

# On characterization for a class of pseudo-convex domains with positive constant pseudo-scalar curvature on their boundaries

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For any compact strictly pseudoconvex pseudo-Hermitian CR manifold  $(M, \theta_0)$  in the sense of Webster, the solution of CR-Yamabe problem concludes that there is a contact form  $\theta$  which conformally equivalent to  $\theta_0$  so that the pseudo scalar curvature for  $(M, \theta)$  is a constant. The current article gives a natural and easily verified sufficient condition, and proves that if  $(M, \theta)$  satisfies the condition and has a positive constant pseudo scalar curvature on  $M$ , then  $(M, \theta)$  must be CR equivalent to the unit sphere.

## 1. Introduction and main results

Let  $(M, \theta)$  be a strictly pseudo-convex pseudo-Hermitian compact hypersurface in  $\mathbb{C}^{n+1}$  in the sense of Webster [36] with a pseudo-Hermitian real one-form  $\theta$  on  $M$ . Let  $R_\theta$  be the Webster pseudo-scalar curvature for  $M$  with respect to  $\theta$ . By the solution of the CR Yamabe problem given by Jerison and Lee [18], Gamara and Yacoub [10] and Gamara [9] (for  $n = 1$ ), there is a pseudo-Hermitian real one-form  $\theta$  so that  $(M, \theta)$  has constant Webster pseudo-scalar curvature  $R_\theta$ . Let  $\rho$  be a defining function for  $M$ . Then  $\theta = \frac{1}{2i}(\partial\rho - \bar{\partial}\rho)$  is a pseudo-Hermitian one-form for  $M$ , and any Hermitian one-form can be constructed in this way by using defining function of  $M$ . When  $M = S^{2n+1}$ , the unit sphere in  $\mathbb{C}^{n+1}$ , if  $\rho(z) = |z|^2 - 1$ , then  $R_\theta = n(n+1)$  on  $M$ . The main purpose of the paper is to give some characterizations on  $\rho$  so that the pseudo-scalar curvature  $R_\theta$  is a positive constant on  $M$  if and only if  $M$  is CR-equivalent to the sphere  $S^{2n+1}$ .

Let  $D$  be a smoothly bounded pseudo-convex domain in  $\mathbb{C}^n$ . Let  $u$  be a strictly plurisubharmonic exhaustion function for  $D$  ( $u = +\infty$  on  $\partial D$ ). Let  $\rho(z) = -e^{-u(z)}$ . Then the Fefferman's functional  $J(\rho)$  of  $\rho$  is defined as:

$$(1.1) \quad J(\rho) = -\det \begin{bmatrix} \rho & \bar{\partial}\rho \\ (\bar{\partial}\rho)^* & H(\rho) \end{bmatrix},$$

where  $H(\rho)$  is the complex Hessian matrix of  $\rho$ ,  $\bar{\partial}\rho = (\partial_{\bar{1}}\rho, \dots, \partial_{\bar{n}}\rho)$  is a  $1 \times n$  matrix and  $(\bar{\partial}\rho)^*$  is its adjoint matrix.

Let  $f(z)$  be a positive function on  $D$ . It was known (see, for example, [7, 25]) that

$$(1.2) \quad J(\rho) = f(z) > 0 \text{ in } D; \quad \rho = 0 \text{ on } \partial D$$

if and only if

$$(1.3) \quad \det H(u)(z) = f(z)e^{(n+1)u} \text{ in } D; \quad u = +\infty \text{ on } \partial D.$$

When  $f(z) \in C^\infty(\bar{D})$  is positive for all  $z \in \bar{D}$ , the existence and uniqueness of a strictly plurisubharmonic solution of (1.3) were given by Cheng and Yau in [7]. In particular, when  $f(z) \equiv 1$ , they proved that the plurisubharmonic solution  $u$  of (1.3) defines a complete Kähler–Einstein metric on  $D$ :  $\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} dz_i \otimes d\bar{z}_j$ .

When  $D$  is strictly pseudo-convex, uniqueness and a formal approximation solution  $\rho$  of (1.2) with  $u = -\log(-\rho)$  being plurisubharmonic in  $D$  were given by Fefferman [8] earlier; the existence of such a solution was proved by Cheng and Yau [7] with  $\rho \in C^{n+3/2}(\bar{D})$ . Lee and Melrose [22] gave an asymptotic expansion solution for  $\rho$ . In particular, they proved that  $\rho \in C^{n+2-\epsilon}(\bar{D})$  for any  $\epsilon > 0$ . In general,  $\rho$  fails to be in  $C^{n+2}(\bar{D})$ . Further informations about complex Monge–Ampère equations and pseudo-Hermitian manifolds can be found (for examples) in: [1–6, 11–17, 23, 26–35] and [38].

When  $M = \{z \in \mathbb{C}^{n+1} : \rho(z) = 0\}$  is hypersurface in  $\mathbb{C}^{n+1}$ , using the notation of  $J(\rho)$ , the following formula for the Webster pseudo-Ricci curvature of  $(M, \theta)$  was given by Li and Luk in [30]:

$$(1.4) \quad \text{Ric}_z(w, \bar{v}) = -L_{\log J(\rho)(z)}(w, \bar{v}) + (n+1) \frac{\det H(\rho)(z)}{J(\rho)(z)} L_{\rho(z)}(w, \bar{v})$$

for  $v, w \in H_z(M)$ , holomorphic tangent space of  $M$  at  $z$ , where

$$(1.5) \quad L_{u(z)}(w, \bar{v}) = \sum_{j,k=1}^{n+1} \frac{\partial^2 u(z)}{\partial z_j \partial \bar{z}_k} w_j \bar{v}_k.$$

Assume that  $\phi : D \rightarrow B_{n+1}$  is a biholomorphic mapping, and if  $\rho(z) = |\phi(z)|^2 - 1$ , then  $\det H(\rho) = J(\rho) = |\det \phi'(z)|^2$  on  $D$  and  $\log J(\rho)$  is pluriharmonic in  $D$ . By (1.4), we have  $R_\theta = n(n+1)$  on  $\partial D$ .

The main purpose of the current paper is to prove the converse is also true, we state it as the following theorem.

**Theorem 1.1.** *Let  $\rho \in C^3(\overline{D})$  be a defining function for  $D \subset \mathbb{C}^{n+1}$  so that  $u(z) = -\log(-\rho(z))$  is strictly plurisubharmonic in  $D$ . Let  $M = \partial D$  and  $\theta = \frac{1}{2i}(\partial\rho - \bar{\partial}\rho)$ . Assume  $\log J(\rho)$  is harmonic in the Kähler metric  $\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} dz_i \otimes \bar{d}\bar{z}_j$ , we have the following two statements hold:*

- (a) *If  $R_\theta \equiv c > 0$ , constant on  $\partial D$ , then  $D$  is biholomorphic to the unit ball  $B_{n+1}$  in  $\mathbb{C}^{n+1}$ .*
- (b) *Webster pseudo-scalar curvature*

$$(1.6) \quad R_\theta = n(n+1) \frac{\det H(\rho)}{J(\rho)} \quad \text{on } \partial D.$$

Notice that if  $u(z) = -\log(-\rho(z))$  is the potential function for the Kähler–Einstein metric for  $D$ , then  $J(\rho) = 1$  on  $D$ . By Theorem 1.1 and Theorem 3.1 in [24], we have the following corollary.

**Corollary 1.2.** *Let  $D$  be a smoothly bounded strictly pseudo-convex domain in  $\mathbb{C}^{n+1}$ . Assume that  $u(z) = -\log(-\rho(z))$  is the potential function for the Kähler–Einstein metric for  $D$  (defined by (1.3) with  $f \equiv 1$ ) and  $\theta = \frac{1}{2i}(\partial\rho - \bar{\partial}\rho)$  on  $M = \partial D$ . If  $R_\theta \equiv c > 0$ , constant on  $\partial D$ , then there is a biholomorphic map  $\phi : D \rightarrow B_{n+1}$  so that  $\det \phi'(z)$  is a constant on  $D$ .*

The paper is organized as follows: In Section 2, we provide several main lemmas. In Section 3, we will prove part (b) of Theorem 1.1. Finally, the proofs of Part(a) of Theorem 1.1 and Corollary 1.2 are given in Section 4.

## 2. Main lemmas

Let  $D = \{z \in \mathbb{C}^n : \rho(z) < 0\}$  with  $C^2$  defining function  $\rho$ . Let  $u(z) = -\log(-\rho(z))$  be plurisubharmonic in  $D$ . Then the relation between  $J(\rho)$  and  $\det H(u)$  is given by the following lemma.

**Lemma 2.1.** *Let  $u \in C^2(D)$  and let  $\rho(z) = -e^{-u}$ . Then*

$$(2.1) \quad \det H(u) = J(\rho)e^{(n+1)u}.$$

*Proof.* By viewing  $\bar{\partial}u$  as a 1 by  $n$  matrix and  $(\bar{\partial}u)^*$  is its adjoint matrix, one has that

$$(2.2) \quad \bar{\partial}\rho = -\rho\bar{\partial}u, \quad H(\rho)(z) = -\rho(z)H(u) + \rho(z)(\bar{\partial}u)^*\bar{\partial}u.$$

Thus

$$\begin{aligned} J(\rho) &= -\det \begin{bmatrix} \rho & -\rho\bar{\partial}u \\ -\rho(\bar{\partial}u)^* & -\rho H(u) + \rho(\bar{\partial}u)^*\bar{\partial}u \end{bmatrix} \\ &= -(\rho)^{n+1} \det \begin{bmatrix} 1 & -\bar{\partial}u \\ -(\bar{\partial}u)^* & -H(u) + (\bar{\partial}u)^*\bar{\partial}u \end{bmatrix} \\ &= -(\rho)^{n+1} \det \begin{bmatrix} 1 & -\bar{\partial}u \\ 0 & -H(u) + (\bar{\partial}u)^*\bar{\partial}u - (\bar{\partial}u)^*\bar{\partial}u \end{bmatrix} \\ &= -(\rho)^{n+1} \det(-H(u)) \\ &= e^{-(n+1)u} \det H(u). \end{aligned}$$

Therefore, (2.1) holds and the lemma is proved.  $\square$

Assume that  $u$  is strictly plurisubharmonic in  $D$ , we let  $[u^{i\bar{j}}]^t = H(u)^{-1}$ , and use the following notations:

$$(2.3) \quad u_{i\bar{j}} = \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}, \quad u_i = \frac{\partial u}{\partial z_i}, \quad u_{\bar{j}} = \frac{\partial u}{\partial \bar{z}_j}$$

and

$$(2.4) \quad |\partial u|_u^2 := u^{i\bar{j}} \partial_i u \partial_{\bar{j}} u, \quad T(z) := |\partial u|_u^2 + e^{-u}.$$

Then if  $\rho(z) = -e^{-u(z)}$ , then by Lemma 2.2 in [24],

$$(2.5) \quad \det H(\rho) = e^{-nu} \det H(u) (1 - |\partial u|_u^2) = J(\rho) e^u (1 - |\partial u|_u^2).$$

Since  $u$  is strictly plurisubharmonic in  $D$ , by (2.1), we have  $J(\rho) > 0$  on  $D$ . Thus

$$(2.6) \quad \frac{\det H(\rho)}{J(\rho)} - 1 = e^u (1 - T(z)).$$

Then the following lemma was included in the proof of part(b) of Theorem 2.4 in [24], given in pages 468–470.

**Lemma 2.2.** *Let  $D$  be a bounded pseudo-convex domain in  $\mathbb{C}^n$ . Let  $u \in C^4(D)$  be strictly plurisubharmonic in  $D$ , and let  $\rho(z) = -e^{-u}$ . If  $-\log J(\rho)$  is subharmonic in the metric  $u_{i\bar{j}}dz_i \otimes d\bar{z}_j$ , then*

$$(2.7) \quad \sum_{i,j=1}^n u^{i\bar{j}} \partial_{i\bar{j}} \left( -\frac{\det H(\rho)}{J(\rho)} \right) (z) \geq 0, \quad z \in D.$$

The next result is the main theorem of this section.

**Theorem 2.3.** *Let  $\rho \in C^3(\bar{D})$  be a defining function for  $D \subset \mathbb{C}^n$  so that  $u(z) = -\log(-\rho(z))$  is strictly plurisubharmonic in  $D$ . If  $-\log J(\rho)$  is subharmonic in Kähler metric  $u_{i\bar{j}}dz_i \otimes d\bar{z}_j$ , and if there is a positive constant  $c$  so that*

$$(2.8) \quad \frac{\det H(\rho)}{J(\rho)} \equiv c \quad \text{on } \partial D,$$

then

$$(2.9) \quad \frac{\det H(\rho)}{J(\rho)} \equiv c \quad \text{on } D.$$

*Proof.* Since  $\det H(\rho)/J(\rho) \equiv c > 0$  on  $\partial D$ , replacing  $\rho$  by  $c\rho$  if it is necessary, without loss of generality, we may assume that  $c = 1$ . We can write

$$(2.10) \quad \frac{\det H(\rho)}{J(\rho)} := 1 + A(z)\rho(z), \quad z \in D.$$

Since  $-\log J(\rho)$  is subharmonic in the metric  $u_{i\bar{j}}dz_i \otimes d\bar{z}_j$ , by Lemma 2.2, we have that  $\det H(\rho)/J(\rho)$  attains its minimum over  $\bar{D}$  at some point in  $\partial D$ . Thus (2.10) holds with  $A(z) \leq 0$  in  $D$ . By (2.6), one has

$$(2.11) \quad 1 - T(z) = \frac{\det H(\rho)}{J(\rho)} e^{-u} - e^{-u} = (1 + A(z)\rho)(-\rho) + \rho.$$

Thus, by (2.10),

$$(2.12) \quad T(z) = 1 - \rho(z) + (1 + A\rho)\rho = 1 + A(z)\rho(z)^2.$$

We claim that  $T(z) \equiv 1$  on  $D$  (i.e.,  $\det H(\rho)/J(\rho) \equiv 1$  on  $D$ ).

By Lemma 2.2, if there is  $z_0 \in D$  so that  $\det H(\rho)(z_0)/J(\rho)(z_0) = 1$ , then  $\det H(\rho)/J(\rho) \equiv 1$  on  $D$  (i.e.,  $A \equiv 0$ ). Otherwise, we have  $A(z) < 0$  on  $D$ , we will prove there is a contradiction.

Let  $C = \max\{-A(z) : z \in \overline{D}\}$ . Then  $C \in (0, \infty)$ , and by (2.12), we have

$$(2.13) \quad 0 < (1 - T) \leq C\rho(z)^2 = Ce^{-2u}, \quad \text{on } D.$$

By (2.12) again, one has  $T(z) := |\partial u|_u^2 + e^{-u} < 1$  on  $D$ . Since  $e^{-u} = 0$  on  $\partial D$  and  $e^{-u} > 0$  in  $D$ , one can easily see that

$$(2.14) \quad b := \max\{e^{-u(z)} : z \in D\} < 1, \quad z \in D.$$

By (2.13), we have

$$(2.15) \quad (1 - T)^{-3/4} \geq C^{-3/4}e^{3u/2}, \quad \frac{1}{1 - e^{u(z)}} \leq \frac{1}{1 - b}.$$

Note: This is a key place we use the condition  $\det H(\rho)/J(\rho) = 1$  on  $\partial D$ .

For any fixed positive integer  $m$  satisfying

$$(2.16) \quad m \geq C^{3/4} \frac{(n+1)^2}{(1-b)^2},$$

it is easy to see that

$$(2.17) \quad m(1 - T)^{-3/4} - \frac{(1+n)}{(1 - e^{-u})} - \frac{(1+n)^2 |\partial u|^2}{4(1 - e^{-u})^2} e^u \geq 0, \quad z \in D.$$

Let

$$(2.18) \quad \mathcal{L}_n := u^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} - n \operatorname{Re}(u^{i\bar{j}} u_i \partial_{\bar{j}})$$

and

$$(2.19) \quad \mathcal{L}_{m,n} = \mathcal{L}_n + m(1 - T)^{-3/4} \operatorname{Re}(\partial_i T u^{i\bar{j}} \partial_{\bar{j}}).$$

For any  $z_0 \in D$ , we will show  $\mathcal{L}_{m,n}T(z_0) \geq 0$ . By a holomorphic change of coordinates, without loss of generality, we may assume that

$$(2.20) \quad u_{i\bar{j}k}(z_0) = 0, \quad 1 \leq i, j, k \leq n.$$

It was proved by the author in [24] that if  $T(z) \leq 1$  on  $D$  and if  $u$  is the strictly plurisubharmonic solution for the Monge–Ampère equation:

$$(2.21) \quad \det H(u) = J e^{(n+1)u} \text{ in } D; \quad u = \infty \text{ on } \partial D$$

with  $-\log J$  is subharmonic in the metric  $u_{i\bar{j}}dz_i \otimes d\bar{z}_j$ , then

$$(2.22) \quad \begin{aligned} u^{i\bar{j}}\partial_{i\bar{j}}T(z_0) &= u^{i\bar{j}}(u^{k\bar{\ell}}u_k u_{\bar{\ell}} + e^{-u}) \\ &\geq n(1-T) - |\partial u|_u^2(1-e^{-u}) + u^{i\bar{j}}u^{k\bar{\ell}}u_{ik}u_{\bar{j}\bar{\ell}}. \end{aligned}$$

Note: This is (2.24) in [24] while (2.21) in [24] becomes inequality with our assumption  $u^{i\bar{j}}\partial_{i\bar{j}}\log J(\rho)(z_0) \leq 0$ .

By (2.20), one has that  $\partial_k u^{i\bar{j}}(z_0) = 0$ . Thus, for  $1 \leq j \leq n$ , one has

$$(2.23) \quad \partial_{\bar{j}}T(z_0) = u^{k\bar{\ell}}u_k u_{\bar{j}\bar{\ell}} + u_{\bar{j}}(1-e^{-u}), \quad \partial_i T(z_0) = u^{k\bar{\ell}}u_{ik}u_{\bar{j}} + u_i(1-e^{-u}).$$

Thus,

$$(2.24) \quad u^{i\bar{j}}u_i\partial_{\bar{j}}T(z_0) = u^{i\bar{j}}u_i u^{k\bar{\ell}}u_k u_{\bar{j}\bar{\ell}} + |\partial u|_u^2(1-e^{-u})$$

and

$$(2.25) \quad u^{i\bar{j}}\partial_{i\bar{j}}T(z_0) \geq n(1-T) - u^{i\bar{j}}u_i\partial_{\bar{j}}T + u^{i\bar{j}}u_i u^{k\bar{\ell}}u_k u_{\bar{j}\bar{\ell}} + u^{i\bar{j}}u^{k\bar{\ell}}u_{ik}u_{\bar{j}\bar{\ell}}.$$

Thus,

$$(2.26) \quad \mathcal{L}_n T \geq n(1-T) - \operatorname{Re}(1+n)u^{i\bar{j}}u_i\partial_{\bar{j}}T + \operatorname{Re}u^{i\bar{j}}u_i u^{k\bar{\ell}}u_k u_{\bar{j}\bar{\ell}} + u^{i\bar{j}}u^{k\bar{\ell}}u_{ik}u_{\bar{j}\bar{\ell}}.$$

By (2.23), one has

$$(2.27) \quad u_i(z_0) = \frac{1}{1-e^{-u}}[\partial_i T(z_0) - u^{k\bar{\ell}}u_{\bar{\ell}}u_{ik}(z_0)].$$

Thus,

$$(2.28) \quad \begin{aligned} &-(1+n)\operatorname{Re}u^{i\bar{j}}u_i\partial_{\bar{j}}T \\ &= \frac{1+n}{1-e^{-u}}[-u^{i\bar{j}}\partial_i T\partial_{\bar{j}}T + u^{i\bar{j}}u^{k\bar{\ell}}u_{\bar{\ell}}u_{ik}\partial_{\bar{j}}T] \\ &\geq -\left[\frac{(1+n)}{(1-e^{-u})} + \frac{(1+n)^2|\partial u|_u^2}{4(1-e^{-u})^2}e^u\right]u^{i\bar{j}}\partial_i T\partial_{\bar{j}}T - e^{-u}u^{i\bar{j}}u^{k\bar{\ell}}u_{ik}u_{\bar{j}\bar{\ell}} \end{aligned}$$

and

$$(2.29) \quad \operatorname{Re}u^{i\bar{j}}u_i u^{k\bar{\ell}}u_k u_{\bar{j}\bar{\ell}}(z_0) \geq -|\partial u|_u^2 u^{i\bar{j}}u^{k\bar{\ell}}u_{ik}u_{\bar{j}\bar{\ell}}(z_0).$$

Combining (2.26), (2.28) and (2.29), one has

$$(2.30) \quad \begin{aligned} \mathcal{L}_n T(z_0) &\geq n(1-T) + (1 - |\partial u|^2 - e^{-u}) u^{i\bar{j}} u^{k\bar{\ell}} u_{ik} u_{\bar{j}\bar{\ell}} \\ &\quad - \left[ \frac{(1+n)}{(1-e^{-u})} + \frac{(1+n)^2 |\partial u|^2}{4(1-e^{-u})^2} e^u \right] u^{i\bar{j}} \partial_i T \partial_{\bar{j}} T. \end{aligned}$$

Therefore,

$$(2.31) \quad \begin{aligned} \mathcal{L}_{m,n} T(z_0) &\geq n(1-T) + (1-T) u^{i\bar{j}} u^{k\bar{\ell}} u_{ik} u_{\bar{j}\bar{\ell}} \\ &\quad + \left[ \frac{m}{(1-T)^{3/4}} - \frac{(1+n)}{(1-e^{-u})} - \frac{(1+n)^2 |\partial u|^2}{4(1-e^{-u})^2} e^u \right] u^{i\bar{j}} \partial_i T \partial_{\bar{j}} T \\ &\geq \left[ \frac{m}{(1-T)^{3/4}} - \frac{(1+n)}{(1-e^{-u})} - \frac{(1+n)^2 |\partial u|^2}{4(1-e^{-u})^2} e^u \right] u^{i\bar{j}} \partial_i T \partial_{\bar{j}} T \\ &\geq 0. \end{aligned}$$

Since  $\det H(u) = J(\rho) e^{(n+1)u}$ , we have  $\sum_{i=1}^n \partial_i [J(\rho) e^{(n+1)u} u^{i\bar{j}}] = 0$  for all  $1 \leq j \leq n$ . Thus,

$$(2.32) \quad \begin{aligned} &\operatorname{Re} \sum_{i,j=1}^n \partial_i [e^{-nu} e^{-4m(1-T)^{1/4}} J(\rho) e^{(n+1)u} u^{i\bar{j}} \partial_{\bar{j}} T] \\ &= e^{(n+1-n)u} e^{-4m(1-T)^{1/4}} [\mathcal{L}_n T(z_0) + m(1-T)^{-3/4} u^{i\bar{j}} \partial_i T \partial_{\bar{j}} T] \\ &= e^u e^{-4m(1-T)^{1/4}} \mathcal{L}_{m,n} T. \end{aligned}$$

Since

$$(2.33) \quad u^{i\bar{j}} = (-\rho) \left( \rho^{i\bar{j}} - \frac{\rho^i \rho^{\bar{j}}}{|\partial \rho|_\rho^2 - \rho} \right), \quad \rho^i = \rho^{i\bar{\ell}} \rho_{\bar{\ell}},$$

we have

$$(2.34) \quad e^u \sum_{i=1}^n \rho_i u^{i\bar{j}} \partial_{\bar{j}} T = -\frac{\rho \rho^{\bar{j}}}{|\partial \rho|_\rho^2 - \rho} \partial_{\bar{j}} T = 0 \quad \text{on } \partial D.$$



Let  $|\partial\rho|_0^2 = \sum_{j=1}^n |\partial_j\rho(z)|^2$ . Then,

$$\begin{aligned} 0 &= \operatorname{Re} \sum_{i,j=1}^n \int_{\partial\Omega} \frac{\rho_i}{|\partial\rho|_0} e^{-nu} e^{-4m(1-T)^{1/4}} J(\rho) e^{(n+1)u} u^{i\bar{j}} \partial_{\bar{j}} T d\sigma(z) \\ &= \operatorname{Re} \int_{\Omega} \sum_{i,j=1}^n \partial_i (e^{-nu} e^{-4m(1-T)^{1/4}} J(\rho) e^{(n+1)u} u^{i\bar{j}} \partial_{\bar{j}} T) dv \\ &= \int_{\Omega} e^u e^{-4m(1-T)^{1/4}} \mathcal{L}_{m,n} T(z) dv(z) \end{aligned}$$

(2.31) shows that the integrand in the last integral is non-negative on  $D$ . Thus, the above identity implies that

$$(2.35) \quad e^u e^{-4m(1-T)^{1/4}} \mathcal{L}_{m,n} T(z) = 0, \quad z \in D.$$

Therefore,

$$\mathcal{L}_{m,n} T(z) \equiv 0, \quad \text{in } D.$$

Maximum principle (for both  $T$  and  $-T$ ) implies that

$$\max\{T(z) : z \in \bar{D}\} = \max\{T(z) : z \in \partial D\} = 1$$

and

$$\min\{T(z) : z \in \bar{D}\} = \min\{T(z) : z \in \partial D\} = 1.$$

Therefore,

$$T(z) \equiv 1, \quad \text{and} \quad \frac{\det H(\rho)}{J(\rho)} \equiv 1 \quad \text{in } D.$$

This contradicts to  $A(z) < 0$  on  $D$  (i.e.,  $T(z) < 1$  on  $D$ ). Therefore, our claim  $T(z) \equiv 1$  is proved, and so

$$(2.36) \quad \frac{\det H(\rho)}{J(\rho)} \equiv 1 \quad \text{in } D.$$

Therefore, the proof of Theorem 2.3 is complete.  $\square$

The following result essentially is the corollary of the main theorem of Stoll [33], Burns [3] and Wong [37] in our notation:

**Corollary 2.4.** *Let  $\rho \in C^3(\overline{D})$  be a defining function for  $D \subset \mathbb{C}^n$  so that  $u(z) = -\log(-\rho(z))$  is strictly plurisubharmonic in  $D$ . Let  $M = \partial D$  and  $\theta = \frac{1}{2i}(\partial\rho - \bar{\partial}\rho)$ . If  $\det H(\rho)/J(\rho)$  is a positive constant on the boundary  $\partial D$  of  $D$ , then  $D$  is biholomorphic to the unit ball  $B_n$  in  $\mathbb{C}^n$ .*

*Proof.* Since  $u = -\log(-\rho)$  is plurisubharmonic in  $D$ , we have  $J(\rho) > 0$  on  $D$ . Notice that  $\det H(\rho)/J(\rho)$  is a positive constant on  $D$ , we have  $\rho$  is strictly plurisubharmonic in  $D$ . Let  $z_0 \in D$  be such that

$$(2.37) \quad m = -\min\{\rho(z) : z \in \overline{D}\} = -\rho(z_0).$$

Then  $\partial\rho(z_0) = 0$  and

$$(2.38) \quad J(\rho)(z_0) = m \det H(\rho)(z_0).$$

Let

$$(2.39) \quad \tau(z) = \rho(z) + m.$$

Then  $\tau : D \rightarrow [0, m)$  is smooth, onto and strictly plurisubharmonic in  $D$ . Moreover, since  $|\partial\tau|_\tau^2 = |\partial\rho|_\rho^2$  and  $\det H(\rho) = \det H(\tau)$ , we have

$$J[\tau] = -\det H(\tau)[\tau - |\partial\tau|_\tau^2] = -m \det H(\rho) + J(\rho).$$

Therefore,

$$(2.40) \quad \frac{\det H(\rho)}{J(\rho)} \equiv \text{constant on } D \iff J(\tau) \equiv 0 \text{ on } D.$$

Notice that

$$(2.41) \quad J(\tau) = -\tau^{n+1} \det H(\log \tau).$$

Thus

$$(2.42) \quad J(\tau) = 0 \iff \det H(\log \tau) = 0, \quad D \setminus \tau^{-1}(0).$$

Combining the above relations and the main theorem of Stoll [33], Burns in [3] and Wong [37], the proof of Corollary 2.4 is complete.  $\square$

### 3. Proof of part(b) of Theorem 1.1

First let us recall some notions and a formula for the Webster pseudo Ricci curvature and pseudo-scalar curvature proved in [30]. Let  $M$  be a real hypersurface in  $\mathbb{C}^{n+1}$  with a defining function  $\rho \in C^3(\mathbb{C}^{n+1})$ . Let  $D = \{z \in \mathbb{C}^{n+1} : \rho(z) < 0\}$  and let  $U$  be a neighborhood of  $M$ . Assume that  $u(z) = -\log(-\rho(z))$  is strictly plurisubharmonic in  $D \cap U$ . From now on, we always use  $\rho$  to denote a defining function for  $M$  and  $\theta = \frac{1}{2i}(\partial\rho - \bar{\partial}\rho)$  is the pseudo-Hermitian form for  $M$  associated to the defining function  $\rho$ . We define a second-order differential operator  $\mathcal{D}_{\alpha\bar{\beta}}$  associated to  $\rho$  for  $1 \leq \alpha, \beta \leq n$  as follows:

$$(3.1) \quad \mathcal{D}_{\alpha\bar{\beta}} = \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} - \frac{\rho_\alpha}{\rho_{n+1}} \frac{\partial^2}{\partial z_{n+1} \partial \bar{z}_\beta} - \frac{\rho_{\bar{\beta}}}{\rho_{n+1}} \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_{n+1}} + \frac{\rho_\alpha \rho_{\bar{\beta}}}{|\rho_{n+1}|^2} \frac{\partial^2}{\partial z_{n+1} \partial \bar{z}_{n+1}}.$$

The following explicit formula for the Webster pseudo-Ricci curvature and pseudo-scalar curvature in terms of defining function  $\rho$  for  $(M, \theta)$  was proved in [30]:

**Theorem 3.1.** *Let  $M = \partial D$  be a strictly pseudo-convex hypersurface in  $\mathbb{C}^{n+1}$ . Let  $\rho \in C^3(\bar{D} \cap U) \cap C^\infty(D \cap U)$  be a defining function for  $M$  with that  $J(\rho) > 0$  on  $\bar{D} \cap U$  and  $u(z) = -\log(-\rho)$  is plurisubharmonic in  $D \cap U$ . Let  $\theta = (\partial\rho - \bar{\partial}\rho)/(2i)$ . Then for  $v, w \in H(M) = T_{1,0}(M)$ , we have:*

$$(3.2) \quad \text{Ric}(w, \bar{v}) = -L_{\log J(\rho)}(w, \bar{v}) + (n+1) \frac{\det H(\rho)}{J(\rho)} L_\rho(w, \bar{v}),$$

where  $L_g(w, \bar{v}) = \sum_{k,j=1}^{n+1} g_{k\bar{j}}(z) w_k \bar{v}_j$  is the Levi form associated to  $g$ .

In a local coordinates, at those  $z \in M_1 = \{z \in M : \rho_{n+1}(z) \neq 0\}$ , we have the Webster pseudo-scalar curvature

$$(3.3) \quad R_\theta(z) = - \sum_{\alpha, \beta=1}^n h^{\alpha\bar{\beta}} \mathcal{D}_{\alpha\bar{\beta}} \log J(\rho) + n(n+1) \frac{\det H(\rho)}{J(\rho)},$$

where the pseudo-Hermitian metric  $h_{\alpha\bar{\beta}} = \mathcal{D}_{\alpha\bar{\beta}}(\rho)$  and  $[h^{\alpha\bar{\beta}}] = ([h_{\alpha\bar{\beta}}]^{-1})^t$ .

**Proposition 3.2.** *With the notation above,  $h_{\alpha\bar{\beta}} = \mathcal{D}_{\alpha\bar{\beta}}(\rho)$ , we have*

$$(3.4) \quad h^{\alpha\bar{\beta}}(z) = \rho^{\alpha\bar{\beta}} - \frac{\rho^\alpha \rho^{\bar{\beta}}}{|\partial\rho|_\rho^2}, \quad z \in M_1, \quad 1 \leq \alpha, \beta \leq n.$$

*Proof.* Since, for  $1 \leq i, k \leq n$  with  $h_{k\bar{j}} = \mathcal{D}_{k\bar{j}}(\rho)$

$$\begin{aligned}
& \sum_{j=1}^n \left( \rho^{i\bar{j}} - \frac{\rho^i \rho^{\bar{j}}}{|\partial\rho|_\rho^2} \right) h_{k\bar{j}} \\
&= \sum_{j=1}^n \left( \rho^{i\bar{j}} - \frac{\rho^i \rho^{\bar{j}}}{|\partial\rho|_\rho^2} \right) \left( \rho_{k\bar{j}} - \frac{\rho_k}{\rho_{n+1}} \rho_{n+1\bar{j}} - \frac{\rho_{\bar{j}}}{\rho_{n+1}} \rho_{k\bar{n}+1} + \frac{\rho_k \rho_{\bar{j}}}{|\rho_{n+1}|^2} \rho_{n+1\bar{n}+1} \right) \\
&= \sum_{j=1}^n \rho^{i\bar{j}} \left( \rho_{k\bar{j}} - \frac{\rho_k}{\rho_{n+1}} \rho_{n+1\bar{j}} - \frac{\rho_{\bar{j}}}{\rho_{n+1}} \rho_{k\bar{n}+1} + \frac{\rho_k \rho_{\bar{j}}}{|\rho_{n+1}|^2} \rho_{n+1\bar{n}+1} \right) \\
&\quad - \sum_{j=1}^n \frac{\rho^i \rho^{\bar{j}}}{|\partial\rho|_\rho^2} \left( \rho_{k\bar{j}} - \frac{\rho_k}{\rho_{n+1}} \rho_{n+1\bar{j}} - \frac{\rho_{\bar{j}}}{\rho_{n+1}} \rho_{k\bar{n}+1} + \frac{\rho_k \rho_{\bar{j}}}{|\rho_{n+1}|^2} \rho_{n+1\bar{n}+1} \right) \\
&= \delta_{ik} - \rho^{i\bar{n}+1} \rho_{k\bar{n}+1} + \frac{\rho_k}{\rho_{n+1}} \rho^{i\bar{n}+1} \rho_{n+1\bar{n}+1} - \frac{\rho^i}{\rho_{n+1}} \rho_{k\bar{n}+1} + \rho^{i\bar{n}+1} \rho_{k\bar{n}+1} \\
&\quad + \frac{\rho^i \rho_k}{|\rho_{n+1}|^2} \rho_{n+1\bar{n}+1} - \rho^{i\bar{n}+1} \frac{\rho_k}{\rho_{n+1}} \rho_{n+1\bar{n}+1} \\
&\quad - \frac{\rho^i \rho_k}{|\partial\rho|_\rho^2} + \frac{\rho^i \rho^{\bar{n}+1}}{|\partial\rho|_\rho^2} \rho_{k\bar{n}+1} + \frac{\rho^i \rho_{n+1}}{|\partial\rho|_\rho^2} \frac{\rho_k}{\rho_{n+1}} - \frac{\rho^i \rho^{\bar{n}+1}}{|\partial\rho|_\rho^2} \frac{\rho_k}{\rho_{n+1}} \rho_{n+1\bar{n}+1} \\
&\quad + \frac{\rho^i}{\rho_{n+1}} \rho_{k\bar{n}+1} - \frac{\rho^i \rho^{\bar{n}+1}}{|\partial\rho|_\rho^2} \rho_{k\bar{n}+1} - \frac{\rho^i \rho_k}{|\rho_{n+1}|^2} \rho_{n+1\bar{n}+1} + \frac{\rho^i \rho^{\bar{n}+1}}{|\partial\rho|_\rho^2} \frac{\rho_k}{\rho_{n+1}} \rho_{n+1\bar{n}+1} \\
&= \delta_{ik}.
\end{aligned}$$

Therefore, (3.4) holds, and the proof of the proposition is complete.  $\square$

Let  $\Delta_u$  be the Beltrami–Laplacian with respect to the metric  $u_{i\bar{j}} dz_i \otimes d\bar{z}_j$ . Then

$$(3.5) \quad \Delta_u = \sum_{i,j=1}^{n+1} u^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j}.$$

Since  $u = -\log(-\rho)$ , we have (see [7, 24]) that

$$(3.6) \quad u_{i\bar{j}} = \frac{\rho_{i\bar{j}}}{-\rho} + \frac{\rho_i \rho_{\bar{j}}}{\rho^2}, \quad \text{and} \quad u^{i\bar{j}} = (-\rho) \left( \rho^{i\bar{j}} - \frac{\rho^i \rho^{\bar{j}}}{-\rho + |\partial\rho|_\rho^2} \right).$$

Then,

$$(3.7) \quad \Delta_u = (-\rho) \sum_{i,j=1}^{n+1} \left( \rho^{i\bar{j}} - \frac{\rho^i \rho^{\bar{j}}}{-\rho + |\partial\rho|_\rho^2} \right) \frac{\partial^2}{\partial z_i \partial \bar{z}_j}.$$

Thus if  $f$  is harmonic in the metric  $u_{i\bar{j}} dz_i \otimes d\bar{z}_j$ , then

$$(3.8) \quad \Delta_u f = \sum_{i,j=1}^{n+1} \left( \rho^{i\bar{j}} - \frac{\rho^i \rho^{\bar{j}}}{-\rho + |\partial\rho|_\rho^2} \right) \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} = 0.$$

**Proposition 3.3.** *With the notation above, for any  $f \in C^2(D)$ , we have*

$$(3.9) \quad \sum_{i,j=1}^{n+1} \left( \rho^{i\bar{j}} - \frac{\rho^i \rho^{\bar{j}}}{|\partial\rho|_\rho^2} \right) f_{i\bar{j}} = \sum_{i,j=1}^n \left( \rho^{i\bar{j}} - \frac{\rho^i \rho^{\bar{j}}}{|\partial\rho|_\rho^2} \right) \mathcal{D}_{i\bar{j}}(f).$$

*Proof.* Since

$$\begin{aligned} & \sum_{i,j=1}^n \left( \rho^{i\bar{j}} - \frac{\rho^i \rho^{\bar{j}}}{|\partial\rho|_\rho^2} \right) f_{i\bar{j}} - \sum_{i,j=1}^n \left( \rho^{i\bar{j}} - \frac{\rho^i \rho^{\bar{j}}}{|\partial\rho|_\rho^2} \right) \mathcal{D}_{i\bar{j}}(f) \\ &= \sum_{i,j=1}^n \left( \rho^{i\bar{j}} - \frac{\rho^i \rho^{\bar{j}}}{|\partial\rho|_\rho^2} \right) \left( \frac{\rho_i}{\rho_{n+1}} f_{n+1\bar{j}} + \frac{\rho_{\bar{j}}}{\rho_{n+1}} f_{i\bar{n}+1} - \frac{\rho_i \rho_{\bar{j}}}{|\rho_{n+1}|^2} f_{n+1\bar{n}+1} \right) \\ &= \sum_{j=1}^n \frac{\rho^{\bar{j}}}{\rho_{n+1}} f_{n+1\bar{j}} - \sum_{j=1}^n \rho^{n+1\bar{j}} f_{n+1\bar{j}} + \sum_{i=1}^n \frac{\rho^i}{\rho_{n+1}} f_{i\bar{n}+1} - \sum_{i=1}^n \rho^{i\bar{n}+1} f_{i\bar{n}+1} \\ &+ \sum_{i=1}^n \left( -\frac{\rho_i \rho^i}{|\rho_{n+1}|^2} + \rho^{i\bar{n}+1} \frac{\rho_i}{\rho_{n+1}} \right) f_{n+1\bar{n}+1} \\ &- \frac{\sum_{i=1}^n \rho^i \rho_i}{|\partial\rho|_\rho^2} \sum_{j=1}^n \frac{\rho^{\bar{j}}}{\rho_{n+1}} f_{n+1\bar{j}} - \frac{\sum_{j=1}^n \rho^j \rho_{\bar{j}}}{|\partial\rho|_\rho^2} \sum_{i=1}^n \frac{\rho^i}{\rho_{n+1}} f_{i\bar{n}+1} \\ &+ \frac{(\sum_{i=1}^n \rho^i \rho_i)(\sum_{j=1}^n \rho^{\bar{j}} \rho_{\bar{j}})}{|\partial\rho|_\rho^2 |\rho_{n+1}|^2} f_{n+1\bar{n}+1} \end{aligned}$$

$$\begin{aligned}
&= \left(1 - \frac{\sum_{i=1}^n \rho^i \rho_i}{|\partial\rho|_\rho^2}\right) \sum_{j=1}^n \frac{\rho^{\bar{j}}}{\rho_{n+1}} f_{n+1\bar{j}} - \sum_{j=1}^n \rho^{n+1\bar{j}} f_{n+1\bar{j}} \\
&\quad + \left(1 - \frac{\sum_{j=1}^n \rho^{\bar{j}} \rho_{\bar{j}}}{|\partial\rho|_\rho^2}\right) \sum_{i=1}^n \frac{\rho^i}{\rho_{n+1}} f_{i\overline{n+1}} - \sum_{i=1}^n \rho^{i\overline{n+1}} f_{i\overline{n+1}} + \left(-\frac{\sum_{i=1}^n \rho_i \rho^i}{|\rho_{n+1}|^2}\right. \\
&\quad \left. + \frac{\rho^{\overline{n+1}}}{\rho_{n+1}} - \rho^{n+1\overline{n+1}} + \frac{(\sum_{i=1}^n \rho^i \rho_i)(\sum_{j=1}^n \rho^{\bar{j}} \rho_{\bar{j}})}{|\partial\rho|_\rho^2 |\rho_{n+1}|^2}\right) f_{n+1\overline{n+1}} \\
&= -\sum_{j=1}^n \left[\rho^{n+1\bar{j}} - \frac{\rho^{n+1}\rho^{\bar{j}}}{|\partial\rho|_\rho^2}\right] f_{n+1\bar{j}} - \sum_{i=1}^n \left[\rho^{i\overline{n+1}} - \frac{\rho^{\overline{n+1}}\rho^i}{|\partial\rho|_\rho^2}\right] f_{i\overline{n+1}} \\
&\quad - \left[\rho^{n+1\overline{n+1}} - \frac{\rho^{n+1}\rho^{\overline{n+1}}}{|\partial\rho|_\rho^2}\right] f_{n+1\overline{n+1}}.
\end{aligned}$$

Moving the right side to the left hand side, we have

$$(3.10) \quad \sum_{i,j=1}^{n+1} \left(\rho^{i\bar{j}} - \frac{\rho^i \rho^{\bar{j}}}{|\partial\rho|_\rho^2}\right) f_{i\bar{j}} - \sum_{i,j=1}^n \left(\rho^{i\bar{j}} - \frac{\rho^i \rho^{\bar{j}}}{|\partial\rho|_\rho^2}\right) \mathcal{D}_{i\bar{j}}(f) = 0.$$

Therefore, the proof of (3.9) is complete, and so is the proof of the proposition.  $\square$

**Proof of part(b) of Theorem 1.1.** Let  $f(z) = \log J(\rho)(z)$  be harmonic in the metric  $u_{i\bar{j}} dz_i \otimes d\bar{z}_j$ . By (3.8), and then by (3.9), we have for all  $z \in M = \partial D$

$$(3.11) \quad 0 = \sum_{i,j=1}^{n+1} \left(\rho^{i\bar{j}} - \frac{\rho^i \rho^{\bar{j}}}{|\partial\rho|_\rho^2}\right) \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} = \sum_{i,j=1}^n \left(\rho^{i\bar{j}} - \frac{\rho^i \rho^{\bar{j}}}{|\partial\rho|_\rho^2}\right) \mathcal{D}_{i\bar{j}}(f)(z).$$

By (3.4) and  $f = \log J(\rho)$ , we have

$$(3.12) \quad \sum_{\alpha,\beta=1}^n h^{\alpha\bar{\beta}} \mathcal{D}_{\alpha\bar{\beta}}(\log J(\rho)) = 0, \quad z \in M_1.$$

Applying the formula (3.3) for the Webster pseudo-scalar curvature  $R_\theta$ , we have that  $R_\theta = n(n+1) \det H(\rho)/J(\rho)$ , i.e., (1.5) holds, and the proof of part(b) of Theorem 1.1 is complete.  $\square$

#### 4. Proof of Theorem 1.1 and Corollary 1.2

We have proved part(b) of Theorem 1.1. Now we prove part(a) of Theorem 1.1.

**Proof of part(a) of Theorem 1.1.** Since  $\log J(\rho)$  is harmonic in the metric  $u_{i\bar{j}}dz_i \otimes d\bar{z}_j$  in  $D$  and the pseudo-scalar curvature  $R_\theta$  is constant  $c$  on  $\partial D$ , by part(b) of Theorem 1.1, we have

$$(4.1) \quad c = R_\theta = n(n+1) \frac{\det H(\rho)}{J(\rho)} \quad \text{on } \partial D.$$

Therefore,

$$(4.2) \quad \frac{\det H(\rho)}{J(\rho)} = \frac{c}{n(n+1)} > 0 \quad \text{on } \partial D.$$

This with  $-\log J(\rho)$  being harmonic in the metric  $u_{i\bar{j}}dz_i \otimes d\bar{z}_j$ , we have that all conditions of Theorem 2.3 hold. By Theorem 2.3, we have

$$(4.3) \quad \frac{\det H(\rho)}{J(\rho)} = \frac{c}{n(n+1)} > 0 \quad \text{on } D.$$

By Corollary 2.4, we have that  $D$  is biholomorphic to  $B_{n+1}$ , and the proof of part(a) of Theorem 1.1 is complete.  $\square$

**Proof of Corollary 1.2.** Since  $u$  is the potential function for  $D$ , we have  $J(\rho) \equiv 1$ . Hence,  $\log J(\rho) = 0$  is harmonic in  $u_{i\bar{j}}dz_i \otimes d\bar{z}_j$ . By part(b) of Theorem 1.1, we have  $R_\theta = c$  on  $\partial D$ , and then by part(a) of Theorem 1.1 that  $D$  is biholomorphic to  $B_{n+1}$ . Moreover,  $\det H(\rho) \equiv c$  on  $D$ . Let  $\rho^0 = c\rho$ .  $\det H(\rho^0)/J(\rho^0) \equiv 1$  on  $D$ . Then by (2.37) and (2.38), there is  $z_0 \in D$  with

$$(4.4) \quad \rho^0(z_0) = \min\{\rho^0(z) : z \in D\} = -\frac{\det H(\rho^0)(z_0)}{J(\rho^0)(z_0)} = -1.$$

By (2.6), we have  $T(z) \equiv 1$  on  $D$ . By (2.5) in [24], we have

$$\det H(\log(1 + \rho^0))(z) = 0, \quad \text{for all } z \in D \text{ with } \rho^0(z) > -1.$$

Applying part(ii) of Theorem 1.2 in [24], there is a constant Jacobian biholomorphic mapping  $\phi : D \rightarrow B_{n+1}$ . The proof of Corollary 1.2 is complete.  $\square$

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