.12) \qquad \int_A e^{-s/y}y^8|\Delta_{g_n} u_n - n u_n + O(\delta_n^\sigma)u_n|^2\, d\nu_{g_n} \leq C\frac{\delta_n^4}{n}.$$

Let

$$E := \nabla\nabla u_n - \frac{\Delta_{g_n} u_n}{n}\, g_n = \nabla\nabla u_n - u_n\, g_n + \text{ error},$$

where the error term is bounded, after integration, as in (3.12). From (3.6) and (3.12) we conclude that

$$(3.13) \qquad \int_A e^{-s/y}y^8|E + O(\delta_n^\sigma)u_n|^2_{g_n}\, d\nu_{g_n} \leq C\frac{\delta_n^4}{n}.$$

Since E is trace free we have $E_{zz} = -\delta_n^{-2}\widehat{g}_n^{CD}E_{CD}$, so that

$$(3.14)$$
$$|E|^2_{g_n} = z^4\left(\left(1 + \frac{\delta_n^4}{n-1}\right)|E_{zz}|^2 + 2\delta_n^2|E_{zA}|^2_{\widehat{g}_n} + \delta_n^4|E_{AB} - \widehat{g}_n^{CD}E_{CD}\widehat{g}_{nAB}|^2_{\widehat{g}_n}\right),$$

which together with the formulae in Appendix A (recall we have assumed $\sigma < 1$) leads to

$$\int_A e^{-s/y} y^8 (|\partial_z^2 u_n + z^{-1}\partial_z u_n - z^{-2}u_n + O(\delta_n^\sigma)u_n|^2$$
$$+ \delta_n^2 |\partial_z \partial_A u_n + z^{-1}\partial_A u_n + O(\delta_n \partial_\theta u_n) + O(\delta_n^\sigma)u_n|^2_{\widehat{g}_n}$$
$$+ \delta_n^4 |D_A D_B u_n - \widehat{g}^{(n)}(\delta_n)^{CD} D_C D_D u_n (\widehat{g}_n)_{AB} + O(\delta_n^\sigma)u_n|^2_{\widehat{g}_n}) d\nu_{g_n} \le C \frac{\delta_n^4}{n}.$$

(3.15)

Next, (3.6) together with the formula for $(P_{g_\delta}^* u)_{zz}$ in Appendix A gives

$$\int_A e^{-s/y} y^8 \Big| [(n-1)z + O(\delta_n^\sigma)]\partial_z u_n + [(n-1) + O(\delta_n^\sigma)]u_n$$
(3.16) $$- \delta_n^2 \Delta_{\widehat{g}_{\delta_n}} u_n \Big|^2 d\nu_{g_n} \le C \frac{\delta_n^4}{n}.$$

Choose $\delta_{n_0} \ne 0$ and let H^1 and H^2 be the Hilbert spaces with norms defined by the left-hand sides of (3.9) and (3.7) with $n = n_0$, and norms, covariant derivatives and measures related to $b_{\delta_{n_0}}$:

(3.17) $$\|u\|_{H^1} := \int_A e^{-s/y}(y^4|\nabla u|^2_{b_{\delta_{n_0}}} + u^2)\, d\nu_{b_{\delta_{n_0}}},$$

(3.18) $$\|u\|_{H^2} := \int_A e^{-s/y}(y^8|\nabla\nabla u|^2_{b_{\delta_{n_0}}} + y^4|\nabla u|^2_{b_{\delta_{n_0}}} + u^2)\, d\nu_{b_{\delta_{n_0}}}.$$

Now, (3.7) shows that u_n and $y^2\partial_z u_n$ are bounded in $L^2 = L^2(A, e^{-s/y} d\nu_{b_{\delta_{n_0}}})$. Equation (3.12) proves that y^4 times the Laplacian of u_n is bounded in L^2. Further, (3.15) establishes that $y^4\partial_z^2 u_n$ is bounded in L^2. Simple algebra gives then that y^4 times the tangential Laplacian of u_n is bounded in L^2. Coming back to (3.15) we obtain that all tangential derivatives of u_n are L^2-bounded, when multiplied by relevant powers of y. Standard interpolation gives an L^2-bound for y^2 times the first tangential derivatives of u_n. But (3.15) shows now that the functions $y^4\partial_z\partial_A u_n$ are L^2-bounded. Finally, an interpolation will bound every (weighted) first derivatives of u_n.

So, the sequence u_n is bounded in H^2, therefore there exists a subsequence, still denoted by u_n, which converges strongly in H^1. But (3.8) with u_n replaced by $u_n - u_m$ shows that u_n is Cauchy in H^2, hence there exists

$u \in H^2$ such that u_n converges to u in H^2. From (3.15) we infer that

$$(3.19) \quad \begin{aligned} &|\partial_z^2 u + z^{-1}\partial_z u - z^{-2}u|^2 + |\partial_z\partial_A u + z^{-1}\partial_A u|^2_{b_{\delta_{n_0}}} \\ &\quad + |D_A D_B u - \widehat{b}(0)^{CD} D_C D_D u\, \widehat{b}(0)_{AB}|^2_{b_{\delta_{n_0}}} = 0, \end{aligned}$$

while (3.16) implies

$$(3.20) \quad\quad\quad\quad\quad z\partial_z u + u = 0.$$

Solving (3.19) and (3.20), we conclude that u is a linear combination of the $u_{(\mu)}$'s as given by (2.14) for a standard hyperbolic metric, while $u = \text{const}/z$ otherwise. But the integral in (3.5) is continuous on H^2, which implies that (3.2) is satisfied in the limit. Similarly, δ^n times the integral (3.4) is continuous on H^2. Recalling that the family $\{u_{(\mu)}\}$ is orthogonal with respect to the scalar product defined by the integral in (3.2), we obtain $u = 0$. This contradicts (3.9), and proves the result. $\quad\square$

Let $\psi = e^{-s/2y}$, $\phi = y^2$. We will use spaces $H^k_\delta \equiv H^k_{g_\delta}$ of tensor fields on A (cf. [8]) for which the norms

$$(3.21) \quad \|u\|_{H^k_\delta} := \left(\int_A \left(\sum_{i=0}^{k} \phi^{2i}|\nabla^{(i)}u|^2_{g_\delta} \right) \psi^2 \delta^{(n-1)} d\mu_{g_\delta} \right)^{1/2}$$

are finite, where $\nabla^{(i)}$ stands for the tensor $\underbrace{\nabla...\nabla}_{i \text{ times}} u$, with ∇ — the Levi–Civita covariant derivative of g_δ; we assume throughout that the metric is at least $W^{1,\infty}_{\text{loc}}$; higher differentiability will be usually indicated whenever needed. The factor $\delta^{(n-1)}$ in front of the measure $d\mu_{g_\delta}$ has been included so that $\delta^{(n-1)}d\mu_{g_\delta}$ is equivalent to the Lebesgue coordinate measure $dz\,d\theta$, uniformly in δ.

Note that $H^0_{g_\delta}$ involves weighs, but L^2 does not.

An equivalent norm, and therefore the same space, is obtained if g_δ in (3.21) is replaced by b_δ.

We will need the following:

Lemma 3.1. *Let $c_0, \sigma > 0$ and $s \geq 0$. There exist constants $C = C(n, \ell, s, c_0, \sigma) > 0$ and $\delta_0 = \delta_0(n, \ell, s, c_0, \sigma) > 0$ such that for all $(c_0, \ell + 2, \sigma)$-asymptotically hyperbolic metrics g and for all $0 < \delta \leq \delta_0$*

$$\|u\|_{H^{\ell+2}_{g_\delta}} \leq C\left(\|\phi^2 P^*_{g_\delta} u\|_{H^\ell_{g_\delta}} + \|u\|_{H^0_{g_\delta}} \right).$$

Proof. For $\ell = 0$ the result has been established in the course of the proof of Proposition 3.1, see the first line of (3.8). For $\ell = 1$ we start the calculation that follows with $k = 2$ and we stop at the second line, invoking weighted interpolation and the result for $\ell = 0$ to conclude. Otherwise, suppose that the result is true for $k - 1 \leq \ell_0$ with $\ell_0 \geq 1$. Using [8, Equation (A.4)] (one can check that the constants in Equations (A.2) and (A.3) there, thus also in (A.4), do not depend on δ) to control the first term when passing from the second to the third line below, we find for $2 \leq k - 1 + 2 \leq \ell_0 + 2$

$$\|\phi^{k+1}\nabla^{(k-1)}(\nabla^{(2)}u - \Delta u g_\delta)\|_{H^0_{g_\delta}}$$
$$= \|\phi^{k+1}\nabla^{(k-1)}(P^*u - (n-1)u + O(\delta^\sigma)u)\|_{H^0_{g_\delta}}$$
$$\leq \|\phi^{k+1}\nabla^{(k-1)}P^*u\|_{H^0_{g_\delta}} + C_1\|\underbrace{\phi^2}_{\leq C}\phi^{k-1}\nabla^{(k-1)}[(1 + O(\delta^\sigma))u]\|_{H^0_{g_\delta}}$$
$$\leq C\left(\|\phi^2 P^*u\|_{H^{k-1}_{g_\delta}} + C_2 \underbrace{\|u\|_{H^{k-1}_{g_\delta}}}_{\leq C(\|\phi^2 P^*u\|_{H^{k-3}_{g_\delta}} + \|u\|_{H^0_{g_\delta}})}\right)$$
$$\leq C(1 + CC_2)\|\phi^2 P^*u\|_{H^{k-1}_{g_\delta}} + C^2 C_2\|u\|_{H^0_{g_\delta}}.$$

This is the desired inequality, to see this set

$$T := \nabla^{(k+1)}u, \quad S := \nabla^{(k-1)}(\nabla^2 u - \Delta u g_\delta),$$

or, in index notation,

$$T_{i_1\ldots i_{k-1}jk} := \nabla_{i_1} \cdots \nabla_{i_{k-1}} \nabla_j \nabla_k u$$
$$S_{i_1\ldots i_{k-1}jk} := \nabla_{i_1} \cdots \nabla_{i_{k-1}} \nabla_j \nabla_k u - \nabla_{i_1} \cdots \nabla_{i_{k-1}} \nabla^\ell \nabla_\ell u (g_\delta)_{jk},$$

straightforward algebra shows that

$$(3.22) \qquad\qquad |T|^2_{g_\delta} \leq |S|^2_{g_\delta},$$

and the lemma follows. $\qquad\square$

As in [8] we set

$$(3.23) \qquad\qquad L_{g_\delta} := \psi^{-2} P_{g_\delta} \phi^4 \psi^2 P^*_{g_\delta},$$

and

$$(3.24) \qquad\qquad K_{b_\delta} = \ker P^*_{b_\delta}.$$

The proof of [8, Theorem 3.6] shows that

$$L_{g_\delta}^{-1} : H_{g_\delta}^k \cap K_\delta^{\perp_{H_{g_\delta}^0}} \to H_{g_\delta}^{k+4}$$

exists for δ small enough. However, uniform boundedness in δ of $L_{g_\delta}^{-1}$ does not hold in our case, instead we have:

Corollary 3.1. *Let $k \in \mathbb{N}$, $c_0, \sigma > 0$ and $s \geq 0$. There exist constants $C = C(n, k, s, c_0, \sigma) > 0$ and $\delta_0 = \delta_0(n, k, s, c_0, \sigma) > 0$ such that for all $(c_0, k + 4, \sigma)$-asymptotically hyperbolic metrics g, for all $0 < \delta \leq \delta_0$, and for all u satisfying (3.2) or (3.4)*

$$\|L_{g_\delta}^{-1} u\|_{H_{g_\delta}^{k+4}} \leq C \Big(\|u\|_{H_{g_\delta}^k} + \delta^{-4} \|u\|_{H_{g_\delta}^0} \Big).$$

Proof. By Proposition 3.1 we have (recall that H^0 is weighted but L^2 is not)

$$c\delta^4 \|u\|_{H_{g_\delta}^2}^2 \leq \|\phi^2 P_{g_\delta}^* u\|_{H_{g_\delta}^0}^2$$
$$= \langle \psi^2 \phi^2 P_{g_\delta}^* u, \phi^2 P_{g_\delta}^* u \rangle_{L_{g_\delta}^2} = \langle \psi^2 u, \underbrace{\psi^{-2} P_{g_\delta} \phi^4 \psi^2 P_{g_\delta}^*}_{L_{g_\delta}} u \rangle_{L_{g_\delta}^2}$$
$$= \langle \psi u, \psi L_{g_\delta} u \rangle_{L_{g_\delta}^2} \leq \|\psi L_{g_\delta} u\|_{L_{g_\delta}^2} \|\psi u\|_{L_{g_\delta}^2} = \|L_{g_\delta} u\|_{H_{g_\delta}^0} \|u\|_{H_{g_\delta}^0}.$$

Replacing u by $L_{g_\delta}^{-1} u$ we conclude that

$$(3.25) \qquad\qquad c\delta^4 \|L_{g_\delta}^{-1} u\|_{H_{g_\delta}^2} \leq \|u\|_{H_{g_\delta}^0}.$$

In order to finish the proof we will use the following elliptic estimate, which is standard except for the uniformity in δ; the proof can be found in Appendix B.

Lemma 3.2. *Under the conditions of Corollary 3.1, there exists a constant C, independent of g and δ, such that for δ small*

$$(3.26) \qquad\qquad \|u\|_{H_{g_\delta}^{k+4}} \leq C \Big(\|L_{g_\delta} u\|_{H_{g_\delta}^k} + \|u\|_{H_{g_\delta}^0} \Big).$$

Returning to the proof of Corollary 3.1, we replace u by $L_{g_\delta}^{-1} u$ in (3.26) to obtain

$$(3.27) \qquad\qquad \|L_{g_\delta}^{-1} u\|_{H_{g_\delta}^{k+4}} \leq C \Big(\|u\|_{H_{g_\delta}^k} + \|L_{g_\delta}^{-1} u\|_{H_{g_\delta}^0} \Big),$$

and the corollary follows from (3.25). □

Summarizing, we have proved:

Theorem 3.1. *Let $k \in \mathbb{N}$, $\sigma > 0$, $c_0 > 0$ and $s \geq 0$. There exist constants $C = C(n, s, \sigma, c_0) > 0$ and $\delta_0 = \delta_0(n, s, \sigma, c_0) > 0$ such that for all $(c_0, k + 4, \sigma)$-asymptotically hyperbolic metrics g, for all $0 < \delta \leq \delta_0$ and for any $u \in H_{g_\delta}^{k+4} \cap K_{b_\delta}^{\perp_{g_\delta}}$,*

$$C\delta^4 \|u\|_{H_{g_\delta}^{k+4}} \leq \|L_{g_\delta} u\|_{H_{g_\delta}^k}.$$

In particular the operator $\Pi_{K_{b_\delta}^{\perp_{g_\delta}}} L_{g_\delta}$, where $\Pi_{K_{b_\delta}^{\perp_{g_\delta}}}$ denotes orthogonal projection on $K_{b_\delta}^{\perp_{g_\delta}}$ in $H_{g_\delta}^0$, is an isomorphism from $H_{g_\delta}^{k+4} \cap K_{b_\delta}^{\perp_{g_\delta}}$ to $H_{g_\delta}^k \cap K_{b_\delta}^{\perp_{g_\delta}}$ such that the norm of its inverse is bounded by $C^{-1}\delta^{-4}$.

At this point, we have established Step 2 of the Introduction, as well as some elements of Step 3. We continue with further details of Step 3.

4. The gluing construction on a moving annulus

In this section we prove Theorem 1.2. We set $k = \ell - 4$. We consider conformally compact asymptotically hyperbolic metrics g which asymptote, with $k + 2$ derivatives, to a fixed AH metric b. We fix a small $\delta_0 > 0$ and define the space $W_b^{k+4,\infty}(M_{4\delta_0})$ of symmetric two tensors with $k + 4$ b-covariant derivatives bounded on $M_{4\delta_0}$, relatively to the norm of b. Following [8], we assume that $g - b$ is close to zero in $W_b^{k+4,\infty}(M_{4\delta_0})$.

Similarly to (2.7), we denote by g_δ the metric on $A_{1,4}$ obtained by restricting g to $A_{\delta,4\delta}$, and rescaling the ρ coordinate to $A_{1,4}$. Unless explicitly specified otherwise, covariant derivatives on $A_{1,4}$ are related to g_δ.

As in [8], consider the map

$$
\begin{array}{rccl}
f_{g_\delta} : & \psi^2 \phi^2 H_\delta^{k+2} & \longrightarrow & H_\delta^k \cap K_{b_\delta}^\perp \\
& h & \longmapsto & \Pi\{\psi^{-2}[R(g_\delta + h) - R(g_\delta)]\},
\end{array}
$$

where $K_{b_\delta} = \mathrm{Ker}\, P_{b_\delta}^*$, where $P_{b_\delta}^*$ is as in (2.9), and Π is the H_δ^0 projection onto $K_{b_\delta}^\perp$, the H_δ^0-orthogonal of K_{b_δ}; all the spaces here are spaces of tensors on $A_{1,4}$.

One should keep in mind that we are interested in h's of the form $h = \psi^2 \phi^4 P^* u$, $u \in H_\delta^{k+4} \cap K_{b_\delta}^\perp$, with u small in the last space.

Near $h = 0$ the map f_{g_δ} is a smooth map between Hilbert spaces. We consider now [8, Proposition G.1] with $x = g_\delta$ so that f_x there equals f_{g_δ} here. One checks that f satisfies conditions (2) and (3) of [8, Proposition

G.1] with the set A there being

(4.1) $\quad A = \{g_\delta,\ 0 < \delta \leq \delta_0,\ (g - b) \text{ sufficiently small in } W_b^{k+4,\infty}(M_{4\delta_0})\}.$

Furthermore,

$$V_x = \psi^2\phi^2 H_\delta^{k+2},\ \ W_x = H_\delta^k \cap K_{b_\delta}^\perp.$$

Now, f_{g_δ} satisfies a modified version of condition (1) there: here, by Theorem 3.1, we have that $Df_x(0)$ has a right inverse $\psi^2\phi^4 P_{g_\delta}^* L_{g_\delta}^{-1}$ bounded by $C_1\delta^{-4}$, where C_1 does not depend upon $x \in A$. For the sake of notational legibility, we present the argument *without* using the smoothing operators of [9]; the latter provide what is needed to obtain the differentiability claimed. Note that we haven't assumed any uniformity in δ on the modulus of the Hölder continuity of g, as the solution will exist, and will have Hölder regularity, without any such assumptions. Any further hypotheses about uniformity of that modulus would be reflected in associated uniformity for the metrics obtained by the gluing procedure, but such uniformity is irrelevant for our purposes.

A repetition of the proof of Proposition G.1 of [8] with C_1 there replaced with $C_1\delta^{-4}$ yields:

Theorem 4.1. *There exist constants $\varepsilon > 0$ and $C > 0$ such that for all δ sufficiently small and for all functions $f \in H_\delta^k$ with*

$$\|f\|_{H_\delta^k} \leq \varepsilon\delta^4,$$

there exists a unique $h = \psi^2\phi^4 P_{g_\delta}^* u$, with $\|\psi^{-2}\phi^{-2}h\|_{H_\delta^{k+2}}$ close to zero, satisfying $f_{g_\delta}(h) = \Pi f$ and

$$\|\psi^{-2}\phi^{-2}h\|_{H_\delta^{k+2}} \leq C'\|u\|_{H_\delta^{k+4}} \leq C\delta^{-4}\|f\|_{H_\delta^k} \leq C\varepsilon.$$

We will use Theorem 4.1 to glue an AH metric g, with timelike energy-momentum vector, with a Kottler one $b_{p_{(\mu)}}$, on an annulus $A_{\delta,4\delta}$. Let χ be a cutoff function equal to zero on $A_{1,2}$ and to one on $A_{3,4}$. We define a first glued metric on $A_{1,4}$ as

(4.2) $$g_{\delta,p_{(\mu)}} := \chi g_\delta + (1 - \chi)b_{\delta,p_{(\mu)}}.$$

It is clear that the metric $g_{\delta,p_{(\mu)}}$ belongs to the set A of (4.1). Set

(4.3) $$f := \psi^{-2}[R(b_\delta) - R(g_{\delta,p_{(\mu)}})] = \psi^{-2}[R(b_{\delta,p_{(\mu)}}) - R(g_{\delta,p_{(\mu)}})].$$

Let $p^0_{(\mu)}$ be the momentum vector of g. We will assume that g has the following asymptotic behaviour:

$$(4.4) \qquad |g - b_{p^0_{(\mu)}}|_b + |\mathring{D}(g - b_{p^0_{(\mu)}})|_b + \ldots + |\mathring{D}^{(k+2)}(g - b_{p^0_{(\mu)}})|_b = O(\rho^\alpha),$$

for some $\alpha > 0$, to be restricted shortly; here $\mathring{D}$ is the covariant derivative of b. Recall that

$$(4.5) \qquad |b_{p_{(\mu)}} - b|_b + |\mathring{D}b_{p_{(\mu)}}|_b + \ldots + |\mathring{D}^{(k+2)}b_{p_{(\mu)}}|_b = O(\rho^n).$$

Under (4.4) we have

$$|g_{\delta,p_{(\mu)}} - b_{\delta,p^0_{(\mu)}}|_{b_\delta} + |\nabla(g_{\delta,p_{(\mu)}} - b_{\delta,p^0_{(\mu)}})|_{b_\delta} + \ldots + |\nabla^{(k+2)}(g_{\delta,p_{(\mu)}} - b_{\delta,p^0_{(\mu)}})|_{b_\delta}$$
$$= O(\delta^\alpha),$$
$$(4.6)$$

where the norm and covariant derivatives are defined by b_δ. This implies

$$|g_{\delta,p_{(\mu)}} - b_{\delta,p_{(\mu)}}|_{b_\delta} \le |g_{\delta,p_{(\mu)}} - b_{\delta,p^0_{(\mu)}}|_{b_\delta} + |b_{\delta,p_{(\mu)}} - b_{\delta,p^0_{(\mu)}}|_{b_\delta}$$
$$= O(\delta^\alpha) + O(|p_{(\mu)} - p^0_{(\mu)}|\delta^n)$$
$$(4.7) \qquad\qquad\qquad = O(\delta^\alpha) + O(\delta^{\beta+n}),$$

provided that $p_{(\mu)}$ is assumed to satisfy

$$(4.8) \qquad\qquad\qquad |p_{(\mu)} - p^0_{(\mu)}| = O(\delta^\beta),$$

for some $\beta > 0$. An inequality similar to (4.7) holds for derivatives of order up to $k + 2$. It follows that the function f defined in (4.3) satisfies

$$\|f\|_{H^k_\delta} = O(\delta^\alpha) + O(\delta^{\beta+n}).$$

By Theorem 4.1 if

$$(4.9) \qquad\qquad\qquad \alpha > 4, \quad \beta + n > 4,$$

then for all δ small enough there exists a solution $h_{\delta,p_{(\mu)}}$ to the equation

$$f_{g_{\delta,p_{(\mu)}}}(h_{\delta,p_{(\mu)}}) = \Pi f,$$

with

$$(4.10) \qquad \|\psi^{-2}\phi^{-2}h_{\delta,p_{(\mu)}}\|_{H^{k+2}_\delta} = O(\delta^{\alpha-4}) + O(\delta^{\beta+n-4}).$$

Summarizing, for all $p_{(\mu)}$, we have constructed a solution $h_{\delta,p_{(\mu)}}$, modulo kernel, to the equation

$$R(g_{\delta,p_{(\mu)}} + h_{\delta,p_{(\mu)}}) - R(b) = 0,$$

satisfying (4.10). This finishes Step 3 of the Introduction.

We now proceed to Step 4. Set

$$\widetilde{g}_{\delta,p_{(\mu)}} = g_{\delta,p_{(\mu)}} + h_{\delta,p_{(\mu)}}.$$

We consider now the projection onto the kernel as follows. For all δ small, and for all $p_{(\mu)}$ satisfying (4.8), we define

$$(4.11) \qquad I_\delta(p_{(\mu)}) = \frac{1}{\delta^n}\pi[\psi^{-2}(R(\widetilde{g}_{\delta,p_{(\mu)}}) - R(b_\delta))],$$

where π is the H_δ^0 orthogonal projection onto K_{b_δ}. We want to show that we can choose $p_{(\mu)}$ such that $I_\delta(p_{(\mu)}) = 0$. We need the following identity, from [10]:

$$(4.12) \qquad \sqrt{\det g}\,\mathring{N}(R_g - R_b) = \partial_i\left(\mathbb{U}^i(\mathring{N})\right) + \sqrt{\det g}\,(\rho + Q),$$

where

$$(4.13) \qquad \mathbb{U}^i(\mathring{N}) := 2\sqrt{\det g}\,\left(\mathring{N}g^{i[k}g^{j]l}\mathring{D}_j g_{kl} + D^{[i}\mathring{N}g^{j]k}e_{jk}\right),$$

$$(4.14) \qquad \rho := (-\mathring{N}\operatorname{Ric}(b)_{ij} + \mathring{D}_i\mathring{D}_j\mathring{N} - \Delta_b\mathring{N}b_{ij})g^{ik}g^{j\ell}e_{k\ell},$$

$$(4.15) \qquad Q := \mathring{N}(g^{ij} - b^{ij} + g^{ik}g^{j\ell}e_{k\ell})\operatorname{Ric}(b)_{ij} + Q'.$$

Brackets over a symbol denote anti-symmetrization, with an appropriate numerical factor ($1/2$ in the case of two indices), and $\mathring{D}$ denotes the covariant derivative operator of the metric b; note that ρ here should not be confused with the defining function of the boundary. Here Q' denotes an expression that is bilinear in

$$e \equiv e_{ij}\,dx^i\,dx^j := (g_{ij} - b_{ij})\,dx^i\,dx^j,$$

and in $\mathring{D}_k e_{ij}$, linear in $\mathring{N}$, $d\mathring{N}$ and Hess $\mathring{N}$, with coefficients that are constants in any ON frame for b. The key is that ρ vanishes when $\mathring{N}$ is in the kernel of P_b^*, and then Q is at least quadratic in e near $e = 0$. Indeed, the first term on the right-hand side of (4.15) does not contain any terms linear in e_{ij} when Taylor expanded at $g_{ij} = b_{ij}$.

The integral of $\mathbb{U}$ at the boundary at infinity provides the momentum vector, and we need to know how fast the limit is approached. The simplest case arises when g is a Kottler metric b_m with mass parameter m, so that (see (1.2) and (1.3))

$$(4.16) \qquad b_m = \frac{dr^2}{W^2} + r^2 \mathring{h}, \quad b = \frac{dr^2}{\mathring{W}^2} + r^2 \mathring{h},$$

where $\mathring{h}$ is the unit round metric on the sphere $\mathbb{S}^{n-1}$. If $\{r\} \times \mathbb{S}^{n-1}$ is positively oriented, a calculation gives

$$(4.17)$$
$$\int_{\{r\} \times \mathbb{S}^{n-1}} \mathbb{U}^i dS_i = 2\omega_{n-1}(n-1)\mathring{W}W^{-1}m = 2\omega_{n-1}(n-1)m + O(r^{-n}),$$

where ω_{n-1} is the volume of $\mathbb{S}^{n-1}$. An identical formula, with ω_{n-1} replaced by the $\widehat{b}$–volume of N, holds for the non-spherical Kottler metrics (2.5). Next, assume that $|g - b|_b = O(r^{-\alpha})$, where r is a coordinate for b as in (4.16), and $\mathring{h}$ is an (r–independent) metric on the compact conformal boundary N, with the same decay rate for first derivatives, and with $R(g) = R(b)$. Integrating (4.12) over $[r, \infty) \times N$ one finds, for $\alpha > n/2$,

$$(4.18) \qquad \int_{\{r\} \times N} \mathbb{U}^i(V_{(\mu)}) \, dS_i = p_{(\mu)} + O(r^{n-2\alpha}),$$

which coincides of course with (4.17) if $\alpha = n$; we note that $\alpha = n$ is the appropriate rate for Kottler metrics, whether boosted or not.

To calculate (4.11) explicitly, let $V_{\delta,(\mu)}$ be a basis of K_{b_δ}, the vanishing of (4.11) is then equivalent to the vanishing of the collection of integrals $J_\delta(p) = (J_{\delta,(\nu)})$, where $p = (p_{(\mu)})$ and

$$J_{\delta,(\nu)} := \int_{A_{1,4}} \psi^2 \psi^{-2}(R(\widetilde{g}_{\delta,p_{(\mu)}}) - R(b_\delta))V_{\delta,(\nu)} d\mu_{b_\delta}.$$

In order to use (4.12) we need to change the measure $d\mu_{b_\delta}$ to $d\mu_{\widetilde{g}_{\delta,p_{(\mu)}}}$, the following estimates are useful for that:

$$R(\widetilde{g}_{\delta,p_{(\mu)}}) - R(b_\delta) = O(\delta^\alpha) + O(\delta^{n+\beta}),$$
$$d\mu_{b_\delta} = \left(1 + O(\delta^{\alpha-4}) + O(\delta^{n+\beta-4}) + O(\delta^n)\right) d\mu_{\widetilde{g}_{\delta,p_{(\mu)}}}.$$

Keeping in mind that $V_{\delta,(\nu)}$ behaves as δ^{-1}, and that the volume form grows as δ^{1-n}, this leads to

$$
\begin{aligned}
J_{\delta,(\nu)} &= \int_{A_{1,4}} \psi^2 \psi^{-2} (R(\widetilde{g}_{\delta,p_{(\mu)}}) - R(b_\delta)) V_{\delta,(\nu)} d\mu_{\widetilde{g}_{\delta,p_{(\mu)}}} \\
&\quad + O(\delta^{2\alpha-4-n}) + O(\delta^{2(\beta+n)-4-n}) + O(\delta^\alpha) + O(\delta^{n+\beta}) \\
&= p^0_{(\nu)} - p_{(\nu)} + O(\delta^{2(\alpha-4)-n}) + O(\delta^{2(\beta+n-4)-n}) + O(\delta^n) + O(\delta^{2\alpha-n}) \\
&= \delta^\beta \left(\frac{p^0_{(\nu)} - p_{(\nu)}}{\delta^\beta} + O(\delta^{2\alpha-8-n-\beta}) + O(\delta^{\beta+n-8}) + O(\delta^{n-\beta}) + O(\delta^{2\alpha-\beta-n}) \right).
\end{aligned}
$$

Here the terms $O(\delta^{2(\alpha-4)-n})$ and $O(\delta^{2(\beta+n-4)-n})$ in the third line arise from the terms quadratic in (4.12) and from the first two terms in the second line; while the terms $O(\delta^n)$ and $O(\delta^{2\alpha-n})$ arise from the difference between the boundary term and its limit (namely $p^0_{(\mu)} - p_{(\mu)}$) when δ goes to zero, cf. (4.17) and (4.18), and also contain the last two terms in the second line. To close the argument all error terms should go to zero as δ tends to zero, thus

$$(4.19) \qquad 2\alpha - 8 - n - \beta > 0, \quad \beta + n - 8 > 0, \quad n - \beta > 0.$$

(Note that (4.9) does not impose any further restrictions.) This is equivalent to

$$(4.20) \qquad n \geq 5, \quad \alpha > \frac{8+n+\beta}{2}, \quad \max(8-n,0) < \beta < n.$$

So β can be chosen consistently with those bounds provided that (1.4) holds.

If the kernel of P_b^* is one dimensional, with $p^0_{(\mu)} = m_0$, then for δ small, using the intermediate value theorem, there exists $p_{(0)} = m$ in an interval $[-m_0, 2m_0]$ such that $J_\delta(m) = 0 = I_\delta(m)$, proving existence of a solution.

Otherwise, under (4.20), we can use a Brouwer fixed point theorem as in Lemma 3.18 of [8] with:

- U: a bounded open ball of centre 0 in $\mathbb{R}^{n+1}$;
- $G = \mathrm{Id}$: $q_{(\mu)} \mapsto q_{(\mu)}$,
- $V = U$,
- $\lambda = 1/\delta$ and $G_\lambda = G_{1/\delta} = \delta^{-\beta} J_\delta(p^0_{(\mu)} + \delta^\beta q_{(\mu)})$,
- $y = 0$.

This shows that for small δ, we can choose $p_{(\mu)}$ so that $I_\delta(p_{(\mu)}) = 0$, which again proves existence of a solution.

Regularity follows from [9, Theorem 4.9]. This completes the proof of Theorem 1.2. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

5. The gluing construction on a fixed annulus

The question addressed in Theorem 1.1 is a special case of the following: In dimension $n \geq 3$, consider an n-dimensional submanifold $M \subset M_{\text{ext}}$, where M_{ext} has been defined in (1.1), with compact, connected, non-empty boundary ∂M which separates M_{ext} into two components, one of which is bounded. We further suppose that M is included in the bounded component, and that M is equipped with a metric $g \in C^{\ell,\lambda}$, $\ell > \lfloor \frac{n}{2} \rfloor + 4$, $\lambda \in (0,1)$, of constant negative scalar curvature.

Let $M_1 \subset M$ be a one-sided collar neighbourhood of ∂M contained in M, we will refer to M_1 as the *interior collar*.

Let $M_2 \subset M_{\text{ext}}$ be a one-sided collar neighbourhood of ∂M which lies in the unbounded component of M_{ext}, we refer to M_2 as the *exterior collar*.

The *extension problem* is to find a constant scalar curvature metric on $M \cup M_2$ which coincides with g on M, and which coincides with a Kottler metric near $\partial M_2 \smallsetminus \partial M$. In this case we set $M' = M$ and $M'' = M \cup M_2$. In this problem one would presumably want M_2 to be small: a solution with a small M_2 provides a solution for any bigger one.

The *deformation problem* is to find a constant scalar curvature metric on M which coincides with g on $M \smallsetminus M_1$, and which coincides with a Kottler metric near ∂M. In this case we set $M' = M \smallsetminus M_1$ and $M'' = M$. Similarly to the previous problem, one would presumably want M_1 to be small.

Let $p^0_{(\mu)}$ denote the energy-momentum vector of ∂M, defined as

$$(5.1) \qquad\qquad p^0_{(\mu)} = \int_{\partial M} \mathbb{U}^i(V_{(\mu)}) \, dS_i,$$

with $\mathbb{U}$ as in (4.13), and $V_{(\mu)}$ defined in Section 2. If $(N, \widehat{b})$ is the round sphere, then $p^0_{(\mu)}$ is a vector in $\mathbb{R}^{n+1}$. Otherwise $p^0_{(\mu)}$ is simply a number, say m_0.

Denote by

$$\psi_{\partial M} : b_{p_{(\mu)}} \mapsto \int_{\partial M} \mathbb{U}^i(V_{(\mu)}) \, dS_i,$$

the map which to a Kottler metric $b_{p_{(\mu)}}$ associates the energy-moment vector of ∂M, where $\mathbb{U}^i$ is calculated using (4.13) with g there replaced by $b_{p_{(\mu)}}$.

Remark 5.1. As an illustration, assume that $\partial M = \{r\} \times N$ for some r. Suppose that $(N, \widehat{b})$ is *not* a round sphere, then $\psi_{\partial M}^{-1}$ is a smooth diffeomorphism between an interval of masses around m_0 and its image; this follows immediately from the (non-spherical equivalent of the) first equality in (4.17). The result remains true in the spherical case when one restricts $\psi_{\partial M}$ to the standard, *unboosted* Kottler metrics b_m as given by (1.2).

Similarly, consider $\partial M = \{r\} \times N$ for some r, with $(N, \widehat{b})$ — a round sphere, and assume moreover that $p^0_{(\mu)}$ lies in the image of $\psi_{\partial M}$. It is then easily seen from (4.18) that $\psi_{\partial M}^{-1}$ provides a smooth diffeomorphism between a neighbourhood of $p^0_{(\mu)}$ and its image *provided that r is large enough*.

We have the following:

Theorem 5.1. *Let $n \geq 3$, $\mathbb{N} \ni \ell > \lfloor \frac{n}{2} \rfloor + 4$, $\lambda \in (0,1)$. Assume that the map $\psi_{\partial M}^{-1}$ is a homeomorphism of a neighbourhood of $p^0_{(\mu)}$ and its image. There exists $\varepsilon > 0$ such that if*

$$\|g - b_{\psi^{-1}(p^0_{(\mu)})}\|_{C^{\ell,\lambda}(M)} < \varepsilon,$$

then there exists a $C^{\ell,\lambda}$ metric of constant negative scalar curvature which coincides with g on M', and which is a Kottler metric on the unbounded component of $M_{ext} \smallsetminus \partial M$ away from M''. If g is smooth, then so is the solution of the deformation problem.

Proof. We proceed as in [8, Section 8.6] but we use the refined versions of Theorem 5.6 and Proposition 5.7 of [8] used there, as given by Theorems 3.1 and 4.9 of [9] (cf. Section 6.3 of [9]). The gluing is done on the collar neighbourhood $[0,1] \times \partial M$, with g_0 there being $b_{p_{(\mu)}}$, $K = \delta J = 0$, g there equal to $\chi g + (1 - \chi) b_{p_{(\mu)}}$ where χ is a cutoff function that vanishes near ∂M, which we identify with $\{1\} \times \partial M$; finally, $p_{(\mu)}$ is close to $\psi_{\partial M}^{-1}(p^0_{(\mu)})$. One thus obtains a solution modulo kernel (note that for the estimates on pp. 53–54 of [8], we have to replace ρ there with $R + n(n-1)$). For the kernel projection (see [8, Equation (8.24), p. 55]) we proceed as in [8, p. 55], where Q there is replaced by p here. By Lemma 3.3 in [11] the kernel at $b_{\psi^{-1}(p^0_{(\mu)})}$ is one dimensional except in the spherical case with $b_{\psi^{-1}(p^0_{(\mu)})} = b$, so except for this last case this is a straightforward continuity argument by varying masses in an interval around m_0. The Hölder regularity of the final metric follows from Theorem 4.9 of [9]. $\qquad\square$

Proof of Theorem 1.1. The result follows immediately from Theorem 5.1 and Remark 5.1 except in the spherical case with $b_{\psi^{-1}(p^0_{(\mu)})} = b$. In this last

situation, the supplementary hypothesis of parity insures that all constructions can be made within the class of parity symmetric metrics. The kernel within this class is one dimensional, and the solution can be adjusted by changing b_m in the exterior region within the family of unboosted Kottler metrics; cf. the proof of Theorem 2.1 in [7]. □

6. b-conformal deformations near infinity

Let $\overline{M}$ be a compact manifold with boundary, set $M = \overline{M} \setminus \partial M$, and let ρ be a defining function for ∂M. Let $\bar{b}$ be a $C^{k+2,\alpha}$ metric on $\overline{M}$. Let h be covariant symmetric two tensor field such that $g = b + h$ is positive definite, and for functions $v > -1$ set $u = 1 + v$. For $\delta > 0$ and $h \in C_\delta^{k+2,\alpha}$ we consider the function F_g defined on a neighbourhood of zero in $C_\delta^{k+2,\alpha}$ to $C_\delta^{k,\alpha}$ as

$$F_g(v) = -4\frac{n-1}{n-2}\nabla^k\nabla_k u + R(g)u + n(n-1)u^{(n+2)/(n-2)},$$

where covariant derivatives are related to $g = b + h$, with the spaces $C_\delta^{k,\alpha}$ of tensors fields or functions as in [16]. Note that $F_g(v) = 0$ if and only if the scalar curvature of $u^{4/(n-2)}g$ equals $-n(n-1)$. The map F_g is smooth near zero and, if $R(g) = -n(n-1)$, the derivative at $v = 0$ given by

$$F_g'(0)w = 4\frac{n-1}{n-2}(-\nabla^k\nabla_k + n)w.$$

The map $F_g'(0)$ is an isomorphism from $C_\delta^{k+2,\alpha}$ to $C_\delta^{k,\alpha}$ when $\delta \in (-1, n)$ [2, Theorem 7.2.1], so in particular when $\delta \in (0, n)$. The implicit function theorem then shows:

Proposition 6.1. *Let $k \in \mathbb{N}$, $\delta \in (0, n)$, $\alpha \in (0, 1)$, and for $\mathring{h} \in C_\delta^{k+2,\alpha}$ let $\mathring{g} = b + \mathring{h}$ be a metric on M as described above with constant scalar curvature $-n(n-1)$. There exists $\varepsilon > 0$ and a constant C such that for any $h \in C_\delta^{k+2,\alpha}$ with norm less than ε there exists a unique $v \in C_\delta^{k+2,\alpha}$ satisfying*

$$F_{\mathring{g}+h}(v) = 0, \quad v > -1, \quad \|v\|_{C_\delta^{k+2,\alpha}} \le C\|h\|_{C_\delta^{k+2,\alpha}},$$

so that the tensor field $u^{4/(n-2)}(\mathring{g} + h)$ defines a Riemannian metric with constant scalar curvature $-n(n-1)$. The map $h \mapsto v$ is smooth near zero.

Given a Riemannian metric $\mathring{g}$ as in the statement of Proposition 6.1, we will use that proposition to construct metrics that are arbitrarily close to $\mathring{g}$

on compact sets, and which are conformal to the background b near infinity, as follows. Let $0 < \delta < \sigma$, $\delta < n$, and let g_a be the metric interpolating between $\mathring{g}$ and b on the annulus $A_{a,4a}$, as in (4.2). Then $g_a = \mathring{g} + h_a$, where $h_a \in C^{k+2,\alpha}_\sigma \subset C^{k+2,\alpha}_\delta$, with $\|h_a\|_{C^{k+2,\alpha}_\delta}$ going to zero when a does. Proposition 6.1 shows that for all a small enough there exists $v_a \in C^{k+2,\alpha}_\delta$ satisfying $v_a > -1$ such that the metric

$$(6.1) \qquad\qquad \widehat{g}_a := (1 + v_a)^{4/(n-2)} g_a$$

has scalar curvature $-n(n-1)$, and v_a goes to zero in $C^{k+2,\alpha}_\delta$ when a does. In particular, v_a goes to zero with $(k+2, \alpha)$ derivatives uniformly on any compact subset of M.

The metric $\widehat{g}_a$ is conformal to b near the conformal boundary at infinity. If we assume that $\rho^2 b$ is smooth up to boundary at ∂M, then the conformal factor u is polyhomogeneous at the conformal boundary [2, 3]. Further [3], the asymptotic expansion of $u_a = 1 + v_a$ is identical to that of the background metric b up to terms $O(\rho^n)$. This implies that u_a is in fact smooth up to boundary and, for small ρ,

$$(6.2) \qquad\qquad |\widehat{g}_a - b|_b = O(\rho^n).$$

If b has the form (1.2), with $\widehat{b}$–Einstein, and if $\sigma > n/2$, then the energy-momentum vector $p_{(\mu)}$ of $\mathring{g}$ is well defined. We can then choose $\delta > n/2$, in which case it immediately follows from the definition of $p_{(\mu)}$ and from (4.12) to (4.15) that the energy-momentum vector of $(1 + v_a)^{4/(n-2)} g_a$ tends to that of $\mathring{g}$ as a goes to zero.

As we have seen, the construction can be done rather generally, resulting in a small conformal deformation of the metric on compact sets. It turns out that the deformation can be localized to the asymptotic region if one supposes, moreover, that $\mathring{g}$ is *not* static in the asymptotic region; by this we mean that $P^*_{\mathring{g}}$ has no kernel on M_ε for all ε small enough. Then the deformation can be localized to the exterior region, in the sense that for any $\varepsilon > 0$ we can find a constant scalar curvature metric $\tilde{g}_\varepsilon$ which coincides with $\mathring{g}$ on $M \setminus M_\varepsilon$, with M_ε as in (2.1), and which is conformal to b near the conformal boundary. The construction goes as follows: By the arguments in [4] there exists a sequence of annuli $A_{a_i,4a_i}$ on which $P^*_{\mathring{g}}$ has no kernel. Choose $a_{i_0} < \varepsilon/4$, for all $a < a_{i_o}$ small enough let $\widehat{g}_a$ be as in (6.1), then $\widehat{g}_a$ restricted to $A_{a_{i_0},4a_{i_0}}$ approaches zero in $C^{k+2,\alpha}$. By the gluing results of [8, 9], for $k > \lfloor \frac{n}{2} \rfloor + 2$ and for a small enough $\widehat{g}_a$ can be deformed within $A_{a_{i_0},4a_{i_0}}$ to a metric $\tilde{g}_\varepsilon$ with constant scalar curvature which coincides with

$\mathring{g}$ on $M \smallsetminus M_{4a_{i_0}} \subset M \smallsetminus M_\varepsilon$, and which coincides with $\widehat{g}_a$ on $M_{a_{i_0}}$, hence is conformal to b near the conformal boundary at infinity. In particular $\tilde{g}_\varepsilon$ approaches b as $O(\rho^n)$ by (6.2).

Summarizing, we have proved the following result, somewhat reminiscent of [21, Proposition 4.1]:

Proposition 6.2. *Let* $\dim M = n \geq 3$, $C, \sigma > 0$, $\ell \in \mathbb{N}$, $\ell \geq 2$, $\alpha \in (0,1)$ *and suppose that* $\mathring{g} = b + \mathring{h}$ *is a Riemannian metric with scalar curvature* $-n(n-1)$ *with* $\rho^2 b \in C^{l,\alpha}(\overline{M})$ *and* $\mathring{h} \in C_\sigma^{l,\alpha}$. *Then:*

(1) *For all* $\varepsilon > 0$ *there exists a metric* $\mathring{g}_\varepsilon$ *with scalar curvature* $-n(n-1)$, *conformal to* $\mathring{g}$ *away from* M_ε, *and conformal to* b *near the conformal boundary.*

(2) *Furthermore* $\mathring{g}_\varepsilon$ *converges to* $\mathring{g}$ *in* $C^{\ell,\alpha}(\mathscr{U})$ *topology on any relatively compact open subset* $\mathscr{U}$ *of* M.

(3) *If* $\rho^2 b$ *is sufficiently differentiable at the conformal boundary (e.g., smooth), then the metrics* $\mathring{g}_\varepsilon$ *approach* b *as* $O(\rho^n)$ *for small* ρ.

(4) *If* b *is of the form* (1.2) *with* $\widehat{b}$ *Einstein, and if* $\sigma > n/2$, *then the energy-momentum of* $\mathring{g}_\varepsilon$ *approaches that of* $\mathring{g}$ *as* ε *tends to zero.*

(5) *If* $\ell > \lfloor \frac{n}{2} \rfloor + 4$ *and if there exists* $\varepsilon_0 > 0$ *such that* $P_{\mathring{g}}^*$ *has no kernel on* $A_{\varepsilon_0/4, \varepsilon_0}$, *then* $\mathring{g}_\varepsilon$ *can be chosen to coincide with* $\mathring{g}$ *away from* M_{ε_0}, *but then the convergence of point* (2) *to* $\mathring{g}$ *is in* $C^{\ell-2,\alpha}(\mathscr{U})$ *topology only.*

Appendix A. The asymptotics of P^*

A.1. Conformally compact metrics

In this section, we study the behaviour of the operator P_g^*, when rescaled from $A_{\delta, 4\delta}$ to $A_{1,4}$, with δ tending to zero, for conformally compact metric, asymptotically hyperbolic in the sense of [18]. We consider on $M_{4\delta}$ a metric of the form

$$(A.1) \qquad\qquad g = \rho^{-2}(d\rho^2 + \widehat{g}(\rho)) =: \rho^{-2}\overline{g}.$$

This metric is conformally compact and $|d\rho|_{\overline{g}} = 1$ at infinity, so

$$\mathrm{Ric}(g) = -(n-1)g + O(\rho),$$

where $O(\rho)$ is a symmetric covariant two tensor with g-norm of order $O(\rho)$ (equivalently $\bar{g}$-norm of order $O(\rho^{-1})$).

We study the metric on $A_{\delta,4\delta}$, of course this calculation is valid for $g = b$ with b as is (2.3) or (1.3). This can be pulled-back to $A = A_{1,4}$ using the change of variable $\rho = \delta z$ to

$$g_\delta = z^{-2}(dz^2 + \delta^{-2} \, {}^\delta\widehat{g}(z)),$$

where $\,{}^\delta\widehat{g}(z) = \widehat{g}(\delta z)$. The determinant reads

$$\det(g_\delta) = z^{-2n}\delta^{-2(n-1)} \det(\,{}^\delta\widehat{g}).$$

The non-trivial Christoffel symbols of g_δ are

$${}^\delta\Gamma^z_{zz} = -z^{-1},$$

$${}^\delta\Gamma^z_{AB} = -\frac{z^2}{2}(-2z^{-3}\delta^{-2} \, {}^\delta\widehat{g}_{AB} + z^{-2}\delta^{-2}\partial_z \, {}^\delta\widehat{g}_{AB}),$$

$${}^\delta\Gamma^C_{Az} = \,{}^\delta\Gamma^C_{zA} = \frac{1}{2}(-2z^{-1}\delta^C_A + \,{}^\delta\widehat{g}^{BC}\partial_z \, {}^\delta\widehat{g}_{AB}),$$

$${}^\delta\Gamma^C_{AB} = \Gamma^C_{AB}(\,{}^\delta\widehat{g}) =: \,{}^\delta\widehat{\Gamma}^C_{AB}.$$

We note that $\partial_z \, {}^\delta\widehat{g}_{AB}(z) = \delta\partial_\rho\widehat{g}_{AB}(\delta z) = O(\delta)$. The Hessian of a function u takes the form

$${}^\delta\nabla_z\partial_z u = \partial_z^2 u + z^{-1}\partial_z u,$$

$${}^\delta\nabla_z\partial_A u = \partial_z\partial_A u + (z^{-1}\delta^C_A + O(\delta)^C_A)\partial_C u,$$

$${}^\delta\nabla_A\partial_B u = \,{}^\delta\widehat{\nabla}_A\partial_B u - (z^{-1}\delta^{-2} \, {}^\delta\widehat{g}_{AB} + O(\delta^{-1})_{AB})\partial_z u,$$

thus

$${}^\delta\nabla^k\partial_k u = z^2\partial_z^2 u - [(n-2)z + O(\delta)]\partial_z u + z^2\delta^{2\delta}\widehat{\nabla}^A\partial_A u.$$

This gives

$$(P^*_{g_\delta}u)_{zz} = [(n-1)z^{-1} + O(\delta)]\partial_z u - u\,\mathrm{Ric}(g_\delta)_{zz} - \delta^{2\delta}\widehat{\nabla}^A\partial_A u,$$

$$(P^*_{g_\delta}u)_{zA} = \partial_z\partial_A u + (z^{-1}\delta^C_A + O(\delta)^C_A)\partial_C u - u\,\mathrm{Ric}(g_\delta)_{zA},$$

$$(P^*_{g_\delta}u)_{AB} = \,{}^\delta\widehat{\nabla}_A\partial_B u - \,{}^\delta\widehat{\nabla}^C\partial_C u \, {}^\delta\widehat{g}_{AB} - \delta^{-2}\partial_z^2 u \, {}^\delta\widehat{g}_{AB}$$

$$+ [(n-3)z^{-1}\delta^{-2} \, {}^\delta\widehat{g}_{AB} + O(\delta^{-1})_{AB}]\partial_z u - u\,\mathrm{Ric}(g_\delta)_{AB}.$$

Now, recall that $\mathrm{Ric}(g) = -(n-1)g + \rho^{-1}T$, where T is $\bar{g}$ bounded. As $\bar{g} = d\rho^2 + \widehat{g}(\rho) = \delta^2\,dz^2 + \,{}^\delta\widehat{g}(z)$, we have that $T_{zz} = O(\delta^2)$, $T_{zA} = O(\delta)$ and

$T_{AB} = O(1)$, thus the coordinate components of $P_{g_\delta}^* u$ are

$$(P_{g_\delta}^* u)_{zz} = [(n-1)z^{-1} + O(\delta)]\partial_z u + [(n-1)z^{-2} + O(\delta)]u - \delta^2 \,^\delta\widehat{\nabla}^A \partial_A u,$$
$$(P_{g_\delta}^* u)_{zA} = \partial_z \partial_A u + (z^{-1}\delta_A^C + O(\delta)_A^C)\partial_C u - uO(1)_{zA},$$
$$(P_{g_\delta}^* u)_{AB} = \,^\delta\widehat{\nabla}_A \partial_B u - \,^\delta\widehat{\nabla}^C \partial_C u \,^\delta\widehat{g}_{AB} - \delta^{-2}\partial_z^2 u \,^\delta\widehat{g}_{AB}$$
$$+ [(n-3)z^{-1}\delta^{-2} \,^\delta\widehat{g}_{AB} + O(\delta^{-1})_{AB}]\partial_z u$$
$$+ u(n-1)(z^{-2}\delta^{-2} \,^\delta\widehat{g}_{AB} + O(\delta^{-1})_{AB}).$$

A.2. The (C, k, σ)-asymptotically hyperbolic case

In this section we compare the behaviour of the operator P_g^* with that of P_b^*, when rescaled from $A_{\delta,4\delta}$ to $A_{1,4}$, for (C, k, σ)-asymptotically hyperbolic metrics of the form (A.1). We also give an explicit formula for P_b^* and its kernel for metrics of the form (2.3).

If $k \in \mathbb{N}$, $\sigma > 0$, and g is (C, k, σ)-asymptotically hyperbolic with b of the form (2.3), we have

$$\mathrm{Ric}(g) = \mathrm{Ric}(b) + O(\rho^\sigma) = -(n-1)b + O(\rho^\sigma) = -(n-1)g + O(\rho^\sigma),$$

where $O(\rho^\sigma)$ is a symmetric covariant two tensor with g-norm (or b-norm) of order $O(\rho^\sigma)$ (equivalently $\bar{g}$-norm of order $O(\rho^{\sigma-2})$) .

First, we have $\widehat{g}(\rho) - \widehat{b}(\rho) = O(\rho^\sigma)$ and $\partial_\rho[\widehat{g} - \widehat{b}](\rho) = O(\rho^{\sigma-1})$, so that

$$\,^\delta\widehat{g}(z) - \,^\delta\widehat{b}(z) = O(\delta^\sigma),$$
$$\partial_z[\,^\delta\widehat{g} - \,^\delta\widehat{b}\,](z) = \delta\, O(\delta^{\sigma-1}) = O(\delta^\sigma).$$

The non-trivial Christoffel symbols of g_δ are

$$\,^\delta\Gamma_{zz}^z = -z^{-1} = \,^\delta\Gamma_{zz}^z(b_\delta),$$
$$\,^\delta\Gamma_{AB}^z = \,^\delta\Gamma_{AB}^z(b_\delta) + O(\delta^{\sigma-2})_{AB},$$
$$\,^\delta\Gamma_{Az}^C = \,^\delta\Gamma_{zA}^C = \,^\delta\Gamma_{Az}^C(b_\delta) + O(\delta^\sigma)_A^C,$$
$$\,^\delta\Gamma_{AB}^C = \Gamma_{AB}^C(\,^\delta\widehat{g}) = \,^\delta\widehat{\Gamma}_{AB}^C(\,^\delta\widehat{b}) + O(\delta^\sigma)_{AB}^C.$$

Let ν be a one-form on A. To make things clear, let ${}^{\delta}\widetilde{\nabla}$ denote the covariant derivative operator of the metric b_{δ} and ${}^{\delta}\nabla$ the one for g_{δ}, we have

$$
\begin{aligned}
{}^{\delta}\nabla_z \nu_z &= \partial_z \nu_z + z^{-1}\nu_z, \\
{}^{\delta}\nabla_z \nu_A &= {}^{\delta}\widetilde{\nabla}_z \nu_A + O(\delta^{\sigma})_A^C \nu_C, \\
{}^{\delta}\nabla_A \nu_z &= {}^{\delta}\widetilde{\nabla}_A \nu_z + O(\delta^{\sigma})_A^C \nu_C, \\
{}^{\delta}\nabla_A \nu_B &= {}^{\delta}\widetilde{\nabla}_A \nu_B + O(\delta^{\sigma-2})_{AB}\nu_z + O(\delta^{\sigma})_{AB}^C \nu_C,
\end{aligned}
$$

where the error terms are measured with any fixed metric on the compact set $\overline{A}$, e.g. $dz^2 + \widehat{b}(0)$.

If $\nu = du$, then

$$
\begin{aligned}
{}^{\delta}\nabla_z \partial_z u &= \partial_z^2 u + z^{-1}\partial_z u = {}^{\delta}\widetilde{\nabla}_z \partial_z u, \\
{}^{\delta}\nabla_z \partial_A u &= {}^{\delta}\widetilde{\nabla}_z \partial_A u + O(\delta^{\sigma})_A^C \partial_C u, \\
{}^{\delta}\nabla_A \partial_B u &= {}^{\delta}\widetilde{\nabla}_A \partial_B u + O(\delta^{\sigma-2})_{AB}\partial_z u + O(\delta^{\sigma})_{AB}^C \partial_C u,
\end{aligned}
$$

thus

$$
{}^{\delta}\nabla^k \partial_k u = {}^{\delta}\widetilde{\nabla}^k \partial_k u + O(\delta^{\sigma})\partial_z u + O(\delta^{\sigma+2})^{AB}\, {}^{\delta}\widehat{\widetilde{\nabla}}_B \partial_A u + O(\delta^{\sigma+2})^C \partial_C u.
$$

We obtain for the components of $P^*_{g_{\delta}} u$:

$$
\begin{aligned}
(P^*_{g_{\delta}} u)_{zz} &= (P^*_{b_{\delta}} u)_{zz} + O(\delta^{\sigma})\partial_z u + O(\delta^{\sigma})u, \\
&\quad + O(\delta^{\sigma+2})^{AB}\, {}^{\delta}\widehat{\widetilde{\nabla}}_B \partial_A u + O(\delta^{\sigma+2})^C \partial_C u \\
(P^*_{g_{\delta}} u)_{zA} &= (P^*_{b_{\delta}} u)_{zA} + O(\delta^{\sigma})_A^C \partial_C u + O(\delta^{\sigma-1})_A u, \\
(P^*_{g_{\delta}} u)_{AB} &= (P^*_{b_{\delta}} u)_{AB} + O(\delta^{\sigma-2})_{AB}\partial_z u + O(\delta^{\sigma})_B^C\, {}^{\delta}\widehat{\widetilde{\nabla}}_C \partial_A u \\
&\quad + O(\delta^{\sigma})_{AB}^C \partial_C u + O(\delta^{\sigma-2})_{AB} u.
\end{aligned}
$$

Next, we compute the explicit expression of $P^*_{b_{\delta}}$ for a metric of the form (2.3). In that case we have

$$
{}^{\delta}\widehat{b}(z) = \left[1 - k\left(\frac{\delta z}{2}\right)^2\right]^2 \widehat{b},
$$

then

$$
\partial_z({}^{\delta}\widehat{b})(z) = -k\delta^2 z \left[1 - k\left(\frac{\delta z}{2}\right)^2\right]\widehat{b}.
$$

The non-trivial Christoffel symbols of b_δ are

$$^\delta\Gamma^z_{zz} = -z^{-1},$$

$$^\delta\Gamma^z_{AB} = \delta^{-2}z^{-1}\left[1 - k\left(\frac{\delta z}{2}\right)^2\right]\left[1 + k\left(\frac{\delta z}{2}\right)^2\right]\widehat{b}_{AB},$$

$$^\delta\Gamma^C_{Az} = -z^{-1}\left[1 - k\left(\frac{\delta z}{2}\right)^2\right]^{-1}\left[1 + k\left(\frac{\delta z}{2}\right)^2\right]\delta^C_A,$$

$$^\delta\Gamma^C_{AB} = \Gamma^C_{AB}(\widehat{b}).$$

We thus obtain for the components of the Hessian of u:

$$^\delta\widetilde{\nabla}_z\partial_z u = \partial^2_z u + z^{-1}\partial_z u$$

$$^\delta\widetilde{\nabla}_z\partial_A u = \partial_z\partial_A u + z^{-1}\left[1 - k\left(\frac{\delta z}{2}\right)^2\right]^{-1}\left[1 + k\left(\frac{\delta z}{2}\right)^2\right]\partial_A u,$$

$$^\delta\widetilde{\nabla}_A\partial_B u = \widehat{\nabla}_A\partial_B u - \delta^{-2}z^{-1}\left[1 - k\left(\frac{\delta z}{2}\right)^2\right]\left[1 + k\left(\frac{\delta z}{2}\right)^2\right]\widehat{b}_{AB}\,\partial_z u,$$

thus

$$^\delta\widetilde{\nabla}^k\partial_k u = z^2\partial^2_z u + \left\{1 - (n-1)\left[1 - k\left(\frac{\delta z}{2}\right)^2\right]^{-1}\left[1 + k\left(\frac{\delta z}{2}\right)^2\right]\right\}z\partial_z u$$

$$+ z^2\delta^2\left[1 - k\left(\frac{\delta z}{2}\right)^2\right]^{-2}\widehat{\nabla}^A\partial_A u.$$

One checks that $\mathrm{Ric}(b) = -(n-1)b$, which implies

$$\mathrm{Ric}(b_\delta) = -(n-1)b_\delta.$$

We obtain for the components of $P^*_{b_\delta}u$:

$$(P^*_{b_\delta}u)_{zz} = (n-1)\left\{\left[1 - k\left(\frac{\delta z}{2}\right)^2\right]^{-1}\left[1 + k\left(\frac{\delta z}{2}\right)^2\right]z^{-1}\partial_z u + z^{-2}u\right.$$

$$\left. - \frac{1}{n-1}\delta^2\left[1 - k\left(\frac{\delta z}{2}\right)^2\right]^{-2}\widehat{\nabla}^A\partial_A u\right\},$$

$$(P_{b_\delta}^* u)_{zA} = \partial_z \partial_A u + z^{-1} \left[1 - k \left(\frac{\delta z}{2} \right)^2 \right]^{-1} \left[1 + k \left(\frac{\delta z}{2} \right)^2 \right] \partial_A u,$$

$$(P_{b_\delta}^* u)_{AB} = \widehat{\nabla}_A \partial_B u - \widehat{\nabla}^C \partial_C u \, \widehat{b}_{AB} - \delta^{-2} z^{-2} \left[1 - k \left(\frac{\delta z}{2} \right)^2 \right]^2 \left\{ z^2 \partial_z^2 u \right.$$

$$\left. + z \left(1 - (n-2) \frac{1 + k \left(\frac{\delta z}{2} \right)^2}{1 - k \left(\frac{\delta z}{2} \right)^2} \right) \partial_z u - (n-1)u \right\} \widehat{b}_{AB}.$$

Now, a function u is in the kernel of $P_{b_\delta}^*$ if and only if

$$^\delta \nabla \partial u = u \, b_\delta.$$

One checks that

$$u_0 = z^{-1} \left[1 + k \left(\frac{\delta z}{2} \right)^2 \right],$$

is indeed in this kernel. Further, if v is a non-trivial solution of the Obata-type equation

$$\widehat{\nabla}_A \partial_B v = -kv \, \widehat{b}_{AB}$$

on the boundary at infinity, then the function

$$u = vz^{-1} \left[1 - k \left(\frac{\delta z}{2} \right)^2 \right]$$

satisfies $(P_{b_\delta}^* u)_{zz} = (P_{b_\delta}^* u)_{zA} = 0$, with $((P_{b_\delta}^* u)_{AB} = 0$ if and only if $k = 1$). Finally, it is an easy exercise to show that these functions generate the kernel of $P_{b_\delta}^*$. Here one can use the well-known fact that the kernel of P^* has dimension at most $n + 1$ (see, e.g., [12]).

Appendix B. Proof of Lemma 3.2

Throughout this appendix we write A_δ for $A_{\delta,4\delta}$ and A for $A_{1,4}$; we hope that the clash of notation with the A-spaces occasionally used elsewhere will not confuse the reader. We start by scaling A_δ to $A = (1,4) \times \partial M$. Recall that the weight function $\phi = y^2$ on A relevant for the calculation at hand equals $(z-1)^2(z-4)^2/9$, where z runs along the $(1,4)$ factor of A. The argument that follows actually applies to any non-negative function $\phi = \phi(z)$ which

vanishes precisely at the boundary of A and satisfies

$$\phi(1) = \phi(4) = \phi'(1) = \phi'(4) = 0, \quad \phi''(1) > 0, \phi''(4) > 0.$$

The idea of the proof is to cover $(1, 4)$ by intervals I_i, with sizes chosen so that on each interval the ratio $\sup \phi / \inf \phi$ is bounded independently of i. Furthermore, the size of each interval should be of the order of the value of ϕ on the interval, to ensure good scaling properties. We then use interior elliptic estimates after a cube decomposition of $I_i \times \partial M$; this requires a second family of thickened intervals $\widehat{I}_i$, with properties similar to the ones satisfied by the I_i's. Summing over the cubes provides the desired estimate, after having ensured that the $\widehat{I}_i$'s do not overlap too much. We note that the scalings in z and θ^A are different; the former is tailored to account for the degeneracy in the "radial" z-direction, measured by ϕ, and the latter accounting for the $\phi\delta$-dependent degeneracy in the "angular" θ^A-direction.

So we divide $(1, 4)$ into intervals

(B.1) $$I_k \subset \widehat{I}_k \subset (1, 4), \quad \cup_k I_k = (1, 4),$$

as follows: There exists $1 < z_1 < 5/2$ such that $\phi : [1, z_1] \to \mathbb{R}^+$ is strictly increasing. Choose $a > 0$ small enough so that

$$z - 2a\phi(z) > 1 \text{ on } (1, z_1] \text{ and } z_1 + a(\phi(z_1)) \le 4.$$

Define z_i by induction using

$$z_{i+1} = z_i - a\phi(z_i),$$

thus $1 < z_{i+1} < z_i$, and $\lim_{i \to \infty} z_i = 1$. For any function $f \in L^1(A)$ we thus have

(B.2) $$\int_{[1,z_1] \times \partial M} f = \sum_i \int_{[z_{i+1}, z_i] \times \partial M} f.$$

We want to show that there exists a constant C such that for all a small enough and for all positive integrable functions f we have

(B.3) $$\sum_i \int_{[z_i - 2a\phi(z_i), z_i + a\phi(z_i)] \times \partial M} f \le C \int_{[1,4] \times \partial M} f.$$

In order to do that we need to count how many of the intervals $[z_i - 2a\phi(z_i), z_i + a\phi(z_i)]$ overlap. Letting $b := a\phi''(1)/2$, one easily finds

$$z_i - 2a\phi(z_i) - 1 = (z_i - 1)\Big(1 - 2b(z_i - 1)\Big) + O\Big((z_i - 1)^3\Big),$$
$$z_{i+k} - 1 = (z_i - 1)\Big(1 - kb(z_i - 1)\Big) + O\Big((z_i - 1)^3\Big),$$
$$z_{i+k} + a\phi(z_{i+k}) - 1 = (z_i - 1)\Big(1 - (k-1)b(z_i - 1)\Big) + O\Big((z_i - 1)^3\Big),$$

where the error terms in the second and third equations depend upon k. Choosing $k = 4$, it follows that

$$z_{i+k} + a\phi(z_{i+k}) < z_i - 2a\phi(z_i)$$

for all i large enough. So for i large enough $[z_i - 2a\phi(z_i), z_i + a\phi(z_i)]$ will intersect at most six such other intervals, and (B.3) with a constant $C \geq 6$ follows.

   An obvious modification of the above construction, decreasing a if necessary, will lead to a sequence $z'_k \to 4$ satisfying

$$5/2 < z'_1 \leq z'_i = z'_{i+1} - a\phi(z'_{i+1}) < z'_{i+1} < 4,$$

with, for $f \in L^1(A)$,

(B.4) $$\int_{[z'_1, 4] \times \partial M} f = \sum_i \int_{[z'_i, z'_{i+1}] \times \partial M} f,$$

and if moreover f is positive then

(B.5) $$\sum_i \int_{[z'_i - 2a\phi(z'_i), z'_i + a\phi(z'_i)] \times \partial M} f \leq C \int_{[1,4] \times \partial M} f.$$

   Letting $\{I_k\}_{k \in \mathbb{N}}$ be the collection, without repetitions, of the intervals

$$\{\underbrace{[z_1, z'_1]}_{=:I_1}, [z_{i+1}, z_i], [z'_j, z'_{j+1}]\}_{i,j \in \mathbb{N}^*},$$

and letting $\{\widehat{I}_k\}_{k \in \mathbb{N}}$ be the collection, without repetitions, of the intervals

$$\{\underbrace{[z_1 - a\phi(z'_1), z'_1 + a\phi(z'_1)]}_{=:\widehat{I}_1}, [z_i - 2a\phi(z_i), z_i + a\phi(z_i)],$$
$$[z'_j - 2a\phi(z'_j), z'_j + a\phi(z'_j)]\}_{i,j \in \mathbb{N}^*},$$

we obtain (B.1) together with

$$(B.6) \qquad \int_{[1,4]\times\partial M} f = \sum_k \int_{I_k\times\partial M} f,$$

$$(B.7) \qquad C^{-1}\int_{[1,4]\times\partial M} f \le \sum_k \int_{\widehat{I}_k\times\partial M} f \le C\int_{[1,4]\times\partial M} f,$$

for positive $f \in L^1(A)$. We set

$$\tilde{z}_k = \sup I_k.$$

The above construction provides a δ-independent decomposition of A into stripes $I_k \times \partial M$, the size of which in the z-direction is comparable to $\phi(z)$ for any $(z, v) \in I_k \times \partial M$; similarly, the sizes of $\widehat{I}_k \times \partial M$ are comparable to $\phi(z)$ for any $(z, v) \in \widehat{I}_k \times \partial M$. Mapping A to A_δ provides an associated decomposition of A_δ into stripes $I_k^\delta \times \partial M$ with sizes uniformly comparable to $\phi(\rho/\delta)$ for any $(\rho, v) \in I_k^\delta \times \partial M$; similarly for $\widehat{I}_k^\delta \times \partial M$.

We continue with a δ-dependent, and stripe dependent, cube decomposition of ∂M, as follows: Let $\{(\mathcal{O}_i, \psi_i)\}_{i=1,\dots,N}$ be a covering of M by coordinate charts with each coordinate system ψ_i^{-1} mapping $\mathcal{O}_i$ smoothly and diffeomorphically to a neighbourhood of $[0,1]^{n-1}$; the local coordinates on $[0,1]^{n-1}$ will be denoted by θ^A. We further assume that $\cup\psi_i([0,1]^{n-1})$ covers ∂M as well. Let φ_i be an associated decomposition of unity, thus $\sum_i \varphi_i = 1$. Setting $f_i = (\varphi_i f) \circ \psi_i$, for any integrable function f we have

$$\int_{[1,4]\times\partial M} f = \sum_{i=1}^N \int_{[1,4]\times[0,1]^{n-1}} f_i.$$

Given δ satisfying $0 < \delta < 1/\sup_{[1,4]} \phi$ and given an interval I_k define $m = m(k,\delta) \in \mathbb{N}$ by the inequality

$$(B.8) \qquad \frac{1}{m+1} \le \phi(\tilde{z}_k)\delta < \frac{1}{m}.$$

Let $\{K_j\}$ be the collection of closed $(n-1)$-cubes, with pairwise disjoint interiors, and with edges of size $1/m$, covering $[0,1]^{n-1}$. For any K_j let $\widehat{K}_j$ be the union of those cubes K_i which have non-empty intersection with K_j. Note that there exists a number $\widehat{N}(n)$ such that $\widehat{K}_j$ consists of at most $\widehat{N}(n)$

cubes K_i. It follows that for any integrable function f_i we have

$$\int_{[1,4]\times[0,1]^{n-1}} f_i = \sum_k \int_{[1,4]\times K_k} f_i,$$

and if $f_i \geq 0$ then

$$\int_{[1,4]\times[0,1]^{n-1}} f_i \leq \sum_k \int_{[1,4]\times \widehat{K}_k} f_i \leq \widehat{N}(n) \int_{[1,4]\times[0,1]^{n-1}} f_i.$$

We are ready now to pass to the heart of our argument. Let $\{\mathscr{U}_\ell\}_{\ell\in\mathbb{N}}$ be the collection, without repetitions, of the sets

$$\{I_k \times \psi_i(K_j)\}_{k\in\mathbb{N},i=1,\ldots,N,j=0,\ldots,m^{n-1}}.$$

Similarly let $\{\widehat{\mathscr{U}}_\ell\}_{\ell\in\mathbb{N}}$ be the collection, without repetitions, of the sets

$$\{\widehat{I}_k \times \psi_i(\widehat{K}_j)\}_{k\in\mathbb{N},\ i=1,\ldots,N,\ j=0,\ldots,m^{n-1}}.$$

From what has been said we have, for any positive integrable function f,

$$\int_{[1,4]\times\partial M} f \leq \sum_\ell \int_{\mathscr{U}_\ell} f \leq N \int_{[1,4]\times\partial M} f,$$

$$\int_{[1,4]\times\partial M} f \leq \sum_\ell \int_{\widehat{\mathscr{U}}_\ell} f \leq N\widehat{N}(n) \int_{[1,4]\times\partial M} f.$$

If $\mathscr{U}_\ell = I_k \times \psi_i(K_j)$ set $\phi_\ell = \phi(\tilde{z}_k)$. Scale the local coordinates (z,θ^A) in $\widehat{\mathscr{U}}_\ell$ as

$$(z,\theta^A) \mapsto (z/\phi_\ell, m\theta^A).$$

Up to translations, this maps all $\mathscr{U}_\ell \subset \widehat{\mathscr{U}}_\ell$'s to fixed cubes

$$\mathscr{U}_\ell \longrightarrow [0,a]\times[0,1]^n \subset [-a,2a]\times[-1,2]^n \longleftarrow \widehat{\mathscr{U}}_\ell,$$

except for those which correspond to $I_1 \times \psi_i(K_j)$, which are mapped to

$$\mathscr{U}_\ell \longrightarrow [0,a(z_1'-z_1)/\phi(z_1')]\times[0,1]^n$$
$$\subset [-a, a+a(z_1'-z_1)/\phi(z_1')]\times[-1,2]^n \longleftarrow \widehat{\mathscr{U}}_\ell,$$

By construction there exists a constant $C > 0$, independent of i, such that we have

$$\sup_{I_i\times\partial M} \phi \leq C \inf_{I_i\times\partial M} \phi, \quad \sup_{\widehat{I}_i\times\partial M} \phi \leq C \inf_{\widehat{I}_i\times\partial M} \phi,$$

hence the same is true on each $\mathscr{U}_\ell$ and $\widehat{\mathscr{U}}_\ell$. Let $\psi = \mathrm{e}^{-s/2y}$; it is shown at the end of [8, Appendix B] that one also has

$$\sup_{I_i \times \partial M} \psi \le C \inf_{I_i \times \partial M} \psi, \qquad \sup_{\widehat{I}_i \times \partial M} \psi \le C \inf_{\widehat{I}_i \times \partial M} \psi$$

(with perhaps a different constant C), and again such ℓ-independent inequalities hold on the $\mathscr{U}_\ell$'s and $\widehat{\mathscr{U}}_\ell$'s. At this stage in is important to realize that

$$L_{g_\delta} = B(\phi \partial_z, \phi \delta \partial_{\theta^A}),$$

where B is *uniformly elliptic* of order 4 (see Equation (A.4) in [8]) on the relevant cubes. We can also write

$$g_\delta = z^{-2}(dz^2 + \delta^2 \widehat{g}_{AB}(\delta z) d\theta^A d\theta^B) = z^{-2}(dz^2 + \widehat{g}_{AB}(\delta z) d(\delta\theta)^A d(\delta\theta)^B).$$

It then follows from the usual elliptic interior estimates [19, p. 246] for the operator B and scaling that (here $g = g_\delta$)

$$\sum_{i \le k+4} \int_{\mathscr{U}_\ell} \psi^2 \phi^{2i} |\nabla^{(i)} u|_g^2 \le C\left(\sum_{i \le k} \int_{\widehat{\mathscr{U}}_\ell} \psi^2 \phi^{2i} |\nabla^{(i)} Lu|_g^2 + \int_{\widehat{\mathscr{U}}_\ell} \psi^2 |u|^2 \right),$$

where C does not depend upon δ nor g close to b.

Summing over ℓ, Lemma 3.2 follows.

Acknowledgments

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