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Riemannian moment map

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In this paper, we introduce a notion called moment map for a real reductive group acting on a Riemannian manifold, and set up the foundation for the theory of Riemannian moment map. This extends the well-known theory in Kähler geometry. Both finite and infinite dimensional examples are presented.

1. Introduction

In 1982, Atiyah and Bott [1] discovered that the curvature as a function on the space of connections on a principal G-bundle over a Riemann surface can be interpreted as the momentum for the gauge group action on the space of connections. This observation, together with its several extensions, has proved to be an extremely useful framework to study various kinds of geometries. To name a few, in the study of the geometry of moduli space of flat connections over a Riemann surface by Kirwan, Jeffrey, Meinrenken, etc. (c.f. [2]) and in the searching of necessary and sufficient conditions for the existence of constant scalar curvature Kähler metric by Fujiki, and Donaldson (c.f. [3–5]). All the works mentioned above have been substantially impacted by Atiyah and Bott's point of view. So it is desirable to find such a framework in the Riemannian world which extends its counterpart in the Kählerian world. This is our motivation of this paper, we took the first step in developing a moment map theory in the world of Riemannian geometry. The main body of the paper is an extension of the results in Sections 2 and 3 in [6] to the Riemannian setting, and as a consequence, we obtained simplified proof of some of the results in [6]. This will take up Section 2.

Start from Section 3, we mainly concentrate on the application of the theory developed in Sections 2 and 3. In particular, we supply two major families of finite dimensional examples, one is the Kähler moment map and the other one is coming from anti-holomorphic involution. In Section 4, we present two infinite dimensional examples which were (although implicitly) our original motivation, that is to extend Donaldson's framework [4] to the study of conformal geometry. And these will be discussed in a separate paper.

2. Riemannian moment map

In this section, we will set up the foundation of the theory of Riemannian moment map.

2.1. Definition

Let G be a real reductive Lie group and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to an involution $\theta:\mathfrak{g}\to\mathfrak{g}$, where $\mathfrak{k}$ is the Lie algebra of a maximal compact subgroup $K\subset G$. So we have relations $[\mathfrak{k},\mathfrak{p}]\subset\mathfrak{p}$, $[\mathfrak{p},\mathfrak{p}]\subset\mathfrak{k}$. Before we move on, let us give three important examples of real reductive group that will be used later.

Example 2.1. $G=K^{\mathbb{C}}$, the complexification of a compact Lie group K. G can be treated as a real reductive group as follows: let $\mathfrak{k}=\mathrm{Lie}(K)$, the Lie algebra of K and

$$J:=\begin{bmatrix} & \mathrm{id}\\ -\mathrm{id} & \end{bmatrix}\in\mathrm{Hom}(\mathfrak{k}\oplus\mathfrak{k}).$$

The real Lie algebra structure on $\mathfrak{k}\oplus\mathfrak{k}$ is then induced from the complex Lie algebra structure $\mathfrak{k}^{\mathbb{C}}=\mathfrak{k}+\sqrt{-1}\mathfrak{k}$ via the $\mathbb{R}$-homomorphism

$$\begin{array}{rcc}\varphi:\mathfrak{k}+\sqrt{-1}\mathfrak{k} & \longrightarrow & \mathfrak{k}\oplus\mathfrak{k}\\ \xi+\sqrt{-1}\eta & \longmapsto & (\xi,0)+J\,(\eta,0)\,.\end{array}$$

In particular, we have $\mathfrak{p}=\{0\}\oplus\mathfrak{k}$ with $[(0,\xi),(0,\eta)]=(-[\xi,\eta],0)\in\mathfrak{k}\oplus\{0\}$.

Example 2.2. $G=\mathrm{GL}(n,\mathbb{R}),\mathfrak{k}=\mathfrak{o}(n,\mathbb{R})$ and $\mathfrak{p}$ can be identified with real symmetric matrices.

Example 2.3. $G=O(p,q;\mathbb{R}),\mathfrak{k}=\mathfrak{o}(p)\oplus\mathfrak{o}(q)$ and $\mathfrak{p}=\mathbb{R}^{pq}$.

Now we are ready to introduce the Riemannian moment map. Let (M,g) be a Riemannian manifold with metric g, and G acts on M with $K\subset\mathrm{Isom}(M,g)$, it induces an infinitesimal action: for $x\in M$

$$\begin{array}{rcc}\sigma_x:\mathfrak{g} & \longrightarrow & T_xM\\ \xi & \longmapsto & \sigma_x\left(\xi\right):=\left.\frac{d}{dt}\right|_{t=0}\exp t\xi\cdot x.\end{array}$$

Definition 2.4. Let G be a real reductive group acting on a Riemannian manifold (M, g) and K be its maximal compact subgroup. We say the action is Hamiltonian if there is a moment map $\mu : M \to \mathfrak{p}^*$ satisfying, for any $\xi \in \mathfrak{p}, X \in \Gamma(TM)$ and $k \in K$

$$\begin{cases} X\langle \mu(x), \xi\rangle = \langle \sigma_x \xi, X\rangle_{T_x M}, \\ \langle \mu(k \cdot x), \xi\rangle = \langle \mu(x), \mathrm{Ad}_{k^{-1}} \xi\rangle, \end{cases} \tag{2.1}$$

where $\langle \cdot, \cdot\rangle : \mathfrak{p}^* \times \mathfrak{p} \to \mathbb{R}$ is the natural pairing. In particular, we have

$$\nabla\langle \mu(x), \xi\rangle = \sigma_x \xi.$$

Let us fix a G-invariant bi-linear form $\langle \cdot, \cdot\rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ such that

$$(\cdot, \cdot)_{\mathfrak{g}} := -\langle \cdot, \theta(\cdot)\rangle_{\mathfrak{g}}$$

is an inner product on $\mathfrak{g}$, where θ is the Cartan involution. Such a form always exists, for instance, we may use Killing form on the semi-simple part and any inner produce on the Abelian factors. With this understood, we may identify $\mathfrak{p}$ with $\mathfrak{p}^*$ using the inner product $(\cdot, \cdot)_{\mathfrak{g}}$ and under this inner product we have $\mathfrak{p} \perp \mathfrak{k}$. For the rest of the paper, we will identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ via the inner product $(\cdot, \cdot)_{\mathfrak{g}}$, in particular, we will treat our moment map μ as $\mathfrak{p}$-valued function.

Let us introduce the operator

$$Q_x := \sigma_x^* \sigma_x : \mathfrak{g} \to \mathfrak{g}$$

satisfying the identity

$$(\xi, Q_x(\eta))_{\mathfrak{g}} = \langle \sigma_x(\xi), \sigma_x(\eta)\rangle_{T_x M} = (\xi, d\mu \circ \sigma_x(\eta))_{\mathfrak{g}}$$

for all $\xi \in \mathfrak{p}$ and $\eta \in \mathfrak{g}$. As a consequence, we have the following the uniqueness result for the Riemannian moment map provided it exists.

Proposition 2.5. *Let (M, g) be a Riemannian manifold with a real reductive group G acting on it. Suppose that the action is Hamiltonian and $H^1(\mathfrak{g}, \mathbb{R}) = 0$. Then the moment map $\mu : M \to \mathfrak{p}^*$ is unique.*

Proof. Notice that for any $\eta \in \mathfrak{k}$ and $\xi \in \mathfrak{p}$, we have

$$(\mu(x), [\xi, \eta])_{\mathfrak{g}} = (d\mu(\sigma_x \eta), \xi)_{\mathfrak{g}} = \langle \sigma_x \xi, \sigma_x \eta \rangle_{T_x M},$$

so μ is uniquely determined up to $\mathfrak{p}/[\mathfrak{k}, \mathfrak{p}]$. In particular, if $0 = H^1(\mathfrak{g}, \mathbb{R}) = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$, then $[\mathfrak{k}, \mathfrak{p}] = \mathfrak{p}$. This means $\mathfrak{p}/[\mathfrak{k}, \mathfrak{p}] = 0$, hence μ is unique provided it exists. □

Remark 2.6. This is parallel to the fact that the moment map for a Hamiltonian action of a simple Lie group on a symplectic manifold is always unique.

Example 2.7. Let (M, g) be a Riemannian manifold and $f \in C^\infty(M)$ be a smooth function. And $\phi(t, x)$ is the integral curve of the vector field ∇f passing through $x \in M$, that is

$$\begin{cases} \dot{\phi}(t, x) = \nabla f(\phi(t, x)), \\ \phi(0, x) = x. \end{cases}$$

This induces an action of $\mathbb{R}_+ := \{r \in \mathbb{R} | r > 0\}$ on M as following

$$\begin{array}{rcl} \Phi : \mathbb{R}_+ \times M & \longrightarrow & M \\ (t, x) & \longmapsto & \phi(\ln t, x). \end{array}$$

Notice that the Cartan decomposition of $\mathbb{R} = \mathrm{Lie}(\mathbb{R}_+)$ is $\{0\} \oplus \mathbb{R}$ with $\theta = -1$. This $\mathbb{R}_+$-action on M is Hamiltonian with moment map $f : M \to \mathbb{R}$, since $df(\cdot) = \langle \nabla f, \cdot \rangle_{TM}$.

Example 2.8. $\mathbb{R}^n$. Let $G = \mathrm{GL}(n, \mathbb{R})$ and $K = O(n, \mathbb{R})$. Then the moment map for the standard action of G on $\mathbb{R}^n$ is given by

$$\mu : \mathbb{R}^n \longrightarrow \mathfrak{p} \cong \mathfrak{p}^*,$$
$$x \longmapsto \frac{xx^t}{2},$$

where we have identified $\mathfrak{p}^*$ with $\mathfrak{p} = \{A \in \mathfrak{gl}(n, \mathbb{R}) | A^t = A\}$ via the invariant metric $\langle \xi, \eta \rangle_{\mathfrak{gl}(n,\mathbb{R})} := \mathrm{Tr}(\xi \eta^t)$.

2.2. Geometric quantization

Suppose we have a Hamiltonian G-action on a Riemannian manifold (M, g). Then it will induce a natural representation of G on the vector space $C^\infty(M)$

which is different from the restriction of $\mathrm{Diff}(M)$ to G. To construct it, let us define an infinitesimal action of $\mathfrak{g}$ on $M \times \mathbb{R}$ as following: for any $\xi \in \mathfrak{g}$

$$\tilde{\xi} \circ (x,r) := \sigma_x(\xi) + (\mu(x), \xi_{\mathfrak{p}})_{\mathfrak{g}} \partial_r \in T_{(x,r)}(M \times \mathbb{R}), \tag{2.2}$$

where $\xi_{\mathfrak{p}} \in \mathfrak{p}$ is the $\mathfrak{p}$ component of ξ under Cartan decomposition and ∂_r is the unit vector field on $\mathbb{R}$. This action is actually integrable.

Proposition 2.9.

$$[\tilde{\xi}, \tilde{\eta}] = -\widetilde{[\xi, \eta]}.$$

Proof. The K-equivariance of μ implies $\sigma(\eta_{\mathfrak{k}})\langle \mu, \xi_{\mathfrak{p}}\rangle = \langle \mu, [\xi_{\mathfrak{p}}, \eta_{\mathfrak{k}}]\rangle$. So we have

$$\begin{aligned}
[\tilde{\xi}, \tilde{\eta}] &= [\sigma(\xi) + (\mu, \xi_{\mathfrak{p}})_{\mathfrak{g}}\partial_r, \sigma(\eta) + (\mu, \eta_{\mathfrak{p}})_{\mathfrak{g}}\partial_r] \\
&= [\sigma(\xi), \sigma(\eta)] + \sigma(\xi)(\mu, \eta_{\mathfrak{p}})_{\mathfrak{g}}\partial_r - \sigma(\eta)(\mu, \xi_{\mathfrak{p}})_{\mathfrak{g}}\partial_r \\
&= [\sigma(\xi), \sigma(\eta)] + \sigma(\xi_{\mathfrak{k}} + \xi_{\mathfrak{p}})(\mu, \eta_{\mathfrak{p}})_{\mathfrak{g}}\partial_r - \sigma(\eta_{\mathfrak{k}} + \eta_{\mathfrak{p}})(\mu, \xi_{\mathfrak{p}})_{\mathfrak{g}}\partial_r \\
&= [\sigma(\xi), \sigma(\eta)] + \langle \sigma(\xi_{\mathfrak{p}}), \sigma(\eta_{\mathfrak{p}})\rangle_{TM}\partial_r - \langle \sigma(\eta_{\mathfrak{p}}), \sigma(\xi_{\mathfrak{p}})\rangle_{TM}\partial_r \\
&\quad + (\mu, [\eta_{\mathfrak{p}}, \xi_{\mathfrak{k}}])_{\mathfrak{g}}\partial_r - (\mu, [\xi_{\mathfrak{p}}, \eta_{\mathfrak{k}}])_{\mathfrak{g}}\partial_r \\
&= [\sigma(\xi), \sigma(\eta)] + (\mu, [\eta_{\mathfrak{p}}, \xi_{\mathfrak{k}}])_{\mathfrak{g}}\partial_r - (\mu, [\xi_{\mathfrak{p}}, \eta_{\mathfrak{k}}])_{\mathfrak{g}}\partial_r \\
&= -\sigma[\xi, \eta] - (\mu, [\xi, \eta]_{\mathfrak{p}})_{\mathfrak{g}}\partial_r \\
&= -\widetilde{[\xi, \eta]}
\end{aligned}$$

thanks to the fact that

$$[\xi, \eta]_{\mathfrak{p}} = [\xi_{\mathfrak{k}} + \xi_{\mathfrak{p}}, \eta_{\mathfrak{k}} + \eta_{\mathfrak{p}}]_{\mathfrak{p}} = [\xi_{\mathfrak{k}}, \eta_{\mathfrak{p}}] + [\xi_{\mathfrak{p}}, \eta_{\mathfrak{k}}],$$

and $\sigma[\xi, \eta] = -[\sigma(\xi), \sigma(\eta)]$ (because G acts on M on the left). $\square$

The action defined in (2.2) give rise to a representation of $\mathfrak{g}$ on $C^\infty(M)$ as following:

$$\tilde{\xi} \circ f := \sigma(\xi) f + (\mu(x), \xi_{\mathfrak{p}})_{\mathfrak{g}} f,$$

which is exactly the $\mathfrak{g}$-representation on the smooth sections of the pre-quantum line bundle L over the symplectic manifold T^*M restricted to M. Now suppose that the action can be lifted to a G-representation (e.g., G is simply connected), then for fixed $(x,r) \in M \times \mathbb{R}$, we may define a potential function P_x associated to G as following:

$$\begin{array}{cccccc} P_x : G & \longrightarrow & M \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ g & \longmapsto & g \circ (x, r) & \longmapsto & \log \frac{|g \circ r|}{r}, \end{array}$$

which clearly depends only on x.

Proposition 2.10. *Following the above notation, we have*

1. *For any $\xi \in \mathfrak{p}$, we have*

$$\begin{aligned} \frac{dP_x(\exp t\xi)}{dt} &= (\mu(\exp t\xi \circ x), \xi)_{\mathfrak{g}}, \\ \frac{d^2 P_x(\exp t\xi)}{dt^2} &= \langle \sigma(\xi), \sigma(\xi) \rangle_{T_{\exp\, t\xi \circ x} M} \geq 0. \end{aligned}$$

2. *Suppose $\mu(x_0) = 0$, then*

$$\mu^{-1}(0) \cap G \cdot x_0 = K \cdot x_0$$

3. *Let G_x be the stabilizer of $x \in M$, it acts on G on the right. Then $dP_x \in \Omega^1(G)$ is G_x-invariant with respect to the action.*

Proof. The first part follows from the following identities

$$P_x(\exp t\xi \circ x) = \log \frac{r(t)}{r} \text{ with } r(t) := \exp t\xi \circ r$$

and

$$\frac{dr(t)}{dt} := (\mu(\exp t\xi \circ x), \xi)_{\mathfrak{g}} r(t).$$

For the second part, we notice that $\mu^{-1}(0) \cap G \cdot x_0 \supset K \cdot x_0$. To show the uniqueness, let $\mathfrak{g}_{x_0} \subset \mathfrak{g}$ be the Lie algebra of the stabilizer of x_0, all we need to show is that for any $\xi \in \mathfrak{p} \backslash \mathfrak{g}_{x_0}$ there is at most one zero for $\langle \mu(x(t)), \xi \rangle$, where $x(t) := \exp t\xi \cdot x_0$. But this follows from the fact that

$$\frac{d}{dt}(\mu(x(t)), \xi)_{\mathfrak{g}} = \langle \sigma(\xi), \sigma(\xi) \rangle_{T_{x(t)} M} \geq 0$$

and for $\xi \in \mathfrak{p} \backslash \mathfrak{g}_{x_0}, |\sigma_{x_0}(\xi)|^2 > 0$.

For the third part, let $\xi \in \mathfrak{g}_x$, with $\mathfrak{g}_x$ being the Lie algebra of G_x. We have for any $g \in G$

$$\left.\frac{dP_x(g \exp t\xi)}{dt}\right|_{t=0} = \left.(\mu(g \exp t\xi g^{-1} g \circ x), \mathrm{Ad}_g\, \xi)_{\mathfrak{g}}\right|_{t=0} = (\mu(g \circ x), \mathrm{Ad}_g\, \xi)_{\mathfrak{g}}.$$

This implies

$$L_{\sigma(\xi)} dP_x = d i_{\sigma(\xi)} dP_x = 0,$$

where the last identity follows from Proposition 2.11 of the next subsection.

□

2.3. Structure of $\mathfrak{g}_x$

In this subsection, we will extend Calabi–Matsushima's theorem to our Riemannian setting.

Proposition 2.11. *Let $\mathfrak{g}_x \subset \mathfrak{g}$ be the Lie algebra of the stabilizer of $x \in M$. Then for any $\xi \in \mathfrak{g}_x$, the function*

$$\begin{array}{ccc} f : G & \longrightarrow & \mathbb{R} \\ g & \longrightarrow & \left(\mu\left(g \circ x\right), Ad_g \xi\right)_{\mathfrak{g}} \end{array}$$

is constant, where we have extended

$$\mathrm{Im}\, \mu \subset \mathfrak{p} \subset \mathfrak{g}$$

by requiring

$$(\mu, \mathfrak{k})_{\mathfrak{g}} = 0.$$

As a consequence, the function $(\mu(x), \cdot)_{\mathfrak{g}} : \mathfrak{g}_x \to \mathbb{R}$ is a character.

Proof. First, for g lies in K, the claim follows from the equivariance of the moment map. So let us assume $g = \exp t\eta$ with $\eta \in \mathfrak{p}$, then

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} (\mu(\exp t\eta \cdot x), \mathrm{Ad}_{\exp t\eta}\, \xi)_{\mathfrak{g}} &= (d\mu \circ \sigma_x(\eta), \xi)_{\mathfrak{g}} + (\mu(x), [\eta, \xi])_{\mathfrak{g}} \\ &= (d\mu \circ \sigma_x(\eta), \xi_{\mathfrak{p}})_{\mathfrak{g}} + (\mu(x), [\eta, \xi_{\mathfrak{k}}])_{\mathfrak{g}} \\ &= \langle \sigma_x(\eta), \sigma_x(\xi_{\mathfrak{p}}) \rangle_{TM} + \langle \sigma_x(\xi_{\mathfrak{k}}), \sigma_x(\eta) \rangle_{TM} \\ &= \langle \sigma_x(\eta), \sigma_x(\xi) \rangle_{TM} \\ &= 0 \end{aligned}$$

since

$$(\mu(x),[\eta,\xi_{\mathfrak{k}}])_{\mathfrak{g}}=\left.\frac{d}{dt}\right|_{t=0}(\mu(\exp t\xi_{\mathfrak{k}}\cdot x),\eta)_{\mathfrak{g}}=(d\mu\circ\sigma_x(\xi_{\mathfrak{k}}),\eta)_{\mathfrak{g}}$$
$$=\langle\sigma_x(\xi_{\mathfrak{k}}),\sigma_x(\eta)\rangle_{TM}.$$

□

Lemma 2.12. *Let $\tau\in\mathfrak{p}$, then for any $\xi,\eta\in\mathfrak{g}$, we have*

$$([\tau,\xi],\eta)_{\mathfrak{g}}=(\xi,[\tau,\eta])_{\mathfrak{g}}.$$

Proof.

$$\begin{aligned}([\tau,\xi],\eta)_{\mathfrak{g}}=-\langle[\tau,\xi],\theta(\eta)\rangle_{\mathfrak{g}}&=-\langle\xi,[\theta(\eta),\tau]\rangle_{\mathfrak{g}}\\&=\langle\xi,[\theta(\eta),\theta(\tau)]\rangle_{\mathfrak{g}}\\&=\langle\xi,\theta([\eta,\tau])\rangle_{\mathfrak{g}}\\&=-\langle\xi,\theta([\tau,\eta])\rangle_{\mathfrak{g}}\\&=(\xi,[\tau,\eta])_{\mathfrak{g}},\end{aligned}$$

where we have used the fact that θ is a Lie algebra homomorphism for the fourth identity. □

Definition 2.13. We call the critical points of the function $|\mu(x)|^2:M\longrightarrow\mathbb{R}$ extremal points. Notice that

$$d\langle\mu(x),\mu(x)\rangle_{\mathfrak{p}}=\langle\sigma_x(\mu(x)),\cdot\rangle_{T_xM}$$

implies

$$\nabla|\mu(x)|^2=2\sigma_x(\mu(x))\in T_xM.$$

Thus x_0 is an extremal point if and only if $\mu(x_0)\in\mathfrak{p}_{x_0}:=\mathfrak{p}\cap\mathfrak{g}_{x_0}$.

Theorem 2.14. (*Real version of Calabi–Matsushima's theorem*). *Suppose (M,g) is a Riemannian manifold with a Hamiltonian action of a real reductive group G and*

$$\mu:M\longrightarrow\mathfrak{p}$$

is the moment map. Suppose x_0 is a critical point of $|\mu(x)|^2$, then we have decomposition

$$\mathfrak{g}_{x_0}=\mathfrak{h}_0\oplus\bigcup_{\lambda>0}\mathfrak{h}_\lambda$$

where

$$\mathfrak{h}_\lambda := \{\xi \in \mathfrak{g}_{x_0} |\ [\mu(x_0), \xi] = \lambda \cdot \xi\}$$

and $\mathfrak{h}_0$ *is the reductive part of* $\mathfrak{g}_{x_0}$. *Moreover,*

$$[\mathfrak{h}_{\lambda_1}, \mathfrak{h}_{\lambda_2}] \subset \mathfrak{h}_{\lambda_1+\lambda_2}.$$

and

$$[\mu(x_0), \mathfrak{h}_0] = 0.$$

Proof. Suppose x_0 is a critical point for $|\mu(x)|^2$, then $\mu(x_0) \in \mathfrak{g}_{x_0}$, hence Lemma 2.12 implies that

$$[\mu(x_0), \cdot] : \mathfrak{g}_{x_0} \longrightarrow \mathfrak{g}_{x_0}$$

is a self-adjoint endomorphism with respect to the inner product $(\cdot, \cdot)_{\mathfrak{g}}$.

So we have decomposition

$$\mathfrak{g}_{x_0} = \mathfrak{h}_0 \oplus \bigcup_{\lambda > 0} \mathfrak{h}_\lambda$$

with

$$\mathfrak{h}_\lambda := \{\xi \in \mathfrak{g}_{x_0} |\ [\mu(x_0), \xi] = \lambda \cdot \xi\}.$$

Suppose $\xi \in \mathfrak{g}_{x_0}$ satisfying $[\mu(x_0), \xi] = \lambda\xi$, we have

$$\begin{aligned}
\lambda(\xi, \xi)_{\mathfrak{g}} &= ([\mu(x_0), \xi], \xi)_{\mathfrak{g}} \\
&= ([\mu(x_0), \xi_{\mathfrak{k}} + \xi_{\mathfrak{p}}], \xi_{\mathfrak{k}} + \xi_{\mathfrak{p}})_{\mathfrak{g}} \\
&= 2([\mu(x_0), \xi_{\mathfrak{k}}], \xi_{\mathfrak{p}})_{\mathfrak{g}} \\
&= 2(\mu(x_0), [\xi_{\mathfrak{k}}, \xi_{\mathfrak{p}}])_{\mathfrak{g}} \\
&= 2 \left.\frac{d}{dt}\right|_{t=0} (\mu(\exp(-t\xi_{\mathfrak{k}})\, x_0), \xi_{\mathfrak{p}})_{\mathfrak{g}} \\
&= -2(d\mu \circ \sigma_{x_0}(\xi_{\mathfrak{k}}), \xi_{\mathfrak{p}})_{\mathfrak{g}} \\
&= -2\langle \sigma_{x_0}(\xi_{\mathfrak{k}}), \sigma_{x_0}(\xi_{\mathfrak{p}}) \rangle_{T_{x_0}M} \\
&= 2\langle \sigma_{x_0}(\xi_{\mathfrak{p}}), \sigma_{x_0}(\xi_{\mathfrak{p}}) \rangle_{T_{x_0}M} \\
&= 2|\sigma_{x_0}(\xi_{\mathfrak{p}})|^2_{T_{x_0}M}
\end{aligned}$$

since $[\mathfrak{k}, \mathfrak{p}] \in \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \in \mathfrak{k}$ and $0 = \sigma_{x_0}(\xi) = \sigma_{x_0}(\xi_{\mathfrak{p}}) + \sigma_{x_0}(\xi_{\mathfrak{k}})$. Hence $\lambda \geq 0$ with equality if and only if $\xi_{\mathfrak{p}} \in \mathfrak{g}_{x_0}$, which means $\mathfrak{h}_0$ is θ-invariant hence reductive . Finally,

$$[\mathfrak{h}_{\lambda_1}, \mathfrak{h}_{\lambda_2}] \subset \mathfrak{h}_{\lambda_1+\lambda_2}.$$

follows from the Jacobi identity. $\square$

Corollary 2.15. *We have the following easy consequences:*

1. *If* $\mu(x) = 0$, *then* $\mathfrak{g}_x$ *is reductive.*
2. *If* x *is an extremal point and for any* $\xi \in \mathfrak{h}_0$, $(\mu(x), \xi)_{\mathfrak{g}} = 0$, *then* $\mu(x) = 0$.

Remark 2.16. The above results are usually proved under the assumption that the G-action can be complexified to a $G^{\mathbb{C}}$-action on a Kähler manifold.

3. Examples

In this section, we will introduce two important families of examples of Riemannian moment map.

3.1. Moment map for Kähler manifold

Let us first recall the classical moment map theory for a compact Lie group acting on a Kähler manifold. Suppose (M, ω, J) is a Kähler manifold with ω being the Kähler form and J being the complex structure compatible with ω, that is, $\omega(\cdot,\cdot) = \omega(J\cdot, J\cdot)$ and suppose $\langle\cdot,\cdot\rangle_{TM} := \omega(J\cdot,\cdot)$ is a Riemannian metric. Let K be a compact Lie group that acts on M in a holomorphic Hamiltonian fashion. Since K has a natural complexification $G := K^{\mathbb{C}}$ with Lie algebra $\mathfrak{k}^{\mathbb{C}} = \mathfrak{k} + \sqrt{-1}\mathfrak{k}$, this allows us to complexify the K-action to a G-action by requiring

$$\sigma_x(\sqrt{-1}\xi) := J\sigma_x(\xi), \quad \forall \xi \in \mathfrak{k}.$$

Notice that G can also be viewed as a real reductive Lie group with Cartan decomposition $\mathfrak{g} := \mathfrak{k} + \mathfrak{p}$ with $\mathfrak{p} = \sqrt{-1}\mathfrak{k}$. Since the K-action on M is Hamiltonian, there is a moment map

$$\mu : M \longrightarrow \mathfrak{k}$$

satisfying

$$\begin{cases} X\langle \mu(x), \xi\rangle = \omega(\sigma_x(\xi), X) & \text{for any } X \in \Gamma(TM), \\ \langle \mu(g\cdot x), \xi\rangle = (\mu(x), Ad_{g^{-1}}\xi)_{\mathfrak{g}} & \text{for any } g \in G. \end{cases}$$

By applying the fact $\omega(\sigma_x(\xi), X) = \langle J\sigma_x(\xi), X\rangle_{TM} = \langle \sigma_x(\sqrt{-1}\xi), X\rangle_{TM}$, we found that $\sqrt{-1}\mu : M \to \mathfrak{p}$ can be regarded as a special case of the Riemannian moment map for the G-action on the Riemannian manifold $(M, \langle\cdot,\cdot\rangle_{TM})$.

As an application, we apply Proposition 2.11 to the above setting to get a simpler and more transparent proof of the following corollary, which is Proposition 6 in [6].

Corollary 3.1. *Suppose $\xi \in \mathfrak{g}_x$, Then $(\mu(g\cdot x), Ad_g\,\xi)_{\mathfrak{g}}$ is a constant function in g. In particular, $(G\cdot x)\cap\mu^{-1}(0) \neq \varnothing$ implies $(\mu(x), \xi)_{\mathfrak{g}} = 0$.*

Proof. Let $\xi^{\mathbb{C}} = \operatorname{Re}\xi^{\mathbb{C}} + \sqrt{-1}\operatorname{Im}\xi^{\mathbb{C}} \in \mathfrak{g}_x$, with $\operatorname{Re}\xi^{\mathbb{C}}, \operatorname{Im}\xi^{\mathbb{C}} \in \mathfrak{k}$. If we complexified the K-action on (M,ω) to a G-action and treat G as a real reductive group (c.f. Example 2.1), then Proposition 2.11 implies that for any $\eta \in \mathfrak{k}$,

$$\begin{aligned} 0 &= \left.\frac{d}{dt}\right|_{t=0} (\mu(\exp tJ\eta\cdot x), \mathrm{Ad}_{\exp tJ\eta}(\operatorname{Re}\xi^{\mathbb{C}}, 0) + J(\operatorname{Im}\xi^{\mathbb{C}}, 0))_{\mathfrak{g}} \\ &= \left.\frac{d}{dt}\right|_{t=0} (\mu(\exp tJ\eta\cdot x), \mathrm{Ad}_{\exp tJ\eta}(\operatorname{Re}\xi^{\mathbb{C}}, -\operatorname{Im}\xi^{\mathbb{C}}))_{\mathfrak{g}} \\ &= \left.\frac{d}{dt}\right|_{t=0} (\mu(\exp tJ\eta\cdot x), \mathrm{Ad}_{\exp tJ\eta}\operatorname{Re}\xi^{\mathbb{C}})_{\mathfrak{g}} \end{aligned}$$

since $\mathrm{Ad}_{\exp tJ\eta}(0, -\operatorname{Im}\xi^{\mathbb{C}}) \in \mathfrak{k}$.

On the other hand, $\sqrt{-1}\xi^{\mathbb{C}} \in \mathfrak{g}_x$ implies

$$\sigma_x(\operatorname{Re}\xi^{\mathbb{C}}) + J\sigma_x(\operatorname{Im}\xi^{\mathbb{C}}) = J\sigma_x(\operatorname{Re}\xi^{\mathbb{C}}) - \sigma_x(\operatorname{Im}\xi^{\mathbb{C}}) = 0,$$

hence

$$0 = \left.\frac{d}{dt}\right|_{t=0} (\mu(\exp tJ\eta\cdot x), \mathrm{Ad}_{\exp tJ\eta}\operatorname{Im}\xi^{\mathbb{C}})_{\mathfrak{g}}. \qquad \square$$

3.2. Moment maps coming from anti-holomorphic involution

Definition 3.2. A Kähler manifold $(M^{\mathbb{C}}, \omega, J)$ with an anti-holomorphic involution θ_M is called a complexification of a Riemannian manifold $(M, \langle\cdot,\cdot\rangle_{TM} := \omega(J\cdot,\cdot)|_{TM})$ if $M \subset M^{\mathbb{C}}$ is the fixed locus of $\theta_M : M^{\mathbb{C}} \to M^{\mathbb{C}}$.

Let G be a real reductive group with a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and $\hat{G}$ be the compact dual with Lie algebra $\widehat{\mathfrak{g}} = \mathfrak{k} + \sqrt{-1}\mathfrak{p}$. So G and $\hat{G}$ are two distinct real forms of the complexification $G^{\mathbb{C}}$ with Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} + \sqrt{-1}\mathfrak{g}$. Notice that the inner product $(\cdot,\cdot)_{\mathfrak{g}} = -\langle \cdot, \theta(\cdot)\rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ extends naturally to an inner product on $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} + \sqrt{-1}\mathfrak{g}$ and hence an inner product on $\widehat{\mathfrak{g}} \subset \mathfrak{g}^{\mathbb{C}}$.

Suppose G acts on (M, g) such that the maximal compact subgroup $K \subset \mathrm{Isom}(M, g)$ and let $(M^{\mathbb{C}}, \omega, \theta_M)$ be a complexification of (M, g). Then we introduce the following.

Definition 3.3. The G-action on (M, g) can be complexified if it is the restriction of a holomorphic $G^{\mathbb{C}}$-action on $(M^{\mathbb{C}}, \omega, J)$ such that G leaves $M \subset M^{\mathbb{C}}$ invariant and $\hat{G} \subset \mathrm{Isom}(M^{\mathbb{C}}, \omega)$.

Remark 3.4. Notice the condition $\hat{G} \subset \mathrm{Isom}(M^{\mathbb{C}}, \omega)$ is not quite restrictive, since we can always achieve that by averaging the metric over the compact group $\hat{G}$.

Proposition 3.5. *Let (M, g) be a Riemannian manifold with an action of a real reductive group G. Suppose that the G-action* can be complexified *and the action of the compact dual $\hat{G}$ on $(M^{\mathbb{C}}, \omega, J)$ is Hamiltonian with moment map*

$$\mu^{\mathbb{C}} : M^{\mathbb{C}} \longrightarrow \widehat{\mathfrak{g}} = \mathfrak{k} + \sqrt{-1}\mathfrak{p}$$

satisfying $\mu(x_0) \in \sqrt{-1}\mathfrak{p}$ for some $x_0 \in M \subset M^{\mathbb{C}}$. Then

$$\mu := \sqrt{-1}\mu^{\mathbb{C}}|_M : M \longrightarrow \mathfrak{p}$$

is a moment map for the G-action on M. Moreover, under the moment map $\mu^{\mathbb{C}}$, the anti-holomorphic involution θ is compatible with the Cartan involution.

Proof. Since the $\hat{G}$-action on $(M^{\mathbb{C}}, \omega, J)$ is Hamiltonian, for any $\sqrt{-1}\xi \in \sqrt{-1}\mathfrak{p}$ and $x \in M$, we have

$$d\langle \mu^{\mathbb{C}}, \sqrt{-1}\xi\rangle_{\widehat{\mathfrak{g}}} = \omega(\sigma_x(\sqrt{-1}\xi), \cdot)|_M.$$

By our assumption, $\omega|_M = 0$, this implies that for any $\eta \in \mathfrak{k}$,

$$d\langle \mu, \eta\rangle|_M = \sqrt{-1}d\langle \mu^{\mathbb{C}}, \eta\rangle_{\widehat{\mathfrak{g}}}|_M = \omega(\sigma_x(\eta), \cdot)|_M = 0,$$

from which we deduce that $\langle \mu(x), \eta\rangle_{\mathfrak{g}} = \langle \mu(x_0), \eta\rangle = 0$, hence $\mu : M \to \mathfrak{p}$.

Now for any $\xi \in \mathfrak{p}$, we have $\sqrt{-1}\xi \in \widehat{\mathfrak{g}}$. By our assumption

$$\begin{aligned} d\langle \mu, \xi \rangle &= d\langle \mu^{\mathbb{C}}, \sqrt{-1}\xi \rangle_{\widehat{g}} \\ &= \omega(\sigma_x(\sqrt{-1}\xi), \cdot) \\ &= \langle \sigma_x(\xi), \cdot \rangle_{TM} \end{aligned}$$

with $\sigma_x(\xi) \in T_xM$. □

Corollary 3.6. *Under the same assumption as above. Then $x_0 \in M$ is an extremal point if and only if it is an extremal point of $M^{\mathbb{C}}$, that is, the critical point of $|\mu^{\mathbb{C}}|^2$.*

Proof. Since $x_0 \in M$ is an extremal point for the G-action if and only if $\mu(x_0) \in \mathfrak{p}_{x_0}$, this implies $\mu^{\mathbb{C}}(x_0) \in \sqrt{-1}\mathfrak{p}_{x_0} \subset \widehat{\mathfrak{g}}_{x_0}$, which is equivalent to x_0 being an extremal point of $|\mu^{\mathbb{C}}|^2$. The converse is obvious. □

Corollary 3.7. (*Real version of Kempf–Ness*). *Suppose (M, g) is a Riemannian manifold with an action of a real reductive group G such that the action can be complexified to a Hamiltonian $\hat{G}$-action on $(M^{\mathbb{C}}, \omega, \theta)$ with moment map*

$$\mu : M^{\mathbb{C}} \longrightarrow \widehat{\mathfrak{g}} = \mathfrak{k} + \sqrt{-1}\mathfrak{p}.$$

Then

1. *If the critical set of $|\mu(x)|^2$ in $G \cdot x$ is nonempty, then it consists of a unique K-orbit.*
2. *If $x, y \in M$ are two extremal points of $|\mu(x)|^2$ on $G \cdot x_0$, then $x \in K \cdot y$.*

Proof. Suppose x is a critical point of $|\mu|^2$, then the previous corollary implies that it is an extremal point of $\mu^{\mathbb{C}}$, by Ness's theorem (Theorem 6.2 in [7]), we have $\mu(x) = 0$. Part one then follows from the fact that $G^{\mathbb{C}} \cdot x \cap M = G \cdot x$.

Suppose the critical set of $|\mu|^2$ inside $G \cdot x$ is nonempty, then the critical set of $|\mu(x)|^2$ inside $G^{\mathbb{C}} \cdot x$ is nonempty too. By Theorem 6.2 in [7], the critical set of $|\mu|^2$ contained in a $G^{\mathbb{C}}$-orbit consists of a unique G'-orbit. Part two then follows from the fact that $\hat{G} \cdot x \cap M = K \cdot x$.

Suppose $x, y \in G \cdot x_0$ are extremal points of $|\mu(x)|^2$, then the previous corollary implies that both x and y are extremal points for $G^{\mathbb{C}}$-action. By Theorem 7.1 in [7], they are in the same $\hat{G} \cdot y \cap M = K \cdot y$. □

3.3. Adjoint orbit

Now we are ready to construct a family of examples that fit into the picture we have described in the previous subsection. Let G be a real reductive group and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition associated to an involution θ. Let $\alpha \in \mathfrak{p}$, we denote $\mathcal{O}_\alpha := \mathrm{Ad}_K\,\alpha$ to be the adjoint orbit of K in $\mathfrak{p}$. Let $G^{\mathbb{C}}$ be the complexification of G and $\hat{G}$ be the compact dual of G with Lie algebra $\widehat{\mathfrak{g}} := \mathfrak{k} + i\mathfrak{p}$. Let $\theta^{\mathbb{C}}$ denote the complex linear extension of θ to $G^{\mathbb{C}}$.

Lemma 3.8.

$$\theta^{\mathbb{C}}(\mathrm{Ad}_{\exp\,\xi}\eta) = Ad_{\exp\,\theta^{\mathbb{C}}(\xi)}\,\theta^{\mathbb{C}}(\eta) \textit{ for any } \xi, \eta \in \mathfrak{g}^{\mathbb{C}}.$$

Proof. Since $\theta^{\mathbb{C}}$ is a Lie algebra homomorphism, this implies that $\theta^{\mathbb{C}}(\mathrm{Ad}_{\exp\,t\xi}\cdot) : \mathfrak{g} \to \mathfrak{g}$ and $\mathrm{Ad}_{\exp\,t\theta^{\mathbb{C}}(\xi)}\theta^{\mathbb{C}}(\cdot) : \mathfrak{g} \to \mathfrak{g}$ are both 1-parameter subgroup of the $\mathrm{Int}(\mathfrak{g})$, the adjoint group of $\mathfrak{g}$, and they both pass through $\theta^{\mathbb{C}}$ and

$$\left.\frac{d}{dt}\right|_{t=0} \theta^{\mathbb{C}}(\mathrm{Ad}_{\exp\,t\xi}\cdot) = \theta^{\mathbb{C}}(\mathrm{ad}_\xi\cdot) = \mathrm{ad}_{\theta^{\mathbb{C}}(\xi)}\theta^{\mathbb{C}}(\cdot) = \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\exp\,t\theta^{\mathbb{C}}(\xi)}\,\theta^{\mathbb{C}}(\cdot).$$

$\square$

Proposition 3.9. *Let $\mathcal{O}^{\mathbb{C}}_\alpha := \mathrm{Ad}_{G'}(i\alpha)$ with $\alpha \in \mathfrak{p}$, then*

$$-\theta^{\mathbb{C}} : \mathcal{O}^{\mathbb{C}}_\alpha \longrightarrow \mathcal{O}^{\mathbb{C}}_\alpha$$

is an anti-holomorphic involution of $\mathcal{O}^{\mathbb{C}}_\alpha$ with fixed point set $(\mathcal{O}^{\mathbb{C}}_\alpha)^\theta = \mathcal{O}_\alpha$. In particular, $\mathcal{O}_\alpha \subset \mathcal{O}^{\mathbb{C}}_\alpha$ is Lagrangian and totally geodesic with respect to the invariant metric.

Proof. First, it follows from the lemma above that for $\forall \xi \in \widehat{\mathfrak{g}}$

$$\begin{aligned} -\theta^{\mathbb{C}}(\mathrm{Ad}_{\exp\,\xi}\sqrt{-1}\alpha) &= -\mathrm{Ad}_{\exp\,\theta^{\mathbb{C}}(\xi)}\sqrt{-1}\theta^{\mathbb{C}}(\alpha) \\ &= \mathrm{Ad}_{\exp\,\theta^{\mathbb{C}}(\xi)} i\alpha \end{aligned}$$

since $\theta^{\mathbb{C}}(\alpha) = -\alpha$. This guarantees that $-\theta^{\mathbb{C}}$ leaves $\mathcal{O}^{\mathbb{C}}_\alpha \subset \widehat{\mathfrak{g}}$ invariant.

Second, let $\omega_{\mathcal{O}_\alpha^{\mathbb{C}}}$ be the Kostant–Kirillov–Souriau Kähler form on $\mathcal{O}_\alpha^{\mathbb{C}}$ and $\mathrm{ad}_\xi\beta$, $\mathrm{ad}_\eta\beta$ are the vector fields induced by $\xi, \eta \in \widehat{\mathfrak{g}}$ at $\beta \in \mathcal{O}_\alpha^{\mathbb{C}}$. Then

$$\begin{aligned}((-\theta^{\mathbb{C}})^*\omega_{\mathcal{O}_\alpha^{\mathbb{C}}})(\mathrm{ad}_\xi\,\beta, \mathrm{ad}_\eta\,\beta)|_\beta &:= \omega_{\mathcal{O}_\alpha^{\mathbb{C}}}((-\theta^{\mathbb{C}})_*(\mathrm{ad}_\xi\beta), (-\theta^{\mathbb{C}})_*(\mathrm{ad}_\eta\beta))|_{-\theta^{\mathbb{C}}(\beta)}\\ &= \omega_{\mathcal{O}_\alpha^{\mathbb{C}}}(\mathrm{ad}_{\theta^{\mathbb{C}}(\xi)}\,\theta_*^{\mathbb{C}}(\beta), \mathrm{ad}_{\theta^{\mathbb{C}}(\eta)}\,\theta_*^{\mathbb{C}}(\beta))|_{-\theta^{\mathbb{C}}(\beta)}\\ &= (-\theta^{\mathbb{C}}(\beta), [\theta^{\mathbb{C}}(\xi), \theta^{\mathbb{C}}(\eta)])_{\widehat{\mathfrak{g}}}\\ &= -(\theta^{\mathbb{C}}(\beta), \theta^{\mathbb{C}}[\xi,\eta])_{\widehat{\mathfrak{g}}}\\ &= -(\beta, [\xi,\eta)_{\widehat{\mathfrak{g}}}\\ &= -\omega_{\mathcal{O}_\alpha^{\mathbb{C}}}(\mathrm{ad}_\xi\,\beta, \mathrm{ad}_\eta\,\beta)|_\beta\end{aligned}$$

where we have used the fact that $\theta^{\mathbb{C}}$ is an isometry with respect to the inner product $(\cdot,\cdot)_{\widehat{\mathfrak{g}}}$, hence an isometry of $\mathcal{O}_\alpha^{\mathbb{C}}$. So $\theta^{\mathbb{C}}$ is an anti-holomorphic involution and $\mathcal{O}_\alpha = \mathrm{Ad}_K\sqrt{-1}\alpha = \mathcal{O}_\alpha^{\mathbb{C}} \cap \sqrt{-1}\mathfrak{p}$. $\square$

Remark 3.10. Soon after I realize the above family of examples, I learned that they were also studied before by O'shea and Sjammar in [8].

Remark 3.11. Notice that $\mathrm{ad}_\alpha : \mathfrak{g} \to \mathfrak{g}$ induce an isomorphism

$$\mathfrak{k}/\mathfrak{k}_\alpha \cong \mathfrak{p}/\mathfrak{p}_\alpha,$$

where $\mathfrak{k}_\alpha = \ker\,\mathrm{ad}_\alpha|_{\mathfrak{k}}$ and $\mathfrak{p}_\alpha = \ker\,\mathrm{ad}_\alpha|_{\mathfrak{p}}$.To see this, we first notice that $\alpha \in \mathfrak{p}$ implies $\mathrm{ad}_\alpha : \mathfrak{g} \to \mathfrak{g}$ is skew-selfadjoint with respect to the inner product on $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, hence

$$\mathrm{ad}_\alpha : \mathfrak{k}_\alpha \oplus \mathfrak{k}_\alpha^\perp = \mathfrak{t} \longrightarrow \mathfrak{p} = \mathfrak{p}_\alpha \oplus \mathfrak{p}_\alpha^\perp$$

implies $\mathfrak{k}_\alpha^\perp \cong \mathrm{Im}\,\mathrm{ad}_\alpha|_{\mathfrak{k}} \subset \mathfrak{p}_\alpha^\perp$. Conversely,

$$\mathrm{ad}_\alpha : \mathfrak{p}_\alpha \oplus \mathfrak{p}_\alpha^\perp = \mathfrak{p} \longrightarrow \mathfrak{t} = \mathfrak{k}_\alpha \oplus \mathfrak{k}_\alpha^\perp$$

implies $\mathfrak{p}_\alpha^\perp \cong \mathrm{Im}\,\mathrm{ad}_\alpha|_{\mathfrak{p}} \subset \mathfrak{k}_\alpha^\perp$. Thus we have

$$\mathrm{ad}_\alpha^2 : \mathfrak{k}_\alpha^\perp \xrightarrow{\cong} \mathfrak{p}_\alpha^\perp \xrightarrow{\cong} \mathfrak{k}_\alpha^\perp.$$

Corollary 3.12. *The natural embedding $\mu : \mathcal{O}_\alpha \to \mathfrak{p}$ is the moment map of G-action on $\mathcal{O}_\alpha$.*

Remark 3.13. Notice that $\mathcal{O}_\alpha := \mathrm{Ad}_K\, \alpha = G/M_\alpha AN$, with

$$M_\alpha := \{\xi \in \mathfrak{k} | [\xi, \alpha] = 0\},$$

A being the maximal Abelian subgroup and N being the unipotent part. For $G = K^{\mathbb{C}}$, we have $\mathcal{O}_\alpha = K/M_\alpha = K^{\mathbb{C}}/P_\alpha$, where P_α is the parabolic subgroup.

Example 3.14. Let $G = \mathbb{R}_+, K = \{1\} \subset G$ and $(M, g) = (\mathbb{R}^n, dx^2)$. Consider the G-action

$$\begin{array}{ccc} G \times M & \longrightarrow & M, \\ (t, x) & \longmapsto & t \cdot x. \end{array}$$

Then the map $|x|^2/2 : M \to \mathbb{R}$ is the moment map of the G-action, since for any $\xi \in \mathbb{R}$,

$$d\left(\frac{|x|^2}{2}, \xi\right)_{\mathbb{R}} = \xi x \cdot dx = \left\langle \xi x \cdot \frac{d}{dx}, \cdot \right\rangle = \left\langle \left.\frac{d}{dt}\right|_{t=0} e^{t\xi} x, \cdot \right\rangle.$$

Example 3.15. Let $M = S^n \in \mathbb{R}^{n+1}$ and $G = \mathrm{SO}_{\mathbb{R}}(n+1, 1)$, with the involution defined by

$$\begin{array}{ccc} \theta : \mathfrak{g} & \longrightarrow & \mathfrak{g}, \\ \xi & \longmapsto & -\xi^t, \end{array}$$

it is easy to see that θ is an automorphism of $\mathfrak{g}$ with its associated Cartan decomposition $\mathfrak{g} = \mathfrak{so}(n+1) + \mathbb{R}^{n+1}$. The compact dual $\widehat{\mathfrak{g}} = \mathfrak{so}(n+1) + \sqrt{-1}\mathbb{R}^{n+1} \subset \mathfrak{g}^{\mathbb{C}}$ is isomorphic to $\mathfrak{so}(n+2)$ via

$$\begin{array}{ccc} \tau : \widehat{\mathfrak{g}} & \xrightarrow{\cong} & \mathfrak{so}(n+2) \\ \begin{bmatrix} A & \sqrt{-1}v \\ \sqrt{-1}v^t & B \end{bmatrix} & & \begin{bmatrix} A & -v \\ v^t & B \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} \sqrt{-1} & \\ & 1 \end{bmatrix} \begin{bmatrix} A & \sqrt{-1}v \\ \sqrt{-1}v^t & B \end{bmatrix} \begin{bmatrix} -\sqrt{-1} & \\ & 1 \end{bmatrix}.$$

Then $\theta^\tau := \tau \circ \theta \circ \tau^{-1}$ is an involution on $\mathfrak{so}(n+2)$, given by

$$\theta^\tau\left(\begin{bmatrix} A & -v \\ v^t & B \end{bmatrix}\right) = \begin{bmatrix} A & v \\ -v^t & B \end{bmatrix} = J \begin{bmatrix} A & v \\ -v^t & B \end{bmatrix} J \quad \text{with } J = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}.$$

It clearly descends to $\hat{G} = \mathrm{SO}(n+2)$ with $\mathrm{SO}(n+2)^{\theta^\tau} = S(O(n+1) \times O(1)) = K$.

Now we extend θ to $\mathfrak{g}^{\mathbb{C}}$ complex linearly, which we denote by $\theta^{\mathbb{C}}$. Let $\alpha = [1, 0, \ldots, 0] \in \mathbb{R}^{n+1}$, then the adjoint orbit

$$\mathcal{O}^{\mathbb{C}}_{i\alpha} = \mathrm{Ad}_{\widehat{G}}\, i\alpha \subset \widehat{\mathfrak{g}} = \tau^{-1}(\mathfrak{so}(n+2))$$

is exactly the complex hyperquadric $Q^n \subset \mathbb{CP}^{n+1}$, which is the complexification of S^n.

Example 3.16. Let $M = \mathbb{HP}^n$, if we identify $\mathbb{H}^n$ with $\mathbb{C}^n \oplus j\mathbb{C}^n$, then we have an embedding $\mathrm{GL}(n, \mathbb{H}) \subset \mathrm{GL}(2n, \mathbb{C})$. To see this, we notice that a $\mathbb{C}$-linear mapping

$$\varphi : \mathbb{H}^n = \mathbb{C}^n \oplus j\mathbb{C}^n \longrightarrow \mathbb{H}^n = \mathbb{C}^n \oplus j\mathbb{C}^n$$

is $\mathbb{H}$-linear precisely when it commutes with $j : \varphi(vj) = \varphi(v)j$. Let

$$G = \mathrm{SL}(n, \mathbb{H}) := \left\{ A \in \mathrm{GL}(2n, \mathbb{C}) | AJ = J\bar{A} \text{ and } \det A = 1 \right\},$$

where $J = \begin{bmatrix} & I_n \\ -I_n & \end{bmatrix}$. We have $\mathfrak{k} = \mathfrak{u}(n+1, \mathbb{H})$ or $\mathfrak{sp}(n+1)$, and $\mathfrak{p} = \mathfrak{k}^{\perp} \subset \mathfrak{g}$ with respect to $\mathrm{Sp}(n+1)$-invariant inner product on $\mathfrak{g}$. Then $\mathbb{HP}^n = \mathrm{Ad}_K\, e$ with $e = [1, 0, \ldots, 0] \in \mathbb{H}^{n+1}$, and G acts on $\mathbb{HP}^n$ with moment map

$$\begin{aligned} \mu(v + jw) &= \begin{bmatrix} v & w \\ w & -v \end{bmatrix} \left(\begin{bmatrix} \bar{v}^t & \bar{w}^t \\ \bar{w}^t & -\bar{v}^t \end{bmatrix} \begin{bmatrix} v & w \\ w & -v \end{bmatrix} \right)^{-1} \begin{bmatrix} \bar{v}^t & \bar{w}^t \\ \bar{w}^t & -\bar{v}^t \end{bmatrix} \\ &= \frac{1}{|v + jw|^2} \begin{bmatrix} v & w \\ w & -v \end{bmatrix} \begin{bmatrix} \bar{v}^t & \bar{w}^t \\ \bar{w}^t & -\bar{v}^t \end{bmatrix} \end{aligned}$$

for $v + jw \in \mathbb{C}^n \oplus j\mathbb{C}^n$, which is the orthogonal projection.

3.4. Moment map for mapping space

Let M be a smooth n-dimensional manifold and (X, Ω) be an N-dimensional Riemannian manifold with a calibrated n-form Ω. Let $\mathrm{Map}_{\Omega}(M, X)$ be the space of Ω-calibrated immersion of M into X. Suppose G is a real reductive group acting on X with Riemannian moment map $\mu : X \to \mathfrak{p}$, and in addition we assume that G preserves Ω. Let $\mathrm{Map}^0_{\Omega}(M, X) := \mathrm{Map}_{\Omega}(M, X)/\mathrm{Diff}$ (M) denote the space of unparameterized mapping from M to X. Then we may introduce a Riemannian structure on $\mathrm{Map}^0_{\Omega}(M, X)$ such that the induced G-action on $\mathrm{Map}^0_{\Omega}(M, X)$ is equipped with a Riemannian moment map. To see this, let us first introduce the metric on $\mathrm{Map}^0_\Omega(M,X)$. For any $f \in \mathrm{Map}^0_\Omega(M,X)$, we define

$$\langle \delta f, \delta f\rangle_f := \int_M \left\langle \delta f^\perp, \delta f^\perp \right\rangle_h f^*\Omega.$$

Notice that G acts on $\mathrm{Map}^0_\Omega(M,X)$ naturally via $g\circ f := g\cdot f(\cdot)$.

Lemma 3.17. *Let $f_t : M \to X$ be a family of immersions with $f = f_0$ and*

$$v := \left.\frac{df_t(x)}{dt}\right|_{t=0} \in f^*TX|_{f(x)}.$$

Then

$$\left.\frac{d}{dt}\right|_{t=0} Vol_X(f_t^*h) = div_{(M,f^*h)}\, v^\top - \langle v, H\rangle_{f^*h},$$

where H is the mean curvature vector of $f(M)$ in X and $v^\top$, $v^\perp$ are the tangential and normal components of v.

Proof. Let $\{e_i\}_{i=1}^n$ be a local orthonormal frame of T_xM and ∇ be the Levi–Civita connection of X, then we have

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} Vol_X(f_t^*h)(x) &= \sum_{i=1}^n \langle \nabla_{e_i} v, e_i\rangle \\
&= \sum_{i=1}^n \left\langle \nabla_{e_i} v^\top, e_i\right\rangle + \sum_{i=1}^n \left\langle \nabla_{e_i} v^\perp, e_i\right\rangle \\
&= \sum_{i=1}^n \left\langle \nabla_{e_i} v^\top, e_i\right\rangle - \sum_{i=1}^n \left\langle v^\perp, \nabla_{e_i} e_i\right\rangle \\
&= \mathrm{div}_{(M,f^*h)}\, v^\top - \langle v, H\rangle_{f^*h}.
\end{aligned}$$

□

Now we are ready to state the following generalization of Proposition 18 of [6].

Proposition 3.18. *The natural action of G on $\mathrm{Map}^0_\Omega(M,X)$ possesses a moment map.*

$$\mu(f) := \int_{f(M)} \mu\Omega.$$

Proof. Let $\delta f \in T_f \mathrm{Map}^0_\Omega(M,X) \subset \Gamma(f^*TX)$. Since $f(M)$ is Ω-calibrated, it is area minimizing, which means the mean curvature H of $f(M)$ vanishes. We deduce

$$\begin{aligned}
&\left\langle \delta f^\perp, \sigma(\xi) \right\rangle_f \\
&= \int_M \left\langle \delta f^\perp, \sigma(\xi) \right\rangle_h Vol_X(f^*h) \\
&= \int_M \delta f \left\langle \mu, \xi \right\rangle_{\mathfrak{p}} Vol(f^*h) + \int_M f^* \left\langle \mu, \xi \right\rangle_{\mathfrak{p}} \Big(\mathrm{div}_{(M,f^*h)} \delta f^\top \\
&\quad - \left\langle \delta f^\perp, H \right\rangle_{f^*h} \Big) Vol(f^*h) \\
&= \int_M \delta f \left\langle \mu, \xi \right\rangle_{\mathfrak{p}} Vol(f^*h) + \int_M f^* \left\langle \mu, \xi \right\rangle_{\mathfrak{p}} \mathcal{L}_{\delta f} Vol(f^*h) \\
&= \delta_f \int_M f^* \left\langle \mu, \xi \right\rangle_{\mathfrak{p}} Vol(f^*h) \\
&= \delta_f \int_{f(M)} \mu \Omega,
\end{aligned}$$

where $\delta f^\perp$ and $\delta f^\top$ are the normal and tangential components of δf with respect to the metric h on X. ☐

Example 3.19. Let M, X be Kähler manifold and $\mathrm{Map}_h(M,X)$ denote the space of holomorphic immersions. Suppose there is a Hamiltonian holomorphic K-action on X with moment map $\mu_X : X \to \mathfrak{k}$, then

$$\int_{f(M)} \mu_X \frac{\omega_X^n}{n!}$$

is the moment map for the $G = K^{\mathbb{C}}$-action on $\mathrm{Map}_h(M,X)$.

Remark 3.20. Another example we may consider is that (X,Ω) is a quaternion Kähler manifold, with Ω being the quaternion Kä hler form. And then we may apply the above proposition and study the space of quaternion submanifold of X.

4. Moment map with infinite dimensional reductive group

In this section, we will present several infinite dimensional example of a Riemannian moment map, at least on the formal level. To do that, first we have to introduce two types of infinite dimensional real reductive groups.

4.1. Group $\mathcal{C} := \mathbf{Diff}(M) \ltimes C_+^\infty(M)$

Let M be an n-dimensional compact smooth orientable manifold and $\wedge^n M$ be the line bundle of top form. We introduce the group

$$\mathcal{C} := \left\{ \begin{array}{ccc} \wedge^n M & \xrightarrow{\rho} & \wedge^n M \\ \downarrow & \circlearrowleft & \downarrow \\ M & \xrightarrow{\rho_M} & M \end{array} \middle| \rho_M \in \mathrm{Diff}(M) \text{ and } \rho : \wedge^n M|_{x\in M} \right.$$
$$\left. \to \wedge^n M|_{x\in M} \text{ is positive linear.} \right\}$$

Notice that $\mathcal{C}$ can be identified with the semi-direct product between $\mathrm{Diff}(M)$ and the space of positive functions on M, that is,

$$\mathcal{C} = \{(\rho_M, u) \in \mathrm{Diff}(M) \times C^\infty(M) | u > 0\},$$

with multiplication being defined by

$$(\phi, u) \cdot (\psi, v) = (\psi \circ \phi, ((\psi^*)^{-1} u) \cdot v).$$

Then it is clear that the Lie algebra $\mathfrak{c}$ of $\mathcal{C}$ is

$$\mathfrak{c} := \{(\xi, \delta u) \in \Gamma(TM) \times C^\infty(M)\},$$

with the Lie bracket given by

$$[(\xi, \delta u), (\eta, \delta v)] = (-[\xi, \eta], -\eta\delta u + \xi\delta v).$$

In particular, we have

$$[(\xi, 0), (0, \delta v)] = (0, \xi\delta v) \quad \text{and} \quad [(0, \delta u), (0, \delta v)] = (0, 0),$$

hence we have a Cartan-type decomposition

$$\mathfrak{c} = \mathfrak{diff} + \mathfrak{p},$$

with $[\mathfrak{p}, \mathfrak{p}] = 0$ and $[\mathfrak{diff}, \mathfrak{p}] \subset \mathfrak{p}$, where $\mathfrak{p} := C^\infty(M)$.

4.1.1. Linear moment map. $\mathcal{C}$ acts naturally on the Hilbert space $\mathcal{H} := (\Gamma(|\wedge^n|^{1/2}, \langle\cdot,\cdot\rangle))$ of half density with the inner product

$$\langle f, g\rangle = \int_M fg$$

as follows.

$$\text{For any} f \in \mathcal{H},\ \text{ we define } (\phi, u) \circ f = \phi^*(uf),$$

and the infinitesimal action is given by

$$(\xi, \delta u) \circ f = \xi f + \delta u f$$

for any $(\xi, \delta u) \in \mathfrak{c}$ and $f \in C^\infty(M)$. Since

$$\begin{aligned} &\langle (\xi, 0) \circ f, g\rangle \\ &\quad = \int_M (\xi f) g \\ &\quad = \int_M -f(\xi g) \\ &\quad = \langle f, (-\xi, 0) \circ g\rangle , \end{aligned}$$

this implies that $\mathrm{Diff}(M) \subset O(\mathcal{H})$, the orthogonal group of $\mathcal{H}$. With this understood, then we have the following.

Proposition 4.1. *The moment map for the action of $\mathcal{C}$ on $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is given by*

$$\begin{array}{rcl} \mu : \mathcal{H} & \longrightarrow & \wedge^n M = C^\infty(M)^*, \\ f & \longmapsto & f^2/2, \end{array}$$

clearly, it is $\mathrm{Diff}(M)$*-equivariant.*

Proof. For any $\xi \in C^\infty(M)$, the infinitesimal action

$$\sigma_f(\xi) = \left.\frac{d}{dt}\right|_{t=0} \exp t\xi \cdot f = \xi \cdot f.$$

Hence,

$$\delta \langle \mu(f), \xi\rangle = \delta \int_M \frac{\xi f^2}{2} = \int_M \delta f(\xi \cdot f) = \langle \delta f, \sigma_f(\xi)\rangle . \qquad \square$$

4.1.2. Moment map of $\mathcal{C}$-action on the space of metrics. We present another example of a Riemannian moment map for the group $\mathcal{C}$, at least on the formal level. Let

$$\mathfrak{G} := \left\{ g(\cdot,\cdot) \in \Gamma(\mathrm{Sym}^2(TM)) | g > 0 \right\}$$

be the space of Riemannian metrics. Then there is a natural Riemannian metric on $\mathfrak{G}$ given by

$$\langle \delta_1 g, \delta_2 g \rangle_{\mathfrak{G}} = \int_M (\delta_1 g)_{ij} (\delta_2 g)_{kl} g^{ik} g^{jl} Vol(g).$$

Let us introduce an action of the group $\mathcal{C}$ on $\mathcal{G}$ as follows:

$$(\phi, u) g = \phi^*(ug).$$

Then it is clear that $\mathrm{Diff}(M)$-action on $(\mathfrak{G}, \langle \cdot,\cdot \rangle_{\mathfrak{G}})$ is isometric.

Proposition 4.2. *The moment map for the $\mathcal{C}$-action on $\mathfrak{G}$ is given by*

$$\begin{array}{rcl} \mu : \mathfrak{G} & \longrightarrow & \wedge^{\mathrm{top}}(M), \\ g & \longmapsto & Vol(g)/2. \end{array}$$

Proof. Let $\xi \in C^\infty(M)$, then we have

$$\begin{aligned}
&\delta \langle \mu(g), \xi \rangle_{\mathfrak{p}} \\
&= \int_M \xi (\delta \, Vol(g)) \\
&= \frac{1}{2} \int_M \xi \ \mathrm{Tr}_g(\delta g) \, Vol(g) \\
&= \frac{1}{2} \int_M \xi g^{ij} (\delta g)_{ij} \, Vol(g) \\
&= \frac{1}{2} \int_M \xi g^{il} g^{jk} g_{lk} (\delta g)_{ij} \, Vol(g) \\
&= \frac{1}{2} \int_M g^{il} g^{jk} (\xi g_{lk}) (\delta g)_{ij} \, Vol(g) \\
&= \frac{1}{2} \langle \sigma_g(\xi), \delta g \rangle_{\mathcal{G}},
\end{aligned}$$

where $\sigma_g(u)$ is the infinitesimal action of u on $g \in \mathfrak{G}$.

Finally, the Diff-equivariance follows from $\mu(\phi^* g) = Vol(\phi^* g) = \phi^*(Vol(g))$ for $\forall \phi \in \mathrm{Diff}(M)$. $\square$

4.2. Real reductive gauge group

Finally, we will describe another family of infinite dimensional Riemannian moment map first studied by Corlette [9]. This framework was applied by Corlette in the study of rigidity of lattices in real reductive groups. To set the scene, let

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \downarrow \\ & & M \end{array}$$

be a principal G-bundle over a Riemannian manifold (M, h) and

$$\begin{array}{c} \operatorname{Ad} P := P \times_{\operatorname{Ad}} \mathfrak{g} \\ \downarrow \\ M \end{array}$$

be its associated adjoint bundle.

Definition 4.3. We define

$$\mathcal{A} := \left\{ \omega \in T^*P \otimes \mathfrak{g} \,\middle|\, \begin{array}{c} \omega(\sigma_p(\xi)) = \xi, \forall \xi \in \mathfrak{g} \\ R_g^* \omega = \operatorname{Ad}_{g^{-1}} \omega, \forall g \in G \end{array} \right\},$$

where R_g is the right action of G on P and

$$\sigma_p : \mathfrak{g} \longrightarrow T_p P$$

is the infinitesimal action of G, that is, $\mathcal{A}$ is the space of G-connection 1-form. So $\mathcal{A}$ is an affine space modeled on $\Omega^1(M, \operatorname{Ad} P)$. For each $\omega \in \mathcal{A}$, the associated connection D_ω on $\operatorname{Ad} P$ is defined as follows: for any $s \in \Gamma(\operatorname{Ad} P), D_\omega s = ds + [\omega, s]$, where we have used the identification

$$\Gamma(\operatorname{Ad} P) = \{ s : P \to \mathfrak{g} | s(g \cdot p) = \operatorname{Ad}_g s(p), \forall g \in G \}.$$

Now we fix a Cartan decomposition of $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ associated to an involution $\theta : \mathfrak{g} \to \mathfrak{g}$ and let $K \subset G$ be the maximal compact subgroup of G with Lie algebra $\mathfrak{k}$. And we also fix a G-invariant trace form $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ as before such that

$$(\cdot, \cdot)_{\mathfrak{g}} := -\langle \cdot, \theta(\cdot) \rangle_{\mathfrak{g}}$$

is an inner product on $\mathfrak{g}$. This together with the Riemannian structure on M induces a metric on $\Omega^1(M, \operatorname{Ad} P)$. Since for any $\omega \in \mathcal{A}$ we have identification

$T_\omega \mathcal{A} = \Omega^1(M, \operatorname{Ad} P)$, this equips $\mathcal{A}$ with a Riemannian structure as follows:

$$(\delta_1 D, \delta_2 D)_{\mathcal{A}} := \int_M (\delta_1 D, \delta_2 D)_{\mathfrak{g}\otimes T^*M}\, dVol_M \text{ for any } \delta_i D \in T_D\mathcal{A}, \quad i = 1, 2.$$

Suppose P has a reduction to Q, a principal K-bundle

$$\begin{array}{ccc} K & \longrightarrow & Q \\ & & \downarrow \\ & & M \end{array}$$

i.e., $P = Q \times_K G$, where K acts on G on the left.

The gauge group

$$\mathcal{G} := \left\{ \begin{array}{ccc} P & \xrightarrow{\rho} & P \\ \downarrow & & \downarrow \\ M & = & M \end{array} \middle|\, \rho \text{ is } G\text{-equivariant} \right\}$$

can be identified with the space of sections of the bundle

$$P \times_{\operatorname{Ad}} G := P \times G/(p, h)\tilde{}\,(p \cdot g, \operatorname{Ad}_{g^{-1}} h).$$

And the action is defined as follows:

$$\begin{array}{cccc} (p, h): & P & \longrightarrow & P, \\ & p & \longmapsto & p \cdot h. \end{array}$$

It is well-defined since for $p \cdot g \in P$, we have

$$(p \cdot h) \cdot g = (p \cdot g) \cdot (g^{-1} h g) = (p \cdot g) \cdot \operatorname{Ad}_{g^{-1}} h.$$

With this understood, we may identify the $\operatorname{Lie}(\mathcal{G})$ with $\Gamma(\operatorname{Ad}(P))$, and the Cartan decomposition of $\mathfrak{g}$ induces Cartan decomposition

$$\operatorname{Lie}(\mathcal{G}) = \Gamma(\operatorname{Ad} P^+) + \Gamma(\operatorname{Ad} P^-)$$

with

$$\begin{aligned} \operatorname{Ad} P^+ &= Q \times_{\operatorname{Ad}_K} \mathfrak{k} = \operatorname{Ad} Q, \\ \operatorname{Ad} P^- &= Q \times_{\operatorname{Ad}_K} \mathfrak{p}, \end{aligned}$$

and

$$\operatorname{Ad} P = \operatorname{Ad} P^+ \oplus \operatorname{Ad} P^-.$$

In particular, we have the relation

$$\Gamma(\operatorname{Ad} P^{\pm}) = \{\xi \in \Gamma(\operatorname{Ad} P) | \theta(\xi) = \pm\xi\}$$

and for any $\omega \in \mathcal{A}$, there is a unique way to write $\omega = \omega^+ + \omega^-$ with $\omega^+ \in \Gamma(\operatorname{Ad} P^+), \omega^- \in \Gamma(\operatorname{Ad} P^-)$. In particular, the maximal compact subgroup can be identified with

$$\mathcal{K} = \{\rho \in \mathcal{G} | \rho \text{ preserves } Q\} \subset \mathcal{G},$$

hence $\operatorname{Lie}(\mathcal{K}) = \operatorname{Ad} Q$.

$\mathcal{G}$-action on P induces a natural action of $\mathcal{G}$ on $\mathcal{A}$ given by

$$D \longmapsto g \circ D \circ g^{-1} = -Dg \circ g^{-1},$$

or infinitesimally

$$\sigma_D(\xi) := -\left.\frac{d}{dt}\right|_{t=0} D \exp t\xi \circ \exp - t\xi = -D\xi \in \Gamma(T_D\mathcal{A})$$

for any $\xi \in \Gamma(\operatorname{Ad}(P))$. Then we have the following.

Proposition 4.4. *The $\mathcal{G}$-action on $\mathcal{A}$ is Hamiltonian with Riemannian moment map*

$$\mu(D) = D^{+*}(\omega^-) \in \Gamma(\operatorname{Ad} P^-),$$

where D^+ is the composition of D with the orthogonal projection to Ad P^+ *with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$, and D^{+*} is the adjoint of D^+.*

Proof. For any $\xi \in \Gamma(\operatorname{Ad} P^-)$,

$$\begin{aligned}
(\mu(D), \xi)_{\mathfrak{g}} &= (\omega^-, D^+\xi), \\
(\delta_1 D, \delta_2 D)_{\mathcal{A}} &= \int_M (\delta_1 D, \delta_2 D)_{\mathfrak{g}}\, dVol_M, \\
(\delta D, \tilde{\xi})_{\mathfrak{g}} &= (\delta\omega^+ + \delta\omega^-, D\xi)_{\mathfrak{g}} \\
&= (\delta\omega^+ + \delta\omega^-, D^+\xi + [\omega^-, \xi])_{\mathfrak{g}} \\
&= (\delta\omega^+, [\omega^-, \xi])_{\mathfrak{g}} + (\delta\omega^-, D^+\xi)_{\mathfrak{g}} \\
&= ([\delta\omega^+, \xi], \omega^-)_{\mathfrak{g}} + (\delta\omega^-, D^+\xi)_{\mathfrak{g}} \\
&= \delta(\omega^-, D^+\xi)_{\mathfrak{g}},
\end{aligned}$$

where in the fourth identity, we have used the fact that for $\xi|_x \in \mathfrak{p}$,

$$(\delta\omega^+, [\omega^-, \xi])_{\mathfrak{g}} = ([\delta\omega^+, \xi], \omega^-)_{\mathfrak{g}}.$$

by Lemma 2.12. □

In particular, we have an immediate consequence of Proposition 2.10.

Corollary 4.5. *If D is simple, i.e., $Ds \equiv 0$ implies $s \equiv 0$, then there is at most one $\mathcal{K}$-orbit in its $\mathcal{G}$-orbit on which the moment map vanishes.*

The above results were originally proved by Corlette [9], which were derived via complexifying G first and then restrict to the fixed locus of an anti-holomorphic involution, as what we did in Section 3.2. Our derivation seems more direct and more natural.

The importance of the moment map interpretation of the equation

$$D^{+*}(\omega^-) = 0$$

lies in the fact that the existence of the solutions to the above equation is equivalent to the existence of harmonic metric over certain G-flat bundle, which can be used to derive some vanishing results via Bochner technique. The readers are encouraged to find more details in [9].

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References

[1] M. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Philos. Trans. Roy. Soc. London Ser. A **308**(**1505**) (1983), 523–615.

[2] F. Kirwan, *Momentum maps and reduction in algebraic geometry; Symplectic geometry*, Diff. Geom. Appl. **9**(**1**–**2**) (1998), 135–171.

[3] A. Fujiki, *Moduli space of polarized algebraic manifolds and Kähler metrics*, Sugaku Expositions **5**(**2**) (1992), 173–191 [translation of Sugaku 42(**3**) (1990), 231–243; MR1073369 (92b:32032)].

[4] S.K. Donaldson, *Scalar curvature and projective embeddings. I*, J. Diff. Geom. **59**(**3**) (2001), 479–522, 393–411.

[5] ———, *Remarks on gauge theory, complex geometry and 4-manifold topology*, Fields Medallists' Lectures, World Sci. Ser. 20th Century Math., **5**, World Sci. Publ., River Edge, NJ, 384–403, 1997.

[6] X. Wang, *Moment map, Futaki invariant and stability of projective manifolds*, Comm. Anal. Geom. **12**(**5**) (2004), 1009–1037.

[7] L. Ness, *Stratification of the null cone via the moment map. With an appendix by David Mumford*, Amer. J. Math. **106**(6) (1984), 1281–1329.

[8] L. O'Shea, and R. Sjamaar, *Moment maps and Riemannian symmetric pairs*, Math. Ann. **317**(**3**) (2000), 415–457.

[9] K. Corlette, *Flat G-bundles with canonical metrics*, J. Diff. Geom. **28**(**3**) (1988), 361–382.

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