Large-sphere and small-sphere limits of the Brown–York mass

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In this paper, we will study the limiting behavior of the Brown–York mass of the coordinate spheres in an asymptotically flat manifold. Limiting behaviors of volumes of regions related to coordinate spheres are also obtained, including a discussion on the isoperimetric mass introduced by Huisken [14]. We will also study expansions of the Brown–York mass and the Hawking mass of geodesic spheres with center at a fixed point p of a 3-manifold. Some geometric consequences will be derived.

1. Introduction

In this work, we will discuss the large-sphere limit of the Brown–York mass in an asymptotically flat (AF) manifold and the small-sphere limit of the Brown–York mass near a point in a three-dimensional manifold. We will also discuss the behaviors of large-sphere limit and small-sphere limit of other interesting quantities.

Let us first recall some definitions. In general relativity, AF manifolds have great interests in many problems. In this paper, we adopt the following definition of AF manifolds.

Definition 1.1. A complete 3-manifold (M, g) is said to be AF of order τ (with one end) if there is a compact subset K such that $M \setminus K$ is diffeomorphic to $\mathbb{R}^3 \setminus B_R(0)$ for some $R > 0$ and in the standard coordinates in $\mathbb{R}^3$, the metric g satisfies:

(1.1)
$$g_{ij} = \delta_{ij} + \sigma_{ij}$$

with

(1.2)
$$|\sigma_{ij}| + r|\partial\sigma_{ij}| + r^2|\partial\partial\sigma_{ij}| + r^3|\partial\partial\partial\sigma_{ij}| = O(r^{-\tau}),$$

for some constant $1 \geq \tau > \frac{1}{2}$, where r and ∂ denote the Euclidean distance and standard derivative operator on $\mathbb{R}^3$, respectively.

A coordinate system of M near infinity so that the metric tensor in these coordinates satisfies the decay conditions in the definition is said to be *admissible*. Note that some of the results in the following do not need decays of the third-order derivatives of σ_{ij}.

Definition 1.2. The Arnowitt–Deser–Misner (ADM) mass (see [1]) of an AF manifold M is defined as:

$$(1.3) \qquad m_{\text{ADM}}(M) = \lim_{r \to \infty} \frac{1}{16\pi} \int_{S_r} (g_{ij,i} - g_{ii,j}) \, \nu^j d\Sigma_r^0,$$

where S_r is the Euclidean sphere with Euclidean radius r and center at the origin, $d\Sigma_r^0$ is the volume element induced by the Euclidean metric, ν is the outward unit normal of S_r in $\mathbb{R}^3$ and the derivative is the ordinary partial derivative.

We always assume that the scalar curvature is in $L^1(M)$ so that the limit exists in the definition. Under the decay conditions in the definition of AF manifold, the definition of ADM mass is independent of the choice of admissible coordinates by the result of Bartnik [2].

Let (Ω, g) be a compact 3-manifold with smooth boundary $\partial\Omega$. Suppose the Gauss curvature of $\partial\Omega$ is positive, then the Brown–York quasi-local mass of $\partial\Omega$ is defined as (see [6, 7]):

Definition 1.3.

$$(1.4) \qquad m_{\text{BY}}(\partial\Omega) = \frac{1}{8\pi} \int_{\partial\Omega} (H_0 - H) d\Sigma,$$

where H is the mean curvature of $\partial\Omega$ with respect to the outward unit normal and the metric g, $d\Sigma$ is the volume element induced on $\partial\Omega$ by g and H_0 is the mean curvature of $\partial\Omega$ with respect to the outward unit normal when embedded in $\mathbb{R}^3$.

The Brown–York mass is well-defined because by the result of Nirenberg [18], $\partial\Omega$ can be isometrically embedded in $\mathbb{R}^3$ and the embedding is unique by Cohn–Vossen [8] (see also [13, 19, 20]). In particular, H_0 is completely determined by the metric on $\partial\Omega$. However, this is a global property. In contrast, the norm of the mean curvature vector of an embedding of $\partial\Omega$ into the light cone in the Minkowski space can be expressed explicitly in terms of the Gauss curvature, see [5]. Hence, in the study of Brown–York mass, one of the difficulties is to estimate $\int_{\partial\Omega} H_0 d\Sigma$. We will use the Minkowski formulae [16] and the estimates of Nirenberg [18] in this regard.

In the first part of this paper, we want to study limiting behaviors of Brown–York mass on large spheres. We will verify the following:

Theorem 1.1. *Let (M, g) be an AF manifold of order $\tau > \frac{1}{2}$ with one end and let S_r be the coordinate spheres in some admissible coordinates. Then*

$$\lim_{r \to \infty} m_{\mathrm{BY}}(S_r) = m_{\mathrm{ADM}}(M).$$

Here $m_{\mathrm{BY}}(S_r)$ is the Brown–York quasi-local mass of S_r, and $m_{\mathrm{ADM}}(M)$ is the ADM mass of M.

Theorem 1.1 was observed and proved to be true by many people, see the works of Brown–York [7], Hawking–Horowitz [12], Braden–Brown–Whiting–York [4] and Baskaran-Lau-Petrov [3], see also [23]. However, in this paper, we will use a different method to derive Theorem 1.1. Interestingly, our method leads to the following volume comparison result. Let $V(r)$ be the volume with respect to an AF metric g of the region inside S_r and let $V_0(r)$ be the Euclidean volume inside the surface S_r when embedded in $\mathbb{R}^3$.

Theorem 1.2. *Let (M, g) be an AF manifold of order $\tau > \frac{1}{2}$ with one end. Then*

$$(1.5) \qquad V_0(r) - V(r) = -2m_{\mathrm{ADM}}(M)\pi r^2 + o(r^2).$$

Hence if the ADM mass is non-negative, then $\lim_{r \to \infty} r^{-2}(V(r) - V_0(r)) \geq 0$. Combining this with positive mass theorem, if we further assume that the scalar curvature is non-negative, then the limit is zero if and only if M is isometric to $\mathbb{R}^3$.

In [14], a notion of *isoperimetric mass* $m_{\mathrm{ISO}}(M)$ of an AF manifold is introduced by Huisken. It is defined as:

$$m_{\mathrm{ISO}} = \limsup_{r \to \infty} \frac{2}{\mathcal{A}(r)} \left(V(r) - \frac{1}{6\pi^{\frac{1}{2}}} \mathcal{A}^{\frac{3}{2}}(r) \right),$$

where $V(r)$ is as before and $\mathcal{A}(r)$ is the area of the coordinate sphere with respect to the AF metric. Using the method of the proof of Theorem 1.2, Miao [17] proves that the isoperimetric mass and the ADM mass of an AF manifold are equal. We would like to thank Miao for allowing us to include his result in this work.

In the second part of the paper, we will consider the small-sphere limit of the Brown–York mass. Let r be the distance to the fixed point p, and $R(p)$ is the scalar curvature evaluated at p. We have the following:

Theorem 1.3. *Let (N, g) be a Riemannian manifold of dimension 3, p be a fixed interior point on N and S_r be the geodesic sphere of radius r center at p. For r small enough, we have*
(1.6)
$$m_{\mathrm{BY}}(S_r) = \frac{r^3}{12} R(p) + \frac{r^5}{1440} \left[24 |\mathrm{Ric}|^2(p) - 13 R^2(p) + 12 \Delta R(p) \right] + O(r^6),$$

where Δ is Laplacian operator of (M, g) and $|\mathrm{Ric}|$ is the norm of the Ricci curvature.

Let M be an AF manifold with non-negative scalar curvature. Suppose the Brown–York mass of the coordinate spheres converge to zero, then M must be the Euclidean space by Theorem 1.1 and the positive mass theorem in [21, 24]. By Theorem 1.3, we have similar result near a point p. Namely, assume $R \geq 0$ in a neighborhood of p, then

(1.7)
$$\lim_{r \to 0} \frac{m_{\mathrm{BY}}(S_r)}{r^5} \geq 0.$$

Equality holds if and only if (N, g) is flat at p and R vanishes up to second order at p.

There are results on the small-sphere limits obtained by Brown–Lau–York [5]. They consider a cut S_r with an affine radius r of the light cone at a point p in a Lorentz manifold. Using the light cone of reference, they show that the expansion of the quasi-local energy is:

$$E = \frac{4\pi r^3}{3} T_{ab} n^a n^b + o(r^3),$$

where T_{ab} is the energy momentum tensor and n is the unit future pointing time like vector defining the choice of the affine parameter. In our case, if we consider the Lorentz manifold $\mathbb{R} \times N$ with metric $\widetilde{g} = -dt^2 + g$, and suppose the metric satisfies the Einstein equation:

$$\widetilde{R}_{ab} - \frac{1}{2} \widetilde{R} \widetilde{g}_{ab} = 8\pi T_{ab}.$$

Let $n = \frac{\partial}{\partial t}$ be the future pointing unit normal, then

$$\frac{R(p)}{12} = \frac{4\pi r^3}{3} T_{ab} n^a n^b.$$

Hence, r^3 term of the expansion in our case is similar to that in [5]. However, we are using Euclidean reference and we only consider the time symmetric case.

In the case of vacuum space-time, Brown–Lau–York [5] also obtain the r^5 term in the expansion of E as follows:

$$E_5 = \frac{r^5}{90} T_{abcd} n^a n^b a^c n^d,$$

where T_{abcd} is the Bel–Robinson tensor, which depends only on the curvature tensor (and the metric). In Theorem 3.1, the space-time is not vacuum in general and is time symmetric. The coefficient of the term r^5 depends not only on the curvature tensor, but also on the derivative of the scalar curvature. For the sake of comparison, in our case, one can compute that $T_{0000} = \frac{1}{8}(4|\text{Ric}|^2 - R^2)$. We use the definition of Bel–Robinson tensor as in (5) of [9].

Next we want to compare the expansion of the Hawking mass with the expansion of the Brown–York mass for small spheres. Recall the definition of the Hawking mass. Let (Ω, g) be a smooth 3-manifold with boundary $\partial\Omega$ and let H be the mean curvature on $\partial\Omega$ with respect to the outward unit normal, the Hawking quasi-local mass is defined as (see [11]):

Definition 1.4.

(1.8) $$m_{\text{H}}(\partial\Omega) = \frac{|\partial\Omega|^{1/2}}{(16\pi)^{3/2}} \left(16\pi - \int_{\partial\Omega} H^2 \, d\Sigma \right),$$

where $d\Sigma$ is the volume element induced on $\partial\Omega$ by g and $|\partial\Omega|$ is the area of $\partial\Omega$.

With the same notations and assumptions in Theorem 1.3, the expansion of $m_{\text{H}}(S_r)$ is given by:

(1.9) $$m_{\text{H}}(S_r) = \frac{r^3}{12} R(p) + \frac{r^5}{720} \left(6\Delta R(p) - 5R^2(p) \right) + O(r^6).$$

One can see that $m_{\text{BY}}(S_r)$ and $m_{\text{H}}(S_r)$ are equal up to the term with order r^3. However, the terms of order r^5 are different. In particular, if the scalar curvature is zero near p, but it is non-flat at p, then $r^{-5}m_{\text{BY}}(S_r) > 0$ and $m_{\text{H}}(S_r) = O(r^6)$ for small r.

As in the large-sphere case, one can also compare $V(r)$ and $V_0(r)$, where $V(r)$ is the volume of the geodesic ball of radius r at p and $V_0(r)$ which is the volume of the region bounded by S_r when embedded in $\mathbb{R}^3$.

The paper is organized as follows. In Section 2, the limit of behavior of Brown–York mass in large spheres and volume comparison are proved; in Section 3, small-sphere limit of the Brown–York mass and the Hawking mass and small-sphere volume comparison are proved.

2. Large-sphere limit

In this section, we will first prove the following theorem (Theorem 1.1).

Theorem 2.1. *Let (M, g) be an AF manifold of order $\tau > \frac{1}{2}$ with one end and let S_r be the coordinate spheres in some admissible coordinates. Then*

$$\lim_{r \to \infty} m_{\mathrm{BY}}(S_r) = m_{\mathrm{ADM}}(M).$$

Here $m_{\mathrm{BY}}(S_r)$ is the Brown–York quasi-local mass of S_r and $m_{\mathrm{ADM}}(M)$ is the ADM mass of M.

Consider an AF manifold (M, g) with coordinates (x^1, x^2, x^3) so that g_{ij} satisfies the decay conditions in Definition 1.3. Let $n = n^i \frac{\partial}{\partial x^i}$ be the unit outward normal of S_r and $n_i = g_{ij} n^j$. Then

$$(2.1) \qquad n^i = \frac{g^{ij} x^j}{r |\nabla r|} \quad \text{and} \quad n_i = \frac{x^i}{r |\nabla r|},$$

where $r = \left(\sum_{i=1}^{3} \left(x^i \right)^2 \right)^{1/2}$. The metric induced on S_r is $h_{ij} = g_{ij} - n_i n_j$ and the second fundamental form is $A_{ij} = h_i^k h_j^l n_{k;l}$, where $n_{k;l}$ is the covariant derivative of n_k with respect to g.

Lemma 2.1. *With the above notations and assumptions, on S_r we have the following:*

(i)

$$A_{ij} = \frac{h_{ij}}{r} + O(r^{-1-\tau}), \quad H = \frac{2}{r} + O(r^{-1-\tau}), K = \frac{1}{r^2} + O(r^{-2-\tau}),$$

where H is the mean curvature and K is the Gauss curvature of S_r.

(ii)

$$d\Sigma_r = \left(1 + h^{ij}\sigma_{ij} + O(r^{-2\tau})\right)^{1/2} d\Sigma_r^0.$$

Hence,

$$\mathcal{A}(r) = 4\pi r^2 + \frac{1}{2}\int_{S_r} h^{ij}\sigma_{ij}\, d\Sigma_r + O(r^{2-2\tau}),$$

where $\mathcal{A}(r)$ is the area of S_r with respect to g.

Proof. (i) is well-known, see [15]. For the sake of completeness, we derive it as follows:

(2.2)
$$|\nabla r|^2 = 1 - \frac{\sigma_{ij}x^i x^j}{r^2} + O(r^{-2\tau})$$

and

(2.3)
$$\begin{aligned}
\frac{\partial}{\partial x^k}\left(|\nabla r|^2\right) &= \frac{\partial}{\partial x^k}\left(g^{ij}\frac{x^i x^j}{r^2}\right)\\
&= \frac{\partial}{\partial x^k}\left[1 + \left(g^{ij} - \delta_{ij}\right)\frac{x^i x^j}{r^2}\right]\\
&= O(r^{-1-\tau}).
\end{aligned}$$

So

(2.4)
$$n^i = \frac{x^i}{r} + O(r^{-\tau})$$

and

(2.5)
$$\begin{aligned}
n_{i;j} &= \frac{\partial n_i}{\partial x^j} - \Gamma_{ij}^k n_k\\
&= \frac{\partial}{\partial x^j}\left(\frac{x^i}{r|\nabla r|}\right) + O(r^{-1-\tau})\\
&= \left(\frac{\delta_{ij}}{r} - \frac{x^i x^j}{r^3}\right) + O(r^{-1-\tau}),
\end{aligned}$$

where Γ_{ij}^k are the Christoffel symbols. Let $h_i^j = g^{jk}h_{ki}$. Using the fact that n has unit length, we have

$$
\begin{aligned}
A_{ij} - \frac{h_{ij}}{r} &= h_i^k h_j^l n_{k;l} - \frac{h_{ij}}{r} \\
&= h_j^l n_{i;l} - \frac{h_{ij}}{r} \\
&= n_{i;j} - \left(\frac{\delta_{ij}}{r} - \frac{x^i x^j}{r^3} \right) + O(r^{-1-\tau}) \\
&= O(r^{-1-\tau}).
\end{aligned}
$$

(2.6)

From this and the fact that the curvature of M decays like $r^{-2-\tau}$, the estimates of H and K follow.

(ii) Let e_1 and e_2 be orthonormal frames on S_r with respect to the Euclidean metric, then

$$
\begin{aligned}
d\Sigma_r &= \left(g(e_1, e_1)g(e_2, e_2) - g^2(e_1, e_2) \right)^{1/2} d\Sigma_r^0 \\
&= \left(1 + \sigma(e_1, e_1) + \sigma(e_2, e_2) + O(r^{-2\tau}) \right)^{1/2} d\Sigma_r^0 \\
&= \left[1 + \left(e_1(x^i)e_1(x^j) + e_2(x^i)e_2(x^j) \right) \sigma_{ij} + O(r^{-2\tau}) \right]^{1/2} d\Sigma_r^0 \\
&= \left[1 + \left(\nabla_0 x^i \cdot \nabla_0 x^j - \frac{\partial x^i}{\partial r}\frac{\partial x^j}{\partial r} \right) \sigma_{ij} + O(r^{-2\tau}) \right]^{1/2} d\Sigma_r^0 \\
&= \left[1 + \left(\delta_{ij} - \frac{\partial x^i}{\partial r}\frac{\partial x^j}{\partial r} \right) \sigma_{ij} + O(r^{-2\tau}) \right]^{1/2} d\Sigma_r^0 \\
&= \left(1 + h^{ij}\sigma_{ij} + O(r^{-2\tau}) \right)^{1/2} d\Sigma_r^0,
\end{aligned}
$$

(2.7)

where ∇_0 is the derivative with respect to the Euclidean metric and '$\cdot$' is the standard inner product in $\mathbb{R}^3$. The last statement follows from this immediately. $\qquad\square$

Lemma 2.2.

(2.8) $$ \int_{S_r} H \, d\Sigma_r = \frac{\mathcal{A}(r)}{r} + 4\pi r - 8\pi m_{\mathrm{ADM}}(M) + o(1) $$

as $r \to \infty$.

Proof. Let $m = m_{\text{ADM}}(M)$. By Lemma 2.1 and the first variational formula, we have

(2.9)
$$\frac{d}{dr}\mathcal{A}(r) = \int_{S_r} \frac{1}{|\nabla r|} H \, d\Sigma_r$$
$$= \int_{S_r} H \, d\Sigma_r + \int_{S_r} \frac{\sigma_{ij} x^i x^j}{r^3} \, d\Sigma_r + O(r^{1-2\tau})$$

where we have used (2.2).

On the other hand, by Lemma 2.1, we have

(2.10)

$$\frac{d}{dr}\mathcal{A}(r) = 8\pi r + \frac{1}{2}\int_{S_r} \frac{\partial}{\partial r}\left(h^{ij}\sigma_{ij}\right) d\Sigma_r + \frac{1}{r}\int_{S_r} h^{ij}\sigma_{ij} \, d\Sigma_r + O(r^{1-2\tau})$$
$$= 8\pi r + \frac{1}{2}\int_{S_r} h^{ij}\sigma_{ij,k}\frac{x^k}{r} \, d\Sigma_r + \frac{1}{r}\int_{S_r} h^{ij}\sigma_{ij} \, d\Sigma_r + O(r^{1-2\tau})$$
$$= 8\pi r + \frac{1}{2}\int_{S_r} \frac{\sigma_{ii,k}x^k}{r} \, d\Sigma_r^0 - \frac{1}{2}\int_{S_r} \frac{\sigma_{ij,k}x^i x^j x^k}{r^3} \, d\Sigma_r^0$$
$$+ \frac{1}{r}\int_{S_r} h^{ij}\sigma_{ij} \, d\Sigma_r + O(r^{1-2\tau}),$$

where $\sigma_{ij,k} = \frac{\partial \sigma_{ij}}{\partial x^k}$. Now, as in [15, (5.17)]:

(2.11)
$$\int_{S_r} \frac{\sigma_{ij,k}x^i x^j x^k}{r^3} d\Sigma_r^0$$
$$= \int_{S_r} \frac{\partial}{\partial x^k}\left(\frac{\sigma_{ij}x^j}{r}\right)\frac{x^i x^k}{r^2} d\Sigma_r^0$$
$$= -\int_{S_r}\left(\delta_{ik} - \frac{x^i x^k}{r^2}\right)\frac{\partial}{\partial x^k}\left(\frac{\sigma_{ij}x^j}{r}\right) d\Sigma_r^0 + \int_{S_r}\frac{\partial}{\partial x^i}\left(\frac{\sigma_{ij}x^j}{r}\right) d\Sigma_r^0$$
$$= -2\int_{S_r}\frac{\sigma_{ij}x^i x^j}{r^3} d\Sigma_r^0 + \int_{S_r}\frac{\sigma_{ij,i}x^j}{r} d\Sigma_r^0 + \int_{S_r}\sigma_{ij}\left(\frac{\delta_{ij}}{r} - \frac{x^i x^j}{r^3}\right) d\Sigma_r^0$$
$$= -2\int_{S_r}\frac{\sigma_{ij}x^i x^j}{r^3} d\Sigma_r + \int_{S_r}\frac{\sigma_{ij,i}x^j}{r} d\Sigma_r^0 + \frac{1}{r}\int_{S_r} h^{ij}\sigma_{ij} d\Sigma_r + O(r^{1-2\tau}).$$

Combining this with (2.10), by Lemma 2.1 and the definition of ADM mass, we have

$$(2.12) \qquad \frac{d}{dr}\mathcal{A}(r) = \frac{\mathcal{A}(r)}{r} + 4\pi r + \int_{S_r} \frac{\sigma_{ij}x^i x^j}{r^3}\, d\Sigma_r^0 - 8\pi m + o(1).$$

By (2.9) and (2.12), the lemma follows. □

By Lemma 2.1, if r is large enough, then the Gauss curvature of S_r is positive. So S_r can be isometrically embedded in $\mathbb{R}^3$ uniquely up to an isometry of $\mathbb{R}^3$ by [13, 18–20]. The following lemma says that the embedded surface (rescaled) is very close to the standard sphere as $r \to \infty$.

Lemma 2.3. *Let (M, g) be an AF 3-manifold with (1.1) and (1.2) for $\tau > \frac{1}{2}$, and let S_r be coordinate spheres. For r large enough, there is an isometrical embedding X_r of S_r in $\mathbb{R}^3$ such that:*

$$(2.13) \qquad \begin{aligned} X_r \cdot n_0 &= r + O\left(r^{1-\tau}\right) \\ H_0 &= \frac{2}{r} + H_1 \ \text{with} \ H_1 = O\left(r^{-1-\tau}\right) \end{aligned}$$

as $r \to +\infty$, where n_0 is the unit outward normal to the surface X_r, '$\cdot$' is the inner product in $\mathbb{R}^3$ and H_0 is the mean curvature of X_r.

Proof. For $r > 0$, define a map $x = ry$ and pull back the metric to the y space. Let the pull back metric be $\hat{g}$. Let $\hat{h}$ be the induced metric on the coordinate spheres in y.

$$(2.14) \qquad \begin{aligned} \hat{h}_{ij} &= \hat{g}_{ij} - \hat{n}_i \hat{n}_j \\ &= r^2 g_{ij} - \hat{n}_i \hat{n}_j \end{aligned}$$

where $\hat{h}_{ij} = \hat{h}(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})$, etc. and $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ etc. Also $\hat{n}_i = y^i/(\rho|\hat{\nabla}\rho|_{\hat{g}})$ is the unit normal on $\left\{ \rho = \left(\sum_{i=1}^{3}\left(y^i\right)^2\right)^{1/2} = \text{constant} \right\}$. Then

$$|\hat{\nabla}\rho|_{\hat{g}}^2 = r^{-2}g^{ij}\frac{y^i y^j}{\rho^2}.$$

Consider the following metric on $\Sigma_\rho = \{y|\ |y| = \rho\}$:

$$\begin{aligned}
ds_r^2 &= r^{-2}\hat{h}_{ij} \\
&= g_{ij} - r^{-2}\hat{n}_i\hat{n}_j \\
&= g_{ij} - \frac{y^i y^j}{g^{kl}y^k y^l}.
\end{aligned}$$

(2.15)

Clearly, the standard metric h_{ij}^0 on Σ_ρ is

(2.16)
$$ds_0^2 = h_{ij}^0 = \delta_{ij} - \frac{y^i y^j}{\rho^2}.$$

Direct computations show

(2.17)
$$\|ds_r^2 - ds_0^2\|_{C^3(\Sigma_\rho)} = O\left(r^{-\tau}\right)$$

for $\frac{1}{2} \le \rho \le 2$. Note that Σ_1 is the unit sphere. By [18, p. 353], we can find an isometric embedding $\hat{X}_r$ of $(\mathbb{S}^2, ds_r^2)$ into $\mathbb{R}^3$ such that

(2.18)
$$\|\hat{X}_r - X_0\|_{C^2(\mathbb{S}^2)} = O\left(r^{-\tau}\right),$$

where X_0 is the identity map. Since $X_0 \cdot n_0 = 1$, where n_0 is the unit outward normal of the unit sphere, we have $\hat{X}_r \cdot n_{0,r} = 1 + O(r^{-\tau})$, where $n_{0,r}$ is the unit outward normal of the surface $\hat{X}_r$. If we identify S_r with metric induced by g with $(\mathbb{S}^2, \hat{h})$, then $X_r = r\hat{X}_r$ is an isometric embedding of S_r with metric induced by g. From this it is easy to see that the first part of (2.13) is true.

By (2.18), we know that $\hat{H}_0 - 2 = O\left(r^{-\tau}\right)$, where $\hat{H}_0$ is the mean curvature of $\hat{X}_r$. After rescaling $r\hat{X}_r$, we can get the second part of (2.13). $\square$

Lemma 2.4. *Let (M, g) be an AF manifold with the properties* (1.1) *and* (1.2), *and let S_r be coordinate spheres. We have*

(2.19)
$$\int_{S_r} H_0 \, d\Sigma_r = 4\pi r + \frac{\mathcal{A}(r)}{r} + O(r^{1-2\tau}).$$

Proof. By Lemma 2.3, for r large enough, we can find an isometric embedding X_r of S_r in $\mathbb{R}^3$ such that $X_r \cdot n_0 = r + O(r^{1-\tau})$. Let H_0 be the mean curvature with respect to the outward unit normal when S_r is embedded

in $\mathbb{R}^3$. By Lemma 2.1(i),

$$\bar{K} \doteq K - \frac{1}{r^2} = O(r^{-2-\tau}).$$

By one of the Minkowski integral formulae [16, Lemma 6.2.9], we have

$$
\begin{aligned}
\int_{S_r} H_0 \, d\Sigma_r &= 2 \int_{S_r} K X_r \cdot n_0 \, d\Sigma_r \\
&= 2 \int_{S_r} \left(\frac{1}{r^2} + \bar{K} \right) X_r \cdot n_0 \, d\Sigma_r \\
&= \frac{2}{r^2} \int_{S_r} X_r \cdot n_0 \, d\Sigma_r + 2 \int_{S_r} \bar{K} X_r \cdot n_0 \, d\Sigma_r \\
&= \frac{6V_0(r)}{r^2} + 2 \int_{S_r} \bar{K} \left(r + O\left(r^{1-\tau} \right) \right) d\Sigma_r \\
&= \frac{6V_0(r)}{r^2} + 2r \int_{S_r} \bar{K} \, d\Sigma_r + O(r^{1-2\tau}) \\
&= \frac{6V_0(r)}{r^2} + 2r \int_{S_r} \left(K - \frac{1}{r^2} \right) d\Sigma_r + O(r^{1-2\tau}) \\
&= \frac{6V_0(r)}{r^2} + 8\pi r - \frac{2\mathcal{A}(r)}{r} + O(r^{1-2\tau}),
\end{aligned}
$$

(2.20)

where $V_0(r)$ is the volume of the interior of the surface X_r in $\mathbb{R}^3$. On the other hand, from Lemma 2.3, $H_0 = \frac{2}{r} + H_1$ with $H_1 = O\left(r^{-1-\tau} \right)$. By another Minkowski integral formula, we have

$$
\begin{aligned}
2\mathcal{A}(r) &= \int_{S_r} H_0 X \cdot n_0 \, d\Sigma_r \\
&= \frac{6V_0(r)}{r} + \int_{S_r} H_1 X \cdot n_0 \, d\Sigma_r \\
&= \frac{6V_0(r)}{r} + r \int_{S_r} H_1 \, d\Sigma_r + O\left(r^{2-2\tau} \right) \\
&= \frac{6V_0(r)}{r} - 2\mathcal{A}(r) + r \int_{S_r} H_0 \, d\Sigma_r + O\left(r^{2-2\tau} \right).
\end{aligned}
$$

(2.21)

So

$$\int_{S_r} H_0 \, d\Sigma_r = -\frac{6V_0(r)}{r^2} + \frac{4\mathcal{A}(r)}{r} + O(r^{1-2\tau}).$$

(2.22)

From (2.20) and (2.22), the lemma follows. $\qquad\square$

Proof of Theorem 2.1. The theorem follows immediately from Lemmas 2.2 and 2.4. □

In Theorem 2.1, S_r can be replaced by slightly deformed spheres. More precisely, we have:

Corollary 2.1. *Same assumptions as in Theorem 2.1. Suppose ρ is a smooth function on M such that*

$$(2.23) \quad |\rho - r| + r|\partial(\rho - r)| + r^2|\partial\partial(\rho - r)| + r^3|\partial\partial\partial(\rho - r)| = O(r^\kappa)$$

for some $0 < \kappa < 1 - \tau$. Then

$$\lim_{\rho \to \infty} m_{\mathrm{BY}}(\Sigma_\rho) = m_{\mathrm{ADM}}(M),$$

where Σ_ρ is the level set of the smooth function ρ.

Proof. Let $y = \frac{\rho}{r}x = F(x)$. Then one can show that y is also a coordinate system of M at infinity so that the metric tensor in this coordinates satisfies the decay conditions (1.1) and (1.2). Note that Σ_ρ is nothing but the coordinate spheres in the y-coordinates. Hence, the corollary follows from the uniqueness of ADM mass by [2]. □

With the notations as in the proof of Theorem 2.1. Let $V(r)$ be the volume with respect to an AF metric g of the region inside S_r. We can compare $V(r)$ and $V_0(r)$ (Theorem 1.2):

Theorem 2.2. *With the above notations. Let (M, g) be an AF manifold of order $\tau > \frac{1}{2}$ with one end. Then*

$$(2.24) \qquad V_0(r) - V(r) = -2m_{\mathrm{ADM}}\pi r^2 + o(r^2).$$

Proof. Let $m = m_{ADM}$. With the same notations as in the proof of Theorem 2.1, by (2.2) and the co-area formula we have

$$
\begin{aligned}
(2.25) \qquad V'(r) &= \int_{S_r} \frac{1}{|\nabla r|} d\Sigma_r \\
&= \mathcal{A}(r) + \frac{1}{2}\int_{S_r} \frac{\sigma_{ij}x^i x^j}{r^2} d\Sigma_r + O(r^{2-2\tau}).
\end{aligned}
$$

Here and below is the derivative with respect to r. On the other hand, by (2.8) and (2.9) we have

$$(2.26) \qquad \mathcal{A}'(r) = \frac{\mathcal{A}(r)}{r} + 4\pi r - 8\pi m + \int_{S_r} \frac{\sigma_{ij} x^i x^j}{r^3} \, d\Sigma_r + o(1).$$

Eliminating the term $\int_{S_r} \frac{\sigma_{ij} x^i x^j}{r^3} d\Sigma_r$ from (2.25) and (2.26), we have

$$(2.27) \qquad \mathcal{A}'(r) = \frac{\mathcal{A}(r)}{r} + 4\pi r - 8\pi m + \frac{1}{r} \left(2V'(r) - 2\mathcal{A}(r) \right) + o(1).$$

Hence,

$$(r\mathcal{A}(r))' = 4\pi r^2 - 8\pi m r + 2V'(r) + o(r)$$

and

$$(2.28) \qquad V(r) = \frac{1}{2} r \mathcal{A}(r) - \frac{2\pi r^3}{3} + 2\pi m r^2 + o(r^2).$$

On the other hand, by (2.20) and (2.21), we have

$$(2.29) \qquad V_0(r) = \frac{1}{2} r \mathcal{A}(r) - \frac{2\pi r^3}{3} + O(r^{3-2\tau}).$$

Hence

$$V_0(r) - V(r) = -2\pi m r^2 + o(r^2)$$

because $\tau > \frac{1}{2}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Combine this with positive mass theorem, we have the following:

Corollary 2.2. *With above notations, let (M, g) be an AF manifold of order $\tau > \frac{1}{2}$. If the scalar curvature is non-negative, then*

$$\lim_{r \to +\infty} \frac{V(r) - V_0(r)}{r^2} \geq 0,$$

and equality holds if and only if (M, g) is isometric to $\mathbb{R}^3$.

From the proof of Theorem 2.2, Miao [17] is able to obtain the following result. Thanks to Pengzi Miao, we include the result and the proof here.

Corollary 2.3. *In an AF manifold M, the ADM mass and the isoperimetric mass introduced by Huisken [14] are equal.*

Proof. Recall that the isoperimetric mass of M is defined as

$$m_{\text{ISO}} = \limsup_{r \to \infty} \frac{2}{\mathcal{A}(r)} \left(V(r) - \frac{1}{6\pi^{1/2}} \mathcal{A}^{3/2}(r) \right).$$

Now by (2.28)

(2.30)
$$\frac{2}{\mathcal{A}(r)} \left(V(r) - \frac{1}{6\pi^{1/2}} \mathcal{A}^{3/2}(r) \right)$$

$$= r + \frac{1}{\mathcal{A}(r)} \left(4\pi m r^2 - \frac{4\pi r^3}{3} \right) - \frac{1}{3\pi^{1/2}} \mathcal{A}^{1/2}(r) + o(1)$$

$$= r + \left(m - \frac{r}{3} \right) \left(1 - \mathcal{I} + O(r^{-2\tau}) \right) - \frac{2r}{3} \left(1 + \frac{1}{2}\mathcal{I} + O(r^{-2\tau}) \right) + o(1)$$

$$= m + o(1),$$

where

$$\mathcal{I} = \frac{1}{8\pi r^2} \int_{S_r} h^{ij} \sigma_{ij} \, d\Sigma_r = O(r^{-\tau})$$

so that

$$\mathcal{A}(r) = 4\pi r^2 \left(1 + \mathcal{I} + O(r^{-2\tau}) \right),$$

see Lemma 2.1(ii). From this the result follows. $\qquad\square$

3. Small-sphere limit

In this section, we will first study the small-sphere limit of the Brown–York mass of geodesic spheres up to order r^5, where r is the geodesic distance from a fixed point. Let (N^3, g) be a three-dimensional manifold and let $p \in N$. Let $\{x^i\}$ be the normal coordinates near p. By [22, Chapter 5], we have the following expansion of g near p:

Lemma 3.1. *For any point x close to p, the metric components of g in the normal coordinates can be expressed as*

$$g_{ij}(x) = \delta_{ij} + \frac{1}{3} R_{iklj}(p) x^k x^l + \frac{1}{6} R_{iklj;m}(p) x^k x^l x^m$$

(3.1)
$$+ \left(\frac{1}{20} R_{iklj;mn}(p) + \frac{2}{45} R_{ikls}(p) R_{jmns}(p) \right) x^k x^l x^m x^n + O\left(r^5\right)$$

and

$$g = \det(g_{ij})$$

$$= 1 - \frac{1}{3}R_{ij}(p)x^i x^j - \frac{1}{6}R_{ij;k}(p)x^i x^j x^k - \left(\frac{1}{20}R_{ij;kl}(p)\right.$$

(3.2) $$+ \left. \frac{1}{90}R_{hijm}(p)R_{hklm}(p) - \frac{1}{18}R_{ij}(p)R_{kl}(p)\right) x^i x^j x^k x^l + O(r^5),$$

where r is the geodesic distance from p, R_{ijkl} is the Riemannian curvature tensor, R_{ij} is the Ricci curvature and R is the scalar curvature with respect to the metric g, and $R_{iklj;m}$ is the covariant derivative of R_{ijkl} etc.

In our notations, the sectional curvature is non-negative if $R_{ijij} \geq 0$. In the following, we always assume that the normal coordinates are chosen so that at p the Ricci curvature is of the form $R_{ij} = \lambda_i \delta_{ij}$ where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of R_{ij}.

Lemma 3.2. *Let $\mathcal{A}(r)$ be the area of geodesic sphere $S_r = \{|x| = r\}$ with radius r in (N, g) with center at p, then:*

(3.3) $$\mathcal{A}(r) = 4\pi r^2 + A_4 + A_6 + O(r^7),$$

where

(3.4) $$A_4 = -\frac{2\pi r^4}{9}R, \quad A_6 = \frac{\pi r^6}{675}\left(4R^2 - 2|\mathrm{Ric}|^2 - 9\Delta R\right),$$

where Δ is the Laplacian operator with respect to metric g and $|\mathrm{Ric}|$ is the norm of the Ricci tensor. Here all the terms involving curvature are evaluated at p.

Proof. By (3.2)

$$\sqrt{g} = 1 + \frac{1}{2}(b_2 + b_3 + b_4) - \frac{1}{8}b_2^2 + O(r^5),$$

where

$$b_2 = -\frac{1}{3}R_{ij}x^i x^j,$$

(3.5) $$b_3 = -\frac{1}{6}R_{ij,k}x^i x^j x^k,$$

$$b_4 = -\left(\frac{1}{20}R_{ij,kl} + \frac{1}{90}R_{hijm}R_{hklm} - \frac{1}{18}R_{ij}R_{kl}\right)x^i x^j x^k x^l.$$

Hence

$$V(r) = \int_{B_r} \sqrt{g}\, dv_0,$$

where dv_0 is the volume element with respect to Euclidean metric and $B_r = \{x|\ |x| < r\}$. Since $|\frac{\partial}{\partial r}| = 1$ in g metric,

(3.6)
$$\begin{aligned}
\mathcal{A}(r) &= V'(r) \\
&= 4\pi r^2 + \int_{S_r}\left[\frac{1}{2}(b_2 + b_3 + b_4) - \frac{1}{8}b_2^2 + O(r^5)\right] d\Sigma_r^0 \\
&= 4\pi r^2 + \frac{1}{2}\int_{S_r} b_2\, d\Sigma_r^0 + \frac{1}{2}\int_{S_r}\left(b_4 - \frac{1}{4}b_2^2\right) d\Sigma_r^0 + O(r^7),
\end{aligned}$$

where $d\Sigma_r^0$ is the area element of S_r with respect to the Euclidean metric. Since $\int_{S_r}\left(x^i\right)^2 d\Sigma_r^0 = \frac{4}{3}\pi r^4$, by (3.5) and the fact that $R_{ij}x^i x^j = \sum_{i=1}^3 \lambda_i(x^i)^2$,

(3.7)
$$\frac{1}{2}\int_{S_r} b_2\, d\Sigma_r^0 = -\frac{2\pi r^4}{9} R.$$

Noting that

(3.8)

$$\begin{aligned}
b_4 &= -\frac{1}{20}R_{ij;kl}x^i x^j x^k x^l - \frac{1}{90}\sum_{h,m}\left(\sum_{ij}R_{hijm}x^i x^j\right)^2 + \frac{1}{18}\left(\sum_{ij}R_{ij}x^i x^j\right)^2 \\
&= -\frac{1}{20}R_{ij;kl}x^i x^j x^k x^l - \frac{1}{90}\sum_{i,j}\left(\sum_{k,l}R_{iklj}x^k x^l\right)^2 + \frac{1}{2}b_2^2.
\end{aligned}$$

Let us first compute $\int_{S_r}\left(R_{ij}x^i x^j\right)^2 d\Sigma_r^0$. By symmetry,

(3.9)
$$\int_{S_r}\left(x^i\right)^4 d\Sigma_r^0 = \frac{4}{5}\pi r^6, \text{ for } i = 1, 2, 3,$$

and for $i \neq j$,

(3.10)
$$\int_{S_r}\left(x^i x^j\right)^2 d\Sigma_r^0 = \frac{4}{15}\pi r^6.$$

We have

$$
\int_{S_r} \left(R_{ij} x^i x^j \right)^2 d\Sigma_r^0 = \int_{S_r} \left(\sum_i \lambda_i (x^i)^2 \right)^2 d\Sigma_r^0
$$

(3.11)
$$
= \frac{4}{5} \pi r^6 \sum_i \lambda_i^2 + \frac{4}{15} \pi r^6 \sum_{i \neq j} \lambda_i \lambda_j
$$

$$
= \frac{4}{15} \pi r^6 \left(R^2 + 2|\mathrm{Ric}|^2 \right).
$$

Since the dim $N = 3$, by [10, p. 276], at p

(3.12)
$$
R_{ijkl} = g_{ik} R_{jl} - g_{il} R_{jk} - g_{jk} R_{il} + g_{jl} R_{ik} - \frac{1}{2} R \left(g_{ik} g_{jl} - g_{il} g_{jk} \right)
$$

$$
= \delta_{ik} R_{jl} - \delta_{il} R_{jk} - \delta_{jk} R_{il} + \delta_{jl} R_{ik} - \frac{1}{2} R \left(\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} \right),
$$

and hence on S_r:

(3.13)
$$
R_{ijkl} x^j x^k = \left(\lambda_i + \lambda_l - \frac{R}{2} \right) x^i x^l - \delta_{il} \left(\sum_k \lambda_k (x^k)^2 + \left(\lambda_i - \frac{R}{2} \right) r^2 \right).
$$

Using (3.10), we have

(3.14)
$$
\int_{S_r} \sum_{i \neq l} \left(\sum_{j,k} R_{ijkl} x^j x^k \right)^2 d\Sigma_r^0 = \int_{S_r} \sum_{i \neq l} \left(\lambda_l + \lambda_i - \frac{R}{2} \right)^2 \left(x^i x^l \right)^2 d\Sigma_r^0
$$

$$
= \frac{4}{15} \pi r^6 \sum_{i \neq l} \left(\lambda_l + \lambda_i - \frac{R}{2} \right)^2
$$

$$
= \frac{4}{15} \pi r^6 \left(2|\mathrm{Ric}|^2 - \frac{R^2}{2} \right).
$$

Clearly, by (3.13), (3.9) and (3.10), we have

$$\int_{S_r} \left(\sum_{jk} R_{1jk1} x^j x^k \right)^2 d\Sigma_r^0$$

(3.15)
$$= \int_{S_r} \left(\left(2\lambda_1 - \frac{R}{2}\right)(x^1)^2 - \sum_k \lambda_k (x^k)^2 - \left(\lambda_1 - \frac{R}{2}\right) r^2 \right)^2 d\Sigma_r^0$$

$$= \pi r^6 \left(-\frac{4}{15}\lambda_1^2 + \frac{8}{15}R\lambda_1 - \frac{4}{15}R^2 + \frac{8}{15}|\mathrm{Ric}|^2 \right).$$

We have similar formula for the case $i = l = 2$ or 3. So

(3.16)
$$\int_{S_r} \sum_{i=l} \left(\sum_{j,k} R_{ijkl} x^j x^k \right)^2 d\Sigma_r^0 = \pi r^6 \left(\frac{20}{15}|\mathrm{Ric}|^2 - \frac{4}{15}R^2 \right).$$

By (3.14) and (3.16) we have

(3.17)
$$\int_{S_r} \sum_{i,l} \left(\sum_{j,k} R_{ijkl} x^j x^k \right)^2 d\Sigma_r^0 = \frac{1}{15} \left(28|\mathrm{Ric}|^2 - 6R^2 \right) \pi r^6.$$

Finally, let us compute $\int_{S_r} \sum_{i,j,k,l} R_{ij;kl} x^i x^j x^k x^l d\Sigma_r^0$. By symmetry $\int_{S_r} R_{ij;kl} x^i x^j x^k x^l d\Sigma_r^0 = 0$ unless $x^i x^j x^k x^l$ is of the form $(x^m)^4$, or $(x^m)^2(x^n)^2$ with $m \neq n$.

(3.18)
$$\sum_{j,k,l} \int_{S_r} R_{1j;kl} x^1 x^j x^k x^l d\Sigma_r^0$$

$$= \int_{S_r} R_{11;11}(x^1)^4 d\Sigma_r^0 + \int_{S_r} R_{11;22}(x^1 x^2)^2 d\Sigma_r^0 + \int_{S_r} R_{11;33}(x^1 x^3)^2 d\Sigma_r^0$$

$$+ \int_{S_r} (R_{12;12} + R_{12;21})(x^1 x^2)^2 d\Sigma_r^0 + \int_{S_r} (R_{13;13} + R_{13;31})(x^1 x^3)^2 d\Sigma_r^0$$

$$= \frac{4}{5}\pi r^6 R_{11;11} + \frac{4}{15}\pi r^6 (R_{12;12} + R_{12;21} + R_{13;13} + R_{13;31} + R_{11;22} + R_{11;33}).$$

Similarly, one can prove that

$$\sum_{j,k,l} \int_{S_r} R_{2j;kl} x^2 x^j x^k x^l d\Sigma_r^0 = \frac{4}{5}\pi r^6 R_{22;22} + \frac{4}{15}\pi r^6 (R_{21;21} + R_{21;12}$$

(3.19)
$$+ R_{23;23} + R_{23;32} + R_{22;11} + R_{22;33}),$$

and

$$\sum_{j,k,l} \int_{S_r} R_{3j,kl} x^3 x^j x^k x^l \, d\Sigma_r^0 = \frac{4}{5}\pi r^6 R_{33;33} + \frac{4}{15}\pi r^6 (R_{31;31} + R_{31;13}$$

$$\text{(3.20)} \qquad\qquad\qquad\qquad + R_{32;32} + R_{32;23} + R_{33;11} + R_{33;22}).$$

Hence,

$$\text{(3.21)} \qquad \sum_{i,j,k,l} \int_{S_r} R_{ij;kl} x^i x^j x^k x^l \, d\Sigma_r^0 = \frac{4}{15}\pi r^6 \sum_{i,j} (R_{ii;jj} + 2R_{ij;ij}).$$

By the second Bianchi identity, we see that

$$\sum_i R_{ij;ij} = \frac{1}{2} R_{;jj}.$$

Therefore, we have

$$\text{(3.22)} \qquad \sum_{i,j,k,l} \int_{S_r} R_{ij,kl} x^i x^j x^k x^l \, d\Sigma_r^0 = \frac{8\pi r^6}{15} \Delta R(p).$$

The lemma follows from (3.5), (3.6), (3.7), (3.8), (3.11), (3.15), (3.17) and (3.22). $\qquad\square$

Corollary 3.1. *With the notations and assumptions as in Lemma 3.2, let H be the mean curvature of S_r with respect to g, then*

$$\text{(3.23)} \qquad \int_{S_r} H \, d\Sigma_r = 8\pi r + \frac{4A_4 + 6A_6}{r} + O(r^6).$$

Proof. By the fact that $|\nabla r| = 1$, we have

$$\int_{S_r} H \, d\Sigma_r = \frac{d}{dr} \mathcal{A}(r).$$

The corollary then follows from Lemma 3.2. $\qquad\square$

By [18], and the fact that for r small $(S_r, g|_{S_r})$ has positive Gauss curvature, one can isometrically embed $(S_r, g|_{S_r})$ in $\mathbb{R}^3$.

Lemma 3.3. *For r small enough, there is an isometric embedding Z of geodesic sphere S_r into $\mathbb{R}^3$ such that*

$$(3.24) \qquad Z \cdot n = r + \frac{r^3}{6}\left(\frac{R}{2} - 2R_{ij}\frac{x^i x^j}{r^2}\right) + O(r^4),$$

where n is the outward unit normal vector of $Z(S_r)$ in $\mathbb{R}^3$ and '·' is the inner product in $\mathbb{R}^3$.

Proof. For $r > 0$, we define a map $x = ry$ and pull back the metric g to the y space and let h be the metric $r^{-2}g$ induced on the unit sphere $\mathbb{S}^2$ in the y space. As in the proof of Lemma 2.3, in order to prove the lemma, it is sufficient to prove that for r small, we can find an isometric embedding Z_r of $(\mathbb{S}^2, h)$ in $\mathbb{R}^3$ such that

$$(3.25) \qquad Z_r \cdot n_r = 1 + \frac{r^2}{6}\left(\frac{R}{2} - 2\sum \lambda_k (y^k)^2\right) + O(r^3),$$

where n_r is the unit outward normal of $Z_r(\mathbb{S}^2)$.

Let $\tilde{h}$ be the induced metric of $r^{-2}\tilde{g}$ on the unit sphere $\mathbb{S}^2$, where

$$\tilde{g}_{ij} = \tilde{g}\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = r^2\left(\delta_{ij} + \frac{r^2}{3}R_{iklj}y^k y^l\right).$$

Let $\hat{h}$ be the metric on $\mathbb{S}^2$ induced by the pull back of the Euclidean metric given by the embedding $\hat{Z} = (z^1, z^2, z^3)$ in $\mathbb{R}^3$ where

$$z^1 = y^1\left(1 + \frac{r^2}{6}\left(\frac{R}{2} - \lambda_1 - \sum_i \lambda_i (y^i)^2\right)\right)$$

$$z^2 = y^2\left(1 + \frac{r^2}{6}\left(\frac{R}{2} - \lambda_2 - \sum_i \lambda_i (y^i)^2\right)\right)$$

$$z^3 = y^3\left(1 + \frac{r^2}{6}\left(\frac{R}{2} - \lambda_3 - \sum_i \lambda_i (y^i)^2\right)\right).$$

We claim that

$$(3.26) \qquad ||h - \tilde{h}||_{C^3} + ||\hat{h} - \tilde{h}||_{C^3} = O(r^3),$$

where the norm is computed with respect to the standard metric. Suppose the claim is true, then by [18], we can conclude that there are isometric

embeddings Z_r, $\tilde{Z}$ and $\hat{Z}$ for $(\mathbb{S}^2, h)$, $(\mathbb{S}^2, \tilde{h})$ and $(\mathbb{S}^2, \hat{h})$, respectively, such that

$$
(3.27) \qquad ||Z_r \cdot n_r - \hat{Z} \cdot \hat{n}||_{C^0(\mathbb{S}^2)} = O(r^3),
$$

where $\hat{n}$ is the unit outward normal of $\hat{Z}(\mathbb{S}^2)$. Then we can prove the lemma by computing $\hat{Z} \cdot \hat{n}$.

Let us first prove the claim and then compute $\hat{Z} \cdot \hat{n}$. It is easy to see that $||h - \tilde{h}||_{C^3} = O(r^3)$ by the expression of g in Lemma 3.1 and the definition of $\tilde{g}$.

To find $\tilde{h}$, let $r^{-2}\tilde{g}_{ij} = \delta_{ij} + \tilde{\sigma}_{ij}$, and let $\lambda_{ij} = -(\lambda_i + \lambda_j) + \frac{R}{2}$. By (3.13), we have

$$
\tilde{\sigma}_{11} = \frac{r^2}{3}(\lambda_{12}(y^2)^2 + \lambda_{13}(y^3)^2),
$$
$$
\tilde{\sigma}_{22} = \frac{r^2}{3}(\lambda_{12}(y^1)^2 + \lambda_{23}(y^3)^2),
$$
$$
\tilde{\sigma}_{33} = \frac{r^2}{3}(\lambda_{13}(y^1)^2 + \lambda_{23}(y^2)^2),
$$
$$
\tilde{\sigma}_{ij} = -\frac{r^2}{3}\lambda_{ij}y^i y^j, \ i \neq j.
$$

In the above last equation, the repeated indices is not taken summation.

Let $e_1 = \partial_\theta = a_i\partial_i$, $e_2 = (\sin\theta)^{-1}\partial_\phi = b_i\partial_i$. Then

$$
a_1 = \cos\theta\cos\phi, \quad a_2 = \cos\theta\sin\phi, \quad a_3 = -\sin\theta;
$$
$$
b_1 = -\sin\phi, \quad b_2 = \cos\phi, \quad b_3 = 0.
$$

Note that

$$
y^1 = \sin\theta\cos\phi, \quad y^2 = \sin\theta\sin\phi, \quad y^3 = \cos\theta.
$$

Hence, in the basis $\{e_1, e_2\}$, $\tilde{h}$ is given by

$$
\tilde{h}_{11} = 1 + \frac{r^2}{3}\left(\lambda_{23}\sin^2\phi + \lambda_{31}\cos^2\phi\right),
$$
$$
(3.28) \quad \tilde{h}_{12} = \frac{r^2}{3}\left(-\lambda_{13} + \lambda_{23}\right)\cos\theta\cos\phi\sin\phi,
$$
$$
\tilde{h}_{22} = 1 + \frac{r^2}{3}\left(\lambda_{12}\sin^2\theta + \lambda_{13}\cos^2\theta\sin^2\phi + \lambda_{23}\cos^2\theta\cos^2\phi\right).
$$

Next we want to compute $\hat{h}$.

$$(z^1)_\theta = (y^1)_\theta \left(1 + \frac{r^2}{6}\left(\frac{R}{2} - \lambda_1 - \sum_i \lambda_i (y^i)^2\right)\right) - \frac{r^2}{6} y^1 \left(\sum_i \lambda_i (y^i)^2\right)_\theta,$$

$$(z^2)_\theta = (y^2)_\theta \left(1 + \frac{r^2}{6}\left(\frac{R}{2} - \lambda_2 - \sum_i \lambda_i (y^i)^2\right)\right) - \frac{r^2}{6} y^2 \left(\sum_i \lambda_i (y^i)^2\right)_\theta,$$

$$(z^3)_\theta = (y^3)_\theta \left(1 + \frac{r^2}{6}\left(\frac{R}{2} - \lambda_3 - \sum_i \lambda_i (y^i)^2\right)\right) - \frac{r^2}{6} y^3 \left(\sum_i \lambda_i (y^i)^2\right)_\theta,$$

and

$$(z^1)_\phi = (y^1)_\phi \left(1 + \frac{r^2}{6}\left(\frac{R}{2} - \lambda_1 - \sum_i \lambda_i (y^i)^2\right)\right) - \frac{r^2}{6} y^1 \left(\sum_i \lambda_i (y^i)^2\right)_\phi,$$

$$(z^2)_\phi = (y^2)_\phi \left(1 + \frac{r^2}{6}\left(\frac{R}{2} - \lambda_2 - \sum_i \lambda_i (y^i)^2\right)\right) - \frac{r^2}{6} y^2 \left(\sum_i \lambda_i (y^i)^2\right)_\phi,$$

$$(z^3)_\phi = (y^3)_\phi \left(1 + \frac{r^2}{6}\left(\frac{R}{2} - \lambda_3 - \sum_i \lambda_i (y^i)^2\right)\right) - \frac{r^2}{6} y^3 \left(\sum_i \lambda_i (y^i)^2\right)_\phi.$$

Hence,

$$\begin{aligned}
\hat{Z}_\theta \cdot \hat{Z}_\theta &= (z^1)_\theta^2 + (z^2)_\theta^2 + (z^3)_\theta^2 \\
&= \sum_i (y^i)_\theta^2 \left(1 + \frac{r^2}{3}\left(\frac{R}{2} - \lambda_i - \sum_k \lambda_k (y^k)^2\right)\right) \\
&\quad - \frac{r^2}{3} \sum_i y^i (y^i)_\theta \left(\sum_i \lambda_i (y^i)^2\right)_\theta + O(r^4) \\
&= 1 + \frac{r^2}{3}\left[\frac{R}{2} - \sum_i \lambda_i \left((y^i)_\theta^2 + (y^i)^2\right)\right] + O(r^4) \\
&= 1 + \frac{r^2}{3}\left(\frac{R}{2} - \lambda_1 \cos^2\phi - \lambda_2 \sin^2\phi - \lambda_3\right) + O(r^4) \\
&= 1 + \frac{r^2}{3}\left(\lambda_{13} \cos^2\phi + \lambda_{23} \sin^2\phi\right) + O(r^4),
\end{aligned}$$

$$\hat{Z}_\phi \cdot \hat{Z}_\phi = (z^1)_\phi^2 + (z^2)_\phi^2 + (z^3)_\phi^2$$

$$= \sum_i (y_i)_\phi^2 \left(1 + \frac{r^2}{3}\left(\frac{R}{2} - \lambda_i - \sum_k \lambda_k y_k^2\right)\right)$$

$$- \frac{r^2}{3}\sum_i y_i(y_i)_\phi \left(\sum_i \lambda_i y_i^2\right)_\phi + O(r^4)$$

$$= \sin^2\theta + \frac{r^2}{3}\left[\frac{R}{2}\sin^2\theta - \sum_i \lambda_i\left((y_i)_\phi^2 + y_i^2 \sin^2\theta\right)\right] + O(r^4)$$

$$= \sin^2\theta\left[1 + \frac{r^2}{3}\left(\frac{R}{2} - \lambda_1\left(\sin^2\phi + \cos^2\phi\sin^2\theta\right)\right.\right.$$

$$\left.\left. - \lambda_2\left(\cos^2\phi + \sin^2\phi\sin^2\theta\right) - \lambda_3\cos^2\theta\right)\right] + O(r^4)$$

$$= \sin^2\theta\left[1 + \frac{r^2}{3}\left(\lambda_{12}\sin^2\theta + \lambda_{13}\cos^2\theta\sin^2\phi + \lambda_{23}\cos^2\theta\cos^2\phi\right)\right]$$

$$+ O(r^4),$$

and

$$\hat{Z}_\theta \cdot \hat{Z}_\phi = (z^1)_\theta(z^1)_\phi + (z^2)_\theta(z^2)_\phi + (z^3)_\theta(z^3)_\phi$$

$$= \sum_i (y^i)_\theta(y^i)_\phi + \frac{r^2}{3}\sum_i (y^i)_\theta(y^i)_\phi\left(\frac{R}{2} - \lambda_i - \sum_k \lambda_k(y^k)^2\right)$$

$$- \frac{r^2}{6}\left(\sum_i y^i(y^i)_\theta\right)\left(\sum_i \lambda_i(y^i)^2\right)_\phi - \frac{r^2}{6}\left(\sum_i y^i(y^i)_\phi\right)$$

$$\times \left(\sum_i \lambda_i(y^i)^2\right)_\theta + O(r^4)$$

$$= -\frac{r^2}{3}\sum_i (y^i)_\theta(y^i)_\phi\lambda_i + O(r^4)$$

$$= \frac{r^2}{3}(-\lambda_{13} + \lambda_{23})\sin\theta\cos\theta\cos\phi\sin\phi + O(r^4).$$

Thus, we see that

$$(3.29) \qquad\qquad \|\tilde{h} - \hat{h}\|_{C^3} = O(r^4).$$

This completes the proof of the claim. Next we want to compute $\hat{Z}\cdot\hat{n}$.

Let $A = (y^1, y^2, y^3)$, $B = A_\theta$, $C = \frac{1}{\sin\theta}A_\phi$, $\bar{A} = (\bar{A}_1, \bar{A}_2, \bar{A}_3)$, $\bar{B} = (\bar{B}_1, \bar{B}_2, \bar{B}_3)$ and $\bar{C} = (\bar{C}_1, \bar{C}_2, \bar{C}_3)$, where

$$\bar{A}_i = \frac{r^2 y^i}{6}\left(\frac{R}{2} - \lambda_i - \sum_i \lambda_k (y^k)^2\right),$$

$$\bar{B}_i = \frac{(y^i)_\theta r^2}{6}\left(\frac{R}{2} - \lambda_i - \sum_k \lambda_k (y^k)^2\right) - \frac{r^2}{6}y^i\left(\sum_k \lambda_k (y^k)^2\right)_\theta$$

$$\bar{C}_i = \frac{(y^i)_\phi r^2}{6\sin\theta}\left(\frac{R}{2} - \lambda_i - \sum_k \lambda_k (y^k)^2\right) - \frac{r^2}{6\sin\theta}y^i\left(\sum_k \lambda_k (y^k)^2\right)_\phi.$$

Note that A, B, C are orthonormal and positively oriented in $\mathbb{R}^3$ for $A \in \mathbb{S}^2$. Let $e_1 = \partial_\theta$ and $e_2 = \frac{1}{\sin\theta}\partial_\phi$ as before. Then,

$$\hat{Z}\cdot\hat{Z}_1 \wedge \hat{Z}_2 = (A + \bar{A})\cdot(B + \bar{B})\wedge(C + \bar{C})$$
$$= A\cdot B\wedge C + A\cdot B\wedge\bar{C} + A\cdot\bar{B}\wedge C + \bar{A}\cdot B\wedge C + O(r^4)$$
$$= 1 + C\cdot\bar{C} + B\cdot\bar{B} + A\cdot\bar{A} + O(r^4).$$

Now

$$A\cdot\bar{A} = \frac{r^2}{6}\left(\frac{R}{2} - 2\sum_k \lambda_k(y^k)^2\right).$$

$$B\cdot\bar{B} = \frac{r^2}{6}\left(\frac{R}{2} - \sum_k \lambda_k\left((y^k)_\theta^2 + (y^k)^2\right)\right)$$
$$= \frac{r^2}{6}\left(\frac{R}{2} - \left(\lambda_1\cos^2\phi + \lambda_2\sin^2\phi + \lambda_3\right)\right).$$

$$C\cdot\bar{C} = \frac{r^2}{6\sin^2\theta}\left(\frac{R}{2}\sin^2\theta - \sum_k \lambda_k\left((y^k)_\phi^2 + (y^k)^2\sin^2\theta\right)\right)$$
$$= \frac{r^2}{6}\left(\frac{R}{2} - \left(\lambda_1\left(\sin^2\phi + \sin^2\theta\cos^2\phi\right)\right.\right.$$
$$\left.\left. + \lambda_2\left(\cos^2\phi + \sin^2\theta\sin^2\phi\right) + \lambda_3\cos^2\theta\right)\right).$$

So

(3.30) $$\hat{Z}\cdot\hat{Z}_1 \wedge \hat{Z}_2 = 1 + \frac{r^2}{6}\left(\frac{R}{2} - 3\sum \lambda_k(y^k)^2\right) + O\left(r^4\right).$$

Noting that

$$|\hat{Z}_1 \wedge \hat{Z}_2|^2 = 1 + \frac{r^2}{3}[(\lambda_{13}(\cos^2\phi + \cos^2\theta\sin^2\phi) + \lambda_{12}\sin^2\theta$$
$$+ \lambda_{23}(\sin^2\phi + \cos^2\theta\cos^2\phi))] + O\left(r^4\right)$$
$$= 1 - \frac{r^2}{3}\sum\lambda_k(y^k)^2 + O\left(r^4\right),$$

we have

$$(3.31) \qquad\qquad |\hat{Z}_1 \wedge \hat{Z}_2|^{-1} = 1 + \frac{r^2}{6}\sum\lambda_k(y^k)^2 + O\left(r^4\right).$$

Combining (3.30) and (3.31)

$$(3.32) \qquad \hat{Z}\cdot\hat{n} = \frac{\hat{Z}\cdot\hat{Z}_1 \wedge \hat{Z}_2}{|\hat{Z}_1 \wedge \hat{Z}_2|} = 1 + \frac{r^2}{6}\left(\frac{R}{2} - 2\sum\lambda_k(y^k)^2\right) + O\left(r^4\right).$$

This completes the proof of the lemma. $\qquad\qquad\qquad\qquad\qquad\Box$

Lemma 3.4. *Let K and H be the Gauss curvature and the mean curvature of S_r in g and H_0 be the mean curvature of $(S_r, g|_{S_r})$ when embedded in $\mathbb{R}^3$. Then,*

$$(3.33) \qquad\qquad K = \frac{1}{r^2} + \frac{R}{2} - \frac{4}{3}R_{ij}\frac{x^i x^j}{r^2} + O(r),$$

$$(3.34) \qquad\qquad H = \frac{2}{r} - \frac{1}{3}R_{ij}\frac{x^i x^j}{r} + O(r^2)$$

and

$$(3.35) \qquad\qquad H_0 = \frac{2}{r} + r\left(\frac{R}{2} - \frac{4}{3}R_{ij}\frac{x^i x^j}{r^2}\right) + O(r^2).$$

Proof. We continue to use the normal coordinates as in Lemma 3.1. Then $n = \frac{\partial}{\partial r}$ is the outward normal of S_r. Let $h_{ij} = g_{ij} - n_i n_j$ be the induced metric on S_r with $n_i = \frac{x^i}{r}$. By Lemma 3.1, the Christoffel symbols are given by:

$$(3.36) \qquad\qquad \Gamma_{ij}^k = \frac{1}{3}\left(R_{kimj} + R_{kjmi}\right)x^m + O(r^2).$$

where the curvature is evaluated at p. Since $\nabla_n n = 0$, the second fundamental form A in these coordinates is given by

$$
\begin{aligned}
A_{ij} &= n_{j;i} \\
(3.37) \qquad &= \frac{\partial n_j}{\partial x^i} - \Gamma_{ij}^k n_k \\
&= \frac{\delta_{ij}}{r} - \frac{x^i x^j}{r^3} - \frac{2}{3} R_{kimj} \frac{x^k x^m}{r} + O(r^2).
\end{aligned}
$$

Let $\{e_1, e_2\}$ be an orthonormal frame with respect to the Euclidean metric on S_r and let λ_1 and λ_2 be the eigenvalues of A. Then

$$
\begin{aligned}
\lambda_1 \lambda_2 &= \frac{A(e_1, e_1) A(e_2, e_2) - A^2(e_1, e_2)}{g(e_1, e_2) g(e_2, e_2) - g^2(e_1, e_2)} \\
&= \left(\frac{1}{r^2} - \frac{2}{3} R_{kimj} \frac{x^k x^m}{r^2} \left(e_1(x^i) e_1(x^j) + e_2(x^i) e_2(x^j) \right) + O(r) \right) \\
(3.38) \qquad &\times \left(1 - \frac{1}{3} R_{ikmj} x^k x^m \left(e_1(x^i) e_1(x^j) + e_2(x^i) e_2(x^j) \right) + O(r^3) \right) \\
&= \frac{1}{r^2} - \frac{1}{3} R_{kimj} \frac{x^k x^m}{r^2} \left(e_1(x^i) e_1(x^j) + e_2(x^i) e_2(x^j) \right) + O(r) \\
&= \frac{1}{r^2} - \frac{1}{3} R_{km} \frac{x^k x^m}{r^2} + O(r),
\end{aligned}
$$

where we have used the fact that $\sum_i \left(e_a(x^i) \right)^2 = 1$ and $e_a(\sum_i (x^i)^2) = 0$ on S_r for $a = 1, 2$, and the fact that

$$
e_1(x^i) e_1(x^j) + e_2(x^i) e_2(x^j) = \delta_{ij} - \frac{x^i x^j}{r^2}.
$$

Hence by the Gauss equation, for $x \in S_r$,

$$
\begin{aligned}
K(x) &= \lambda_1 \lambda_2 + \frac{1}{2} h^{ik} h^{jl} R_{ijkl}(x) \\
(3.39) \qquad &= \frac{1}{r^2} - \frac{1}{3} R_{km} \frac{x^k x^m}{r^2} + \frac{1}{2} h^{ik} h^{jl} R_{ijkl}(p) + O(r) \\
&= \frac{1}{r^2} + \frac{1}{2} R(p) - \frac{4}{3} R_{ij}(p) \frac{x^i x^j}{r^2} + O(r),
\end{aligned}
$$

where $h^{ij} = g^{ij} - n^i n^j$. On the other hand, for $x \in S_r$,

(3.40)
$$
\begin{aligned}
H(x) &= h^{ij} A_{ij} \\
&= h^{ij} \left(\frac{\delta_{ij}}{r} - \frac{2}{3} R_{kimj} \frac{x^k x^m}{r} \right) + O(r^2) \\
&= \left(\delta_{ij} - \frac{1}{3} R_{ikmj} x^k x^m - \frac{x^i x^j}{r^2} \right) \left(\frac{\delta_{ij}}{r} - \frac{2}{3} R_{kimj} \frac{x^k x^m}{r} \right) + O(r^2) \\
&= \frac{2}{r} - \frac{1}{3} R_{ij} \frac{x^i x^j}{r} + O(r^2),
\end{aligned}
$$

where we have used the fact that $h^{ij} x^i x^j = 0$.

It remains to prove the last assertion. Let Z_r be the embedding as in the proof of Lemma 3.3. One may conclude that by an isometry of $\mathbb{R}^3$, we have $||Z_r - \mathrm{Id}||_{C^2(\mathbb{S}^2)} = O(r^2)$, where Id is the identity map of $\mathbb{S}^2$. Let H_r and K_r be the mean curvature and the Gauss curvature of $Z_r(\mathbb{S}^2)$. Let $\{e_1, e_2\}$ be an orthonormal frames on $\mathbb{S}^2$ with respect to the standard metric, then the metric tensor h and the second fundamental form A of the surface $Z_r(\mathbb{S}^2)$ satisfy:

(3.41) $$ h(e_a, e_b) = \delta_{ab} + \alpha_{ab}, \quad A(e_a, e_b) = \delta_{ab} + \beta_{ab}, $$

where $\alpha_{ab} = O(r^2)$ and $\beta_{ab} = O(r^2)$. Hence, we have

$$
\begin{aligned}
K_r &= 1 - \alpha_{11} - \alpha_{22} + \beta_{11} + \beta_{22} + O(r^4), \\
H_r &= 2 - \alpha_{11} - \alpha_{22} + \beta_{11} + \beta_{22} + O(r^4).
\end{aligned}
$$

After rescale to an embedding of $(S_r, g|_{S_r})$ in $\mathbb{R}^3$, we conclude that

$$
K = \frac{1}{r^2} (1 - \alpha_{11} - \alpha_{22} + \beta_{11} + \beta_{22}) + O(r^2)
$$

and

$$
H_0 = \frac{1}{r} (2 - \alpha_{11} - \alpha_{22} + \beta_{11} + \beta_{22}) + O(r^3).
$$

From these and (3.33), (3.35) follows. $\qquad\square$

We are ready to prove the following (Theorem 1.3):

Theorem 3.1. *Let (N, g) be a Riemannian manifold of dimension 3, p be a fixed interior point on N and S_r be the geodesic sphere of radius r center at p. For r small enough, we have*
(3.42)
$$m_{\text{BY}}(S_r) = \frac{r^3}{12} R(p) + \frac{r^5}{1440} \left[24|\text{Ric}|^2(p) - 13R^2(p) + 12\Delta R(p) \right] + O(r^6);$$

here Δ is Laplacian operator of (M, g).

Proof. For r small, let Z be the isometric embedding of $(S_r, g|_{S_r})$ in $\mathbb{R}^3$ as in Lemma 3.3 and let H_0 be the mean curvature of $Z(S_r)$ in $\mathbb{R}^3$. Let

$$k_0 = \frac{R}{2} - \frac{4}{3} R_{ij} \frac{x^i x^j}{r^2}, \quad h_1 = rk_0, \quad n_3 = \frac{r^3}{6} \left(\frac{R}{2} - 2R_{ij} \frac{x^i x^j}{r^2} \right).$$

By Lemmas 3.4 and 3.3, we have

$$K = \frac{1}{r^2} + k_0 + O(r), \quad H_0 = \frac{2}{r} + h_1 + O(r^2), \quad Z \cdot n = r + n_3 + O(r^4).$$

As in the proof of Theorem 2.1 in Section 2, by one of the Minkowski integral formulae [16, Lemma 6.2.9] and Lemma 3.4, we have

(3.43)
$$\int_{S_r} H_0 \, d\Sigma_r = 2 \int_{S_r} K Z \cdot n \, d\Sigma_r$$
$$= \frac{1}{r^2} \int_{S_r} Z \cdot n \, d\Sigma_r + 2 \int_{S_r} \left(K - \frac{1}{r^2} \right) (r + n_3) d\Sigma_r + O(r^6)$$
$$= 6r^{-2} V_0(r) + 2r \int_{S_r} \left(K - \frac{1}{r^2} \right) d\Sigma_r + 2 \int_{S_r} k_0 n_3 \, d\Sigma_r + O(r^6)$$
$$= 6r^{-2} V_0(r) + 8\pi r - \frac{2\mathcal{A}(r)}{r} + 2 \int_{S_r} k_0 n_3 \, d\Sigma_r + O(r^6),$$

where $V_0(r)$ is the volume inside $Z(S_r)$ in $\mathbb{R}^3$.

By another Minkowski integral formula, we obtain

(3.44)
$$
\begin{aligned}
2\mathcal{A}(r) &= \int_{S_r} H_0 Z \cdot n \, d\Sigma_r \\
&= \int_{S_r} \frac{2}{r} Z \cdot n \, d\Sigma_r + \int_{S_r} \left(H_0 - \frac{2}{r} \right)(r + n_3) d\Sigma_r + O(r^7) \\
&= 6r^{-1} V_0(r) + r \int_{S_r} H_0 \, d\Sigma_r - 2\mathcal{A}(r) + \int_{S_r} h_1 n_3 \, d\Sigma_r + O(r^7) \\
&= 6r^{-2} V_0(r) + r \int_{S_r} H_0 \, d\Sigma_r - 2\mathcal{A}(r) + r \int_{S_r} k_0 n_3 \, d\Sigma_r + O(r^7).
\end{aligned}
$$

Hence,

(3.45) $$\int_{S_r} H_0 \, d\Sigma_r = -6r^{-2} V_0(r) + 4r^{-1}\mathcal{A}(r) - \int_{S_r} k_0 n_3 \, d\Sigma_r + O(r^6).$$

By (3.43) and (3.45), we have

(3.46)
$$
\begin{aligned}
\int_{S_r} H_0 \, d\Sigma_r &= 4\pi r + r^{-1}\mathcal{A}(r) + \int_{S_r} k_0 n_3 \, d\Sigma_r - \frac{1}{2} \int_{S_r} k_0 n_3 \, d\Sigma_r + O(r^6) \\
&= 8\pi r + \frac{A_4 + A_6}{r} + \frac{1}{2} \int_{S_r} k_0 n_3 \, d\Sigma_r + O(r^6)
\end{aligned}
$$

where we have used Lemma 3.2. Combining this with Lemma 3.1, we have

(3.47) $$\int_{S_r} (H_0 - H) d\Sigma_r = -\frac{3A_4 + 5A_6}{r} + \frac{1}{2} \int_{S_r} k_0 n_3 \, d\Sigma_r + O(r^6).$$

Now by (3.7) and (3.11)

(3.48)
$$
\begin{aligned}
\int_{S_r} k_0 n_3 \, d\Sigma_r &= \frac{r^3}{6} \int_{S_r} \left(\frac{R}{2} - \frac{4}{3} R_{ij} \frac{x^i x^j}{r^2} \right) \left(\frac{R}{2} - 2R_{ij} \frac{x^i x^j}{r^2} \right) d\Sigma_r \\
&= \frac{r^3}{6} \int_{S_r} \left(\frac{R}{2} - \frac{4}{3} R_{ij} \frac{x^i x^j}{r^2} \right) \left(\frac{R}{2} - 2R_{ij} \frac{x^i x^j}{r^2} \right) d\Sigma_r^0 + O(r^6) \\
&= \frac{\pi r^5}{270} \left(64|\mathrm{Ric}|^2 - 23R^2 \right).
\end{aligned}
$$

The theorem follows from (3.47),(3.48) and Lemma 3.2. $\square$

As a corollary, we have

Corollary 3.2. *With the assumptions and notation as in Theorem* 3.1, *suppose $R \geq 0$ in a neighborhood of p, then*

$$(3.49) \qquad \lim_{r \to 0} \frac{m_{\mathrm{BY}}(S_r)}{r^5} \geq 0.$$

Equality holds if and only if (N, g) is flat at p and R vanishes up to second order at p.

Proof. By the result of [23] on the positivity of the Brown–York mass, we know that (3.49) is true. However, in this special case, one can deduce this from the theorem. In fact, if $R(p) > 0$, then by (3.42), we have

$$\lim_{r \to 0} \frac{m_{\mathrm{BY}}(S_r)}{r^5} = \infty > 0.$$

In case $R(p) = 0$, then $R(p)$ is a minimum of R because $R \geq 0$. It is easy to see that (3.49) is still true.

It is obvious that if (N, g) is flat at p and R vanishes up to second order at p, then equality holds in (3.49). Conversely, if the equality holds in (3.49), then we must have $R(p) = 0$, $\nabla R(p) = 0$, $\Delta R(p) = 0$ and $|\mathrm{Ric}|(p) = 0$. Since R has a minimum at p, the Hessian of R has non-negative eigenvalues. So the Hessian of R must be zero at p because $\Delta R(p) = 0$. Moreover, since N has dimension 3, $|\mathrm{Ric}|(p) = 0$ implies that (N, g) is flat at p. $\square$

Remark 3.1. From the proof, it is easy to see that (3.49) is true if $R(p) = 0$ and $\Delta R(p) \geq 0$ and the equality holds only if g is flat at p.

One should compare the corollary to the following fact: If M is an AF manifold with non-negative scalar curvature, suppose the Brown–York mass of the coordinate spheres converge to zero, then M must be the Euclidean space. This follows from Theorem 2.1 and the positive mass theorem in Schoen–Yau [21] and Witten [24].

For the expansion of the Hawking mass, we have:

Theorem 3.2. *With the same notations and assumptions in Theorem* 3.1, *we have*

$$(3.50) \qquad m_{\mathrm{H}}(S_r) = \frac{r^3}{12} R(p) + \frac{r^5}{720} \left(6 \Delta R(p) - 5 R^2(p) \right) + O(r^6).$$

Proof. By Lemma 3.4, we have

$$H = \frac{2}{r} + H_1 + O(r^2),$$

where $H_1 = -\frac{1}{3r}R_{ij}x^i x^j$. Hence

$$H^2 = -4r^{-2} + 4r^{-1}H + H_1^2 + O\left(r^3\right).$$

Then

$$\int_{S_r} H^2 \, d\Sigma_r = -\frac{4\mathcal{A}(r)}{r^2} + \frac{4}{r}\int_{S_r} H \, d\Sigma_r + \int_{S_r} H_1^2 \, d\Sigma_r^0 + O(r^5)$$

$$= -\frac{4(4\pi r^2 + A_4 + A_6)}{r^2} + \frac{4}{r}\cdot\left(8\pi r + \frac{4A_4}{r} + \frac{6A_6}{r}\right)$$

$$+ \int_{S_r} H_1^2 \, d\Sigma_r^0 + O(r^5)$$

$$= 16\pi + \frac{12A_4}{r^2} + \frac{20A_6}{r^2} + \int_{S_r} H_1^2 \, d\Sigma_r^0 + O(r^5).$$

Hence,

$$16\pi - \int_{S_r} H^2 \, d\Sigma_r = -\frac{12A_4}{r^2} - \frac{20A_6}{r^2} - \int_{S_r} H_1^2 \, d\Sigma_r^0 + O(r^5).$$

On the other hand,

$$\frac{\mathcal{A}^{1/2}(r)}{(16\pi)^{3/2}} = \frac{2\pi^{1/2}r}{(16\pi)^{3/2}}\left(1 + \frac{A_4}{8\pi r^2} + O(r^4)\right)$$

$$= \frac{r}{32\pi}\left(1 + \frac{A_4}{8\pi r^2} + O(r^4)\right).$$

So

$$m_{\mathrm{H}}(S_r) = -\frac{3A_4}{8\pi r} - \frac{5A_6}{8\pi r} - \frac{r}{32\pi}\int_{S_r} H_1^2 \, d\Sigma_r^0 - \frac{3A_4^2}{64\pi^2 r^3} + O(r^6).$$

By (3.11) and Lemma 3.2, the result follows. □

Hence, the expansions of the Brown–York mass and the Hawking mass are equal up to order r^3. However, they differ on the term of order r^5.

As in the case of large-sphere limit, we can compare $V(r)$ and $V_0(r)$, where $V(r)$ is the volume of the geodesic ball of radius r at p and $V_0(r)$ is the volume of the region bounded by S_r when embedded in $\mathbb{R}^3$.

Theorem 3.3. *With the above notations, we have*

(3.51)
$$V_0(r) - V(r) = -\frac{\pi}{15}Rr^5 + \frac{\pi r^7}{5670}(173R^2 - 454|\text{Ric}|^2 - 27\Delta R(p)) + O\left(r^8\right).$$

Proof. By (3.43) and (3.45)

$$\int_{S_r} H_0\, d\Sigma_r = 6r^{-2}V_0(r) + 8\pi r - \frac{2\mathcal{A}(r)}{r} + 2\int_{S_r} k_0 n_3\, d\Sigma_r + O(r^6),$$

and

$$\int_{S_r} H_0\, d\Sigma_r = -6r^{-2}V_0(r) + 4r^{-1}\mathcal{A}(r) - \int_{S_r} k_0 n_3\, d\Sigma_r,$$

where k_0 is as in (3.43).

We have

$$V_0(r) = \frac{r}{2}\mathcal{A}(r) - \frac{r^2}{4}\int_{S_r} k_0 n_3\, d\Sigma_r^0 - \frac{2}{3}\pi r^3 + O\left(r^8\right)$$

$$= \frac{4}{3}\pi r^3 + \frac{r}{2}A_4 + \frac{r}{2}A_6 - \frac{r^2}{4}\int_{S_r} k_0 n_3\, d\Sigma_r^0 + O\left(r^8\right).$$

On the other hand,

$$V(r) = \int_0^r \mathcal{A}(t)dt$$

$$= \frac{4}{3}\pi r^3 + \int_0^r A_4\, dt + \int_0^r A_6\, dt + O\left(r^8\right)$$

$$= \frac{4}{3}\pi r^3 + \frac{r}{5}A_4 + \frac{r}{7}A_6 + O\left(r^8\right).$$

Hence,

$$V_0(r) - V(r) = \frac{3}{10}rA_4 + \frac{5}{14}rA_6 - \frac{r^2}{4}\int_{S_r} k_0 n_3\, d\Sigma_r^0 + O\left(r^8\right).$$

By (3.48) and Lemma 3.2, the result follows. □

By Theorem 3.3, we see that if scalar curvature is positive at p, then $V_0(r) < V(r)$, for sufficiently small r. More precisely,

Corollary 3.3. *With the assumptions and notations as in Theorem 3.3, suppose $R \geq 0$ in a neighborhood of p, then*

$$\lim_{r \to 0} \frac{V_0(r) - V(r)}{r^7} \leq 0.$$

Equality holds if and only if (N, g) is flat at p and R vanishes up to second order at p.

Proof. Similar to the argument of Corollary 3.2, one can derive the result from Theorem 3.3. □

Acknowledgments

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