Tunnel number one, genus-one fibered knots

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We determine the genus-one fibered knots in lens spaces that have tunnel number one. We also show that every tunnel number one, once-punctured torus bundle is the result of Dehn filling a component of the Whitehead link in the 3-sphere.

1. Introduction

A null homologous knot K in a 3-manifold M is a genus-one fibered knot (GOF-knot for short), if $M - N(K)$ is a once-punctured torus bundle over the circle whose monodromy is the identity on the boundary of the fiber and K is ambient isotopic in M to the boundary of a fiber.

We say the knot K in M has *tunnel number one* if there is a properly embedded arc τ in $M - N(K)$ such that $M - N(K) - N(\tau)$ is a genus-two handlebody. An arc such as τ is called an *unknotting tunnel*. Similarly, a manifold with toroidal boundary is *tunnel number one* if it admits a genus-two Heegaard splitting. Thus a knot is tunnel number one if and only if its complement is tunnel number one.

In the genus 1 Heegaard surface of $L(p,1)$, $p \neq 1$, there is a unique link that bounds an annulus in each solid torus. This two-component link is called the *p-Hopf link* and is fibered with monodromy p Dehn twists along the core curve of the fiber. We refer to the fiber of a p-Hopf link as a *p-Hopf band*. In this terminology, the standard positive and negative Hopf bands in $S^3 = L(+1,1) = L(-1,1)$ are the $(+1)$- and (-1)-Hopf bands, respectively.

González-Acuña [10] shows that the trefoil (and its mirror) and the figure eight knot are the only GOF-knots in S^3. These knots arise as the boundary of the plumbing of two (± 1)-Hopf bands. For both knots, a transverse arc on one of the plumbed Hopf bands is an unknotting tunnel, so both have tunnel number one.

GOF-knots in lens spaces were first studied by Morimoto in [14], and were classified by the first author of the present article in [2]. Unlike in S^3, these knots do not all have tunnel number one. In particular we prove the following:

Theorem 1.1. *Every tunnel number one, GOF-knot in a lens space is the plumbing of an r-Hopf band and a (± 1)-Hopf band which is contained in the lens space $L(r, 1)$, with one exception. Up to mirroring, this exception is the GOF-knot in $L(7, 2)$ that arises as (-1)-Dehn surgery on the plumbing of a 7-Hopf band and a $(+1)$-Hopf band.*

Every GOF-knot has tunnel number 1 or 2. Thus Theorem 1.1 determines the tunnel number of every GOF-knot in a lens space.

We also obtain:

Theorem 1.2. *Every tunnel number one, once-punctured torus bundle is the complement of a GOF-knot in a lens space of the form $L(r, 1)$.*

Theorem 1.3. *Every tunnel number one, once-punctured torus bundle is the $(r/1)$-Dehn filling of a boundary component of the exterior of the Whitehead link for some integer r.*

This article begins by determining the monodromy of a tunnel number one, once-punctured torus bundle in Section 2. We then translate these monodromies into the language of closed 3-braids in Section 3. In particular we determine which 3-braids in a solid torus have double branched covers producing tunnel number one, once-punctured torus bundles. In Section 4, we discuss the presentations of two bridge links as closed 3-braids in S^3. In Section 5, we determine which of the braids from Section 3 are two bridge links. The proof of Theorem 1.1 is presented in Section 6, and Theorems 1.2 and 1.3 are proved in Section 7.

We refer the reader to [4] for background regarding fibered knots, braids, two bridge links, lens spaces, and double coverings of S^3 branched over a link. Also, recall that plumbing is a local operation generalizing the connect sum. In particular, the plumbing of a fibered link in S^3 with a fibered link in another 3-manifold M produces a new fibered link in M. See for example Gabai's geometric description of the yet more generalized Murasugi sum in [9].

The authors would like to thank Alan Reid for a helpful conversation regarding the Whitehead link.

2. The monodromy of a tunnel number one, once-punctured torus bundle

Heegaard splittings of closed torus bundles over the circle were studied in detail by Cooper and Scharlemann [7]. In particular, their work characterized genus-two Heegaard splittings of such bundles. Their method transfers almost directly to the case of once-punctured torus bundles which are tunnel number one.

Lemma 2.1. *Let M be a once-punctured torus bundle over the circle with once-punctured torus fiber T and monodomy ϕ. Further assume M allows a genus-two Heegaard splitting. Then there is a pair of simple closed curves α_1, α_2 in T such that $\alpha_1 \cap \alpha_2$ is a single point and ϕ sends α_1 onto α_2. The map ϕ is isotopic to $(s_2 s_1 s_2)^{\pm 1} s_1^n$ where s_1 is a Dehn twist along α_1 and s_2 is a Dehn twist along α_2.*

Proof. We will show that the proof for closed torus bundles in Theorem 4.2 of [7] works equally well in the once-punctured case. Because this method has been described in detail elsewhere, we will give only an outline of the setup and leave many of the details to the reader. A similar exposition for general surface bundles can also be found in [1].

Let (Σ, H_1, H_2) be a genus-two Heegaard splitting for M. Assume H_1 is a compression body and H_2 a handlebody. A spine K_1 for H_1 consists of $\partial_- H_1$ and an arc properly embedded in H_1 such that the complement in H_1 of K_1 is homeomorphic to $\partial_+ H_1 \times (-1, 0]$. A spine K_2 for H_2 is a graph whose complement is homeomorphic to $\partial H_2 \times [0, 1)$. The Heegaard splitting determines a continuous one-parameter family of embedded, pairwise disjoint surfaces $\{\Sigma_x : x \in (-1, 1)\}$ such that as x approaches -1, the surfaces limit to K_1 and as x approaches 1, the surfaces limit onto K_2. This family of surfaces is called a *sweep-out*. For each x, the surface Σ_x separates M into a compression body H_1^x and a handlebody H_2^x.

The fibers of the bundle structure on M form a continuous one-parameter family of embedded, pairwise disjoint essential surfaces $\{T_y : y \in S^1\}$. Assume the surfaces $\{\Sigma_x\}$ and $\{T_y\}$ are in general position. The *Rubinstein–Scharlemann graphic* is the subset R of $(-1, 1) \times S^1$ consisting of pairs (x, y) such that Σ_x and T_y are tangent at some point in M. General position implies that this set will be a graph.

At each point (x, y) in the complement of the graphic, the corresponding surfaces Σ_x and T_y are transverse. Label each point (x, y) (in $(-1, 1) \times S^1$) with a 1 if some loop component of $\Sigma_x \cap T_y$ is essential in Σ_x and is the

boundary component of a disk or an essential annulus in H_1^x. Note that for such an essential annulus, its other boundary component is a curve on ∂M isotopic to ∂T_y. Label (x, y) with a 2 if a loop is essential in Σ_x and bounds a disk in H_2^x. Given a second point (x', y') in the same component of the complement of the graphic as (x, y), a piecewise vertical and horizontal path from (x, y) to (x', y') determines an ambient isotopy of M taking Σ_x to $\Sigma_{x'}$ and T_y to $T_{y'}$. Thus any two points in the same component of the complement of the graphic have the same labels.

Claim 2.2. *For a fixed x, there cannot be values y, y' such that (x, y) is labeled with a 1 and (x, y') is labeled with a 2. (In particular, a point cannot have both labels 1 and 2.)*

Proof. Let M' be the result of Dehn filling ∂M along the slope of the boundary of a level surface T_y. Then M' is a closed torus bundle and the image of a loop which is boundary parallel in T_y will be trivial in M'. The induced Heegaard splitting of M' will be irreducible because a torus bundle cannot be a lens space. Moreover, an irreducible, genus-two Heegaard splitting is strongly irreducible so the induced Heegaard splitting of M' is strongly irreducible.

Now assume for contradiction the point (x, y) is labeled with a 1 and (x, y') is labeled with a 2. Then there is a loop in $\Sigma_x \cap T_y$ that bounds an essential disk in the filling of H_1^x (after the filling, an essential annulus is capped off and becomes a disk) and a loop $\Sigma_x \cap T_{y'}$ that bounds an essential disk in H_2^x. Because T_y and $T_{y'}$ are disjoint, each loop in $\Sigma_x \cap T_{y'}$ is disjoint from each loop in $\Sigma_x \cap T_y$. Thus the Heegaard surface Σ_x for M' is weakly reducible and therefore reducible. This contradicts the assumption that M' is not a lens space. □

Claim 2.3. *No fiber T_y can be made disjoint from H_2^x.*

Proof. If a fiber T_y were disjoint from H_2^x, then it would be contained in H_1^x. Since T_y is essential, a non-separating compressing disk of H_1^x must be disjoint from T_y. Therefore compressing H_1^x along such a disk yields a manifold which is homeomorphic to $T^2 \times I$ and contains a properly embedded, essential, once-punctured torus, which cannot occur. □

Claim 2.4. *In a component of $(-1, 1) \times S^1 - R$ that intersects $(-1, -1 + \epsilon) \times S^1$ for suitably small ϵ, each point is labeled with a 1. In a component of $(-1, 1) \times S^1 - R$ that intersects $(1 - \epsilon, 1) \times S^1$ for suitably small ϵ, each point is labeled with a 2.*

Proof. Fix y. Because Σ_x limits onto K_1, if x is near -1, then $T_y \cap H_1^x$ will be a collection of disks and an essential annulus. Hence (x, y) has label 1.

If x is near 1, then $T_y \cap H_2^x$ consists of disks. If none of these disks are essential in H_2^x, then T_y can be pushed into H_1^x, contradicting Claim 2.3. Hence (x, y) has label 2. $\qquad\square$

Claim 2.5. *For some value x, the point (x, y) is unlabeled for every y.*

Proof. The complement of the graphic is an open set, as is each component of the complement. The union of all the components labeled with a 1 projects to an open set in $(-1, 1)$ as does the union of the components labeled with a 2. By Claim 2.4 the projection of each set is non-empty. By Claim 2.2 the images of these projections are disjoint. Because $(-1, 1)$ is connected, it cannot be written as the union of two non-empty open sets. Thus there is a point x in the complement of the projections. This x has the desired property. $\qquad\square$

Fix an x such that for every y, (x, y) is unlabeled as guaranteed by Claim 2.5. If for some y, T_y is transverse to Σ_x and each loop of $\Sigma_x \cap T_y$ is trivial in T_y then T_y can be isotoped into H_1^x. As noted in Claim 2.3 above, this is impossible. Therefore for each $y \in S^1$, if T_y and Σ_x intersect transversely, then the intersection $\Sigma_x \cap T_y$ must contain a loop which is essential in T_y. Since two disjoint, essential loops in a once-punctured torus are parallel, the essential loops in $\Sigma_x \cap T_y$ are all parallel in T_y, for each y.

For each point $p \in \Sigma_x$ there is a $y \in S^1$ such that p is contained in T_y. Define the function $f_x : \Sigma_x \to S^1$ such that p is contained in $T_{f(p)}$ for each $p \in \Sigma_x$. By general position, either f_x is a circle-valued Morse function or f_x is a near-Morse function such that the critical points consist of either a single isolated degenerate critical point or just two critical points at the same level.

Assume for contradiction f_x is a Morse function. Then for each $y \in S^1$, the level set $f_x^{-1}(y)$ contains a loop which is essential in Σ_x. Thus $\Sigma_x \cap T_y$ contains a loop which is essential in Σ_x and therefore essential in T_y. By continuity, the isotopy class in Σ_x of the essential loops cannot change as y varies, so the monodromy ϕ must send this essential loop onto itself. A quick calculation shows the homology of this manifold would have rank 3, contradicting the fact that M admits a genus two Heegaard splitting.

If f_x has an isolated degenerate critical point, then once again $f_x^{-1}(y)$ must contain an essential loop for each y. This leads to the same contradiction as when f_x is a Morse function; see [7]. Therefore we conclude that
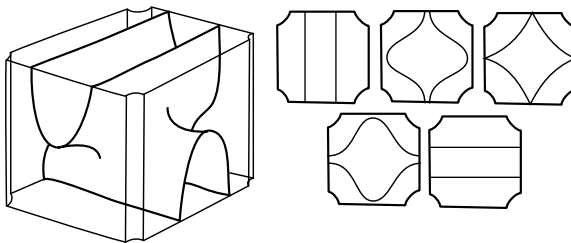
Figure 1: Σ_x in the once-punctured torus bundle M cut open along a fiber T_y, and the sequence of intersections of Σ_x with the fibers T_y as y passes through the critical value of f_x.

f_x must have two critical points on the same level. As described in [7], this implies that Σ_x is embedded in M as shown in figure 1.

The right side of the figure shows how the loops of $\Sigma_x \cap T_y$ sit in each T_y. As y passes through the critical value c of f_x from $c + \epsilon$ to $c - \epsilon$, the isotopy class $\ell_{c+\epsilon}$ of the essential loops of $\Sigma_x \cap T_{c+\epsilon}$ is replaced by a new isotopy class $\ell_{c-\epsilon}$ of the loops of $\Sigma_x \cap T_{c-\epsilon}$ which, under the projection $T \times (c - \epsilon, c + \epsilon) \to T$, intersects the original in a single point. Because this is the only level at which this happens, we conclude that there are essential loops $\alpha_1, \alpha_2 \subset T$ such that $\alpha_1 \cap \alpha_2$ is a single point and $\phi(\alpha_1) = \alpha_2$.

Let s_1 be a Dehn twist along α_1 and s_2 a Dehn twist along α_2. Either composition $(s_2 s_1 s_2)^{\pm 1}$ takes α_1 to α_2 and α_2 to α_1. By choosing $+1$ or -1 appropriately, we can ensure that the map sends α_1 to α_2 with the same orientation as ϕ. Composing further with s_1^n for some n will take α_2 to $\phi(\alpha_2)$. Thus $(s_2 s_1 s_2)^{\pm 1} s_1^n$ is isotopic to ϕ. □

3. GOF-knots via closed 3-braids

A GOF-knot may be described in terms of a double branched covering of a closed 3-braid whose word encodes the monodromy of the fibering. This viewpoint allows us to describe tunnel number one once-punctured torus bundles.

Lemma 3.1. *Every GOF-knot is the image of the braid axis of a closed 3-braid in S^3 in the double branched cover of the closed 3-braid. Conversely, the image of the braid axis of a closed 3-braid in the double branched cover of the closed 3-braid is always a GOF-knot.*

By considering the standard involution of the once-punctured torus T, we will show that the set of GOF-knots in 3-manifolds and the set of braid

axes of closed 3-braids in S^3 are in one-to-one correspondence. We sketch the passage from GOF-knots to axes of closed 3-braids in S^3 below. The return passage is then clear. (See Section 5 of [3], Sections 4 and 5 of [12] and Section 2 of [2].)

Sketch. The standard involution ι with three fixed points of the once-punctured torus T extends to an involution of the once-punctured torus bundle M that takes a fiber to a fiber. Since ι commutes with the monodromy of M, quotienting M by this involution yields a closed 3-braid $\hat{\beta}$ in a solid torus W where the braid $\hat{\beta}$ is the image of the fixed set of ι and a meridional disk of W is the image of a fiber.

The extension of ι is a fixed point free involution on the boundary of M, so ι further extends to an involution of the filling M' of M by a solid torus V' whose meridians intersect fibers of M just once. In M' the core of V' is a GOF-knot K. Since the extension of ι acts as a free involution of V', in the quotient V' descends to a solid torus V. The meridian of V intersects the meridian of W once, as the meridian of V' intersects the boundary of a fiber of M once. Hence under this quotient, K descends to the core of V which is the axis of the closed 3-braid $\hat{\beta}$ in $V \cup W \cong S^3$. This defines the correspondence. □

Lemma 3.2. *Every tunnel number one, GOF-knot is the lift of the braid axis of the closure $\hat{\beta}_{k,n}$ of the braid $\beta_{k,n} = (\sigma_2\sigma_1\sigma_2)^k\sigma_1^n$ in the double cover of S^3 branched over $\hat{\beta}_{k,n}$, where k is odd and n is an arbitrary integer.*

A depiction of the braid $\beta_{k,n} = (\sigma_2\sigma_1\sigma_2)^k\sigma_1^n$ is shown in figure 2.

Proof. Let $\widehat{M}$ be a 3-manifold that contains a tunnel number one, GOF-knot K. Let M be the once-punctured torus bundle exterior $\widehat{M} - N(K)$ with fiber T.
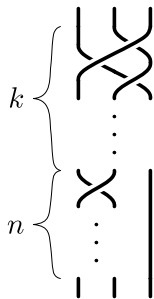

Figure 2: The braid $\beta_{k,n} = (\sigma_2\sigma_1\sigma_2)^k\sigma_1^n$.

By Lemma 2.1, the monodromy of M may be given by $(s_2 s_1 s_2)^{\pm 1} s_1^n$ up to Dehn twists along the boundary of a fiber T. Write a single Dehn twist along ∂T as $(s_2 s_1 s_2)^4$. Then the monodromy $T \to T$ fixing ∂T with an orbit on ∂M bounding a meridional curve for K is $(s_2 s_1 s_2)^{4\ell \pm 1} s_1^n$ for some $\ell \in \mathbb{Z}$.

By Lemma 3.1, $\widehat{M}$ is the double branched cover of a closed 3-braid in S^3 with braid axis lifting to K. Accordingly, M is the double branched cover of the closed 3-braid in the solid torus exterior of the braid axis. A meridian of K in ∂M corresponds to a meridian of the braid axis in the solid torus.

Under the quotient of the covering involution, the Dehn twist s_i along α_i corresponds to the braid σ_i, a right-handed crossing between the ith and $(i+1)$th strands. Thus K in $\widehat{M}$ corresponds to the braid axis of the closed braid $(\sigma_2 \sigma_1 \sigma_2)^k \sigma_1^n$ where $k = 4\ell \pm 1$. $\qquad\square$

Remark 3.3. A meridian of the braid axis is a longitudinal curve on the solid torus containing the closed 3-braid. In the double branched cover it lifts to two meridians of the GOF-knot. The longitude of the braid axis is the meridian of the solid torus containing the closed 3-braid. It lifts to the longitude of the GOF-knot.

In these coordinates for the braid axis, a slope p/q lifts to the slope $2p/q$. If q is even (and p and q are coprime) this slope is to be interpreted as two parallel curves of slope $p/(q/2)$. It follows that $1/q$ surgery on a GOF-knot corresponds to inserting $2q$ full twists (right-handed if $q < 0$, left-handed if $q > 0$) into the 3-braid.

If q is odd then a p/q slope lifts to a single curve of slope $2p/q$. Hence p/q surgery on the braid axis cannot lift to surgery on the GOF-knot in the double branched cover unless the core of the surgery solid torus is added to the branch locus.

4. GOF-knots in lens spaces

By Corollary 4.12 of [11], the lens space $L(\alpha, \beta)$ is the double cover of S^3 branched over a link L if and only if L is equivalent to the two bridge link $\mathfrak{b}(\alpha, \beta)$. Thus to understand GOF-knots in lens spaces using Lemma 3.1 we must consider representations of two bridge links as closed 3-braids; see [2].

Murasugi and later Stoimenow describe which two bridge links admit such closed braid representations. These two descriptions may be seen to be equivalent by working out their corresponding continued fractions. Let $\mathbf{b}(L)$ denote the braid index of the link L.

Proposition 4.1 (Proposition 7.2, [16]). *Let L be a two bridge link of type $\mathfrak{b}(\alpha, \beta)$, where $0 < \beta < \alpha$ and β is odd. Then*

(1) $\mathbf{b}(L) = 2$ *iff* $\beta = 1$.

(2) $\mathbf{b}(L) = 3$ *iff either*
   (a) *for some $p, q > 1$, $\alpha = 2pq + p + q$ and $\beta = 2q + 1$, or*
   (b) *for some $q > 0$, $\alpha = 2pq + p + q + 1$ and $\beta = 2q + 1$.*

Lemma 4.2 (Corollary 8, [17]). *If L is a two bridge link of braid index 3, then L has Conway notation $(p, 1, 1, q)$ or $(p, 2, q)$ for some $p, q > 0$.*

The two bridge link with Conway notation $(p, 1, 1, q)$ is shown in figure 3. The link with Conway notation $(p, 2, q)$ is shown in figure 4.

In the other direction, we can determine which 3-braids have closures that are two bridge links.

Lemma 4.3. *The closure of a 3-braid β is a two bridge link if and only if β or its mirror image is conjugate to $\sigma_2^{-1}\sigma_1^p\sigma_2^2\sigma_1^q$ for some $p, q \in \mathbb{Z}$.*
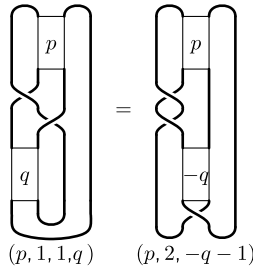

Figure 3: The two bridge link $(p, 1, 1, q)$ is equivalent to $(p, 2, -q - 1)$.
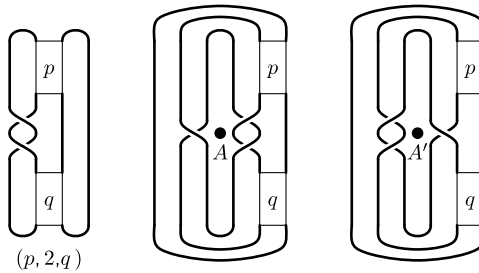

Figure 4: Closed 3-braid representatives of the two bridge link $(p, 2, q)$.

Proof. Assume the closure $\hat{\beta}$ of a 3-braid β is a two bridge link. By Lemma 4.2 $\hat{\beta}$ may be described with Conway notation $(p, 1, 1, q)$ or $(p, 2, q)$ for some integers p, q. Figure 3 shows that a two bridge link with Conway notation $(p, 1, 1, q)$ is equivalent to one with Conway notation $(p, 2, -q - 1)$. Figure 4 shows the expression of a two bridge link with Conway notation $(p, 2, q)$ as the closure of the braid $\sigma_2^{-1}\sigma_1^p\sigma_2^2\sigma_1^q$. By Theorem 2.4 in [2] the braid axis is unique in Case (2) of Proposition 4.1. For Case (1), the braid axes are classified and have the desired form (see the proof of Theorem 2.4 in [2]).

If a 3-braid β is conjugate to $\sigma_2^{-1}\sigma_1^p\sigma_2^2\sigma_1^q$ for some $p, q \in \mathbb{Z}$, then its closure can readily be identified with a two bridge link; see figure 4. $\square$

Lemma 4.4. *The double branched cover of the closure of the braid* $\sigma_2^{-1}\sigma_1^p\sigma_2^2\sigma_1^q$ *is the lens space* $L(2pq + p + q, 2q + 1)$.

Proof. Since the closure of the braid $\sigma_2^{-1}\sigma_1^p\sigma_2^2\sigma_1^q$ is a two bridge link with Conway notation $(p, 2, q)$ (see figure 4), it corresponds to the two bridge link $\mathfrak{b}(\alpha, \beta)$ where $\alpha/\beta = p + 1/(2 + 1/q) = \frac{2pq+p+q}{2q+1}$. Since the double branched cover of the two bridge link $\mathfrak{b}(\alpha, \beta)$ is the lens space $L(\alpha, \beta)$ the result follows. $\square$

5. Tunnel number one, GOF-knots in lens spaces

The monodromy of tunnel number one once-punctured torus bundle has a special form, which can now be connected with lens spaces via two bridge links.

Lemma 5.1. *The closure* $\hat{\beta}_{k,n}$ *of the braid* $\beta_{k,n} = (\sigma_2\sigma_1\sigma_2)^k\sigma_1^n$, *where* k *is odd, is a two bridge link if and only if* $k = \pm 1$, $(k, n) = \pm(-3, 3)$, *or* $(k, n) = \pm(-3, 5)$.

Proof. If $k = \pm 1$ then

$$\beta_{\pm 1, n} = (\sigma_2\sigma_1\sigma_2)^{\pm 1}\sigma_1^n = (\sigma_1\sigma_2\sigma_1)^{\pm 1}\sigma_1^n \equiv \sigma_2^{\pm 1}\sigma_1^{n \pm 2}.$$

Thus $\hat{\beta}_{\pm 1, n}$ is the two bridge link $\mathfrak{b}(n \pm 2, 1)$ (the $(2, n \pm 2)$-torus link). The reader may check that $\hat{\beta}_{k,n}$ is the two bridge knot $\mathfrak{b}(\mp 5, 1)$ for $(k, n) = \pm(-3, 3)$ and the two bridge knot $\mathfrak{b}(\pm 7, 2)$ for $(k, n) = \pm(-3, 5)$.

If n is even then $\hat{\beta}_{k,n}$ has one unknotted component and one potentially knotted component (a $(2, k)$-torus knot). Since a two bridge link has either one component or two unknotted components, the $(2, k)$-torus knot must be the unknot. Hence $k = \pm 1$.

Assume that n is an odd integer. By taking mirror images, we only need consider the case that $k \equiv 1$ mod 4. We will deal with the three cases $n = 1, n = -1$, and $|n| > 2$ separately.

Since $k = 4\ell + 1$ for some $\ell \in \mathbb{Z}$ we have $\beta_{k,n} = (\sigma_2\sigma_1\sigma_2)^{4\ell}\beta_{1,n}$. By Remark 3.3, the double branched cover of $\hat{\beta}_{k,n}$ may be obtained from the double branched cover $L(n+2, 1)$ of $\hat{\beta}_{1,n}$ by $(1/\ell)$-Dehn surgery on the lift of its braid axis.

If $n = -1$, then the braid axis of $\hat{\beta}_{1,n}$ lifts to the right handed trefoil. A $(1/\ell)$-Dehn surgery on this knot is a lens space only when $\ell = 0$ [15]; hence $k = 1$.

To handle the cases when $n = +1$ and $|n| > 2$, we must further understand the relationship between the order of the lens space and the braid structure of $\hat{\beta}_{k,n}$. The exponent sum of a braid is the sum of the exponents of a word in the letters $\{\sigma_1, \sigma_2\}$ representing the braid. By Lemma 4.3, $\hat{\beta}_{k,n}$ is a two bridge link if and only if it is conjugate to $\sigma_2^{-1}\sigma_1^p\sigma_2^2\sigma_1^q$. Since an exponent sum is invariant under conjugation, we must have:

$$(5.1) \qquad\qquad 3k + n = p + q + 1.$$

Since the double branched cover of $\hat{\beta}_{4\ell+1,n}$ is obtained from $L(n+2, 1)$ by $(1/\ell)$-Dehn surgery on the lift of the braid axis, its first homology group must be cyclic of order $n + 2$. By Lemma 4.4, the double branched cover of closure of the braid $\sigma_2^{-1}\sigma_1^p\sigma_2^2\sigma_1^q$ is the lens space $L(2pq + p + q, 2q + 1)$. Thus the orders of first homology of the double branched covers $\hat{\beta}_{4\ell+1,n}$ and the closure of $\sigma_2^{-1}\sigma_1^p\sigma_2^2\sigma_1^q$ must agree:

$$(5.2) \qquad\qquad |n + 2| = |2pq + p + q|.$$

Since n is assumed to be odd, Equation (5.2) implies that the integers p and q have different parity.

Putting Equations (5.1) and (5.2) together, we obtain $|p + q + 3 - 3k| = |2pq + p + q|$. Hence either

$$(5.3) \qquad\qquad 3k - 3 = 2(pq + p + q) \text{ or } 3k - 3 = -2pq.$$

Now consider the case when $n = +1$. For $\hat{\beta}_{4\ell+1,1}$ to be a two bridge knot, Equation (5.2) implies that $\{p, q\} = \{0, 3\}, \{0, -3\}, \{-1, 2\},$ or $\{-1, -4\}$. Since $n = +1$, Equation (5.1) then implies that $k = 1, -1, 1/3,$ or $-5/3$, respectively. Because k must be an integer, $k = \pm 1$.

Lastly consider the case when $|n| > 2$. Denote the lift of the braid axis of $\hat{\beta}_{1,n}$ by $K_{1,n}$. The action of the monodromy on the homology of the fiber of $K_{1,n}$ is given by

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & n+2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n+2 \\ -1 & -n-1 \end{pmatrix}.$$

This has trace $-n$. By [5] this monodromy is pseudo-Anosov for $|n| > 2$ and therefore by [18] the complement of $K_{1,n}$ is hyperbolic. According to the Cyclic Surgery Theorem [8], $1/\ell$ surgery on $K_{1,n}$ is a lens space only if $\ell = 0$, $\ell = +1$, or $\ell = -1$. Thus only for $k = 1$, $k = 5$ or $k = -3$, respectively, could $\hat{\beta}_{k,n}$ with $|n| > 2$ be a two bridge link. We complete the proof with an examination of the latter two possibilities.

Case A: $k = 5$

By Equation (5.3) either $-6 = pq$ or $6 = pq + p + q$. Thus $-6 = pq$ or $7 = (p+1)(q+1)$. Since p and q have opposite parity, $-6 = pq$, and so $\{p, q\} = \{-2, 3\}, \{2, -3\}, \{1, -6\}$, or $\{-1, 6\}$. Hence by Equation (5.1) $n = -13, -15, -19$, or -9 respectively. `Braid Group Calculator` [6] shows that none of the braids $\beta_{5,n}$ are conjugate to $\sigma_2^{-1}\sigma_1^p\sigma_2^2\sigma_1^q$ for corresponding choices of n and $\{p, q\}$. By Lemma 4.3 this implies the braid closure $\hat{\beta}_{5,n}$ with $|n| > 2$ is not a two bridge link.

Case B: $k = -3$

By Equation (5.3) either $6 = pq$ or $-6 = pq + p + q$. Thus $6 = pq$ or $-5 = (p+1)(q+1)$. Since p and q have opposite parity, $6 = pq$, and so $\{p, q\} = \{2, 3\}, \{-2, -3\}, \{1, 6\}$, or $\{-1, -6\}$. Hence by Equation (5.1) $n = 15$, 5, 17 or 3, respectively. `Braid Group Calculator` [6] shows that the braids $\beta_{-3,n}$ are not conjugate to $\sigma_2^{-1}\sigma_1^p\sigma_2^2\sigma_1^q$ for $(n, \{p, q\}) = (15, \{2, 3\})$ and $(n, \{p, q\}) = (17, \{1, 6\})$. However, these braids are conjugate if $(n, \{p, q\}) = (5, \{-2, -3\})$ or $(n, \{p, q\}) = (3, \{-1, -6\})$. By Lemma 4.3 this implies the braid closure $\hat{\beta}_{-3,n}$ with $|n| > 2$ is not a two bridge link unless $n = 3$ or $n = 5$. $\square$

Lemma 5.2. *The braid $\beta_{-3,5}$ is not conjugate to a braid $\beta_{\epsilon,n}$ for any choice of $\epsilon = +1$ or -1 and integer n; neither is $\beta_{3,-5}$.*

Proof. The proof of Lemma 5.1 shows that $\beta_{-3,5}$ is conjugate to $\sigma_2^{-1}\sigma_1^{-2}\sigma_2^2\sigma_1^{-3}$. By Lemma 4.4, the double branched cover of its closure is the lens space $L(7, 2)$. For $\epsilon = \pm 1$ the double branched cover of the closure of $\beta_{\epsilon,n}$ is the lens space $L(n + 2\epsilon, 1)$. If $\beta_{-3,5}$ were conjugate to $\beta_{\epsilon,n}$ then the double branched covers of their closures would be equal, but this is not the case. Similarly, $\beta_{3,-5}$ is not conjugate to $\beta_{\epsilon,n}$ for any choice of ϵ and n. $\square$

Remark 5.3. As one may check, the braids $\beta_{-3,3}$ and $\beta_{-1,-3}$ are conjugate as are $\beta_{3,-3}$ and $\beta_{1,3}$.

6. Proof of Theorem 1.1

Using our understanding of the braid structure associated to the knot, we can now determine the precise lens space in which the knot lies, as well as describe the structure of the knot.

Theorem 1.1. *Every tunnel number one, GOF-knot in a lens space is the plumbing of an r-Hopf band and a (± 1)-Hopf band in the lens space $L(r, 1)$ with one exception. Up to mirroring, this exception is the GOF-knot in $L(7, 2)$ that arises as (-1)-Dehn surgery on the plumbing of a 7-Hopf band and a $(+1)$-Hopf band.*

Proof. Let K be a tunnel number one, GOF-knot in a lens space. By Lemma 3.1 K corresponds to the braid axis of the closure of a 3-braid β. As K lies in a lens space, $\hat{\beta}$ must be a 2–bridge link [11]. Since K is tunnel number one, Lemma 3.2, Lemma 5.1 and Remark 5.3 together imply that β must be conjugate to $\beta_{k,n}$ for either $k = \pm 1$ and $n \in \mathbb{Z}$ or $(k, n) = \pm(-3, 5)$.

Let us first assume $\beta = \beta_{\pm 1,n}$. Then β is conjugate to $\sigma_2^{\pm 1}\sigma_1^{n\pm 2}$. Setting $r = n \pm 2$, the closure $\hat{\beta}$ is then the $(2, r)$-torus link and its double branched cover is the lens space $L(r, 1)$.

We may now view K as the plumbing of an r-Hopf band and a (± 1)-Hopf band as follows. First observe that β is a Markov stabilization of the 2-braid σ_1^r: a third string and a single crossing $(\sigma_2^{\pm 1})$ is added to σ_1^r to form the 3-braid β preserving its closure.

As in Proposition 12.3 of [4], the lens space $L(r, 1)$ can be decomposed along a torus arising as the double branched cover of a bridge sphere S^2 for the closure $\widehat{\sigma_1^r}$. Isotope the braid axis h of σ_1^r to lie on S^2, and let D be a component of $S^2 - h$.

The link $\widehat{\sigma_1^r}$ punctures D twice, and thus in the double branched covering, D lifts to an annulus A bounded by the preimage of h. By Lemma 11.8 of [4], the braid σ_1^r corresponds to r Dehn twists about the core of A. In other words, the preimage of h lifts to an r-Hopf link.

As demonstrated in Theorem 5.3.1 of [13], the Markov stabilization β of σ_1^r corresponds to the plumbing of a Hopf band onto A in the double branched cover. Thus we see K as the boundary of the plumbing of an r-Hopf band and a (± 1)-Hopf band in the lens space $L(r, 1)$.

Now assume β is conjugate to $\beta_{-3,5}$. Since $\hat{\beta}_{-3,5} = \mathfrak{b}(7,2)$, K is a knot in $L(7,2)$. By Lemma 5.2, β is not conjugate to $\beta_{\pm 1,n}$. Therefore β is not a Markov stabilization of any 2-braid, and so K is not the boundary of the plumbing of an r-Hopf band and a (± 1)-Hopf band. (Indeed, K is not the Murasugi sum of any two fibered links.) Nevertheless, observe that adding two full positive twists to β gives the braid $(\sigma_2\sigma_1\sigma_2)^4\beta_{-3,5} = \beta_{1,5}$. Remark 3.3 thus implies that K is the core of a (-1)-Dehn surgery on the lift of the braid axis of $\hat{\beta}_{1,5}$ to the double branched cover, i.e. the boundary of the plumbing of a 7-Hopf band and a $(+1)$-Hopf band. $\square$

7. Punctured torus bundles as knot complements

Theorem 1.2. *Every tunnel number one once-punctured torus bundle is the complement of a GOF-knot in a lens space of the form $L(r,1)$.*

Proof. Let M be a once-punctured torus bundle with tunnel number one. Filling ∂M along a slope that intersects each fiber once produces a closed 3-manifold $\widehat{M}$ in which the core of the filling solid torus is a tunnel number one GOF-knot K. By Lemma 3.2, K is the lift of the braid axis of the closure of the braid $(\sigma_2\sigma_1\sigma_2)^k\sigma_1^n$ where $k = 4\ell \pm 1$ is odd. Following Remark 3.3, we may perform $-1/\ell$ surgery on K to produce a manifold $\widehat{M}'$ in which the core of the surgery solid torus is a tunnel number one GOF-knot K'. Hence K' is the lift of the braid axis of the closure of the braid $(\sigma_2\sigma_1\sigma_2)^{\pm 1}\sigma_1^n$. This closed braid is equivalent to the two bridge link $\mathfrak{b}(r,1)$ where $r = n \pm 2$. Therefore $\widehat{M}'$, the double cover of this link, is the lens space $L(r,1)$, and M is the complement of K'. $\square$

Theorem 1.3. *Every tunnel number one once-punctured torus bundle is the $(r/1)$-Dehn filling of a boundary component of the exterior of the Whitehead link for some integer r.*

Proof. By Theorem 1.2, a tunnel number one once-punctured torus bundle is the complement of a GOF-knot K in a lens space $L(r,1)$. By Theorem 1.1, K is the plumbing of a r-Hopf band and a (± 1)-Hopf band.

The r-Hopf band is obtained by $r/1$ surgery on a circle C that links an annulus A whose boundary is the unlink in S^3. Plumbing a (± 1)-Hopf band onto A produces the unknot U whose union with C forms the Whitehead link. The $r/1$ surgery on C transforms U into the plumbing of a (± 1)-Hopf band onto the r-Hopf band in the lens space $L(r,1)$. Thus the image in

$L(r,1)$ of U is our knot K. Therefore the complement of K in $L(r,1)$ is obtained from the complement of the Whitehead link $U \cup C$ by the filling of the boundary component coming from C. □

Acknowledgments

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