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Blow-up time for a nonlocal diffusion problem with dirichlet boundary conditions

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This paper concerns the study of the following initial-boundary value problem

$$\begin{cases} u_t = \varepsilon(J * u - u) + f(u) \quad \text{in} \quad \Omega \times (0, T), \\ u = 0 \quad \text{in} \quad (\mathbb{R}^N - \Omega) \times (0, T), \\ u(x, 0) = u_0(x) > 0 \quad \text{in} \quad \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $J * u(x, t) = \int_{\mathbb{R}^N} J(x - y)u(y, t)dy$, J: $\mathbb{R}^N \longrightarrow \mathbb{R}$ is nonnegative, symmetric ($J(z) = J(-z)$), bounded and $\int_{\mathbb{R}^N} J(z)dz = 1$, $f(s)$ is positive, increasing, convex function for positive values of s and $\int^{\infty} \frac{ds}{f(s)} < \infty$. The initial data $u_0 \in C^1(\overline{\Omega})$. We show that if ε is small enough, the solution of the above problem blows up in a finite time and its blow-up time goes to the one of the solution of the following differential equation

$$\begin{cases} \alpha'(t) = f(\alpha(t)), \quad t > 0, \\ \alpha(0) = M, \end{cases}$$

as ε goes to zero, where $M = \sup_{x\in\Omega} u_0(x)$.

Finally, we give some numerical results to illustrate our analysis.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Consider the following initial-boundary value problem

$$u_t = \varepsilon(J * u - u) + f(u) \quad \text{in } \Omega \times (0, T), \tag{1.1}$$

$$u = 0 \quad \text{in } (\mathbb{R}^N - \Omega) \times (0, T), \tag{1.2}$$

$$u(x, 0) = u_0(x) > 0 \quad \text{in } \Omega, \tag{1.3}$$

where $J * u(x, t) = \int_{\mathbb{R}^N} J(x - y)u(y, t)dy$, J: $\mathbb{R}^N \longrightarrow \mathbb{R}$ is nonnegative, bounded, symmetric ($J(z) = J(-z)$) and $\int_{\mathbb{R}^N} J(z)dz = 1$, $f(s)$ is a positive, increasing, convex function for the positive values of s, $\int^{\infty} \frac{ds}{f(s)} < +\infty$. The initial data $u_0 \in C^1(\overline{\Omega})$. Here $(0, T)$ is the maximal time interval on which the solution u of (1.1) to (1.3) exists. The time T may be finite or infinite. When T is infinite, we say that the solution u exists globally. When T is finite, the solution u develops a singularity in a finite time, namely

$$\lim_{t \to T} \|u(.,t)\|_\infty = +\infty,$$

where $\|u(.,t)\|_\infty = \max_{x \in \Omega} |u(x,t)|$. In this last case, we say that the solution u blows up in a finite time and the time T is called the blow-up time of the solution u. Recently nonlocal diffusion problems have been the subject of investigations of many authors (see [3–8, 14–21, 24, 25, 27, 33, 39] and the references cited therein). Nonlocal evolution equations of the form

$$u_t = \int_{\mathbb{R}^N} J(x-y)(u(y,t) - u(x,t))dy$$

and variations of it have been used by several authors to model diffusion processes (see [5, 8, 14, 20, 21]). The solution $u(x,t)$ can be interpreted as the density of a single population at the point x, at time t and $J(x-y)$ as the probability distribution of jumping from location y to location x. Then the convolution $(J * u)(x,t) = \int_{\mathbb{R}^N} J(y-x)u(y,t)dy$ is the rate at which individuals are arriving to position x from all other places and $-u(x,t) = -\int_{\mathbb{R}^N} J(y-x)u(x,t)dy$ is the rate at which they are leaving location x to travel to any other site (see [20]). Solutions of nonlinear parabolic equations (local diffusion) which blow up in a finite time have been the subject of investigation of many authors (see [9, 10, 13, 23, 29, 34, 36, 38] and the references cited therein). One may also find in [33] some results on blow-up for nonlocal diffusion with Neumann boundary conditions. In this paper, we are interested in the asymptotic behavior of the blow-up time when ε is small enough. Our work was motivated by the paper of Friedman and Lacey in [23], where they have considered the following problem

$$\begin{aligned}
&u_t = \varepsilon \Delta u + f(u) \quad \text{in } \Omega \times (0,T),\\
&u = 0 \quad \text{on } \partial\Omega \times (0,T),\\
&u(x,0) = u_0(x) \quad \text{in } \Omega,
\end{aligned}$$

where Δ is the Laplacian, $f(s)$ is a positive, increasing and convex function for the nonnegative values of s, $\int_0^{+\infty} \frac{ds}{f(s)} < +\infty$. The initial data u_0 is a positive and continuous function in Ω. Under some additional conditions on the initial data, they have proved that when ε is small enough, the solution

u of the above problem blows up in a finite time and its blow-up time goes to the one of the solution of the following differential equation

$$\alpha'(t) = f(\alpha(t)), \quad \alpha(0) = M, \tag{1.4}$$

as ε tends to zero where $M = \sup_{x\in\overline{\Omega}} u_0(x)$.

Let us notice that the result of Friedman and Lacey holds when $f(0) > 0$, but they have noticed that if the solution increases with respect to the second variable, it is possible that their result holds for $f(0) = 0$. The proof in [23] is based on the construction of upper and lower solutions, and it is difficult to extend the method in [23] to our problem. In this paper, we obtain a similar result for the problem described in (1.1) to (1.3) using both a modification of Kaplan's method (see [29]) and a method based on the construction of upper solutions. Our paper is written in the following manner. In the next section, we prove the local existence and uniqueness of the solution of (1.1) to (1.3). We also give some results about the maximum principle for nonlocal problems. In the third section, under some conditions, we show that the solution u of (1.1) to (1.3) blows up in a finite time and its blow-up time goes to the one of the solution of the differential equation defined in (1.4) as ε goes to zero. Finally, in the last section, we give some numerical results to illustrate our analysis.

2. Local existence

In this section, we shall establish the existence and uniqueness of the solution of (1.1) to (1.3) in $\Omega \times (0, T)$ for small T and certain initial data.

Let t_0 be fixed and define the function space $Y_{t_0} = \{u \in C([0, t_0], C(\overline{\Omega}))\}$ equipped with the norm defined by $\|u\|_{Y_{t_0}} = \max_{0\le t\le t_0} \|u\|_\infty$ for $u \in Y_{t_0}$. It is easy to see that Y_{t_0} is a Banach space. Introduce the set $X_{t_0} = \{u/u \in Y_{t_0}, \|u\|_{Y_{t_0}} \le b_0\}$, where $b_0 = 2\|u_0\|_\infty + 1$. We observe that X_{t_0} is a nonempty bounded closed convex subset of Y_{t_0}. Define the map R as follows

$$R : X_{t_0} \longrightarrow X_{t_0},$$

$$R(v)(x,t) = u_0(x) + \varepsilon \int_0^t \int_{\mathbb{R}^N} J(x-y)(v(y,s) - v(x,s))dy\,ds + \int_0^t f(v(x,s))ds,$$

where we impose

$$v(x,t) = 0 \quad \text{for } x \in \mathbb{R}^N - \Omega.$$

Theorem 2.1. *Assume that $u_0 \in Y_{t_0}$. Then R maps X_{t_0} into X_{t_0} and R is strictly contractive if t_0 is appropriately small relative to $\|u_0\|_\infty$.*

Proof. We get

$$|R(v) - u_0| \leq 2\varepsilon \|v\|_{Y_{t_0}} t + f(\|v\|_{Y_{t_0}})t,$$

which implies that $\|R(v)\|_{Y_{t_0}} \leq \|v_0\|_\infty + 2\varepsilon b_0 t_0 + f(b_0)t_0$. If

$$t_0 \leq \frac{b_0 - \|u_0\|_\infty}{2\varepsilon b_0 + f(b_0)} \tag{2.1}$$

then

$$\|R(v)\|_{Y_{t_0}} \leq b_0.$$

Therefore, if (2.1) holds, then R maps X_{t_0} into X_{t_0}. Now we are going to prove that the map R is strictly contractive. Let $t_0 > 0$ and let $v, z \in X_{t_0}$ and let $\alpha = v - z$, we discover that

$$\begin{aligned} |(R(v) - R(z))(x,t)| \leq \varepsilon & \left| \int_0^t \int_{\mathbb{R}^N} J(x-y)(\alpha(y,s) - \alpha(x,s)) dy\, ds \right| \\ & + \left| \int_0^t (f(v(x,s)) - f(z(x,s))) ds \right|. \end{aligned}$$

Use Taylor's expansion to obtain

$$|R(v) - R(z)| \leq 2\varepsilon \|\alpha\|_{Y_{t_0}} t + t\|v - z\|_{Y_{t_0}} f'(\|\beta\|_{Y_{t_0}}),$$

where β is an intermediate value between v and z. We deduce that

$$\|R(v) - R(z)\|_{Y_{t_0}} \leq 2\varepsilon \|\alpha\|_{Y_{t_0}} t_0 + t_0 \|v - z\|_{Y_{t_0}} f'(\|\beta\|_{Y_{t_0}}),$$

which implies that

$$\|R(v) - R(z)\|_{Y_{t_0}} \leq (2\varepsilon t_0 + t_0 f'(b_0)\|v - z\|_{Y_{t_0}}.$$

If $t_0 \leq \frac{1}{4\varepsilon + 2f'(b_0)}$, then $\|R(v) - R(z)\|_{Y_{t_0}} \leq \frac{1}{2}\|v - z\|_{Y_{t_0}}$. Hence, we see that $R(v)$ is a strict contraction in Y_{t_0} and the proof is complete. □

It follows from the contraction mapping principle that for appropriately chosen t_0, R has a unique fixed point $u(x,t) \in Y_{t_0}$ which is a solution of (1.1) to (1.3).

If $\|u\|_{Y_{t_0}} < \infty$, taking as initial data $u(x, t_0) \in C(\overline{\Omega})$ and arguing as before, it is possible to extend the solution up to some interval $[0, t_1)$ for certain $t_1 > t_0$.

Now, let us give some results about the maximum principle for nonlocal problems.

The following lemma is a version of the maximum principle for nonlocal problems.

Lemma 2.1. *Let $a \in C^0(\overline{\Omega} \times [0,T))$ and let $u \in C^{0,1}(\Omega \times [0,T))$ satisfying the following inequalities*

$$\begin{aligned} u_t - \int_{\mathbb{R}^N} J(x-y)(u(y,t)-u(x,t))dy + a(x,t)u(x,t) \geq 0 \quad &in\ \Omega \times (0,T), \\ u(x,t) \geq 0 \quad in\ (\mathbb{R}^N - \Omega) \times (0,T),& \\ u(x,0) \geq 0 \quad in\ \Omega.& \end{aligned}$$

Then we have $u(x,t) \geq 0$ in $\overline{\Omega} \times (0,T)$.

Proof. Let $T_0 < T$. Since $a(x,t)$ is bounded in $\overline{\Omega} \times [0,T_0]$, let λ be such that $a(x,t) - \lambda > 0$ in $\overline{\Omega} \times [0,T_0]$. Introduce the function $z(x,t) = e^{\lambda t}u(x,t)$ and let $m = \min_{x\in\overline{\Omega},t\in[0,T_0]} z(x,t)$. Then there exists $(x_0,t_0) \in \overline{\Omega} \times [0,T_0]$ such that $m = z(x_0,t_0)$. If $x_0 \in \mathbb{R}^N - \Omega$, then $m \geq 0$. If $x_0 \in \Omega$, we get $z(x_0,t_0) \leq z(x_0,t)$ for $t \leq t_0$ and $z(x_0,t_0) \leq z(y,t_0)$ for y$\in \Omega$, which implies that

$$z_t(x_0,t_0) \leq 0 \tag{2.2}$$

and

$$\int_{\mathbb{R}^N} J(x_0-y)(z(y,t_0)-z(x_0,t_0))dy \geq 0. \tag{2.3}$$

Using the first inequality of the lemma, it is not hard to see that

$$z_t(x_0,t_0) - \int_{\mathbb{R}^N} J(x_0-y)(u(y,t_0)-u(x_0,t_0))dy + (a(x_0,t_0)-\lambda)z(x_0,t_0) \geq 0.$$

It follows from (2.2) and (2.3) that $(a(x_0,t_0)-\lambda)z(x_0,t_0) \geq 0$, which implies that $z(x_0,t_0) \geq 0$ because $a(x_0,t_0) - \lambda > 0$. We deduce that $u(x,t) \geq 0$ in $\overline{\Omega} \times [0,T_0]$, which leads us to the result. $\square$

A direct consequence of the above result is the following comparison lemma.

Lemma 2.2. *Let $u, v \in C^{1,0}(\overline{\Omega} \times [0, T))$ be such that*

$$\begin{aligned}
&u_t - \int_{\mathbb{R}^N} J(x-y)(u(y,t) - u(x,t))dy - f(u(x,t)) \geq v_t \\
&\quad - \int_{\mathbb{R}^N} J(x-y)(v(y,t) - v(x,t))dy - f(v(x,t)) \quad \text{in } \Omega \times (0,T), \\
&u(x,t) \geq v(x,t) \quad \text{in } (\mathbb{R}^N - \Omega) \times (0,T), \\
&u(x,0) \geq v(x,0) \quad \text{in } \Omega.
\end{aligned}$$

Then we have $u(x,t) \geq v(x,t)$ in $\overline{\Omega} \times (0,T)$.

Proof. Let $z(x,t) = u(x,t) - v(x,t)$ in $\overline{\Omega} \times (0,T)$. Applying the mean value theorem, a routine computation reveals that

$$\begin{aligned}
&z_t - \int_{\mathbb{R}^N} J(x-y)(z(y,t) - z(x,t))dy + f'(\xi(x,t))z(x,t) \geq 0 \\
&\quad \text{in } \Omega \times (0,T), \\
&z(x,t) \geq 0 \quad \text{in } (\mathbb{R}^N - \Omega) \times (0,T), \\
&z(x,0) \geq 0 \quad \text{in } \Omega,
\end{aligned}$$

where $\xi(x,t)$ is an intermediate value between $u(x,t)$ and $v(x,t)$. Use Lemma 2.1 to complete the rest of the proof. □

3. Blow-up times

In this section, we show that if ε is small enough, the solution u of (1.1) to (1.3) blows up in a finite time and its blow-up time goes to the one of the solution of the differential equation defined in (1.4). Before starting, let us recall a result which may be found in [24, 25]. Consider the eigenvalue problem below:

$$\int_{\mathbb{R}^N} J(x-y)(\varphi(y) - \varphi(x))dy = -\lambda\varphi(x) \quad \text{in } \Omega, \tag{3.1}$$

$$\varphi(x) = 0 \quad \text{in } \mathbb{R}^N - \Omega, \tag{3.2}$$

$$\varphi(x) > 0 \quad \text{in } \Omega. \tag{3.3}$$

It is well known that the above problem admits a solution (φ, λ) such that $0 < \lambda < 1$. We can normalize φ so that $\int_{\mathbb{R}^N} \varphi \, dx = 1$.

Now, let us state our first result on the blow-up time.

Theorem 3.1. *Suppose that $u_0(x) = 0$ and $f(0) > 0$ and let $A = \lambda \int_0^\infty \frac{ds}{f(s)}$. If $\varepsilon < \frac{1}{A}$ then the solution u of* (1.1) *to* (1.3) *blows up in a finite time T which obeys the following estimates*

$$0 \le T - T_0 \le \varepsilon T_0 A + o(\varepsilon),$$

where $T_0 = \int_0^\infty \frac{ds}{f(s)}$ is the blow-up time of the solution $\alpha(t)$ of the differential equation defined in (1.4).

Proof. Since $(0, T)$ is the maximal time interval of existence of the solution u, our aim is to show that T is finite and satisfies the above estimates. Since the initial data u_0 is nonnegative in Ω, from Lemma 2.1, u is also nonnegative in $\overline{\Omega} \times (0, T)$.

Introduce the function $v(t)$ defined as follows:

$$v(t) = \int_{\mathbb{R}^N} u(x,t)\varphi(x)dx \quad \text{for } t \in (0,T).$$

Take the derivative of v in t and use (1.1) to obtain

$$v'(t) = \varepsilon \int_{\mathbb{R}^N} \varphi(x) \left(\int_{\mathbb{R}^N} J(x-y)u(y,t)dy \right) dx - \varepsilon v(t) + \int_{\mathbb{R}^N} f(u(x,t))\varphi(x)dx.$$

From Fubini's theorem, we have

$$v'(t) = \varepsilon \int_{\mathbb{R}^N} u(y,t) \left(\int_{\mathbb{R}^N} \varphi(x)J(x-y)dx \right) dy - \varepsilon v(t) + \int_{\mathbb{R}^N} f(u(x,t))\varphi(x)dx.$$

Since J is symmetric, we arrive at

$$v'(t) = \varepsilon \int_{\mathbb{R}^N} u(y,t) \left(\int_{\mathbb{R}^N} J(y-x)\varphi(x)dx \right) dy - \varepsilon v(t) + \int_{\mathbb{R}^N} f(u(x,t))\varphi(x)dx.$$

It follows from (3.1) that

$$v'(t) = \varepsilon \int_{\mathbb{R}^N} u(y,t)(\varphi(y) - \lambda\varphi(y))dy - \varepsilon v(t) + \int_{\mathbb{R}^N} f(u(x,t))\varphi(x)dx,$$

which implies that

$$v'(t) = -\varepsilon\lambda v(t) + \int_{\mathbb{R}^N} f(u(x,t))\varphi(x)dx.$$

Using Jensen's inequality, we find that

$$v'(t) \geq -\varepsilon\lambda v(t) + f(v(t)) \quad \text{for } t \in (0, T).$$

Obviously, we have

$$v'(t) \geq f(v(t))\left(1 - \frac{\varepsilon\lambda v(t)}{f(v(t))}\right) \quad \text{for } t \in (0, T).$$

It is not hard to see that

$$\int_0^\infty \frac{ds}{f(s)} \geq \sup_{t\geq 0} \int_0^t \frac{ds}{f(s)} \geq \sup_{t\geq 0} \frac{t}{f(t)}$$

because $f(s)$ is an increasing function for $s > 0$. We deduce that

$$v'(t) \geq (1 - \varepsilon A) f(v(t)) \quad \text{for } t \in (0, T).$$

This estimate may be rewritten as follows

$$\frac{dv}{f(v)} \geq (1 - \varepsilon A) dt \quad \text{for } t \in (0, T).$$

Integrate the above inequality over $(0, T)$ to obtain

$$T \leq \frac{1}{1 - \varepsilon A} \int_0^\infty \frac{ds}{f(s)}, \tag{3.4}$$

which implies that the solution u blows up at the time T because the quantity on the right hand side of the above inequality is finite. On the other hand, setting

$$z(x, t) = \alpha(t) \quad \text{in } \mathbb{R}^N \times (0, T_0),$$

it is not hard to see that

$$\begin{cases} z_t(x,t) - \varepsilon \displaystyle\int_{\mathbb{R}^N} J(x-y)(z(y,t) - z(x,t))dy - f(z(x,t)) = 0 & \text{in } \Omega \times (0, T), \\ z(x,t) \geq 0 \quad \text{in } (\mathbb{R}^N - \Omega) \times (0, T), \\ z(x, 0) = 0 \quad \text{in } \Omega. \end{cases}$$

Comparison Lemma 2.2 implies that $0 \leq u(x, t) \leq z(x, t) = \alpha(t)$ in $\overline{\Omega} \times (0, T_*)$, where $T_* = \min\{T, T_0\}$. It follows that $T \geq T_0$. Indeed, suppose that $T < T_0$, which implies that $0 \leq \|u(., T)\|_\infty \leq \alpha(T) < +\infty$. But this is a contradiction because $(0,T)$ is the maximal time interval of existence of the solution u. We deduce that

$$T \geq T_0 = \int_0^\infty \frac{ds}{f(s)}. \tag{3.5}$$

Apply Taylor's expansion to obtain $\frac{1}{1-\varepsilon A} = 1 + \varepsilon A + o(\varepsilon)$. Use (3.4), (3.5) and the above relation to complete the rest of the proof. $\square$

Now let us consider the case where the initial data are not null. We have the following result.

Theorem 3.2. *Let $f(0) = 0$. Suppose that $\sup_{x\in\overline{\Omega}} u_0(x) = M > 0$ and let ε be such that $\varepsilon < \frac{1}{A}$, where $A = \frac{M}{2f(M/2)}$. Then the solution u of* (1.1) *to* (1.3) *blows up in a finite time and its blow-up time T satisfies the following estimates*

$$0 \leq T - T_0 \leq \varepsilon T_0 A + \frac{\varepsilon}{f(M/2)} + o(\varepsilon),$$

where $T_0 = \int_M^\infty \frac{ds}{f(s)}$ is the blow-up time of the solution $\alpha(t)$ of the differential equation defined in (1.4).

Proof. Since $(0,T)$ is the maximal time interval on which u exists, our goal is to prove that T is finite and obeys the above relation. Since the initial data u_0 are nonnegative in Ω, from Lemma 2.1, u is also nonnegative in $\Omega \times (0,T)$. Let $a \in \Omega$ such that $u_0(a) = M$. There exists $\delta > 0$ such that

$$u_0(x) \geq M - \varepsilon \quad \text{for } x \in B(a,\delta) \subset \Omega.$$

Consider the following eigenvalue problem

$$\int_{\mathbb{R}^N} J(x-y)(\varphi(y) - \varphi(x))dy = -\lambda_\delta \varphi(x) \quad \text{in } B(a,\delta), \tag{3.6}$$

$$\varphi(x) = 0 \quad \text{in } \mathbb{R}^N - B(a,\delta), \tag{3.7}$$

$$\varphi(x) > 0 \quad \text{in } B(a,\delta). \tag{3.8}$$

We know that the above problem admits a solution $(\varphi, \lambda_\delta)$ with $0 < \lambda_\delta < 1$ [24,25]. Let w be the solution of the following initial-boundary value problem

$$\begin{aligned}
&w_t - \varepsilon \int_{\mathbb{R}^N} J(x-y)(w(y,t) - w(x,t))dy - f(w) = 0 \quad \text{in } B(a,\delta) \\
&\quad \times (0, T^*), \\
&w = 0 \quad \text{in } (\mathbb{R}^N - B(a,\delta)) \times (0, T^*), \\
&w(x,0) = u_0(x) \quad \text{in } B(a,\delta),
\end{aligned}$$

where $(0, T^*)$ is the maximal time interval of existence of the solution $w(x,t)$.

Introduce the function $v(t)$ defined as follows

$$v(t) = \int_{\mathbb{R}^N} \varphi(x) w(x,t) dx \quad \text{for } t \in (0, T^*).$$

As in the proof of Theorem 3.1, we find that

$$v'(t) \geq -\varepsilon \lambda_\delta v(t) + f(v(t)) \geq -\varepsilon v(t) + f(v(t)) \quad \text{for } t \in (0, T^*)$$

because $0 < \lambda_\delta < 1$. We deduce that

$$v'(t) \geq f(v(t)) \left(1 - \frac{\varepsilon v(t)}{f(v(t))}\right) \quad \text{for } t \in (0, T^*). \tag{3.9}$$

Since $f(s)$ is a convex function for the positive values of s and $f(0) = 0$, then $\frac{f(s)}{s}$ is an increasing function for the positive values of s. Due to the fact that $v(0) \geq M - \varepsilon \geq \frac{M}{2}$, we have

$$v'(0) \geq f(v(0)) \left(1 - \frac{\varepsilon v(0)}{f(v(0))}\right) \geq f(v(0)) \left(1 - \frac{\varepsilon M}{2f(M/2)}\right) > 0.$$

Therefore, we have $v'(t) > 0$ for $t \in (0, T^*)$. Indeed let t_0 be the first $t > 0$ such that $v'(t) > 0$ for $t \in (0, t_0)$ but $v'(t_0) = 0$, which implies that

$$v'(t_0) \geq f(v(t_0)) \left(1 - \frac{\varepsilon v(t_0)}{f(v(t_0))}\right).$$

The fact that $v(t_0) \geq v(0) \geq M - \varepsilon \geq \frac{M}{2}$ implies that

$$0 = v'(t_0) \geq f(v(t_0)) \left(1 - \frac{\varepsilon M}{2f(M/2)}\right) > 0,$$

which is a contradiction. We deduce that $v(t) \geq v(0) \geq \frac{M}{2}$ for $t \in (0, T^*)$, which implies that

$$v'(t) \geq f(v(t)) \left(1 - \frac{\varepsilon M}{2f(M/2)}\right).$$

Obviously, we have

$$v'(t) \geq (1 - \varepsilon A) f(v(t)) \quad \text{in } (0, T^*). \tag{3.10}$$

This estimate may be rewritten as follows:

$$\frac{dv}{f(v)} \geq (1 - \varepsilon A)dt \quad \text{for } t \in (0, T^*).$$

Integrate this inequality over $(0, T^*)$ to obtain

$$T^* \leq \frac{1}{1 - \varepsilon A} \int_{v(0)}^{\infty} \frac{ds}{f(s)} \leq \frac{1}{1 - \varepsilon A} \int_{M-\varepsilon}^{\infty} \frac{ds}{f(s)}.$$

This implies that the solution w blows up in a finite time because the quantity on the right hand side of the second inequality is finite. On the other hand, from Lemma 2.1, we have $u \geq 0$ in $\overline{\Omega} \times (0, T)$, which implies that

$$\begin{aligned}
&u_t - \varepsilon \int_{\mathbb{R}^N} J(x-y)(u(y,t) - u(x,t))dy - f(u) \geq w_t \\
&\quad -\varepsilon \int_{\mathbb{R}^N} J(x-y)(w(y,t) - w(x,t))dy - f(w) \quad \text{in } B(a,\delta) \times (0, T_*), \\
&u \geq w \quad \text{in} (\mathbb{R}^N - B(a,\delta)) \times (0, T_*), \\
&u(x,0) \geq w(x,0) \quad \text{in } B(a,\delta),
\end{aligned}$$

where $T_* = \min\{T, T^*\}$. It follows from Lemma 2.2 that

$$u(x,t) \geq w(x,t) \quad \text{in } \overline{B(a,\delta)} \times (0, T_*),$$

which implies that

$$T \leq T^* \leq \frac{1}{1 - \varepsilon A} \int_{M-\varepsilon}^{\infty} \frac{ds}{f(s)}. \tag{3.11}$$

Indeed, suppose that $T > T^*$. We have $\|u(.,T^*)\|_\infty \geq \|w(\cdot,T^*)\|_\infty = +\infty$. But this is a contradiction because $(0, T)$ is the maximal time interval of existence of the solution u. We observe that

$$\int_{M-\varepsilon}^{\infty}\frac{ds}{f(s)}=\int_{M}^{\infty}\frac{ds}{f(s)}+\int_{M-\varepsilon}^{M}\frac{ds}{f(s)}\leq\int_{M}^{\infty}\frac{ds}{f(s)}+\frac{\varepsilon}{f(M-\varepsilon)}$$

because $f(s)$ is an increasing function for the positive values of s. The fact that $f(M-\varepsilon)\geq f(\frac{M}{2})$ implies that

$$\int_{M-\varepsilon}^{\infty}\frac{ds}{f(s)}\leq\int_{M}^{\infty}\frac{ds}{f(s)}+\frac{\varepsilon}{f(\frac{M}{2})}. \tag{3.12}$$

Setting $z(x,t)=\alpha(t)$ in $\mathbb{R}^N\times(0,T_0)$, it is not hard to see that

$$\begin{cases} z_t-\varepsilon\displaystyle\int_{\mathbb{R}^N}J(x-y)(z(y,t)-z(x,t))dy-f(z)=0 \quad \text{in } \Omega\times(0,T),\\ z(x,t)\geq 0 \quad \text{in } (\mathbb{R}^N-\Omega)\times(0,T),\\ z(x,0)\geq u_0(x) \quad \text{in } \Omega. \end{cases}$$

Comparison Lemma 2.2 implies that $0\leq u(x,t)\leq z(x,t)=\alpha(t)$ in $\overline{\Omega}\times(0,T^0)$, where $T^0=\min\{T_0,T\}$. We deduce that

$$T\geq T_0=\int_{M}^{\infty}\frac{ds}{f(s)}. \tag{3.13}$$

Indeed, suppose that $T_0>T$, which implies that $\alpha(T)\geq\|u(\cdot,T)\|_\infty=+\infty$. But this is a contradiction because $(0,T_0)$ is the maximal time interval of existence of the solution $\alpha(t)$. Apply Taylor's expansion to obtain $\frac{1}{1-\varepsilon A}=1+\varepsilon A+o(\varepsilon)$. Use (3.11)–(3.13) and the above relation to complete the rest of the proof. □

Remark 3.1. *Theorem* 3.2 *remains valid when* $f(0)>0$ *if we take* $A=\int_0^\infty\frac{ds}{f(s)}$. *Indeed using* (3.9) *and the fact that* $\int_0^{+\infty}\frac{ds}{f(s)}\geq\sup_{s\geq 0}\frac{s}{f(s)}$, *we obtain the inequality in* (3.10). *Now, reasoning as in the proof of Theorem* 3.2, *we obtain the desired result.*

4. Numerical results

In this section, we give some numerical results to illustrate our analysis. We consider the problem (1.1) to (1.3) in the case where $\Omega=(-3,3)$,

$$J(x)=\begin{cases}\frac{1}{4} & \text{if } |x|<2,\\ 0 & \text{if } |x|\geq 2.\end{cases}$$

Let I be a positive integer and let $h = \frac{3}{I}$. Define the grid $x_i = ih, -I \leq i \leq I$, and approximate the solution $u(x,t)$ of the problem (1.1) to (1.3) by the solution $U_h^{(n)} = (U_{-I}^{(n)}, \ldots, U_I^{(n)})^T$ of the following discrete equations

$$\begin{aligned}
&\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \varepsilon \sum_{j=-I}^{I} hJ(x_i - x_j)(U_j^{(n)} - U_i^{(n)}) + f(U_i^{(n)}), \\
&\quad -I \leq i \leq I, \\
&U_{-I}^{(n)} = 0, \quad U_I^{(n)} = 0, \\
&U_i^{(0)} = u_0(x_i), \quad -I \leq i \leq I,
\end{aligned}$$

where $x_i = ih$, $-I \leq i \leq I$. If $f(u) = e^u$, then

$$\Delta t_n = \min\left\{\frac{4}{\varepsilon(6+h)}, h^2 e^{-\|U_h^{(n)}\|_\infty}\right\}.$$

If $f(u) = u^2$, then

$$\Delta t_n = \min\left\{\frac{4}{\varepsilon(6+h)}, \frac{h^2}{\|U_h^{(n)}\|_\infty}\right\}.$$

Here $\|U_h^{(n)}\|_\infty = \sup_{-I\leq i\leq I} |U_i^{(n)}|$.

We need the following definition.

Definition 4.1. We say that the discrete solution $U_h^{(n)}$ of the explicit scheme blows up in a finite time if $\lim_{n\to+\infty} \|U_h^{(n)}\|_\infty = +\infty$ and the series $\sum_{n=0}^{+\infty} \Delta t_n$ converges. The quantity $\sum_{n=0}^{+\infty} \Delta t_n$ is called the numerical blow-up time of the solution $U_h^{(n)}$.

In the following tables, in rows, we present the numerical blow-up times, the numbers of iterations, CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128. We take for the numerical blow-up time $T^n = \sum_{j=0}^{n-1} \Delta t_j$, which is computed at the first time when $|T^{n+1} - T^n| \leq 10^{-16}$. The order(s) of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

Numerical experiments for $u_0(x) = 0, f(u) = e^u$.
First case: $\varepsilon = \frac{1}{10}$.

Table 1: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method.

I	T^n	n	CPU time	s
16	1.030317	325	9	—
32	1.007324	1208	128	—
64	1.001738	4476	1032	2.04
128	1.000344	16885	8216	2.00

Second case: $\varepsilon = \frac{1}{50}$.

Table 2: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method.

I	T^n	n	CPU time	s
16	1.031544	325	10	—
32	1.007590	1208	142	—
64	1.001947	4476	1932	2.01
128	1.000536	16885	8421	2.06

Numerical experiments for $u_0(x) = 20\sin(\pi x),\ f(u) = u^2$.
First case: $\varepsilon = \frac{1}{10}$.

Table 3: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method.

I	T^n	n	CPU time	s
16	0.053430	269	8	—
32	0.050902	1010	708	—
64	0.050305	3705	1622	2.08
128	0.050182	10124	7845	2.12

<u>Second case: $\varepsilon = \frac{1}{50}$.</u>

Table 4: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method.

I	T^n	n	CPU time	s
16	0.053342	269	8	—
32	0.050815	109	117	—
64	0.050217	3703	1548	2.07
128	0.050091	10124	6314	2.14

Remark 4.1. *If we consider the problem* (1.1) *to* (1.3) *in the case where the initial data are null and the reaction term* e^u*, it is not hard to see that the blow-up time of the solution of the differential equation defined in Theorem* 2.1 *equals one. We observe from tables 1 and 2 that when* ε *diminishes, the numerical blow-up time tends to one. This result has been proved in Theorem* 2.1*. When the initial data* $u_0(x) = 20\sin(x\pi)$ *and the reaction term* u^2*, we find that the blow-up time of the solution of the differential equation defined in Theorem* 2.2 *equals* 0.05*. We discover from tables 3 and 4 that when* ε *diminishes, the numerical blow-up time goes to* 0.05*, which is a result proved in Theorem* 2.2*.*

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