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On complete mean curvature $\frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$

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Dedicated to David Hoffman and Bill Meeks on the occasion of their 60th birthdays

For a complete embedded surface with compact boundary and constant mean curvature $\frac{1}{2}$ in $\mathbb{H}^2 \times \mathbb{R}$ lying on one side of a horocylinder, we prove an analogue of the Hoffman-Meeks half-space theorem. As an application, we show that a complete immersed surface of constant mean curvature $\frac{1}{2}$ which is transverse to the vertical killing field must be an entire graph. Moreover, to each holomorphic quadratic differential on the unit disk or $\mathbb{C}$ we can associate an entire graph of constant mean curvature $\frac{1}{2}$.

1. Introduction

In this paper, we study complete constant mean curvature $\frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$. Recall that the famous half-space theorem of Hoffman–Meeks says that a properly immersed minimal surface in $\mathbb{R}^3$ that lies in a half-space must be a plane. More precisely, a properly immersed minimal surface, which may have compact boundary, that is asymptotic to a plane, must intersect the plane. Our first result is an analogous result for a complete properly embedded constant mean curvature $\frac{1}{2}$ surface Σ in $\mathbb{H}^2 \times \mathbb{R}$.

Theorem 1.1. *Let Σ be a properly embedded constant mean curvature $\frac{1}{2}$ surface in $\mathbb{H}^2 \times \mathbb{R}$. Suppose Σ is asymptotic to a horocylinder C, and on one side of C. If the mean curvature vector of Σ has the same direction as that of C at points of Σ converging to C, then Σ is equal to C (or a subset of C if $\partial\Sigma \neq \emptyset$). Here $\partial\Sigma$ may be compact or $\partial\Sigma$ may be non-compact and properly embedded in a one-sided tubular neighborhood of C bounded by C and an equidistant horocylinder $\tilde{C}$, with $\partial\Sigma$ contained in $\tilde{C}$.*

Our second main result concerns complete $H = 1/2$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$ transverse to the vertical Killing field $Z = \frac{\partial}{\partial t}$. We prove such surfaces are entire graphs.

Theorem 1.2. *Let Σ be a complete immersed surface in $\mathbb{H}^2 \times \mathbb{R}$ of constant mean curvature $H = 1/2$. If Σ is transverse to Z, then Σ is an entire vertical graph over $\mathbb{H}^2$.*

Finally, we apply Theorem 1.2, together with work of Fernandez–Mira [4], and Wan–Au [3] and Wan [9], to understand such entire graphs.

Theorem 1.3. *To each quadratic holomorphic differential on $\mathbb{C}$ or the unit disk, one associates an entire $H = 1/2$ graph.*

Our proof of Theorem 1.1 while in the same spirit as that of [5] is technically more complicated as we must prove the existence of continuous families of catenoid like $H = \frac{1}{2}$ surfaces that converge nicely to a horocylinder. For this purpose, it is convenient to use the half-space model $\mathbb{H}^2 = \{(x, y) : y > 0\}$ with metric $ds^2 = \frac{dx^2+dy^2}{y^2}$ so that the product space $\mathbb{H}^2 \times \mathbb{R}$ with coordinates (x, y, t) is endowed with the metric $d\sigma^2 = ds^2 + dt^2$. Following Sa Earp [7], we will consider "horizontal" graphs $y = g(x, t), g > 0$ with $H = \frac{1}{2}$. Some standard computations show that on $S = \text{graph } g$,

Lemma 1.4. *The coefficients of the metric are given by*

$$g_{11} = \frac{1}{g^2}(1 + {g_x}^2), \quad g_{12} = \frac{g_x g_t}{g^2}, \quad g_{22} = \frac{g^2 + g_t^2}{g^2},$$

$$g^{11} = \frac{g^2}{W^2}(g^2 + g_t^2), \quad g^{12} = -\frac{g^2}{W^2}(g_x g_t), \quad g^{22} = \frac{g^2}{W^2}(1 + g_x^2).$$

The coefficients of the second fundamental form are

$$b_{11} = \frac{1}{W}\left(g_{xx} + \frac{1 + g_x^2}{g}\right), \quad b_{12} = \frac{g_{xt}}{W}, \quad b_{22} = \frac{1}{W}\left(g_{tt} - \frac{g_t^2}{g}\right).$$

In addition,

$$|\nabla_S g|^2 = g^2\left(1 - \frac{g^2}{W^2}\right).$$

The mean curvature equation is given by

$$(1.1) \quad 1 = g^{ij} b_{ij} = \frac{g^2}{W^3}\{(g^2 + g_t^2)g_{xx} - 2g_x g_t g_{xt} + (1 + g_x^2)g_{tt} + g(1 + g_x^2)\},$$

and finally we have

$$\Delta_S g = \frac{g^2}{W}\left(1 - \frac{g}{W} + \frac{g g_x^2}{W}\right) > 0, \tag{1.2}$$

$$\Delta_S \frac{1}{g} = \frac{W - g}{gW} + \frac{g_t^2}{gW^2} > 0, \tag{1.3}$$

where $W^2 = g^2(1 + g_x^2) + g_t^2$.

In particular, we see from (1.1) that g satisfies the strange looking equation

$$(g^2 + g_t^2)g_{xx} - 2g_x g_t g_{xt} + (1 + g_x^2)g_{tt} = -g(1 + g_x^2) + \frac{W^3}{g^2}. \tag{1.4}$$

Also note the somewhat surprising result that both g and $\frac{1}{g}$ are subharmonic on S.

We remark that the mean curvature vector of S is given by

$$\vec{H} = \frac{1}{2W}(-g^2 g_x, g^2, -g_t)$$

and that the constant solutions $g = \tau > 0$ correspond to the horocylinders $C(\tau) = \{(x, y, t) : y = \tau\}$. The induced metric on each $C(\tau)$ is complete and isometric to the flat $\mathbb{R}^2$, and the mean curvature of each $C(\tau)$ is $H = 1/2$. Our catenoid-like horizontal graphs are given by the following theorem.

Theorem 1.5. *Let U be the annulus $U = B_{R_2} \backslash B_{R_1}$ with $R_2 \geq 4R_1$. Then for $\varepsilon > 0$ sufficiently small (depending only on R_1), there exists constant mean curvature $H = 1/2$ horizontal graphs g^+ and g^- satisfying (1.4) in U with Dirichlet boundary data $g^\pm = 1 \pm \varepsilon$ on $\partial B_{R_1}, g^\pm = 1$ on ∂B_{R_2}. Moreover, $g^\pm$ is unique and varies continuously with the parameters ε, R_1, R_2 and $g^\pm$ tends to $1 \pm \varepsilon$ uniformly on compact subsets as R_2 tends to ∞.*

Assuming Theorem 1.5, the proof of Theorem 1.1 goes as follows. After an isometry, we can assume that there is a sequence of points $p_i = (x_i, y_i, t_i) \in \Sigma$ with $y_i \to 1$ and $\langle \overrightarrow{H}, \frac{\partial}{\partial y} \rangle > 0$. Suppose that $\Sigma \subseteq \{y \geq 1\}$; then either Σ is contained in $C(1)$ or is contained in $y > 1$. For $\varepsilon > 0$ small, we consider the slab S^+ bounded by $C(1)$ and $C(1 + \varepsilon)$. Then by the maximum principle $\Sigma \cap S^+$ has a non-compact component Σ^+ with boundary $\partial \Sigma^+ \subset C(1 + \varepsilon)$. More precisely, let Σ^+ be any connected component of Σ in the slab with

$\partial\Sigma^+ \subset C(1+\varepsilon)$. We can assume Σ^+ is asymptotic to $C(1)$. Otherwise, one considers $\tau = \sup\{t > 0 : C(s) \cap \Sigma^+ = \emptyset \text{ for } s < t\}$. Then if $C(\tau)$ intersects Σ^+, they are equal by the usual maximum principle, a contradiction. Hence Σ^+ is asymptotic to $C(\tau)$ and we do the following argument with $C(1)$ replaced by $C(\tau)$. So we may assume Σ^+ is asymptotic to $C(1)$.

Let $D(\tau, R)$ denote the disk in $C(\tau)$ defined by $D(\tau, R) = \{(x, \tau, t) : x^2 + t^2 \le R^2\}$. We can find a disk $D(1, 3R_1)$ such that $D(1, 3R_1) \times [1, 1+\varepsilon] \cap S^+ = \{\emptyset\}$. Let $\Gamma(1, R) = \partial D(1, R)$. Then for each $R \ge 4R_1$, there is a horizontal graph g_R^+ bounded by $\Gamma(1+\varepsilon, R_1) \cup \Gamma(1, R)$ in the slab S^+. By the maximum principle, this family of graphs foliates the unbounded component of $S^+ \backslash \mathrm{graph}(g_{2R_1}^+)$ but converges to $C(1+\varepsilon)$. Thus there is a first point of contact at an interior point. Since the mean curvature vectors are pointing up, this violates the maximum principle and Σ^+ cannot exist. If $\Sigma \subseteq \{0 < y \le 1\}$, we redo exactly the same argument exchanging the roles of $C(1-\varepsilon)$ and $C(1+\varepsilon)$.

We will prove the existence part of Theorem 1.5 in Section 2 using the Schauder fixed point theorem. Because of the complicated dependence of Equation (1.4) on g, the uniqueness of the solutions is not obvious. This will be proved in Section 3 from which the continuous dependence follows by standard elliptic theory. The proof of Theorem 1.2 is given in Section 4 using compactness and analytic continuation arguments. Finally, in Section 5, we describe the construction of Fernandez and Mira [4] of entire $H = \frac{1}{2}$ vertical graphs starting from holomorphic quadratic differentials on $\mathbb{C}$ or the unit disk U.

2. The existence part of Theorem 1.5

Let $U = B_{R_2} \backslash B_{R_1}$ be an annulus with $R_2 \ge 4R_1$ and fix

$$h = 1 \pm \frac{\varepsilon}{\log(R_2/R_1)} \log \frac{R_2}{r}, \text{ where } r^2 = x^2 + t^2.$$

We expect the solution g to be close to h, so we define the weighted $C^{2+\alpha}$ norm

$$|v|^*_{2,\alpha;U} = \sup_X \{|v(X)| + r(X)|Dv(X)| + r^2(X)|D^2v(X)| + r_X^{2+\alpha}[D^2v]_{\alpha;X}\},$$

where $X = (x, t)$ and $[D^2v]_{\alpha;X}$ is the Hölder coefficient of D^2v at X.

Definition 2.1. We say g is an admissible solution of (1.4) if $g \in \mathcal{A}_\varepsilon$, where

$$\mathcal{A}_\varepsilon = \{g \in C^{2,\alpha}(U), g = h \text{ on } \partial U : |g-h|^*_{2,\alpha;U} \leq \sqrt{\varepsilon}\}.$$

We note that $\mathcal{A}_\varepsilon$ is a convex and compact subset of the Banach space $\mathcal{B} = C^{2,\beta}(U), \beta < \alpha$. We will reformulate our existence problem as a fixed point of a continuous operator $T : \mathcal{A}_\varepsilon \to \mathcal{A}_\varepsilon$ by rewriting Equation (1.4) in the form

(2.1)
$$(g^2+g_t^2)g_{xx} - 2g_xg_tg_{xt} + (1+g_x^2)g_{tt} + gg_x{}^2 - \left(\frac{W}{g^2} + \frac{1}{W+g}\right)(g^2g_x^2 + g_t^2) = 0$$

Remark 2.2. Note that (2.1) implies that any solution $g^\pm$ solving the Dirichlet problem of Theorem 1.5 satisfies $1-\varepsilon \leq g^- \leq 1$ and $1 \leq g^+ \leq 1+\varepsilon$ in U.

We now define the operator $w = Tg$ as the solution of the linear Dirichlet problem

$$\begin{aligned} L_g w &:= aw_{xx} + 2bw_{xt} + cw_{tt} + dw_x + ew_t = 0 \text{ in } U, \\ w &= h \text{ on } \partial U, \end{aligned} \tag{2.2}$$

where $a = g^2 + g_t^2, b = -g_xg_t, c = 1 + g_x^2, d = gg_x - g^2(\frac{W}{g^2} + \frac{1}{W+g})g_x$ and $e = -(\frac{W}{g^2} + \frac{1}{W+g})g_t$. Note that for $g \in \mathcal{A}_\varepsilon$, if $L_g u = f$ in $D \subset \overline{U}$, where D is "of scale R", (i.e., if $X \in D$, then $c_1R \leq |X| \leq c_2R$ for uniform constants c_1, c_2), then $\tilde{u} = u(RX)$ satisfies

$$\tilde{L}\tilde{u} = \tilde{a}\tilde{u}_{xx} + 2\tilde{b}\tilde{u}_{xt} + \tilde{c}\tilde{u}_{tt} + R\tilde{d}\tilde{u}_x + R\tilde{e}\tilde{u}_t = R^2\tilde{f} \text{ in } \tilde{D}, \tag{2.3}$$

where $\tilde{D}$ is of scale 1 and $\tilde{a}(X) = a(RX), \tilde{b}(X) = b(RX)$, etc. Hence for ε sufficiently small, $\tilde{L}$ is uniformly close to Δ with Hölder continuous coefficients.

Proposition 2.3. *Let $w = Tg$ for $g \in \mathcal{A}_\varepsilon$. Then for ε sufficiently small, $w \in \mathcal{A}_\varepsilon$.*

Proof. Set $u = w - h$; then

$$L_g u = [(1-g^2-g_t^2)h_{xx} + 2g_xg_th_{xt} - g_x^2h_{tt} - dh_x - eh_t] := f. \tag{2.4}$$

By the maximum principle, $1 \leq w \leq 1+\varepsilon$ (or $1-\varepsilon \leq w \leq 1$) so $|u| \leq \varepsilon$.

We now write $U = U_1 \cup U_2 \cup U_3$ where

$$U_1 = \{X : R_1 \leq |X| \leq \tfrac{3}{2}R_1\},$$
$$U_2 = \{X : \tfrac{3}{2}R_1 < |X| < \tfrac{3}{4}R_2\},$$
$$U_3 = \{X : \tfrac{3}{4}R_2 \leq |X| \leq R_2\}.$$

Fix $Y \in U$. If $Y \in U_1$, then $B_{R_1/4}(R_1\frac{Y}{|Y|}) \cap U \subset B_{R_1/2}(R_1\frac{Y}{|Y|}) \cap U \subset U_1$. If $Y \in U_3$, then $B_{R_2/4}(R_2\frac{Y}{|Y|}) \cap U \subset B_{R_2/2}(R_2\frac{Y}{|Y|}) \cap U \subset U_3$. Finally if $Y \in U_2$, then $B_{R/16}(Y) \subset B_{R/8}(Y) \subset U$ for $R = |Y|$. So each of the three domains is of scale R and we can apply Schauder interior or boundary estimates to $\tilde{L}\tilde{u} = R^2\tilde{f}$ in $\tilde{D}$ to obtain

$$\|\tilde{u}\|_{2,\alpha;\tilde{D}} \leq C(\|\tilde{u}\|_{0;\tilde{D}} + \|R^2\tilde{f}\|_{0,\alpha;\tilde{D}}\|) \leq C\varepsilon, \tag{2.5}$$

since from (2.4) follows $||\tilde{f}||_{0,\alpha;\tilde{D}} \leq C\varepsilon^{3/2}$. Undoing the scaling gives

$$\|u\|^*_{2,\alpha;D} \leq C\varepsilon.$$

Since $u = w - h$, it follows that for ε small enough, $w \in \mathcal{A}_\varepsilon$ and the proposition is proved.

We are now in a position to apply the Schauder fixed point theorem to our operator $w = Tg$ to find a solution $g^\pm \in \mathcal{A}$ to (2.1) which is equivalent to our original Equation (1.4). □

3. Completion of the proof of Theorem 1.5

In this section, we refine our estimates in order to prove uniqueness, continuous dependence and convergence to a constant as $R_2 \to \infty$.

Let $g^\pm$ be an admissible solution of (1.4) and let ϕ satisfy

$$\Delta_S \phi = 0 \text{ in } U,$$
$$\phi = 1 \text{ on } \partial B_{R_1},$$
$$\phi = 0 \text{ on } \partial B_{R_2}.$$

Then since $g^\pm = 1 \pm \varepsilon\phi$ on ∂U and both $g^\pm$ and $\frac{1}{g^\pm}$ are subharmonic on S (see Lemma 1.4), we have

$$\frac{1}{1 \mp \varepsilon\phi/(1 \pm \varepsilon)} \leq g^\pm \leq 1 \pm \varepsilon\phi. \tag{3.1}$$

Proposition 3.1. $0 \le 1 - \phi \le C\frac{\log(r/R_1)}{\log(R_2/R_1)}$ *where* C *is independent of* R_2.

Proof. On U, ϕ satisfies the uniformly elliptic divergence form equation

$$L\phi := \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}(a^{ij}\phi_{x_j}) = 0, \tag{3.2}$$

where $a^{ij} = g^{-2}Wg^{ij}$ is close to δ_{ij}. We extend $g \equiv 1$ for $|X| > R_2$ so L is also extended as a uniformly elliptic divergence form operator with Lipschitz continuous coefficients. We now recall the generalized Kelvin transform (see [8] p.262). Let $Y = \frac{X}{|X|^2}$ be inversion in the unit circle mapping U to the annulus $\tilde{U} = \{Y : \frac{1}{R_2} \le |Y| \le \frac{1}{R_1}\}$ and define $\tilde{\phi}(Y) = \phi(X)$. Then $\tilde{\phi}$ satisfies the uniformly elliptic divergence form equation

$$\tilde{L}\tilde{\phi} := \sum_{i,j=1}^{2} \frac{\partial}{\partial y_i}(\tilde{a}^{ij}\tilde{\phi}_{y_j}) = 0 \tag{3.3}$$

in $\tilde{U}$, where

$$\tilde{a}^{kl}(Y) = a^{ij}(X)\left(\delta_{ki} - 2\frac{x_k x_i}{|X|^2}\right)\left(\delta_{lj} - 2\frac{x_l x_j}{|X|^2}\right). \tag{3.4}$$

Note that because the matrix $(\delta_{ki} - 2\frac{x_k x_i}{|X|^2})$ is orthogonal, the eigenvalues of $\tilde{a}^{kl}(Y)$ are the same as those of $a^{ij}(X)$.

Now let $\tilde{G}(Y)$ be the positive Green's function for $\tilde{L}$ in B_{1/R_1} with pole at the origin. Then by the maximum principle,

$$1 - \tilde{\phi} \le \frac{\tilde{G}(Y)}{\min_{|Y|=1/R_2}\tilde{G}} \le C\frac{\log 1/R_1|Y|}{\log R_2/R_1}, \tag{3.5}$$

where C is universal. This last inequality is the classical comparison theorem of Littman, Stampacchia and Weinberger for fundamental solutions [6]. Returning to the original variables gives the desired inequality. □

Combining inequality (3.1) and Proposition 3.1 gives

Corollary 3.2. Let $g^{\pm}$ be an admissible solution of (1.4). Then

$$(1+\varepsilon)\left(1 - C\varepsilon\frac{\log(r/R_1)}{\log(R_2/R_1)}\right) \le g^+ \le 1+\varepsilon,$$
$$1-\varepsilon \le g^- \le 1-\varepsilon + C\varepsilon\frac{\log(r/R_1)}{\log(R_2/R_1)}$$

We use Corollary 3.2 to prove the uniqueness of admissible solutions completing the proof of Theorem 1.5. (Note that Corollary 3.2 shows that $g^{\pm}$ tends to $1 \pm \varepsilon$ uniformly on compact subsets as R_2 tends to ∞.)

Proposition 3.3. *For $\varepsilon = \varepsilon(R_1)$ sufficiently small, the solutions $g^{\pm}$ are unique.*

Proof. Note that if g is a solution of (1.4) in U, then $g^{\lambda} = \frac{1}{\lambda} g(\lambda x, t)$ is also a solution in $U^{\lambda} = \{(x,t) : R_1^2 \le \lambda^2 x^2 + t^2 \le R_2^2\}$. To fix the ideas, we give the proof of uniqueness for g^+, which is straightforward. Suppose that g_1 and g_2 are two admissible solutions (we drop the + for convenience). Then for $\lambda = \frac{1}{1+\varepsilon}$, $g_2^{\lambda} > g_1$ in $U \cap U^{\lambda}$. As we increase λ back toward $\lambda = 1$, $g_2^{\lambda} > g_1$ on $\partial(U \cap U^{\lambda})$. Thus a first contact may only occur in the interior, which is impossible by the maximum principle. Thus $g_2 > g_1$ in U. Reversing the roles of g_1 and g_2 proves uniqueness.

Observe that this argument seems to have a problem when we consider g^- because it might happen that there is a first point of contact on the inner boundary of $U \cap U^{\lambda}$, i.e.,

$$g_2^{\lambda} = \frac{1-\varepsilon}{\lambda} = g_1 \text{ for some point on } \lambda^2 x^2 + t^2 = R_1^2, \tag{3.6}$$

We show this cannot happen. For at such a contact point, $\lambda^2(x^2+t^2) = R_1^2 + (\lambda^2 - 1)t^2 \le R_1^2$, so $r \le \frac{R_1}{\lambda}$. Hence by (3.6) and Corollary 3.2, we have

$$\frac{1-\varepsilon}{\lambda} = g_1 \le 1 - \varepsilon + C\varepsilon\frac{\log(1/\lambda)}{\log(R_2/R_1)} \le 1-\varepsilon + \frac{C\varepsilon}{\log(R_2/R_1)}\left(\frac{1}{\lambda} - 1\right).$$

Therefore,

$$1-\varepsilon \le \frac{C\varepsilon}{\log(R_2/R_1)}, \tag{3.7}$$

which is impossible if ε is chosen so that $\frac{\varepsilon}{1-\varepsilon} < \frac{\log 2}{C}$. Thus there is no first contact on the boundary and $g_2 > g_1$ in U. This proves uniqueness and also the continuous dependence in the parameters. In particular, we can now let $g^{\pm}$ be an arbitrary solution which is not necessarily admissible. We do know however (see Remark 2.2) that $1-\varepsilon < g^- < 1$ and $1 < g^+ < 1+\varepsilon$ in U. Now let $g_2^{\pm}$ be our continuous family of admissible solutions. If we make R_2 very large, $g_2^+ > g^+$ and $g_2^- < g^-$ when restricted to the original annulus by Corollary 3.2. Thus decreasing R_2 back to its original value shows these inequalities persist. Similarly we can decrease the parameter ε close to zero and obtain the reverse inequalities proving the general uniqueness of solutions. $\square$

4. Proof of Theorem 1.2

Theorem 1.2. *Let Σ be a complete immersed surface in $\mathbb{H}^2 \times \mathbb{R}$ of constant mean curvature $H = 1/2$. If Σ is transverse to Z, then Σ is an entire vertical graph over $\mathbb{H}^2$.*

Proof. In the following, it is convenient to think of $\mathbb{H}^2$ as the unit disk $B_1(0)$ with the Poincaré metric. The mean curvature vector of Σ never vanishes, so Σ is orientable. Let ν be a unit vector field along Σ in $\mathbb{H} \times \mathbb{R}$. The function $u = \langle \nu, Z \rangle$ is a non-zero Jacobi function on Σ, so Σ is strongly stable and thus has bounded curvature [10]. We can assume $u > 0$ and $\left\langle \nu, \overrightarrow{H} \right\rangle > 0$.

Hence there is $\delta > 0$ such that for each $p \in \Sigma$, Σ is a graph (in exponential coordinates) over the disk $D_\delta \subset T_p\Sigma$ of radius δ, centered at the origin of $T_p\Sigma$. This graph, denoted by $G(p)$, has bounded geometry. The δ is independent of p and the bound on the geometry of $G(p)$ is uniform as well.

We denote by $F(p)$ the surface $G(p)$ translated to the origin $O \in \mathbb{H}^2 \equiv \mathbb{H}^2 \times \{0\}$ (The translation that takes p to O).

For $q \in \mathbb{H}^2 \times \mathbb{R}$, we denote by $\Gamma_\delta(q)$ a horizontal horocycle arc of length 2δ, centered at q.

Claim 1. Let $p_n \in \Sigma$, satisfy $u(p_n) \to 0$ as $n \to \infty$ ($T_{p_n}(\Sigma)$ are becoming vertical). There is a subsequence of p_n (which we also denote by $\{p_n\}$) such that $F(p_n)$ converges to $\Gamma_\delta(O) \times [-\delta, \delta]$, for some horocycle $\Gamma_\delta(O)$. The convergence is in the C^2-topology.

Proof of Claim 1. Choose a subsequence p_n so that the oriented tangent planes $T_O(F(p_n))$ converge to a vertical plane P. Let $\Gamma_\delta(O)$ be the horocycle arc through O whose curvature vector has the same direction as the curvature vector of the (limit) curvature vectors of $F(p_n)$.

Since the $F(p_n)$ have bounded geometry and they are graphs over $D_\delta(p_n) \subset T_{p_n}(F(p_n))$, the surfaces $F(p_n)$ are bounded horizontal graphs over $\Gamma_\delta(O) \times [-\delta, \delta]$ for n large. Thus a subsequence of these graphs converges to an $H = 1/2$ surface F; F is tangent to $\Gamma_\delta(O) \times [-\delta, \delta]$ at O and a horizontal graph over this. It suffices to show $F = \Gamma_\delta(O) \times [-\delta, \delta]$.

If this was not the case, then the intersection near O, of F and $\Gamma_\delta(O) \times [-\delta, \delta]$ would consist of m smooth curves passing through O, $m \geq 2$, meeting transversally at O. In a neighborhood of O, these curves separate F into $2m$ components. Adjacent components lie on opposite sides of $\Gamma_\delta(O) \times [-\delta, \delta]$.

Hence in a neighborhood of O in F, the mean curvature vector of F alternates from pointing up in $\mathbb{H}^2 \times \mathbb{R}$ to pointing down (or vice-versa), as one goes from one component to the other. But $F(p_n)$ converges to F in the C^2-topology, so $F(p_n)$, n large, would also have points where the mean curvature vector points up and down in $\mathbb{H}^2 \times \mathbb{R}$. This contradicts that $F(p_n)$ is transverse to Z, and Claim 1 is proved. Notice that we have proved that whenever $F(p_n)$ converges to a local surface F, F is necessarily some $\Gamma_\delta(O) \times [-\delta, \delta]$. This proves Claim 1. $\square$

Now let $p \in \Sigma$ and assume Σ in a neighborhood of p is a vertical graph of a function f defined on B_R, B_R the open ball of radius R of $\mathbb{H}^2$, centered at $O \in \mathbb{H}^2$. Denote by $S(R)$ the graph of f over B_R. If Σ is not an entire graph, then we let R be the largest such R so that f exists. Since Σ has constant mean curvature, f has bounded gradient on relatively compact subsets of B_R.

Let $q \in \partial B_R$ be such that f does not extend to any neighborhood of q (to an $H = 1/2$ graph).

Claim 2. For any sequence $q_n \in B_R$, converging to q, the tangent planes $T_{p_n}(S(R))$, $p_n = (q_n, f(q_n))$, converge to a vertical plane P. P is tangent to ∂B_R at q (after vertical translation to height zero in $\mathbb{H}^2 \times \mathbb{R}$).

Proof of Claim 2. Let $F(n)$ denote the image of $G(p_n)$ under the vertical translation taking p_n to q_n. Observe first, that $T_{q_n}(F(n))$ converges to the vertical, for any subsequence of the q_n. Otherwise the graph of bounded geometry $G(p_n)$, would extend to a vertical graph beyond q, for q_n close enough to q, hence f would extend, a contradiction.

Now we can prove $T_{q_n}(F_n)$ converges to the vertical plane P passing through q and tangent to ∂B_R at q. Suppose some subsequence q_n satisfies $T_{q_n}(F_n)$ converges to a vertical plane Q, $Q \neq P$, $q \in Q$. By Claim 1, the F_n converge in the C^2-topology, to $\Gamma_\delta(q) \times [-\delta, \delta]$, where $\Gamma_\delta(q)$ is a horocycle arc centered at q. Since $Q \neq P$, and $\Gamma_\delta(q)$ is tangent to Q at q, there are points of $\Gamma_\delta(q)$ in B_R. Such a point is the limit of points on F_n. Then the gradient of f at these points of F_n diverges, which contradicts interior gradient estimates of f. This proves Claim 2. $\square$

Now applying Claim 1 and Claim 2, we know that for any sequence $q_n \in B_R$ converging to q, the $F(q_n)$ converge to $\Gamma_\delta(q) \times [-\delta, \delta]$.

Claim 3. For any $q_n \longrightarrow q$, $q_n \in B_R$, we have $f(q_n) \longrightarrow +\infty$ or $f(q_n) \longrightarrow -\infty$.

Proof of Claim 3. Let γ be a compact horizontal geodesic of length ε starting at q, entering B_R at q, and orthogonal to ∂B_R at q. Let C be the graph of f over γ. Notice that C has no horizontal tangents at points near q since the tangent planes of $S(R)$ are converging to P. So assume f is increasing along γ as one converges to q. If f were bounded above, then C would have a finite limit point (q, c) and C would have finite length up till (q, c). Since Σ is complete, $(q, c) \in \Sigma$. But then Σ has a vertical tangent plane at (q, c), a contradiction. This proves Claim 3. $\square$

Now choose $q_n \in \gamma$, $q_n \longrightarrow q$, and $F(q_n)$ converges to $\Gamma_\delta(q) \times [-\delta, \delta]$. Let Γ be the horocycle containing $\Gamma_\delta(q)$, and parameterize Γ by arc length; denote $q(s) \in \Gamma$ the point at distance s on Γ from $q = q(0)$, $-\infty < s < +\infty$. Denote by $\gamma(s)$ a horizontal geodesic arc orthogonal to Γ at $q(s)$, $q(s)$ the mid-point of $\gamma(s)$. Assume the length of each $\gamma(s)$ is 2ε and $\cup_{s\in\mathbb{R}}\gamma(s) = N_\varepsilon(\Gamma)$ is the ε-tubular neighborhood of Γ.

Let $\gamma^+(s)$ be the part of $\gamma(s)$ on the mean convex side of Γ; so $\gamma = \gamma^+(0)$. More precisely, the mean curvature vector of Σ points up in $\mathbb{H}^2 \times \mathbb{R}$, and $f \longrightarrow +\infty$ as one approaches q along γ, so Γ is convex towards B_R.

Claim 4. For n large, each $F(q_n)$ is disjoint from $\Gamma \times \mathbb{R}$. Also, for $|s| \leq \delta$, $F(q_n) \cap \gamma^+(s)$ is a vertical graph over an interval of $\gamma^+(s)$.

Proof of Claim 4. Choose n_0 so that for $n \geq n_0$, $C_n(s) = F(q_n) \cap (\gamma(s) \times \mathbb{R})$ is one connected curve of transverse intersection, for each $s \in [-\delta, \delta]$. Since the $F(q_n)$ are C^2-close to $\Gamma_\delta(q) \times [-\delta, \delta]$, $C_n(s)$ has no horizontal or vertical tangents and is a graph over an interval in $\gamma(s)$.

We now show this interval is in $\gamma^+(s) - q(s)$. Suppose not, so $C_n(s)$ goes beyond $\Gamma \times \mathbb{R}$ on the concave side. Recall that $C = \gamma \cap P^\perp$ is the graph of f and $f \longrightarrow +\infty$ as one goes up on C. We have $p_n = (q_n, f(q_n))$. Fix $n \geq n_0$ and choose new points q_k, $k \geq n$, so that $f(q_{k+1}) - f(q_k) = \delta$; clearly $q_k \longrightarrow q$ as $k \longrightarrow \infty$. Lift each $C_k(s)$ to $G(p_k)$ by the vertical translation of $F(q_k)$ by $f(q_k)$. By construction, $C_{k+1}(s)$ is the analytic continuation of $C_k(s)$ in $\Sigma \cap (\gamma(s) \times \mathbb{R})$, for each $s \in [-\delta, \delta]$, and for all $k \geq n+1$. The curve $C(s) = \cup_{k \geq n} C_k(s)$ is a vertical graph over an interval in $\gamma(s)$. It has points on the concave side of $\Gamma \times \mathbb{R}$ for some $s_0 \in [-\delta, \delta]$. For $s = 0$, $C(0) = C$ stays on the convex side of $\Gamma \times \mathbb{R}$. So for some s_1, $0 < s_1 \leq s_0$, $C(s_1)$ has a point on $\Gamma \times \mathbb{R}$ and also inside the concave side of $\Gamma \times \mathbb{R}$.

But the $F(q_k)$ converge uniformly to $\Gamma_\delta(q) \times [-\delta, \delta]$ as $k \longrightarrow \infty$, so the curve $C(s_1)$ converges to $q(s_1) \times \mathbb{R}$ as the height goes to ∞. This obliges $C(s_1)$ to have a vertical tangent on the concave side of $\Gamma \times \mathbb{R}$, a contradiction. This proves Claim 4. □

Now we choose an $\varepsilon_1 < \varepsilon$ (which we call ε as well) so that $\cup_{s \in [-\delta, \delta]} C(s)$ is a vertical graph of a function g on $\cup_{s \in [-\delta, \delta]} (\gamma^+(s) - q(s))$, (the $\gamma^+(s)$ now have length ε_1); g is an extension of f.

The graph of g on each $\gamma^+ \times \mathbb{R}$ is the curve $C(s)$, and the graph of g converges to $\Gamma_\delta(q) \times \mathbb{R}$ as the height goes to infinity.

Now we begin this process again replacing C by the curves $C(\delta)$ and then $C(-\delta)$. Analytic continuation yields an extension h of g to a domain Ω contained in the open ε-tubular neighborhood of $\Gamma \times \mathbb{R}$, on the convex side of Γ. Ω is an open neighborhood of Γ in this mean convex side. The graph $h \longrightarrow \infty$ as one approaches Γ in Ω; it converges to $\Gamma \times \mathbb{R}$ as the height goes to infinity.

Claim 5. There is an $\varepsilon > 0$, such that Ω contains the ε tubular neighborhood of Γ on the convex side.

Proof of Claim 5. The surface Σ contains a graph over Ω, composed of curves $C(q)$, $q \in \Gamma$, where each curve $C(q)$ is a graph over an interval $\gamma^+(q)$, $\gamma^+(q)$ orthogonal to Γ at q. Also $C(q)$ is a strictly monotone increasing graph with no horizontal tangents and $C(q)$ converges to $\{q\} \times \mathbb{R}^+$, as one goes up to $+\infty$; cf. figures 1 and 2.

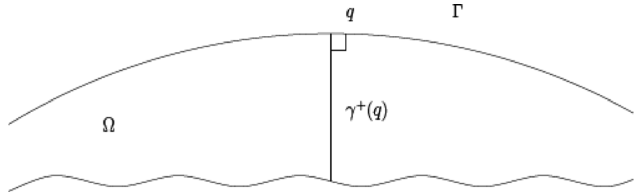

Figure 1

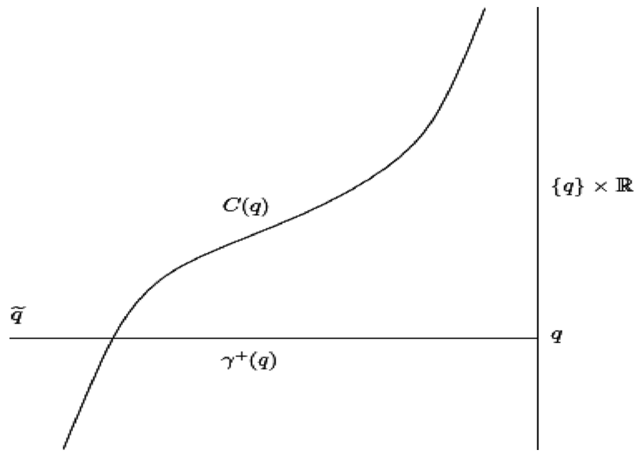

Figure 2

The graph over Ω is converging uniformly to $\Gamma \times \mathbb{R}^+$ as one goes up.

Now suppose that for some $q \in \Gamma$, $\gamma^+(q)$ is of length less than ε. Then $C(q)$ diverges to $-\infty$ as one approaches the end-point $\tilde{q}$ of $\gamma^+(q)$, $\tilde{q} \neq q$; cf. figure 2.

The previous discussion where we showed the graph over Ω exists and converges to $\Gamma \times \mathbb{R}^+$, now applies to show that there is a horocycle $\widetilde{\Gamma}$ passing through $\widetilde{q}$, $C(q)$ converges to $\{\widetilde{q}\} \times \mathbb{R}^-$ as one tends to $\widetilde{q}$ on $\gamma^+(q)$. Also a δ-neighborhood of $C(q)$ in Σ converges uniformly to $\widetilde{\Gamma}_\delta(\widetilde{q}) \times \mathbb{R}^-$, as one goes down to $-\infty$. We know this δ-neighborhood of $C(q)$ in Σ converges uniformly to $\Gamma_\delta(q) \times \mathbb{R}^+$ as one goes up to $+\infty$. $\square$

For each $q(s) \in \Gamma$, a distance s from q on Γ, $|s| \leq \delta$, the curve $C(q(s))$ converges uniformly to some $\{\tilde{q}(\tilde{s})\} \times \mathbb{R}^-$ as one goes down to $-\infty$. By analytic continuation of the δ-neighborhoods, one continues this process along γ.

If $\Gamma \cap \tilde{\Gamma} = \emptyset$, then the process continues along all Γ, and Ω is the region bounded by $\Gamma \cup \tilde{\Gamma}$, which has has constant width; Γ and $\tilde{\Gamma}$ are equidistant.

So we can assume $\Gamma \cap \tilde{\Gamma} = \{p\}$. Consider the curves $C(q(s))$ as $q(s)$ goes from q to p along Γ. They are graphs that become vertical both at $+\infty$ and $-\infty$. Hence the graphs $C(q(s))$ become vertical at every point as $q(s) \to p$; cf. figure 3.

Consider the point of $C(q(s))$ at height 0 in $\mathbb{H}^2 \times \mathbb{R}$. As $q(s) \to p$, these points converge to a point of Σ and the tangent plane of Σ is vertical at this point, a contradiction.

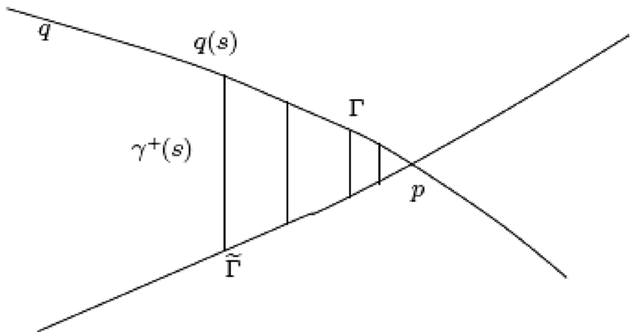

Figure 3

We remark that in the case $f(q_n) \longrightarrow -\infty$ (see Claim 3), one works on the concave side of the horocycle $\Gamma(q)$ and Claims 4 and 5 show there is an $\varepsilon > 0$ and a graph $G \subset \Sigma$ over the domain $\Omega(\varepsilon)$ between $\Gamma(\varepsilon)$ (the equidistant horocycle to Γ on the concave side of Γ) and Γ. The graph G converges uniformly to $\Gamma \times \mathbb{R}$ as one approaches Γ in $\Omega(\varepsilon)$. □

Now Claim 5 contradicts Theorem 1.1 since the graph over Ω contains a properly embedded H $= 1/2$ surface in the slab between $C(1)$ and $C(1+\varepsilon)$ with boundary contained in $C(1+\varepsilon)$, which is asymptotic to $C(1)$.

5. Quadratic holomorphic differentials, harmonic maps and entire graphs $H = 1/2$

We will now describe how to obtain entire $H = 1/2$ graphs starting with a holomorphic quadratic differential $Q = \phi(z)dz^2$. This originates from the work of Fernandez–Mira [4], Wan [9] and Au–Wan [3].

Abresch and Rosenberg [1,2] constructed a holomorphic quadratic differential Q_0 associated to the surface; this Q_0 generalizes the Hopf differential associated to constant mean curvature surfaces of $\mathbb{R}^3$. When $H = 1/2$ and the surface is a graph, Fernandez–Mira [4] proved there exists a harmonic map from the surface to $\mathbb{H}^2$ whose associated holomorphic quadratic differential is $Q = -Q_0$. In addition, given a harmonic map G from a surface to $\mathbb{H}^2$ plus some additional data (described below), they construct graphs $H = 1/2$ on $\mathbb{H}^2 \times \mathbb{R}$ with this harmonic map as Gauss map.

For a given holomorphic $Q = \phi(dz)^2$, Wan [9] on the disk, Wan and Au [3] for $\mathbb{C}$, construct a unique harmonic map $G : \Sigma \longrightarrow \mathbb{H}^2$ such that the

Jacobian $J(G) \geq 0$ and the metric $\tau|dz| := 4(\sigma \circ G)^2|G_z|^2$ is complete. To do that, they construct a CMC $H = 1/2$ in $M^{2,1}$, the Minkovski space with Gauss map G and metric $\tau|dz|^2$.

Let $G : \Sigma \longrightarrow \mathbb{H}^2$ be a harmonic map where Σ is $\mathbb{C}$ or the unit disk. Then $Q(z) = \phi\, dz^2$ is a quadratic holomorphic differential associated to G by the relation $\phi = (\sigma \circ G)^2 G_z G_{\bar{z}}$. Here we note $\mathbb{H}^2 = (D^2, \sigma)$, where σ is the conformal factor of the hyperbolic metric on the disk. We define the function $\omega = \frac{1}{2}\log\frac{|G_z|}{|G_{\bar{z}}|}$ and we express the Jacobian:

$$J(G) = \sigma^2(|G_z|^2 - |G_{\bar{z}}|^2) = 2\sinh(2\omega)|\phi|.$$

Fernandez–Mira construct multi-graph immersions $\psi : \Sigma \longrightarrow \mathbb{H}^2 \times \mathbb{R}$ with $H = 1/2$, depending on the data $\{Q, \tau\}$; τ as above. We note the unit normal vector of ψ by $\eta = (\widehat{N}, u)$, with $0 < |u| \leq 1$. They show that the metric $ds^2 = \lambda|dz|^2$ can be expressed as

$$\lambda = \frac{2\tau}{u^2} = 2\tau + 4|h_z|^2 \quad \text{and} \quad u = \sqrt{\frac{\tau}{\tau + 2|h_z|^2}},$$

where h is the solution of a differential equation depending on τ and ϕ. By the above relation between λ and τ, it is clear that the metric $ds = \lambda|dz|^2$ is complete.

Thus associated to a holomorphic quadratic differential $Q = \phi(z)dz^2$, one obtains a complete multi-graph $H = 1/2$ in $\mathbb{H}^2 \times \mathbb{R}$; hence an entire graph by Theorem 1.2. We give an independent proof below that the curvature K_λ is bounded (using the fact that the Jacobian of G is non-negative). This condition is $\omega \geq 0$ on Σ.

Lemma 5.1. *If G satisfies $J(G) > 0$ and $\tau = 4(\sigma \circ G)^2|G_z|^2$ is non-zero, then the curvature of the associate constant mean curvature $H = 1/2$ immersion ψ in $\mathbb{H}^2 \times \mathbb{R}$ is bounded*:

$$|K_\lambda| \leq C.$$

Proof. In the Fernandez–Mira paper we have (Formula (2.5)), for the metric $ds^2 = \lambda|dz|^2$ of the immersion ψ, with mean curvature H:

$$\lambda(\log\lambda)_{z\bar{z}} = 2(|p|^2 - \lambda^2(H^2 - 1)/4 - \lambda|h_z|^2).$$

Here $p\,dz^2 = -\langle \psi_z, \eta_z \rangle\, dz^2$ is the Hopf differential of ψ (the (2,0)-part of its complexified second fundamental form). Moreover, we have $\phi = 2Hp + h_z^2$ (see [4]). Then with $H = 1/2$ and $\frac{|h_z|^2}{\lambda} = \frac{1-u^2}{4} \leq 1/4$:

$$\begin{aligned} |K_\lambda| &= \frac{1}{2\lambda}|(\log\lambda)_{z\bar{z}}| \leq \frac{|p|^2}{\lambda^2} + \frac{3}{16} + \frac{|h(z)|^2}{\lambda} \\ &\leq \frac{|p + h_z^2|^2}{\lambda^2} + \frac{7}{16} + \frac{|h(z)|^4}{\lambda^2} \leq \frac{1}{2} + \frac{4u^4|\phi|^2}{\tau^2}. \end{aligned}$$

Notice that $\frac{4u^4|\phi|^2}{\tau^2} = \frac{u^4}{4e^{4\omega}} \leq C$ since $\omega \geq 0$. $\square$

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