Cayley cones ruled by 2-planes: desingularization and implications of the twistor fibration

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Cayley cones in the octonions $\mathbb{O}$ that are ruled by oriented 2-planes are equivalent to pseudoholomorphic curves in the Grassmannian of oriented 2-planes $G(2, \mathbb{O})$. The well known twistor fibration $G(2, \mathbb{O}) \to \mathbb{S}^6$ is used to prove the existence of immersed higher-genus pseudoholomorphic curves in $G(2, \mathbb{O})$. Equivalently, this produces Cayley cones whose links are $\mathbb{S}^1$-bundles over genus-g Riemann surfaces. When the degree of an immersed pseudoholomorphic curve is large enough, the corresponding 2-ruled Cayley cone is the asymptotic cone of a non-conical 2-ruled Cayley 4-fold.

1. Introduction

We begin by showing that r-oriented 2-ruled Cayley cones are equivalent to pseudoholomorphic curves in the Grassmannian of oriented 2-planes in $\mathbb{O}$. Results about such Cayley cones are then deduced by studying the geometry of the corresponding pseudoholomorphic curves. Most significantly, the twistor fibration $\mathcal{J} : G(2,\mathbb{O}) \to \mathbb{S}^6$ decomposes every pseudoholomorphic curve into an oriented branched minimal surface in $\mathbb{S}^6$, a branched cover of Riemann surfaces and a holomorphic line subbundle. This decomposition allows an easy proof of the existence of immersed higher-genus pseudoholomorphic curves, and thus Cayley cones whose links have complicated topology. By comparison, the proof of the existence of higher-genus embedded special Legendrian surfaces in $\mathbb{S}^5$ by Haskins and Kapouleas [17] requires very involved and delicate analysis. Neither the techniques nor the results of the current article seem to have any implications for special Legendrian surfaces in $\mathbb{S}^5$.

The geometry of a pseudoholomorphic curve in $G(2,\mathbb{O})$ defines a (possibly empty) class of deformations of the Cayley cone that preserve the properties of being Cayley and 2-ruled but destroy the cone structure. When the pseudoholomorphic curve has large negative degree, then such deformations are abundant and will generically produce a smooth Cayley 4-fold.

The method for generating new pseudoholomorphic curves from old ones can also be viewed as a type of Bäcklund transformation in the sense of [3, 11, 12]. An article describing this is under preparation. Similar transformations for certain harmonic maps have been described by Burstall in [6].

2. 2-Ruled Cayley cones

Cayley submanifolds are a special class of minimal 4-folds defined by first-order partial differential equations. We recall some of the basic facts and refer the reader to Appendix A and [16] for details.[1]

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[1]Beware that the choice of Cayley 4-form in this article differs from [16] and most other conventions.

Let $\mathbb{O}$ denote the octonions with standard coordinates $(x^1, \ldots, x^8) : \mathbb{O} \to \mathbb{R}^8$. There is a quadruple cross-product on $\mathbb{O}$

$$\mathbb{O} \times \mathbb{O} \times \mathbb{O} \times \mathbb{O} \to \mathbb{O}$$
$$(x, y, z, w) \mapsto x \times y \times z \times w$$

that is totally skew and whose components therefore define 4-forms on $\mathbb{O}$. Let $e_i^0 = \frac{\partial}{\partial x^i}$, so that $\{e_i^0\}$ provide an orthonormal basis of $\mathbb{O}$. Assume that the octonionic structure is chosen so that $e_8^0 = 1 \in \mathbb{O}$ and the rest span $\mathrm{Im}(\mathbb{O})$. Define the 4-forms

$$\Phi(x, y, z, w) = \langle 1, \, x \times y \times z \times w \rangle,$$
$$\psi_m(x, y, z, w) = \langle e_m^0, \, x \times y \times z \times w \rangle \ \text{ for } m = 1, \ldots, 7.$$

The stabilizer of Φ is $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$ and so Φ determines a $\mathrm{Spin}(7)$ structure on $\mathbb{O}$.[2] The structure equations for the $\mathrm{Spin}(7)$-principal bundle

$$\mathrm{Spin}(7) \longrightarrow \mathbb{O} \times \mathrm{Spin}(7)$$
$$\Big\downarrow \mathbf{x}$$
$$\mathbb{O}$$

take the standard form

(2.1)
$$d\mathbf{x} = e_i \omega_i,$$
$$de_i = e_j \omega_{ji},$$
$$d\omega_i = -\omega_{ij} \wedge \omega_j,$$
$$d\omega_{ij} = -\omega_{ik} \wedge \omega_{kj},$$

where the ω_{ij} satisfy the $\mathfrak{spin}(7)$-relations given in Equation (A.3) of Appendix A.

Definition 2.1. A smooth immersion $f : M^4 \to \mathbb{O}$ is a Cayley 4-fold if

(2.2) $$f^*(\psi_m) = 0.$$

Cayley 4-folds are calibrated by Φ and this implies that they are oriented. Neither the perspective of calibrations nor the orientability of Cayley 4-folds

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[2]In fact, Φ determines the Octonionic structure, a metric and an orientation on $\mathbb{R}^8$.

is needed in this article and so we refer the reader to [16] for the original definition of Cayley 4-folds in terms of calibrations. This article is concerned with the r-oriented Cayley cones ruled by 2-planes, which are now defined.

Definition 2.2. $M^4 \subset \mathbb{O}$ admits a *smooth ruling by 2-planes* (or is *2-ruled*) if there exists a smooth surface Σ and a smooth map $\pi : M \to \Sigma$ such that for all $\sigma \in \Sigma$, $\pi^{-1}(\sigma)$ is a 2-plane in $\mathbb{O}$. Such a triple (M, π, Σ) is said to be a *2-ruled* 4-fold. If there exists a continuous choice of orientation for $\pi^{-1}(\sigma)$, then M is said to be *r-oriented*.

Using parallel translation, each 2-plane in $\{\pi^{-1}(\sigma) : \sigma \in \Sigma\}$ can be translated so that it contains the origin. Thus π^{-1} defines a map

$$(2.3) \qquad\qquad \gamma : \Sigma \to G(2, \mathbb{O}),$$

where $G(2, \mathbb{O})$ is the Grassmannian of oriented 2-planes in $\mathbb{O}$. The triple (M, π, Σ) will be referred to as *non-degenerate* if the associated map γ is an immersion. The 2-dimensional family of 2-planes $\gamma(\Sigma)$ forms a cone $M_0 \subset \mathbb{O}$.

Definition 2.3. M_0 is the *asymptotic cone* of the 2-ruled 4-fold (M, π, Σ).

For a justification of the term *asymptotic cone*, see Definition 3.2 in [22] and the surrounding text. In Section 10, it will also be pointed out how each 2-ruled Cayley 4-fold comes in a 1-parameter family that has its asymptotic cone as a singular limit.

We now restrict our attention to the case in which (M, π, Σ) is already a cone. In Section 10, we return to the general case in order to prove that non-degenerate 2-ruled Cayley cones are the asymptotic cones of smooth 2-ruled Cayley 4-folds whenever their associated surface $\gamma : \Sigma \to G(2, \mathbb{O})$ satisfies a topological condition.

For a 2-ruled cone (M, π, Σ) the Cayley equations (2.2) reduce to equations on the surface Σ. A 2-ruled cone in $\mathbb{O}$ can locally be parametrized on an open set $U \subset \Sigma$ as

$$(2.4) \qquad\qquad \mathbf{x} : \mathbb{R}^2 \times U \to \mathbb{O},$$

$$(2.5) \qquad\qquad \mathbf{x}(r_1, r_2, \sigma) = r_1 e_1 + r_2 e_2,$$

where e_1 and e_2 are the first two legs of a Spin(7)-adapted frame e_i. Such a parametrization is always possible because Spin(7) acts transitively on

oriented orthonormal pairs of vectors in $\mathbb{O}$. This 4-fold will be Cayley if $\mathbf{x}^*(\psi_m) = 0$ for $m = 1\ldots7$. One computes that

$$\mathbf{x}^*(\psi_m) = 12\,dr_1 \wedge dr_2 \wedge \psi_m(e_1,\,e_2,\,r_1 e_i\omega_{i1} + r_2 e_j\omega_{j2},\,r_1 e_k\omega_{k1} + r_2 e_l\omega_{l2}).$$
(2.6)

Expanding in r_1 and r_2 leads to the following system on Σ:

(2.7)
$$\begin{aligned}
\psi_m(e_1,\,e_2,\,e_i,\,e_k)\omega_{i1} \wedge \omega_{k1} &= 0, \\
\psi_m(e_1,\,e_2,\,e_i,\,e_l)\omega_{i1} \wedge \omega_{l2} + \psi_m(e_1,\,e_2,\,e_j,\,e_k)\omega_{j2} \wedge \omega_{k1} &= 0, \\
\psi_m(e_1,\,e_2,\,e_j,\,e_l)\omega_{j2} \wedge \omega_{l2} &= 0.
\end{aligned}$$

For any such adapted coframe, define the complex forms

(2.8)
$$\begin{aligned}
\zeta_3 &= \omega_{31} + i\omega_{41}, & \zeta_4 &= \omega_{32} + i\omega_{42}, \\
\zeta_6 &= \omega_{61} - i\omega_{71}, & \zeta_7 &= \omega_{62} - i\omega_{72}, \\
\zeta_5 &= \omega_{51} - i\omega_{81}, & \zeta_8 &= \omega_{52} - i\omega_{82}.
\end{aligned}$$

All of the 2-forms that appear in Equation (2.7) can be written as linear combinations of the real and imaginary parts of $\zeta_i \wedge \zeta_j$. This indicates that, once the auxiliary directions of a 2-ruled Cayley cone are boiled off, one is left with a geometry of pseudoholomorphic curves, which we now explain.

Definition 2.4. Let (X^{2n}, J) be a smooth manifold with a smooth almost complex structure and Σ a smooth surface. A *pseudoholomorphic curve* is a smooth map

(2.9)
$$\gamma : \Sigma \to (X, J)$$

for which $\gamma_*(T_\sigma\Sigma)$ is a complex line in $(T_xX, J_x) \cong \mathbb{C}^n$ whenever it is of real dimension 2. The image $\gamma(\Sigma)$ inherits a complex structure and so Σ acquires the structure of a Riemann surface.

The linear map J (which also acts on T_x^*X) has eigenvalues $\pm i$ and so the module of smooth $\mathbb{C}$-valued 1-forms $\Omega_{\mathbb{C}}^1(X)$ splits into the $+i$ and $-i$ eigenbundles $\Omega^{(1,0)}(X)$ and $\Omega^{(0,1)}(X)$, each with fiber isomorphic to $\mathbb{C}^n$. Alternatively, one can define an almost complex structure on X be choosing

a splitting

$$(2.10) \qquad \Omega^1_{\mathbb{C}}(X) = \Omega^{(1,0)}(X) \oplus \Omega^{(0,1)}(X)$$

such that $\overline{\Omega^{(1,0)}(X)} = \Omega^{(0,1)}(X)$. An immersion of a (real) surface $\gamma : \Sigma \to X$ will be a pseudoholomorphic curve if and only if $\gamma^*(\alpha \wedge \beta) = 0$ for all $\alpha, \beta \in \Omega^{(1,0)}(X)$, since this is equivalent to the tangent space of $\gamma(\Sigma)$ spanning a complex line in $T^{(1,0)}_{\gamma(\sigma)}X$. It is this characterization of a pseudoholomorphic curve that we will use most often.

Similarly, for any real vector bundle V with a complex structure, one can define the $(1,0)$-bundle inside $V \otimes \mathbb{C}$. It will be denoted as V' and V'' will denote the $(0,1)$-bundle.

In the case at hand, define a form $\alpha \in \Omega^1_{\mathbb{C}}(G(2, \mathbb{O}))$ to be of type $(1,0)$ if for some (and thus any) $\mathrm{Spin}(7)$-adapted coframe ω_{ij}, the 1-form α is a section of the subbundle spanned by the ζ_i defined in Equation (2.8). This almost complex structure is $\mathrm{Spin}(7)$-invariant:

Proposition 2.1.

- $G(2, \mathbb{O}) = \mathrm{Spin}(7)/U(3)$ *and the* $U(3)$ *structure defines a* $\mathrm{Spin}(7)$-*invariant, non-integrable almost complex structure;*

- *non-degenerate* r-*oriented* 2-*ruled Cayley cones are equivalent to immersed pseudoholomorphic curves in* $G(2, \mathbb{O})$

Proof. The homogeneous structure is derived in [16]. The non-integrability of the $\mathrm{Spin}(7)$-invariant almost complex structure follows from the existence of the $\overline{\theta}_i \wedge \overline{\theta}_j$ terms in the structure equations derived below, Equation (4.3). This almost complex structure is referred to as J_2 in [25] and the non-integrability is given in Proposition 3.2 of [25].

The rest of the proof is similar to that of the analogous result for 2-ruled coassociative cones, Proposition 7.2 of [14], so here we only sketch the proof of the second statement. The Cayley conditions reduce to the vanishing of certain 2-forms on the surface $\gamma(\Sigma) \subset G(2, \mathbb{O})$ (Equation (2.7)) and these 2-forms are the real and imaginary parts of $(2,0)$-forms for the almost complex structure on $G(2, \mathbb{O})$ that is defined by declaring the ζ_i of Equation (2.8) to be of type $(1,0)$. This shows that pseudoholomorphic curves in $G(2, \mathbb{O})$ define r-oriented 2-ruled Cayley cones. The other direction is a little bit more involved. Suppose we have an r-oriented 2-ruled Cayley cone. If there exists a coframe so that none of the ζ_i vanish on the surface Σ, then the Cayley equations (2.7) imply that $\zeta_i \wedge \zeta_j$ vanish on the surface,

and thus it is a pseudoholomorphic curve on $G(2, \mathbb{O})$. If some ζ_i vanishes on Σ, then the vanishing of $d\zeta_i$ must be used to show that $\zeta_j \wedge \zeta_k = 0$ on Σ for all j and k. $\qquad\square$

The non-integrable Spin(7)-invariant almost complex structure is closely related to an integrable one, as will be described in Section 5.

There is another perspective on pseudoholomorphic curves in $G(2, \mathbb{O})$ which we will point out but not elaborate upon. The almost complex structure used on $G(2, \mathbb{O}) = \text{Spin}(7)/U(3)$ arises from its 3-symmetric space structure. Any k-symmetric space acts as the target for primitive maps [7] and in the case at hand the primitive maps are the same as the pseudoholomorphic curves. Primitive maps are a natural generalization of harmonic maps and have a similarly elaborate theory in case that the domain is $\mathbb{C}$, $\mathbb{S}^2$ or T^2. This article is aimed at exploiting the exceptionally rich geometric structure of $G(2, \mathbb{O})$ in order to obtain results about primitive maps from higher-genus domains.

3. The tautological bundles $\mathcal{H}$ and $\mathcal{E}$

On $G(2, \mathbb{O})$ there is a tautological complex line bundle $\mathcal{E}$ and a tautological complex 3-plane bundle $\mathcal{H}$

$$\mathbb{C} \oplus \mathbb{C}^3 \longrightarrow \mathcal{E} \oplus \mathcal{H}$$
$$\downarrow {\scriptstyle (\pi_\mathcal{E}, \pi_\mathcal{H})}$$
$$G(2, \mathbb{O})$$

which are defined as follows. For $E \in G(2, \mathbb{O})$ let $H = E^\perp$. Then

$$(3.1) \qquad \pi_\mathcal{E}^{-1}(E) = E, \quad \text{and} \quad \pi_\mathcal{H}^{-1}(E) = H$$

The transitive action of Spin(7) on $G(2, \mathbb{O})$ allows the frame to be chosen so that $E = e_1 \wedge e_2$. Let

$$(3.2) \qquad f_0 = e_1 - ie_2,$$

and $f = (f_1, f_2, f_3)$ where

$$f_1 = e_3 - ie_4,$$
$$f_2 = e_6 + ie_7,$$
$$f_3 = e_5 + ie_8.$$

Declaring the f_0, f_i to be of type $(1,0)$ defines a complex structure on $\mathcal{E} \otimes \mathbb{C}$ and $\mathcal{H} \otimes \mathbb{C}$.

We now calculate the structure equations for the tautological bundles $\mathcal{E}$ and $\mathcal{H}$. A unitary framing of $\mathcal{E}$ (of length 2) is given by f_0 and $f = (f_1, f_2, f_3)$ is a unitary framing of $\mathcal{H}$ of length 2. The $\mathfrak{spin}(7)$-structure equations (2.1) can be organized as

$$(3.3) \quad d(f_0, f) = (f_0, f) \begin{pmatrix} -\text{tr}(\kappa) & -\dfrac{1}{2}{}^t\bar{\theta}_{\text{ev}} \\ \dfrac{1}{2}\theta_{\text{ev}} & \kappa \end{pmatrix} + (\bar{f}_0, \bar{f}) \begin{pmatrix} 0 & -\dfrac{1}{2}{}^t\bar{\theta}_{\text{od}} \\ \dfrac{1}{2}\bar{\theta}_{\text{od}} & -\dfrac{i}{2}[\theta_{\text{od}}] \end{pmatrix},$$

where the symbols have the following definitions:

$$\theta_{\text{ev}} = {}^t\left(\theta_2 \quad \theta_4 \quad \theta_6\right),$$
$$\theta_{\text{od}} = {}^t\left(\theta_1 \quad \theta_3 \quad \theta_5\right),$$

$$\theta_1 = \zeta_3 + i\zeta_4, \quad \theta_2 = \zeta_3 - i\zeta_4,$$
$$\theta_3 = \zeta_6 + i\zeta_7, \quad \theta_4 = \zeta_6 - i\zeta_7,$$
$$\theta_5 = \zeta_5 + i\zeta_8, \quad \theta_6 = \zeta_5 - i\zeta_8.$$

$$\kappa = \begin{pmatrix} i\omega_{43} & -(\omega_{63} + i\omega_{64}) - \dfrac{i}{2}\bar{\theta}_5 & -(\omega_{53} + i\omega_{54}) + \dfrac{i}{2}\bar{\theta}_3 \\ (\omega_{63} - i\omega_{64}) - \dfrac{i}{2}\theta_5 & -i\omega_{76} & (\omega_{65} - i\omega_{75}) - \dfrac{i}{2}\bar{\theta}_1 \\ (\omega_{53} - i\omega_{54}) + \dfrac{i}{2}\theta_3 & -(\omega_{65} + i\omega_{75}) - \dfrac{i}{2}\theta_1 & i(\omega_{76} - \omega_{43} - \omega_{21}) \end{pmatrix}$$

and satisfies

$$^*\kappa = -\kappa,$$

and for any vector $v = {}^t\left(v_1 \quad v_2 \quad v_3\right)$,

$$(3.4) \qquad\qquad [v] = \begin{pmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{pmatrix}.$$

The map $v \to [v]$ gives an explicit isomorphism $\mathbb{C}^3 \to \Lambda^2\mathbb{C}^3$.

The form of the structure equations (3.3) modulo $\theta_{\mathrm{od}}, \theta_{\mathrm{ev}}, \bar\theta_{\mathrm{od}}, \bar\theta_{\mathrm{ev}}$ indicate that the stabilizer of E is isomorphic to $U(3)$ and that it acts on H in the standard way and on E as $\det^{-1}$.

Define the metric compatible connections $\nabla^{\mathcal{H}}$ and $\nabla^{\mathcal{E}}$ as

$$\nabla^{\mathcal{H}} f = \kappa f,$$
$$\nabla^{\mathcal{E}} f_0 = -\mathrm{tr}(\kappa) f_0.$$

Both $\mathcal{H}'$ and $\mathcal{E}'$ pull back to any pseudoholomorphic curve to be Hermitian using these connections. The curvature equations for these connections will also be used. For $\mathcal{E}$,

$$(3.5) \quad d(-\mathrm{tr}(\kappa)) = \frac{1}{4}(\theta_2 \wedge \bar\theta_2 + \theta_4 \wedge \bar\theta_4 + \theta_6 \wedge \bar\theta_6 - \theta_1 \wedge \bar\theta_1 - \theta_3 \wedge \bar\theta_3 - \theta_5 \wedge \bar\theta_5),$$

and for $\mathcal{H}$,

$$(3.6) \qquad\qquad d\kappa + \kappa \wedge \kappa = \frac{1}{4}\Omega,$$

where Ω is skew-hermitian with components

$$\Omega_{11} = \theta_1 \wedge \bar\theta_1 - \theta_3 \wedge \bar\theta_3 - \theta_5 \wedge \bar\theta_5 + \theta_2 \wedge \bar\theta_2, \quad \Omega_{21} = 2\theta_3 \wedge \bar\theta_1 + \theta_4 \wedge \bar\theta_2,$$
$$\Omega_{22} = -\theta_1 \wedge \bar\theta_1 + \theta_3 \wedge \bar\theta_3 - \theta_5 \wedge \bar\theta_5 + \theta_4 \wedge \bar\theta_4, \quad \Omega_{31} = 2\theta_5 \wedge \bar\theta_1 + \theta_6 \wedge \bar\theta_2,$$
$$\Omega_{33} = -\theta_1 \wedge \bar\theta_1 - \theta_3 \wedge \bar\theta_3 + \theta_5 \wedge \bar\theta_5 + \theta_6 \wedge \bar\theta_6, \quad \Omega_{32} = 2\theta_5 \wedge \bar\theta_3 + \theta_6 \wedge \bar\theta_4.$$

4. Structure equations for the tangent bundle of $G(2, \mathbb{O})$

We next derive the structure equations for the $U(3)$-structure on $G(2, \mathbb{O})$. Let

$$(4.1) \qquad \mathbf{p} = e_1 \wedge e_2 = -\frac{i}{2} f_0 \wedge \bar f_0 : \mathrm{Spin}(7) \to G(2, \mathbb{O}).$$

We differentiate:

$$d\mathbf{p} = \frac{1}{4}\left[i f_1 \wedge f_0 \theta_1 - i \bar f_1 \wedge \bar f_0 \bar\theta_1 \right] - \frac{1}{4}\left[i f_1 \wedge \bar f_0 \theta_2 - i \bar f_1 \wedge f_0 \bar\theta_2 \right]$$
$$+ \frac{1}{4}\left[i f_2 \wedge f_0 \theta_3 - i \bar f_2 \wedge \bar f_0 \bar\theta_3 \right] - \frac{1}{4}\left[i f_2 \wedge \bar f_0 \theta_4 - i \bar f_2 \wedge f_0 \bar\theta_4 \right]$$
$$+ \frac{1}{4}\left[i f_3 \wedge f_0 \theta_5 - i \bar f_3 \wedge \bar f_0 \bar\theta_5 \right] - \frac{1}{4}\left[i f_3 \wedge \bar f_0 \theta_6 - i \bar f_3 \wedge f_0 \bar\theta_6 \right].$$

Thus the $(1,0)$-vectors are spanned by $f_i \wedge f_0$ and $f_i \wedge \bar f_0$ for the dual $(1,0)$-coframe θ_i. This shows that the $(1,0)$-subbundle of the complexified tangent

bundle $TG(2, \mathbb{O}) \otimes \mathbb{C}$ splits,

$$(4.2) \qquad\qquad T^{(1,0)} G(2, \mathbb{O}) = \mathcal{V}_1' \oplus \mathcal{V}_2',$$

where each bundle $\mathcal{V}_i'$ has fiber $\mathbb{C}^3$. By comparing the respective bases of the $(1, 0)$-vectors in $\mathcal{H}$, $\mathcal{E}$, $\mathcal{V}_1$ and $\mathcal{V}_2$ we find that

$$\mathcal{V}_1 \cong \mathcal{H} \otimes \mathcal{E},$$
$$\mathcal{V}_2 \cong \mathcal{H} \otimes \mathcal{E}^*.$$

The structure equations for a coframe of $G(2, \mathbb{O})$ are

$$(4.3) \qquad\qquad d \begin{bmatrix} \theta_{\mathrm{od}} \\ \theta_{\mathrm{ev}} \end{bmatrix} = - \begin{bmatrix} \Psi & 0 \\ 0 & \tilde{\Psi} \end{bmatrix} \wedge \begin{bmatrix} \theta_{\mathrm{od}} \\ \theta_{\mathrm{ev}} \end{bmatrix} + \begin{bmatrix} \tau_{\mathrm{od}} \\ \tau_{\mathrm{ev}} \end{bmatrix},$$

where

$$\Psi = \kappa - \mathrm{tr}(\kappa),$$
$$\tilde{\Psi} = \kappa + \mathrm{tr}(\kappa),$$

and

$$\tau_1 = \frac{i}{2} \left(\bar{\theta}_4 \wedge \bar{\theta}_5 + \bar{\theta}_3 \wedge \bar{\theta}_6 \right), \quad \tau_2 = i\bar{\theta}_3 \wedge \bar{\theta}_5,$$
$$\tau_3 = \frac{i}{2} \left(\bar{\theta}_6 \wedge \bar{\theta}_1 + \bar{\theta}_5 \wedge \bar{\theta}_2 \right), \quad \tau_4 = i\bar{\theta}_5 \wedge \bar{\theta}_1,$$
$$\tau_5 = \frac{i}{2} \left(\bar{\theta}_2 \wedge \bar{\theta}_3 + \bar{\theta}_1 \wedge \bar{\theta}_4 \right), \quad \tau_6 = i\bar{\theta}_1 \wedge \bar{\theta}_3.$$

5. The fibration $G(2, \mathbb{O}) \to \mathbb{S}^6$ and the distributions $\mathcal{V}_1$ and $\mathcal{V}_2$

The results of this section are well known in the twistor theory literature. For example, see [8, 13, 24, 25] and chapter 9 of [21].

The splitting $TG(2, \mathbb{O}) = \mathcal{V}_1 \oplus \mathcal{V}_2$ is induced by the fibration

$$\mathbb{CP}^3 \longrightarrow G(2, \mathbb{O})$$
$$\Big\downarrow {\scriptstyle \mathscr{J}}$$
$$\mathbb{S}^6$$

and plays a fundamental role in the geometry of pseudoholomorphic curves in $G(2, \mathbb{O})$. In this section, the fibration and its most basic implications will

be described. In Section 7, a more detailed investigation of the fibration will locate the truly *holomorphic* aspects of the *pseudoholomorphic* curves in $G(2, \mathbb{O})$.

Using the exceptional isomorphism

$$\text{(5.1)} \qquad \text{Spin}(6) \cong \text{SU}(4),$$

the map $\mathcal{J}$ can be defined as follows. Each 2-plane $E \in G(2, \mathbb{O})$ has stabilizer $U(3) \subset \text{Spin}(7)$ and thus is given by a coset $g\,U(3) \in \text{Spin}(7)/U(3)$. There is a unique $\text{SU}(4) \subset \text{Spin}(7)$ containing that stabilizer. The map is

$$\mathcal{J} : \text{Spin}(7)/U(3) \to \text{Spin}(7)/\text{SU}(4),$$
$$\mathcal{J} : g\,U(3) \mapsto g\text{SU}(4).$$

The exceptional isomorphism (Equation (5.1)) provides the identification

$$\text{Spin}(7)/\text{SU}(4) = \text{Spin}(7)/\text{Spin}(6) = \mathbb{S}^6.$$

This gives $\mathbb{S}^6$ the interpretation as the space of possible reductions from the $\text{Spin}(7)$-structure to an $\text{SU}(4)$-structure.

It will be useful to understand how the fibration is manifested in the infinitesimal geometry.

Lemma 5.1.

- *The distribution $\mathcal{V}_1$ is not integrable. The map $\mathcal{J}_* : \mathcal{V}_1 \to T\mathbb{S}^6$ is an isomorphism.*

- *The distribution $\mathcal{V}_2$ is integrable and its maximal leaves are the $\mathbb{CP}^3$-fibers of $\mathcal{J}$.*

Proof. The first distribution is defined by the ideal

$$\text{(5.2)} \qquad \mathcal{I}_1 = \langle \text{Re}(\theta_{\text{ev}}), \text{Im}(\theta_{\text{ev}}) \rangle$$

and the second by the ideal

$$\text{(5.3)} \qquad \mathcal{I}_2 = \langle \text{Re}(\theta_{\text{od}}), \text{Im}(\theta_{\text{od}}) \rangle.$$

Structure equations (4.3) show that the first ideal is not Frobenius while the second is.

The maximal leaves of $\mathcal{I}_2$ are six-dimensional manifolds. On a leaf, the structure equations from (3.3) become

$$(5.4) \qquad\qquad d(f_0, f) = (f_0, f) \begin{pmatrix} -\operatorname{tr}(\kappa) & -\frac{1}{2}\,{}^t\overline{\theta}_{ev} \\ \frac{1}{2}\theta_{ev} & \kappa \end{pmatrix}.$$

Thus they are SU(4)-orbits with stabilizer $U(3)$, i.e., they are isomorphic to $\mathbb{CP}^3$. Since there is a unique SU(4) $\subset$ Spin(7) containing any given $U(3)$, these leaves must be the fibers of the map $\mathcal{J}$.

One checks that $\ker(\mathcal{J}_*) = \mathcal{V}_2$ and then the isomorphism $\mathcal{J}_* : \mathcal{V}_1 \to T\mathbb{S}^6$ follows from a dimension count. $\qquad\square$

The fibration distinguishes two special types of pseudoholomorphic curves in $G(2, \mathbb{O})$: those that are in a single fiber of $\mathcal{J}$ (tangent to $\mathcal{V}_2$) and those that are *horizontal* (tangent to $\mathcal{V}_1$). It is clear that the first type consists of actual algebraic curves in $\mathbb{CP}^3 \subset G(2, \mathbb{O})$. Let $P \in \mathbb{S}^6$ be the image under $\mathcal{J}$ of such a pseudoholomorphic curve. It defines an SU(4)-structure on $\mathbb{O}$ and thus a complex structure J. The pseudoholomorphic curves that are contained in a fiber $\mathcal{J}^{-1}(P)$ correspond to the 2-ruled Cayley cones in $\mathbb{O}$ that are actually holomorphic cones in $\mathbb{C}^4 \cong \mathbb{O}_J$.

We will now see that the second type also consists of algebraic curves. Let

$$(5.5) \qquad\qquad Q_6 = \{[z] \in \mathbb{CP}^7 \mid z_1^2 + \cdots + z_8^2 = 0\} \subset \mathbb{CP}^7$$

denote the six quadric, which is diffeomorphic to $G(2, \mathbb{O})$ via the map

$$G(2, \mathbb{O}) \to Q_6,$$
$$e_1 \wedge e_2 \mapsto [e_1 - ie_2].$$

The Spin(7)-invariant almost complex structure is not integrable and so does not induce the holomorphic structure on Q_6. However the two almost complex structures do agree on $\mathcal{V}_1$.

Lemma 5.2.

- *The complex 3-plane distribution $\mathcal{V}_1'$ is holomorphic with respect to the integrable complex structure on Q_6.*

- *The complex 3-plane distribution $\overline{\mathcal{V}_2'}$ is holomorphic with respect to the integrable complex structure on Q_6.*

To prove Lemma 5.2, we need to relate the holomorphic structure on Q_6 and the Spin(7)-invariant almost complex structure. In an SO(8)-adapted coframe (and so also in a Spin(7)-adapted coframe), the $(1, 0)$-forms of the holomorphic structure on Q_6 are spanned by

(5.6) $$\xi_j = \omega_{j1} + i\omega_{j2}$$

and satisfy

(5.7) $$d\xi_j = -i\omega_{21} \wedge \xi_j - \omega_{jk} \wedge \xi_k \quad (j, k > 2).$$

The closure of the algebraic ideal $\langle \xi_i \rangle$ under exterior differentiation implies the integrability of the corresponding complex structure. Equations (2.8) and (5.6) allow one to relate the ξ_i and the ζ_i.

Proof. We begin with $\mathcal{V}_1'$. The vanishing of the θ_{2j} is equivalent to

$$\xi_4 = -i\xi_3,$$
$$\xi_7 = i\xi_6,$$
$$\xi_8 = i\xi_5,$$

which define a holomorphic distribution on Q_6. One checks that the complex structures on $\mathcal{V}_1'$ and Q_6 agree.

Now consider $\overline{\mathcal{V}_2'}$. The vanishing of the θ_{2j-1} is equivalent to

(5.8) $$\xi_4 = i\xi_3,$$
$$\xi_7 = -i\xi_6,$$
$$\xi_8 = -i\xi_5,$$

which define a holomorphic distribution on Q_6. One can check that the complex structure of $\overline{\mathcal{V}_2'}$ agrees with that of Q_6.   □

Corollary 5.3. *Let* $\gamma : \Sigma \to G(2, \mathbb{O})$ *be a pseudoholomorphic curve in* $G(2, \mathbb{O})$.

- *If it is tangent to* $\mathcal{V}_1$, $\gamma(\Sigma)$ *is algebraic in* Q_6.
- *If it is tangent to* $\mathcal{V}_2$, *then* $\overline{\gamma(\Sigma)}$ *is algebraic in* $\mathbb{CP}^3 \subset Q_6$.

6. First fundamental forms

There are many invariants for pseudoholomorphic curves in $G(2, \mathbb{O})$. We now introduce two basic first-order invariants. Throughout this section, we

will make use of the natural identifications

$$(6.1) \qquad \mathbb{P}(\mathcal{V}_i' \otimes (T^*\Sigma)') \cong \mathbb{P}(\mathcal{V}_i') \cong \mathbb{P}(\mathcal{H}')$$

that arise from the natural isomorphisms $\mathcal{V}_1 = \mathcal{H} \otimes \mathcal{E}$ and $\mathcal{V}_2 = \mathcal{H} \otimes \mathcal{E}^{-1}$. Correspondingly, a (complex) line in any of these vector spaces will be freely identified with a line in the others.

Let

$$(6.2) \qquad \gamma : \Sigma \to G(2, \mathbb{O})$$

be a pseudoholomorphic curve. Denote the tangent bundle $T\Sigma$ by T.

Definition 6.1. The first fundamental forms of γ are

$$(6.3) \qquad I_1 = \gamma^*(f_0 \otimes f \otimes \theta_{\mathrm{od}}) \in \Gamma(\gamma^*(\mathcal{E}' \otimes \mathcal{H}') \otimes T'^*),$$
$$(6.4) \qquad I_2 = \gamma^*(\bar{f}_0 \otimes f \otimes \theta_{\mathrm{ev}}) \in \Gamma(\gamma^*(\mathcal{E}'' \otimes \mathcal{H}') \otimes T'^*).$$

Lemma 6.1. I_1 and I_2 are holomorphic sections and I_1 (resp. I_2) vanishes identically if and only if $\gamma(\Sigma)$ is tangent to $\mathcal{V}_2$ (resp. $\mathcal{V}_1$).

The proof of the lemma, which is just a calculation in local coordinates, takes the same shape as the proof of Lemma 4.2 in [4] or Lemma 3.1 in [26], and we direct the reader to either proof for details. The holomorphicity of I_2 is also a special case of Lemma 6.18 of [8]

As long as (Σ, γ) is not tangent to one of the holomorphic distributions, I_1 and I_2 only vanish at isolated points. Thus there exist holomorphic line subbundles of $\gamma^*(\mathcal{V}_i')$, of which the I_i are non-zero sections. For later use (see Lemma 7.1, Section 9 and Section 10), we define R_1 and R_2 to be the vanishing loci of I_1 and I_2.

7. The relationship to minimal surfaces in $\mathbb{S}^6$

We now consider in more detail the implications of the fibration

$$\begin{array}{ccc} \mathbb{CP}^3 & \longrightarrow & G(2, \mathbb{O}) \\ & & \downarrow {\scriptstyle \mathscr{J}} \\ & & \mathbb{S}^6 \end{array}$$

on the geometry of pseudoholomorphic curves in $G(2, \mathbb{O})$. The structure equations for $\mathbb{S}^6$ are related to those of $G(2, \mathbb{O})$ as follows. Let

$$(7.1) \qquad \eta_i = \text{Re}(\theta_i), \quad \sigma_i = \text{Im}(\theta_i), \quad \varphi = \text{Re}(\kappa), \quad \psi = \text{Im}(\kappa).$$

Then

$$(7.2) \qquad d \begin{pmatrix} \eta_{\text{od}} \\ \sigma_{\text{od}} \end{pmatrix} = - \begin{pmatrix} \phi + \frac{1}{2}[\sigma_{\text{ev}}] & -{}^t(\psi + \frac{1}{2}[\eta_{\text{ev}}]) \\ \psi + \frac{1}{2}[\eta_{\text{ev}}] & \phi - \frac{1}{2}[\sigma_{\text{ev}}] \end{pmatrix} \wedge \begin{pmatrix} \eta_{\text{od}} \\ \sigma_{\text{od}} \end{pmatrix}$$

are the structure equations for the standard $\text{SO}(6)$-structure on $\mathbb{S}^6$.

The fibration relates pseudoholomorphic curves in $G(2, \mathbb{O})$ to minimal surfaces in $\mathbb{S}^6$.

Lemma 7.1. *Let* $\gamma : \Sigma \to G(2, \mathbb{O})$ *be a pseudoholomorphic curve. Then*

$$\phi = \mathcal{J} \circ \gamma : \Sigma \to \mathbb{S}^6$$

is a minimal surface with a finite number of branch points at R_1, *the zero locus of the first fundamental form of* γ.

This is due to Salamon ([25], Theorem 3.5) so we only sketch a proof.

Proof. Locally choose an adapted coframe on the pseudoholomorphic curve such that

$$(7.3) \qquad\qquad\qquad \theta_3 = \theta_5 = \theta_6 = 0.$$

On the one hand, this implies that $\eta_3 = \eta_5 = \sigma_3 = \sigma_5 = 0$, so that η_1, σ_1 are a coframe for $\phi(\Sigma) \subset \mathbb{S}^6$. In the standard way, this allows one to express the second fundamental form in terms of certain connection forms from (7.2). On the other hand, differentiating the vanishing conditions in (7.3) implies that $\theta_1, \theta_2, \theta_4, \kappa_{21}, \kappa_{31}, \kappa_{32}$ are all of type $(1, 0)$. This now implies that the second fundamental form of $\phi(\Sigma)$ is trace-free.

The fact concerning branch points follows from the definition of I_1 and the fact that the real and imaginary parts of θ_{od} span the semi-basic forms for the fibration $\mathcal{J} : G(2, \mathbb{O}) \to \mathbb{S}^6$. □

Remark 7.2. As we will see below, in general the map

$$(7.4) \qquad\qquad\qquad\qquad \phi : \Sigma \to \mathbb{S}^6$$

is a multiple-sheeted branched cover onto its image $\phi(\Sigma) \subset \mathbb{S}^6$.

Lemma 7.1 invites the question: Is every minimal surface in $\mathbb{S}^6$ the image of a pseudoholomorphic curve? The minimal surfaces in $\mathbb{S}^6$ locally depend on eight functions of one variable, whereas the pseudoholomorphic curves in $G(2, \mathbb{O})$ depend on ten functions of one variable, which suggests an affirmative answer. Moreover, this function count suggests that there should be two functions of one variable's worth of pseudoholomorphic curves sitting over any minimal surface, at least locally. To understand this better, it is useful to consider the fibration from a slightly different perspective.

Let

$$\mathbb{C}^4 \longrightarrow \Delta$$
$$\downarrow$$
$$\mathbb{S}^6$$

be the spinor bundle on $\mathbb{S}^6$ induced from the homogeneous structure

$$\mathbb{S}^6 = \mathrm{Spin}(7)/\mathrm{Spin}(6).$$

The action of $\mathrm{Spin}(6)$ on the fibers is via the isomorphism $\mathrm{Spin}(6) = \mathrm{SU}(4)$ and the standard action of $\mathrm{SU}(4)$ on $\mathbb{C}^4$.

Lemma 7.3. *As fiber bundles over* $\mathbb{S}^6$, *$G(2, \mathbb{O}) \cong \mathbb{P}(\Delta)$.*

Proof. The preimage under $\mathcal{J}$ of a point $P \in \mathbb{S}^6$ is an $\mathrm{SU}(4)$ orbit since $\mathrm{Stab}(P) = \mathrm{Spin}(6) \cong \mathrm{SU}(4) \subset \mathrm{Spin}(7)$. Let J be the complex structure on $\mathbb{O}$ defined by P. The orbit is a $\mathbb{CP}^3 = \mathrm{SU}(4)/U(3)$ since the stabilizer of a point in $G(2, \mathbb{O})$ is $U(3)$. So any point in $\mathcal{J}^{-1}(P)$ is a complex line in $\mathbb{C}^4 = \mathbb{O}_J$, which is the same as the fiber of $\mathbb{P}(\Delta)$. $\qquad\square$

Now we begin to work out how the geometry of the spinor bundle captures the geometry of a minimal surface. Let

$$(7.5) \hspace{4cm} \phi : \Sigma \to \mathbb{S}^6$$

be a branched minimal immersion of an *oriented* surface. This induces a conformal structure on Σ. Let T denote the tangent bundle of $\phi(\Sigma)$, T' the corresponding $(1, 0)$ component of the complexified tangent bundle, and let N denote the normal bundle of $\phi(\Sigma) \subset \mathbb{S}^6$, a real rank-4 bundle.

The product formula for Stiefel–Whitney classes implies that the normal bundle of an oriented surface in a six-dimensional spin manifold has a spin structure. Denote the principal spin bundle as $\mathrm{Spin}(N) = \mathrm{Spin}(N)^+ \times \mathrm{Spin}(N)^-$ and the associated spinor bundles as $\Delta_\pm$. A change in orientation

on Σ will interchange Δ_+ and Δ_-, so we can focus our attention on Δ_+ and similar results follow for Δ_- by changing the orientation of Σ.

Lemma 7.4. *There is a decomposition*

$$(7.6) \qquad\qquad \phi^*(\Delta) = \mathcal{W}_+ \oplus \mathcal{W}_-,$$

where

$$\mathcal{W}_+ = \Delta_+ \otimes T',$$
$$\mathcal{W}_- = \Delta_- \otimes (T')^*,$$

and their principal bundles are respectively

$$\mathrm{Spin}^c(N)^+ = (\mathrm{Spin}(N)^+ \times \mathrm{Spin}(T))/\mathbb{Z}_2,$$
$$\mathrm{Spin}^c(N)^- = (\mathrm{Spin}(N)^- \times \mathrm{Spin}(T))/\mathbb{Z}_2.$$

Proof. The decomposition

$$(7.7) \qquad\qquad \phi^*(T\mathbb{S}^6) = T\Sigma \oplus N\Sigma$$

corresponds to the reduction of the structure group

$$\mathrm{Spin}(6)$$
$$\cup$$
$$(\mathrm{Spin}(2) \times \mathrm{Spin}(3)^+ \times \mathrm{Spin}(3)^-)/\mathbb{Z}_2.$$

Using the exceptional isomorphisms, this is equivalent to the reduction

$$\mathrm{SU}(4)$$
$$\cup$$
$$S(U(2) \times U(2)),$$

which implies the splitting of the spinor bundle Δ given above. $\qquad\square$

Lemma 7.5. *Suppose that $\phi : \Sigma \to \mathbb{S}^6$ is minimal and oriented. Any holomorphic line subbundle $L \subset \mathcal{W}_+$ defines a* pseudoholomorphic *lift*

$$(7.8) \qquad\qquad \gamma_{(\phi,L)} : \Sigma \to G(2, \mathbb{O}).$$

Proof. The prescription defines a lift because $L \in \mathbb{P}(\mathcal{W}_+) \subset \mathbb{P}(\phi^*\Delta) = \phi^* G(2, \mathbb{O})$. We must check that it is holomorphic, and to do so we calculate locally. In a neighborhood of any point, there exists a coframe in which $\theta_3 = \theta_5 = 0$, and then the minimality implies that θ_4, θ_6, κ_{21}, κ_{31} are of type $(1,0)$ with respect to the holomorphic structure induced by θ_1.

Locally we have

$$(7.9) \qquad\qquad\qquad \mathcal{W}_+ = \mathbb{C} \cdot \{f_0, f_1\},$$

and this bundle has the induced connection

$$\nabla^+(f_0, f_1) = (f_0, f_1) \begin{pmatrix} -\mathrm{tr}(\kappa) & -\frac{1}{2}\bar{\theta}_2 \\ \frac{1}{2}\theta_2 & \kappa_{11} \end{pmatrix}.$$

A further frame adaptation allows $L = \mathbb{C} \cdot \{f_0\}$. It then follows that locally

$$\mathrm{I\!I}_L = f_1 \otimes f_0^* \otimes \theta_2,$$

where $\mathrm{I\!I}_L$ is the second fundamental form of the line subbundle L. In Appendix B the second fundamental form of a line subbundle is defined and Lemma B.1 implies that θ_2 is of type $(1,0)$ since L is holomorphic. This implies that

$$(7.10) \qquad\qquad\qquad \gamma_{(\phi,L)} = [f_0] \circ \phi : \Sigma \to G(2, \mathbb{O})$$

is a pseudoholomorphic curve since $\theta_i \wedge \theta_j = 0$ for all i, j. $\square$

Remark 7.6. Choosing a holomorphic line subbundle of $\mathcal{W}_+$ is locally equivalent to choosing a holomorphic map $\Sigma \to \mathbb{CP}^1$ which depends on two functions of one variable. This clarifies the function count made earlier.

Corollary 7.7. *The pseudoholomorphic curve $(\Sigma, \gamma_{(\phi,L)})$ projects to $\phi(\Sigma)$ in a one-to-one fashion, and so every minimal surface in $\mathbb{S}^6$ is the image of a pseudoholomorphic curve in $G(2, \mathbb{O})$. This is true even if ϕ has branch points.*

This corollary is also a special case of a theorem of Rawnsley ([24], Theorem 9.10).

The results of this section are summarized in the following theorem, which decomposes every pseudoholomorphic curve in $G(2, \mathbb{O})$ into a (possible branched) minimal surface in $\mathbb{S}^6$ and holomorphic data.

Theorem 7.8. *Any pseudoholomorphic curve* $\gamma : \Sigma \to G(2, \mathbb{O})$ *is equivalent to the following data:*

- $\phi' : \Sigma' \to \mathbb{S}^6$, *a branched minimal immersion of an oriented surface;*

- $\psi : \Sigma \to \Sigma'$, *a branched cover of Riemann surfaces;*

- $\phi : \Sigma \to \mathbb{S}^6$, *a branched minimal immersion of an oriented surface;*

- L, *a holomorphic line subbundle of* $\phi^*(\mathcal{W}_+)$;

- $\gamma = \gamma_{(\phi, L)}$, *where this map is defined as in Lemma* 7.5,

where $\phi = \phi' \circ \psi$ *and* $\phi(\Sigma') = \mathcal{J} \circ \gamma(\Sigma)$.

Remark 7.9. The bulk of the geometry of a pseudoholomorphic curve in $G(2, \mathbb{O})$ arises from a minimal surface in $\mathbb{S}^6$. Much is known about minimal spheres in $\mathbb{S}^6$. Minimal spheres are always isotropic and thus lift to $G(2, \mathbb{O})$ to be tangent to $\mathcal{V}_1$. They are therefore algebraic curves in Q_6 and it has been shown that the moduli space of minimal spheres of fixed degree deg is an algebraic variety of complex dimension $9 + 2\deg$. The short survey by Bolton and Woodward [2] addresses these main points but is a little bit out of date by now.

Less is known about minimal tori in $\mathbb{S}^6$, though there is a spectral curve. It would be nice to have an existence theorem for such tori along the lines of the work of Carberry and McIntosh [9, 10].

For higher-genus minimal surfaces, it may be fruitful to mimic the twistor methods of Bryant [4]. He shows that every compact Riemann surface admits a *branched* immersion into $\mathbb{S}^6 = G_2/\mathrm{SU}(3)$ as a null-torsion pseudoholomorphic curve. The null-torsion condition is similar to the isotropic condition of [2]. It seems likely that Bryant's approach could be extended to prove the existence of branched minimal surfaces of every genus in $\mathbb{S}^6$ that are not pseudoholomorphic for any G_2-invariant almost complex structure. These would all arise from pseudoholomorphic curves in $G(2, \mathbb{O})$ that are tangent to $\mathcal{V}_1$. Xu [26] used this method to prove the existence of higher-genus pseudoholomorphic curves in the nearly Kähler $\mathbb{CP}^3$.

Without going into detail, I want to describe an interesting degenerate case. From this degenerate case one recovers the surprising duality between minimal surfaces in $\mathbb{S}^5$ and ruled minimal Lagrangian 3-folds in $\mathbb{CP}^3$, which appeared in the work of Bolton and Vrancken [1]. They work directly with a moving frame instead of using the 2-ruled special Lagrangian cones as a

link between the two geometries. At the most basic level, this equivalence is a consequence of the exceptional isomorphism $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$.

Choosing a point $P \in \mathbb{S}^6$ defines both the perpendicular 5-sphere $\iota : \mathbb{S}^5_P \to \mathbb{S}^6$ (if P is a pole, $\mathbb{S}^5_P$ is the equator) and an $\mathrm{SU}(4)$-structure on $\mathbb{O}$. Let (Ω, Υ) be the Kähler and holomorphic volume forms for the $\mathrm{SU}(4)$-structure. One finds that $\Phi = \frac{\Omega^2}{2} + \mathrm{Re}(\Upsilon)$. If $\mathrm{Re}(\Upsilon)$ vanishes on a 2-ruled Cayley cone, then it is a 2-dimensional holomorphic cone in $\mathbb{C}^4$, and this is equivalent to the corresponding pseudoholomorphic curve being contained in the fiber $\mathbb{CP}^3_P = \mathcal{J}^{-1}(P)$.

If $\frac{\Omega^2}{2}$ vanishes on a 2-ruled Cayley cone, then it is a special Lagrangian cone and the associated pseudoholomorphic curve is contained in $\mathcal{J}^{-1}(\mathbb{S}^5_P)$. In fact, $\mathcal{J}^{-1}(\mathbb{S}^5_P) = \mathrm{Gr}^0(2, \mathbb{C}^4)$, where $\mathrm{Gr}^0(2, \mathbb{C}^4)$ is the Grassmannian of oriented isotropic real 2-planes in $\mathbb{C}^4$. The image of the pseudoholomorphic curve in $\mathbb{S}^5_P$ is necessarily minimal. One can check that there is at most a discrete family of pseudoholomorphic curves in $\mathrm{Gr}^0(2, \mathbb{C}^4)$ that project to the same minimal surface in $\mathbb{S}^5_P$ even locally.

Given a minimal surface in $\mathbb{S}^5_P$, one can lift it to a pseudoholomorphic curve in $\mathrm{Gr}^0(2, \mathbb{C}^4) \subset G(2, \mathbb{O})$, and this defines a 2-ruled special Lagrangian cone $L \subset \mathbb{C}^4$. Then the link $M^3 = L^4 \cap \mathbb{S}^7$ projects down to $\mathbb{CP}^3 = \mathbb{S}^7/\mathbb{S}^1$ to be a minimal Lagrangian submanifold that is fibered by geodesic $\mathbb{S}^1$'s. This process can clearly be reversed, and this gives a correspondence between minimal surfaces in $\mathbb{S}^5$ and minimal Lagrangian 3-folds (ruled by geodesic $\mathbb{S}^1$'s) in $\mathbb{CP}^3$.

8. The degree

We quickly review the definition of the degree of a pseudoholomorphic curve in $G(2, \mathbb{O})$. It will be used in Sections 9 and 10.

The $(1, 1)$-form on $G(2, \mathbb{O})$ is

$$(8.1) \qquad \omega = \frac{i}{2}[\theta_1 \wedge \bar{\theta}_1 + \theta_3 \wedge \bar{\theta}_3 + \theta_5 \wedge \bar{\theta}_5 + \theta_2 \wedge \bar{\theta}_2 + \theta_4 \wedge \bar{\theta}_4 + \theta_6 \wedge \bar{\theta}_6].$$

The splitting $\Omega^{(1,0)}(G(2, \mathbb{O})) \cong \mathcal{V}_1^{*\prime} \oplus \mathcal{V}_2^{*\prime}$ provides a decomposition

$$(8.2) \qquad \omega = \omega_1 + \omega_2$$

into the odd and even halves. These $(1, 1)$-forms are not closed:

$$(8.3) \qquad d\omega_1 = d\omega_2 = -\mathrm{Re}(\theta_2 \wedge \theta_3 \wedge \theta_5 + \theta_1 \wedge \theta_4 \wedge \theta_5 + \theta_1 \wedge \theta_3 \wedge \theta_6),$$

but their difference is. One can check that $\omega_1 - \omega_2$ is the standard Kähler form on $G(2, \mathbb{O})$ induced by the diffeomorphism $G(2, \mathbb{O}) \cong Q_6$.

A topological measure of the complexity of a pseudoholomorphic curve in $G(2, \mathbb{O})$ is its *degree*.

Definition 8.1. The degree of a pseudoholomorphic curve $\gamma : \Sigma \to G(2, \mathbb{O})$ is defined to be

$$(8.4) \qquad\qquad d_\gamma := \int_\Sigma \gamma^*(c_1(\mathcal{E})),$$

where $c_1(\mathcal{E})$ is the first Chern class of $\mathcal{E}$.

The connection on $\mathcal{E}$ is $-\mathrm{tr}(\kappa)$, so that its first chern class is

$$(8.5) \quad c_1(\mathcal{E}) = \left[\frac{i}{2\pi} d(-\mathrm{tr}(\kappa)) \right]$$

$$(8.6) \qquad = \frac{1}{4\pi} \frac{i}{2} [(\theta_1 \wedge \bar{\theta}_1 + \theta_3 \wedge \bar{\theta}_3 + \theta_5 \wedge \bar{\theta}_5) - (\theta_2 \wedge \bar{\theta}_2 + \theta_4 \wedge \bar{\theta}_4 + \theta_6 \wedge \bar{\theta}_6)]$$

or

$$(8.7) \qquad\qquad c_1(\mathcal{E}) = \frac{1}{4\pi}(\omega_1 - \omega_2).$$

9. The existence of higher-genus immersed curves

The characterization of pseudoholomorphic curves in $G(2, \mathbb{O})$ provided by Theorem 7.8 leads to a simple method for producing immersed higher-genus pseudoholomorphic curves from those of lower genus, or simply from a minimal surface in $\mathbb{S}^6$. Begin with a (possibly branched) minimal surface

$$(9.1) \qquad\qquad \phi' : \Sigma' \to \mathbb{S}^6.$$

Now choose a higher-genus Riemann surface Σ that admits a branched cover $\psi : \Sigma \to \Sigma'$ and let $\phi = \phi' \circ \psi$. This defines a branched minimal immersion

$$(9.2) \qquad\qquad \phi : \Sigma \to \mathbb{S}^6$$

that multiply covers itself. This induces the rank 2 Hermitian vector bundle $\mathcal{W}_+$ over Σ. To lift Σ to a pseudoholomorphic curve in $G(2, \mathbb{O})$ requires choosing a holomorphic line subbundle of $\mathcal{W}_+$. Such subbundles are bountiful.

Theorem 9.1. *Let (Σ', ϕ') be a minimal immersion, $\psi : \Sigma \to \Sigma'$ a branched cover of Riemann surfaces and $\phi = \phi' \circ \psi : \Sigma \to \mathbb{S}^6$.*

- *There exist (possibly branched) immersions (Σ, γ) as pseudoholomorphic curves such that $\phi = \mathcal{J} \circ \gamma$ and for which (Σ, γ) has arbitrarily large negative degree.*

- *There exist immersions $\gamma : \Sigma \to G(2, \mathbb{O})$ as pseudoholomorphic curves such that $\phi = \mathcal{J} \circ \gamma$.*

Remark 9.2. One could choose $\Sigma' = \mathbb{CP}^1$ and then every compact Riemann surface becomes a branched cover of it by choosing a meromorphic function on the Riemann surface. Therefore, every compact Riemann surface appears as an immersed pseudoholomorphic curve in $G(2, \mathbb{O})$.

The proof of Theorem 9.1 follows from the following sequence of lemmas. From the discussion prior to the statement of the theorem, we must find a holomorphic line subbundle of $\mathcal{W}_+$ with the desired properties. We will first point out why every Hermitian vector bundle over a Riemann surface admits holomorphic line subbundles of arbitrarily large negative degree.

Lemma 9.3. *Let $\mathbb{C}^2 \to E \to \Sigma$ be a holomorphic vector bundle over a holomorphic curve. For every line bundle $\mathcal{L}$ on Σ of degree $k > 0$, there exists $n \gg 1$ such that $\mathcal{L}^{-n} \subset E$ is a holomorphic subbundle of degree $-kn < 0$.*

Proof. Let $E_n = E \otimes \mathcal{L}^n$. For n large enough, there exist holomorphic sections $(s_1, \ldots, s_r)$ of E_n that span $(E_n)_\sigma$ for all $\sigma \in \Sigma$. In this case, there exists a nowhere vanishing section s (see exercise 8.2 in [15]) which defines an inclusion

$$(9.3) \qquad\qquad 0 \to \mathcal{O}_\Sigma \to E_n$$

of the trivial line bundle $\mathcal{O}_\Sigma$. In turn, this defines a holomorphic inclusion

$$(9.4) \qquad\qquad 0 \to \mathcal{L}^{-n} \to E.$$

$\square$

The holomorphic line subbundle $\mathcal{L} \subset \mathcal{W}_+$ used to lift $\phi : \Sigma \to \mathbb{S}^6$ to a pseudoholomorphic curve $\gamma : \Sigma \to G(2, \mathbb{O})$ will be the pullback of the tautological bundle of $G(2, \mathbb{O})$, $\mathcal{L} = \gamma^*(\mathcal{E})$. Thus the degree of γ is the degree of $\mathcal{L}$. By Lemma 9.3, $\mathcal{L}$ can be chosen to have arbitrarily large negative degree.

This shows that there is always a lift of $\phi : \Sigma \to \mathbb{S}^6$ to a pseudoholomorphic curve with arbitrarily large negative degree, proving the first statement of Theorem 9.1.

Now we turn to the second statement. Let $\phi : \Sigma \to \mathbb{S}^6$ be a branched minimal immersion and $\mathcal{L} \subset \mathcal{W}_+$ a holomorphic subbundle. Let $\gamma := \mathcal{L} \circ \phi : \Sigma \to G(2, \mathbb{O})$ be the holomorphic lift given by Lemma 7.5. Recall that $R_i = \{x \in \Sigma : \mathrm{I}_i(x) = 0\}$. Then γ is an immersion when

$$(9.5) \qquad\qquad R_1 \cap R_2 = \varnothing.$$

Locally one can adapt frames such that $\mathcal{L} = [f_0]$. Since I_1 only depends on θ_{od}, it is an invariant of ϕ and so is independent of the choice of $\mathcal{L}$. In contrast, I_2 very much depends upon the geometry of $\mathcal{L}$. For example, the second fundamental form of $\mathcal{L}$ is

$$(9.6) \qquad\qquad \mathrm{II}_{\mathcal{L}} = \pi_{\mathcal{L}}^{\perp} \circ \nabla^+ = f_1 \otimes f_0^* \otimes \frac{1}{2}\theta_2$$

and I_2 also involves θ_2. To show that $R_1 \cap R_2 = \varnothing$, it is sufficient to find a holomorphic line subbundle $\mathcal{L}$ such that $\mathrm{II}_{\mathcal{L}}$ does not vanish on the finite set of points R_1. The existence of such an $\mathcal{L}$ is a consequence of the following lemma.

Lemma 9.4. *Let $\mathbb{C}^2 \to E \to \Sigma$ be a hermitian vector bundle on a holomorphic curve with connection ∇ and hermitian inner product $(\, , \,)$. Let*

$$(9.7) \qquad\qquad D = \{x_1', \ldots, x_m'\}$$

be any finite set of points in Σ. For any holomorphic line subbundle $L \subset E$, let

$$(9.8) \qquad\qquad D_L = \{\sigma \in \Sigma : \mathrm{II}_L(\sigma) = 0\}.$$

Then there exists a holomorphic line subbundle $L \subset E$ such that $D \cap D_L = \varnothing$.

Proof. Choose a meromorphic section s of E for which $s(x_i') \neq 0, \infty$ for all $i = 1, \ldots, m$. Let $V_i \subset E_{x_i'}$ be the subspaces spanned by $s(x_i')$. Let

$$(9.9) \qquad Z = \{t \in \Gamma(E) : t \text{ is meromorphic}, \ t(x_i') = 0, \ \nabla t(x_i') \pitchfork V_i\}^3.$$

---

[3]This set is always non-empty. For instance, let t' be a meromorphic section of E such that $\nabla t(x_i') \pitchfork V_i$ for all $i = 1, \ldots, m$. The set of such sections is open in

Let $s_\epsilon = s + \epsilon t$ for any $t \in Z$, let $L_\epsilon \subset E$ be the line subbundle that it defines, let $\mathbb{II}_\epsilon$ be its second fundamental form and let $D_\epsilon = D_{L_\epsilon}$. Define a map

$$M_\epsilon : Z \to \oplus_{i=1}^m \left(E_{x_i'} \otimes T_{x_i'}^* \Sigma \right),$$

where $M_\epsilon = (M_\epsilon^1, \ldots, M_\epsilon^m)$ and

(9.10) $$M_\epsilon^i(t) = \mathbb{II}_{L_\epsilon}(s_\epsilon)(x_i').$$

Due to the vanishing property of the t, we have

(9.11) $$M_\epsilon^i(t) = \mathbb{II}_L(s)(x_i') + \epsilon(\pi_L^\perp(\nabla t(x_i'))).$$

By definition, $\pi_L^\perp(\nabla t(x_i')) \neq 0$. Thus for each $i = 1, \ldots, m$ there is a unique ϵ_i such that $M_{\epsilon_i}^i(t) = 0$. Choose $\epsilon' \neq \epsilon_i$ for all $i = 1, \ldots, m$. Then $M_{\epsilon'}^i(t) \neq 0$ and thus $D_\epsilon \cap D = \varnothing$. □

10. 2-Ruled Cayley 4-folds with a fixed asymptotic cone

In this section, we build upon the work of Bryant [5], Joyce [19] and Lotay [22] on ruled calibrated submanifolds. We study the space of 2-ruled Cayley 4-folds (M, π, Σ) that have a fixed asymptotic cone M_0. This is a linear space, in fact the kernel of a first-order linear differential operator [22]. We will show that when the pseudoholomorphic curve defining a 2-ruled Cayley cone M_0 has sufficiently negative degree, then M_0 is the asymptotic cone of some non-conical 2-ruled Cayley 4-fold. Lotay obtained an analogous result in the case that the pseudoholomorphic curve is a T^2 of degree 0. If the pseudoholomorphic curve is embedded (so that the cone is smooth away from the origin), then this results in a smooth 2-ruled Cayley 4-fold M with asymptotic cone M_0. In fact, due to the linear structure, it gives rise to a 1-parameter family of smooth 2-ruled Cayley 4-folds that degenerate to M_0 (see the paragraph after Lemma 10.1). Finding a smooth Cayley 4-fold M with a fixed asymptotic cone is also one of the ingredients used in the process of desingularizing calibrated submanifolds with conical singularities [18, 20, 23].

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the set of all meromorphic sections. Choose a meromorphic function u such that $u(x_i') = 0$ but $du(x_i') \neq 0$. Then $t = t'u$ is in Z.

Let

$$(10.1) \qquad \gamma : \Sigma \to G(2, \mathbb{O})$$

be a pseudoholomorphic curve and $U \subset \Sigma$ an open neighborhood on which one can write

$$(10.2) \qquad \gamma = e_1 \wedge e_2$$

for e_1, e_2, the first two legs of a Spin(7)-adapted framing. A 2-ruled 4-fold asymptotic to this cone can locally be parametrized as

$$\mathbf{x} : \mathbb{R}^2 \times U \to \mathbb{O},$$
$$(r_1, r_2, \sigma) \mapsto r_1 e_1 + r_2 e_2 + s,$$

where $s \in C^\infty(\mathcal{H})$. This 4-fold will be Cayley if

$$(10.3) \qquad \mathbf{x}^*(\psi_m) = 0.$$

Let $s = e_a s^a$ for $8 \geq a, b, c \geq 3$ and also define the 1-forms $s'^a := ds^a + \omega^a_b s^b$. Using the fact that the asymptotic cone is already Cayley, the condition for $\mathbf{x}$ to be Cayley simplifies to

$$(10.4) \qquad \begin{aligned} \psi_m(e_1, e_2, ds, ds) &= 0, \\ \psi_m(e_1, e_2, de_i, ds) &= 0, \end{aligned}$$

for $i = 1, 2$. Notice that we may replace ds by $\nabla^\mathcal{H} s$ due to the congruence $\nabla^\mathcal{H} s \equiv ds$ modulo e_1, e_2.

The solutions to these quadratic equations are the same as the solutions to a *linear $\bar\partial$-system* which we now introduce. Let

$$\begin{aligned} a_1 &= s^3 + is^4, \\ a_2 &= s^6 - is^7, \\ a_3 &= s^5 - is^8, \end{aligned}$$

so that if $\sigma = f\,a$ is a local section of $\mathcal{H}'$, then $\mathrm{Re}(\sigma) = s$ is the corresponding section of $\mathcal{H}$. Similarly, define the complex 1-forms

$$\begin{aligned} \alpha_1 &= s'^3 + is'^4, \\ \alpha_2 &= s'^6 - is'^7, \\ \alpha_3 &= s'^5 - is'^8. \end{aligned}$$

Now recall that $\langle\,,\,\rangle$ is the positive definite symmetric bilinear form on $\mathbb{O}$ (extended to $\mathbb{O}\otimes_\mathbb{R}\mathbb{C}$ as needed), so that $(v,w)=\langle\bar{v},\,w\rangle$ is a Hermitian inner product. In terms of a local coframe, we find that for $i=1,2,3$

$$\alpha_i = \langle\bar{f}_i,\,\nabla^{\mathcal{H}}s\rangle = \langle\bar{f}_i,\,d\operatorname{Re}(\sigma)\rangle$$

$$= \left\langle\bar{f}_i,\operatorname{Re}\left(f(\kappa a+da)-\frac{i}{2}\bar{f}[\theta_{od}]a\right)\right\rangle$$

$$= da_i + \kappa_{ij}a_j + \frac{i}{2}[\bar{\theta}_{od}]_{ij}\bar{a}_j.$$

Lemma 10.1. *Still using the notation above, the condition for* $\mathbf{x}=s+r_1e_1+r_2e_2$ *to be Cayley (Equation (10.4)) is equivalent to the* α_i *being of type* $(1,0)$ *on* Σ.

Proof. Here is an outline. Since (Σ,γ) is pseudoholomorphic, there exist smooth functions $A_i:\Sigma\to\mathbb{C}$ such that $\theta_i=A_i\,dz$, where $z:\Sigma\to\mathbb{C}$ is a local holomorphic coordinate. One must check that the equations in (10.4) are equivalent to the vanishing of the real and imaginary parts of $\alpha_i\wedge dz$, which is a routine calculation. $\square$

From this, we see that the space of 2-ruled Cayley 4-folds with fixed asymptotic cone is the kernel of a first-order linear differential operator. Locally the equation is

$$\bar{\partial}a_i + \pi^{(0,1)}(\kappa_{ij})a_j + \frac{i}{2}[\bar{\theta}_{\mathrm{od}}]_{ij}\bar{a}_j = 0,$$

where $\pi^{(0,1)}:\Omega^1\to\Omega^{(0,1)}$ is the standard projection. So if $\mathbf{x}=s+r_1e_1+r_2e_2$ is Cayley, then so is $\mathbf{x}_\lambda=\lambda s+r_1e_1+r_2e_2$ for all $\lambda\in\mathbb{R}$. Thus the limit of the image of $\mathbf{x}_\lambda$ as $\lambda\to 0$ is M_0. It is in this sense that $\mathbf{x}=\mathbf{x}_{\lambda=1}$ is a desingularization of its asymptotic cone.

The fact that the Cayley equations reduce to a system of linear equations that are $\bar{\partial}$ to highest order makes one hope that they are equivalent to some functions being holomorphic. This does not appear to be the case. For instance, σ is a holomorphic section of $\mathcal{H}'$ if

(10.5) $0 = \bar{\partial}\sigma = \pi^{(0,1)}(da+\kappa a).$

Unfortunately, the condition for s to define a Cayley 4-fold is not equivalent to this due to the $[\bar{\theta}_{\mathrm{od}}]$ term in the expression for α_i. However, there is always a holomorphic line subbundle of $\mathcal{H}'$ whose holomorphic sections do define Cayley 4-folds.

Let $\mathcal{L} \subset \mathcal{H}'$ be the holomorphic line subbundle defined by $I_1 \in \Gamma(\mathcal{V}_1')$ (see Section 6) and the equivalence $\mathbb{P}(\mathcal{H}') = \mathbb{P}(\mathcal{V}_1')$. In an adapted frame for which $\mathcal{L} = \mathbb{C} \cdot f_1$, we have $\theta_3 = \theta_5 = 0$ and the simplification

$$\alpha_1 = \kappa_{11}a_1 + da_1,$$
$$\alpha_2 = \kappa_{21}a_1,$$
$$\alpha_3 = \kappa_{31}a_1,$$

showing that the condition for such an $s \in \Gamma(\mathcal{L})$ to define a Cayley 4-fold is that σ be a holomorphic section of $\mathcal{L} \subset \mathcal{H}'$.

Theorem 10.2. *Fix a minimal immersion*

$$(10.6) \qquad\qquad\qquad \phi : \Sigma \to \mathbb{S}^6.$$

Then any pseudoholomorphic curve

$$(10.7) \qquad\qquad\qquad \gamma : \Sigma \to G(2, \mathbb{O})$$

with $\phi = \mathcal{J} \circ \gamma$ and sufficiently large negative degree always has a non-conical 2-ruled Cayley 4-fold with asymptotic cone defined by $\gamma(\Sigma)$.

Proof. We will see that when $\deg(\gamma)$ is sufficiently large and negative, then $\mathcal{L}$ admits many holomorphic sections.

There is a decomposition

$$(10.8) \qquad\qquad\qquad \gamma^*(\mathbb{O}) = \gamma^*(\mathcal{E}) \oplus \mathcal{L} \oplus \mathcal{W},$$

where $\mathcal{W} = \mathcal{L}^\perp \subset \gamma^*(\mathcal{H})$. Let

$$(10.9) \qquad\qquad\qquad k = \gamma^*(c_1(\mathcal{W}')).$$

We have the relation

$$(10.10) \qquad\qquad\qquad c_1(\mathcal{L}) = -\deg(\gamma) - k.$$

As we choose different curves γ that cover ϕ, the integer k remains unchanged. Thus by requiring $\deg(\gamma)$ to be sufficiently negative, which we can achieve by Theorem 9.1, Riemann–Roch implies that $h^0(\mathcal{L})$ can be made as large as may be desired. This guarantees holomorphic sections and thus deformations of the cone that are not conical. $\qquad\square$

Remark 10.3. Equation (9.5) and the proof of the Theorem 9.1 indicate that the generic pseudoholomorphic curve of large negative degree will be immersed. One even expects it to be embedded since it is a real surface in a real 12-dimensional space. Thus we can expect that for $\deg(\gamma)$ sufficiently large and negative (and thus $h^0(\mathcal{L}) \gg 0$), the generic holomorphic section will deform the cone into an *immersed* 4-fold. In other words, we can expect that the generic 2-ruled Cayley cone of large negative degree is the asymptotic cone of a smooth 2-ruled Cayley 4-fold. This construction coincides with the one introduced by Lotay [22] in the case that $\gamma(\Sigma)$ is a torus and $\gamma^*(\mathcal{E})$ is trivial.

A. Spin(7) algebra

Here are the details of the Spin(7) conventions used in this article. The Cayley 4-form used here is

(A.1)
$$\begin{aligned}
\Phi = {} & dx^{5678} - dx^{5128} - dx^{5348} - dx^{6138} - dx^{6428} - dx^{7148} - dx^{7238} \\
& + dx^{1234} - dx^{3467} - dx^{1267} - dx^{2457} + dx^{1357} - dx^{2356} - dx^{1456},
\end{aligned}$$

where dx^{ijkl} is shorthand for $dx^i \wedge dx^j \wedge dx^k \wedge dx^l$. Φ is the real part of the quadruple cross-product. The imaginary components of the cross-product

(A.2)
$$\psi_m(e_i^0, e_j^0, e_k^0, e_l^0) = \langle e_m^0, \, e_i^0 \times e_j^0 \times e_k^0 \times e_l^0 \rangle$$

for $1 \le m \le 7$ are

$$\begin{aligned}
\psi_1 &= dx^{5137} + dx^{5864} + dx^{7123} + dx^{6421} + dx^{1675} + dx^{7538} + dx^{2786} + dx^{4283} \\
\psi_2 &= dx^{3612} + dx^{2354} + dx^{4578} + dx^{7214} + dx^{5863} + dx^{3418} + dx^{5762} + dx^{1876} \\
\psi_3 &= dx^{7314} + dx^{4821} + dx^{8476} + dx^{6342} + dx^{3756} + dx^{1253} + dx^{1578} + dx^{8256} \\
\psi_4 &= dx^{1364} + dx^{6574} + dx^{4521} + dx^{3812} + dx^{2587} + dx^{6518} + dx^{7234} + dx^{6783} \\
\psi_5 &= dx^{7352} + dx^{2847} + dx^{6814} + dx^{8632} + dx^{3817} + dx^{1653} + dx^{4571} + dx^{4265} \\
\psi_6 &= dx^{5346} + dx^{6521} + dx^{5328} + dx^{4387} + dx^{1647} + dx^{7128} + dx^{2763} + dx^{4158} \\
\psi_7 &= dx^{8463} + dx^{8531} + dx^{6274} + dx^{6137} + dx^{2485} + dx^{3574} + dx^{7521} + dx^{1628}.
\end{aligned}$$

The connection ω_{ij} is $\mathfrak{spin}(7)$-valued if it satisfies

$$
\begin{aligned}
\omega_{87} &= -\omega_{65} + (\omega_{41} + \omega_{32}), \\
\omega_{86} &= \omega_{75} + (\omega_{31} - \omega_{42}), \\
\omega_{85} &= -\omega_{76} + (\omega_{43} + \omega_{21}), \\
\omega_{81} &= \omega_{74} + \omega_{63} + \omega_{52}, \\
\omega_{82} &= \omega_{73} - \omega_{64} - \omega_{51}, \\
\omega_{83} &= -\omega_{72} - \omega_{61} + \omega_{54}, \\
\omega_{84} &= -\omega_{71} + \omega_{62} - \omega_{53}.
\end{aligned}
$$

(A.3)

B. Holomorphic line subbundles over curves

Let $(\mathcal{W}, \nabla)$ be a hermitian vector bundle over a holomorphic curve Σ. Any line subbundle $\mathcal{L} \subset \mathcal{W}$ has a second fundamental form whose definition we now recall. Let $\pi : \Gamma(\mathcal{W}) \to \Gamma(\mathcal{L})$ be the orthogonal projection and $\pi^{\perp}$ the complementary projection.

Definition B.1. The second fundamental form of $\mathcal{L} \subset \mathcal{W}$ is a $C^{\infty}(\Sigma)$-linear bundle map

$$\mathbb{II}_{\mathcal{L}} : \Gamma(L) \to \Gamma(\mathcal{L}^{\perp} \otimes T^*\Sigma).$$

Given a section $s \in \Gamma(\mathcal{L})$,

$$\mathbb{II}_{\mathcal{L}}(s) = \pi^{\perp} \circ \nabla(s).$$

The main fact that will be used is contained in the following lemma.

Lemma B.1. $\mathcal{L} \subset \mathcal{W}$ *is a* holomorphic *line subbundle if and only if the image of* $\mathbb{II}_{\mathcal{L}}$ *is contained in* $\mathcal{L}^{\perp} \otimes \Omega^{(1,0)}(\Sigma)$.

The criterion is equivalent to requiring that a holomorphic section of $\mathcal{L} \subset \mathcal{W}$ (with respect to the natural holomorphic structure induced by the hermitian structure $\mathcal{L}$ inherits from $\mathcal{W}$) is also a holomorphic section of $\mathcal{W}$ under the natural inclusion

$$\Gamma(\mathcal{L}) \hookrightarrow \Gamma(\mathcal{W}).$$

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