

Integral cohomology of Hilbert schemes of points on surfaces

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Using Heisenberg algebra operators and interplay with the ring of symmetric functions, we verify a conjecture raised in [15]. As applications, we obtain linear bases and ring generators for the integral cohomology of the Hilbert schemes of points on a smooth projective surface with no odd cohomology.

1. Introduction

An additive basis for the *rational* cohomology group

$$H^*(X^{[n]}) = \bigoplus_{i=0}^{4n} H^i(X^{[n]})$$

of the Hilbert scheme $X^{[n]}$ of n points on a projective surface X can be given using the results of Göttsche, Grojnowski and Nakajima ([1–3]). When X is a projective plane, Ellingsrud and Strømme [4] found a set of ring generators for the *integral* cohomology ring $H^*(X^{[n]}; \mathbb{Z})$, which essentially comes from the Chern classes of certain tautological bundles over $X^{[n]}$ (see [5] for a generalization). If X is an arbitrary projective surface, a set of ring generators for the rational cohomology ring $H^*(X^{[n]})$ is found in [6] (see also [7–11]). The result is for rational coefficients since the Chern characters of the tautological bundles were used in [6]. Recently, Markman [12–14] used the integral cohomology ring $H^*(X^{[n]}; \mathbb{Z})$ to study the weight-2 Hodge structure on $H^2(Y; \mathbb{Z})$ when X is a $K3$ surface and Y is a hyperkähler manifold deformation equivalent to $X^{[n]}$. It is interesting to search for additive bases and ring generators of the integral cohomology ring $H^*(X^{[n]}; \mathbb{Z})$ for a general surface X . In [15], an effective method for finding integral additive bases was developed. The main idea there is to study *integral operators*, i.e., linear operators on $H^*(X^{[n]})$ which send integral classes in $H^*(X^{[n]})$ to integral ones.

Our goal in this paper is to determine both integral additive bases and integral ring generators when the odd Betti numbers of the surface X vanish. To state our results, we introduce some notations first. Let

$$\mathfrak{a}_i(\alpha), \quad i \in \mathbb{Z}, \alpha \in H^*(X)$$

be the Heisenberg operators constructed in [16]. For a partition λ of n and a smooth curve C in X , we can define the subvariety $L^\lambda C \subset X^{[n]}$ (see (3.1) and [2, 16]). By Corollary 9.15 in [16], the fundamental class $[L^\lambda C]$ can be written as $\mathfrak{m}_{\lambda,C}|0\rangle$ where $\mathfrak{m}_{\lambda,C}$ is a polynomial of creation operators $\mathfrak{a}_{-i}(C)$, $i > 0$. Therefore we can define the operator $\mathfrak{m}_{\lambda,\alpha}$ for an arbitrary class $\alpha \in H^2(X)$ simply by replacing the creation operators $\mathfrak{a}_{-i}(C)$, $i > 0$ appearing in $\mathfrak{m}_{\lambda,C}$ by the corresponding creation operators $\mathfrak{a}_{-i}(\alpha)$, $i > 0$. By Theorem 4.5 in [15], $\mathfrak{m}_{\lambda,\alpha}$ is integral when α is a divisor. It was conjectured in [15] that the linear operator $\mathfrak{m}_{\lambda,\alpha}$ is integral whenever $\alpha \in H^2(X)$ is an integral cohomology class.

In Section 3, we confirm the above conjecture, i.e., we prove that the operator $\mathfrak{m}_{\lambda,\alpha}$ is integral for every partition λ and every integral class $\alpha \in H^2(X)$. The main idea is to use topological K -theory and relations between the Hilbert schemes and the ring of symmetric functions. As an immediate consequence, we obtain the following (see the conventions below, (2.6) and (2.7) for notations).

Theorem 1.1. *Let X be a smooth projective surface with vanishing odd Betti numbers. Let $\alpha_1, \dots, \alpha_k$ be an integral basis for $H^2(X; \mathbb{Z})/\text{Tor}$. Then the classes*

$$(1.1) \quad \frac{1}{\mathfrak{z}^\lambda} \mathfrak{a}_{-\lambda}(1) \mathfrak{a}_{-\mu}(x) \mathfrak{m}_{\nu^1, \alpha_1} \cdots \mathfrak{m}_{\nu^k, \alpha_k} |0\rangle, \quad |\lambda| + |\mu| + \sum_{i=1}^k |\nu^i| = n$$

are integral, and furthermore, they form an integral basis for $H^*(X^{[n]}; \mathbb{Z})/\text{Tor}$.

Theorem 1.1, which was also conjectured in [15], generalizes the corresponding statement in the Theorem 5.4 of [15] under the assumption that $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$.

In Section 4, we study the integral ring generators. Instead of using the Chern characters as in [6], we use the Chern classes of the tautological vector bundles $L^{[n]}$ over the Hilbert scheme $X^{[n]}$ where L denotes line bundles over X (see (3.10)). We investigate in details various relations among the Chern

characters and Chern classes of $\mathcal{O}_X^{[n]}$, Heisenberg operators, and monomial symmetric functions. Using the techniques developed in [15,17,18], we prove our second theorem.

Theorem 1.2. *Suppose that the smooth projective surface X has vanishing odd Betti numbers. Let $\alpha_1, \dots, \alpha_k$ be an integral basis for $H^2(X; \mathbb{Z})/\text{Tor}$. Then the integral ring $H^*(X^{[n]}; \mathbb{Z})/\text{Tor}$ is generated over \mathbb{Z} by all the integral classes:*

$$(1.2) \quad c_i(\mathcal{O}_X^{[n]}), \quad \mathbf{1}_{-(n-j)}\mathbf{m}_{(1^j), \alpha_s}|0\rangle, \quad \mathbf{1}_{-(n-j)}\mathbf{a}_{-j}(x)|0\rangle,$$

where $1 \leq i \leq n-1$, $1 \leq j \leq n$, and $1 \leq s \leq k$.

The tautological rank- n bundle $\mathcal{O}_X^{[n]}$ is defined by (3.10). The fundamental reason for the appearance of the classes $\mathbf{1}_{-(n-j)}\mathbf{m}_{(1^j), \alpha_s}|0\rangle$ in the list (1.2) is the well-known fact (see [19]) that the monomial symmetric functions $m_{(1^i)}, i \geq 1$ corresponding to the partitions $(1^i), i \geq 1$ generate the (integral) ring of symmetric functions in infinitely many variables. We refer to Remark 4.2 for the geometric representations of the cohomology classes $\mathbf{1}_{-(n-j)}\mathbf{m}_{(1^j), \alpha_s}|0\rangle$ and $\mathbf{1}_{-(n-j)}\mathbf{a}_{-j}(x)|0\rangle$.

Finally, in view of the results in [4], we propose a conjecture concerning the ring generators of the integral ring $H^*(X^{[n]}; \mathbb{Z})/\text{Tor}$ in terms of certain Chern classes. Note that if $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$, then the odd Betti numbers of X are equal to zero. Moreover, we can choose an integral basis $\alpha_1, \dots, \alpha_k$ for $H^2(X; \mathbb{Z})/\text{Tor}$ such that $\alpha_i \cdot \alpha_j = 1$ for some i and j with the possibility that $i = j$.

Conjecture 1.3. *Assume that $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$. Let $\alpha_1, \dots, \alpha_k$ be an integral basis for $H^2(X; \mathbb{Z})/\text{Tor}$ such that $\alpha_i \cdot \alpha_j = 1$ for some i and j . Put $\alpha_{k+1} = \alpha_i + \alpha_j$. Then the integral ring $H^*(X^{[n]}; \mathbb{Z})/\text{Tor}$ is generated over \mathbb{Z} by:*

$$(1.3) \quad c_i(\mathcal{O}_X^{[n]}), \quad c_j(\mathcal{O}_X(\alpha_s)^{[n]}),$$

where $1 \leq i \leq n-1$, $1 \leq j \leq n$, and $1 \leq s \leq (k+1)$.

Conventions. Unless otherwise indicated, all the cohomology groups in this paper are in \mathbb{Q} -coefficients. For a continuous map $p: Y_1 \rightarrow Y_2$ between two smooth compact manifolds and for $\alpha_1 \in H^*(Y_1)$, we define $p_*(\alpha_1)$ to be $\text{PD}^{-1}p_*(\text{PD}(\alpha_1))$ where PD stands for the Poincaré duality. For a smooth

projective surface X , by abusing notations, we let $1 \in H^0(X; \mathbb{Z})$ be the fundamental cohomology class of X ; in addition, we let x denote either a point in X , or the class in $H^4(X; \mathbb{Z})$ which is the Poincaré dual of the homology class represented by a point in X .

2. Basics on Hilbert schemes of points on surfaces

Let X be a complex smooth projective surface, and $X^{[n]}$ be the Hilbert scheme of points on X . An element in $X^{[n]}$ is represented by a length- n 0-dimensional closed subscheme ξ of X . It is well known that $X^{[n]}$ is smooth. Sending an element in $X^{[n]}$ to its support in the symmetric product $\text{Sym}^n(X)$, we obtain the Hilbert-Chow morphism $\pi_n : X^{[n]} \rightarrow \text{Sym}^n(X)$, which is a resolution of singularities. Let $H^*(X^{[n]})$ be the total cohomology of $X^{[n]}$ with \mathbb{Q} -coefficients, and put

$$(2.1) \quad \mathbb{H}_X = \bigoplus_{n=0}^{\infty} H^*(X^{[n]}).$$

There exists a non-degenerate super-symmetric bilinear form $\langle -, - \rangle$ on the space \mathbb{H}_X which is induced from the standard one on $H^*(X^{[n]})$ defined by

$$\langle \alpha, \beta \rangle = \int_{X^{[n]}} \alpha \beta, \quad \alpha, \beta \in H^*(X^{[n]}).$$

Recall that Heisenberg algebra action on \mathbb{H}_X was defined in [2, 3, 16]. Namely, we have the operators $\mathfrak{a}_{-n}(\alpha) \in \text{End}(\mathbb{H}_X)$ with $n \in \mathbb{Z}$ and $\alpha \in H^*(X)$ which satisfy a Heisenberg algebra commutation relation:

$$(2.2) \quad [\mathfrak{a}_m(\alpha), \mathfrak{a}_n(\beta)] = -m \delta_{m,-n} \cdot \langle \alpha, \beta \rangle \cdot \text{Id}_{\mathbb{H}_X}.$$

When $n > 0$, we often refer to $\mathfrak{a}_{-n}(\alpha)$ (resp. $\mathfrak{a}_n(\alpha)$) as the *creation* (resp. *annihilation*) operator. The space \mathbb{H}_X is an irreducible module over the Heisenberg algebra generated by the operators $\mathfrak{a}_n(\alpha)$ with a highest weight vector

$$|0\rangle = 1 \in H^0(X^{[0]}) \cong \mathbb{Q}.$$

It follows that \mathbb{H}_X is linearly spanned by all the *Heisenberg monomial classes*:

$$(2.3) \quad \mathfrak{a}_{-n_1}(\alpha_1) \cdots \mathfrak{a}_{-n_k}(\alpha_k)|0\rangle,$$

where $k \geq 0, n_1, \dots, n_k > 0$, and $\alpha_1, \dots, \alpha_k$ run over a linear basis of $H^*(X)$.

For $n > 0$ and a homogeneous class $\alpha \in H^*(X)$, let $|\alpha| = s$ if $\alpha \in H^s(X)$, and let $G_i(\alpha, n)$ be the homogeneous component in $H^{|\alpha|+2i}(X^{[n]})$ of

$$(2.4) \quad G(\alpha, n) = p_{1*}(\text{ch}(\mathcal{O}_{\mathcal{Z}_n}) \cdot p_2^* \text{td}(X) \cdot p_2^* \alpha) \in H^*(X^{[n]}),$$

where \mathcal{Z}_n is the universal codimension-2 subscheme defined by

$$(2.5) \quad \mathcal{Z}_n = \{(\xi, x) \subset X^{[n]} \times X \mid x \in \text{Supp}(\xi)\} \subset X^{[n]} \times X,$$

$\text{ch}(\mathcal{O}_{\mathcal{Z}_n})$ denotes the Chern character of the structure sheaf $\mathcal{O}_{\mathcal{Z}_n}$, $\text{td}(X)$ denotes the Todd class, and p_1 and p_2 are the projections of $X^{[n]} \times X$ to $X^{[n]}$ and X , respectively. We extend the notion $G_i(\alpha, n)$ linearly to an arbitrary class $\alpha \in H^*(X)$, and set $G_i(\alpha, 0) = 0$. These *Chern character classes* $G_i(\alpha, n)$ play essential roles in understanding the cup product on $X^{[n]}$.

Let $\lambda = (1^{m_1} 2^{m_2} \dots)$ be a partition of the integer $n = \sum_r r m_r$, whose part r has multiplicity m_r . We write $\lambda \vdash n$. Let $\ell(\lambda) = \sum_r m_r$ and

$$(2.6) \quad \begin{aligned} |\lambda| &= \sum_r r m_r = n, \\ \mathfrak{z}\lambda &= \prod_{r \geq 1} (r^{m_r} m_r!). \end{aligned}$$

In addition, for a cohomology class $\alpha \in H^*(X)$ and $m \geq 0$, we define

$$(2.7) \quad \begin{aligned} \mathbf{a}_{-\lambda}(\alpha) &= \prod_{r \geq 1} \mathbf{a}_{-r}(\alpha)^{m_r}, \\ \mathbf{1}_{-m} &= \frac{1}{m!} \cdot \mathbf{a}_{-(1^m)}(1) = \frac{1}{m!} \cdot \mathbf{a}_{-1}(1)^m. \end{aligned}$$

The geometric meaning of $\mathbf{1}_{-m}$ is that $\mathbf{1}_{-m}|0\rangle$ is equal to the fundamental class of the Hilbert scheme $X^{[m]}$. For simplicity, we set $\mathbf{1}_{-m} = 0$ when $m < 0$.

The following is the Stability Theorem 5.1 proved in [18], which will be a key technical tool used in later sections.

Theorem 2.1. *Let $s \geq 1$ and $k_i \geq 1$ for $1 \leq i \leq s$. Fix $n_{i,j} \geq 1$ and $\alpha_{i,j} \in H^*(X)$ for $1 \leq j \leq k_i$, and fix n with $n \geq \sum_{j=1}^{k_i} n_{i,j}$ for all $1 \leq i \leq s$. Then*

the cup product

$$(2.8) \quad \prod_{i=1}^s \left(\mathbf{1}_{-(n-\sum_{j=1}^{k_i} n_{i,j})} \left(\prod_{j=1}^{k_i} \mathbf{a}_{-n_{i,j}}(\alpha_{i,j}) \right) |0\rangle \right)$$

in $H^*(X^{[n]})$ is equal to a finite linear combination of monomials of the form

$$(2.9) \quad \mathbf{1}_{-(n-\sum_{p=1}^N m_p)} \left(\prod_{p=1}^N \mathbf{a}_{-m_p}(\gamma_p) \right) |0\rangle,$$

whose coefficients are independent of $X, \alpha_{i,j}$ and n . Here $\sum_{p=1}^N m_p \leq \sum_{i=1}^s \sum_{j=1}^{k_i} n_{i,j}$, and $\gamma_1, \dots, \gamma_N$ depend only on $\alpha_{i,j}$, $1 \leq i \leq s, 1 \leq j \leq k_i$, and the canonical class and the Euler class of X . In addition, the expression (2.9) satisfies the upper bound

$$\sum_{p=1}^N m_p = \sum_{i=1}^s \sum_{j=1}^{k_i} n_{i,j}$$

if and only if it is $\mathbf{1}_{-(n-\sum_{i=1}^s \sum_{j=1}^{k_i} n_{i,j})} (\prod_{i=1}^s \prod_{j=1}^{k_i} \mathbf{a}_{-n_{i,j}}(\alpha_{i,j})) |0\rangle$ with coefficient 1.

In particular, Theorem 2.1 says that the cup product (2.8) is of the form:

$$\mathbf{1}_{-(n-\sum_{i=1}^s \sum_{j=1}^{k_i} n_{i,j})} \left(\prod_{i=1}^s \prod_{j=1}^{k_i} \mathbf{a}_{-n_{i,j}}(\alpha_{i,j}) \right) |0\rangle + \mathbf{1}_{-(n-\sum_{i=1}^s \sum_{j=1}^{k_i} n_{i,j}+1)} A$$

for some cohomology class A in $H^*(X^{[\sum_{i=1}^s \sum_{j=1}^{k_i} n_{i,j}-1]})$.

3. Integral basis of the integral cohomology

In this section, using the method developed in [15] and topological K -theory, we will prove a conjecture raised in [15]. As a consequence, we will find an integral basis for $H^*(X^{[n]}; \mathbb{Z})/\text{Tor}$, and hence prove Theorem 1.1.

3.1. Integral operators and the definition of $\mathfrak{m}_{\lambda, \alpha}$

In this subsection, we review the basic results and methods from [15].

Definition 3.1.

(i) A class $A \in H^*(X^{[n]})$ is *integral* if it is contained in

$$H^*(X^{[n]}; \mathbb{Z})/\text{Tor} \subset H^*(X^{[n]}).$$

(ii) A linear basis of $H^*(X^{[n]})$ is *integral* if its members are integral classes and form a \mathbb{Z} -basis of the lattice $H^*(X^{[n]}; \mathbb{Z})/\text{Tor}$.

(iii) A linear operator $f \in \text{End}(\mathbb{H}_X)$ is *integral* if $f(A) \in \mathbb{H}_X$ is an integral class whenever $A \in \mathbb{H}_X$ is an integral cohomology class.

The following lemma compiles various results proved in [15].

Lemma 3.2.

(i) The operators $\mathbf{a}_n(\alpha), n \in \mathbb{Z}$ are integral if $\alpha \in H^*(X)$ is integral;

(ii) The operator $1/\mathfrak{z}_\lambda \cdot \mathbf{a}_{-\lambda}(1)$ is integral for every partition λ ;

(iii) Let $A \in \mathbb{H}_X$ be an integral class. Write A as $\mathbf{a}_A|0\rangle$ where \mathbf{a}_A is a unique polynomial of creation operators. Then, \mathbf{a}_A is an integral operator. \square

Next, let C be a smooth irreducible curve in the surface X . By abusing notations, we also use C to denote the corresponding divisor and the corresponding cohomology class. Following Subsection 9.3 of [16] (see also [2]), for every partition $\lambda = (\lambda_1, \lambda_2, \dots)$ of n , we define the subvariety of $X^{[n]}$:

$$(3.1) \quad L^\lambda C = \overline{(\pi_n)^{-1}(S_\lambda^n C)},$$

where $S_\lambda^n C = \{ \sum_i \lambda_i x_i \mid x_i \in C, x_i \neq x_j \text{ for } i \neq j \}$, and

$$\pi_n : X^{[n]} \rightarrow \text{Sym}^n(X)$$

is the Hilbert–Chow morphism. For $n \geq 0$, let $\mathbb{H}_{n,C}$ be the \mathbb{Q} -linear span:

$$(3.2) \quad \text{Span}\{\mathbf{a}_{-\lambda}(C)|0\rangle \mid \lambda \vdash n\}$$

in $H^*(X^{[n]})$. We see from the Theorem 9.14 in [16] that the fundamental class $[L^\lambda C] \in H^{2n}(X^{[n]}; \mathbb{Z})$ is contained in $\mathbb{H}_{n,C} \subset H^*(X^{[n]})$. Define

$$(3.3) \quad \mathbb{H}_C = \bigoplus_{n=0}^{\infty} \mathbb{H}_{n,C}.$$

Let Λ be the ring of symmetric functions in infinitely many variables (see [19, p.19]), and $\Lambda_{\mathbb{Q}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. For a partition λ , let p_{λ} and m_{λ} be the power-sum symmetric function and the monomial symmetric function, respectively.

In [16], Nakajima defined a linear isomorphism

$$(3.4) \quad \Phi_C : \Lambda_{\mathbb{Q}} \longrightarrow \mathbb{H}_C,$$

which satisfies the following two properties:

$$(3.5) \quad \Phi_C(p_{\lambda}) = \mathbf{a}_{-\lambda}(C)|0\rangle, \quad \Phi_C(m_{\lambda}) = [L^{\lambda}C].$$

By Lemma 3.2 (iii), the operator $\mathbf{a}_{[L^{\lambda}C]}$ is integral. Put

$$\mathbf{m}_{\lambda,C} = \mathbf{a}_{[L^{\lambda}C]} \in \text{End}(\mathbb{H}_X).$$

Since $[L^{\lambda}C] \in \mathbb{H}_{|\lambda|,C}$, we see from (3.2) that the integral operator $\mathbf{m}_{\lambda,C}$ is a polynomial of the creation Heisenberg operators $\mathbf{a}_{-i}(C)$, $i > 0$.

Definition 3.3. For a partition λ and an arbitrary class $\alpha \in H^2(X)$, we define

$$\mathbf{m}_{\lambda,\alpha} \in \text{End}(\mathbb{H}_X)$$

by replacing the creation operators $\mathbf{a}_{-i}(C)$ in $\mathbf{m}_{\lambda,C}$ by the corresponding creation operators $\mathbf{a}_{-i}(\alpha)$. Define the cohomology class $[L^{\lambda}\alpha] \in H^*(X^{[|\lambda|]})$ by putting

$$(3.6) \quad [L^{\lambda}\alpha] = \mathbf{m}_{\lambda,\alpha}|0\rangle.$$

Similarly, we can define the subspaces $\mathbb{H}_{n,\alpha}$ and \mathbb{H}_{α} of \mathbb{H}_X as in (3.3). Moreover, in view of (3.4) and (3.5), we have a linear isomorphism

$$(3.7) \quad \Phi_{\alpha} : \Lambda_{\mathbb{Q}} \longrightarrow \mathbb{H}_{\alpha}$$

which satisfies the following two properties:

$$(3.8) \quad \Phi_{\alpha}(p_{\lambda}) = \mathbf{a}_{-\lambda}(\alpha)|0\rangle, \quad \Phi_{\alpha}(m_{\lambda}) = [L^{\lambda}\alpha] = \mathbf{m}_{\lambda,\alpha}|0\rangle.$$

Since the monomial symmetric function $m_{(1^n)}$ is the same as the n th elementary symmetric function, we see from formula (2.14') in [19] that

$$\mathbf{m}_{(1^n)} = \sum_{|\mu|=n} \frac{(-1)^{n-\ell(\mu)} p_{\mu}}{\mathfrak{z}_{\mu}}.$$

Therefore, applying Φ_α to both sides and using (3.8), we obtain

$$(3.9) \quad \mathfrak{m}_{(1^n),\alpha}|0\rangle = [L^{(1^n)}\alpha] = \sum_{|\mu|=n} \frac{(-1)^{n-\ell(\mu)} \mathfrak{a}_{-\mu}(\alpha)|0\rangle}{\mathfrak{z}_\mu}.$$

3.2. Integrality of $\mathfrak{m}_{\lambda,\alpha}$ for integral α

In this subsection, we shall prove that if $\alpha \in H^2(X)$ is integral, then the class $[L^\lambda\alpha]$ and the operator $\mathfrak{m}_{\lambda,\alpha}$ are integral. This was proved in Theorem 4.5 of [15] when α is a divisor, and conjectured in Remark 5.6 of [15] for a general integral class α .

First of all, we recall some tautological vector bundles over the Hilbert scheme $X^{[n]}$. For a line bundle L over the surface X , we define

$$(3.10) \quad L^{[n]} = (p_1)_*(\mathcal{O}_{\mathcal{Z}_n} \otimes p_2^*L),$$

where \mathcal{Z}_n is from (2.5), and p_1 and p_2 are the projections of $X^{[n]} \times X$ to $X^{[n]}$ and X , respectively. Then, $L^{[n]}$ is a rank- n bundle over $X^{[n]}$.

Lemma 3.4. *Let $0 \leq i \leq n$. Let L be a line bundle on X with $\alpha = c_1(L)$. Then,*

$$(3.11) \quad c_i(L^{[n]}) = \sum_{\substack{|\lambda|+|\mu|=n \\ \ell(\lambda)=n-i}} \frac{(-1)^{i-\ell(\mu)} \mathfrak{a}_{-\lambda}(1)\mathfrak{a}_{-\mu}(\alpha)|0\rangle}{\mathfrak{z}_\lambda \mathfrak{z}_\mu}.$$

Proof. Let $c_{\hbar}(L^{[n]}) = \sum_{i=0}^n c_i(L^{[n]})\hbar^i$ be the generating function for the Chern classes of $L^{[n]}$, where \hbar is a formal variable. Then the Theorem 2.4 of [20], which is a variant of the Theorem 4.6 in [7], states that

$$(3.12) \quad \sum_{n=0}^{\infty} c_{\hbar}(L^{[n]})w^n = \exp \left(\sum_{r \geq 1} \frac{(-\hbar)^{r-1}}{r} \mathfrak{a}_{-r}(c_{\hbar}(L))w^r \right) \cdot |0\rangle,$$

where w is another formal variable. Expanding the right-hand side yields

$$\begin{aligned}
 & \exp \left(\sum_{r \geq 1} \frac{(-\hbar)^{r-1}}{r} \mathbf{a}_{-r}(c_{\hbar}(L)) w^r \right) \cdot |0\rangle \\
 &= \exp \left(\sum_{r \geq 1} \frac{(-1)^{r-1}}{r} \mathbf{a}_{-r}(1) \hbar^{r-1} w^r \right) \cdot \exp \left(\sum_{r \geq 1} \frac{(-1)^{r-1}}{r} \mathbf{a}_{-r}(\alpha) \hbar^r w^r \right) \cdot |0\rangle \\
 &= \sum_{\lambda} \frac{(-1)^{|\lambda|-\ell(\lambda)} \mathbf{a}_{-\lambda}(1)}{\mathfrak{z}_{\lambda}} \hbar^{|\lambda|-\ell(\lambda)} w^{|\lambda|} \cdot \sum_{\mu} \frac{(-1)^{|\mu|-\ell(\mu)} \mathbf{a}_{-\mu}(\alpha)}{\mathfrak{z}_{\mu}} \hbar^{|\mu|} w^{|\mu|} |0\rangle \\
 &= \sum_{(\lambda, \mu)} \frac{(-1)^{|\lambda|+|\mu|-\ell(\lambda)-\ell(\mu)} \mathbf{a}_{-\lambda}(1) \mathbf{a}_{-\mu}(\alpha) |0\rangle}{\mathfrak{z}_{\lambda} \mathfrak{z}_{\mu}} \hbar^{|\lambda|+|\mu|-\ell(\lambda)} w^{|\lambda|+|\mu|}.
 \end{aligned}
 \tag{3.13}$$

Now the Chern class $c_i(L^{[n]})$ is the coefficients of $\hbar^i w^n$ in (3.13). □

Lemma 3.5. *If the class $\alpha \in H^2(X)$ is integral, then so is the operator $\mathfrak{m}_{(1^n), \alpha}$.*

Proof. Let p_1 and p_2 be the projections of $X^{[n]} \times X$ to $X^{[n]}$ and X , respectively. By Theorem in [21, p. 20], p_1 induces a homomorphism

$$(p_1)! : K_{\text{top}}^0(X^{[n]} \times X) \rightarrow K_{\text{top}}^0(X^{[n]}),$$

where $K_{\text{top}}^0(\cdot)$ denotes the ring of *complex* vector bundles; moreover, for an element $V \in K_{\text{top}}^0(X^{[n]} \times X)$, the Riemann–Roch formula holds:

$$(3.14) \quad \text{ch}((p_1)!(V)) = (p_1)_*(\text{ch}(V) \cdot p_2^* \text{td}(X)).$$

Next, by Theorem E.5 in [22], there exists a *complex* (not necessarily algebraic) line bundle L_{α} on X with $c_1(L_{\alpha}) = \alpha$. Applying (3.14) to the element

$$V = V_{\alpha} := \mathcal{O}_{\mathcal{Z}_n} \otimes p_2^* L_{\alpha} \in K_{\text{top}}^0(X^{[n]} \times X)$$

where \mathcal{Z}_n is the universal codimension-2 subscheme defined in (2.5), we obtain

$$\begin{aligned}
 (3.15) \quad \text{ch}((p_1)!(V_{\alpha})) &= p_{1*}(\text{ch}(\mathcal{O}_{\mathcal{Z}_n}) \cdot p_2^* \text{td}(X) \cdot p_2^* \text{ch}(L_{\alpha})) \\
 &= G(1, n) + G(\alpha, n) + \langle \alpha, \alpha \rangle / 2 \cdot G(x, n),
 \end{aligned}$$

where we have used definition (2.4) and the fact that

$$\text{ch}(L_\alpha) = 1 + \alpha + \langle \alpha, \alpha \rangle / 2 \cdot x.$$

In particular, we conclude that the integral class $c_i((p_1)!(V_\alpha))$ is a polynomial P_i of $G_{i_1}(1, n)$, $G_{i_2}(\alpha, n)$, $\langle \alpha, \alpha \rangle$ and $G_{i_3}(x, n)$ with $0 \leq i_1, i_2, i_3 \leq 2n$, i.e.,

$$c_i((p_1)!(V_\alpha)) = P_i(G_{i_1}(1, n), G_{i_2}(\alpha, n), \langle \alpha, \alpha \rangle, G_{i_3}(x, n) \mid 0 \leq i_1, i_2, i_3 \leq 2n). \tag{3.16}$$

Note that the coefficients in P_i are independent of α and X .

When α is a divisor, L_α is an algebraic line bundle on X , and the element

$$(p_1)!(V_\alpha) \in K_{\text{top}}^0(X^{[n]})$$

is represented precisely by the tautological rank- n vector bundle $(L_\alpha)^{[n]}$ defined in (3.10). Therefore, we obtain from Lemma 3.4 that

$$c_i((p_1)!(V_\alpha)) = \sum_{\substack{|\lambda|+|\mu|=n \\ \ell(\lambda)=n-i}} \frac{(-1)^{i-\ell(\mu)} \mathbf{a}_{-\lambda}(1) \mathbf{a}_{-\mu}(\alpha) |0\rangle}{\mathfrak{z}\lambda \mathfrak{z}\mu}. \tag{3.17}$$

Finally, we conclude from (3.16) and the universality results in [18] (e.g., the Theorem 4.1 there) that (3.17) holds for a general integral class $\alpha \in H^2(X)$. Setting $i = n$ in (3.17) and using (3.9), we obtain the integral class

$$c_n((p_1)!(V_\alpha)) = \sum_{|\mu|=n} \frac{(-1)^{n-\ell(\mu)} \mathbf{a}_{-\mu}(\alpha) |0\rangle}{\mathfrak{z}\mu} = [L^{(1^n)}\alpha].$$

By Lemma 3.2 (iii) and (3.6), the operator $\mathfrak{m}_{(1^n),\alpha}$ is integral. □

Theorem 3.6. *Let the class $\alpha \in H^2(X)$ be integral. Then for every partition λ , the class $[L^\lambda\alpha]$ and the operator $\mathfrak{m}_{\lambda,\alpha}$ are integral.*

Proof. It is well known (see [19]) that the monomial symmetric functions

$$m_{(1^i)}, \quad i \geq 1$$

generate the integral ring Λ . So the monomial symmetric function m_λ can be written as an integral polynomial P of the functions $m_{(1^i)}$, $i \geq 1$, i.e.,

$$m_\lambda = P(m_{(1^i)} \mid i \geq 1).$$

Combining this with the two formulas in (3.8), we see that

$$(3.18) \quad [L^\lambda \alpha] = \Phi_\alpha(m_\lambda) = \Phi_\alpha(P(m_{(1^i)} | i \geq 1)) = P(\mathbf{m}_{(1^i), \alpha} | i \geq 1) | 0 \rangle.$$

Hence, the class $[L^\lambda \alpha]$ is integral by Lemma 3.5. Finally, we conclude from Lemma 3.2 (iii) and (3.6) that the operator $\mathbf{m}_{\lambda, \alpha}$ is integral. □

3.3. Proof of Theorem 1.1

Let $\alpha_1, \dots, \alpha_k \in H^2(X)$ be integral cohomology classes. By Lemma 3.2 and Theorem 3.6, we have three types of integral operators:

$$\frac{1}{\mathfrak{z}^\lambda} \mathbf{a}_{-\lambda}(1), \quad \mathbf{a}_{-\mu}(x), \quad \mathbf{m}_{\nu_i, \alpha_i},$$

where λ, μ, ν_i stand for partitions. Therefore, we obtain integral classes:

$$(3.19) \quad \frac{1}{\mathfrak{z}^\lambda} \mathbf{a}_{-\lambda}(1) \mathbf{a}_{-\mu}(x) \mathbf{m}_{\nu^1, \alpha_1} \cdots \mathbf{m}_{\nu^k, \alpha_k} | 0 \rangle.$$

Recall from the conditions in Theorem 1.1 that

$$H^1(X) = H^3(X) = 0.$$

By (2.3), the number of classes in (1.1) is equal to the rank of $H^*(X^{[n]}; \mathbb{Z})/\text{Tor}$. Now consider the intersection number of two classes from (1.1):

$$\begin{aligned} & \frac{1}{\mathfrak{z}^\lambda} \mathbf{a}_{-\lambda}(1) \mathbf{a}_{-\mu}(x) \mathbf{m}_{\nu^1, \alpha_1} \cdots \mathbf{m}_{\nu^k, \alpha_k} | 0 \rangle, \\ & \frac{1}{\mathfrak{z}^{\tilde{\lambda}}} \mathbf{a}_{-\tilde{\lambda}}(1) \mathbf{a}_{-\tilde{\mu}}(x) \mathbf{m}_{\tilde{\nu}^1, \alpha_1} \cdots \mathbf{m}_{\tilde{\nu}^k, \alpha_k} | 0 \rangle. \end{aligned}$$

By the Heisenberg algebra commutation relation (2.2), the intersection number is

$$(3.20) \quad \pm \delta_{\lambda, \tilde{\mu}} \cdot \delta_{\tilde{\lambda}, \mu} \cdot \langle \mathbf{m}_{\nu^1, \alpha_1} \cdots \mathbf{m}_{\nu^k, \alpha_k} | 0 \rangle, \mathbf{m}_{\tilde{\nu}^1, \alpha_1} \cdots \mathbf{m}_{\tilde{\nu}^k, \alpha_k} | 0 \rangle \rangle.$$

Recall from the Modularity Theorem 4.6 in [15] that for every fixed $\tilde{n} \geq 0$, the intersection matrix formed by the pairings of the classes

$$\mathbf{m}_{\nu^1, \alpha_1} \cdots \mathbf{m}_{\nu^k, \alpha_k} | 0 \rangle, \quad \sum_{i=1}^k |\nu^i| = \tilde{n}$$

has determinant ± 1 . Hence we conclude from (3.20) that the intersection matrix formed by the pairings of the integral classes in (1.1) is unimodular. Therefore, the classes in (1.1) form an integral basis of $H^*(X^{[n]}; \mathbb{Z})/\text{Tor}$.

4. Generators of the integral cohomology ring

In this section, we shall assume that the odd Betti numbers of the surface X are equal to zero, and study generators for the integral cohomology ring of $X^{[n]}$.

Let $\alpha_1, \dots, \alpha_k$ be an integral basis for $H^2(X; \mathbb{Z})/\text{Tor}$. By (1.1), we have the following integral cohomology classes in $H^*(X^{[n]})$:

$$\begin{aligned} & \mathbf{1}_{-(n-j)} \mathbf{m}_{(1^j), \alpha_s} | 0 \rangle, \\ & \mathbf{1}_{-(n-j)} \mathbf{a}_{-j}(x) | 0 \rangle, \end{aligned}$$

where $1 \leq j \leq n$ and $1 \leq s \leq k$. Recall the tautological rank- n vector bundle $\mathcal{O}_X^{[n]}$ defined by (3.10), and note from (3.11) that $c_n(\mathcal{O}_X^{[n]}) = 0$.

Definition 4.1. For $n \geq 1$, we define \mathcal{R}_n to be the subring of the integral cohomology ring $H^*(X^{[n]}; \mathbb{Z})/\text{Tor}$ generated over \mathbb{Z} by all the integral classes:

$$(4.1) \quad c_i(\mathcal{O}_X^{[n]}), \quad \mathbf{1}_{-(n-j)} \mathbf{m}_{(1^j), \alpha_s} | 0 \rangle, \quad \mathbf{1}_{-(n-j)} \mathbf{a}_{-j}(x) | 0 \rangle,$$

where $1 \leq i \leq n - 1$, $1 \leq j \leq n$, and $1 \leq s \leq k$.

Remark 4.2. Fix a point $x \in X$ and a smooth irreducible curve $C \subset X$. Then the integral class $\mathbf{1}_{-(n-j)} \mathbf{a}_{-j}(x) | 0 \rangle$ is represented by the subvariety

$$\{ \xi \in X^{[n]} \mid m(\xi, x) \geq j \},$$

where $m(\xi, x)$ denotes the multiplicity of ξ at the point x . Similarly, the integral class $\mathbf{1}_{-(n-j)} \mathbf{m}_{(1^j), C} | 0 \rangle$ is represented by the closure of the subvariety in $X^{[n]}$:

$$\left\{ \sum_{i=1}^{n-j} x_i + \sum_{i=1}^j x'_i \mid x_i \notin C, x'_i \in C, \text{ and } x_1, \dots, x_{n-j}, x'_1, \dots, x'_j \text{ are distinct} \right\}.$$

Our goal in this section is to show that

$$\mathcal{R}_n = H^*(X^{[n]}; \mathbb{Z})/\text{Tor},$$

i.e., we shall prove that the ring $H^*(X^{[n]}; \mathbb{Z})/\text{Tor}$ is generated by the integral classes in (4.1). We begin with two lemmas which enable us to use certain induction.

Lemma 4.3. *If λ is a partition with $|\lambda| \leq n$, then*

$$(4.2) \quad \mathbf{1}_{-(n-|\lambda|)} \mathbf{m}_{\lambda, \alpha_i} |0\rangle \equiv A \pmod{\mathcal{R}_n},$$

where A is an integral linear combination of classes of the form

$$(4.3) \quad \frac{1}{3^{\tilde{\lambda}}} \mathbf{a}_{-\tilde{\lambda}}(1) \mathbf{a}_{-\mu}(x) \mathbf{m}_{\nu^1, \alpha_1} \cdots \mathbf{m}_{\nu^k, \alpha_k} |0\rangle$$

such that the partitions $\tilde{\lambda} = (1^{\tilde{m}_1} 2^{\tilde{m}_2} \dots)$ satisfy $\tilde{m}_1 > (n - |\lambda|)$, and

$$|\tilde{\lambda}| + |\mu| + |\nu^1| + \cdots + |\nu^k| = n.$$

Proof. Combining formulas (3.6) and (3.18), we obtain

$$\mathbf{m}_{\lambda, \alpha_i} |0\rangle = [L^\lambda \alpha_i] = P(\mathbf{m}_{(1^j), \alpha_i} |j \geq 1\rangle |0\rangle),$$

where P is an integral polynomial. It follows that

$$\mathbf{1}_{-(n-|\lambda|)} \mathbf{m}_{\lambda, \alpha_i} |0\rangle = \mathbf{1}_{-(n-|\lambda|)} P(\mathbf{m}_{(1^j), \alpha_i} |j \geq 1\rangle |0\rangle).$$

Therefore, it remains to prove that if $j_1 + \cdots + j_s = |\lambda|$, then

$$(4.4) \quad \mathbf{1}_{-(n-|\lambda|)} \mathbf{m}_{(1^{j_1}), \alpha_i} \cdots \mathbf{m}_{(1^{j_s}), \alpha_i} |0\rangle \equiv A_1 \pmod{\mathcal{R}_n},$$

where A_1 is an integral linear combination of classes of the form (4.3).

By definition, the classes $\mathbf{1}_{-(n-j_u)} \mathbf{m}_{(1^{j_u}), \alpha_i} |0\rangle$, $1 \leq u \leq s$, are contained in \mathcal{R}_n . By (3.9), $\mathbf{1}_{-(n-j_u)} \mathbf{m}_{(1^{j_u}), \alpha_i} |0\rangle$ is a linear combination of classes of the

form

$$\mathbf{1}_{-(n-j_u)} \mathbf{a}_{-\gamma}(\alpha_i) |0\rangle, \quad |\gamma| = j_u.$$

By Theorem 2.1, the cup product of the classes $\mathbf{1}_{-(n-j_u)} \mathbf{m}_{(1^{j_u}), \alpha_i} |0\rangle$, $1 \leq u \leq s$, is

$$\mathbf{1}_{-(n-|\lambda|)} \mathbf{m}_{(1^{j_1}), \alpha_i} \cdots \mathbf{m}_{(1^{j_s}), \alpha_i} |0\rangle + \mathbf{1}_{-(n-|\lambda|+1)} A_2$$

for some cohomology class $A_2 \in H^*(X^{[|\lambda|-1]})$. Hence, we obtain

$$\mathbf{1}_{-(n-|\lambda|)} \mathbf{m}_{(1^{j_1}), \alpha_i} \cdots \mathbf{m}_{(1^{j_s}), \alpha_i} |0\rangle \equiv -\mathbf{1}_{-(n-|\lambda|+1)} A_2 \pmod{\mathcal{R}_n}.$$

Since $\mathbf{1}_{-(n-|\lambda|)} \mathbf{m}_{(1^{j_1}), \alpha_i} \cdots \mathbf{m}_{(1^{j_s}), \alpha_i} |0\rangle$ is an integral class, we conclude from Theorem 1.1 that the class $A_1 := -\mathbf{1}_{-(n-|\lambda|+1)} A_2$ is an integral linear combination of classes of the form (4.3). This completes the proof of (4.4). \square

Lemma 4.4. *Let $\lambda = (1^{m_1} 2^{m_2} \dots)$ be a partition of n . Then,*

$$(4.5) \quad \frac{1}{\mathfrak{z}_\lambda} \mathbf{a}_{-\lambda}(1) |0\rangle \equiv A \pmod{\mathcal{R}_n},$$

where A is an integral linear combination of classes of the form

$$(4.6) \quad \frac{1}{\mathfrak{z}_{\tilde{\lambda}}} \mathbf{a}_{-\tilde{\lambda}}(1) \mathbf{a}_{-\mu}(x) \mathbf{m}_{\nu^1, \alpha_1} \cdots \mathbf{m}_{\nu^k, \alpha_k} |0\rangle$$

such that $\mu \neq \emptyset$ or $\nu^i \neq \emptyset$ for some i , $\tilde{\lambda} = (1^{\tilde{m}_1} 2^{\tilde{m}_2} \dots)$ with $\tilde{m}_1 > m_1$, and

$$|\tilde{\lambda}| + |\mu| + |\nu^1| + \cdots + |\nu^k| = n.$$

Proof. For simplicity, let $G_i = G_i(1, n)$ and $c_i = c_i(\mathcal{O}_X^{[n]})$. By Corollary 4.8 in [17], the cohomology class G_{r-1} is equal to

$$\frac{(-1)^{r-1}}{r!} \cdot \mathbf{1}_{-(n-r)} \mathbf{a}_{-r}(1) |0\rangle + \mathbf{1}_{-(n-r+1)} A_1$$

for some class $A_1 \in H^*(X^{[r-1]})$. By Theorem 2.1, the cup product $\prod_{r \geq 2} G_{r-1}^{m_r}$ equals

$$\prod_{r \geq 2} \left(\frac{(-1)^{r-1}}{r!} \right)^{m_r} \cdot \frac{1}{m_1!} \cdot \mathbf{a}_{-\lambda}(1) |0\rangle + \mathbf{1}_{-(m_1+1)} A_2$$

for some class A_2 . In view of the basis (1.1), we can write $\mathbf{1}_{-(m_1+1)} A_2$ as

$$\sum_{\lambda'} d_{\lambda'} \mathbf{a}_{-\lambda'}(1) |0\rangle + A_3,$$

where A_3 is a linear combination of classes of the form (4.6), $d_{\lambda'} \in \mathbb{Q}$, and the partitions $\lambda' := (1^{m'_1} 2^{m'_2} \dots)$ satisfy $m'_1 > m_1$. So

$$\prod_{r \geq 2} G_{r-1}^{m_r} = \frac{1}{m_1!} \cdot \prod_{r \geq 2} \left(\frac{(-1)^{r-1}}{r!} \right)^{m_r} \cdot \mathbf{a}_{-\lambda}(1)|0\rangle + \sum_{\lambda'} d_{\lambda'} \mathbf{a}_{-\lambda'}(1)|0\rangle + A_3.$$

Solving $\mathbf{a}_{-\lambda}(1)|0\rangle$ and using induction on m_1 , we conclude that

$$\frac{1}{\mathfrak{z}_\lambda} \mathbf{a}_{-\lambda}(1)|0\rangle = P(G_1, G_2, \dots) + A,$$

where A is a linear combination of classes of the form (4.6), and $P(G_1, G_2, \dots)$ denotes a polynomial in the classes G_1, G_2, \dots . Note from (2.4) that G_i is the i th Chern character of the tautological bundle $(\mathcal{O}_X)^{[n]}$. Replacing the Chern characters G_i of $(\mathcal{O}_X)^{[n]}$ by its Chern classes c_i , we have

$$(4.7) \quad \frac{1}{\mathfrak{z}_\lambda} \mathbf{a}_{-\lambda}(1)|0\rangle = \tilde{P}(c_1, c_2, \dots) + A,$$

where $\tilde{P}(c_1, c_2, \dots)$ denotes a polynomial in the classes c_1, c_2, \dots , and has coefficients independent of the smooth surface X .

Now, from the proof of the Proposition 6.1 in [8], we see that in the cohomology ring $H^*((\mathbb{C}^2)^{[n]})$ of the Hilbert scheme $(\mathbb{C}^2)^{[n]}$, we also have

$$\frac{1}{\mathfrak{z}_\lambda} \mathbf{a}_{-\lambda}(1)|0\rangle = \tilde{P}(\tilde{c}_1, \tilde{c}_2, \dots),$$

where $1 \in H^0(\mathbb{C}^2)$ is the identity element, and \tilde{c}_i denotes the i th Chern class of the tautological bundle $(\mathcal{O}_{\mathbb{C}^2})^{[n]}$ on $(\mathbb{C}^2)^{[n]}$. Moreover, all the coefficients in the polynomial $\tilde{P}(\tilde{c}_1, \tilde{c}_2, \dots)$ are integers. Thus by (4.7), we obtain

$$\frac{1}{\mathfrak{z}_\lambda} \mathbf{a}_{-\lambda}(1)|0\rangle \equiv A \pmod{\mathcal{R}_n}.$$

Finally, note that $1/\mathfrak{z}_\lambda \mathbf{a}_{-\lambda}(1)|0\rangle$ and all the classes in \mathcal{R}_n are integral. Since A is a linear combination of the classes (4.6), we conclude from Theorem 1.1 that all the coefficients in this linear combination are integers. \square

Now we can present a proof of Theorem 1.2.

Proof of Theorem 1.2. By Theorem 1.1, it suffices to show that all the basis classes (1.1) are contained in \mathcal{R}_n . These basis classes can be rewritten as

$$\mathbf{1}_{-(n-n_0)} \frac{1}{\mathfrak{z}_\rho} \mathbf{a}_{-\rho}(1) \mathbf{a}_{-\mu_1}(x) \cdots \mathbf{a}_{-\mu_\ell}(x) \mathbf{m}_{\nu^1, \alpha_1} \cdots \mathbf{m}_{\nu^k, \alpha_k} |0\rangle,$$

where $0 \leq n_0 \leq n$, the partition ρ is of the form $(2^{m_2} 3^{m_3} \dots)$ and

$$|\rho| + (\mu_1 + \cdots + \mu_\ell) + \sum_{i=1}^k |\nu^i| = n_0.$$

In the following, we shall use induction on n_0 to prove that

$$(4.8) \quad A := \mathbf{1}_{-(n-n_0)} \frac{1}{\mathfrak{z}_\rho} \mathbf{a}_{-\rho}(1) \mathbf{a}_{-\mu_1}(x) \cdots \mathbf{a}_{-\mu_\ell}(x) \mathbf{m}_{\nu^1, \alpha_1} \cdots \mathbf{m}_{\nu^k, \alpha_k} |0\rangle \in \mathcal{R}_n.$$

Note that (4.8) is trivially true when $n_0 = 0$. Now fix $n_0 > 0$, and assume that

$$(4.9) \quad \mathbf{1}_{-(n-\tilde{n}_0)} \frac{1}{\mathfrak{z}_{\tilde{\rho}}} \mathbf{a}_{-\tilde{\rho}}(1) \mathbf{a}_{-\tilde{\mu}_1}(x) \cdots \mathbf{a}_{-\tilde{\mu}_\ell}(x) \mathbf{m}_{\tilde{\nu}^1, \alpha_1} \cdots \mathbf{m}_{\tilde{\nu}^k, \alpha_k} |0\rangle \in \mathcal{R}_n$$

holds for all $\tilde{n}_0 < n_0$. We want to verify that (4.8) is also true.

We start with the three special cases when $\mu_1 = n_0$ (so $\ell = 1$), or $|\rho| = n_0$, or $|\nu^i| = n_0$ for some i . First of all, by the definition of \mathcal{R}_n , we have

$$\mathbf{1}_{-(n-n_0)} \mathbf{a}_{-n_0}(x) |0\rangle \in \mathcal{R}_n.$$

Next, we see from Lemma 4.4 and the induction hypothesis that

$$\mathbf{1}_{-(n-n_0)} \frac{1}{\mathfrak{z}_\rho} \mathbf{a}_{-\rho}(1) |0\rangle \in \mathcal{R}_n$$

for every partition $\rho = (2^{m_2} 3^{m_3} \dots)$ with $|\rho| = n_0$. Finally, for every partition ν^i with $|\nu^i| = n_0$, we conclude from Lemma 4.3 and the induction hypothesis that

$$\mathbf{1}_{-(n-n_0)} \mathbf{m}_{\nu^i, \alpha_i} |0\rangle \in \mathcal{R}_n.$$

For general $\rho, \mu_1, \dots, \mu_\ell, \nu^1, \dots, \nu^k$ from (4.8), we consider the classes:

$$\begin{aligned} & \mathbf{1}_{-(n-|\rho|)} \frac{1}{\mathfrak{z}_\rho} \mathbf{a}_{-\rho}(1) |0\rangle, \\ & \mathbf{1}_{-(n-\mu_i)} \mathbf{a}_{-\mu_i}(x) |0\rangle \quad (1 \leq i \leq \ell), \\ & \mathbf{1}_{-(n-|\nu^i|)} \mathbf{m}_{\nu^i, \alpha_i} |0\rangle \quad (1 \leq i \leq k), \end{aligned}$$

which are contained in \mathcal{R}_n by the preceding paragraph. By Theorem 2.1, their cup product is $A + \mathbf{1}_{-(n-n_0+1)}A_1$ for some class $A_1 \in H^*(X^{[n_0-1]})$. Therefore,

$$A \equiv -\mathbf{1}_{-(n-n_0+1)}A_1 \pmod{\mathcal{R}_n}.$$

Note that the integral class $-\mathbf{1}_{-(n-n_0+1)}A_1$ is an integral linear combination of classes of the form (4.9). Hence by induction, we have $A \in \mathcal{R}_n$. \square

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