

Spectral geometry and asymptotically conic convergence

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We define a new conic metric collapse, *asymptotically conic convergence*,¹ in which a family of smooth Riemannian metrics degenerates to have an isolated conic singularity. For a conic metric (M_0, g_0) and an asymptotically conic or “scattering” metric (Z, g_z) , we construct a new non-standard blowup, the *resolution blowup*, in which the conic singularity in M_0 is resolved by Z . This blowup induces a smooth family of metrics $\{g_\epsilon\}$ on the compact resolution space M . (M, g_ϵ) is said to converge *asymptotically conically* to (M_0, g_0) as $\epsilon \rightarrow 0$.

Let Δ_ϵ and Δ_0 be geometric Laplacians on (M, g_ϵ) and (M_0, g_0) , respectively. Our first result is convergence of the spectrum of Δ_ϵ to the spectrum of Δ_0 as $\epsilon \rightarrow 0$. Note that this result implies spectral convergence for the k -form Laplacian under certain geometric hypotheses. This theorem is proven using rescaling arguments, standard elliptic techniques and the b -calculus of [33]. Our second result is technical: we construct a parameter (ϵ) dependent heat operator calculus which contains, and hence describes precisely, the heat kernel for Δ_ϵ as $\epsilon \rightarrow 0$. The consequences of this result include the existence of a polyhomogeneous asymptotic expansion for H_ϵ as $\epsilon \rightarrow 0$, with uniform convergence down to $t = 0$. To prove this result, we construct heat spaces as manifolds with corners using both standard and non-standard blowups on which we construct suitable heat operator calculi. A parametrix construction modeled after Melrose’s heat kernel construction [33] and a maximum principle argument complete the proof.

1. Introduction

In the 1970s and 80s, Cheeger [6] initiated a study of the spectrum of the Laplacian and heat kernel on manifolds with isolated conical singularities. Simultaneously, Melrose [33] developed pseudodifferential techniques

¹The name “asymptotically conic convergence” was chosen because the geometry of the collapsing neighborhood is described by an asymptotically conic space.

to study elliptic operators on manifolds with cylindrical ends and conic singularities. Other works include [2, 3, 24, 34, 35]. Manifolds with isolated conical singularities are instructive models for the more singular Calabi–Yau orbifolds which are of particular interest to physicists, see for example [21]. Although an isolated conical singularity is the simplest example of a metric singularity, the spectral geometry of Riemannian manifolds with isolated conic singularities is still not completely understood; see, for example, [29, 31]. One approach to understanding the spectral geometry of a singular space is to consider a family of smooth manifolds which converge in some sense to the singular space; that is the approach taken here and in [7–9, 11, 13, 25, 26, 30, 32]. The goal of our somewhat lengthy and technical spectral convergence results is to provide new tools for understanding the spectral geometry of manifolds with isolated conic singularities. Our results also apply to the singular spaces studied in [10, 29]. With sufficient control on the metric degeneration, uniform spectral convergence results for k -form Laplacians and their heat kernels are applicable to understanding more subtle spectral invariants such as the η invariant, as in [30]. Our work is in the spirit of [30] and generalizes the work of McDonald [32]. We briefly recall the setup and main results of those authors.

Let M be a fixed compact manifold with Riemannian metric, and let H be an embedded orientable hypersurface with defining function x and smooth metric g_H . Let

$$g_\epsilon := dx^2 + (\epsilon^2 + x^2)g_H, \quad \epsilon \in [0, 1).$$

As $\epsilon \rightarrow 0$, $g_\epsilon \rightarrow dx^2 + x^2g_H$, which has an isolated conic singularity at $x = 0$. Geometrically, M is pinched along the hypersurface H as $\epsilon \rightarrow 0$ and the resulting metric has a conic singularity at $x = 0$. The study of this metric collapse is the content of the 1990 thesis of McDonald [32]. In 1995, Mazzeo and Melrose [30] developed pseudodifferential techniques to describe the behavior of the spectral geometry under another specific type of metric collapse known as *analytic surgery*. As $\epsilon \rightarrow 0$, the metrics

$$\frac{|dx|^2}{x^2 + \epsilon^2} + h = g_\epsilon \longrightarrow g_0 = \frac{|dx|^2}{x^2} + h.$$

g_0 is an exact b -metric on the compact manifold with boundary \overline{M} obtained by cutting M along H and compactifying as a manifold with boundary, hence the name, *analytic surgery*. Under certain assumptions on the associated

Dirac operators,² they proved

$$\lim_{\epsilon \rightarrow 0} \eta(\partial_\epsilon) = \eta_b(\partial_{\overline{M}}),$$

where $\eta_b(\partial_{\overline{M}})$ is the b -version of the eta invariant introduced by Melrose. These results were proven by analyzing the resolvent family of the Dirac operators ∂_ϵ uniformly near zero. This led to a precise description of the behavior of the small eigenvalues. In our work, we obtain uniform convergence results for heat kernels under conic metric degeneration. Since many interesting spectral invariants (including the η invariant) are computable via the heat kernel and its trace, it is reasonable to expect that our results will yield applications like [30]. Our results are necessarily quite technical; without such a detailed analysis, it is simply not possible to calculate spectral invariants, see, for example, Perelman's counterexample in [9].

The metric degeneration we consider, *asymptotically conic convergence* (*acc*), encompasses the conic collapse of [32], the analogous smooth collapse of a higher codimension submanifold and the collapse of an open neighborhood with some restrictions on the local geometry. Before stating our spectral convergence results for geometric Laplacians, we recall their definition.

Definition 1.1. Let (E, ∇) be a Hermitian vector bundle over a Riemannian manifold (M, g) with metric-compatible connection ∇ . A geometric Laplacian is an operator Δ acting on sections of E which has the form

$$\Delta = \nabla^* \nabla + R,$$

where R is a non-negative self-adjoint endomorphism of E . By the Weitzenböck Theorem [36], the Laplacian on k -forms is a geometric Laplacian, as is the Hodge Laplacian and the conformal Laplacian; any geometric Laplace-type operator is a geometric Laplacian.

Our results are the following.

Theorem 5.1. Let (M_0, g_0) be a compact Riemannian n -manifold with isolated conic singularity, and let (Z, g_z) be an asymptotically conic (ac)

²In a later collaboration with Hassell *et al.* [18], these hypotheses were removed.

space, with $n \geq 3$. Assume (M, g_ϵ) converges asymptotically conically to (M_0, g_0) . Let (E_0, ∇_0) and (E_z, ∇_z) be rank k Hermitian vector bundles over (M_0, g_0) and (Z, g_z) , respectively, so that each of these bundles in a neighborhood of the boundary is the pullback from a bundle over the boundary (Y, h) . Let Δ_0, Δ_z be the corresponding Friedrichs extensions of geometric Laplacians, and let Δ_ϵ be the induced geometric Laplacian on (M, g_ϵ) . Assume Δ_z has no \mathcal{L}^2 nullspace. Then the accumulation points of the spectrum of Δ_ϵ as $\epsilon \rightarrow 0$ are precisely the points of the spectrum of Δ_0 , counting multiplicity.

The setting for our next result is the acc heat space, a manifold with corners constructed in Section 7. Note that the hypotheses for this result are weaker: Δ_z may have nontrivial \mathcal{L}^2 cohomology and the dimension may be 2 or greater.

Theorem 7.2. Let (M_0, g_0) be a compact Riemannian n -manifold with isolated conic singularity, and let (Z, g_z) be an ac space, with $n \geq 2$. Assume (M, g_ϵ) converges asymptotically conically to (M_0, g_0) . Let (E_0, ∇_0) and (E_z, ∇_z) be rank k Hermitian vector bundles over (M_0, g_0) and (Z, g_z) , respectively, so that each of these bundles in a neighborhood of the boundary is the pullback from a bundle over the boundary (Y, h) . Let Δ_0 and Δ_z be the corresponding Friedrichs extensions of geometric Laplacians, and let Δ_ϵ be the induced geometric Laplacian on (M, g_ϵ) . Then the associated heat kernels H_ϵ have a full polyhomogeneous expansion as $\epsilon \rightarrow 0$ on the acc heat space with the following leading terms:

- At the conic front face, $F_{0101}, H_\epsilon(z, z', t) \rightarrow H_0(z, z', t)$, the heat kernel for (M_0, g_0) .
- At the rescaled b front face, $F_{1010,2}, H_\epsilon(z, z', t) \rightarrow (\rho_{1010,2})(\rho_{1010})(\rho_{0101})^{-1}H_b(\tau)$, the b heat kernel with rescaled time variable τ .
- At the exact conic front face, $F_{1111,2}, H_\epsilon(z, z', t) \rightarrow H_0(\tilde{\tau})$, the heat kernel for the exact cone with rescaled time variable $\tilde{\tau}$.
- At the side faces F_{1001}, F_{0110} and the residual b face F_{1010} , the heat kernel has polyhomogeneous expansion with leading term vanishing to at least second order.

This convergence is uniform in ϵ for bounded time and moreover, the error term is bounded by $C\epsilon t^N$ as $t \rightarrow 0$, for any $N \in \mathbb{N}_0$, where C may depend on N .

Remarks.

- This theorem immediately implies the uniform convergence

$$H_\epsilon(z, z', t) \rightarrow H_0(z, z', t), \quad T > t > 0,$$

as well as the convergence

$$H_\epsilon(z, z', t) \rightarrow H_0(z, z', t) + O(\epsilon), \quad t, \epsilon \rightarrow 0,$$

with explicit error given by the leading terms above in the polyhomogeneous expansion of $H_\epsilon(z, z', t)$ on the acc heat space as $\epsilon \rightarrow 0$.

- We have dropped the half-density factor,

$$(\rho_{1111,2})^{n+1}(\rho_{1010,2})^n(\rho_{1010})^{n-1}(\rho_{d2})^{(n+1)/2}(\rho_{1001}\rho_{0110})^{(n-1)/2}\sqrt{\nu},$$

where above ν is a smooth non-vanishing density on the acc heat space.

These theorems are proven in Sections 5 and 7, respectively. In Section 2, we define the resolution blowup and ac convergence. Sections 3 and 4 contain a brief review of geometric and analytic results and terminology on manifolds with corners. In Section 6, we construct the heat spaces and heat operator calculi that will be used to prove the main theorem in Section 7.

This work is based on the author’s doctoral dissertation completed at Stanford University in June 2006 under the supervision of Rafe Mazzeo. The author wishes to thank Rafe Mazzeo for excellent advising, Andras Vasy for many helpful conversations and suggestions, Richard Melrose for enlightening conversations and correspondence and the anonymous referee for useful comments.

2. Asymptotically conic convergence

The conic collapse in ac convergence is locally modeled by an ac scattering metric or ac space. The following definitions, which may be used differently by other authors, were motivated to impose as few restrictions on the geometry of the collapse as possible while simultaneously maintaining sufficient control to uniformly analyze the Laplacian and its heat kernel.

Definition 2.1. An *ac* scattering metric (\bar{Z}, g_z) is a smooth complete metric on a compact manifold with boundary so that a neighborhood of the

boundary is diffeomorphic to a product $(0, r_1)_r \times Y$, where r is a boundary-defining function for the boundary Y . On this neighborhood,

$$g_z \sim \frac{dr^2}{r^4} + \frac{h(r)}{r^2}, \quad h(r) \sim h_0 + rh_1 + r^2h_2 + \dots, \quad \text{as } r \rightarrow 0,$$

where $\{h_k\}_{k \geq 0}$ are a smooth family of Riemannian metrics on Y .³ Uniquely associated to the ac scattering metric (\bar{Z}, g_z) is the complete non-compact manifold Z with ac end; Z is known as an *ac space*.⁴ Letting $\rho = \frac{1}{r}$, there is a compact subset $K_z \subset Z$ so that

$$Z - K_z \cong \left(\frac{1}{r_1}, \infty \right)_\rho \times Y, \quad g_z|_{(Z - K_z)} = d\rho^2 + \rho^2 h(1/\rho).$$

A familiar example of an ac scattering metric is the radial compactification of \mathbb{R}^n with boundary \mathbb{S}^{n-1} . The intuitive geometric picture of an ac space is the “big end of a complete cone”.

Definition 2.2. Let M be a compact metric space with Riemannian metric g . Then, (M, g) has an isolated conical singularity at the point p and g is called a *conic metric* if

1. $(M - \{p\}, g)$ is a smooth, open manifold.
2. There is $x_1 > 0$ and a neighborhood N of p with a function $x : (N - \{p\}) \rightarrow (0, x_1]$ such that $N - \{p\}$ is diffeomorphic to $(0, x_1]_x \times Y$ and

$$g \sim dx^2 + x^2 h(x), \quad h(x) \sim h_0 + xh_1 + x^2h_2 + \dots, \quad \text{as } x \rightarrow 0,$$

where $\{h_k\}_{k \geq 0}$ are a smooth family of metrics on Y .

By “blowing up” the cone point p adding a copy of (Y, h_0) at $x = 0$, a conic metric is an incomplete metric on a compact manifold with boundary whose metric has the above degenerate form in a neighborhood of the boundary defined by $\{x = 0\}$.

³For technical reasons, we require the stronger assumption that h has a formal power series at the boundary rather than the standard regularity assumption that $h(r)$ extends smoothly to a metric on the boundary.

⁴Note that ac spaces are sometimes called “asymptotically locally Euclidean”, or *ALE*. However, that term is often used for the more restrictive class of spaces that are asymptotic at infinity to a cone over a quotient of the sphere by a finite group, so to avoid confusion, we use the term *asymptotically conic*.

The intuitive geometric picture of a conic metric is an open manifold, with a boundary that pinches to the point of a cone. It is an incomplete metric because the cone point is not included in the smooth Riemannian structure.

In the definition of *resolution blowup*, we use the notation $M \cup_\phi N$ for a smooth manifold constructed from the smooth manifolds M and N with a diffeomorphism ϕ from $V \subset N$ to $U \subset M$ that gives the equivalence relation, $V \ni p \sim \phi(p) \in U$. $M \cup_\phi N$ is the disjoint union of M and N modulo the equivalence relation of ϕ . The smooth structure on $M \cup_\phi N$ and the topology are induced by those of M and N .

Definition 2.3. Let (M_0, g_0) be a compact n dimensional manifold with isolated conic singularity and let (Z, g_z) be an ac space of dimension n , so that $\partial M_0 = \partial \bar{Z} = (Y, h_0)$. Then, the interior of M_0 ,

$$M_0^0 = K_0 \cup V_0,$$

where $V_0 \cong (0, x_1)_x \times Y$, and K_0 is compact. With this diffeomorphism

$$g_0 = dx^2 + x^2 \tilde{h}(x) \quad \text{on } (0, x_1)_x \times Y,$$

and the boundary of K_0 in M_0 is of the form $\partial K_0 = \{x = x_1\} \cong Y$. Similarly,

$$Z = K_z \cup V_z,$$

where $V_z \cong (\rho_1, \infty)_\rho \times Y$, and K_z is compact. With this diffeomorphism,

$$g_z = d\rho^2 + \rho^2 h(\rho, y) \quad \text{on } (\rho_1, \infty)_\rho \times Y,$$

and the boundary of K_z in Z is of the form $\partial K_z = \{\rho = \rho_1\} \cong Y$.

Let $\delta = \min \{x_1, 1/\rho_1\}$. Then for $0 < \epsilon < \delta$, and $R > \frac{1}{\delta}$, let

$$M_{0,\epsilon} = \{(x, y) \in M_0 : x > \epsilon\} \cup K_0, \quad Z_R = \{(\rho, y) \in Z : \rho < R\} \cup K_z.$$

The resolution blowup of (M_0, g_0) by (Z, g_z) is

$$M_\epsilon := M_{0,\epsilon} \cup_{\phi_\epsilon} Z_{1/\epsilon},$$

where the joining map ϕ is defined for each ϵ by

$$\phi_\epsilon : M_{0,\epsilon} - M_{0,\delta} \rightarrow Z_{1/\epsilon} - Z_{1/\delta}, \quad \phi_\epsilon(x, y) = \left(\frac{x}{\delta\epsilon}, y \right).$$

For $\delta > \epsilon > \epsilon' > 0$, the manifolds M_ϵ and $M_{\epsilon'}$ are diffeomorphic, and so the resolution blowup of M_0 by Z , which we call M , is unique up to diffeomorphism.

Remark. The resolution blowup is a smooth compact manifold which resolves a conical singularity using an ac space. Instead of resolving the singularity in M_0 using Z , we may equivalently define the resolution blowup to resolve the boundary of \bar{Z} using M_0 as follows. Let $r = 1/\rho$ be the defining function for $\partial\bar{Z}$. The resolution blowup of \bar{Z} by M_0 is

$$M_\epsilon := M_{0,\epsilon} \cup_{\psi_\epsilon} Z_{1/\epsilon}, \quad \psi_\epsilon(x, y) = \left(\frac{\epsilon\delta}{x}, y \right),$$

where the joining map $\psi_\epsilon : M_{0,\epsilon} - M_{0,\delta} \rightarrow Z_{1/\epsilon} - Z_{1/\delta}$ is defined by $\psi_\epsilon(x, y) = (\epsilon\delta/x, y) = (r, y)$. The resulting smooth compact resolution space is diffeomorphic to M .

The ac single space, analogous to the analytic surgery single space in [30], is the setting for the definition of ac convergence.

Definition 2.4. Let (M_0, g_0) be a conic metric and let (\bar{Z}, g_z) be a scattering metric; assume both are dimension n with $\partial M_0 = \partial\bar{Z} = (Y, h)$, and assume $\delta = 1$ (Definition 2.3). Then, $M_0 \cong ((0, 1)_x \times Y) \cup K_0$ and $\bar{Z} \cong ((0, 1)_r \times Y) \cup K_z$ with $\partial K_0 \cong Y \cong \partial K_z$. The *acc single space* \mathcal{S} is

$$\mathcal{S} := [0, 1)_x \times [0, 1)_r \times Y \cup (K_0 \times \{x = 1, r \neq 1\}) \cup (K_z \times \{r = 1, x \neq 1\}).$$

The smooth structure of \mathcal{S} is induced by that of M_0 and \bar{Z} . Namely, smooth functions on \mathcal{S} are functions which are smooth jointly in x and r on $(0, 1)_x \times (0, 1)_r \times Y$ and smoothly extend to a smooth function on K_0 at $x = 1$, on K_z at $r = 1$, on M_0 at $r = 0$ and on \bar{Z} at $x = 0$.

Definition 2.5. Let \mathcal{S} be the acc single space associated to M_0 and \bar{Z} as in Definition 2.4. Let $\epsilon(p) = x(p)r(p) : \mathcal{S} \rightarrow [0, 1)$, where x and r are extended to K_0 and K_z , respectively, to be identically 1. We define the *acc tensor* \mathcal{G} as follows:

$$\mathcal{G} = \begin{cases} \frac{1}{2} \left(dx^2 + x^2 \left(h(x) + r^2 \left(\frac{dr^2}{r^4} + \frac{h(r)}{r^2} \right) \right) \right), & x, r \in (0, 1), \\ (xr)^2 (g_z|_{K_z}), & r = 1, \\ g_0|_{K_0}, & x = 1. \end{cases}$$

For $0 < \epsilon < 1$, let $M_\epsilon = \{xr = \epsilon\} \subset \mathcal{S}$; note that M_ϵ is diffeomorphic to the resolution blowup M of M_0 by Z . The family of metrics $\{g_\epsilon = \mathcal{G}|_{M_\epsilon}\}$ on M is said to converge *asymptotically conically* to (M_0, g_0) .

Remarks.

1. In the definition of the acc single space to simplify calculations, we have assumed $\delta = 1$; no generality is lost by this assumption. On $\{xr = \epsilon\}$, we then have $r = \frac{\epsilon\delta}{x}$, equivalently $\rho = \frac{1}{r} = \frac{x}{\epsilon\delta}$. Letting $\delta = 1$ in the definition of resolution blowup, the identification of $\{xr = \epsilon\} \subset \mathcal{S}$ with the resolution blowup M_ϵ follows immediately.
2. The acc single space has two boundary hypersurfaces at $\epsilon = 0$. These are diffeomorphic to M_0 at $r = 0$ and \bar{Z} at $x = 0$, and they meet in a codimension 2 corner diffeomorphic to Y .
3. The acc single space contains a submanifold diffeomorphic to a cone over $[0, 1)_r \times Y \subset \bar{Z}$ with radial variable x .
4. At $r = 0$, \mathcal{G} restricts to $\mathcal{G}|_{\{r=0\}} = g_0$. On $M_\epsilon \subset \mathcal{S}$ where $0 < r(p)$, $x(p) < 1$,

$$r = \frac{\epsilon}{x} \implies dr^2 = \frac{\epsilon^2}{x^4} dx^2 \implies g_\epsilon = dx^2 + x^2 h(x),$$

which shows that g_ϵ extends smoothly to $g_0|_{K_0}$ as $x \rightarrow 1$. Similarly, we compute

$$g_\epsilon = \epsilon^2 \left(\frac{dr^2}{r^4} + \frac{h(r)}{r^2} \right),$$

so g_ϵ extends smoothly to $(\epsilon^2)(g_z|_{K_z})$ as $r \rightarrow 1$. \mathcal{G} is a smoothly polyhomogeneous symmetric 2-cotensor on \mathcal{S} , and its restriction g_ϵ is also smooth and polyhomogeneous as $\epsilon \rightarrow 0$.

The following lemma is useful for visualizing ac convergence and for proving spectral convergence.

Lemma 2.6. *Let (M_0, g_0) and (Z, g_z) be as in Definitions 2.3 to 2.5 and let (M, g_ϵ) converge asymptotically conically to (M_0, g_0) . Then, there exists a family of diffeomorphisms $\{\phi_\epsilon\}$ from a fixed open proper subset $U \subset M$ to increasing neighborhoods $Z_{1/\epsilon} \subset Z$ such that $g_\epsilon|_U \cong \left(\epsilon^2(\phi_\epsilon)^*(g|_{Z_{1/\epsilon}}) \right)|_U$. Moreover, on $M - U$, $g_\epsilon \rightarrow g_0$ smoothly as $\epsilon \rightarrow 0$, and any $K \subset\subset M_0^0$ is diffeomorphic to some fixed $K' \subset M$ so that $g_\epsilon \rightarrow g_0$ smoothly and uniformly on K' .*

Proof. The existence of ϕ_ϵ and $U \subset M$ follows immediately from the definition of resolution blowup and the diffeomorphism between the resolution blowup M and $\{xr = \epsilon\} \subset \mathcal{S}$. By the above remarks, on the neighborhood $U \subset M$ where this diffeomorphism is defined, $g_\epsilon|_U = (\epsilon^2)(\phi_\epsilon)^* \left(g_z|_{Z_{1/\epsilon}} \right)$.

Since $g_\epsilon = g_0 + \mathcal{O}(\epsilon^2)$, the smooth convergence of g_ϵ to g_0 on $M - U$ follows immediately. Any compact subset $K \subset\subset M_0$ is contained in $M_{0,\epsilon}$ for some $\epsilon > 0$ and so is diffeomorphic to $K_\epsilon \subset\subset M_\epsilon$ and also to $K' \subset\subset (M - U')$. Conversely, any $K \subset\subset (M - U)$ is diffeomorphic to $K_\epsilon \subset M_\epsilon$ for some $\epsilon > 0$ and also to $K' \subset\subset M_{0,\epsilon} \subset M_0$.

3. Geometric preliminaries

This section is a brief review of the theory and terminology of manifolds with corners, b maps, and blowups; see also [28, 33].

3.1. Manifolds with corners

Let X be a manifold with corners. This means that near any of its points, X is modeled on a product $[0, \infty)^k \times \mathbb{R}^{n-k}$, where k depends on the point and is the maximal codimension of the boundary face containing that point. We also assume that all boundary faces of X are embedded so they too are manifolds with corners. The space $\mathcal{V}(X)$ of all smooth vector fields on X is a Lie algebra under the standard bracket operation. It contains the Lie subalgebra

$$(3.1) \quad \mathcal{V}_b(X) := \{V \in \mathcal{V}(X) : V \text{ is tangent to each boundary face of } X\}.$$

Then $\mathcal{V}_b(X)$ is itself the space of all smooth sections of a vector bundle,

$$\mathcal{V}_b(X) = \mathcal{C}^\infty(X; {}^bTX),$$

where bTX is the bundle defined so that the above holds and is called the *b-tangent bundle*.

3.1.1. Blowing up. An embedded codimension k submanifold Y of a manifold with corners X is called a p -submanifold (p for product) if near each point of Y there are local product coordinates so that Y is defined by the vanishing of some subset of them. In other words, X and Y must have consistent local product decompositions. Then one can define a new manifold with corners $[X; Y]$ to be the normal blowup of X around Y . This is

obtained by replacing Y by its inward-pointing spherical normal bundle. The union of this normal bundle and $X - Y$ has a unique minimal differential structure as a manifold with corners so that the lifts of smooth functions on X and polar coordinates around Y are smooth. One can also consider iterated blowups, written $[[X; Y_1]; Y_2]$, where Y_1 and Y_2 are p -submanifolds of X . However, depending on the geometry, this may not yield the same space as $[[X; Y_2]; Y_1]$. In the special case of embedded p -submanifolds,

$$X \supset Y_1 \supset Y_2 \supset Y_3 \supset \dots \supset Y_n$$

the iterated blowup

$$[[X; Y_1]; Y_2]; \dots ; Y_n]$$

can be performed in any order with the same result [30]. Blowups may also be defined using equivalence classes of curves [33]. Let r be a defining function for the p submanifold Y and consider the family of curves $\gamma(t) = (r(t), y(t))$ such that

$$\begin{aligned} \gamma(t) \in Y &\iff t = 0, \\ r(t) &= O(t). \end{aligned}$$

Let E be the set of equivalence classes of all such curves with

$$\gamma \sim \gamma' \iff (y - y')(t) = O(t) \quad \text{and} \quad (r - r')(t) = O(t^2).$$

There is a natural \mathbb{R}^+ action on E given by

$$\mathbb{R}^+ \ni a : \gamma(t) \rightarrow \gamma(at).$$

E modulo this equivalence relation is naturally diffeomorphic to $N^+(Y)$, the inward-pointing spherical normal bundle of Y , so we can define $[X; Y]$ by

$$[X; Y] = (X - Y) \cup E / (\mathbb{R}^+ - \{0\}).$$

We can also define *parabolic* blowups in certain contexts [12]. Let Y be a p -submanifold of codimension k so that there exist local coordinates $(r, y) = (r_1, \dots, r_k, y_1, \dots, y_{n-k})$ in a neighborhood of Y with r_i vanishing precisely at Y and so that dr_1 induces a sub-bundle of the tangent bundle TX . Instead

of the above equivalence classes of curves, we consider $\gamma(t)$ such that

$$\begin{aligned} \gamma(t) = (r_1(t), \dots, r_k(t), y_1(t), \dots, y_{n-k}(t)) \in Y &\iff t = 0, \\ r_i(t) = O(t), \quad i \neq 1, \quad r_1(t) = O(t^2). \end{aligned}$$

Two such curves are equivalent if

$$\begin{aligned} \gamma \sim \gamma' &\iff (y_j - y'_j)(t) = O(t), \quad (r_i - r'_i)(t) = O(t^2) \quad i \neq 1, \\ &\quad (r_1(t) - r'_1(t)) = O(t^3). \end{aligned}$$

Since dr_1 is a sub-bundle of TX , there is a natural \mathbb{R}^+ action on the set of equivalence classes E_2 of all such curves,

$$\mathbb{R}^+ \ni a : \gamma(t) \rightarrow (r_1(a^2t), r_i(at), \dots, y_j(at)).$$

The set of equivalence classes of all such curves modulo this \mathbb{R}^+ action is naturally diffeomorphic to the inward-pointing r_1 -parabolic normal bundle of Y ,

$$E_2/(\mathbb{R}^+ - \{0\}) \cong PN_{r_1}^+(Y).$$

We define the r_1 -parabolic blowup of X around Y as the union of $X - Y$ and this inward-pointing r_1 -parabolic bundle,

$$[X; Y, dr_1] := (X - Y) \cup PN_{r_1}^+(Y).$$

The union of this r_1 -parabolic bundle and $X - Y$ again has a unique minimal differential structure as a manifold with corners so that the lifts of smooth functions on X and r_1 -parabolic coordinates around Y are smooth. By r_1 -parabolic coordinates around Y , we mean the coordinates,

$$\rho = (r_1^2 + r_2^4 + \dots + r_k^4)^{1/4}, \quad \theta = (\theta_1, \dots, \theta_n) \in \mathbb{S}^{n-1},$$

with local coordinates $(r_1, \dots, r_k, y_1, \dots, y_{n-k})$ in a neighborhood of Y satisfying

$$r_i = \rho\theta_i, \quad i \neq 1, \quad r_1 = \rho^2\theta_1, \quad y_j = \rho\theta_j.$$

For any parabolic or spherical blowup, there is a natural blow-down map $\beta_* : [X; Y] \rightarrow X$ and corresponding blowup map $\beta^* : X \rightarrow [X; Y]$, so that the image of Y under β^* is a boundary hypersurface of $[X; Y]$ diffeomorphic to the inward-pointing spherical (or parabolic) normal bundle of Y . As such,

$$[X; Y] = (X - Y) \cup \beta^*(Y).$$

3.1.2. *b*-Maps and *b*-fibrations

Definition 3.1. Let M_1 be a manifold with boundary hypersurfaces, $\{N_j\}_{j=1}^k$, and defining functions r_j . Let M_2 be a manifold with boundary hypersurfaces, $\{L_i\}_{i=1}^l$, and defining functions ρ_i . Then $f : M_1 \rightarrow M_2$ is called a *b-map* if for every i there exist non-negative integers $e(i, j)$ and a smooth non-vanishing function h such that $f^*(\rho_i) = h \prod_{j=1}^k r_j^{e(i, j)}$.

The image under a *b-map* of the interior of each boundary hypersurface of M_1 is either contained in or disjoint from each boundary hypersurface of M_2 , and the order of vanishing of the differential of f is constant along each boundary hypersurface of M_1 . The matrix $(e(i, j))$ is called the lifting matrix for f .

In order for the map f to preserve polyhomogeneity, stronger conditions are required. Associated to a manifold with corners are the *b-tangent* and *cotangent* bundles, bTM (3.1) and ${}^bT^*M$.⁵ The map f may be extended to induce the map ${}^bf_* : {}^bTM_1 \rightarrow {}^bTM_2$.

Definition 3.2. The *b-map*, $f : M_1 \rightarrow M_2$, is called a *b-fibration* if the associated maps bf_* at each $p \in \partial M_1$ are surjective at each $p \in \partial M_1$, and the lifting matrix $(e(i, j))$ has the property that for each j there is at most one i such that $(e(i, j)) \neq 0$. In other words, f does not map any boundary hypersurface of M_1 to a corner of M_2 .

3.1.3. *b*-Manifolds and the *b*-blowup A *b-manifold* is a complete manifold with boundary whose metric is closely related to conic and ac scattering metrics.

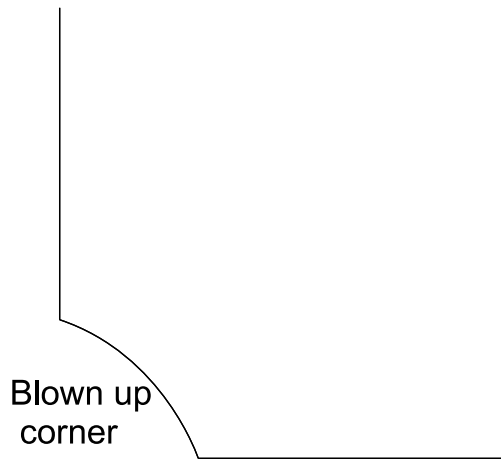
Definition 3.3. Let (X, g) be a smooth Riemannian manifold with boundary (Y, h) and boundary-defining function x so that a neighborhood of the boundary is diffeomorphic to a product $[0, x_1)_x \times Y$ and in this neighborhood

$$g = \frac{dx^2}{x^2} + h(x),$$

where $h(x)$ is a smoothly varying family of metrics on Y that converges smoothly to h as $x \rightarrow 0$. Then (X, g) is said to be a *b-manifold*.

Equivalently, under the change of variables $r \mapsto e^{-t}$, a *b-manifold* is seen to be the compactification as a manifold with boundary of a complete manifold with asymptotically cylindrical end(s). To study the regularity and

⁵These are also called the totally characteristic tangent and cotangent bundles.

Figure 1: The b -blowup X_b^2 .

mapping properties of geometric operators on b -manifolds, melrose [33] introduced the b -double space. This space is obtained from X^2 by performing a radial blowup called the b -blowup along the codimension 2 corner at the boundary in each copy of X , and it is written X_b^2 (Figure 1),

$$(3.2) \quad X_b^2 = [X \times X; \partial X \times \partial X] = [X \times X; Y \times Y].$$

For any manifold M with boundary having a product structure in a neighborhood of the boundary, we may define the b -blowup in the analogous way, $M_b^2 := [M \times M; \partial M \times \partial M]$.

3.2. Asymptotically conic convergence double space

The acc double space is an instructive model for the more complicated acc heat space in Section 7. Let

$$\mathcal{S}_b^2 := [\mathcal{S}^2; Y \times Y].$$

The acc double space \mathcal{D} is the submanifold of \mathcal{S}_b^2 defined by the vanishing set of

$$f(p) = x(p)r(p) - x'(p)r'(p) = \epsilon(p) - \epsilon'(p),$$

so that

$$\mathcal{D} = \{p \in \mathcal{S}_b^2 : f(p) = x(p)r(p) - x'(p)r'(p) = 0\} = \{p \in \mathcal{S}_b^2 : \epsilon(p) = \epsilon'(p)\}.$$

The acc double space has four boundary faces at $\epsilon = 0$, described in the following table. Here and throughout, we label each face F_{wxyz} where the subscript indicates the order to which each of the scalar variables x, r, x', r' vanishes at that face.

Scalar variables	Face	Geometry
$x = 0, x' = 0,$	F_{1010}	$[\bar{Z} \times \bar{Z}; Y \times Y]$
$x = 0, r' = 0$	F_{1001}	$[\bar{Z} \times M_0; Y \times Y]$
$r = 0, x' = 0$	F_{0110}	$[M_0 \times \bar{Z}; Y \times Y]$
$r = 0, r' = 0$	F_{0101}	$[M_0 \times M_0; Y \times Y]$

To see that \mathcal{D} is a smooth submanifold of \mathcal{S}_b^2 , we consider the function

$$(3.3) \quad f(p) = x(p)r(p) - x'(p)r'(p).$$

Away from the boundary faces, f is smooth with non-vanishing differential. In a neighborhood of $S_{11} - F_{1001}$, let

$$f_{0110}(p) = \frac{x(p)}{x'(p)} - \frac{r'(p)}{r(p)}.$$

Since $x'rf_{0110} = f$, we see that f_{0110} is smooth near the $\epsilon = 0$ boundary faces away from where those faces meet F_{0110} . Moreover, wherever defined, f_{0110} has non-vanishing differential and its zero set coincides with that of f away from F_{0110} . Similarly, let

$$f_{1001}(p) = \frac{x(p)}{r'(p)} - \frac{x'(p)}{r(p)}.$$

f_{1001} is smooth with non-vanishing differential and has the same vanishing set as f in a neighborhood of $\{\epsilon = 0\} - F_{1001}$. This shows that \mathcal{D} is a smooth submanifold of \mathcal{S}_b^2 . While the acc double space will not be used here, we note that the acc double space, with an additional blowup along the diagonal for $\epsilon \geq 0$, would be the natural space on which to study the resolvent behavior under ac convergence.

4. Analytic preliminaries

Since we are working on manifolds with singularities corners and boundaries, we briefly review some key features of the analysis in these settings.

4.1. Polyhomogeneous conormal functions

On a manifold M with corners having a consistent local product structure near each boundary and corner, a natural class of functions (or sections) with good regularity near the boundary and corners are the polyhomogeneous conormal functions (or sections); see [28]. In a neighborhood of a corner, we have coordinates $(x_1, \dots, x_k, y_1, \dots, y_{n-k})$, where x_1, \dots, x_k vanish at this corner and (y_1, \dots, y_{n-k}) are smooth local coordinates on a smooth compact $n - k$ manifold Y . The edge tangent bundle \mathcal{V}_e in a neighborhood of this corner is spanned over $\mathcal{C}^\infty(M)$ by the vector fields,

$$\{x_i \partial_{x_i}, \partial_{y^\alpha}\}.$$

The basic conormal space of sections is

$$\mathcal{A}^0(M_0) = \{\phi : V_1 \cdots V_l \phi \in L^\infty(M_0), \forall V_i \in \mathcal{V}_e, \text{ and } \forall l\}.$$

Let α and p be multi-indices with $\alpha_j \in \mathbb{C}$ and $p_j \in \mathbb{N}_0$. Then we define

$$\mathcal{A}^{\alpha,p}(M_0) = x^\alpha (\log x)^p \mathcal{A}^0.$$

The space \mathcal{A}^* is the union of all these spaces for all α and p . The space $\mathcal{A}_{phg}^*(M_0)$ consists of all conormal distributional sections which have an expansion of the form

$$\phi \sim \sum_{\text{Re}(\alpha_j) \rightarrow \infty} \sum_{p=0}^{p_j} x^{\alpha_j} (\log x)^p a_{j,p}(x, y), \quad a_{j,p} \in \mathcal{C}^\infty.$$

We define an index set to be a discrete subset $E \subset \mathbb{C} \times \mathbb{N}_0$ such that

$$(\alpha_j, p_j) \in E, \quad |(\alpha_j, p_j)| \rightarrow \infty \implies \text{Re}(\alpha_j) \rightarrow \infty.$$

Then, the space $\mathcal{A}_{phg}^E(M_0)$ consists of those distributional sections $\phi \in \mathcal{A}_{phg}^*$ having polyhomogeneous expansions with $(\alpha_j, p_j) \in E$.

4.2. Conic differential operators and \mathbf{b} -operators

Let (M_0, g_0) be a Riemannian manifold with isolated conic singularity so that in a neighborhood of the singularity,

$$M_0 \cong (0, x_1)_x \times Y, \quad g_0 = dx^2 + x^2 h(x).$$

A conic differential operator of order m is a smooth differential operator on M_0 which can be expressed in neighborhood of the singularity by

$$A = x^{-m} \sum_{k=0}^m B_k(x)(-x\partial_x)^k$$

with $B_k \in C^\infty((0, x_1), \text{Diff}^{m-k}(Y))$, where $\text{Diff}^j(Y)$ denotes the space of differential operators of order $j \in \mathbb{N}_0$ on Y with smooth coefficients. The cone differential operators are elements of the cone operator calculus; for a detailed description, see [24]. These cone operators are closely related to b -operators. A b -operator of order m is a smooth differential operator such that near the boundary it can be expressed by

$$A = \sum_{k=0}^m B_k(x)(-x\partial_x)^k$$

with $B_k \in C^\infty((0, x_1), \text{Diff}^{m-k}(Y))$. Note that a cone differential operator of order m is equal to a rescaled b -differential operator of order m ; if A is an order m cone differential operator, then $x^m A$ is a b -differential operator. In local coordinates (x, y_1, \dots, y_{n-1}) near the boundary a b -operator may be expressed as

$$A = \sum_{j+|\alpha| \leq m} a_{j,\alpha}(x, y)(-x\partial_x)^j(\partial_{y^\alpha}).$$

The b -symbol of A is

$${}^b\sigma_m(A) = \sum_{j+|\alpha|=m} a_{j,\alpha}(x, y)\lambda^j\eta^\alpha.$$

Here λ and η are linear functions on the b -cotangent bundle defined by the coordinates so that a generic element of the b -cotangent bundle is

$$\lambda \frac{dx}{x} + \sum_{i=1}^n \eta_i dy_i.$$

The b -operator is b -elliptic if the b -symbol ${}^b\sigma_m(A)$ is non-zero off the zero section.

The scalar Laplacian on M_0 is

$$x^{-2}\{(-x\partial_x)^2 + (-n + 1 + xH^{-1}(\partial_x H)(-x\partial_x)) + \Delta_{h(x)}\} = x^{-2}L_b,$$

where L_b is an elliptic order 2 b -operator and H is a smooth function that depends on the metric. Similarly, a geometric Laplacian Δ_0 on M_0 is also of the form

$$\Delta_0 = x^{-2}L_b,$$

for an elliptic order two b -operator acting on sections of the vector bundle. The Schwartz kernel of L_b is a distribution on the b -double space. By the b -calculus theory, [33] L_b has a parametrix G_b such that G_b is a b -operator of order -2 with

$$G_b L_b = I - R,$$

where I is the identity operator and R is a b -operator with polyhomogeneous Schwartz kernel on the b -double space. For any $u \in \mathcal{L}^2(x^{n-1} dx dy)$ with $\Delta_0 u = f \in \mathcal{L}^2(x^{n-1} dx dy)$,

$$(x^2 G_b)(x^{-2} L_b u) = (x^2 G_b)f = u - Ru \implies u = x^2 G_b f + Ru = \alpha + \beta.$$

The first term, $\alpha \in x^2 H_b^2 \subset x^2 \mathcal{L}^2(x^{n-1} dx dy)$. The second term $\beta \in \mathcal{L}^2(x^{n-1} dx dy)$ has a polyhomogeneous expansion as $x \rightarrow 0$,

$$\beta \sim \sum_{j=0}^{\infty} \sum_{k=0}^{N_j} x^{\gamma_j+k} \varphi_j(y).$$

Above γ_j is an indicial root for the operator L_b and φ_j is an eigensection for the induced geometric Laplacian on (Y, h) . Then,

$$(4.1) \quad u = \alpha + \sum_{j=0}^{\infty} \sum_{k=0}^{N_j} x^{\gamma_j+k} \varphi_j(y)$$

where $\alpha \in x^2 \mathcal{L}^2(x^{n-1} dx dy)$. This decomposition plays a key role in our proof of spectral convergence.

4.3. Friedrichs domain of the conic Laplacian

A geometric Laplacian Δ_0 on a conic manifold is an unbounded operator on \mathcal{L}^2 sections of the bundle. It can be extended to various domains in \mathcal{L}^2 .

The minimal domain \mathcal{D}_{\min} is the completion of $\mathcal{C}_c^\infty(M_0)$ with respect to the norm $\|u\| + \|\Delta_0 u\|$, while the maximal domain

$$\mathcal{D}_{\max} = \{u \in \mathcal{L}^2(M_0) \mid \Delta_0 u \in \mathcal{L}^2(M_0)\}.$$

Both \mathcal{D}_{\min} and \mathcal{D}_{\max} are dense in $\mathcal{L}^2(M_0)$, and the extension of the Laplacian to either domain is a closed operator. On complete manifolds $\mathcal{D}_{\min} = \mathcal{D}_{\max}$ by the Gaffney–Stokes theorem [14]. Conic metrics are incomplete and these domains will not in general be equal. The Friedrichs domain \mathcal{D}_F lies between \mathcal{D}_{\min} and \mathcal{D}_{\max} and is the closure of the graph of Δ_0 in \mathcal{L}^2 with respect to the densely defined Hermitian form,

$$Q(u, v) = \int_{M_0} \langle \nabla u, \nabla v \rangle.$$

The extension of the Laplacian to the Friedrichs domain, known as the Friedrichs extension of the Laplacian, preserves the operator’s lower bound and is essentially self adjoint. Here, we work exclusively with the Friedrichs extension of the conic Laplacian.

For elements of \mathcal{D}_{\max} , with $u \in \mathcal{L}^2$ and $\Delta_0 u = f \in \mathcal{L}^2$, we have the expansion (4.1) from the preceding section,

$$u = \alpha + \sum_{j=0}^{\infty} \sum_{k=0}^{N_j} x^{\gamma_j+k} \varphi_j(y).$$

The volume form on M_0 near the singularity is asymptotic to $x^{n-1} dx dy$. Therefore, the exponents γ_j must all be strictly greater than $-\frac{n}{2}$. For $v \in \mathcal{D}_{\min} \subset \mathcal{D}_{\max}$ the decomposition (4.1) and the definition of \mathcal{D}_{\min} imply that $\mathcal{D}_{\min} \subset x^2 \mathcal{L}^2$. The equality of \mathcal{D}_{\min} and \mathcal{D}_{\max} then depends on the indicial roots of $L_b = x^2 \Delta_0$. For further discussion of domains of the conic Laplacian, see [16], whose results include:

$$\mathcal{D}_F = \{f \in \mathcal{L}^2 : \Delta_0 f \in \mathcal{L}^2 \text{ and } f = \mathcal{O}(x^{(2-n+\delta)/2}) \text{ as } x \rightarrow 0, \text{ for some } \delta > 0\}.$$

We will use this characterization of the domain of the Friedrichs extension of the conic Laplacian in the proof of the first theorem.

5. Spectral convergence

We now have all the necessary ingredients to prove spectral convergence.

Theorem 5.1. *Let (M_0, g_0) be a compact Riemannian n -manifold with isolated conic singularity, and let (Z, g_z) be an ac space, with $n \geq 3$. Assume (M, g_ϵ) converges asymptotically conically to (M_0, g_0) . Let (E_0, ∇_0) and (E_z, ∇_z) be rank k Hermitian vector bundles over (M_0, g_0) and (Z, g_z) , respectively, so that each of these bundles in a neighborhood of the boundary is the pullback from a bundle over the boundary (Y, h) . Let Δ_0, Δ_z be the corresponding Friedrichs extensions of geometric Laplacians, and let Δ_ϵ be the induced geometric Laplacian on (M, g_ϵ) . Assume Δ_z has no \mathcal{L}^2 nullspace. Then the accumulation points of the spectrum of Δ_ϵ as $\epsilon \rightarrow 0$ are precisely the points of the spectrum of Δ_0 , counting multiplicity.*

The theorem follows from the inclusion *accumulation* $\sigma(\Delta_\epsilon) \subset \sigma(\Delta_0)$, the reverse inclusion *accumulation* $\sigma(\Delta_\epsilon) \supset \sigma(\Delta_0)$, and correct multiplicities.

5.1. Accumulation $\sigma(\Delta_\epsilon) \subset \sigma(\Delta_0)$

We extract a smoothly convergent sequence of eigensections corresponding to a converging sequence of eigenvalues as $\epsilon \rightarrow 0$ and show that the limit section of this sequence is an eigensection for the conic metric, and its eigenvalue is the accumulation point. For this argument, we work with sequences of metrics $\{g_{\epsilon_j}\}$ which we abbreviate $\{g_j\}$ with corresponding Laplacians Δ_j .

Let $\lambda(\epsilon_j)$ be an eigenvalue of Δ_j , with eigensection f_j . Assume that $\lambda(\epsilon_j) \rightarrow \bar{\lambda}$. Over any compact set $K \subset M_0^0$, $g_j = g_{\epsilon_j}$ converges smoothly to g_0 by Lemma 2.6, thus so do the coefficients of Δ_j . Hence, normalizing f_j by $\sup_M |f_j| = 1$, it follows using standard elliptic estimates and the Arzela–Ascoli theorem that f_j converges in \mathcal{C}^∞ on any compact subset of M_0^0 [22]. Furthermore, the limit section \bar{f} satisfies the limiting equation

$$\Delta_0 \bar{f} = \bar{\lambda} \bar{f}.$$

However, we do not yet know that $\bar{f} \neq 0$, nor, even if this limit is non-trivial, that it lies in the domain of the Friedrichs extension of Δ_0 . This is the content of the arguments to follow.

5.1.1. Weight functions Let $\phi_\epsilon : M_{0,\epsilon} - M_{0,\delta} \rightarrow Z_{1/\epsilon} - Z_{1/\delta}$ as in Definition 2.3. We identify $Z_{1/\delta}$ with a fixed $K \subset U \subset M$ so that $M_{0,\epsilon} - M_{0,\delta} \cong (U - K)$, $K \cong Z_{1/\delta}$.

Let

$$w_\epsilon = \begin{cases} c & \text{on } M - U, \\ \epsilon (\phi_\epsilon^{-1})^* \rho & \text{on } U - K, \\ c\epsilon & \text{on } K. \end{cases}$$

Above, c is a constant, and no generality is lost by assuming $c = 1$. Let $w_j = w_{\epsilon_j}$. For some $\delta > 0$ to be chosen later, replacing f_j by $\frac{f_j}{\|w_j^\delta f_j\|_\infty}$ we assume the supremum of $|f_j w_j^\delta|$ is 1 on M . Since M is compact, $|f_j|$ attains a maximum at some point $p_j \in M$, and we may assume p_j converges to some $\bar{p} \in M$. The argument splits into three cases depending on how and where p_j accumulates in M .

5.1.2. Case 1: $w_j(p_j) \rightarrow c > 0$ as $j \rightarrow \infty$. In this case, the points $\{p_j\}$ accumulate in a compact subset of $M - U$, which we may identify with a compact subset of M_0^0 by Lemma 2.6. So, we may assume that these points converge to some point $\bar{p} \neq p$ (the conic singular point). The maximum of $|f_j w_j^\delta|$ on M is 1 and occurs at p_j , so

$$|f_j| \leq w_j^{-\delta} \text{ on } M \text{ for each } j \implies |f_j(p_j)| \rightarrow c^{-\delta} \text{ as } j \rightarrow \infty.$$

The locally uniform C^∞ convergence of f_j to \bar{f} implies that $|\bar{f}|$ satisfies a similar bound,

$$|\bar{f}| \leq x^{-\delta} \quad \text{as } x \rightarrow 0,$$

and clearly $|\bar{f}(\bar{p})| = c^{-\delta} \neq 0$. By the dimension assumption $n \geq 3$ and the characterization of the Friedrichs domain of the Laplacian, we may choose δ so that

$$\frac{2 - n}{2} < -\delta < 0.$$

Then \bar{f} lies in the Friedrichs domain of the Laplacian and satisfies

$$\Delta_0 \bar{f} = \bar{\lambda} \bar{f},$$

so $\bar{\lambda}$ is an eigenvalue of Δ_0 .

5.1.3. Case 2: $|w_j(p_j)| \leq c\epsilon_j$ as $j \rightarrow \infty$. Analysis on Z in this case leads to a contradiction. Let $\phi_j = \phi_{\epsilon_j}$ and $\tilde{f}_j = f_j(\phi_j^{-1})$. Let $\tilde{p}_j = \phi_j(p_j)$. Because $|f_j w_j^\delta|$ attains its maximum value of 1 at p_j , $|f_j(\tilde{p}_j)| = (w_j(p_j))^{-\delta}$. Rescale f_j and \tilde{f}_j , replacing them respectively with $(w_j(p_j))^\delta f_j$ and $(w_j(p_j))^\delta \tilde{f}_j$, so that the maximum of $|\tilde{f}_j \rho^\delta|$ occurs at the point $\tilde{p}_j \in Z_j = Z_{1/\epsilon_j}$ and is equal to 1. Since $w_j(p_j) = \mathcal{O}(\epsilon_j)$, $\rho(\tilde{p}_j) = \epsilon_j^{-1} w_j(p_j)$ stays bounded for all j , we assume \tilde{p}_j converges to $\tilde{p} \in Z$. By Lemma 2.6, $(Z_j, \epsilon_j^{-2} \phi_j^* g_j|_U)$ converges

smoothly to (Z_j, g_z) . This implies the following equation is satisfied by \tilde{f}_j on Z_j ,

$$\Delta_z \tilde{f}_j = \epsilon_j^2 \lambda(\epsilon_j) \tilde{f}_j + O(\epsilon_j).$$

Since $\lambda(\epsilon_j)$ converge to $\bar{\lambda}$ and $|\tilde{f}_j \rho^\delta| \leq 1$ on Z_j ,

$$\Delta_z \tilde{f}_j \rightarrow 0 \text{ as } j \rightarrow \infty, \text{ on any compact subset of } Z.$$

This implies $f_j \rightarrow \bar{f}$ on M and correspondingly, $\tilde{f}_j \rightarrow \tilde{f}$ locally uniformly C^∞ on Z , and \tilde{f} satisfies

$$\Delta_z \tilde{f} = 0, \quad |\tilde{f} \rho^\delta| \leq 1.$$

Equality holds in the second equation at the point p . This shows that \tilde{f} is not identically zero on Z and $\tilde{f} = O(\rho^{-\delta})$ as $\rho \rightarrow \infty$. Since \tilde{f} is smooth on any compact subset of Z and is therefore in $\mathcal{L}_{loc}^2(Z)$, choosing $\delta > n - 2$ contradicts the assumption that Z has trivial \mathcal{L}^2 nullspace.

5.1.4. Case 3: $w_j(p_j) \rightarrow 0, \frac{\epsilon_j}{w_j(p_j)} \rightarrow 0$ as $j \rightarrow \infty$. In this case, the points $\phi_j(p_j) \rightarrow \infty$ in Z , so we rescale and derive a contradiction on the complete cone over (Y, h) . Consider the coordinates (ρ, y) on Z defined for $\rho \geq \rho_1$. In these coordinates $g_z = d\rho^2 + \rho^2 h(\rho)$. Let $r_j = \frac{\epsilon_j}{w_j(p_j)} \rho$ and \tilde{g}_j on Z_j be defined by

$$\tilde{g}_j = \left(\frac{\epsilon_j}{w_j(p_j)} \right)^2 g_z.$$

Then,

$$(Z_j, \tilde{g}_j) \cong \left(\left(\frac{\rho_1 \epsilon_j}{w_j(p_j)}, \frac{1}{w_j(p_j)} \right)_{r_j} \times Y, dr_j^2 + r_j^2 h \left(\frac{r_j \epsilon_j}{w_j(p_j)} \right) \right).$$

As $j \rightarrow \infty, h \left(\frac{r_j \epsilon_j}{w_j(p_j)} \right)$ converges smoothly to h , and

$$\tilde{g}_j \rightarrow g_C = dr^2 + r^2 h$$

on the complete cone C over (Y, h) . Let $\tilde{f}_j = (w_j(p_j))^\delta f_j(\phi_j^{-1})$. Since $|f_j w_j^\delta| \leq 1$ with equality at p_j ,

$$|\tilde{f}_j r_j^\delta| \leq 1 \text{ on } (Z_j, \tilde{g}_j) \text{ with equality at } \tilde{p}_j = \phi_j(p_j).$$

Let $\tilde{\Delta}_j$ on Z_j be the Laplacian induced by \tilde{g}_j on Z_j ,

$$\tilde{\Delta}_j = \frac{w_j(p_j)^2}{\epsilon_j^2} \Delta_Z.$$

Then,

$$\tilde{\Delta}_j \tilde{f}_j = \epsilon_j^2 \frac{w_j(p_j)^2}{\epsilon_j^2} \lambda(\epsilon_j) \tilde{f}_j + O(\epsilon_j) \text{ on } (Z_j, \tilde{g}_j).$$

Since $w_j(p_j) \rightarrow 0$ as $j \rightarrow \infty$, there is a locally uniform C^∞ limit f_c of $\{\tilde{f}_j\}$ on C which satisfies

$$|f_c r^\delta| \leq 1, \quad \Delta_c f_c = 0.$$

Since the points \tilde{p}_j stay at a bounded radial distance with respect to the radial variable r_j on Z_j , we may assume $\tilde{p}_j \rightarrow p_c$ for some $p_c \in C$. At this point, $|f_c(p_c)r(p_c)^\delta| = 1$, so f_c is not identically zero. By separation of variables (see, for example, [27]), f_c has an expansion in an orthonormal eigenbasis $\{\phi_j\}$ of $\mathcal{L}^2(Y, h)$,

$$f_c = \sum_{j \geq 0} a_{j,+} r^{\gamma_{j,+}} \phi_j(y) + a_{j,-} r^{\gamma_{j,-}} \phi_j(y),$$

where $\gamma_{j,+/-}$ are indicial roots corresponding to ϕ_j and $a_{j,+/-} \in \mathbb{C}$. In order for $|f_c r^\delta| \leq 1$ globally on C , we must have only one term in this expansion; $f_c = a_j r^{-\delta} \phi_j(y)$. Because the indicial roots are discrete, we may choose δ so that $-\delta$ is not an indicial root. This is a contradiction.

5.2. $\sigma(\Delta_0) \subset \text{accumulation } \sigma(\Delta_\epsilon)$

We use the Rayleigh–Ritz characterization of the eigenvalues [4]. Let $\lambda_l(\epsilon_j)$ be the l th eigenvalue of Δ_j and let

$$R_j(f) := \frac{\langle \nabla f, \nabla f \rangle_j}{\langle f, f \rangle_j}.$$

The subscript j indicates that the inner product is taken with respect to the \mathcal{L}^2 norm on M with the g_j metric. The eigenvalues are characterized using Mini-Max by

$$\lambda_l(\epsilon_j) = \inf_{\dim L=l, L \subset C^1(M)} \sup_{f \in L, f \neq 0} R_j(f).$$

Similarly, this characterization holds for the eigenvalues of the Friedrichs extension of the conic Laplacian which are known to be discrete (see [6], for example). Because $C_0^\infty(M_0)$ is dense in $\mathcal{L}^2(M_0)$, we may restrict to subspaces contained in $C_0^\infty(M_0)$. Then, the l th eigenvalue of Δ_0 is

$$\bar{\lambda}_l = \inf_{\dim L=l, L \subset C_0^\infty(M_0)} \sup_{f \in L, f \neq 0} R_0(f).$$

Let $\bar{\lambda}_l$ be the l th eigenvalue in the spectrum of Δ_0 . Fix $\epsilon > 0$. Then there exists $L \subset C_0^\infty$ with $\dim(L) = l$ and

$$\sup_{f \in L, f \neq 0} R_0(f) < \bar{\lambda}_l + \epsilon.$$

Since any $f \in L$ is also in $C_0^\infty(M)$ and because L is finite dimensional, by the local convergence of g_j to g_0 , for large j

$$|R_j(f) - R_0(f)| < \epsilon, \text{ for any } f \in L.$$

Since $\lambda_l(\epsilon_j)$ is the infimum

$$\lambda_l(\epsilon_j) \leq \bar{\lambda}_l + 2\epsilon.$$

This shows $\{\lambda_l(\epsilon_j)\}$ is bounded in j , and so we extract a convergent subsequence and a corresponding convergent sequence of eigensections which exists by our previous arguments. For each l , we take

$$\begin{aligned} \lambda_l(\epsilon_j) &\rightarrow \mu_l \leq \bar{\lambda}_l, \\ f_{j,l} &\rightarrow u_l, \quad \Delta_0 u_l = \mu_l u_l. \end{aligned}$$

These limit eigensections u_l are seen to be orthogonal as follows. Fix l, k , with $f_{j,k} \rightarrow u_k$ and $f_{j,l} \rightarrow u_l$. Since $C_0^\infty(M_0)$ is dense in $\mathcal{L}^2(M_0)$, we may choose a smooth cutoff function χ vanishing identically near the singularity in M_0 such that

$$\begin{aligned} \|\chi u_k - u_k\|_{L^2(M_0)} &< \epsilon, \\ \|\chi u_l - u_l\|_{L^2(M_0)} &< \epsilon, \\ \text{Vol}_j(M - \text{spt}(\chi)) &< \epsilon. \end{aligned}$$

Then on the support of χ , $g_j \rightarrow g_0$ uniformly so for large j ,

$$\begin{aligned} |\langle u_k, u_l \rangle_0 - \langle \chi u_k, \chi u_l \rangle_0| &< \epsilon, \\ |\langle \chi u_k, \chi u_l \rangle_0 - \langle \chi u_k, \chi u_l \rangle_j| &< \epsilon, \\ |\langle \chi u_k, \chi u_l \rangle_j - \langle \chi u_k, \chi f_{j,l} \rangle_j| &< \epsilon, \\ |\langle \chi u_k, \chi f_{j,l} \rangle_j - \langle \chi f_{j,k}, \chi f_{j,l} \rangle_j| &< \epsilon. \end{aligned}$$

Since the eigensections for Δ_j were chosen to be orthonormal and the volume of $(M - \text{support}(\chi))$ is small with respect to g_j ,

$$|\langle \chi f_{j,k}, \chi f_{j,l} \rangle_j| < 2\epsilon.$$

Thus, $\langle u_k, u_l \rangle_0$ can be made arbitrarily small and u_k, u_l are orthogonal for $l \neq k$. We complete this basis to form an eigenbasis of $\mathcal{L}^2(M_0)$. Let \bar{f}_l be an arbitrary element of this eigenbasis with eigenvalue $\bar{\lambda}_l$. We wish to show that this \bar{f}_l is actually the u_l above, defined to be the limit of (a subsequence of) $\{f_{j,l}\}$, and hence the corresponding μ_l is equal to $\bar{\lambda}_l$. Again, assume the smooth cut-off function χ is chosen so that

$$\|\chi \bar{f}_l - \bar{f}_l\|_{L^2(M_0)} < \epsilon.$$

For each j , we expand $\chi \bar{f}_l$ in eigensections of Δ_j ,

$$\chi \bar{f}_l = \sum_{k=0}^{\infty} a_{j,k} f_{j,k}, \quad \text{where } a_{j,k} = \langle \chi \bar{f}_l, f_{j,k} \rangle_j.$$

Now, fix k and choose χ such that

$$\|\chi u_k - u_k\|_{L^2(M_0)} < \epsilon.$$

Then,

$$\begin{aligned} |\langle \chi \bar{f}_l, f_{j,k} \rangle_0 - \langle \chi \bar{f}_l, f_{j,k} \rangle_j| &< \epsilon, \\ |\langle \chi \bar{f}_l, f_{j,k} \rangle_j - \langle \chi \bar{f}_l, u_k \rangle_j| &< \epsilon, \\ |\langle \chi \bar{f}_l, u_k \rangle_j - \langle \chi \bar{f}_l, u_k \rangle_0| &< \epsilon, \\ |\langle \chi \bar{f}_l, u_k \rangle_0 - \langle \bar{f}_l, u_k \rangle_0| &< \epsilon. \end{aligned}$$

By the orthogonality $\langle \bar{f}_l, u_k \rangle_0 = 0$ if $\bar{f}_l \neq u_k$, and otherwise is 1, so for each k , $a_{j,k} \rightarrow 0$ as $j \rightarrow \infty$, for all k with $u_k \neq \bar{f}_l$. Because \bar{f}_l is not identically zero,

there must be some k with $u_k = \bar{f}_l$. This shows that *every* eigensection of Δ_0 is the limit of (a subsequence of) $\{f_{j,k}\}$, and the corresponding eigenvalue $\bar{\lambda}_k$ is the limit of the corresponding eigenvalues.

5.3. Correct multiplicities

We show by induction that the eigenvalues of Δ_0 are achieved as accumulation points with correct multiplicity. The spectrum of Δ_0 listed with multiplicity is

$$\lambda_1 \leq \lambda_2 \leq \cdots, \quad \text{where } \Delta_0 f_i = \lambda_i f_i,$$

and the eigensections are \mathcal{L}^2 orthonormal. For each ϵ , the spectrum of Δ_ϵ is

$$\lambda_{1,\epsilon} \leq \lambda_{2,\epsilon} \leq \cdots, \quad \text{where } \Delta_\epsilon f_{i,\epsilon} = \lambda_{i,\epsilon} f_{i,\epsilon}.$$

By previous arguments, there is a sequence $\{f_{1,j}\}$ which converges to h_1 and $\lambda_{1,j} \rightarrow \lambda_1$. If there is some other $\{f_{1,k}\} \rightarrow h_2$ where h_2 is also an eigensection for Δ_0 with eigenvalue λ_1 , then we have shown in preceding arguments that both h_1 and h_2 are equal to eigensections of Δ_0 listed above, and both have eigenvalue λ_1 . If the multiplicity of λ_1 is one, we must have $h_1 = h_2 = f_1$. By preceding arguments, the limit of any converging sequence $\lambda_{k,j}$ is λ_k , so that when λ_1 is an eigenvalue of multiplicity one, $\lambda_1 < \lambda_2$, so λ_1 is an accumulation point of multiplicity one. If the multiplicity of λ_1 is greater than one, then $\lambda_2 = \lambda_1$, and we have shown that there is a sequence $\lambda_{2,j} \rightarrow \lambda_2$, and we have shown that the corresponding limit eigensection f_2 is orthogonal to f_1 . By these arguments and the preceding orthogonality arguments, it is clear that the multiplicity of λ_1 as an eigenvalue is equal to the multiplicity of λ_1 as an accumulation point. By induction, the same argument shows that all eigenvalues of Δ_0 are achieved as accumulation points with the correct multiplicity, thereby completing the proof of spectral convergence.

6. Heat kernels

The heat kernels for each of the geometries in ac convergence are elements of a pseudodifferential heat operator calculus that is defined on a corresponding heat space. For the details in the construction of these heat calculi, kernels and spaces, see [37].

6.1. b -Heat kernel

Let (M, g) be a b -manifold with local coordinates $z = (x, y)$ in a neighborhood of ∂M so that near the boundary

$$g = \frac{dx^2}{x^2} + h(x, y).$$

Let (z, z') be coordinates on $M \times M$ and let Δ_b be a geometric Laplacian on M . The b -heat kernel $H(z, z', t)$ is the Schwartz kernel of the fundamental solution of the heat operator $\partial_t + \Delta_b$. The heat kernel is a distributional section acting on smooth sections of M and satisfying

$$\begin{aligned} (\partial_t + \Delta_b)H(z, z', t) &= 0, \quad t > 0, \\ H|_{t=0} &= \delta(z - z'). \end{aligned}$$

By self adjointness since we work with the Friedrichs extension of Δ_b ,

$$H(z, z', t) = H(z', z, t)^*.$$

For a smooth section u on M ,

$$u(z, t) := \int_M \langle u(z'), H(z, z', t) \rangle dz'$$

satisfies

$$(\partial_t + \Delta_b)u(z, t) = 0 \text{ for } t > 0, \quad u(z, 0) = u(z).$$

Physically, $u(z, t)$ describes the heat on M at time $t > 0$ where the initial heat applied to M is given by $u(z)$.

Recall the Euclidean heat kernel,

$$G(z, z', t) = (4\pi t)^{-n/2} \exp\left(-\frac{|z - z'|^2}{2t}\right).$$

For a compact manifold without boundary, the heat kernel can be constructed locally using the Euclidean heat kernel, Riemannian normal coordinates, the “transport equations” determined by the local geometry and Duhamel’s principle; see for example [36]. On the interior of a manifold with boundary (or singularity), the Euclidean heat kernel is also a good model; however, near the boundary (or singularity) a different construction is required.

6.1.1. *b*-Heat space It is convenient to study the heat kernel on a manifold with boundary (or singularity) as an element of a heat operator calculus defined on a corresponding heat space (Figure 2). This space is a manifold with corners constructed from $M \times M \times \mathbb{R}^+$ by blowing up along submanifolds at which the heat kernel may have interesting or singular behavior.⁶ For example, the diagonal is always blown up at $t = 0$, since away from the boundary the heat kernel behaves like the Euclidean heat kernel which is singular along the diagonal at $t = 0$. To construct the *b*-heat space, we first blow up the codimension 2 corner at the boundary in both copies of M to create the *b*-double space,

$$M_b^2 = [M \times M; \partial M \times \partial M].$$

The *b*-heat space $M_{b,h}^2$ is then

$$M_{b,h}^2 = [M_b^2 \times \mathbb{R}_t^+; \Delta(M \times M) \times \{t = 0\}, dt],$$

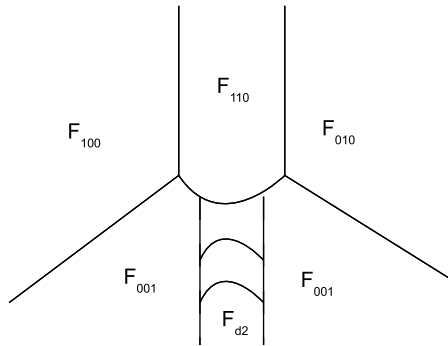
where $\Delta(M \times M)$ is the diagonal in $M \times M$. The *b*-heat space has five boundary faces, two of which result from blowing up. The remaining three boundary faces are at $t = 0$ off the diagonal and at the boundary in each copy of M . More precisely, we have the following.⁷

Face	Geometry of face	Defining function in local coordinates
F_{110}	$N^+(Y \times Y) \times \mathbb{R}^+$	$\rho_{110} = (x^2 + (x')^2)^{1/2}$
F_{d2}	$PN_t^+(\Delta(M \times M))$	$\rho_{d2} = (z - z' ^4 + t^2)^{1/4}$
F_{100}	$Y \times (M - \partial M) \times \mathbb{R}^+$	$\rho_{100} = x$
F_{010}	$Y \times (M - \partial M) \times \mathbb{R}^+$	$\rho_{010} = x'$
F_{001}	$(M - \partial M)^2 - \Delta(M \times M)$	$\rho_{001} = t$

Above PN_t^+ denotes the inward-pointing t parabolic normal bundle, while N^+ denotes the inward-pointing spherical normal bundle. Note that

⁶The key idea is to blow-up appropriate submanifolds and introduce a new “symbol” or “normal operator” for each blown-up face and then solve these normal operators. In the classical microlocal sense, we invert the symbol for each face. In this way, the complexity of the operator on the original space is handled geometrically by blowing up, thereby allowing one to use the powerful tools of classical microlocal analysis in non-classical (non-compact) geometric settings. For a detailed exposition, see [33].

⁷The subscript “ d ” indicates a face created by blowing up along the diagonal, so for example F_{d2} is the face created by blowing up along the diagonal parabolically in the t direction.

Figure 2: Schematic diagram of the b -heat space.

the local coordinates x, x', t lift from $M \times M \times \mathbb{R}^+$ to $M_{b,h}^2$ as follows:

$$\beta^*(x) = \rho_{110}\rho_{100}, \quad \beta^*(x') = \rho_{110}\rho_{010}, \quad \beta^*(t) = (\rho_{d2})^2 \rho_{001},$$

so these coordinates are only *local* defining functions.

6.1.2. b -Heat calculus The b -heat calculus consists of distributional section half-density kernels on $M_+^2 = M \times M \times \mathbb{R}^+$ which are smooth on the interior and lift to be polyhomogeneous on $M_{b,h}^2$ with specified leading orders at the boundary faces. By constructing the b -heat kernel as an element of the b -heat calculus, it is polyhomogeneous on $M_{b,h}^2$ with specified leading orders. Once the calculus is defined and the composition rule is proven, construction of the heat kernel as an element of the heat calculus proceeds like the classical microlocal constructions in [19, 20, 38]. The following definition is from [33].

Definition 6.1. For any $k \in \mathbb{R}$ and index set E_{110} , A is an element of the b -heat calculus $\Psi_{b,H}^{E_{110},k}$ if A is smooth on the interior of $M_{b,h}^2$ and satisfies the following.

1. $A \in \mathcal{A}_{phg}^{-1/2+E_{110}}(F_{110})$.
2. A vanishes to infinite order at F_{001} , F_{100} and F_{010} .
3. $A \in \rho_{d2}^{-((n+3)/2)-k} \mathcal{C}^\infty(F_{d2})$.

Because the heat calculus is defined with half-densities, the normalizing factors at F_{110} and F_{d2} simplify the composition rule. An element A of the b -heat calculus is the Schwartz kernel of an operator acting on a smooth

half-density section f of M by

$$Af(z, t) = \int_M \langle A(z, z', t), f(z') \rangle dz'.$$

Furthermore, A acts by convolution in the t variable so for a smooth half-density section f of $M \times \mathbb{R}_t^+$,

$$Af(z, t) = \int_0^t \int_M \langle A(z, z', t - s), f(z', s) \rangle dz' ds.$$

Two elements of the b -heat calculus compose as follows.

Technical Theorem 6.2. *Let $A \in \Psi_{b,H}^{k_a,A}$ and let $B \in \Psi_{b,H}^{k_b,B}$. Then the composition, $A \circ B$, is an element of $\Psi_{b,H}^{k_a+k_b,A \cup B}$.*

The proof of this composition rule is in [33].⁸

6.1.3. Construction of the b -heat kernel First we construct a model heat kernel H_1 as an element of the b -heat calculus that solves the heat equation up to an error vanishing to positive order at the boundary faces of $M_{b,h}^2$. On the interior of $M_{b,h}^2$ restricting to a coordinate patch with coordinates (z, z', t) , we locally define

$$H_1(z, z', t) := (4\pi t)^{-n/2} e^{(|z-z'|_g)^2/2t},$$

where $|z - z'|_g$ is the distance from z to z' with respect to the metric g . As $t \rightarrow 0$ away from the diagonal, this construction immediately implies infinite order vanishing at F_{001} . At F_{d2} we solve exactly: for each $p \in M$ and for each point $z \in F_{d2}$ in the fiber over $(p, p, 0)$, the heat kernel at that point is determined by the coefficients of the metric (and its derivatives) at p .

The *normal operator* of $\partial_t + \Delta_b$ is the restriction to F_{110} of the lift of $\partial_t + \Delta_b$ to $M_{b,h}^2$. H_1 is defined at F_{110} to be the kernel of a first-order parametrix of this normal operator and is smooth at this face. At F_{100}, F_{010}

⁸Note that in order for the composition to be *defined*, A and B must satisfy certain compatibility conditions. In all our applications, these conditions are a priori satisfied. The notation $\bar{\cup}$ denotes the extended union of index sets, see [28].

and F_{001} , the model kernel vanishes to infinite order. As constructed, H_1 satisfies

$$(\partial_t + \Delta)H_1 = K_1, \quad H_1 \in \Psi_{b,H}^{-2,0}.$$

K_1 vanishes to positive order at the boundary faces of $M_{b,h}^2$. Next, define

$$H_2 = H_1 - H_1 * K_1,$$

where now the error term,

$$K_2 = (\partial_t + \Delta)H_2,$$

vanishes to one order higher on each of the boundary faces of $M_{b,h}^2$ by the composition rule. This construction is iterated and the Borel summation [38] gives $H_\infty \in \Psi_{b,H}^{-2,0}$ with $H_\infty - H_N = O(t^{N-(n+3)/2})$, for $N > 0$, so that

$$(\partial_t + \Delta)H_\infty = K.$$

K vanishes to infinite order on the boundary faces of $M_{b,h}^2$, so we may push K forward to $M \times M \times \mathbb{R}^+$. We solve away the residual error term using the action of elements of the b -heat calculus as t -convolution operators. As a t -convolution operator, the heat kernel is the identity. Above, K as a t -convolution operator is of the form $K = \text{Id} - A$, where A is a Volterra operator, and Id is the identity. An operator of this form has an inverse of the same form, so defining

$$H := H_\infty(\text{Id} - A)^{-1}$$

solves away this residual error term. By construction, the leading order behavior of the b -heat kernel is that of the model heat kernel and is summarized below.

Face	Leading order
F_{110}	$0; O(t^{-1})$ as $t \rightarrow \infty$
F_{d2}	$-(n+3)/2 - (-2)$
F_{100}	∞ order vanishing
F_{010}	∞ order vanishing
F_{001}	∞ order vanishing

6.2. Conic heat kernel

Let (M_0, g_0) be a compact manifold with isolated conic singularity, and let Δ_0 be the Friedrichs extension of a geometric Laplacian associated to a

Hermitian vector bundle over (M_0, g_0) . We construct the conic heat kernel using the same steps for constructing the b -heat kernel: first we construct the conic heat space with its calculus and composition rule, then we solve the normal operators by constructing a model parametrix and use the calculus to solve up to infinite order, and finally we solve away the residual error using the action as a t -convolution operator.

6.2.1. The conic heat space This construction comes from [34]. The conic heat space $M_{0,h}^2$ is a manifold with corners obtained from $M_0 \times M_0 \times \mathbb{R}^+ = M_{0,+}^2$ by blowing up along two submanifolds (Figure 3),

$$M_{0,h}^2 := [[M_0 \times M_0 \times \mathbb{R}^+; \partial M_0 \times \partial M_0 \times \{t = 0\}, dt]; \Delta(M_0^0 \times M_0^0) \times \{t = 0\}, dt].$$

The conic heat space has five boundary faces described in the following table in which $z = (x, y)$ and $z' = (x', y')$ are local coordinates in a neighborhood of the singularity in each copy of M_0 so that $x = 0, x' = 0$ define the singularity as well as the boundary of M_0 .

Face	Geometry of face	Defining function in local coordinates
F_{112}	$PN_t^+(Y \times Y)$	$\rho_{112} = (x^4 + (x')^4 + t^2)^{1/4}$
F_{d2}	$PN_t^+(\Delta(M_0^0 \times M_0^0))$	$\rho_{d2} = (z - z' ^4 + t^2)^{1/4}$
F_{100}	$Y \times \mathbb{R}^+$	$\rho_{100} = x$
F_{010}	$\mathbb{R}^+ \times Y$	$\rho_{010} = x'$
F_{001}	$M_0^0 \times M_0^0 - \Delta(M_0^0 \times M_0^0)$	$\rho_{001} = t$

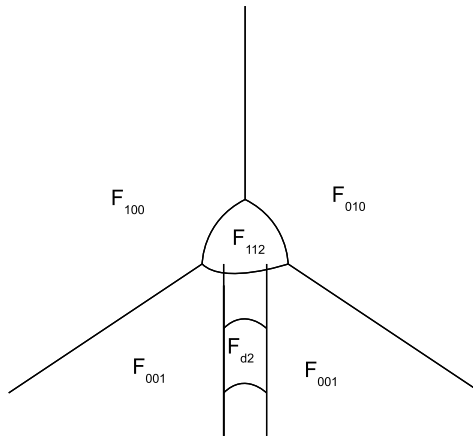


Figure 3: Schematic diagram of the conic heat space.

Note that the coordinates x, x', t lift from $M_0 \times M_0 \times \mathbb{R}^+$ to $M_{0,h}^2$ as follows:

$$\beta^*(x) = \rho_{100}\rho_{112}, \quad \beta^*(x') = \rho_{010}\rho_{112}, \quad \beta^*(t) = \rho_{112}^2\rho_{d2}^2\rho_{001},$$

so again these are only *local* defining functions.

6.2.2. The conic heat calculus Let μ be a conic half-density on $M_{0,+}^2$. We may assume

$$\mu = (xx')^{(n-1)/2} \sqrt{dz dz' dt} = \sqrt{dV_c dt}.$$

Fix also a smooth, non-vanishing half-density ν on $M_{0,h}^2$. Elements of the conic heat calculus are distributional section half densities on $M_{0,+}^2$ which are smooth on the interior and lift to be polyhomogeneous on $M_{0,h}^2$.

Definition 6.3. Let $k \in \mathbb{R}$ and let E_{100}, E_{010} and E_{112} be index sets. Then $A \in \Psi_{0,H}^{k,E_{100},E_{010},E_{112}}$ if A is a smooth half-density section on the interior of $M_{0,h}^2$ and satisfies the following.

1. $A \in \mathcal{A}_{phg}^{E_{100}}$ at F_{100} .
2. $A \in \mathcal{A}_{phg}^{E_{010}}$ at F_{010} .
3. $A \in \mathcal{A}_{phg}^{E_{112}}$ at F_{112} .
4. A vanishes to infinite order at F_{001} .
5. $A \in \rho_{d2}^{(-(n+3)/2)-k} \mathcal{C}^\infty(F_{d2})$.

With this normalization, the conic heat kernel has order $k = -2$, and the composition rule is the following.

Technical Theorem 6.4. Let $A \in \Psi_{0,h}^{A_{100},A_{010},A_{112},k_a}$ and $B \in \Psi_{0,H}^{B_{100},B_{010},B_{112},k_b}$ with leading index terms satisfying

$$\beta_{112} + \alpha_{010} > 0, \quad \alpha_{112} + \beta_{100} > 0, \quad -k_a > 0, \quad -k_b > 0, \quad \beta_{100} + \alpha_{010} > -1.$$

Then, the composition $B \circ A$ is an element of $\Psi_{0,H}^{A_{100},B_{010},\Gamma_{112},k}$ with $\Gamma_{112} = A_{112} \cup B_{112}$ and $k = (k_a + k_b)$.

The proof of this theorem is in [37] and is originally due to [34]; see also [15, 27].

Away from F_{112} , F_{100} and F_{010} , the model heat kernel is locally constructed using the Euclidean heat kernel. At the front face F_{112} , which arises from blowing up the singularity, the coordinates x, x', t vanish. Instead, consider the projective coordinates⁹

$$s = \frac{x}{x'}, \quad s' = x', \quad \tau = \frac{t}{(x')^2}.$$

In these coordinates, the heat operator

$$\partial_t + \Delta = (s')^{-2}(\partial_\tau + (\partial_s)^2 + s^{-2}\Delta_h),$$

where Δ_h is the induced geometric Laplacian on the boundary (Y, h) . Since s' is a defining function for F_{112} , we see that the normal operator is the heat operator for the exact cone over (Y, h) , so we define the model heat kernel at F_{112} to be

$$H_0(s, s', y, y', \tau),$$

where H_0 is the heat kernel for the exact cone over (Y, h) . Using the scaling properties of the heat kernel for an exact cone, this is equivalent to

$$(\rho_{112})^{-n} H_0(s, 1, y, y', \tau),$$

which is well defined away from F_{010} and has a polyhomogeneous expansion up to F_{100} ; see [5, 23]. With the Friedrichs extension, symmetry in space variables defines the model heat kernel on all of F_{112} , and we define it on F_{100} and F_{010} by extending the polyhomogeneous expansion at the corners of these faces with F_{112} .

6.3. Asymptotically conic scattering heat kernel

A summary of the ac scattering heat kernel, space and calculus is given here; details are contained in the appendix. Let (\bar{Z}, g_z) be an ac scattering metric with boundary defined by $\{x = 0\}$ and with local coordinates $z = (x, y)$ near the boundary. Let Δ_z be the Friedrichs extension of a geometric Laplacian on \bar{Z} .

⁹It is generally helpful to work with projective coordinates since differential operators transform nicely under projective coordinate changes.

6.3.1. Asymptotically conic scattering heat space First, we construct the ac scattering double space,

$$\bar{Z}_{sc}^2 := [[\bar{Z} \times \bar{Z}; \partial\bar{Z} \times \partial\bar{Z}]; \Delta(Y \times Y) \cap F_{110}],$$

where F_{110} is the face created by the first blowup. This construction comes from [17, Figure 4]. The ac scattering heat space is

$$\bar{Z}_{sc,h}^2 = [\bar{Z}_{sc}^2 \times \mathbb{R}^+; \Delta(Z \times Z) \times \{t = 0\}, dt].$$

The ac scattering heat space has six boundary faces described in the following table.

Face	Geometry of face	Defining function in local coordinates
F_{220}	$N^+(\Delta(Y \times Y)) \times \mathbb{R}^+$	$\rho_{220} = (x^2 + (x')^2 + y - y' ^2)^{1/2}$
F_{110}	$N^+(Y \times Y)$ $-\Delta(Y \times Y) \times \mathbb{R}^+$	$\rho_{110} = (x^2 + (x')^2)^{1/2}$
F_{100}	$Z \times Y \times \mathbb{R}^+$	$\rho_{100} = x$
F_{010}	$Y \times Z \times \mathbb{R}^+$	$\rho_{010} = x'$
F_{d2}	$PN_t^+(\Delta(Z \times Z))$	$\rho_{d2} = (z - z' ^4 + t^2)^{1/4}$
F_{001}	$(Z \times Z) - \Delta(Z \times Z)$	$\rho_{001} = t$

6.3.2. Asymptotically conic scattering heat calculus Elements of the ac scattering heat calculus are distributional section half- densities of \bar{Z}_+^2 which are smooth on the interior and lift to be polyhomogeneous on $\bar{Z}_{sc,h}^2$. Let μ be a smooth, non-vanishing half- density on \bar{Z}_+^2 , and let ν be a smooth, non-vanishing half- density on $\bar{Z}_{sc,h}^2$.

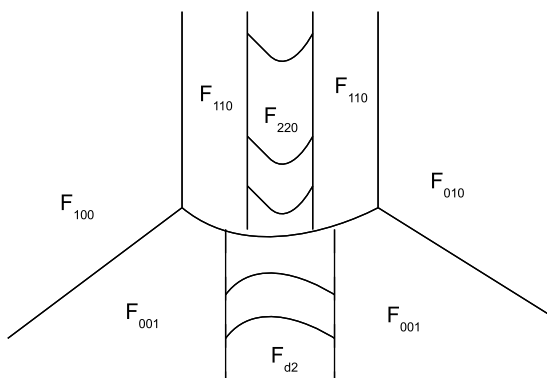


Figure 4: Schematic diagram of the ac scattering heat space.

Definition 6.5. For any $k \in \mathbb{R}$ and index sets E_{110} and E_{220} , $A \in \Psi_{sc,H}^{E_{110}, E_{220}, k}$ if A is smooth on the interior of $\bar{Z}_{sc,h}^2$ and satisfies the following.

1. $A \in \mathcal{A}_{phg}^{-(1/2)+E_{110}}$ at F_{110} .
2. $A \in \mathcal{A}_{phg}^{-((n+2)/2)+E_{220}}$ at F_{220} .
3. A vanishes to infinite order at F_{001} , F_{100} and F_{010} .
4. $A \in \rho_{d2}^{-((n+3)/2)-k} \mathcal{C}^\infty(F_{d2})$.

Two elements of the ac scattering heat calculus compose as follows.

Technical Theorem 6.6. Let $A \in \Psi_{sc,H}^{A_{110}, A_{220}, k_a}$, and $B \in \Psi_{sc,H}^{B_{110}, B_{220}, k_b}$.
 Then, the composition $B \circ A$ is an element of $\Psi_{sc,H}^{A_{110} \cup B_{110}, A_{220} \cup B_{220}, k_a + k_b}$.

The ac scattering heat kernel is constructed analogously to the b and conic heat kernels. The model heat kernel in this case is the lift of the Euclidean heat kernel to $\bar{Z}_{sc,h}^2$.¹⁰

Technical Theorem 6.7. Let (\bar{Z}, g_z) be an ac scattering metric with boundary (Y, h) . Let (E, ∇) be a Hermitian vector bundle over (Z, g_z) which induces a compatible bundle over (Y, h) . Let Δ be a geometric Laplacian on (\bar{Z}, g_z) associated to the bundle (E, ∇) . Then there exists $H \in \Psi_{sc,H}^{E_{110}, E_{220}, -2}$ satisfying

$$(\partial_t + \Delta)H(z, z', t) = 0, \quad t > 0,$$

$$H(z, z', 0) = \delta(z - z').$$

Moreover, H vanishes to infinite order at F_{110} and is smooth up to F_{220} .

The proof of this theorem is in the appendix.

7. Heat kernel convergence

These heat kernels interact on the acc heat space.

¹⁰Since the standard example of an ac scattering metric is the radial compactification of \mathbb{R}^n , it is natural that the ac scattering model heat kernel is the lift of the Euclidean heat kernel to $\bar{Z}_{sc,h}^2$.

7.1. Asymptotically conic convergence heat space

This construction is similar to the heat space constructions in Section 6 and the acc double space construction in Section 3. First, let

$$\mathcal{H}_0 := \{\epsilon = \epsilon'\} \subset \mathcal{S} \times \mathcal{S}.$$

Next, let

$$\mathcal{H}_1 := [\mathcal{H}_0 \times \mathbb{R}_t^+; Y \times Y \times \{t = 0\}, dt].$$

This blowup must be done first to create the F_{112} face in the conic heat space. The scalar variables on $\mathcal{S} \times \mathcal{S} \times \mathbb{R}_t^+$ are (x, r, x', r', t) ,¹¹ so by our notation, the face created by this blowup is $F_{1111,2}$. Let

$$\mathcal{H}_2 := [\mathcal{H}_1; Z \times Z \times \{t = 0\}, dt].$$

This blowup is less obvious: the ac metric is scaled by ϵ^2 , so the corresponding Laplacian is scaled by ϵ^{-2} , and the time should also be scaled. To create a face with rescaled time, we perform this blowup, and the resulting face is $F_{1010,2}$, which has the geometry of a compactified b -heat space at $t, \epsilon = 0$.

Finally, the acc heat space results from blowing up the closure of the lift of the diagonal in $\mathcal{S} \times \mathcal{S}$ at $\{t = 0\}$,

$$\mathcal{H} := [\mathcal{H}_2; \overline{\beta^*(\Delta(\mathcal{S} \times \mathcal{S} - (Y \times Y)) \times \{t = 0\})}, dt].$$

The face created by this last blowup is F_{d2} .

The $\epsilon = 0$ boundary faces of \mathcal{H} are summarized below:

$\mathcal{S} \times \mathcal{S} \times \mathbb{R}_t^+$ corner	\mathcal{H} face	geometry
$x = 0, x' = 0, r = 0,$ $r' = 0, t = 0$	$F_{1111,2}$	$PN^+(Y \times Y)$
$x = 0, x' = 0, t = 0,$	$F_{1010,2}$	$[\bar{Z} \times \bar{Z} \times \mathbb{R}_r^+; Y \times Y];$ $\Delta(Z \times Z) \times \{0\}, d\tau]$
$x = 0, x' = 0,$ $r = 0, r' = 0$	F_{1010}	$[Z \times Z \times \mathbb{R}^+ - \{t = 0\}]$
$x = 0, r' = 0$	F_{0101}	$M_{0,h}^2$
$x = 0, r' = 0$	F_{1001}	$[\bar{Z} \times M_0 \times \mathbb{R}^+; Y \times Y \times \{0\}, dt]$
$r = 0, x' = 0$	F_{0110}	$[M_0 \times \bar{Z} \times \mathbb{R}^+; Y \times Y \times \{0\}, dt]$
$\Delta(\mathcal{S} \times \mathcal{S}) \times \{t = 0\}$	F_{d2}	$PN^+(\Delta(\mathcal{S} \times \mathcal{S}))$
$\{t = 0\}$	$F_{0000,1}$	$(\{\epsilon = \epsilon'\} \subset \mathcal{S} \times \mathcal{S}) - \Delta(\mathcal{S} \times \mathcal{S})$

¹¹Note that these variables are not independent; they are related by $xr = x'r' = \epsilon$.

7.1.1. Asymptotically conic convergence half-density calculations

We calculate the lift to \mathcal{H} of $dV g_\epsilon dV g'_\epsilon dt d\epsilon$. The Jacobian determinant factors which result from blowing up are

$$(\rho_{1111,2})^4(\rho_{1010,2})^2(\rho_{d2})^{n+1}.$$

Next, we calculate the lift of the variables x, x', r, r' to \mathcal{H} ,¹²

$$\begin{aligned} \beta^*(x) &= \rho_{1010}\rho_{1010,2}\rho_{1111,2}\rho_{1001}, \\ \beta^*(x') &= \rho_{1010}\rho_{1010,2}\rho_{1111,2}\rho_{0110}, \\ \beta^*(r) &= \rho_{1111,2}\rho_{0101}\rho_{0110}, \\ \beta^*(r') &= \rho_{1111,2}\rho_{0101}\rho_{1001}. \end{aligned}$$

We calculate the volume form $dV g_\epsilon$ in a neighborhood of the faces of \mathcal{H} at $\epsilon = 0$. At F_{0101} , $dV g_\epsilon \sim dV_0$ and

$$\begin{aligned} dV g_\epsilon dV g'_\epsilon &\sim (\rho_{1111,2}\rho_{1010}\rho_{1010,2})^{2n-2}(\rho_{1001}\rho_{0110})^{n-1} \\ &\quad \times (\rho_{1111,2})^4(\rho_{1010,2})^2(\rho_{d2})^{n+1}\mu. \end{aligned}$$

At F_{1010} and $F_{1010,2}$, $dV g_\epsilon \sim \epsilon^2 dV_z$, consequently

$$\begin{aligned} dV g_\epsilon dV g'_\epsilon &\sim (\rho_{1111,2})^{2n-2}(\rho_{1010,2}\rho_{1010})^{2n}(\rho_{0101})^{-2}(\rho_{1001}\rho_{0110})^{n-1} \\ &\quad \times (\rho_{1111,2})^4(\rho_{1010,2})^2(\rho_{d2})^{n+1}\mu, \end{aligned}$$

where μ is a smooth non-vanishing spatial density on \mathcal{H} .¹³

Then, we arrive at the following half-density calculation at F_{0101} and $F_{1111,2}$,

$$\begin{aligned} \beta^*(\sqrt{dV g_\epsilon dV g'_\epsilon dt d\epsilon}) &\sim (\rho_{1111,2})^{n+1}(\rho_{1010,2})^n(\rho_{1010})^{n-1} \\ &\quad \times (\rho_{1001}\rho_{0110})^{(n-1)/2}(\rho_{d2})^{(n+1)/2}\sqrt{\nu}, \end{aligned}$$

where ν is a smooth non-vanishing density on \mathcal{H} . At F_{1010} and $F_{1010,2}$,

$$\begin{aligned} \beta^*(\sqrt{dV g_\epsilon dV g'_\epsilon dt d\epsilon}) &\sim (\rho_{1111,2})^{n+1}(\rho_{1010,2})^{n+1}(\rho_{1010})^n(\rho_{0101})^{-1} \\ &\quad \times (\rho_{1001}\rho_{0110})^{(n-1)/2}(\rho_{d2})^{(n+1)/2}\sqrt{\nu}. \end{aligned}$$

¹²The recipe for these exponents is: (codimension of space variables -1) + (codimension of parabolic variables $* 2$).

¹³We have used that $dV_0 \sim x^{n-1} dx dy$ and $dV_z \sim \frac{\epsilon^n}{r^{n+1}} dr dy$ and that $\epsilon = xr = x'r'$.

7.2. Asymptotically conic convergence heat calculus

The acc heat calculus is a parameter (ϵ) dependent operator calculus incorporating the smooth, conic, and b -heat calculi.¹⁴

Definition 7.1. The acc heat calculus of order k , written $\Psi_{\text{acc},H}^{k,E_{0101},E_{1010,2},E_{1111,2}}$, consists of half-density kernels A which are smooth on the interior of \mathcal{H} and satisfy the following.

1. For each $\epsilon > 0$, A restricts to an element of $\Psi_{\epsilon,H}^k$, the standard heat calculus of order k for (M, g_ϵ) .¹⁵
2. In a neighborhood of $F_{1010,2}$, A has an asymptotic expansion in $\rho_{1010,2}$ with index set $E_{1010,2}$ and coefficients in the b -heat calculus of order k . Such an expansion is of the form

$$A \sim \sum_{j \geq 1} \sum_{0 \leq p_0 \leq p \leq p_j} (\rho_{1010,2})^{\alpha_j} (\log \rho_{1010,2})^p A_{j,l},$$

with $A_{j,l} \in \Psi_{b,H}^{k,E_{110}^j}$. Above, if for some j , $p_j = 0$, then there are no log terms.

3. In a neighborhood of F_{0101} , A has an asymptotic expansion in ρ_{0101} with index set E_{0101} , and coefficients are elements of the conic heat calculus of order k .
4. In a neighborhood of $F_{1111,2}$, A has an asymptotic expansion in $\rho_{1111,2}$ with index set $E_{1111,2}$ so that the coefficients in the conic heat calculus are of order k for the exact cone over Y .
5. A has a smooth polyhomogeneous expansion up to the side faces $F_{1001}, F_{1010}, F_{0110}$, and these expansions extend smoothly to the corners meeting the other faces of \mathcal{H} .

¹⁴The ac scattering heat calculus was expected to arise in the acc heat space and calculus, but after calculating the normal operators as $\epsilon \rightarrow 0$, it became clear that the b -heat space and calculus rather than the ac scattering heat space and calculus were needed.

¹⁵The heat space for a smooth compact manifold M is $M_h^2 := [M \times M \times \mathbb{R}_t^+; \Delta(M) \times \{t = 0\}, dt]$. The standard heat calculus Ψ_H^k consists of half-density sections which are smooth on the interior of M_h^2 , vanish identically at $t = 0$ away from the blown-up face, and which are $\rho^{-(1/2)(n+3)-k} \mathcal{C}^\infty$ at the blown-up face defined by ρ ; see, for example, [33, 37].

The composition rule is not required for the proof of our main theorem, but we would expect it to follow from the composition rules for the smooth (compact manifolds), b , and conic heat calculi, together with the result for combining polyhomogeneous index sets as in [28].

Theorem 7.2. *Let (M_0, g_0) be a compact Riemannian n -manifold with isolated conic singularity, and let (Z, g_z) be an ac space, with $n \geq 2$. Assume (M, g_ϵ) converges asymptotically conically to (M_0, g_0) . Let (E_0, ∇_0) and (E_z, ∇_z) be rank k Hermitian vector bundles over (M_0, g_0) and (Z, g_z) , respectively, so that each of these bundles in a neighborhood of the boundary is the pullback from a bundle over the boundary (Y, h) . Let Δ_0 and Δ_z be the corresponding Friedrichs extensions of geometric Laplacians, and let Δ_ϵ be the induced geometric Laplacian on (M, g_ϵ) . Then the associated heat kernels H_ϵ have a full polyhomogeneous expansion as $\epsilon \rightarrow 0$ on the acc heat space with the following leading terms:*

- At the conic front face, F_{0101} , $H_\epsilon(z, z', t) \rightarrow H_0(z, z', t)$, the heat kernel for (M_0, g_0) with half-density factor

$$(*) = (\rho_{1111,2})^{n+1}(\rho_{1010,2})^n(\rho_{1010})^{n-1}(\rho_{1001}\rho_{0110})^{(n-1)/2}(\rho_{d2})^{(n+1)/2}\sqrt{\nu}.$$

- At the rescaled b front face, $F_{1010,2}$, $H_\epsilon(z, z', t) \rightarrow (\rho_{1010,2})(\rho_{1010})(\rho_{0101})^{-1}H_b(\tau)$, the b heat kernel with rescaled time variable τ and with half-density $(*)$.
- At the exact conic front face, $F_{1111,2}$, $H_\epsilon(z, z', t) \rightarrow H_0(\tilde{\tau})$, the heat kernel for the exact cone with rescaled time variable $\tilde{\tau}$ with half-density $(*)$.
- At the side faces F_{1001}, F_{0110} and the residual b face F_{1010} , the heat kernel has polyhomogeneous expansion with leading term vanishing to at least second order.

This convergence is uniform in ϵ for bounded time and moreover, the error term is bounded by $C\epsilon t^N$ as $t \rightarrow 0$, for any $N \in \mathbb{N}_0$, where C may depend on N .

7.3. Proof

This proof is modeled after the parametrix construction of [33]. First, we lift the operator $\partial_t + \Delta_\epsilon$ to \mathcal{H} and construct the *acc model heat kernel* as an element of the acc heat calculus which solves the normal operators of $\partial_t + \Delta_\epsilon$ on \mathcal{H} . We use the b -heat calculus and introduce the acc conic triple

heat space to use the conic heat calculus and solve up to infinite order, then we estimate the error term of this solution kernel and finally solve away the residual error.

7.3.1. Lifted heat operator We must carefully choose good coordinates in a neighborhood of each of the front faces $F_{0101}, F_{1010,2}$ and $F_{1111,2}$ and calculate the leading term of $\partial_t + \Delta_\epsilon$ as $\epsilon \rightarrow 0$, or in the terminology of [1, 28, 33], we calculate the “normal operators” at the $\epsilon = 0$ boundary faces of \mathcal{H} . In a neighborhood of F_{0101} , g_ϵ is smoothly approaching the conic metric, so

$$\partial_t + \Delta_\epsilon \rightarrow \partial_t + \Delta_0.$$

In a neighborhood of $F_{1010,2}$, the metric $g_\epsilon \rightarrow \epsilon^{-2}g_z$, so

$$\partial_t + \Delta_\epsilon \rightarrow \partial_t + \epsilon^{-2}\Delta_z = (x)^{-2}(\partial_\tau + \Delta_b),$$

where $\tau = t/(x^2)$. In the above calculation, we have used $xr = \epsilon$ together with the relation between the ac scattering Laplacian and the rescaled b Laplacian $\Delta_z = r^2\Delta_b$. Note that x lifts to define $F_{1010}, F_{1010,2}$ and $F_{1111,2}$. This indicates that the leading part of $\partial_t + \Delta_\epsilon$ at $F_{1010,2}$ is

$$(\rho_{1010,2}\rho_{1010}\rho_{1111,2})^{-2}(\partial_\tau + \Delta_b).$$

In a neighborhood of $F_{1111,2}$, the scalar variables (x, x', r, r', t) are not good coordinates since they all vanish. Better coordinates are the projective $(s, s', \sigma, \sigma', \tau)$, where

$$s = \frac{x}{x'}, \quad s' = x', \quad \sigma = \frac{r}{x'}, \quad \sigma' = \frac{r'}{x'}, \quad \tilde{\tau} = \frac{t}{(x')^2}.$$

Note that

$$(r\partial_r) = (\sigma\partial_\sigma), \quad xr = \epsilon \implies \frac{r^2}{\epsilon^2} = x^{-2}.$$

At $F_{1111,2}$ we see both the conic metric g_0 near the singularity and the rescaled ac scattering metric ϵ^2g_z near the boundary. Using the projective coordinates around the conic singularity, we compute

$$\partial_t + \Delta_\epsilon \rightarrow (s')^{-2}(\partial_{\tilde{\tau}} + (\partial_s)^2 + (s)^{-2}(\Delta_h)) = (ss')^{-2}(\partial_\tau + (s\partial_s)^2 + \Delta_h),$$

where Δ_h is the Laplacian on Y for $h = h(x = 0)$, and we note that $\tau = \frac{t}{(ss')^2}$. Using the projective coordinates near the boundary of \bar{Z} , we compute

$$\partial_t + \Delta_\epsilon \rightarrow (ss')^{-2}(\partial_\tau + (\sigma\partial_\sigma)^2 + \Delta_h) = (s')^{-2}(\partial_{\tilde{\tau}} + (s)^{-2}((\sigma\partial_\sigma)^2 + \Delta_h),$$

where in the last equality we have used $xr = \epsilon = x'r'$, which gives $s\partial_s = \sigma\partial_\sigma$. Since $(ss') = x$ lifts to \mathcal{H} to define $F_{1111,2}, F_{1010}$ and $F_{1010,2}$ as does s' , this indicates that the leading term of $\partial_t + \Delta_\epsilon$ at $F_{1111,2}$ is

$$(\rho_{1111,2}\rho_{1010}\rho_{1010,2})^{-2}(\partial_{\tilde{\tau}} + \Delta_{0,s}),$$

where $\Delta_{0,s}$ is the Laplacian for the exact cone over (Y, h) . Note that these calculations demonstrate that the model operators are consistent at the corners.

These calculations together with the half-density calculation indicate how to define the acc model heat kernel as a parametrix for $\partial_t + \Delta_\epsilon$ as $\epsilon \rightarrow 0$.

7.3.2. Asymptotically conic convergence model heat kernel, H_1

- At F_{0101} , let $H_1(z, z', t, \epsilon) \sim H_0$, the heat kernel for (M_0, g_0) , with half-density factor¹⁶

$$(*) = (\rho_{1111,2})^{n+1}(\rho_{1010,2})^n(\rho_{1010})^{n-1}(\rho_{1001}\rho_{0110})^{(n-1)/2}(\rho_{d2})^{(n+1)/2}\sqrt{\nu}.$$

- At $F_{0101,2}$, let $H_1(z, z', t, \epsilon) \sim (\rho_{1010}\rho_{1010,2})(\rho_{0101})^{-1}H_b(\tau)(*)$, where $H_b(\tau)$ is the b heat kernel with rescaled time variable.
- At $F_{1111,2}$, let $H_1(z, z', t, \epsilon) \sim H_0(\tilde{\tau})(*)$, where $H_0(\tilde{\tau})$ is the heat kernel for the exact cone over (Y, h) with rescaled time.

At the side faces of \mathcal{H} , it is instructive to recall the side faces in the conic heat space. The conic heat kernel has normal operator

$$\partial_t + (\partial_x)^2 + x^{-2}\Delta_h,$$

where Δ_h is the induced Laplacian on the boundary (Y, h) . At the side face F_{100} , x vanishes and there is no natural “normal operator”. However, the side faces meet the front face, so we define the conic model heat kernel on the side faces by extending the asymptotic expansion from the front face to the side faces and using symmetry of the Friedrichs extension. When the conic density is included, the conic model heat kernel vanishes to positive order at the side faces. Similarly, there is no natural normal operator at

¹⁶Note that we do not require any additional vanishing factors to account for the singular factors in the normal operators since the half-density already includes sufficient vanishing factors to remove the singular factors when the normal operators are applied.

F_{1010}, F_{1001} and F_{0110} . However, the acc model heat kernel is defined on F_{0101} which meets F_{1001} and F_{0110} , and so it may be defined on these faces by extending the expansion as in the conic case. At the corner of $F_{1010,2}$ with F_{1010} , the rescaled time variable $\tau \rightarrow \infty$, and the b heat kernel vanishes to positive order as $t \rightarrow \infty$, so we define the acc model heat kernel at F_{1010} by extending the expansion at $F_{1010,2}$ as $\tau \rightarrow \infty$. For $\epsilon > 0$, let $H_1(z, z', t, \epsilon)$ be the heat kernel for $\partial_t + \Delta_\epsilon$.

7.3.3. Asymptotically conic convergence model heat kernel construction along diagonal at $t = 0$ In a neighborhood of the faces diffeomorphic to the parabolic normal bundle of the diagonal, $PN^+(\Delta)$, we carry out a local construction as in [33, chapter 7]. Let X be a manifold; since this construction is the same for $X = M, X = M_0$ and $X = \bar{Z}$, we use X to simplify notation. Let F_X denote the $PN^+(\Delta(X))$ face in \mathcal{H} , where $\Delta(X)$ is the diagonal in $X \times X$. Let $N(\partial_t + \Delta_X)$ be the restriction to F_X of the lift of $(\partial_t + \Delta_X)$ to \mathcal{H} , where Δ_X is our geometric Laplacian on X . With the heat calculus normalization at F_X , an element A of the acc heat calculus of order $-k$ restricts to F_X as follows

$$N(A) = t^{(k+n+2)/2} A|_{F_X}.$$

As in [33] we observe that F_X is naturally diffeomorphic to a radial compactification of the tangent space of X , with each fiber of F_X over $(x, x, 0)$ diffeomorphic to the tangent space at x . From [33, 7.15],

$$(7.1) \quad N(t(\partial_t + \Delta_X)A) = \left[\sigma(\Delta_X) - \frac{1}{2}(R + n + k + 2) \right] N(A),$$

where R is the radial vector field on the fibers of TX . Note that if G_0 satisfies

$$t(\partial_t + \Delta_X)G_0 = O(t^\infty) \text{ as } t \rightarrow 0, \quad G_0|_{t=0} = \delta(x - x'),$$

then G_0 also satisfies

$$(7.2) \quad (\partial_t + \Delta_X)G_0 = O(t^\infty) \text{ as } t \rightarrow 0, \quad G_0|_{t=0} = \delta(x - x').$$

So, we may work with $t(\partial_t + \Delta_X)$ as in [33]. Our initial parametrix G_0 has order $k = -2$ at F_X . From (7.1), we have the following equation for G_0 ,

$$\left[\sigma(\Delta_X) - \frac{1}{2}(R + n) \right] N(G_0) = 0.$$

From [33, 7.13], in order for G_0 to satisfy the initial condition, it must satisfy

$$\int_{\text{fiber}} N(G_0) = 1.$$

Since these conditions are fiber-by-fiber, we introduce local coordinates so that

$$\sigma(\Delta_X) = D_1^2 + \dots + D_n^2 \text{ on } T_x X.$$

Then we have

$$(7.3) \quad \left[D_1^2 + \dots + D_n^2 - \frac{1}{2}(R + n) \right] N(G_0) = 0,$$

so

$$(7.4) \quad N(G_0) = (2\pi)^{-n/2} \exp\left(-\frac{|X|_x^2}{4}\right)$$

is the desired solution, where X is a projective local coordinate on F_x , $X = \frac{x-x'}{t^{1/2}}$ (see [33, 7.36]), and $|\cdot|_x$ is the Riemannian norm on TX induced by the metric at x . To see that this is the desired solution, consider the Fourier transform of (7.3) with $u = N(G_0)|_{T_x X}$,

$$(\xi \partial_\xi + 2|\xi|^2)\hat{u} = 0, \quad \hat{u}(0) = 1.$$

Then by standard results in ordinary differential equations, the expression in (7.4) is the unique decaying solution.

Now we may iterate this to solve up to higher order. Assume we have found G_0, \dots, G_k satisfying

$$t(\partial_t + \Delta_X)G_j = R_j,$$

where R_j is of order $-3 - j$ at F_X . To find $G_{k+1} = G_k - T_k$, we wish to solve

$$t(\partial_t + \Delta_X)T_k = R_k + R_{k+1},$$

where we have already found R_k of order $-3 - k$, and R_{k+1} will be of order $-4 - k$. Lifting to TX this becomes

$$\left[\sigma(\Delta_X) - \frac{1}{2}(R + n - j - 1) \right] N(T_k) = N(R_k),$$

which we may again solve via Fourier transform. Letting $u = N(T_k)$ and $f = N(R_k)$, we find

$$\hat{u}(\xi) = \int_0^1 \exp((r - 1)|\xi|^2) \hat{f}(r\xi) r^{k+1} dr$$

is the desired solution. This completes the inductive construction for all k . Now the successive $T_j = G_{j+1} - G_j$ give a formal power series at F_X which can be summed by Borel’s lemma so that G is order -2 at F_X , and satisfies (7.2). We then set the acc model heat kernel $H_1 = G$ in a neighborhood of F_X .

7.3.4. Asymptotically conic convergence model heat kernel construction off diagonal at $t = 0$ Consider $F_{1010,2} \cong \bar{Z}_{b,h,\tau}^2$. This face has the following geometry.

Boundary face	Geometry of face	Arising from
F_{bd2}	$PN^+(\Delta(Z \times Z))$	Parabolic blowup of diagonal at $\tau = 0$
F_{b110}	$N^+(Y \times Y) \times \mathbb{R}^+$	Blowup of $Y \times Y$ for all τ
F_{b100}	$Y \times Z \times \mathbb{R}^+$	Boundary in first copy of Z
F_{b010}	$Z \times Y \times \mathbb{R}^+$	Boundary in second copy of Z
F_{b001}	$(Z \times Z) - \Delta(Z \times Z)$	$\tau = 0$ Away from diagonal

At this face, H_1 is asymptotic to $H_b(z, z', \tau)$, which vanishes to infinite order at the side faces F_{b100} and F_{b010} . At the diagonal face F_{bd2} , we have solved H_1 up to error vanishing to infinite order in t . So, we have at this point an approximation H_1 whose error vanishes to infinite order on the interior of $F_{1010,2}$ and at all boundary faces except F_{b110} . The indicial operator for $\Delta_{b,\sigma}$ at F_{b110} is

$$I(\Delta_{b,\sigma}) = (\sigma\partial_\sigma)^2 + \Delta_y$$

on $\mathbb{R}_\sigma^+ \times Y$. We would like to find u which is polyhomogeneous on $\bar{Z}_{b,h,\tau}^2$ and solves

$$(\partial_\tau + \Delta_b)|_{F_{b110}} u = -K|_{F_{b110}},$$

where $K(z, z', t, \epsilon) = (\partial_t + \Delta_\epsilon)H_1$. Since H_1 is polyhomogeneous on \mathcal{H} and smooth up to F_{b110} , so also the error term K . Expanding K , where we use

simply ρ for the projective defining function for F_{b110} ,

$$K|_{F_{b110}} \sim \sum_{j \geq 0} (\rho)^j k_j, \quad k_j \in \mathcal{C}_0^\infty.$$

Expanding the desired solution u ,

$$u \sim \sum_{j \geq 0} (\rho)^j u_j.$$

We may use either separation of variables expanding in eigenfunctions of Δ_y or the Mellin transform [33], to find u_0 satisfying

$$(\partial_\tau + I(\Delta_b))u_0 = -k_0,$$

with u_0 vanishing to infinite order at the side faces F_{b100} and F_{b010} . Since $\Delta_b - I(\Delta_b) = \rho L_1$, where L_1 is also a b -differential operator, we may now iteratively solve for u_1, u_2, \dots , to solve the equation to increasingly higher order. Recall that in the projective coordinates near this face, σ' defines F_{b110} , and since the operator does not differentiate with respect to $\sigma' = r'$, the defining function commutes past the operator. Using Borel summation, we construct u so that

$$(\partial_\tau + \Delta_b)u = -K + K_2,$$

where K_2 vanishes to infinite order at F_{b110} . Using a smooth cutoff function χ supported in a neighborhood of these faces, the second approximation $H_2 = H_1 + \chi u$ now satisfies

$$(\partial_\tau + \Delta_b)H_2 = K_2,$$

where K_2 vanishes to infinite order on both the interior and all boundary faces of $F_{1010,2}$.

A similar exact construction applies to $F_{1111,2}$, since this face is the heat space for the exact cone over (Y, h) .

7.3.5. Asymptotically conic convergence conic triple heat space

Since the error now vanishes to infinite order at all boundary faces except F_{0101} , we restrict attention to this face. It is convenient to use the conic heat calculus composition rule which requires the conic triple space. We construct a partial acc triple heat space,¹⁷ the acc conic triple heat space,

¹⁷The full acc triple heat space is constructed in the second appendix although it is not needed for this proof.

which contains the conic triple heat space, thereby allowing us to solve away the error at F_{0101} using the conic composition rule. The acc conic triple heat space \mathcal{H}_c^3 is a submanifold constructed from $\mathcal{S}^3 \times \mathbb{R}^+ \times \mathbb{R}^+$ by eight blowups. Let X, X', X'' denote the three copies of the submanifold X in X^3 . The blowups are listed in the following table by the order in which the blowups are performed together with the name of the face created.

Blowup	Face
$Y \times Y' \times Y'' \times \{t = 0, t' = 0\}, dt, dt'$	F_{11122}
$Y \times Y' \times \{t = 0\}, dt$	F_{11020}
$Y' \times Y'' \times \{t' = 0\}, dt'$	F_{01102}
$Y \times Y'' \times \{t'' = t - t' = 0\}, dt''$	F_{10122}
$\Delta(\mathcal{S} \times \mathcal{S}' \times \mathcal{S}'') \times \{t = 0, t' = 0\}, dt, dt'$	F_{d3}
$\Delta(\mathcal{S} \times \mathcal{S}') \times \{t = 0\}, dt$	F_{d20}
$\Delta(\mathcal{S}' \times \mathcal{S}'') \times \{t' = 0\}, dt'$	F_{d02}
$\Delta(\mathcal{S} \times \mathcal{S}'') \times \{t'' = 0\}, dt''$	F_{d22}

Let β^*H_2 be the lift of H_2 to \mathcal{H}_c^3 , and let β^*K_2 be the lift of K_2 to \mathcal{H}_c^3 . Then, β^*K_2 vanishes to infinite order at all boundary faces except those arising from the lift of F_{0101} . Now let

$$H_3 := \beta_*(\beta^*H_2 - \beta^*H_2\beta^*K_2),$$

where β_* is the push forward to \mathcal{H} from \mathcal{H}_c^3 . Since β^*K_2 vanishes to infinite order at all boundary faces except F_{0101} , the push forward of $(\beta^*H_2)(\beta^*K_2)$ to \mathcal{H} vanishes to infinite order at all boundary faces except F_{0101} , where the result is given by the conic heat calculus composition rule. Consequently,

$$H_3 = H_2 - \beta_*(\beta^*H_2\beta^*K_2)$$

vanishes to higher order at the boundary faces of F_{0101} by the conic heat calculus composition rule. Continuing this construction and using the Borel summation, we arrive at H_∞ with expansion asymptotic to H_2, H_3, \dots and satisfying

$$(\partial_\tau + \Delta_0)H_\infty = K_\infty,$$

where K_∞ now vanishes to infinite order on F_{0101} . Using a smooth cutoff function, we now have H_∞ defined on all of \mathcal{H} satisfying

$$(\partial_t + \Delta_\epsilon)H_\infty = K_\infty,$$

where K_∞ vanishes to infinite order at all boundary faces of \mathcal{H} .

7.3.6. Error term approximation For each $\epsilon > 0$, let $E(z, z', t, \epsilon) = H_\epsilon(z, z', t) - H_1(z, z', t, \epsilon)$. Let K be defined for each $\epsilon > 0$ by

$$(\partial_t + \Delta_\epsilon)E(z, z', t, \epsilon) = K(z, z', t, \epsilon).$$

By construction of H_1 , $K = O(\epsilon t^\infty)$ as $\epsilon, t \rightarrow 0$, so for any $N \in \mathbb{N}$, there is $C > 0$ such that for any $(z, z') \in M \times M$,

$$|K(z, z', t, \epsilon)| < C\epsilon t^N.$$

Moreover, K has a polyhomogeneous expansion down to $\epsilon = 0$.

For each $\epsilon > 0$, E is smooth on \mathcal{H} for $t > 0$ by parabolic regularity applied for each $\epsilon > 0$ since K is $O(t^\infty)$. By construction, E is smooth down to $t = 0$, so $E(z, z', t, \epsilon)$ is smooth on the blown-down space, $M \times M \times \mathbb{R}^+ \times (0, 1]_\epsilon$. The following maximum principle argument on $M \times [0, T]_t$ shows that E is also $O(\epsilon t^\infty)$ as $\epsilon, t \rightarrow 0$ in the same sense as K .

Fix $\epsilon > 0$, $z' \in M$. Since $K = O(\epsilon t^\infty)$, fix $C > 1$ and $\mathbb{N} \ni N \gg 1$ such that $|K(z, z', t, \epsilon)|^2 \leq C\epsilon^2 t^{2N}$ for all $z \in M$. Let $u(z, t) = |E(z, z', t, \epsilon)|^2$. Let Δ be the scalar Laplacian for (M, g_ϵ) . Then u satisfies

$$\begin{aligned} (\partial_t + \Delta)u &= 2\langle (\partial_t + \nabla^* \nabla)E, E \rangle - |\nabla E|^2 = 2\langle K - \mathcal{R}E, E \rangle - |\nabla E|^2 \\ &\leq 2\langle K, E \rangle \leq 2|K||E| \leq |K|^2 + |E|^2 = |K|^2 + u. \end{aligned}$$

Above we have used the positivity of \mathcal{R} and the compatibility of the bundle connection with the metric. Now, let $\tilde{u} = e^{-t}u$. Then \tilde{u} satisfies

$$(\partial_t + \Delta)\tilde{u} \leq e^{-t}|K|^2 \leq C\epsilon^2 t^{2N}.$$

Let $w = \tilde{u} - C\epsilon^2 t^{2N+1}$. Since E and hence u and \tilde{u} vanish at $t = 0$, $w|_{t=0} = 0$, and w satisfies

$$(\partial_t + \Delta)w \leq C\epsilon^2 t^{2N} - C(2N + 1)\epsilon^2 t^{2N} < 0.$$

Fix $T > 0$ and consider w on $M \times [0, T]_t$. If w has a local maximum for $z \in M$ and $t \in (0, T)$, then

$$(\partial_t + \Delta)w > 0,$$

and this is a contradiction. If w has a maximum at $t = T$, then $\partial_t w \geq 0$, and

$$(\partial_t + \Delta)w > 0,$$

which is again a contradiction. Therefore, the maximum of w occurs at $t = 0$ and so

$$w \leq C\epsilon^2 t^{2N+1}.$$

This implies

$$u \leq e^T C\epsilon^2 t^{2N}, \quad \text{for } 0 < t \leq T,$$

which in turn implies that $E = O(\epsilon t^N)$ as $\epsilon, t \rightarrow 0$, for any $N \in \mathbb{N}$.

7.3.7. Solving away the residual error term To complete this construction, we must remove the residual error term which vanishes to infinite order at the boundary faces of \mathcal{H} . It is now convenient to consider the elements of the acc heat calculus as t -convolution operators acting on $\mathcal{S} \times \mathbb{R}^+$. For an element A which vanishes to infinite order at the boundary faces of \mathcal{H} and u , a smooth half-density section of $\mathcal{S} \times \mathbb{R}^+$, the t -action of A on u is

$$(7.5) \quad Au(t) = \int_0^t \langle Au(t-s), u(s) \rangle ds,$$

where the spatial variables have been suppressed. As a function of s' , $s \geq 0$, $[Au(s')](s)$ vanishes to infinite order at $s' = 0$. Restricting to $s' = t - s$,

$$[Au(t-s)](s) = s^{-k/2-1} (t-s)^j u_{k,j}(t-s, s),$$

for any $-k, j \in \mathbb{N}_0$, where $u_{k,j}$ is a smooth half-density, so for any $-k \geq 1$ this is integrable. Consequently, $Au(t)$ as in (7.5) is smooth in t and vanishes rapidly as $t \rightarrow 0$. So, an element A of the acc heat calculus which vanishes to infinite order at all boundary faces of \mathcal{H} gives rise to a Volterra operator. Since as a t -convolution operator we have

$$(\partial_t + \Delta)H_\infty = Id - K_\infty,$$

we would like to invert $(Id - K_\infty)$. Formally, the inverse should be

$$(Id - K_\infty)^{-1} = \sum_{j \geq 0} K_\infty^j,$$

where K_∞^j is the j -fold composition of K_∞ . To show that this Neumann series converges, we estimate the kernel of K_∞^j . Since K_∞ vanishes to infinite order at all boundary faces of \mathcal{H} , we may restrict to submanifolds of \mathcal{H} , estimating as in [33] and then combine these estimates to estimate K_∞^j

on \mathcal{H} . The kernel k_∞^j of the restriction of K_∞^j to $M \times M \times \mathbb{R}^+ \times \{\epsilon\}$ is bounded by

$$|k_\infty^j(z, z', t, \epsilon)| \leq C_{\epsilon, j} \frac{t^j}{(j+1)!}, \quad t < T.$$

This follows from the composition rule for the heat calculus on M and the analogous bound in [33, 7.3], where we have taken $k = -2$, with k as above, which we are free to choose since A vanishes to infinite order. Similarly, by the composition rule for the heat calculus on M_0 and the same estimate of [33], the kernel of the restriction of K_∞^j to $M_0 \times M_0 \times \mathbb{R}^+$ is bounded by

$$|k_\infty^j(z, z', t)|_{F_{0101}} \leq C_{0, j} \frac{t^j}{(j+1)!}, \quad t < T.$$

Similarly, the kernel of the restriction of K_∞^j to $\bar{Z} \times \bar{Z} \times \mathbb{R}_+^d$ is bounded by

$$|k_\infty^j(z, z', t)|_{F_{1010, 2}} \leq C_{z, j} \frac{t^j}{(j+1)!}, \quad t < T.$$

These three bounds imply that the constants $C_{\epsilon, j}$ stay bounded as $\epsilon \rightarrow 0$, and so we have the following global bound for the kernel of K_∞^j on both \mathcal{H} and the blown-down space $\{\epsilon = \epsilon'\} \subset \mathcal{S} \times \mathcal{S} \times \mathbb{R}^+$

$$|k_\infty^j(z, z', t, \epsilon)| \leq C_j \frac{t^j}{(j+1)!}, \quad t < T.$$

It follows that the Neumann series for $(\text{Id} - K_\infty)^{-1}$ is summable and has an inverse which as a t -convolution operator is also of the form $(\text{Id} - A)$, where A is an element of the acc heat calculus that vanishes to infinite order at the boundary faces of \mathcal{H} . Then, the full acc heat kernel is

$$H = H_\infty(\text{Id} - K_\infty)^{-1}.$$

As a consequence of this construction, H has a fully polyhomogeneous expansion down to $\epsilon = 0$ with leading order terms given by the acc model heat kernel, H_1 .

Remarks. A consequence of this theorem is the uniform convergence for $0 < t < T$,

$$H_\epsilon \rightarrow H_0 + O(\epsilon) \text{ as } \epsilon \rightarrow 0.$$

This extends the convergence results of [11] for scalar heat kernels to heat kernels for geometric Laplacians acting on vector bundles. Due to the rescaled time variables at the faces contained in $\epsilon, t = 0$, we expect to see interesting applications involving the short time asymptotic behavior of the heat trace; see [31]. We have not computed the $t \rightarrow \infty$ asymptotics but expect this will also yield interesting applications.

A. Asymptotically conic scattering heat kernel

Let (\bar{Z}, g_z) be an ac scattering metric with boundary defined by $\{x = 0\}$ and local coordinates (x, y) near the boundary. Let Δ_z be the Friedrichs extension of a geometric Laplacian on \bar{Z} . We motivate the definition of the acc heat space by lifting the Euclidean heat kernel to \bar{Z}_+^2 .

Recall the Euclidean heat kernel for \mathbb{R}^n ,

$$G(z, z', t) = (4\pi t)^{-n/2} e^{-|z-z'|^2/2t}.$$

Consider the coordinates $z = (r, y)$ on the complete ac space. With the compactification given by $x = \frac{1}{r}$ in the local coordinates (x, y, x', y', t) on \bar{Z}_+^2 near the boundary of \bar{Z} , the Euclidean heat kernel is

$$G(x, y, x', y', t) = (4\pi t)^{-n/2} \exp\left(-\frac{(|(1/x) - (1/x')|^2 + |y - y'|^2)}{2t}\right).$$

This motivates blowing up

$$\{(x, y, x', y', t) : x = 0, x' = 0\}.$$

In the projective coordinates $s = \frac{x}{x'}$ and $s' = x'$, the Euclidean heat kernel is

$$G(s, y, s', y', t) = (4\pi t)^{-n/2} \exp\left(-\frac{(|(s - 1)/ss'|^2 + |y - y'|^2)}{2t}\right).$$

This motivates a second blowup at $s = 1$, along the submanifold where the diagonal in $\bar{Z} \times \bar{Z}$ meets the first blown-up face

$$\{(x, y, x', y', t) : x = 0, x' = 0, y = y'\}.$$

A.1. Asymptotically conic scattering heat space

As motivated above, the ac scattering heat space is constructed from \bar{Z}_+^2 by performing three blowups.

The scattering double space, \bar{Z}_{sc}^2 , originally constructed by [17], is

$$\bar{Z}_{sc}^2 := [[\bar{Z} \times \bar{Z}; \partial\bar{Z} \times \partial\bar{Z}]; \Delta(Y \times Y) \cap F_{110}],$$

where F_{110} is the face created by the first blowup. Including the time variable, we perform one more blowup to construct the ac scattering heat space,

$$\bar{Z}_{sc,h}^2 = [\bar{Z}_{sc}^2 \times \mathbb{R}^+; \Delta(Z \times Z) \times \{t = 0\}, dt].$$

The ac scattering heat space has six boundary faces described in the following table.

Face	Geometry of face	Defining function in local coordinates
F_{220}	$N^+(\Delta(Y \times Y)) \times \mathbb{R}^+$	$\rho_{220} = (x^2 + (x')^2 + y - y' ^2)^{1/2}$
F_{220}	$N^+(Y \times Y)$	
	$-\Delta(Y \times Y) \times \mathbb{R}^+$	$\rho_{110} = (x^2 + (x')^2)^{1/2}$
F_{100}	$Z \times Y \times \mathbb{R}^+$	$\rho_{100} = x$
F_{010}	$Y \times Z \times \mathbb{R}^+$	$\rho_{010} = x'$
F_{d2}	$PN_t^+(\Delta(Z \times Z))$	$\rho_{d2} = (z - z' ^4 + t^2)^{1/2}$
F_{001}	$(Z \times Z) - \Delta(Z \times Z)$	$\rho_{001} = t$

A.2. Asymptotically conic scattering heat calculus

Elements of the ac scattering heat calculus are distributional section half-densities on \bar{Z}_+^2 which are smooth on the interior and lift to be polyhomogeneous on $\bar{Z}_{sc,h}^2$. Let μ be a smooth, non-vanishing half-density on $\bar{Z} \times \bar{Z} \times \mathbb{R}^+$, and let ν be a smooth, non-vanishing half-density on $\bar{Z}_{sc,h}^2$.

Definition A.1. For any $k \in \mathbb{R}$ and index sets E_{110} and E_{220} , $A \in \Psi_{sc,H}^{E_{110}, E_{220}, k}$, if A is smooth on the interior of $\bar{Z}_{sc,h}^2$ and satisfies the following.

1. $A \in \mathcal{A}_{phg}^{-(1/2)+E_{110}}$ at F_{110} .
2. $A \in \mathcal{A}_{phg}^{(-(n+2)/2)+E_{220}}$ at F_{220} .

- 3. A vanishes to infinite order at F_{001} , F_{100} and F_{010} .
- 4. $A \in \rho_{d2}^{-((n+3)/2)-k} C^\infty(F_{d2})$.

Elements of the ac scattering heat calculus act on sections of \bar{Z} in the usual way and on sections of $Z \times \mathbb{R}_t^+$ by t -convolution. The composition rule is proven using the ac scattering triple heat space, $\bar{Z}_{sc,h}^3$. This space has partial blow-down and projection maps to three identical copies of the ac scattering heat space as well as full blow-down and projection maps to three identical copies of \bar{Z}_+^2 ; these are called the left, right and center. Formally, two elements of the ac heat calculus are composed by lifting from the left and right copies of $\bar{Z}_{sc,h}^2$ to $\bar{Z}_{sc,h}^3$, then multiplying and blowing down/projecting to the center copy of $\bar{Z}_{sc,h}^2$. It is key that the triple space be constructed so that these lifts and push-forward maps are b -fibrations in order that polyhomogeneity be preserved.

A.2.1. Asymptotically conic scattering triple heat space In a neighborhood of the boundary in each copy of \bar{Z} , we have the local coordinates (x, y) , which provide the local coordinates (x, y, x', y', x'', y'') on \bar{Z}^3 . First, we blow up the codimension 3 corner defined by $\{x = 0, x' = 0, x'' = 0\}$, resulting in the face F_{11100} with defining function locally given by

$$\rho_{11100} = (x^2 + (x')^2 + (x'')^2)^{1/2}.$$

Next, we blow up the three codimension 2 corners corresponding to the F_{110} faces in each of the three copies of $\bar{Z}_{sc,h}^2$. These faces are as follows.

Face	Submanifold to be blown up	Defining function
F_{11000}	$S_{11000} = \{x = 0, x' = 0\} - F_{11100}$	$\rho_{11000} = (x^2 + (x')^2)^{1/2}$
F_{01100}	$S_{01100} = \{x' = 0, x'' = 0\} - F_{11100}$	$\rho_{01100} = ((x')^2 + (x'')^2)^{1/2}$
F_{10100}	$S_{10100} = \{x = 0, x'' = 0\} - F_{11100}$	$\rho_{10100} = ((x)^2 + (x'')^2)^{1/2}$

Next we blow up the codimension $2n + 1$ corner where the diagonals meet F_{11100} . After the F_{11100} blowup, we have coordinates $(\theta, \theta', \theta'', y, y', y'', \rho_{11100})$, with

$$x = (\rho_{11100})\theta, \quad x' = (\rho_{11100})\theta', \quad x'' = (\rho_{11100})\theta'', \quad (\theta)^2 + (\theta')^2 + (\theta'')^2 = 1.$$

Using these coordinates, we next blow up

$$S_{22200} = \{\theta = \theta' = \theta'', y = y' = y'', r_0 = 0\}.$$

The face created by this blowup is called F_{22200} with defining function

$$\rho_{22200} = ((\theta - \theta')^2 + (\theta' - \theta'')^2 + |y - y'|^2 + |y' - y''|^2 + r_0^2)^{1/2}.$$

After this, we blow up the three codimension n corners corresponding to the F_{220} faces in the three copies of the double heat space. These are as follows.

Face	Submanifold to be blown up	Defining function
F_{22000}	$S_{22000} = \{\theta = 0, \theta' = 0, y = y'\}$	$\rho_{22000} = (\theta^2 + (\theta')^2 + y - y' ^2)^{1/2}$
F_{02200}	$S_{02200} = \{\theta' = 0, \theta'' = 0, y' = y''\}$	$\rho_{02200} = ((\theta')^2 + (\theta'')^2 + y' - y'' ^2)^{1/2}$
F_{20200}	$S_{20200} = \{\theta = 0, \theta'' = 0, y = y''\}$	$\rho_{20200} = ((\theta)^2 + (\theta'')^2 + y - y'' ^2)^{1/2}$

We have now constructed the ac scattering triple space, \bar{Z}_{sc}^3 . We next introduce the time variables and perform the parabolic temporal diagonal blowups. We must first blow up the codimension 2 corner of $\mathbb{R}^+ \times \mathbb{R}^+$ to preserve symmetry. Let

$$\mathcal{T}_0^2 = [\mathbb{R}^+ \times \mathbb{R}^+; t = t' = 0].$$

The defining function for the blowup of $\{t = t' = 0\}$ is ρ_{00011} , which we call t'' because it plays the role of the third time variable. We now take $Z_{sc}^3 \times \mathcal{T}_0^2$ and blow up the temporal diagonal faces. First, we blow up the codimension $2n + 3$ triple diagonal, S_{d3} , defined by

$$\{z = z' = z'', t'' = 0\}.$$

The defining function of this face is ρ_{d3} ,

$$\rho_{d3} = (|z - z'|^4 + |z - z''|^4 + (t'')^2)^{1/4}.$$

Next, we blow up the three temporal diagonals corresponding to the diagonal faces in the three copies of the double heat space. These are as follows.

Face	Submanifold to be blown up	Defining function
F_{d20}	$S_{d20} = \{z = z'\}$	$\rho_{d20} = (z - z' ^4 + t^2)^{1/4}$
F_{d02}	$S_{d02} = \{z' = z''\}$	$\rho_{d02} = (z' - z'' ^4 + (t')^2)^{1/4}$
F_{d22}	$S_{d22} = \{z = z''\}$	$\rho_{d22} = (z - z'' ^4 + (t'')^2)^{1/4}$

We have now constructed the ac scattering triple heat space and proceed with the composition rule.

Technical Theorem A.2. *Let $A \in \Psi_{sc,H}^{A_{110},A_{220},k_a}$ and $B \in \Psi_{sc,H}^{B_{110},B_{220},k_b}$. Then, the composition $B \circ A$ is an element of $\Psi_{sc,H}^{A_{110} \cup B_{110}, A_{220} \cup B_{220}, k_a + k_b}$.*

A.2.2. Proof. Formally we have,

$$(A.1) \quad \kappa_{B \circ A} \nu = (\beta_C)_* ((\beta_R)^* (\kappa_{A\nu}) (\beta_L)^* (\kappa_{B\nu})).$$

Multiplying both sides of (A.1) by ν and using the fact that $(\beta_c)_* (\beta_c)^* (\nu) = \nu$,

$$(A.2) \quad \kappa_{B \circ A} \nu^2 = (\beta_C)_* ((\beta_R)^* (\kappa_{A\nu}) (\beta_L)^* (\kappa_{B\nu}) (\beta_c)^* (\nu)).$$

Next we calculate the lifts of the defining functions and half densities from $\bar{Z}_{sc,h}^2$ to $\bar{Z}_{sc,h}^3$. A calculation gives the half density on the heat space ν in terms of the half density μ on \bar{Z}_+^2 ,

$$\nu = (\beta_h)^* \left((\rho_{110})^{-1/2} (\rho_{220})^{-n/2} (\rho_{d2})^{-n+1/2} \mu \right).$$

The ac scattering triple heat space has partial blow-down/projection maps β_L, β_R and β_C to three identical copies of $\bar{Z}_{sc,h}^2$. If we denote the three copies of \bar{Z} by $\bar{Z}, \bar{Z}', \bar{Z}''$ and the three time variables (t, t', t'') , where t'' is from the blowup of $\mathbb{R}^+ \times \mathbb{R}^+$, then the three copies of $\bar{Z}_{sc,h}^2$ are as follows.

Copy of $\bar{Z}_{sc,h}^2$	Associated to in $\bar{Z}_{sc,h}^3$
Left	$\bar{Z} \times \bar{Z}' \times \mathbb{R}_t^+$
Right	$\bar{Z}' \times \bar{Z}'' \times \mathbb{R}_{t'}^+$
Center	$\bar{Z} \times \bar{Z}'' \times \mathbb{R}_{t''}^+$

Next, we compute the lifts of the defining functions for the boundary faces of the heat space to the triple heat space.

Lifting map	Defining function on $\bar{Z}_{sc,h}^2$	Lift to $\bar{Z}_{sc,h}^3$
$(\beta_L)^*$	ρ_{100}	$\rho_{10000}\rho_{10100}$
$(\beta_L)^*$	ρ_{010}	$\rho_{01000}\rho_{01100}$
$(\beta_L)^*$	ρ_{110}	$\rho_{11100}\rho_{11000}$
$(\beta_L)^*$	ρ_{220}	$\rho_{22200}\rho_{22000}$
$(\beta_L)^*$	ρ_{d2}	$\rho_{d3}\rho_{d20}$
$(\beta_L)^*$	ρ_{001}	$\rho_{00010}\rho_{00011}\rho_{d22}$
$(\beta_R)^*$	ρ_{100}	$\rho_{01000}\rho_{01100}$
$(\beta_R)^*$	ρ_{010}	$\rho_{00100}\rho_{10100}$
$(\beta_R)^*$	ρ_{110}	$\rho_{11100}\rho_{01100}$
$(\beta_R)^*$	ρ_{220}	$\rho_{22200}\rho_{02200}$
$(\beta_R)^*$	ρ_{d2}	$\rho_{d3}\rho_{d02}$
$(\beta_R)^*$	ρ_{001}	$\rho_{00001}\rho_{00011}\rho_{d22}$
$(\beta_C)^*$	ρ_{100}	$\rho_{10000}\rho_{11000}$
$(\beta_C)^*$	ρ_{010}	$\rho_{001000}\rho_{01100}$
$(\beta_C)^*$	ρ_{110}	$\rho_{11100}\rho_{10100}$
$(\beta_C)^*$	ρ_{220}	$\rho_{22200}\rho_{20200}$
$(\beta_C)^*$	ρ_{d2}	$\rho_{d3}\rho_{d22}$
$(\beta_C)^*$	ρ_{001}	$\rho_{00022}\rho_{00011}\rho_{d22}$

Then,

$$(\beta_L)^*(\nu) = (\beta_L)^*((\rho_{110})^{-1/2}(\rho_{220})^{-n/2}(\rho_{d2})^{-n+1/2}\mu).$$

Next, we use the fact that

$$(\beta_L)^*(\mu)(\beta_R)^*(\mu)(\beta_C)^*(\mu) = \mu_3^2.$$

Here, μ_3^2 is a smooth density on $\bar{Z} \times \bar{Z} \times \bar{Z} \times \mathbb{R}^+ \times \mathbb{R}^+$, so we may assume

$$\mu_3^2 = dz dz' dz'' dt dt'.$$

A Jacobian calculation gives the lift of μ_3^2 to the triple heat space. First note

$$(\beta_3)^*(x) = (\rho_{11100})(\rho_{11000})(\rho_{10100})(\rho_{10000}),$$

$$\begin{aligned}
 (\beta_3)^*(x') &= (\rho_{11100})(\rho_{11000})(\rho_{01100})(\rho_{01000}), \\
 (\beta_3)^*(x'') &= (\rho_{11100})(\rho_{01100})(\rho_{10100})(\rho_{00100}).
 \end{aligned}$$

This implies

$$\begin{aligned}
 (\beta_3)^*(\mu_3^2) &= (\rho_{11100})^2(\rho_{11000}\rho_{01100}\rho_{10100})(\rho_{22000}\rho_{02200}\rho_{20200})^n \\
 &\quad (\rho_{22200})^{2n+1}(\rho_{d20}\rho_{d02}\rho_{d22})^{n+1}\rho_{d3}^{2n+3}(t'')\nu_3^2.
 \end{aligned}$$

Here, ν_3^2 is a smooth, non-vanishing density on the triple heat space. Combining this with the above lifts, we arrive at the following formula

$$\begin{aligned}
 (\beta_L)^*(\nu)(\beta_R)^*(\nu)(\beta_C)^*(\nu) &= (\rho_{11100})^{1/2}(\rho_{10100}\rho_{01100}\rho_{10100})^{1/2} \\
 (\rho_{22000}\rho_{02200}\rho_{20200})^{n/2}(\rho_{22200})^{(n+1)/2}(\rho_{d3})^{(n+3)/2}(\rho_{d20}\rho_{d02}\rho_{d22})^{(n+1)/2}(t'')\nu_3^2.
 \end{aligned}$$

To use the push-forward theorem of [28], we need to write each of these in terms of b -densities. First, we have on the center copy of $\bar{Z}_{sc,h}^2$,

$${}^b\nu^2 = (\rho_{100}\rho_{010}\rho_{110}\rho_{220}\rho_{001}\rho_{d2})^{-1}\nu^2.$$

Then, we have

$${}^b\nu^2 = (\beta_c)_*(\beta_c)^*((\rho_{100}\rho_{010}\rho_{110}\rho_{220}\rho_{001}\rho_{d2})^{-1}\nu^2).$$

We observe

$$\begin{aligned}
 (\beta_c)^*((\rho_{100}\rho_{010}\rho_{110}\rho_{220}\rho_{001}\rho_{d2})^{-1}) &= \\
 (\rho_{10000}\rho_{00100}\rho_{11000}\rho_{01100}\rho_{10100}\rho_{11100}\rho_{22200}\rho_{02200}\rho_{20200}\rho_{d3}\rho_{d22}\rho_{00011})^{-1}.
 \end{aligned}$$

So now we multiply both sides of (A.2) by $(\beta_c)_*(\beta_c)^*(\rho_{100}\rho_{010}\rho_{110}\rho_{220}\rho_{001}\rho_{d2})^{-1}$ and inside the right side of (A.2) we have

$$\begin{aligned}
 (\rho_{11100}\rho_{11000}\rho_{01100}\rho_{10100})^{-1/2}(\rho_{22000}\rho_{02200}\rho_{22200})^{n/2}(\rho_{20200})^{(n-2)/2} \\
 (\rho_{d3})^{(n+1)/2}(\rho_{d20}\rho_{d02})^{(n+1)/2}(\rho_{d22})^{n/2}(\rho_{10000}\rho_{00100})^{-1}\nu_3^2.
 \end{aligned}$$

To use the push-forward theorem, we must change the density ν_3^2 to a b -density. We observe

$$\begin{aligned}
 {}^b\nu_3^2 &= (\rho_{11100}\rho_{11000}\rho_{01100}\rho_{10100}\rho_{22200}\rho_{22000}\rho_{02200}\rho_{20200} \\
 &\quad \rho_{10000}\rho_{01000}\rho_{00100}\rho_{d3}\rho_{d20}\rho_{d02}\rho_{d22}\rho_{00011}\rho_{00010}\rho_{00001})^{-1}\nu_3^2.
 \end{aligned}$$

So, we now have for the composition formula

$$(\beta_c)_*(\tilde{\kappa}_A \tilde{\kappa}_B (\rho_{11100} \rho_{11000} \rho_{01100} \rho_{10100})^{1/2} (\rho_{22200} \rho_{22000} \rho_{02200})^{(n+2)/2} (\rho_{20200})^{n/2} (\rho_{d3} \rho_{d20} \rho_{d02})^{(n+3)/2} (\rho_{d22})^{(n+1)/2} \rho_{01000} \rho_{00011} \rho_{00010} \rho_{00001} ({}^b\nu_3^2)).$$

We observe the following orders of $\tilde{\kappa}_A$ on $\bar{Z}_{sc,h}^3$.

Face	$\tilde{\kappa}_A$ Index set/leading order
F_{11100}	$-1/2 + A_{110}$
F_{11000}	$-1/2 + A_{220}$
$F_{01100}, F_{10100}, F_{02200}, F_{20200}, F_{d22}$	∞
F_{22200}, F_{22000}	$-(n+2)/2 + A_{220}$
F_{d3}, F_{d20}	$-(n+3)/2 - k_a$
$F_{10000}, F_{01000}, F_{00100}, F_{00010}, F_{00011}$	∞

Similarly, for $\tilde{\kappa}_B$, we have orders as follows.

Face	$\tilde{\kappa}_B$ Index set/leading order
F_{11100}	$-1/2 + B_{110}$
F_{01100}	$-1/2 + B_{220}$
$F_{11000}, F_{10100}, F_{22000}, F_{20200}, F_{d22}$	∞
F_{22200}, F_{02200}	$-(n+2)/2 + B_{220}$
F_{d3}, F_{d02}	$-(n+3)/2 - k_b$
$F_{10000}, F_{01000}, F_{00100}, F_{00010}, F_{00011}$	∞

Now, recalling the formula:

$$(\beta_c)_*(\tilde{\kappa}_A \tilde{\kappa}_B (\rho_{11100} \rho_{11000} \rho_{01100} \rho_{10100})^{1/2} (\rho_{22200} \rho_{22000} \rho_{02200})^{(n+2)/2} (\rho_{20200})^{n/2} (\rho_{d3} \rho_{d20} \rho_{d02})^{(n+3)/2} (\rho_{d22})^{(n+1)/2} \rho_{01000} \rho_{00011} \rho_{00010} \rho_{00001} ({}^b\nu_3^2)).$$

We see that the quantity on the right-hand side to be pushed forward by $(\beta_c)_*$ has the following indices on the boundary faces.

Face	Index set/leading order
F_{11100}	$-1/2 + A_{110} \cup B_{110}$
$F_{11000}, F_{01100}, F_{10100}, F_{22000}, F_{02200}, F_{20200}$	∞
F_{22200}	$-(n+2)/2 + A_{220} \cup B_{220}$
F_{d3}	$-(n+3)/2 - (k_a + k_b)$
$F_{d20}, F_{d02}, F_{d22}$	∞
$F_{10000}, F_{01000}, F_{00100}, F_{00010}, F_{00001}, F_{00011}$	∞

The push forward under $(\beta_c)^*$ sends the boundary faces of $\bar{Z}_{sc,h}^3$ to $\bar{Z}_{sc,h}^2$ as follows.

\bar{Z}_h^3 face	Boundary face of $\bar{Z}_{sc,h}^2$ or interior
F_{11100}	F_{110}
F_{10100}	F_{110}
F_{22200}, F_{20200}	F_{220}
F_{d3}, F_{d22}	F_{d2}
F_{10000}	F_{100}
F_{00100}	F_{010}
F_{00011}	F_{001}
$F_{11000}, F_{01100}, F_{22000}, F_{02200},$	Interior
$F_{d20}, F_{d02}, F_{01000}, F_{00010}, F_{00001}$	Interior

The quantity to be pushed forward is integrable with respect to ${}^b\nu_3^2$ at the faces that are mapped to the interior, so we may apply the push-forward theorem [28] to arrive at the result of the composition rule. The kernel, $\kappa_{B \circ A}$, will have the following polyhomogeneous index sets and leading orders on \bar{Z}_h^2 .

Face of \bar{Z}_h^2	Index set/leading order
F_{110}	$-1/2 + A_{110} \cup B_{110}$
F_{220}	$-(n + 2)/2 + A_{220} \cup B_{220}$
F_{d2}	$-(n + 3)/2 - (k_a + k_b)$
F_{100}	∞
F_{010}	∞
F_{001}	∞

This concludes the proof of the composition rule.

Technical Theorem A.3. *Let (Z, g_z) be an ac scattering space with boundary (Y, h) . Let (E, ∇) be a Hermitian vector bundle over (Z, g_z) so that near the boundary E is the pullback of a bundle over (Y, h) . Let Δ be a geometric Laplacian on (Z, g_z) associated to the bundle (E, ∇) . Then there exists $H \in \Psi_{ac,H}^{E_{110}, E_{220}, -2}$ satisfying*

$$\begin{aligned}
 (\partial_t + \Delta)H(z, z', t) &= 0, \quad t > 0, \\
 H(z, z', 0) &= \delta(z - z').
 \end{aligned}$$

Moreover, H vanishes to infinite order at F_{110} and is smooth up to F_{220} .

On the interior of $\bar{Z}_{ac,h}^2$, the ac scattering model heat kernel is locally defined by the Euclidean heat kernel and a partition of unity. At F_{d2} , we construct the model heat kernel explicitly using the jet of the metric at the base point of each fiber. At F_{001} , the model heat kernel vanishes to infinite order. At F_{110} and F_{220} , the model heat kernel is the lift of the Euclidean heat kernel. Then the ac scattering model heat kernel H_1 satisfies

$$(\partial_t + \Delta)H_1 = K_1,$$

where K_1 vanishes to positive order on the boundary faces of $\bar{Z}_{sc,h}^2$. We now define

$$H_2 = H_1 - H_1 K_1,$$

with

$$(\partial_t + \Delta)H_2 = K_2,$$

where K_2 vanishes to one order higher on the boundary faces of $\bar{Z}_{sc,h}^2$. Similarly,

$$H_3 := H_2 - H_2 K_2.$$

Using the Borel summation, we construct H_∞ with expansion asymptotic to H_1, H_2, H_3, \dots and satisfying

$$(\partial_t + \Delta)H_\infty = K,$$

where now K vanishes to infinite order on the boundary faces of $\bar{Z}_{sc,h}^2$. As a t -convolution operator, we wish to have

$$H_\infty = \text{Id}.$$

However, we currently have

$$H_\infty = \text{Id} + K,$$

but this is not a problem since $(\text{Id} + K)$ is invertible with inverse of the same form. Then the ac scattering heat kernel

$$H = H_\infty(\text{Id} - K)^{-1}$$

is an element of the ac scattering heat calculus with leading orders on the boundary faces of $\bar{Z}_{sc,h}^2$ determined by those of the model kernel.

B. Asymptotically conic convergence triple heat space

Let

$$T := \mathbb{[[[\mathcal{S} \times \mathcal{S} \times \mathcal{S}; Y \times Y' \times Y'']; Y \times Y']; Y' \times Y'']; Y \times Y''],$$

where we have used Y, Y', Y'' to denote the three copies of Y in \mathcal{S}^3 . Let

$$\begin{aligned} \mathcal{T} := \{p \in T : f_i(p) = 0, i = 1, 2\}, \quad f_1(p) &= x(p)r(p) - x'(p)r'(p), \\ f_2(p) &= x'(p)r'(p) - x''(p)r''(p). \end{aligned}$$

Like the acc double and heat space, \mathcal{T} is a smooth manifold with corners.

Let

$$\mathbb{R}_{2,b}^+ := [\mathbb{R}_t^+ \times \mathbb{R}_s^+; \{0\} \times \{0\}].$$

Then the acc triple heat space is constructed from $\mathcal{T} \times \mathbb{R}_{2,b}^+$ by blowing up along 12 submanifolds, creating the following 12 boundary faces. Below, let tD be the lift to \mathcal{T} of the diagonal in $\mathcal{S} \times \mathcal{S}' \times \mathcal{S}''$, let D_{110} be the lift of the diagonal in $\mathcal{S} \times \mathcal{S}'$, D_{011} be the lift of the diagonal in $\mathcal{S}' \times \mathcal{S}''$, and D_{101} be the lift of the diagonal in $\mathcal{S} \times \mathcal{S}''$.

Submanifold blown up	Face created
$Y \times Y' \times Y'' \times \{0\} \times \{0\}, dt, ds$	S_{11122}
$Y \times Y' \times \{t = 0\}, dt$	$S_{11020},$
$Y' \times Y'' \times \{s = 0\}, ds$	S_{01102}
$Y \times Y'' \times \{t = s = 0\}, ds, dt$	S_{10122}
$Y \times Y' \times Y''$	S_{111}
$Y \times Y'$	S_{110}
$Y' \times Y''$	S_{011}
$Y \times Y''$	S_{101}
$tD \times \{t, s = 0\}, ds, dt$	S_{td}
$D_{110} \times \{t = 0\}, dt$	S_{d20}
$D_{011} \times \{s = 0\}, ds$	S_{d02}
$D_{101} \times \{s = t = 0\}, ds, dt$	S_{d22}

As constructed, the acc triple heat space has full and partial projection/blow-down maps to three identical copies of the acc heat space, the left, right and center, and to three corresponding copies of the blown-down space $\{\epsilon = \epsilon'\} \subset \mathcal{S}^2 \times \mathbb{R}^+$. To compose two elements A and B , we view the element A as acting from the left to the right while B acts from the right to the center. Formally, the composition $B \circ A$ is the push forward from the acc triple heat

space of the product of the lifts of A and B . Compatibility assumptions on the leading orders of A and B at boundary faces of the acc heat space are required so that we can push forward. With these assumptions and with the possible inclusion of normalizing factors at boundary faces of the acc heat space, two elements compose as one would expect. The technical details in the proof of this composition rule are expected to be analogous to the technical details in the proof of the ac scattering heat calculus composition rule (Appendix A). Although the full acc triple heat space and composition rule are not necessary in our analysis of the heat kernels, its construction may be an instructive model for employing our methods and those of [33] to construct solution kernels of geometric operators under conic and more general metric degenerations.

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RECEIVED DECEMBER 15, 2006