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Connecting certain rigid birational non-homeomorphic Calabi–Yau threefolds via Hilbert scheme

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We shall give an explicit pair of birational projective Calabi–Yau threefolds which are rigid, non-homeomorphic, but are connected by projective flat deformation over some connected base scheme.

0. Introduction

A *Calabi–Yau manifold* is a compact Kähler simply connected manifold with a nowhere vanishing global n-form but no global i-form with $0 < i < n = \dim X$. By Kodaira's criterion, it is projective if the dimension $n \geq 3$.

As well known, Calabi–Yau manifolds, hyperkähler manifolds and complex tori form the building blocks of compact Kähler manifolds with vanishing first Chern class [2, 6]. A famous theorem of Huybrechts states that two bimeromorphic hyperkähler manifolds are equivalent under smooth deformation [8, 9]. In particular, they are homeomorphic to each other, having the same Betti numbers and Hodge numbers. Clearly, the same holds true for complex tori. Another theorem, originally due to Batyrev and Kontsevich, says that two birational Calabi–Yau manifolds have the same Betti numbers and Hodge numbers [1, 4, 10, 19, 21]. However, there are rigid birational non-isomorphic Calabi–Yau manifolds (cf. Theorem 0.1). Obviously, they are not equivalent under any *smooth* deformation.

The aim of this paper is to remark that there nevertheless exist birational Calabi–Yau threefolds which are rigid, non-homeomorphic, but are connected by (necessarily non-smooth) projective flat deformation:

Theorem 0.1. *There are Calabi–Yau threefolds X and Y such that:*

(1) *X and Y are birational and rigid,*

(2) *X and Y are not homeomorphic but,*

(3) *X and Y are connected by projective flat deformation over some connected scheme.*

Note that any family that connects X and Y necessarily involves very singular spaces (see the toy example in Section 1). This result is experimental but we believe that it is the first attempt to study "deformation" of *rigid* Calabi-Yau threefolds. This work is also motivated by the first named author's recent result on the equivalence of certain Calabi–Yau threefolds with Picard number one, of different topological type, under projective flat deformation [11].

In the proof of our main theorem (Theorem 0.1), the following deep theorem of Hartshorne [7] (see also [16]) plays an important role:

Theorem 0.2 (Hartshorne)**.** *The Hilbert scheme* $\mathrm{Hilb}_{\mathbf{P}^N}^{P(x)}$ *of* $\mathbf{P}^N$ *with fixed Hilbert polynomial* $P(x)$ *is connected.*

So, if two varieties belong to the same Hilbert scheme $\mathrm{Hilb}_{\mathbf{P}^N}^{P(x)}$, then they appear as fibers of the universal family $u : \mathcal{U} \longrightarrow \mathrm{Hilb}_{\mathbf{P}^N}^{P(x)}$, in which $\mathrm{Hilb}_{\mathbf{P}^N}^{P(x)}$ is *connected*. In this way, they are connected by projective flat deformation.

Let Z be a Calabi–Yau threefold and let H be an ample divisor on Z. Then, by the Kodaira vanishing theorem and the Riemann–Roch formula, we have

$$\dim H^0(\mathcal{O}_Z(nH)) = \chi(\mathcal{O}_Z(nH)) = \frac{H^3}{6}n^3 + \frac{H \cdot c_2(Z)}{12}n.$$

Here $c_2(Z) = c_2(T_Z)$ is the second Chern class of Z. It is also known that $10H$ is always very ample on Z [14]. Therefore, as a special case of Theorem 0.2, one obtains the following:

Theorem 0.3. *Two Calabi–Yau threefolds are embedded into a projective space with the same Hilbert polynomial, accordingly belong to the same Hilbert scheme of that projective space and connected by projective flat deformation, if and only if they have ample divisors that have the same values of*

$$H^3 \text{ and } H \cdot c_2.$$

In general, two Calabi–Yau threefolds are unlikely to be connected by projective flat deformation, especially if they are of different topological type. Let X and Y be a complete intersection of two cubics in $\mathbf{P}^5$ and a quintic hypersurface in $\mathbf{P}^4$, respectively. Then we always have

$$9k^3 = (kH_X)^3 \neq (lH_Y)^3 = 5l^3$$

for any positive integers k, l, where H_X and H_Y are the ample generators of the Picard groups of X and Y, respectively. So X and Y cannot be connected by any projective flat deformation.

Our Calabi–Yau threefolds in Theorem 0.1 are the famous rigid Calabi–Yau threefold X_ϕ constructed by Beauville [3] and its birational modification X_T studied by the second named author [13] (see also Section 2).

Our proof (for connectedness) is implicit. So the following question might be interesting:

Question 0.1. Can one describe how X and Y in Theorem 0.1 are connected in more explicit manner?

One can find some relevant work in [20].

The structure of this paper is as follows: we discuss some toy case of elliptic curves in Section 1. This explains some idea behind our consideration. In Section 2, we recall Beauville's rigid Calabi–Yau threefold X_ϕ and its birational modification X_T. Sections 3 and 4 are devoted to the proof of Theorem 0.1.

1. Toy example: connecting two elliptic curves in two ways

Let C_λ ($\lambda \neq 0, 1$) be the elliptic curve defined by the Weierstrass equation

$$y^2 = x(x-1)(x-\lambda).$$

Obviously, any two elliptic curves C_{λ_1} and C_{λ_2} are connected by the following projective smooth family:

$$\psi : \mathcal{X} = \{([x_0 : x_1 : x_2], \lambda) \in \mathbf{P}^2 \times \mathcal{B} \mid x_1^2 x_2 - x_0(x_0 - x_2)(x_0 - \lambda x_2) = 0\} \longrightarrow \mathcal{B}.$$

Here and hereafter, we put $\mathcal{B} = \mathbf{P}^1 \setminus \{0, 1, \infty\}$.

Yet, we can connect C_{λ_1} and C_{λ_2} by another way.

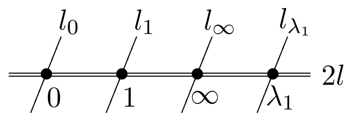
Let D be a hyperelliptic curve with a hyperelliptic involution ι and let Ξ be the set of the branch points of ι in $D/\langle\iota\rangle \simeq \mathbf{P}^1$. We consider the natural morphisms,

$$\varphi_1 : C_{\lambda_1} \times \widetilde{D/\langle(-1,\iota)\rangle} \longrightarrow D/\langle\iota\rangle \simeq \mathbf{P}^1$$

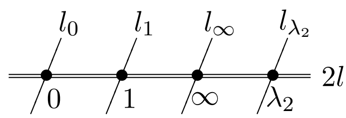
and

$$\varphi_2 : C_{\lambda_2} \times \widetilde{D/\langle(-1,\iota)\rangle} \longrightarrow D/\langle\iota\rangle \simeq \mathbf{P}^1.$$

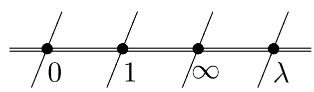
Here $\tilde{\ }$'s are the minimal resolutions. We regard φ_1 and φ_2 as projective flat deformations. Then, for $q \in \Xi$, the scheme-theoretic fiber $\varphi_1^{-1}(q) = 2l + l_0 + l_1 + l_\infty + l_{\lambda_1}$ consists of 5 $\mathbf{P}^1$'s, intersecting like



and $\varphi_1^{-1}(p) \simeq C_{\lambda_1}$ for $p \notin \Xi$. Similarly, the scheme-theoretic fiber $\varphi_2^{-1}(q)$ for $q \in \Xi$ is like



and $\varphi_2^{-1}(p) \simeq C_{\lambda_2}$ for $p \notin \Xi$. The singular schemes $\varphi_1^{-1}(q)$ and $\varphi_2^{-1}(q)$ can be put into a projective flat family, in which the fibers are of the form



For example, the natural projection $\psi : \mathcal{Y} \longrightarrow \mathcal{B}$, where

$$\mathcal{Y} = \{([x_0 : x_1], [y_0 : y_1], \lambda) \in \mathbf{P}^1 \times \mathbf{P}^1 \times \mathcal{B} \,|\, x_0^2 y_0 y_1 (y_0 - y_1)(y_0 - \lambda_1 y_1) = 0\}$$

is such a family. In this way, C_{λ_1} and C_{λ_2} are connected by a chain of three projective flat deformations.

In the second method, smooth fibers in families are only C_{λ_1} and C_{λ_2} and they are connected through *very singular spaces*. So, the method suggests some possibilities to connect two rigid manifolds of different topological structure. This is the idea behind our construction.

2. Beauville's rigid Calabi–Yau threefold and its modification

We briefly recall the two rigid Calabi–Yau threefolds X and Y that appear in Theorem 0.1.

Let $\zeta = \mathrm{e}^{2\pi\sqrt{-1}/3}$. By E_ζ, we denote the elliptic curve whose period is ζ and by $E_\zeta^n/\langle\zeta\rangle$ the quotient of the n-fold product, E_ζ^n by the scalar multiplication by ζ. Let

$$Q_0 = 0, \quad Q_1 = (1-\zeta)/3 \quad \text{and} \quad Q_2 = -(1-\zeta)/3$$

in E_ζ. These are exactly the fixed points of the scalar multiplication by ζ on E_ζ. For $i_k = 0, 1, 2$, let

$$Q_{i_1 i_2 \cdots i_n} = (Q_{i_1}, Q_{i_2}, \ldots, Q_{i_n}) \in E_\zeta^n$$

and let $\overline{Q}_{i_1 i_2 \cdots i_n}$ be its image in $E_\zeta^n/\langle\zeta\rangle$. Then $\overline{X} = E_\zeta^3/\langle\zeta\rangle$ has singularities of type $\frac{1}{3}(1,1,1)$ at $\overline{Q}_{ijk}$'s and the blow-up $\pi : X_\varphi \longrightarrow \overline{X}$ at these 27 singular points gives a Calabi–Yau threefold X_ϕ. This is the famous rigid Calabi–Yau threefold found by Beauville [3]. We denote by E_{ijk} the exceptional divisor lying over $\overline{Q}_{ijk}$. The surfaces E_{ijk} is isomorphic to $\mathbf{P}^2$.

Let

$$p_\phi : X_\phi \longrightarrow B := E_\zeta^2/\langle\zeta\rangle$$

be the morphism, induced by the projection $\mathrm{pr}_{12} : E_\zeta^3 \longrightarrow E_\zeta^2$. Then we have

$$p_\phi^{-1}\overline{Q}_{ij} = l_{ij} \cup E_{ij0} \cup E_{ij1} \cup E_{ij3}.$$

Here l_{ij} is a smooth rational curve meeting E_{ijk} transversally. See figure 1. The normal bundle of l_{ij} in X_ϕ is

$$N_{X_\phi | l_{ij}} = \mathcal{O}_{l_{ij}}(-1)^{\oplus 2}.$$

Performing the elementary transformation along $\bigcup_{i,j} l_{ij}$, we obtain a smooth threefold X_T. This X_T corresponds to that in [13] for $T = \{(i,j) | i,j = 0,1,2\}$. Denote the proper transform of E_{ijk} in X_T by F_{ijk}. Note that F_{ijk} is the first Hirzebruch surface $\mathbf{F}_1$. Compare figure 2 with figure 1.

Now we summarize some properties of X_ϕ and X_T, showed in [3, 13].

Theorem 2.1. (1) *X_ϕ and X_T are both Calabi–Yau threefolds.*

(2) *$h^{1,2}(X_\phi) = h^{1,2}(X_T) = 0$, i.e., X_ϕ and X_T are rigid.*

So, X_ϕ and X_T are birational, rigid Calabi–Yau threefolds. In fact, these X_ϕ and X_T are the Calabi–Yau threefolds X and Y in our Theorem 0.1. We shall show that X_ϕ and X_T are non-homeomorphic in Section 3 and that X_ϕ and X_T are connected by projective flat deformation in Section 4.

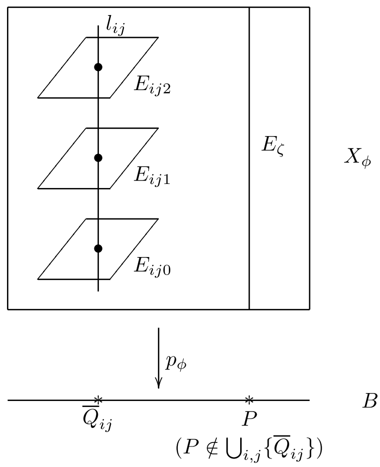


Figure 1: $p_\phi : X_\phi \to B$

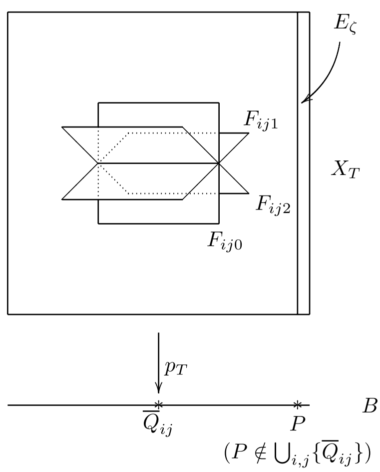


Figure 2: $p_T : X_T \to B$

Here, we summarize notations which will be frequently used in the next two sections:

Notation 2.1.

- $\zeta = e^{2\pi\sqrt{-1}/3}$, the primitive third root of unity in the upper half plane.
- $E_\zeta = \mathbf{C}/(\mathbf{Z} \oplus \mathbf{Z}\zeta)$ is the elliptic curve with period ζ.

- $Q_0 = 0$, $Q_1 = (1-\zeta)/3$ and $Q_2 = -(1-\zeta)/3$ in E_ζ: the fixed points of the scalar multiplication by ζ on E_ζ.
- $Q_{i_1 i_2 \cdots i_n} = (Q_{i_1}, Q_{i_2}, \ldots, Q_{i_n}) \in E_\zeta^n$ for $i_1, i_2, \ldots, i_n \in \{0,1,2\}$.
- $\overline{Q}_{i_1 i_2 \cdots i_n}$ is the image of $Q_{i_1 i_2 \cdots i_n}$ in $E_\zeta^n/\langle\zeta\rangle$.
- $\overline{X} = E_\zeta^3/\langle\zeta\rangle$, $B = E_\zeta^2/\langle\zeta\rangle$. Here E_ζ^2 in the definition of B is the product of the first two factors of E_ζ^3.
- $q : E_\zeta^3 \longrightarrow \overline{X}$ is the quotient map.
- $\mathrm{pr}_i : E_\zeta^3 \longrightarrow E_\zeta$ is the projection to the ith factor.
- $\mathrm{pr}_{ij} : E_\zeta^3 \longrightarrow E_\zeta^2$ is the projection to the product of ith and jth factors.
- $p_{ij} : \overline{X} \longrightarrow E_\zeta^2/\langle\zeta\rangle$ and $p_i : \overline{X} \longrightarrow E_\zeta/\langle\zeta\rangle$ are the the morphisms induced by pr_{ij} and pr_i, respectively.
- $g_i : B \longrightarrow E_\zeta/\langle\zeta\rangle = \mathbf{P}^1$ is the morphism, induced by the projection $E_\zeta^2 \longrightarrow E_\zeta$ to the ith factor ($i = 1, 2$).
- $\pi : X_\phi \longrightarrow \overline{X}$ is the blow-up at $\{\overline{Q}_{ijk} | i, j, k = 0, 1, 2\}$.
- $p_\phi = p_{12} \circ \pi : X_\phi \longrightarrow B$.
- $p_T : X_T \longrightarrow B$ is the projection, induced by p_ϕ.
- $E_{ijk} \simeq \mathbf{P}^2$ is the exceptional divisor over $\overline{Q}_{ijk}$ by the blow-up $\pi : X_\phi \longrightarrow \overline{X}$.
- $F_{ijk} \simeq \mathbf{F}_1$ is the proper transformation of E_{ijk} in X_T.

The next lemma will be also frequently used in the next two sections:

Lemma 2.1. *Let Z be a Calabi–Yau threefold and let D be a smooth divisor on Z. Then, $D^3 = c_1(T_D)^2$ and $D \cdot c_2(Z) = -c_1(T_D)^2 + c_2(T_D)$.*

We also note that $c_1(T_D)^2 = K_D^2$ and that $c_2(T_D) = c_2(D)$ is the topological Euler number of the surface D.

Proof. This follows from the fact that $c_1(Z) = 0$ and the normal sequence

$$0 \longrightarrow T_D \longrightarrow T_Z|D \longrightarrow N_{Z|D} \longrightarrow 0.$$

□

3. Topological difference between X_ϕ and X_T

In this section we shall prove (0.1) of Theorem 0.1, i.e., that X_ϕ and X_T are not homeomorphic. Since the linear form $c_2(Z) : H^2(Z, \mathbf{Z}) \longrightarrow \mathbf{Z}$ and the cubic form $c_Z : \mathrm{Sym}^3 H^2(Z, \mathbf{Z}) \longrightarrow \mathbf{Z}$ are topological invariants, the result follows from:

Theorem 3.1. (1) *The linear form given by $c_2(X_\phi)$ is divisible by* 6, *i.e.* $D \cdot c_2(X_\phi) \equiv 0 \pmod 6$ *for each* $D \in H^2(X_\phi, \mathbf{Z})$, *while the linear form* $c_2(X_T)$ *is not.*

(2) *The cubic form of X_ϕ is divisible by* 3, *i.e.,* $D^3 \equiv 0 \pmod 3$ *for each* $D \in H^2(X_\phi, \mathbf{Z})$, *while the cubic form of X_T is not.*

For the main result, the statement (2) is sufficient but we also add statement (1) for its own interest.

Remark 3.1. As far as we know, Friedman is the first who found a pair of birational projective Calabi–Yau threefolds which are not homeomorphic [5, Example 7.7]. His examples are based on [17] and they are not rigid. Our proof here is inspired by his argument there.

We shall prove Theorem 3.1 in the sequel.

Let $F \simeq \mathbf{F}_1$ be one of F_{ijk} in X_T. Then, by Lemma 2.1,

$$F^3 = K_F^2 = 8, \;\; F \cdot c_2(X_T) = c_2(F) - K_F^2 = 4 - 8 = -4.$$

Clearly, none of them is divisible by 3.

In the rest of this section, we shall show 6-divisibility of the linear form $c_2(X_\phi)$. Three-divisibility of the cubic form then follows from the Riemann–Roch formula (cf. Introduction). Here we note that $\mathrm{Pic}\, X_\phi \simeq H^2(X_\phi, \mathbf{Z})$. *From now until the end of this section, we write*

$$E = E_\zeta, \quad X = X_\phi.$$

For other notations, see Notation 2.1.

Proposition 3.1. (1) *The Néron-Severi group* $\mathrm{NS}(E^2)$ *is generated by the classes of the four divisors,* $\{0\} \times E$, $E \times \{0\}$, Δ *and* Γ. *Here* Δ *is the diagonal and* Γ *is the graph of the automorphism* $\zeta : E \longrightarrow E$.

(2) *The Néron-Severi group* $\mathrm{NS}(E^3)$ *is generated by the subgroups* $\mathrm{pr}_{ij}^*\,\mathrm{NS}(E^2)$ $(1 \le i < j \le 3)$.

Proof. The four classes in (1) are clearly in $\mathrm{NS}(E^2)$, and their intersection matrix is

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 3 \\ 1 & 1 & 3 & 0 \end{pmatrix}.$$

The discriminant of this matrix is 3. On the other hand, the discriminant of the transcendental lattice of E^2 is 3 by Chioda and Mitani [18]. Thus, the discriminant of $\mathrm{NS}(E^2)$ is also 3. Since $\mathrm{NS}(E^2)$ is torsion free, assertion (1) follows.

Let us show (2). By the Künneth formula, we have

$$H^2(E^3, \mathbf{Z}) = \bigoplus_{i=1}^{3} \mathrm{pr}_i^* H^2(E, \mathbf{Z}) \oplus \bigoplus_{1 \le i < j \le 3} \mathrm{pr}_{ij}^*(H^1(E, \mathbf{Z}) \otimes H^1(E, \mathbf{Z})).$$

This decomposition is compatible with the Hodge decomposition. Since $\mathrm{NS}(E^3) = H^2(E^3, \mathbf{Z}) \cap H^{1,1}(E^3)$ by the Lefschetz $(1,1)$-theorem, we have then

$$\mathrm{NS}(E^3) = \bigoplus_{i=1}^{3} \mathrm{pr}_i^* H^2(E, \mathbf{Z}) \oplus \bigoplus_{1 \le i < j \le 3} \mathrm{pr}_{ij}^*(H^1(E, \mathbf{Z}) \otimes H^1(E, \mathbf{Z}) \cap H^{1,1}(E^2)).$$

Again, by the Lefschetz $(1,1)$-theorem, the groups $\mathrm{pr}_k^* H^2(E, \mathbf{Z})$ $(k = i,\ j)$ and $\mathrm{pr}_{ij}^*(H^1(E, \mathbf{Z}) \otimes H^1(E, \mathbf{Z}) \cap H^{1,1}(E^2))$ are subgroups of $\mathrm{pr}_{ij}^* \mathrm{NS}(E^2)$, in which E^2 is the product of ith and jth factors of E^3. This implies (2). □

Recall that $\overline{X} = E^3/\langle \zeta \rangle$. In particular, $\overline{X}$ is $\mathbf{Q}$-factorial. A bit more precisely, the divisor $3D$ is Cartier for any Weil divisor D on $\overline{X}$. Let $N^1(\overline{X})$ be the group of the numerically equivalent classes of Weil divisors on $\overline{X}$. Note that Cartier divisors and Weil divisors are the same on E^3 or on X (as E^3 and X are smooth) and the numerical equivalence and the algebraic equivalence of divisors are also the same on E^3 or on X (as their Néron–Severi groups are torsion free).

Proposition 3.2. *The group homomorphism $q^* : N^1(\overline{X}) \longrightarrow \mathrm{NS}(E^3)$ is an isomorphism.*

Proof. Our argument here is similar to [12]. Since $\overline{X}$ is $\mathbf{Q}$-factorial and q is finite, the group homomorphism q^* is indeed well defined and injective.

Let $[H] \in \mathrm{NS}(E^3)$. We need to find $D \in N^1(\overline{X})$ such that $[H] = q^* D$.

Claim 1. *We can (and will) choose the representative* $H \in \operatorname{Pic} E^3$ *of the class* $[H]$*, such that* $\zeta^* H = H$ *as line bundles.*

Proof. Take the origin of E as polarization of E. One can then identify $\operatorname{Pic}^0(E) = E$ in an equivariant way with respect to the action of ζ. Under this identification, we have an identification $\operatorname{Pic}^0(E^3) = E^3$ in which the action of ζ^* on $\operatorname{Pic}^0(E^3)$ is the same as the diagonal action (ζ, ζ, ζ) on E^3. Note also that $\zeta^* = \mathrm{id}$ on $\mathrm{NS}(E^3)$ as $\zeta^* = \mathrm{id}$ on the wider space $H^{1,1}(E^3)$.

Put $T = \zeta^* H - H$. Here the equality is as line bundles. Then $T = (T_1, T_2, T_3)$ is an element of $\operatorname{Pic}^0(E^3) = E^3$, as $\zeta^*[H] = [H]$. Note that there is a point $P = (P_1, P_2, P_3) \in \operatorname{Pic}^0(E^3) = E^3$ such that

$$(P_1, P_2, P_3) - (\zeta P_1, \zeta P_2, \zeta P_3) = (T_1, T_2, T_3).$$

The line bundle $H + P$ is a desired representative. □

From now, we regard H as an effective divisor in $|H|$ rather than the line bundle.

Claim 2. *We may (and will) assume that there is an effective divisor* H *in* $|H|$ *such that* $\zeta^* H = H$ *as divisors.*

Proof. Since q is finite, the divisor $q^* A$ is ample if A is ample. Thus, by adding $q^* A$ with sufficiently ample A to H, we may assume that $|H|$ is a free linear system. Since $\zeta^* H = H$ as line bundles, ζ acts on the projective space $|H|$. This action certainly has a fixed points. Let H be a divisor corresponding to (one of) the fixed point. Then $\zeta^* H = H$ as divisors on E^3. □

Let $\overline{H} = q_* H$ as Weil divisors. Since $\zeta^* H = H$ as divisors and $(E^3)^{\langle \zeta \rangle}$ consists of finitely many points, there is a divisor D such that $\overline{H} = 3D$ as Weil divisors. For this D, we have

$$3q^* D = q^* \overline{H} = H + \zeta^* H + (\zeta^*)^2 H = 3H.$$

Since $\mathrm{NS}(E^3)$ is torsion free, this implies $q^* D = H$. □

Proposition 3.3. *Let* $\tilde{D}_{ijl}$ $(1 \le l \le 4)$ *be the divisors on* E^3*, which are pull back of the four divisors* $E \times \{0\}$*,* $\{0\} \times E$*,* Δ *and* Γ *on* E^2 *by* p_{ij} $(1 \le i < j \le 3)$*. Let* $\overline{D}_{ijl} := (q_* \tilde{D}_{ijl})_{\mathrm{red}}$*. Then, the (classes of) 12 Weil divisors* $\overline{D}_{ijl}$ *generate* $N^1(\overline{X})$*.*

Proof. We note that $\zeta^* \tilde{D}_{ijl} = \tilde{D}_{ijl}$ as divisors on E^3. Thus $\tilde{D}_{ijl} = \pi^* \overline{D}_{ijl}$ (cf. Proof of Proposition 3.2). Since $\tilde{D}_{ijl}$ generate $\mathrm{NS}(E^3)$ by Proposition 3.1, the result follows from Proposition 3.2. □

Let D_{ijl} be the proper transform of $\overline{D}_{ijl}$ on X by $\pi : X \longrightarrow \overline{X}$.

Proposition 3.4. $\mathrm{NS}(X)$ *is contained in the subgroup of* $\mathrm{NS}(X) \otimes \mathbf{Q}$ *generated by the classes of the following divisors:*

$$D_{ijl},\ E_{ijk},\ T_{\Lambda,\Lambda'} := \frac{1}{3} \sum_{(i,j,k)\in\Lambda} E_{ijk} + \frac{1}{3} \sum_{(i',j',k')\in\Lambda'} 2E_{i'j'k'}$$

where Λ *and* Λ' *are some disjoint subsets (possibly empty) of the product set* $\{0,1,2\}^3$ *such that* $\Lambda \cap \Lambda' = \emptyset$ *and such that both* $|\Lambda|$ *and* $|\Lambda'|$ *are divisible by* 3.

Proof. Let D be a prime divisor on X. Put $\overline{D} = \pi_* D$ as Weil divisors. Then, by Proposition 3.3, there are integers b_{ijl} such that $\overline{D} = \sum_{i,j,l} b_{ijl} \overline{D}_{ijl}$ in $N^1(\overline{X})$. Since $3\overline{D}$ and $3\overline{D}_{ijl}$ are Cartier, there are integers a_{ijk} such that

$$D = \sum_{i,j,l} b_{ijl} D_{ijl} + \frac{1}{3} \sum_{i,j,k} a_{ijk} E_{ijk}$$

in $\mathrm{NS}(X) \simeq \mathrm{Pic}\, X$. So, the result follows from the next lemma. □

Lemma 3.1. *Let* Λ *and* Λ' *be subset of* $\{0,1,2\}^3$ *such that* $\Lambda \cap \Lambda' = \emptyset$*. If*

$$M := \sum_{(i,j,k)\in\Lambda} E_{ijk} + \sum_{(i',j',k')\in\Lambda'} 2E_{i'j'k'}$$

is divisible by 3 *in* $\mathrm{Pic}\, X$*, then both* $|\Lambda|$ *and* $|\Lambda'|$ *are divisible by* 3.

Proof. Let $\alpha \in \{0,1,2\}$. Let D_α be the divisor on X, which is the proper transform of the divisor $\overline{D}_\alpha = (p_3^* Q_\alpha)_{\mathrm{red}}$ on $\overline{X}$ (See figure 3 and Notation 2.1 for Q_α).

Since $\overline{D}_\alpha$ passes through 9-singular points of $\overline{X}$, the surface D_α meets the 9-exceptional divisors, say,

$$E_{00\alpha}, E_{01\alpha}, E_{02\alpha}, E_{10\alpha}, E_{11\alpha}, E_{12\alpha}, E_{20\alpha}, E_{21\alpha}, E_{22\alpha}.$$

We put $l_{ij\alpha} := E_{ij\alpha}|_{D_\alpha}$. These are all (-3)-curves. The surface D_α is a non-relatively minimal rational elliptic surface with 3-singular fibers $l_{i0\alpha} + l_{i1\alpha} +$

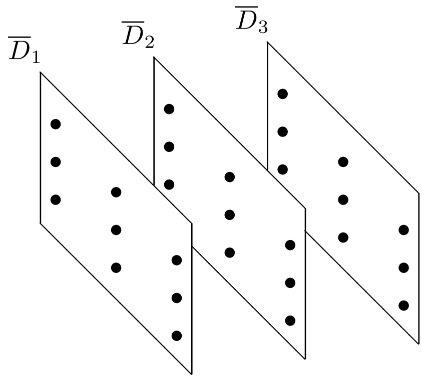


Figure 3: D_α's

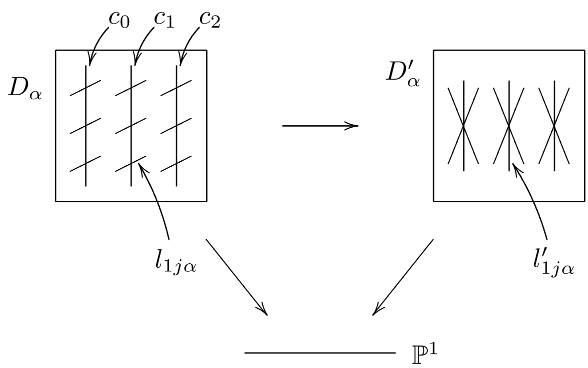


Figure 4: $\nu_\alpha : D_\alpha \to D'_\alpha$

$l_{i2\alpha} + 3c_i$ ($i = 0,\ 1,\ 2$) as in the figure below (figure 4). Here c_i are (-1)-curves. Let $\nu_\alpha : D_\alpha \longrightarrow D'_\alpha$ be the contraction of the three (-1)-curves c_i. Let $l'_{ij\alpha} = \nu(l_{ij\alpha})$. Then D'_α is a relatively minimal rational elliptic surface with 3 singular fibers $l'_{i0\alpha} + l'_{i1\alpha} + l'_{i2\alpha}$ (figure 4). Since M is 3-divisible, so is the divisor

$$M'_\alpha := (\nu_\alpha)_*(M|_{D_\alpha}) = \sum_{i,j} a_{ij\alpha} l'_{ij\alpha}.$$

Our D'_α belongs to No.39 in the list of [15]. In particular, the Mordell–Weil group has a torsion element of order 3. Thus, there are three sections s_0, s_1 and s_2 which meet $l'_{00\alpha}$, $l'_{01\alpha}$, $l'_{02\alpha}$, respectively. On the other hand, since $M'_\alpha \cdot l'_{ij\alpha}$ are divisible by 3, the set of three elements $\{a_{i0\alpha}, a_{i1\alpha}, a_{i2\alpha}\}$ (counted with multiplicities) is either one of $\{0,0,0\}$,$\{1,1,1\}$, $\{2,2,2\}$, $\{0,1,2\}$, for each $i = 0,\ 1,\ 2$. Suppose that for $i = 0$ we have $\{a_{00\alpha}, a_{01\alpha},$

$a_{02\alpha}\} = \{0,1,2\}$. Then, the same holds for $i = 1$ and 2. This is because $s_0 \cdot M'_\alpha$, $s_1 \cdot M'_\alpha$ and $s_2 \cdot M'_\alpha$ are all divisible by 3. Thus both $|\Lambda \cap \{(i,j,\alpha) \mid i,j = 0,1,2\}|$ and $|\Lambda' \cap \{(i,j,\alpha) \mid i,j = 0,1,2\}|$ are divisible by 3 for each $\alpha \in \{0,1,2\}$. This implies the result. □

Now we are ready to prove 6-divisibility of the linear from $c_2(X)$. It suffices to check that $D \cdot c_2(X) \equiv 0 \bmod 6$ for D_{ijl}, E_{ijk} and $T_{\Lambda,\Lambda'}$ in Proposition 3.4.

We have $K^2_{E_{ijk}} = E^3_{ijk} = 9$ and $c_2(E_{ijk}) = 3$, as $E_{ijk} \simeq \mathbf{P}^2$. Thus $E_{ijk} \cdot c_2(X) = -6$ by Lemma 2.1. This also implies 6-divisibility of $T_{\Lambda,\Lambda'} \cdot c_2(X)$ as both $|\Lambda|$ and $|\Lambda'|$ are divisible by 3.

Let us compute $D_{ijl} \cdot c_2(X)$. As we have observed in Lemma 3.1, the surface D_{ijl} is the blow up at three points of a relatively minimal rational elliptic surface. Thus, $K^2_{D_{ijl}} = -3$ and $c_2(D_{ijl}) = 15$, and therefore, $D_{ijl} \cdot c_2(X) = 18$ by Lemma 2.1.

This completes the proof Theorem 3.1.

4. Connecting X_ϕ and X_T by projective flat deformation

In this section we shall prove (0.1) in Theorem 0.1, i.e., X_ϕ and X_T are connected by projective flat deformation. By Theorem 0.3, this follows from:

Theorem 4.1. *There are ample divisors H_ϕ on X_ϕ and H_T on X_T such that*

$$H_\phi \cdot c_2(X_\phi) = H_T \cdot c_2(X_T) \text{ and } H_\phi^3 = H_T^3.$$

We shall prove Theorem 4.1 in the sequel. In the proof, we freely use the notations given in Notation 2.1.

4.1. Construction of a divisor H_ϕ on X_ϕ

Recall that $E_\zeta/\langle\zeta\rangle \simeq \mathbb{P}^1$. Let $\overline{L}_i = p_i^* \mathcal{O}_{E_\zeta/\langle\zeta\rangle}(1)$ and $L_i = \pi^* \overline{L}_i$. Let

$$H_\phi = -\sum_{i,j,k} E_{ijk} + xL_1 + yL_2 + zL_3$$

where x, y and z are positive integers.

Lemma 4.1. (1) *For sufficiently large number C, H_ϕ is ample on X_ϕ when $x > C, y > C, z > C$.*

(2) $H_\phi \cdot c_2(X_\phi) = 162$.

(3) $H_\phi^3 = 54xyz - 243$.

Proof. By construction, the divisor $-\sum_{i,j,k} E_{ijk}$ is π-ample, the divisors $\overline{L}_i$'s are nef on $\overline{X}$ and $\overline{L}_1 + \overline{L}_2 + \overline{L}_3$ is ample on $\overline{X}$. This implies (1). Note that L_i is represented by a smooth abelian surface and $E_{ijk} \simeq \mathbf{P}^2$. Thus, by Lemma 2.1, we have $L_i \cdot c_2(X_\phi) = 0$ and $E_{ijk} \cdot c_2(X_\phi) = -6$. This implies (2). Note also that

$$E_{ijk}^3 = 9, L_1 \cdot L_2 \cdot L_3 = 9, E_{ijk} \cdot L_l = L_i^2 = 0$$

and $E_{ijk} \cdot E_{lmn} = 0$ unless $(i,j,k) = (l,m,n)$. Therefore we have

$$\begin{aligned} H_\phi^3 = & \left(-\sum_{i,j,k} E_{ijk}\right)^3 + 3\left(-\sum_{i,j,k} E_{ijk}\right)^2 (xL_1 + yL_2 + zL_3) \\ & + 3\left(-\sum_{i,j,k} E_{ijk}\right)(xL_1 + yL_2 + zL_3)^2 \\ & + (xL_1 + yL_2 + zL_3)^3 \\ = & -\sum_{i,j,k} E_{ijk}^3 + 0 + 0 + 6xyzL_1 \cdot L_2 \cdot L_3 \\ = & \, 54xyz - 243. \end{aligned}$$

□

4.2. Construction of a divisor H_T on X_T

We recall the following commutative diagram:

$$\begin{array}{ccccc} X_\phi & \xrightarrow{\pi} & \overline{X} & \xrightarrow{p_3} & E_\zeta/\langle\zeta\rangle \simeq \mathbf{P}^1 \\ \vdots & \searrow & \downarrow{\scriptstyle p_{12}} & & \\ X_T & \xrightarrow[p_T]{} & B & \xrightarrow[g_1]{} & E_\zeta/\langle\zeta\rangle \simeq \mathbf{P}^1 \end{array} .$$

Let $l'_i = g_1^{-1}(\overline{Q}_i)$ and $M_i = \overline{p_T^{-1}(l'_i \setminus \{\overline{Q}_{i0}, \overline{Q}_{i1}, \overline{Q}_{i2}\})}$ ($i = 0,\ 1,\ 2$). Then M_i is a relatively minimal rational elliptic surface. We denote a general smooth fiber of the fibration $M_i \longrightarrow \mathbf{P}^1$ by f_{M_i}. By construction, M_i has 3

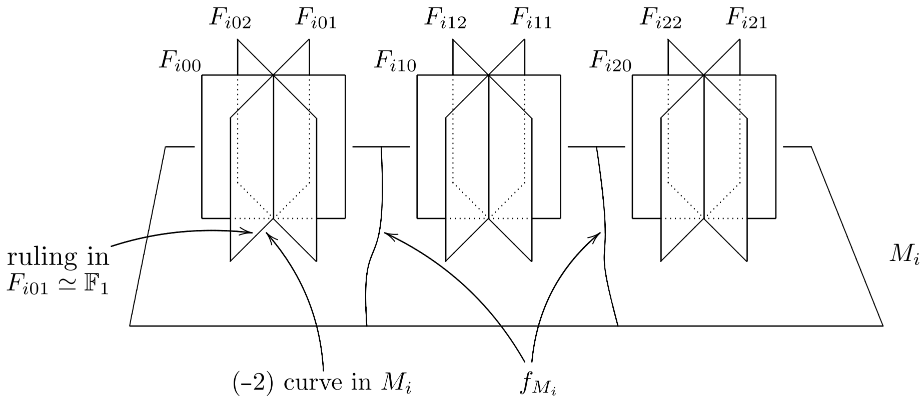


Figure 5: M_i and F_{ijk}'s in X_T

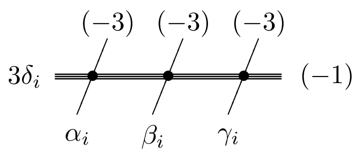
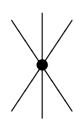


Figure 6: η_i

singular fibers of Kodaira type IV:



See figure 5 for M_i and the way M_i intersects with F_{ijk}'s.

Let $\overline{S}_j = (p_3^*(\overline{Q}_j))_{\text{red}}$ on $\overline{X}$ and S_j be the proper transformation of $\overline{S}_j$ on X_T ($j = 0,\ 1,\ 2$). Then S_j is a (non-relatively minimal) rational elliptic surface with three singular fibers (denote them by η_1, η_2 and η_3) that are composed of one (-1)-curve of multiplicity 3 and three (-3)-curves; $\eta_i = \alpha_i + \beta_i + \gamma_i + 3\delta_i$. See figure 6. We denote by f_{S_j} a general smooth fiber of the fibration $S_j \longrightarrow \mathbf{P}^1$.

See figure 7 for the configuration of S_j, M_i and $F_{\alpha\beta\gamma}$'s.

Lemma 4.2. *The following divisor is p_T-ample;*

$$3(M_0 + M_1 + M_2) + S_0 + S_1 + S_2.$$

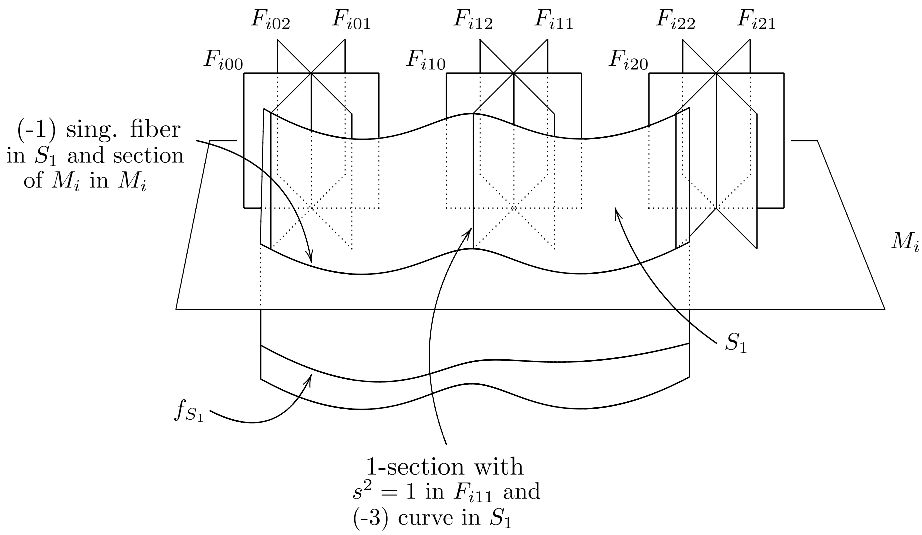


Figure 7: M_i, F_{ijk}'s and S_1 in X_T

Proof. Since S_j are sections of p_T over $B \setminus \{\overline{Q}_{\alpha\beta} \,|\, \alpha, \beta = 0, 1, 2\}$, we only need to check that $3(M_0 + M_1 + M_2) + S_0 + S_1 + S_2$ is ample on $F_{\alpha\beta\gamma}$'s. This, however, follows from the fact that

$$(3(M_0 + M_1 + M_2) + S_0 + S_1 + S_2))\,|_{F_{\alpha\beta\gamma}} = 3f + s$$

where f is the ruling of the ruled surface $F_{\alpha\beta\gamma} \simeq \mathbf{F}_1$ and s is a positive section (with $s^2 = 1$) (see figures 5 and 7). $\square$

Let $\overline{A}_k$ be a general fiber of $g_k : B \longrightarrow E_\zeta/\langle\zeta\rangle$ and let $A_k = p_T^*\overline{A}_k$ ($k = 1$, 2). $\overline{A}_k$ is an elliptic curve and A_k is an abelian surface. Note that $\overline{A}_k \in |g_k^*\mathcal{O}_{E_\zeta/\langle\zeta\rangle}(1)|$. Put

$$H_T := 3(M_0 + M_1 + M_2) + S_0 + S_1 + S_2 + aA_1 + bA_2.$$

Here a and b are positive integers.

Lemma 4.3. (1) *For sufficiently large number C, H_T is ample on X_T when $a > C$, $b > C$.*

(2) $H_T \cdot c_2(X_T) = 162.$

(3) $H_T^3 = 18ab - 27b - 333.$

Proof. Since $\overline{A}_1, \overline{A}_2$ are nef and $\overline{A}_1 + \overline{A}_2$ is an ample on B, the first assertion follows from Lemma 4.2. By using Lemma 2.1, we compute that

$$\begin{aligned} M_i \cdot c_2(X_T) &= - K_{M_i}^2 + c_2(M_i) = 12 \\ S_j \cdot c_2(X_T) &= - K_{S_j}^2 + c_2(S_j) = 18 \\ A_k \cdot c_2(X_T) &= - K_{A_k}^2 + c_2(A_k) = 0. \end{aligned}$$

This implies the second assertion. For the third one, we first expand H_T^3 as

$$\begin{aligned} H_T^3 = {} & (3(M_0 + M_1 + M_2) + S_0 + S_1 + S_2)^3 && (= Q_1) \\ & + (3(M_0 + M_1 + M_2) + S_0 + S_1 + S_2)^2 (aA_1 + bA_2) && (= Q_2) \\ & + (3(M_0 + M_1 + M_2) + S_0 + S_1 + S_2)(aA_1 + bA_2)^2 && (= Q_3) \\ & + (aA_1 + bA_2)^3 && (= Q_4) \\ = {} & Q_1 + Q_2 + Q_3 + Q_4. \end{aligned}$$

We compute Q_1, Q_2, Q_3, Q_4 separately.

Q_1: Note that

$$\begin{aligned} & S_j^3 = K_{S_j}^2 = -3 \\ & M_i^3 = K_{M_i}^2 = 0 \\ & S_i \cdot S_j = M_i \cdot M_j = 0 \text{ for } i \neq j \\ & M_i^2 \cdot S_j = (M_i|_{S_j})^2 = -1 \\ & M_i \cdot S_j^2 = (S_j|_{M_i})^2 = -1 \ . \end{aligned}$$

With these, we have $Q_1 = -333$.

Q_2: We observe that

$$-M_i|_{M_i} \sim -K_{M_i} \sim A_1|_{M_i} \sim f_{M_i} \quad \text{and} \quad f_{M_i} \cdot A_2 = 3.$$

From this we have $M_i^2 \cdot (aA_1 + bA_2) = -3b$. Note also that

$$A_1|_{S_j} \sim f_{S_j}.$$

It follows that

$$\begin{aligned} M_i \cdot S_j \cdot (aA_1 + bA_2) &= M_i|_{S_j} \cdot (aA_1 + bA_2)|_{S_j} \\ &= b(\delta_i \cdot A_2) \\ &= b. \end{aligned}$$

See also figure 6. Finally,

$$\begin{aligned} S_j^2 \cdot (aA_1 + bA_2) &= K_{S_j} \cdot (aA_1 + bA_2)|_{S_j} \\ &= b(-f_{S_j} + \delta_1 + \delta_2 + \delta_3) \cdot A_2 \\ &= b(-3 + 1 + 1 + 1) \\ &= 0. \end{aligned}$$

Thus we have $Q_2 = -27b$.

Q_3**:** Note that $(aA_1 + bA_2)^2 = 6ab(\text{fiber of } p_T)$. So we have

$$\begin{aligned} M_i \cdot (aA_1 + bA_2)^2 &= 0 \\ S_j \cdot (aA_1 + bA_2)^2 &= 6ab \end{aligned}$$

Thus, $Q_3 = 18ab$.

Q_4**:** Clearly, $Q_4 = 0$.

With all these, we obtain $H_T^3 = Q_1 + Q_2 + Q_3 + Q_4 = 18ab - 27b - 333$. □

4.3. Synthesis

Now we are ready to prove Theorem 4.1. By Lemmas 4.1 and 4.3, the divisors H_ϕ and H_T are ample on X_ϕ and X_T, respectively, when x, y, z and a, b are greater than some sufficiently large C. So, it suffices to find integers x, y, z and a, b greater than any given positive integer C that satisfy the following equations:

$$162 = H_\phi \cdot c_2(X_\phi) = H_T \cdot c_2(X_T) = 162$$

and

$$54xyz - 243 = H_\phi^3 = H_T^3 = 18ab - 27b - 333.$$

The first one poses no condition on x, y, z, a, b, and the second one is simplified to

$$6xyz = 2ab - 3b - 10. \tag{1}$$

For a given positive integer C, let

$$x = 12C^2 - 6,\ y = z = 2C,\ a = 6C^2 + 1,\ b = 24C^2 - 10.$$

Then x, y, z and a, b are integers which are greater than C and satisfy the above Equation (1).

This completes the proof of Theorem 4.1.

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