

Homotopy problems for harmonic maps to spaces of nonpositive curvature

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Dedicated to the memory of Professor Aurel Cornea

Two results by N.J. Korevaar and R.M. Schoen (1993) about solvability of the free homotopy problem and the Dirichlet homotopy problem for harmonic maps from a compact Riemannian manifold X to a compact geodesic space Y of nonpositive curvature are strengthened here, and extended, particularly by allowing X to be any compact admissible Riemannian polyhedron. In that setting, the two results were stated in the author's monograph from 2001 with J. Eells, but the short proofs given there were incomplete.

1. Introduction

As shown in the seminal paper by Eells and Sampson [10], if X and Y are compact Riemannian manifolds without boundary, and if Y has nonpositive sectional curvature, then every smooth map $X \rightarrow Y$ is homotopic with a harmonic map which has minimum energy in its homotopy class. This was extended by Hamilton [20] to the case where X has a smooth boundary bX , on which the maps shall be kept fixed throughout the homotopy — the so-called Dirichlet homotopy problem. Uniqueness results were obtained by Hartman [21]. While the stated results were established using the heat equation method, new proofs were given by Schoen [34] by a variational approach. For related results, see Schoen and Yau [39].

A new development was initiated by Gromov and Schoen [19], who extended the above results to the case of maps into a Riemannian space with singularities, more precisely, a compact Riemannian polyhedron of nonpositive curvature in the sense of A.D. Alexandrov, and embedded isometrically in some \mathbb{R}^N . In that setting (and in similar settings below), a harmonic map is defined to be a *continuous locally energy minimizing map* (see Remark 2.6 and Lemma 2.7 in the present paper for similar and, actually, equivalent definitions of harmonicity). Furthermore, local Lipschitz continuity of harmonic maps was established in [19]. These results were further extended

by Korevaar and Schoen [25] to the still more general case of a compact geodesic space target Y , again of nonpositive Alexandrov curvature.

In the present paper we keep the compact geodesic space target Y of nonpositive curvature. As source space we allow any compact Riemannian polyhedron which is admissible in the sense of [9, 18]. Lipschitz continuity of harmonic maps must then necessarily be replaced by Hölder continuity; cf. [6], [9, Example 6.1].

In the case of free homotopy there is no prescribed boundary map, and a minimizing sequence of equivariant maps may diverge to infinity in the universal cover \tilde{Y} of Y ; cf. [25, Remark 2.6.6]. We adapt (for Theorem 3.2) the impressive existence proof by Korevaar and Schoen of their result [25, Theorems 2.6.4 and 2.7.1] to the present case of a polyhedral domain X , but we do not require that $bX = \emptyset$. We further go beyond [25] by proving not only that, as expected from [10] and [19], every homotopy class \mathcal{H} of continuous maps $X \rightarrow Y$ has an energy minimizer which is harmonic, but that indeed every energy minimizer in \mathcal{H} is harmonic. As expected from Hartman [21], two energy minimizers in \mathcal{H} are identical if they agree at a point. As an extension of another uniqueness result of [21] it is shown by Mese [30] that if Y has Alexandrov curvature $\leq \kappa$ for some constant $\kappa < 0$ (and if X is a manifold) then an energy minimizer in \mathcal{H} is unique unless its image is contained in a geodesic of Y .

For Hölder continuity of local energy minimizers we use [9] and [11], drawing on Jost [23]. We now obtain a more uniform estimate involving the energy (Proposition 3.1 and Remark 3.3). For Hölder continuity of harmonic maps in different settings, see [17, 29, 36, 38, 41].

For the Dirichlet homotopy problem it is supposed that $bX \neq \emptyset$. We allow (in Theorem 4.1) $X \setminus bX$ to be replaced more generally (as domain of harmonicity of the solution) by any connected *regular* open set $\Omega \subset X$ such that $\Omega \subset X \setminus bX$, or at least that $\Omega \cap bX$ is not dense in bX . Regularity of Ω is understood in the sense of potential theory — a necessary, fairly mild local condition on $\partial\Omega$; and $X \setminus bX$ is itself always connected, open, and regular. We show that, as expected from Hamilton's theorem [20] and from [21], every homotopy class \mathcal{H} of continuous maps $X \rightarrow Y$ which agree on $X \setminus \Omega$ with a prescribed continuous map $\chi : X \rightarrow Y$ of finite energy has a unique energy minimizer φ_0 , and φ_0 is harmonic and Hölder continuous in Ω . The existence of an energy minimizer φ_0 can be established in the same way as in the free homotopy case, as briefly explained in [25, p. 658] (where the source space X is a compact Riemannian manifold with smooth boundary ∂X , and where $\Omega = X \setminus \partial X$). Our existence proof of Theorem 4.1 (Dirichlet homotopy) is simpler and more elementary. By application of

the Banach–Saks theorem it is shown that the Dirichlet boundary condition prevents a minimizing sequence of equivariant maps from escaping to infinity in the universal cover \tilde{Y} . Continuity of the minimizer φ_0 up to the boundary of Ω , and harmonicity in Ω , are reduced to the same for the solution to the ordinary variational Dirichlet problem for maps into a simply connected space such as \tilde{Y} , as established in [14] (see also [16]) by use of potential theory with respect to the H. Cartan fine topology; this replaces the use of a result of Serbinowski [35] in [25, p. 658] for proving in their setting that φ is Hölder continuous C^α up to the boundary (supposed smooth), provided that χ is so. When specialized to the smooth source manifold in [25, Theorem 2.7.2] our result involving continuous boundary data neither implies the quoted result from [25], nor is it implied thereby.

Theorem 3.2 (free homotopy) and Theorem 4.1 (for the particular case $\Omega = X \setminus bX$) were both stated in [9, Theorems 11.1 and 11.2], but the proofs were incomplete.¹ The case of the free homotopy problem when the source space is an admissible flat 2-complex was treated by Daskalopoulos and Mese [8].

It is understood in Theorems 3.2 and 4.1 that the Riemannian metric g on the source polyhedron X is *simplex-wise smooth* (as defined in [9, p. 151], cf. [19, p. 182]); this is needed for the extension in [9] of the energy concept of [25] (for maps from a Riemannian domain) to the present polyhedral source space (X, g) . However, when confined to maps into a (compact) smooth Riemannian *manifold* Y of nonpositive sectional curvature, Theorems 3.2 and 4.1 remain valid with nearly the same proofs even when g is allowed to be merely measurable; see Theorem 5.2. In that setting there is a well-known concept of energy of maps $X \rightarrow Y$, defined for example in terms of an isometric embedding of Y in some \mathbb{R}^N , as in Nash's theorem; see [9, Lemma 9.3]. The two energy concepts are indeed known to be identical if g happens to be simplex-wise smooth [9, Theorem 9.2]. An application of Theorem 3.2 (with Riemannian manifold target) is given in [24, §6].

For a survey of harmonic maps in the present setting, see [15].

¹The main gap concerning [9, Theorem 11.1] is the assertion that a minimizing sequence $(\tilde{\varphi}_i)$ of equivariant maps $\tilde{\varphi}_i : \tilde{X} \rightarrow \tilde{Y}$ has an L^2 -convergent subsequence; but that may not be true, cf. [25, p. 647 and Remark 2.6.6.]

The same gap seems to be present already in [28, p. 63, paragraph following (7.3)], despite a construction in [27], by which there remains the difficulty that equivariance of a sequence is not necessarily preserved under postcomposition with an automorphism. However, the result in question [27, Théorème 7.1] has been proved by different methods by Sacks and Uhlenbeck [33] and by Schoen and Yau [40] (and is covered by the present Theorem 3.2).

2. Preliminaries

Throughout this article (unless otherwise specified), the *source space* (X, g) is a compact Riemannian polyhedron (with possibly empty boundary bX); cf. [19, p. 182; 9, Chapter 4]. This means that (i) X is a compact connected metric space, (ii) there exists a finite (abstract) simplicial complex K of finite dimension $m \geq 1$ and a bi-Lipschitz bijection θ of the (compact metric) space $(|K|, d)$ of the complex K onto X , and (iii) for every simplex s of K there is given a smooth Riemannian metric g_s on the geometric simplex $|s|$; we then write $g = (g_s)_{s \in K}$. The pair (K, θ) is called a *triangulation* of X . Recall, e.g., from [37, p. 110 f.] (or from [18]), that the space $|K|$ is the set of all maps ξ of the set K^0 of vertices v of K into $[0, 1]$ such that $\sum_{v \in K^0} \xi(v) = 1$ and that $s := \{v \in K^0 : \xi(v) > 0\}$ is an (abstract) simplex of K . The numbers $\xi(v)$ are called the barycentric coordinates of $\xi \in |K|$. A single vertex $v \in K^0$ is identified with the point ξ of $|K|$ for which $\xi(v) = 1$. The *boundary* bK of K is the union of those $(m - 1)$ -simplexes of K which are faces of only one m -simplex, and $bX := \theta(|bK|)$ is called the boundary of X . $|K|$ is a separable compact metric space with the barycentric metric d defined by

$$(2.1) \quad d(\xi, \xi')^2 = \sum_{v \in K^0} (\xi(v) - \xi'(v))^2, \quad \xi, \xi' \in |K|.$$

We denote by $\mu = \mu_g$ the image under θ of the Riemannian volume measure on $(|K|, g)$ (that is, on each m -simplex $(|s|, g_s)$ of K). Furthermore, d_X denotes the *intrinsic Riemannian distance* on X , induced by θ from the intrinsic Riemannian distance on $|K|$; cf. [9, p. 53]. (The intrinsic distance d_X and the metric on X induced by θ from the barycentric metric d on $|K|$ are equivalent, but generally distinct.) In the sequel we often tacitly identify $|K|$ with $X = \theta(|K|)$ and a geometric simplex $|s|$, $s \in K$, with the ‘curved’ simplex $\theta(|s|) \subset X$.

Except in Propositions 2.1 and 2.2 we further require that K and hence X be *admissible* in the sense of [9, p. 45; 18]. This means that (i) K is *dimensionally m -homogeneous*, i.e., every simplex of K is a face of some m -simplex and (ii) K is *locally $(m - 1)$ -chainable* (cf. [6; 43, §2]); i.e., for any simplex σ of K , any two m -simplexes $s, t \in K$ containing σ can be joined by a chain of m -simplexes $s = s_0, \dots, s_k = t$ in K containing σ and such that $\dim(s_{i-1} \cap s_i) = m - 1$ for $i \in \{1, \dots, k\}$.

Every connected open subset U of a polyhedron X can be triangulated so as to become a polyhedron (not necessarily a subpolyhedron of X) such that every simplex of U is a subset of some simplex of X . If X is admissible then so is U ; cf. [18]. When X is Riemannian it is understood (in the absence

of other indication) that U is given the induced Riemannian metric. In particular, the open star $st v$ of a vertex v of X is an admissible polyhedron, and so is clearly the closed star $\overline{st} v$.

The metric *target space* (Y, d_Y) is required throughout (except in Section 5) to be a compact geodesic space of nonpositive Alexandrov curvature; see for example [2].

It is known that (X, d_X) and (Y, d_Y) are connected, path-connected and locally path-connected, and therefore have connected, complete, separable, metric, and locally compact universal covering spaces $(\tilde{X}, d_{\tilde{X}})$ and $(\tilde{Y}, d_{\tilde{Y}})$ with nonexpanding and locally isometric covering projections $p_X : \tilde{X} \rightarrow X$ and $p_Y : \tilde{Y} \rightarrow Y$; cf. e.g. [31, §3.5], [37, p. 80]. Denote by Γ_X and Γ_Y the discrete groups of isometric covering transformations (deck transformations) of \tilde{X} and \tilde{Y} , respectively. Thus Γ_X , respectively Γ_Y , is countable and isomorphic with the fundamental group $\pi_1(X)$, respectively $\pi_1(Y)$; cf. [37, Corollary 4 in §2.6].

Because \tilde{X}/Γ_X is homeomorphic with X there exists a number $r_X > 0$ such that

$$(2.2) \quad p_X \text{ maps } B_{\tilde{X}}(\tilde{x}, \varepsilon) \text{ isometrically onto } B_X(x, \varepsilon)$$

for any $\tilde{x} \in \tilde{X}$, $x = p_X(\tilde{x})$ and $\varepsilon \leq r_X$. (We write for example $B_X(x, r) = \{x' \in X : d_X(x, x') \leq r\}$, $x \in X$, $r > 0$.) See [31, Proposition 3.5.7], noting that X is compact and p_X a local isometry; similarly with X, \tilde{X}, r_X replaced by Y, \tilde{Y}, r_Y . It follows that, say for any $\tilde{x} \in \tilde{X}$, $p_X^{-1}(B_X(x, \varepsilon))$ is the disjoint countable union of the closed balls $B_{\tilde{X}}(\tilde{x}, \varepsilon)$, $\tilde{x} \in p_X^{-1}(x)$, one for each $\gamma \in \Gamma_X$.

We may assume that every open star in X (understood as the image under θ of the star of some vertex in K) is a subset of some ball $B_X(x, r_X)$, $x \in X$; for that can always be arranged by passing to a subdivision of X (that is, of K), as shown by Whitehead [42, Theorem 35, p. 317]; or see [37, Theorem 14, p. 125]. Then $(\tilde{X}, d_{\tilde{X}})$ becomes an admissible locally compact separable polyhedron whose m -simplexes \tilde{s} are the connectivity components of $p_X^{-1}(s)$, with s ranging over all m -simplexes of X . The polyhedron $(\tilde{X}, d_{\tilde{X}})$ is a length space by [9, Proposition 4.1(b)]. Being complete and locally compact, $(\tilde{X}, d_{\tilde{X}})$ is even a *geodesic space* (like $(\tilde{Y}, d_{\tilde{Y}})$), according to the Hopf–Rinow theorem, cf. [2, p. 35; 31, p. 62]. Similarly, the compact length space (X, d_X) is itself a geodesic space. The Riemannian metric g_s on s lifts uniquely to a Riemannian metric $\tilde{g}_{\tilde{s}}$ on \tilde{s} , and we thereby obtain an admissible, locally compact Riemannian polyhedron (\tilde{X}, \tilde{g}) .

After fixing a base point $x_0 \in X$ and above it a point $\tilde{x}_0 \in p_X^{-1}(x_0)$, a map $\varphi \in C(X, Y)$ (the continuous maps) lifts uniquely to a map $\tilde{\varphi} \in C(\tilde{X}, \tilde{Y})$

after choosing $\tilde{\varphi}(\tilde{x}_0) \in p_Y^{-1}(\varphi(x_0))$, the lift $\tilde{\varphi}$ being then characterized by the lifting property

$$(2.3) \quad \varphi \circ p_X = p_Y \circ \tilde{\varphi},$$

cf. [37, p. 67] or [22, p. 62].

In the following two propositions and in Section 5 the Riemannian metric g on X is allowed to be just *measurable* (rather than simplex-wise smooth). Proposition 2.1 is basically well known, but we include a proof. (An equivalence class for some equivalence relation may be indicated by placing one of its elements between square brackets.)

Proposition 2.1. *Let (X, g) be a compact Riemannian polyhedron, and let (Y, d_Y) be a compact geodesic space of nonpositive curvature. The set of homotopy classes $\mathcal{H} = [\varphi]$ of continuous maps $\varphi : X \rightarrow Y$ is mapped bijectively onto the set of conjugacy classes $[\varphi_*]$ of homomorphisms $\varphi_* : \Gamma_X \rightarrow \Gamma_Y$ under the map $[\varphi] \mapsto [\varphi_*]$, whereby*

$$(2.4) \quad \tilde{\varphi} \circ \gamma = \varphi_*(\gamma) \circ \tilde{\varphi} \quad \text{in } \tilde{X} \text{ for } \gamma \in \Gamma_X, \varphi \in \mathcal{H}.$$

The covering transformation $\varphi_*(\gamma) \in \Gamma_Y$ is uniquely determined by (2.4) because $\tilde{\varphi}(\tilde{X}) \neq \emptyset$ and because a covering transformation on \tilde{Y} is determined by its action on any single point, \tilde{Y} being path-connected; cf. e.g. [22, p. 70]. If we replace the chosen point $\tilde{y}_0 = \tilde{\varphi}(\tilde{x}_0) \in p_Y^{-1}(y_0)$ by another such point, that is, by $\beta(\tilde{y}_0)$ for some $\beta \in \Gamma_Y$, then $\tilde{\varphi}$ will be replaced by $\beta \circ \tilde{\varphi}$, and hence $\varphi_*(\gamma)$ by its conjugate $\beta \circ \tilde{\varphi} \circ \beta^{-1}$. The elements of the conjugacy class $[\varphi_*]$ thus correspond to the possible choices of $\tilde{y}_0 \in p_Y^{-1}(y_0)$. For any other map ψ from the given homotopy class \mathcal{H} the corresponding homomorphism ψ_* is, by the proposition, a conjugate to φ_* and, therefore, equal to φ_* by a certain choice of $\tilde{\varphi}(\tilde{x}_0)$ or $\tilde{\psi}(\tilde{x}_0)$.

Proof of Proposition 2.1. The polyhedron X being a CW-complex (cf. [37, p. 400]) we may derive Proposition 2.1 from the corresponding result [37, Theorem 11 in §8.1], in which Γ_X and Γ_Y are replaced by the fundamental groups $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$, respectively. The homomorphisms $\varphi_* : \Gamma_X \rightarrow \Gamma_Y$ shall then be replaced by the usual induced homomorphisms $\varphi_\# : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$, $y_0 = \varphi(x_0)$, given by

$$(2.5) \quad \varphi_\#([\omega]) = [\varphi \circ \omega] \quad \text{for loops } \omega \text{ in } (X, x_0).$$

Denote by $\mathcal{P}(\tilde{X}; \tilde{x}_0, \tilde{x}_1)$ the set of (continuous) paths in \tilde{X} from \tilde{x}_0 to \tilde{x}_1 , and similarly relative to \tilde{Y} . (Loops in (X, x_0) then correspond to $\tilde{x}_0 = \tilde{x}_1$.)

The natural isomorphism $\psi_X : \Gamma_X \rightarrow \pi_1(X, x_0)$ is then given by

$$\psi_X(\gamma) = [p_X \circ \tilde{\omega} : \tilde{\omega} \in \mathcal{P}(\tilde{X}; \tilde{x}_0, \gamma(\tilde{x}_0))], \quad \gamma \in \Gamma_X.$$

See [37, pp. 85–87], and note that $p_X \circ \tilde{\omega}$ indeed is a loop in (X, x_0) since $p_X(\gamma(\tilde{x}_0)) = p_X(\tilde{x}_0) = x_0$; similarly for $\psi_Y : \Gamma_Y \rightarrow \pi_1(Y, y_0)$. It remains to evaluate

$$\varphi_* := \psi_Y^{-1} \circ \varphi_{\#} \circ \psi_X.$$

For $\gamma \in \Gamma_X$ we have (since $\tilde{\varphi}(\tilde{x}_0) = \tilde{y}_0$)

$$(2.6) \quad (\psi_Y \circ \varphi_*)(\gamma) = \psi_Y(\varphi_*(\gamma)) = [p_Y \circ \tilde{\chi} : \tilde{\chi} \in \mathcal{P}(\tilde{Y}; \tilde{y}_0, (\varphi_*(\gamma) \circ \tilde{\varphi})(\tilde{x}_0))]$$

(a homotopy class of loops in (Y, y_0)). Furthermore, by (2.3) and (2.5),

$$\begin{aligned} (\varphi_{\#} \circ \psi_X)(\gamma) &= \varphi_{\#} \circ [p_X \circ \tilde{\omega} : \tilde{\omega} \in \mathcal{P}(\tilde{X}; \tilde{x}_0, \gamma(\tilde{x}_0))] \\ &= [\varphi \circ p_X \circ \tilde{\omega} : \tilde{\omega} \in \mathcal{P}(\tilde{X}; \tilde{x}_0, \gamma(\tilde{x}_0))] \\ &= [p_Y \circ \tilde{\varphi} \circ \tilde{\omega} : \tilde{\omega} \in \mathcal{P}(\tilde{X}; \tilde{x}_0, \gamma(\tilde{x}_0))] \\ (2.7) \quad &\subset [p_Y \circ \tilde{\omega}' : \tilde{\omega}' \in \mathcal{P}(\tilde{Y}; \tilde{y}_0, (\tilde{\varphi} \circ \gamma)(\tilde{x}_0))]. \end{aligned}$$

The inclusion stems from $\tilde{\omega}' := \tilde{\varphi} \circ \tilde{\omega} \in \mathcal{P}(\tilde{Y}; \tilde{y}_0, \tilde{\varphi}(\gamma(\tilde{x}_0)))$ when $\tilde{\omega} \in \mathcal{P}(\tilde{X}; \tilde{x}_0, \gamma(\tilde{x}_0))$. Note that $p_Y \circ \tilde{\omega}'$ is a loop in (Y, y_0) because $p_Y(\tilde{\varphi}(\gamma(\tilde{x}_0))) = \varphi(p_X(\gamma(\tilde{x}_0))) = \varphi(p_X(\tilde{x}_0)) = \varphi(x_0) = y_0$. From (2.6) and (2.7) (whose left-hand members are equal) we have thus altogether obtained

$$\begin{aligned} [p_Y \circ \tilde{\chi} : \tilde{\chi} \in \mathcal{P}(\tilde{Y}; \tilde{y}_0, (\varphi_*(\gamma) \circ \tilde{\varphi})(\tilde{x}_0))] \\ = [p_Y \circ \tilde{\omega}' : \tilde{\omega}' \in \mathcal{P}(\tilde{Y}; \tilde{y}_0, (\tilde{\varphi} \circ \gamma)(\tilde{x}_0))] \end{aligned}$$

because inclusion between homotopy classes (of paths) implies equality. Thus every loop $\chi := p_Y \circ \tilde{\chi}$ as in (2.6) is homotopic (in (Y, y_0)) with any loop $\omega' := p_Y \circ \tilde{\omega}'$ as in (2.7). The lifted paths $\tilde{\chi}$ and $\tilde{\omega}'$ in \tilde{Y} have the same initial point \tilde{y}_0 and therefore also the same endpoint, so (2.4) holds true at \tilde{x}_0 . For any $\tilde{x} \in \tilde{X}$ we have $p_Y((\varphi_*(\gamma) \circ \tilde{\varphi})(\tilde{x})) = p_Y(\tilde{\varphi}(\tilde{x})) = \varphi(p_X(\tilde{x})) = \varphi(p_X(\gamma(\tilde{x}))) = p_Y((\tilde{\varphi} \circ \gamma)(\tilde{x}))$; cf. (2.3). Consequently, (2.4) holds for any $\tilde{x} \in \tilde{X}$, by the unique lifting property (2.3). \square

Proposition 2.2. *Let (X, g) be a compact Riemannian polyhedron, and let (Y, d_Y) be a compact geodesic space of nonpositive curvature.*

(a) *Lip(X, Y) (Lipschitz maps) is uniformly dense in $C(X, Y)$ (continuous maps).*

(b) For any map $\varphi \in C(X, Y)$ and any number $r > 0$ there exists a map $f \in C(X \times [0, 1], Y)$ such that $f(x, 0) = \varphi(x)$ for $x \in X$, that $d_Y(f(x, t), \varphi(x)) < r$ for $(x, t) \in X \times [0, 1]$, and that $f(\cdot, t) \in \text{Lip}(X, Y)$ for each $t \in]0, 1]$.

In particular, every homotopy class $\mathcal{H} \subset C(X, Y)$, as in Proposition 2.1, contains a Lipschitz map. For Y simply connected (and not necessarily compact) Proposition 2.2 was obtained in [14, Proposition 6], inspired by [25, p. 645] where ball averages are used. For the present polyhedral domain X , barycentric averages are preferable; see (2.10) below. The choice of Riemannian metric g on X in Proposition 2.2 is immaterial, by [9, Lemma 4.2].

Proof of Proposition 2.2. (a) Given $\varphi \in C(X, Y)$, consider a lift $\tilde{\varphi} \in C(\tilde{X}, \tilde{Y})$; cf. (2.3). For any prescribed $\varepsilon > 0$ with $\varepsilon \leq r_X$ from (2.2), choose $\delta = \delta(\varepsilon) > 0$ so that $\delta \leq r_X$ and, by uniform continuity of φ ,

$$(2.8) \quad d_Y(\varphi(x), \varphi(x')) < \varepsilon \quad \text{for } x, x' \in X \text{ with } d_X(x, x') \leq \delta.$$

We proceed to show that (with the same ε and δ)

$$(2.9) \quad d_{\tilde{Y}}(\tilde{\varphi}(\tilde{x}), \tilde{\varphi}(\tilde{x}')) \leq \varepsilon \quad \text{for } \tilde{x}, \tilde{x}' \in \tilde{X} \text{ with } d_{\tilde{X}}(\tilde{x}, \tilde{x}') \leq \delta.$$

Consider the constant-speed geodesic $\tilde{\varrho} : [0, d_{\tilde{X}}(\tilde{x}, \tilde{x}')] \rightarrow \tilde{X}$ joining \tilde{x} to \tilde{x}' , and the image $\tilde{\varphi} \circ \tilde{\varrho}$, a path in \tilde{Y} . If $d_{\tilde{Y}}(\tilde{\varphi}(\tilde{x}), \tilde{\varphi}(\tilde{x}')) > \varepsilon$ there exists τ , $0 < \tau < d_{\tilde{X}}(\tilde{x}, \tilde{x}')$, such that $d_{\tilde{Y}}(\tilde{\varphi}(\tilde{x}), \tilde{\varphi}(\tilde{\varrho}(\tau))) = \varepsilon$. From (2.2) and (2.3) we obtain, writing $p_X(\tilde{x}) = x$ and $p_X(\tilde{\varrho}(\tau)) = \varrho(\tau)$,

$$d_Y(\varphi(x), \varphi(\varrho(\tau))) = d_{\tilde{Y}}(\tilde{\varphi}(\tilde{x}), \tilde{\varphi}(\tilde{\varrho}(\tau))) = \varepsilon.$$

This contradicts (2.8) (applied with x' replaced by $\varrho(\tau)$) because

$$d_X(x, \varrho(\tau)) \leq d_{\tilde{X}}(\tilde{x}, \tilde{\varrho}(\tau)) < d_{\tilde{X}}(\tilde{x}, \tilde{x}') \leq \delta,$$

p_X being nonexpanding.

We may assume that, for every simplex $s \in K$, $\theta(|s|)$ has diameter $\leq \delta$ with δ from (2.8). (If this is not the case from the beginning we subdivide K into simplexes s' such that $|s'| \subset \theta^{-1}(B)$ for some open ball $B \subset X$ of diameter δ ; this is possible according to the abovementioned theorem of Whitehead [42] applied to a cover of $|K|$ by pre-images $\theta^{-1}(B)$ of open balls $B \subset X$ of diameter δ covering X).

We proceed to adapt the construction from the proof of [14, Proposition 6(a)] (in which Y was simply connected) to the lifted map $\tilde{\varphi} : \tilde{X} \rightarrow \tilde{Y}$. Given a point $\tilde{x} \in \tilde{X}$ with projection $p_X(\tilde{x}) = x \in X$, the corresponding point $\xi = \theta^{-1}(x) \in |K|$ has barycentric coordinates $\xi(v)$, $v \in K^0$, and we have the probability measure

$$\nu_x = \sum_{v \in K^0} \xi(v) \varepsilon_{\theta(v)}$$

on X , where ε_p denotes the mass 1 at a point $p \in X$. The support of ν_x is the simplex $\theta(|s|)$ in X , where s denotes the abstract simplex in K whose vertices are those $v \in K^0$ for which $\xi(v) > 0$, so $\text{supp } \nu_x \subset B_X(x, \delta)$. Because $\delta \leq r_X$ (cf. the line following (2.2)), p_X maps $B_{\tilde{X}}(\tilde{x}, \delta)$ isometrically onto $B(x, \delta)$, and ν_x therefore lifts to the probability measure

$$(2.10) \quad \nu_{\tilde{x}} = \sum_{v \in K^0} \xi(v) \varepsilon_{\tilde{\theta}(v)}$$

on \tilde{X} , whereby $\tilde{\theta}(v) \in B_{\tilde{X}}(\tilde{x}, \delta)$ is defined by having the projection $\theta(v) \in B_X(x, \delta)$ when $\xi(v) > 0$.

Invoking [25, Lemma 2.5.1] we define

$$\tilde{f}(\tilde{x}) = \tilde{\varphi}_{\nu_{\tilde{x}}}, \quad \tilde{x} \in \tilde{X},$$

$\tilde{\varphi}_{\nu_{\tilde{x}}}$ denoting the average (or centre of mass) of $\tilde{\varphi} : \tilde{X} \rightarrow \tilde{Y}$ with respect to $\nu_{\tilde{x}}$; that is, $\tilde{\varphi}_{\nu_{\tilde{x}}}$ is the unique point of \tilde{Y} which minimizes $\int_{\tilde{X}} d_{\tilde{Y}}^2(\tilde{\varphi}(\tilde{x}'), \tilde{y}) d\nu_{\tilde{x}}(\tilde{x}')$ as a function of $\tilde{y} \in \tilde{Y}$. For points $\tilde{x}, \tilde{x}' \in \tilde{X}$ with $d_{\tilde{X}}(\tilde{x}, \tilde{x}') \leq \delta$ we then have $\tilde{\varphi}(\tilde{x}') \in B_{\tilde{Y}}(\tilde{\varphi}(\tilde{x}), \varepsilon)$, by (2.9), and hence by [25, Proposition 2.4], or see [9, Lemma 10.4], writing $\tilde{f} = \tilde{f}^\varepsilon$ to indicate the dependence of \tilde{f} on ε :

$$(2.11) \quad \tilde{f}(\tilde{x}) = \tilde{f}^\varepsilon(\tilde{x}) = \tilde{\varphi}_{\nu_{\tilde{x}}} \in B_{\tilde{Y}}(\tilde{\varphi}(\tilde{x}), \varepsilon),$$

closed balls $B_{\tilde{Y}}(\tilde{y}, \varepsilon)$ in $(\tilde{Y}, d_{\tilde{Y}})$ being convex. Thus, $\tilde{f}^\varepsilon \rightarrow \tilde{\varphi}$ uniformly in \tilde{X} as $\varepsilon \rightarrow 0$.

We show that $\tilde{f} = \tilde{f}^\varepsilon \in \text{Lip}(\tilde{X}, \tilde{Y})$. Consider two points \tilde{x}, \tilde{x}' of \tilde{X} with $d_{\tilde{X}}(\tilde{x}, \tilde{x}') \leq \delta$, and write $p_X(\tilde{x}) = x = \theta(\xi)$ and $p_X(\tilde{x}') = x' = \theta(\xi')$ with $\xi, \xi' \in |K|$ and $d_X(x, x') \leq \delta$. Apply [25, Proposition 2.5.2] (with $\alpha = 1$) to the corresponding representing measures $\nu_{\tilde{x}}$ and $\nu_{\tilde{x}'}$; cf. (2.10). After

division by $d_{\tilde{Y}}(\tilde{f}(\tilde{x}), \tilde{f}(\tilde{x}'))$ (if nonzero) we obtain by (2.11) with the above notation

$$\begin{aligned}
 d_{\tilde{Y}}(\tilde{f}(\tilde{x}), \tilde{f}(\tilde{x}')) &\leq 2 \int_{\tilde{X}} d_{\tilde{Y}}(\tilde{f}, \tilde{\varphi}) |d\nu_{\tilde{x}} - d\nu_{\tilde{x}'}| \leq 2\varepsilon \|\nu_{\tilde{x}} - \nu_{\tilde{x}'}\| \\
 (2.12) \qquad \qquad \qquad &= 4\varepsilon \sum_{v \in K^0} |\xi(v) - \xi'(v)| \leq 4\varepsilon \sqrt{\#K^0} d(\xi, \xi') \leq C d_{\tilde{X}}(\tilde{x}, \tilde{x}').
 \end{aligned}$$

Here $\|\nu_{\tilde{x}} - \nu_{\tilde{x}'}\|$ denotes total variation. In the third inequality one applies (2.1) and Cauchy’s inequality. Finally, $C = 4\varepsilon c \sqrt{\#K^0}$ with c denoting the bi-Lipschitz constant of $\theta : |K| \rightarrow X$, and $\#K^0$ the number of vertices in K .

The above requirement $d_{\tilde{X}}(\tilde{x}, \tilde{x}') \leq \delta$ can clearly be omitted because $(\tilde{X}, d_{\tilde{X}})$ is a geodesic space. Consider again the geodesic $\tilde{\rho}$ joining \tilde{x} with \tilde{x}' , hence of length $d_{\tilde{X}}(\tilde{x}, \tilde{x}')$. With points $\tilde{x} = \tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_k = \tilde{x}'$ of $\tilde{\rho}$ such that $d_{\tilde{X}}(\tilde{x}_{i-1}, \tilde{x}_i) \leq \delta$ for $i \in \{1, \dots, k\}$, we have from (2.12)

$$\begin{aligned}
 d_{\tilde{Y}}(\tilde{f}(\tilde{x}), \tilde{f}(\tilde{x}')) &\leq \sum_{i=1}^k d_{\tilde{Y}}(\tilde{f}(\tilde{x}_{i-1}), \tilde{f}(\tilde{x}_i)) \leq C \sum_{i=1}^k d_{\tilde{X}}(\tilde{x}_{i-1}, \tilde{x}_i) \\
 &= C d_{\tilde{X}}(\tilde{x}, \tilde{x}').
 \end{aligned}$$

The (Lipschitz) continuous map $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ is φ_* -equivariant in the sense that

$$(2.13) \qquad \qquad \qquad \tilde{f} \circ \gamma = \varphi_*(\gamma) \circ \tilde{f} \quad \text{for } \gamma \in \Gamma_{\tilde{X}}.$$

Indeed, for $\tilde{x} \in \tilde{X}$, the measure $\nu_{\gamma(\tilde{x})} = \sum_{v \in K^0} \xi(v) \varepsilon_{\gamma(\tilde{\theta}(v))}$, (cf. (2.10)) is the image of $\nu_{\tilde{x}}$ under the isometry γ of \tilde{X} . We therefore obtain, by (2.4),

$$\begin{aligned}
 &\int_{\tilde{X}} d_{\tilde{Y}}^2(\tilde{\varphi}(\tilde{x}'), \tilde{y}) d\nu_{\gamma(\tilde{x})}(\tilde{x}') = \int_{\tilde{X}} d_{\tilde{Y}}^2((\tilde{\varphi} \circ \gamma)(\tilde{x}'), \tilde{y}) d\nu_{\tilde{x}}(\tilde{x}') \\
 &= \int_{\tilde{X}} d_{\tilde{Y}}^2((\varphi_*(\gamma) \circ \tilde{\varphi})(\tilde{x}'), \tilde{y}) d\nu_{\tilde{x}}(\tilde{x}') = \int_{\tilde{X}} d_{\tilde{Y}}^2(\tilde{\varphi}(\tilde{x}'), (\varphi_*(\gamma))^{-1}(\tilde{y})) d\nu_{\tilde{x}}(\tilde{x}'),
 \end{aligned}$$

which is minimized (uniquely) as a function of $\tilde{y} \in \tilde{Y}$ both by $\tilde{f}(\gamma(\tilde{x}))$ and by $(\varphi_*(\gamma))^{-1}(\tilde{y}) = \tilde{f}(\tilde{x})$, that is, by $\tilde{y} = (\varphi_*(\gamma))(\tilde{f}(\tilde{x}))$, so (2.13) indeed holds at any point $\tilde{x} \in \tilde{X}$.

Being thus φ_* -equivariant, the continuous map $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ has a continuous projection $f : X \rightarrow Y$, characterized by \tilde{f} being a lift of f , that is,

$f \circ p_X = p_Y \circ \tilde{f}$; cf. (2.3) and the text following (3.2) below. This projection is of class $\text{Lip}(X, Y)$ because

$$d_Y(f(x), f(x')) \leq d_{\tilde{Y}}(\tilde{f}(\tilde{x}), \tilde{f}(\tilde{x}')) \leq C d_{\tilde{X}}(\tilde{x}, \tilde{x}') = C d_X(x, x')$$

for points $x, x' \in X$ with $d_X(x, x') \leq \delta$ ($\leq r_X$), noting that x, x' are the projections of points $\tilde{x}, \tilde{x}' \in \tilde{X}$ with $d_{\tilde{X}}(\tilde{x}, \tilde{x}') = d_X(x, x')$; cf. (2.2). As in the above case of \tilde{f} , the requirement $d_X(x, x') \leq \delta$ can be omitted, (X, d_X) being a geodesic space. From the uniform convergence $\tilde{f}^\varepsilon \rightarrow \tilde{\varphi}$ follows by projection that the map $f^\varepsilon \in \text{Lip}(X, Y)$ converges uniformly to φ as $\varepsilon \rightarrow 0$.

(b) As noted after (2.11), now with $\varepsilon = 1/n$ ($n = 1, 2, \dots$), we have constructed a sequence of maps \tilde{f}^ε , now written $\tilde{f}_{1/n}$, which are φ_* -equivariant and of class $\text{Lip}(\tilde{X}, \tilde{Y})$ and converge uniformly to $\tilde{\varphi}$ as $n \rightarrow \infty$. For $t \in [\frac{1}{n+1}, \frac{1}{n}]$ write $t = \frac{\tau}{n} + \frac{1-\tau}{n+1}$ with $\tau \in [0, 1]$; and consider for each $\tilde{x} \in \tilde{X}$ the geodesic $[\frac{1}{n+1}, \frac{1}{n}] \ni t \mapsto \tilde{f}_t(\tilde{x}) \in \tilde{Y}$ given by

$$(2.14) \quad \tilde{f}_t(\tilde{x}) = \tau \tilde{f}_{1/n}(\tilde{x}) + (1 - \tau) \tilde{f}_{1/(n+1)}(\tilde{x})$$

with the usual abuse of notation for “convex combinations” of points of \tilde{Y} . For given $r > 0$ there is by (2.11) an integer $N > 0$ such that $\tilde{f}_{1/n}(\tilde{x})$ lies in the convex ball $B_{\tilde{Y}}(\tilde{\varphi}(\tilde{x}), r)$ for $n \geq N$ and $\tilde{x} \in \tilde{X}$. So does therefore $\tilde{f}_t(\tilde{x})$ for $t \in [\frac{1}{n+1}, \frac{1}{n}]$, hence for any $t \in]0, 1]$, and even for $t \in [0, 1]$ when we define $\tilde{f}_0 = \tilde{\varphi}$ and $\tilde{f}_t = \tilde{f}_{1/N}$ for $t \in]1/N, 1]$:

$$(2.15) \quad d_{\tilde{Y}}(\tilde{f}_t(\tilde{x}), \varphi(\tilde{x})) \leq r \quad \text{for } \tilde{x} \in \tilde{X}, t \in [0, 1].$$

The metric $d_{\tilde{Y}}$ being convex (cf. e.g. [2, Proposition 2.2]), it follows by (2.12) that each $\tilde{f}_t, t \in]0, 1]$, is of class $\text{Lip}(\tilde{X}, \tilde{Y})$. In terms of a common Lipschitz constant c_n for $\tilde{f}_{1/n}$ and $\tilde{f}_{1/(n+1)}$ we have, in fact, for $\tilde{x}, \tilde{x}' \in \tilde{X}$ and $t \in [\frac{1}{n+1}, \frac{1}{n}]$ in view of (2.14):

$$(2.16) \quad \begin{aligned} d_{\tilde{Y}}(\tilde{f}_t(\tilde{x}), \tilde{f}_t(\tilde{x}')) &\leq \tau d_{\tilde{Y}}(\tilde{f}_{1/n}(\tilde{x}), \tilde{f}_{1/n}(\tilde{x}')) \\ &\quad + (1 - \tau) d_{\tilde{Y}}(\tilde{f}_{1/(n+1)}(\tilde{x}), \tilde{f}_{1/(n+1)}(\tilde{x}')) \\ &\leq \tau c_n d_{\tilde{X}}(\tilde{x}, \tilde{x}') + (1 - \tau) c_n d_{\tilde{X}}(\tilde{x}, \tilde{x}') = c_n d_{\tilde{X}}(\tilde{x}, \tilde{x}'). \end{aligned}$$

And from $\tilde{f}_t \in \text{Lip}(\tilde{X}, \tilde{Y})$ for each $t \in]0, 1]$ follows $f_t \in \text{Lip}(X, Y)$ for each such t by the argument used above to show that $\tilde{f} = \tilde{f}^\varepsilon \in \text{Lip}(\tilde{X}, \tilde{Y})$ implies $f \in \text{Lip}(X, Y)$.

Because geodesics in \tilde{Y} vary continuously with their endpoints (see [2, p. 160]) it follows from (2.14) and (2.16) that $\tilde{f}(\tilde{x}, t) := \tilde{f}_t(\tilde{x})$ depends continuously on $(\tilde{x}, t) \in \tilde{X} \times [\frac{1}{n+1}, \frac{1}{n}]$, hence altogether on $\tilde{X} \times]0, 1]$, and even on $\tilde{X} \times [0, 1]$ by (2.15) together with the continuity of $\tilde{f}(\cdot, 0) = \tilde{f}_0 = \tilde{\varphi}$ on \tilde{X} . The metric $d_{\tilde{Y}}$ being convex, the continuous map \tilde{f}_t is φ_* -equivariant for each $t \in [0, 1]$; that is, (2.13) remains valid with \tilde{f} replaced by \tilde{f}_t . Hence $\tilde{f}_t : \tilde{X} \rightarrow \tilde{Y}$ admits a continuous projection $f_t : X \rightarrow Y$:

$$(2.17) \quad f_t(x) := p_Y(\tilde{f}_t(\tilde{x})), \quad \tilde{x} \in p_X^{-1}(x), \quad x \in X.$$

Moreover, the map $f : X \times [0, 1] \rightarrow Y$ given by $f(x, t) = f_t(x)$ is continuous in view of (2.17) because

$$d_Y(f(x, t), f(x', t')) \leq d_{\tilde{Y}}(f(\tilde{x}, t), f(\tilde{x}', t')) \rightarrow 0 \quad \text{as } (x', t') \rightarrow (x, t)$$

and hence $(x', t') \rightarrow (x, t)$, p_Y being locally isometric. This completes the proof of Proposition 2.2. □

Energy of maps. The concept of energy of maps will be used in the sequel also for maps from suitable locally compact spaces, in particular for maps between the universal covers of the above compact spaces X and Y . In the rest of the present section we accordingly allow (X, g) to be any locally compact separable m -dimensional admissible Riemannian polyhedron with simplex-wise smooth Riemannian metric g and corresponding intrinsic Riemannian distance d_X (as in [9]), and (Y, d_Y) any complete metric space (in the absence of other indication). The definition and basic properties of energy of maps, due to [25] for maps from a Riemannian domain (in a Riemannian manifold), were extended in [9, Chapter 9] to the present polyhedral case. Denote by $L^2(X, Y)$, respectively $L^2_{loc}(X, Y)$, the space of all μ -measurable maps $\varphi : X \rightarrow Y$ having separable essential range and for which $d_Y(\varphi(\cdot), q)$ is of class $L^2(X, \mu)$, respectively $L^2_{loc}(X, \mu)$, for some and hence any $q \in Y$. It is known for example that $L^2(X, Y)$ is a complete metric space with distance D given by (cf. [25, p. 572])

$$D^2(\varphi, \varphi') = \int_X d_Y^2(\varphi(x), \varphi'(x)) d\mu(x) \quad \text{for } \varphi, \varphi' \in L^2(X, Y).$$

For $\varphi \in L^2_{loc}(X, Y)$ the approximate energy density $e_\varepsilon(\varphi) \in L^1_{loc}(X, \mu)$, suitably normalized, is defined for $\varepsilon > 0$ at every point $x \in X$ by

$$(2.18) \quad e_\varepsilon(\varphi)(x) = \frac{1}{c_m} \int_{B_X(x, \varepsilon)} \frac{d_Y^2(\varphi(x), \varphi(x'))}{\varepsilon^{m+2}} d\mu(x'),$$

where $c_m = \omega_m / (m + 2)$, ω_m being the volume of the unit ball in \mathbb{R}^m ; and $B_X(x, \varepsilon) := \{x' \in X : d_X(x, x') \leq \varepsilon\}$. The *energy* of $\varphi \in L^2_{\text{loc}}(X, Y)$ is defined by

$$(2.19) \quad E(\varphi) = \sup_{f \in C_c(X, [0,1])} \left(\limsup_{\varepsilon \rightarrow 0} \int_X f e_\varepsilon(\varphi) d\mu \right) \quad (\leq \infty).$$

For any map $\varphi : X \rightarrow Y$ not of class $L^2_{\text{loc}}(X, Y)$ we put $E(\varphi) = \infty$. The class of all maps $\varphi : X \rightarrow Y$ of finite energy will be denoted here by $\mathcal{E}(X, Y)$ (instead of $W^{1,2}(X, Y)$ as in [9, 11–13, 25, 26], because we do not require that $\varphi \in L^2(X, Y)$ globally). For connected open subsets U, V of X we have by (2.19),

$$(2.20) \quad E(\varphi|U) \leq E(\varphi|V) \quad \text{if } U \subset V$$

because $\varphi \in L^2_{\text{loc}}(V, Y) \Rightarrow \varphi|U \in L^2_{\text{loc}}(U, Y)$ and because $B_U(x, \varepsilon) \subset B_V(x, \varepsilon)$ for any $x \in U$ and $\varepsilon > 0$.

Every map $\varphi \in \mathcal{E}(X, Y)$ has *quasicontinuous* versions [12], to be used throughout, and any two of these are equal quasi-everywhere (q.e.), that is, everywhere except in some polar set [14, paragraph following Definition 1].

A map $\varphi : X \rightarrow Y$ is said to be *locally of finite energy*, and we write $\varphi \in \mathcal{E}_{\text{loc}}(X, Y)$, if it has one of the following three equivalent properties:

- (i) X can be covered by connected open sets $U \subset X$ such that $\varphi|U \in \mathcal{E}(U, Y)$.
- (ii) $\varphi|U \in \mathcal{E}(U, Y)$ for every connected open set $U \Subset X$.
- (iii) $\varphi \in L^2_{\text{loc}}(X, Y)$, and there exists a function $e(\varphi) \in L^1_{\text{loc}}(X, \mu)$ such that

$$(2.21) \quad \lim_{\varepsilon \rightarrow 0} \int_X f e_\varepsilon(\varphi) d\mu = \int_X f e(\varphi) d\mu, \quad f \in C_c(X).$$

In the affirmative case the unique weak limit $e(\varphi)$ from (iii) is termed the *energy density* of φ , and we have from (2.19)

$$(2.22) \quad E(\varphi) = \int_X e(\varphi) d\mu \quad (\leq \infty).$$

Clearly, $\varphi|U \in \mathcal{E}(U, Y)$ can be replaced by $\varphi|U \in \mathcal{E}_{\text{loc}}(U, Y)$ in (i) and (ii).

Slightly more generally, for any open set $A \subset X$, a map $\varphi : A \rightarrow Y$ is said to be locally of finite energy if the restriction of φ to each component of A is locally of finite energy; we then write $\varphi \in \mathcal{E}_{\text{loc}}(A, Y)$. (Recall that

every connected open subset of X is an admissible Riemannian polyhedron, like X itself [18, Theorem 1 and Corollary 3].)

If X can be covered by open sets $U \subset X$ such that $\varphi|U \in \mathcal{E}_{\text{loc}}(U, Y)$ (in analogy with (i) above) then clearly $\varphi \in \mathcal{E}_{\text{loc}}(X, Y)$.

Proof of equivalence of (i), (ii) and (iii). Clearly, (i) or (ii) implies that $\varphi \in L_{\text{loc}}^2(X, Y)$. Furthermore, (ii) \Rightarrow (i) is obvious, X being locally compact. Here, and in what follows, the topological interior of a subset A of X is denoted by A° .

(i) \Rightarrow (iii). Suppose that φ satisfies (i). Then φ has a quasicontinuous version because each $\varphi|U \in \mathcal{E}(U, Y)$ has such, and because quasicontinuity is a local property, X being second countable. According to [42, Theorem 35, p. 317] the given triangulation of X can be refined so that every open star and hence every open m -simplex s° of X becomes a subset of some set U from the given open cover of X , and hence $E(\varphi|s^\circ) \leq E(\varphi|U) < \infty$ by (i), in view of (2.20). From [9, Theorem 9.1, (a) \wedge (b) \Rightarrow (a) \wedge (b') \Rightarrow (9.4)] we therefore deduce (iii).²

(iii) \Rightarrow (ii). Clearly, $\varphi|U \in L_{\text{loc}}^2(U, Y)$ because $\varphi \in L_{\text{loc}}^2(X, Y)$. For $x \in U$ we have $B_U(x, \varepsilon) \subset B_X(x, \varepsilon)$, hence $e_\varepsilon(\varphi|U)(x) \leq e_\varepsilon(\varphi)(x)$. Fix $h \in C_c(X, [0, 1])$ with $h = 1$ in U . For any $f \in C_c(U, [0, 1])$ we obtain

$$\begin{aligned} \int_U f e_\varepsilon(\varphi|U) d\mu &= \int_U f h e_\varepsilon(\varphi|U) d\mu \leq \int_X f h e_\varepsilon(\varphi) d\mu \\ &\rightarrow \int_X f h e(\varphi) d\mu \leq \int_X h e(\varphi) d\mu \end{aligned}$$

for $\varepsilon \rightarrow 0$. Taking supremum over all $f \in C_c(U, [0, 1])$ we conclude by (2.19) that indeed $E(\varphi|U) \leq \int_X h e(\varphi) d\mu < \infty$. \square

Clearly $\mathcal{E}(X, Y) \subset \mathcal{E}_{\text{loc}}(X, Y)$. In the particular case $(Y, d_Y) = \mathbb{R}$ we have

$$(2.23) \quad \mathcal{E}_{\text{loc}}(X, \mathbb{R}) = W_{\text{loc}}^{1,2}(X), \quad E(u) = \int_X |\nabla u|^2 d\mu \text{ for } u \in W_{\text{loc}}^{1,2}(X),$$

see [25, Theorem 1.6.2] (for a Riemannian domain) and [9, Corollary 9.2].

As a consequence of [9, Lemma 4.4] (on volumes of balls), every locally Lipschitz map $\varphi : X \rightarrow Y$ is of class $\mathcal{E}_{\text{loc}}(X, Y)$.

²It may be noted that one may, in that proof, alternatively use simplexes instead of cubes, as in [13, from p. 771 (mid) to p. 772 (top)]. (We shall not use this fact in the present paper.)

If X is compact we have $\mathcal{E}(X, Y) = \mathcal{E}_{\text{loc}}(X, Y)$; and according to (2.19) the energy of an arbitrary map $\varphi \in L^2(X, Y) = L^2_{\text{loc}}(X, Y)$ is then given simply by

$$(2.24) \quad E(\varphi) = \limsup_{\varepsilon \rightarrow 0} \int_X e_\varepsilon(\varphi) d\mu \quad (\leq \infty).$$

Lemma 2.3. (a) For any map $\varphi \in \mathcal{E}_{\text{loc}}(X, Y)$ and any connected open set $A \subset X$ we have $\varphi|_A \in \mathcal{E}_{\text{loc}}(A, Y)$ and

$$(2.25) \quad e(\varphi|_A) = e(\varphi) \quad \mu\text{-a.e. in } A,$$

and hence $E(\varphi|_A) = \int_A e(\varphi) d\mu$.

(b) For any two maps $\varphi, \psi \in \mathcal{E}_{\text{loc}}(X, Y)$ and any μ -measurable set $A \subset X$ such that $\mu(A \setminus A^\circ) = 0$ we have the implications

$$\begin{aligned} \{\varphi = \psi \quad \mu\text{-a.e. in } A\} &\Rightarrow \{e(\varphi) = e(\psi) \quad \mu\text{-a.e. in } A\} \\ &\Rightarrow \left\{ \int_A e(\varphi) d\mu = \int_A e(\psi) d\mu \right\}. \end{aligned}$$

Proof. (a) Clearly, $B_A(x, \varepsilon) \subset B_X(x, \varepsilon)$ for $x \in A$, and equality prevails if $B_X(x, \varepsilon) \subset A$. Likewise, $\varphi|_A \in \mathcal{E}_{\text{loc}}(A, Y)$ by (ii) preceding (2.21). For any $f \in C_c(A)$, any $x \in \text{supp } f$, and any $0 < \varepsilon < d_X(\text{supp } f, X \setminus A)$ we therefore have $B_A(x, \varepsilon) = B_X(x, \varepsilon)$, and hence $e_\varepsilon(\varphi|_A)(x) = e_\varepsilon(\varphi)(x)$. By (2.21) (applied to A in place of X) we infer that

$$\int_A f e(\varphi|_A) d\mu = \lim_{\varepsilon \rightarrow 0} \int_A f e_\varepsilon(\varphi|_A) d\mu = \lim_{\varepsilon \rightarrow 0} \int_X f e_\varepsilon(\varphi) d\mu = \int_X f e(\varphi) d\mu,$$

which implies (2.25). It follows by (2.22) that $E(\varphi|_A) = \int_A e(\varphi|_A) d\mu = \int_A e(\varphi) d\mu$.

(b) Suppose that $\varphi = \psi$ μ -a.e. in A , hence μ -a.e. in the interior A° of A . It follows from (a), applied to each component U of A° , that $e(\varphi|_U) = e(\psi|_U)$ μ -a.e. in U , and hence μ -a.e. in A° , and indeed in A because $\mu(A \setminus A^\circ) = 0$, by hypothesis. \square

Lemma 2.4. (a) Two quasicontinuous functions $X \rightarrow \mathbb{R}$ or maps $X \rightarrow Y$ which are equal μ -a.e. in X are equal quasi-everywhere (q.e.) in X .

(b) Every sequence of quasicontinuous functions $u_n \in W^{1,2}_{\text{loc}}(X)$ converging to 0 in $W^{1,2}_{\text{loc}}(X)$ has a subsequence converging to 0 pointwise q.e. in X .

Proof. Part (a) for functions and Part (b) follow from [7, Théorèmes 5, 7], respectively, because $W^{1,2}(X)$ is a regular Dirichlet space with a kernel of positive type, according to [9, Proposition 5.1]. For two quasicontinuous maps $\varphi, \psi : X \rightarrow Y$ the quasicontinuous function $d_Y(\varphi(\cdot), \psi(\cdot))$ equals 0 μ -a.e., hence q.e. □

The following lemma will be applied in Step 1 of the proof of Theorem 4.1 below.

Lemma 2.5. *For any E -bounded sequence of quasicontinuous maps $\varphi_n \in \mathcal{E}(X, Y)$ converging in $L^2_{\text{loc}}(X, Y)$ to a map φ , we have $E(\varphi) \leq \liminf_n E(\varphi_n) < \infty$, and*

(a) *for any $(m - 1)$ -simplex σ of X ,*

$$\text{tr}_\sigma \varphi_n \rightarrow \text{tr}_\sigma \varphi \quad \text{in } L^2(\sigma) \text{ as } n \rightarrow \infty;$$

(b) *If X is compact there is a subsequence, still denoted (φ_n) , such that (with φ likewise taken to be quasicontinuous)*

$$\frac{d_Y(\varphi_1, \varphi) + \dots + d_Y(\varphi_n, \varphi)}{n} \rightarrow 0 \quad \text{q.e. in } X \text{ as } n \rightarrow \infty.$$

For any $\varphi \in \mathcal{E}_{\text{loc}}(X, Y)$, $\text{tr}_\sigma \varphi$ denotes the trace of φ on σ ; it is of class $L^2(\sigma)$, σ being endowed with $(m - 1)$ -dimensional Hausdorff measure \mathcal{H}_{m-1} , [25, §1.12], [9, p. 153]. In terms of a triangulation $T = (K, \theta)$ of X (see text preceding (2.1)) the measure \mathcal{H}_{m-1} on $\sigma = \theta(|\tau|)$ is equivalent to Lebesgue measure on the geometric $(m - 1)$ -simplex $|\tau|$. For a quasicontinuous version of φ , $\text{tr}_\sigma \varphi$ equals φ \mathcal{H}_{m-1} -a.e. on σ , see [13, p. 760].

Proof of Lemma 2.5. The stated lower semicontinuity of energy as well as property (a) are due to [25, Theorems 1.6.1 and 1.12.2] for a Riemannian domain X ; for the present setting of an admissible Riemannian polyhedron, see [9, Lemma 9.1].

For the proof of (b), the quasicontinuous functions $u_n := d_Y(\varphi_n, \varphi)$ are of class $\mathcal{E}(X, \mathbb{R}) \cap L^2(X, \mu) = W^{1,2}(X)$. By the triangle inequality we have

$$|u_n(x) - u_n(x')| \leq d_Y(\varphi_n(x), \varphi_n(x')) + d_Y(\varphi(x), \varphi(x')),$$

and hence by (2.19) and (2.23) together with $(a + b)^2 \leq 2a^2 + 2b^2$

$$\int_X |\nabla u_n|^2 d\mu = E(u_n) \leq 2E(\varphi_n) + 2E(\varphi).$$

Thus (u_n) is a norm-bounded sequence in the Hilbert space $W^{1,2}(X)$. By passing to a subsequence, still denoted (u_n) , we achieve that u_n converges weakly in $W^{1,2}(X)$ to some $u \in W^{1,2}(X)$. By the Banach–Saks theorem (cf. e.g. [32, p. 80]), we therefore have (after passing to a further subsequence (u_n)) $\sum_{i=1}^n u_i/n \rightarrow u$ in norm in $W^{1,2}(X)$, in particular in $L^2(X)$, and pointwise q.e. in X , by Lemma 2.4(b) applied to the sequence $u_n - u \rightarrow 0$ in $W^{1,2}(X)$, taking u quasicontinuous. By hypothesis,

$$\int_X u_n^2 d\mu = \int_X d_Y^2(\varphi_n, \varphi) d\mu = D^2(\varphi_n, \varphi) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

It follows that $\|\sum_{i=1}^n u_i/n\|_{L^2(X)} \leq \sum_{i=1}^n \|u_i\|_{L^2(X)}/n \rightarrow 0$ because convergence implies Césaro summability. Thus $u = 0$ μ -a.e., and even q.e.; cf. Lemma 2.4(a). We conclude that indeed $\sum_{i=1}^n u_i/n \rightarrow u = 0$ q.e. according to Lemma 2.4(b) after passing to yet a further subsequence. \square

Remark 2.6. Here are three possible definitions of *harmonicity* of a *continuous* map $\varphi \in \mathcal{E}_{\text{loc}}(X, Y)$ (X any locally compact separable admissible Riemannian polyhedron and Y any complete metric space):

- (a) φ is *locally E -minimizing*, that is, the topology on X has a base of connected open sets $U \Subset X$ for each of which $E(\psi|U) \geq E(\varphi|U)$ for every map $\psi \in \mathcal{E}_{\text{loc}}(X, Y)$ with $\psi = \varphi$ in $X \setminus U$ (this notion does not require continuity of φ);
- (b) the same requirement for *continuous* comparison maps $\psi \in \mathcal{E}_{\text{loc}}(X, Y)$;
- (c) φ is *bi-locally E -minimizing*, that is: X has a base of connected open sets $U \Subset X$ for each of which there is an open set $V \supset \varphi(U)$ in Y such that $E(\psi|U) \geq E(\varphi|U)$ for every *continuous* map $\psi \in \mathcal{E}_{\text{loc}}(X, Y)$ with $\psi(U) \subset V$ and $\psi = \varphi$ in $X \setminus U$.

Slightly more generally, for any open subset A of X , a map $\varphi : A \rightarrow Y$ is said to have one of these three properties if the restriction of φ to each component of A has that property (recall that a connected open subset of X has the properties required of X).

Of these three definitions of harmonicity, (a) is the one most commonly used; (b) is used for example in [25, §2.7] (the free homotopy problem); and (c) in [9, Chapter 12] (maps into Riemannian manifolds) and in Step 4 of the proof of Theorem 3.2 below. In each case (a), (b), or (c), one might equally well require that U is an *open star* in a triangulation of X fine enough that U is the interior of its closure; in particular, $\mu(\partial U) = 0$. Indeed, given a point $p \in X$ and a neighbourhood N of p , consider any set U as in (x) (where (x)

stands for (a), (b), or (c) such that $p \in U \subset N$. Choose open sets U_1, U_2 so that $p \in U_1 \Subset U_2 \Subset U$. By [42, Theorem 35], applied to the open cover of X by U_2 and $X \setminus \overline{U}_1$, there exists a subdivision T of the given triangulation of X such that an open star $S \ni p$ satisfies $S \subset U_2 \Subset U$. For the proof that (x) holds with U replaced by S , consider any map $\psi \in \mathcal{E}_{\text{loc}}(X, Y)$ such that $\psi = \varphi$ in $X \setminus S$. [In case (b) or (c), take moreover ψ continuous. In case (c), take an open set $V \subset Y$ so that $\varphi(U) \subset V$, and suppose that $\psi(S) \subset V$; then $\varphi(S) \subset \varphi(U) \subset V$, and $\psi(U) \subset V$ because $\psi(U \setminus S) = \varphi(U \setminus S) \subset V$.] It follows that $\psi = \varphi$ in $X \setminus U (\subset X \setminus S)$, and so $E(\psi|U) \geq E(\varphi|U)$. By Lemma 2.3, with X, A replaced by $U, U \setminus S$, we conclude that indeed $E(\psi|S) \geq E(\varphi|S)$ because $\mu(\partial S) = 0$ and because $e(\psi) = e(\varphi)$ μ -a.e. in $U \setminus S$.

If X is *noncompact* then, by the latter argument, it would suffice in each definition (a), (b), or (c), to assume that the sets U in question form a *cover* of X . (This would be false for compact X since X is covered by itself, and so property (a) would amount to φ being *globally* E -minimizing.)³

Each of the properties (a), (b), (c) is a *local* one. In the first place, if $\varphi : X \rightarrow Y$ has one of these properties, denoted (x), then so has $\varphi|A$ for every open set $A \subset X$ (we may assume that A is connected). To see this, let N be a neighbourhood of a point p in A , and consider a connected open set $U \Subset A$ such that $p \in U \subset N$ and that (x) holds for $\varphi \in \mathcal{E}_{\text{loc}}(X, Y)$ and for that set U [and moreover for some open set $V \supset \varphi(U)$ in case (c)]. Let $\psi \in \mathcal{E}_{\text{loc}}(A, Y)$ be as in (x) with X replaced by A (and φ by $\varphi|A$), whilst keeping U [and V in case (c)]. Then ψ extends to a well-defined map $\psi^* \in \mathcal{E}_{\text{loc}}(X, Y)$ such that $\psi^* = \varphi$ in $X \setminus A$ and hence in $X \setminus U$, in view of (i) preceding (2.21), applied to the open cover of X by $X \setminus \overline{U}$ and A , noting that $\psi^*|X \setminus \overline{U} = \varphi|X \setminus \overline{U} \in \mathcal{E}_{\text{loc}}(X \setminus \overline{U}, Y)$ and that $\psi^*|A = \psi \in \mathcal{E}_{\text{loc}}(A, Y)$ [hence $\psi^*(U) = \psi(U) \subset V$ in case (c)]. By hypothesis, $E(\psi^*|U) \geq E(\varphi|U)$, that is $E(\psi|U) \geq E(\varphi|U)$ since $\psi^* = \psi$ in U . Thus $\varphi|A$ indeed has the property (x) in question.

Secondly, let (A_n) be a sequence (or family) of open subsets of X , and write $A = \bigcup_n A_n$. For a map $\varphi : A \rightarrow Y$, if each $\varphi|A_n$ has property (x) then so has φ . For a given point $p \in A$ and neighbourhood N of p in A , fix n with $p \in A_n$. By hypothesis, there exists an open set $U \Subset A_n$ with $p \in U \subset N$ [and moreover an open set $V \subset Y$ in case (c)] such that (x) holds with X replaced by A_n (and φ by $\varphi|A_n$). Let $\psi \in \mathcal{E}_{\text{loc}}(A, Y)$ be as stated in (x)

³In [9, Definitions 10.1 and 12.1], and elsewhere concerning these definitions of harmonicity, it should thus have been required, as above, that the sets U form a *base* for X (not just a cover), or else that X is *noncompact*.

(now with X replaced by A). In particular, $\psi = \varphi$ in $A \setminus U$ [and $\psi(U) \subset V$ in case (c)]. The map $\psi_n := \psi|_{A_n} \in \mathcal{E}_{\text{loc}}(A_n, Y)$ then satisfies $\psi_n = \varphi$ in $A_n \setminus U$ [and $\psi_n(U) = \psi(U) \subset V$ in case (c)]. It follows that $E(\psi_n|U) \geq E(\varphi|U)$, that is, $E(\psi|U) \geq E(\varphi|U)$, noting that $\psi_n = \psi$ in $U \Subset A_n \subset A$.

If $E(\varphi) < \infty$, the requirement $U \Subset X$ in (a), (b), or (c) (redundant when X is compact) may equally well be replaced by $U \subset X$ and $\mu(\partial U) = 0$ in each of the three definitions in view of Lemma 2.3 (because the relatively compact open sets U from a base likewise constitute a base). Furthermore (again if $E(\varphi) < \infty$), the inequality $E(\psi|U) \geq E(\varphi|U)$ becomes equivalent with $E(\psi) \geq E(\varphi)$ provided that $\mu(\partial U) = 0$; this follows again from Lemma 2.3 because $\psi = \varphi \in X \setminus U$.

Lemma 2.7. *Let (Y, d_Y) be a complete geodesic space of nonpositive curvature. The three definitions of harmonicity (a), (b), and (c) in Remark 2.6 are equivalent for any continuous map $\varphi \in \mathcal{E}_{\text{loc}}(X, Y)$. The property of being a harmonic map from an open subset of X to Y is a local one.*

Proof. Clearly, (a) \Rightarrow (b) \Rightarrow (c) (since we may take $V = Y$ in (c)), so it remains to prove that (c) \Rightarrow (a). Consider first the case that Y is *simply connected*. Suppose that $\varphi \in \mathcal{E}_{\text{loc}}(X, Y)$ is continuous and satisfies (c). For a given point $p \in X$ and a neighbourhood N of p let U and V be as in (c), with $p \in U \subset N$. As shown in Remark 2.6 we may assume that U is an open star in a triangulation T of X and that U is the interior of its closure in X . Because (a) is a local property (see again Remark 2.6) and because the open stars U of T cover X , it suffices to prove that the restriction $\varphi|U$ to each star U has the property (a). Note that U satisfies the *Poincaré inequality* (4.9) below, according to [13, Lemma 1(c)] applied to $X = \bar{U}$. After subdividing T let $S \Subset U$ denote an open star in U such that $p \in S$ and that S is the interior of its closure \bar{S} . By [14, Theorem 1(a)] there exists a unique map $\varphi^* \in \mathcal{E}(U, Y)$ which minimizes the energy among all maps ψ in $\mathcal{E}(U, Y)$ such that $\psi = \varphi$ on $U \setminus S$. In particular, $\varphi^*|S$ is *locally E-minimizing*. Indeed, after a still further subdivision, let s denote an open star in S such that s is the interior of its closure \bar{s} in S . Consider any map $\sigma \in \mathcal{E}_{\text{loc}}(S, Y)$ such that $\sigma = \varphi^*$ on $S \setminus s$. The extension σ^* of σ to U defined by $\sigma^* = \varphi^*$ on $U \setminus S$, and hence on $U \setminus s$, is of class $\mathcal{E}_{\text{loc}}(U, Y)$ (since $\varphi^* \in \mathcal{E}(U, Y)$ and $\sigma \in \mathcal{E}_{\text{loc}}(S, Y)$). Because $\sigma^* = \varphi^* = \varphi$ on $U \setminus S$ we conclude from the definition of φ^* that $E(\sigma^*) \geq E(\varphi^*)$, and so indeed $E(\sigma) \geq E(\varphi)$ by Lemma 2.3(b), noting that $\sigma^* = \varphi^*$ on $U \setminus S$ and that $\mu(\partial s) = 0$. The locally E -minimizing map φ^* has a (Hölder) continuous version, by [9, Theorem 10.1]. The star S is *regular*,

[9, Proposition 7.1]. By [14, Theorem 1(b)], φ^* therefore is continuous on \bar{S} , and hence in all of U because $\varphi^*|U \setminus S = \varphi|U \setminus S$ is continuous.

For the proof that $\varphi|U$ has property (a) (locally E -minimizing) consider, as above, any map $\psi \in \mathcal{E}_{\text{loc}}(U, Y)$ such that $\psi = \varphi$ in $U \setminus S$. Being simply connected, Y is a CAT(0)-space, by the Cartan–Hadamard theorem, cf. [2, p. 193f]. For $t \in [0, 1]$ define on U the map $\varphi_t = (1 - t)\varphi + t\varphi^*$ into Y (with the usual abuse of notation), and note that $\varphi_t = \varphi^* = \varphi$ on $U \setminus S$. Because $\varphi(\bar{S})$ and $\varphi^*(\bar{S})$ are compact and that geodesics in Y vary continuously with their endpoints ([2, p. 160]), it follows from $\varphi(\bar{S}) \subset \varphi(U) \subset V$ that $\varphi_t(\bar{S}) \subset V$ for small t . However, $\varphi_t(U \setminus S) = \varphi(U \setminus S) \subset \varphi(U) \subset V$, and so altogether $\varphi_t(U) \subset V$. Since $\varphi|U$ has property (c) it follows that

$$E(\varphi|U) \leq E(\varphi_t) \leq (1 - t)E(\varphi|U) + tE(\varphi^*),$$

by (c) and by the energy convexity inequality [9, Equation (11.2)]. We conclude that $E(\varphi|U) \leq E(\varphi^*) \leq E(\psi)$, and so $\varphi|U$ indeed has property (a) from Remark 2.6 on each U considered above, and therefore on all of X .⁴

If Y is not simply connected, pass to the universal covers \tilde{X}, \tilde{Y} of X, Y , and note that \tilde{X} can be covered by open subsets \tilde{A} projected isometrically onto open subsets $p_X(\tilde{A}) = A$ of X covering X . If $\varphi : X \rightarrow Y$ satisfies (c) then so does each $\varphi|A$ (see Remark 2.6 above), hence also $\tilde{\varphi}|A$, by isometry, letting the set $p_Y^{-1}(V) \subset \tilde{Y}$ play the role of V . It follows (again by Remark 2.6) that the map $\tilde{\varphi} : \tilde{X} \rightarrow \tilde{Y}$ satisfies (c). Because \tilde{Y} is simply connected, $\tilde{\varphi}$ satisfies (a) as well, and so does each $\tilde{\varphi}|A$, hence each $\varphi|A$, by isometry; and consequently $\varphi : X \rightarrow Y$ indeed has property (a). The latter assertion of Lemma 2.7 holds in view of Remark 2.6. \square

Remark 2.8. As mentioned in [9, Remark 12.1] the implication (c) \Rightarrow (a) in Remark 2.6 likewise holds when (Y, d_Y) is a complete geodesic space of curvature $\leq \kappa$ for some constant $\kappa > 0$, provided that the image $\varphi(X)$ is a subset of some closed convex ball $B = B_Y(o, R)$ in Y of radius $R < \pi/(2\sqrt{\kappa})$

⁴In the justification for Lemma 2.7 with Y simply connected (proposed in [9, Lemma 2.7]), the reference to [9, Theorem 11.3] was insufficient; it should be replaced as above by [13, Theorem 1], which includes continuity of the Dirichlet solution up to the boundary (but that result was not available at the time). This error has no consequences because the lemma was not used in the proof of [13, Theorem 1], nor in earlier work [9, 11, 12]. Moreover, at the quoted place in [9], “ $\varphi^*(S)$ for small t ” should read “ $\varphi_t(S)$ for small t ” (as above).

and such that B is a $\text{CAT}(\kappa)$ -space. The above proof of Lemma 2.7 carries over right away.⁵

Remark 2.9. In the setting of [19], if a harmonic map $\varphi : X \rightarrow Y$ (X a Riemannian manifold) is constant in some nonvoid open subset Ω of X , then φ is constant in all of X (assumed connected), see [19, p. 204]; this was shown there to imply that the order of the map is well defined in all of Ω , which is essential for the theory developed in [19]. In the present setting, where X is a polyhedron, the concept of order does not seem to be available. The stated uniqueness property is not relevant here, and actually fails in the simplest possible situation: Let X be a tripod, obtained from the disjoint union of three copies X_1, X_2, X_3 of $[0, 1]$ by identifying the three endpoints 0, cf. [19, p. 178]. With $Y = \mathbb{R}$, take $\varphi|_{X_1} = 0$, $\varphi|_{X_2} = \text{id}$, and $\varphi|_{X_3} = -\text{id}$, where id denotes the identity map of $[0, 1]$, cf. [9, Remark 5.3]. Then $\varphi : X \rightarrow \mathbb{R}$ is harmonic (see [9, Example 8.1]), and nonconstant, though constant on X_1 , which has nonvoid interior.

3. The free homotopy problem

In the case of free homotopy there is no control of the growth of a minimizing sequence of equivariant maps; such a sequence may in fact converge to infinity in the locally compact space \tilde{Y} , [25, p. 647 and Remark 2.6.6].

The proof of Theorem 3.2 below is an adaptation of that of [25, Theorems 2.6.4 and 2.7.1] for manifold domains. With the present polyhedral domain (X, g) , harmonic maps generally are no longer Lipschitz continuous, but only Hölder continuous (Proposition 3.1 below); cf. [6], [9, Example 6.1]. The idea of Korevaar and Schoen is to modify a given minimizing sequence of equivariant maps into another which is moreover equicontinuous. The sequence of projected maps therefore subconverges uniformly to a map φ which is shown to belong to the given homotopy class \mathcal{H} and to minimize energy within \mathcal{H} . Lemma 2.7 above allows us to go beyond [25] by proving that, as expected from [10, 19], *every* energy minimizer relative to \mathcal{H} is harmonic. And unlike [25], *we do not assume that $bX = \emptyset$* ;

⁵In analogy with the preceding note the reference in [9, Remark 12.1] to [9, Theorem 11.4] shall now be replaced by [13, Theorem 3] (noting that the additional hypothesis there that either Y be locally compact or $R < \pi/(4\sqrt{\kappa})$ can be omitted according to [17, Theorem 2]).

thus X may be, more generally, any compact admissible Riemannian polyhedron (with or without boundary) with simplex-wise smooth Riemannian metric g .

In the following proposition, however, we relax the requirement that X and Y be compact (because the result will be applied to the universal covers \tilde{X} and \tilde{Y}). See also Remark 3.3. For any compact set $X_0 \subset X$ we denote by Λ_{X_0} an ellipticity constant of g restricted to X_0 [9, p. 48 f]. Recall from [9, 11] that every locally energy minimizing map has a (Hölder) continuous and hence harmonic version.

Proposition 3.1. *Let (X, g) be a locally compact admissible Riemannian polyhedron. For any compact set $X_0 \subset X$ there exist constants $C, \alpha, \varrho > 0$ depending only on X, X_0, Λ_{X_0} such that, for any harmonic map φ from X to any complete simply connected geodesic space (Y, d_Y) of nonpositive curvature,*

$$d_Y(\varphi(x), \varphi(y)) \leq C \sqrt{E(\varphi)} d_X^\alpha(x, y) \quad \text{for } x, y \in X_0 \text{ with } d_X(x, y) \leq \varrho.$$

Proof. In view of [9, Chapters 4 and 5] our locally compact separable admissible Riemannian polyhedron (X, g) satisfies the hypotheses of Birolì and Mosco [3, §1]. According to [3, Theorem 5.4] applied with $p = 2$ to the function $d_Y(\varphi(\cdot), q)$, $q \in Y$, which is weakly subharmonic by [9, Lemma 10.2(a)] (or see [17, Lemma 2]), we may choose $\varrho > 0$, depending only on X, X_0, Λ_{X_0} , so that, by the triangle inequality for (Y, d_Y) , the oscillation $\omega(B_X(a, r))$ of φ over $B_X(a, r)$ for $a \in X_0$ and $0 < r \leq \varrho$ satisfies the first two of the following three inequalities

$$\begin{aligned} \omega(B_X(a, r)) &\leq 2 \inf_{q \in Y} \sup_{B_X(a, r)} d_Y(\varphi(\cdot), q) \\ &\leq 2c' \inf_{q \in Y} \left(\int_{B_X(a, 2r)} d_Y^2(\varphi(\cdot), q) d\mu \right)^{1/2} \\ (3.1) \qquad &\leq c'' r^{1-m/2} \sqrt{E(\varphi)} \end{aligned}$$

for constants c', c'' depending on X, X_0, Λ_{X_0} only. The last inequality follows by application of [9, Lemma 4.4] (on ball volumes) and the weak Poincaré inequality [11, Corollary 1, p. 381] which clearly holds also for noncompact X when $a \in X_0 \Subset X$ (again with constants depending on X, X_0, Λ_{X_0}).

Choose $\varrho > 0$ small enough that [11, Lemma 3 and Proposition 1(b)] hold with $\kappa = 5$. These two results from [11] shall then replace, in [9, pp. 186–189], the repeated use of [9, Sublemma 10.1 and Lemma 10.1(a)],

respectively. It follows by inspection of its proof that [9, Lemma 10.3] holds uniformly in the same sense for $a \in X_0 \Subset X$. Local compactness of (Y, d_Y) was only used towards the end of the proof of [9, Theorem 10.1] when a point q was chosen (p. 191) in the intersection of the lower directed family of nonvoid *compact* essential images $\varphi(B_X^e(a, r))$, $r > 0$. If Y is not locally compact, choose instead any point $q_j = q_j(r)$ of $\varphi(B_X^e(a, \beta^j r))$, $j = 1, 2, \dots$. Recall from [9, p. 190] that ε , p , and hence β , are constants > 0 . Take j large enough so that $\beta^j < \varepsilon^p$. Then

$$q_j \in \varphi(B_X^e(a, \beta^j r)) \subset \varphi(B_X^e(a, \varepsilon^p r)).$$

It follows that Equations (10.15) and (10.16) in [9] hold with q replaced by q_j (cf. the text following Equation (10.7), and Equations (10.13), (10.14) there). We then find that there are positive constants R, A, α depending only on X, X_0, Λ_{X_0} such that, for any $r_0 \leq R$ and $r \leq r_0$,

$$\omega(B_X(a, r)) \leq A(r/r_0)^\alpha \omega(B_X(a, r_0)).$$

Taking $r_0 = \varrho$ from the text preceding (3.1) we obtain for $x, y \in X_0$ with $a = y$ and $r := d_X(x, y) \leq \varrho$,

$$\begin{aligned} d_Y(\varphi(x), \varphi(y)) &\leq \omega(B_X(y, r)) \leq A\varrho^{-\alpha} \omega(B_X(y, \varrho))r^\alpha \\ &\leq Ac'' \varrho^{1-m/2-\alpha} \sqrt{E(\varphi)} d_X^\alpha(x, y), \end{aligned}$$

which was to be proved. □

Theorem 3.2. *Let (X, g) be a compact admissible Riemannian polyhedron, and let (Y, d_Y) be a compact geodesic space of nonpositive curvature. Every (free) homotopy class \mathcal{H} of continuous maps $X \rightarrow Y$ has an energy minimizer; and any such is harmonic and uniformly Hölder continuous. Two energy minimizers relative to \mathcal{H} are identical if they agree at a point.*

More generally (as expected from [21] in the classical setting of [9]): for any two energy minimizers φ_0, φ_1 relative to \mathcal{H} there exists a homotopy $[0, 1] \ni t \mapsto \varphi_t \in \mathcal{H}$ such that, for each $x \in X$, the path $[0, 1] \ni t \mapsto \varphi_t(x) \in Y$ is a constant-speed geodesic segment (not necessarily minimizing) of length \tilde{d} independent of x . If $\tilde{d} < r_Y$ (cf. (2.2)) then $\tilde{d} = d_Y(\varphi_0(x), \varphi_1(x))$, $x \in X$, and the geodesic segment $t \mapsto \varphi_t(x)$ becomes minimizing. And if $\varphi_0(x) = \varphi_1(x)$ for some $x \in X$ then $\tilde{d} = 0$ and hence $\varphi_0 = \varphi_1$.

Simple examples such as $X = Y = S^1$ show that there may be infinitely many minimizers in every homotopy class. In [25, Remark 2.6.6] the problem

is posed as to when energy minimizers relative to \mathcal{H} are unique up to domain or codomain isometries (as they are when $X = Y = S^1$).

Proof of Theorem 3.2. In view of Whitehead [42, Theorem 35, p. 317] or see [37, Theorem 14, p. 125] there is a subdivision T of the given triangulation of X such that every open star S of (X, T) is contained in some ball $B_X(x', \frac{1}{2}r_X)$ with r_X as in (2.2). For any $x \in S$ we therefore have $x \in B_X(x', \frac{1}{2}r_X)$, and so $S \subset B_X(x', \frac{1}{2}r_X) \subset B_X(x, r_X)$. Accordingly, the connectivity components of $p_X^{-1}(S)$ have disjoint closures, each mapped isometrically onto \bar{S} by p_X . By a further subdivision T' of T we arrange that, for any $x \in X$, there exists a star $\bar{S} = S(x)$ of (X, T) such that every star S' of (X, T') containing x has closure \bar{S}' contained in S .⁶ In particular, if two stars S'_1, S'_2 of (X, T') have a point x in common then $\bar{S}'_1 \cup \bar{S}'_2 \subset S(x)$.

Step 1. Construction of a modified sequence $(\tilde{\varphi}_i)$. Let $B^j, j \in \{1, \dots, k\}$, denote the open stars of (X, T') . Here $k \geq 2$, for if $k = 1$ we would have $B^1 = X$, hence $X =$ a single vertex (X being compact), contradicting our hypothesis $\dim X \geq 1$ from the beginning of Section 2. Following [25, p. 649], choose a Lipschitz partition of unity $(\eta^j)_{j=1, \dots, k}$ on X , subordinate to the cover (B^j) , cf. [9, p. 41 f]; and for each $j \in \{1, \dots, k\}$ two compact subsets Z^j and W^j of X so that the support $\text{supp } \eta^j$ is contained in the interior of Z^j , that Z^j is contained in the interior of W^j , and that $W^j \subset B^j$. In particular, $\bigcup_j Z^j = X$, and hence $\bigcup_j \tilde{Z}^j = \tilde{X}$ when defining $\tilde{Z}^j = p_X^{-1}(Z^j)$. As in [25, p. 649 (bottom)] (where, however, the B^j are balls), consider the Γ_X -invariant pre-images $p_X^{-1}(B^j)$, and the Γ_X -invariant Lip functions $\tilde{\eta}^j = \eta^j \circ p_X$ on \tilde{X} . Clearly $\sum_1^k \tilde{\eta}^j = 1$.

Fix a Lipschitz map $\chi \in \mathcal{H}$ (see Proposition 2.2(b), which replaces [25, Proposition 2.6.1]), a corresponding homomorphism $\chi_* : \Gamma_X \rightarrow \Gamma_Y$ (see Proposition 2.1), a base point $x_0 \in X$, and a point $\tilde{x}_0 \in p_X^{-1}(x_0)$. For any map $\varphi \in \mathcal{H}$, a χ_* -equivariant lift $\tilde{\varphi} : \tilde{X} \rightarrow \tilde{Y}$ of φ is obtained by choosing $\tilde{\varphi}(\tilde{x}_0) \in p_Y^{-1}(\varphi(x_0))$ (see (2.3) and the end of the paragraph following Proposition 2.1) so that $\varphi_* = \chi_*$ and hence

$$(3.2) \quad \tilde{\varphi} \circ \gamma = \chi_*(\gamma) \circ \tilde{\varphi} \quad \text{for } \gamma \in \Gamma_X.$$

⁶Every point $x \in X$ lies in some star $S(x)$ of (X, T) , and $S(x)$ contains a ball $B_X(x, r_x)$. Finitely many of these balls cover X , and we define $r =$ the least among these finitely many r_x . Again by [42], choose the subdivision T' of T so that every star S' of (X, T') is a subset of some ball $B_X(x', \frac{1}{2}r)$. For any $x \in X$ and any star S' of (X, T') containing x we have $\bar{S}' \subset B_X(x', \frac{1}{2}r)$ for some $x' \in X$. In particular, $x \in B_X(x', \frac{1}{2}r)$, and so indeed $\bar{S}' \subset B_X(x, r) \subset S(x)$.

Conversely, every χ_* -equivariant map $\tilde{\varphi}$ has in view of (3.2) a projection $\varphi : X \rightarrow Y$ of class \mathcal{H} , defined by (2.3), that is, $\varphi(x) = p_Y(\tilde{\varphi}(\tilde{x}))$ for some $\tilde{x} \in p_X^{-1}(x)$. This makes sense because any other point of $p_X^{-1}(x)$ has the form $\gamma(\tilde{x})$ for some $\gamma \in \Gamma_X$, and so $p_Y((\tilde{\varphi} \circ \gamma)(\tilde{x})) = p_Y((\chi_*(\gamma) \circ \tilde{\varphi})(\tilde{x})) = p_Y(\tilde{\varphi}(\tilde{x}))$ as before, according to (3.2). The projection φ is continuous because we may take $\tilde{x} \in p_X^{-1}(x)$ to be a locally defined, continuous function of $x \in X$. Furthermore, $\varphi \in \mathcal{H}$ by Proposition 2.1, or as shown directly by geodesic homotopy: For $\tilde{x} \in \tilde{X}$ and $t \in [0, 1]$ denote $\tilde{\Phi}(\tilde{x}, t) = (1 - t)\tilde{\varphi}(\tilde{x}) + t\tilde{\chi}(\tilde{x})$ (with the usual abuse of notation). Then $\tilde{\Phi}(\tilde{x}, t)$ is continuous on $\tilde{X} \times [0, 1]$ (see [2, p. 160]), and Equation (3.2) holds with $\tilde{\varphi}$ replaced by $\tilde{\Phi}(\cdot, t)$. The projection $(x, t) \mapsto \Phi(x, t) := p_Y(\tilde{\Phi}(\tilde{x}, t))$ is therefore well defined and continuous as a function of $(x, t) \in X \times [0, 1]$. Because $\Phi(x, 0) = \varphi(x)$ and $\Phi(x, 1) = \chi(x)$, φ is indeed homotopic with χ , i.e., $\varphi \in \mathcal{H}$.

Let $s_\nu, \nu \in \{1, \dots, n\}$, denote the m -simplexes of (X, T') , and let \tilde{F} be the union of corresponding m -simplexes \tilde{s}_ν of \tilde{X} having the projections $p_X(\tilde{s}_\nu) = s_\nu, \nu \in \{1, \dots, n\}$, but otherwise selected arbitrarily. The compact (generally disconnected) set \tilde{F} is then a fundamental “domain” of Γ_X , up to a $\tilde{\mu}$ -nullset.

Consider any χ_* -equivariant map $\tilde{\varphi} \in \mathcal{E}_{\text{loc}}(\tilde{X}, \tilde{Y})$ (one such map is $\tilde{\chi}$). The restriction $\tilde{\varphi}|_{\tilde{F}}$ has finite energy, understood as the sum of the (finite) energies of the restrictions of $\tilde{\varphi}$ to the m -simplexes \tilde{s}_ν of \tilde{F} ; see [9, Theorem 9.1(a)]. Equivalently, $E(\tilde{\varphi}|_{\tilde{F}}) = \int_{\tilde{F}} e(\tilde{\varphi}) d\tilde{\mu}$, cf. (2.22). The projection $\varphi : X \rightarrow Y$ has finite energy, and

$$(3.3) \quad E(\tilde{\varphi}|_{\tilde{F}}) = E(\varphi), \quad e(\tilde{\varphi})|_{\tilde{F}} = e(\varphi) \circ p_X|_{\tilde{F}}.$$

Indeed, $E(\tilde{\varphi}|_{\tilde{s}_\nu}) = E(\varphi|_{s_\nu}) < \infty$ because $\tilde{\varphi}|_{\tilde{s}_\nu} = (\varphi|_{s_\nu}) \circ (p_X|_{\tilde{s}_\nu})$, where $p_X|_{\tilde{s}_\nu}$ is an isometry of \tilde{s}_ν on s_ν , cf. the first lines of the present proof. By summation over ν we obtain the former equation of (3.3) by [9, Theorem 9.1(a),(b)] because $E(\varphi) < \infty$ and because $E(\varphi) = \sum_\nu E(\varphi|_{s_\nu})$ by [9, Remark 9.1, p. 153]. The latter equation of (3.3) now follows from $e(\tilde{\varphi})|_{\tilde{s}_\nu} = e(\varphi) \circ p_X|_{\tilde{s}_\nu}$.

Conversely, any χ_* -equivariant lift $\tilde{\varphi}$ of a finite-energy map $\varphi \in \mathcal{H}$ has locally finite energy since $E(\tilde{\varphi}|_{\tilde{s}}) = E(\varphi|_s) < \infty$ for any m -simplex \tilde{s} of \tilde{X} (writing $s = p_X(\tilde{s})$), again because $p_X|_{\tilde{s}}$ is an isometry. It follows that $\tilde{\varphi} \in \mathcal{E}_{\text{loc}}(\tilde{X}, \tilde{Y})$ (again by [9, Theorem 9.1(a),(b)]), so the above applies. We may thus define

$$(3.4) \quad \begin{aligned} E_0 &= \inf\{E(\varphi) : \tilde{\varphi} \text{ is } \chi_*\text{-equivariant and of class } \mathcal{E}_{\text{loc}}(\tilde{X}, \tilde{Y})\} \\ &= \inf\{E(\varphi) : \varphi \in \mathcal{H}\} \quad (< \infty). \end{aligned}$$

Consider a sequence $(\tilde{\psi}_i)$ of χ_* -equivariant maps which is *minimizing* for (3.4) in the sense that the projections ψ_i of $\tilde{\psi}_i$ satisfy the equation

$$(3.5) \quad \lim_{i \rightarrow \infty} E(\psi_i) = E_0.$$

As in [25, p. 650] we begin by constructing for each $i \in \{1, 2, \dots\}$ and $j \in \{1, 2, \dots, k\}$ a unique χ_* -equivariant energy minimizer $\tilde{\varphi}_i^j$ among all χ_* -equivariant maps which agree with $\tilde{\psi}_i$ on $\tilde{X} \setminus p_X^{-1}(B^j)$; so $\tilde{\varphi}_i^j$ solves an ordinary Dirichlet problem. For the moment, fix $j \in \{1, \dots, k\}$ and one of the connectivity components \tilde{B}^j of $p_X^{-1}(B^j)$; thus \tilde{B}^j is an open star in \tilde{X} contained in some ball in \tilde{X} of radius $\frac{1}{2}r_X$ (see the first lines of the proof). The closure \tilde{X}^j of \tilde{B}^j is a closed star and thus a compact admissible Riemannian subpolyhedron of \tilde{X} with nonvoid boundary $b\tilde{X}^j = \tilde{X}^j \setminus \tilde{B}^j$.

According to [13, Theorem 1(a)] applied to the polyhedron \tilde{X}^j and the prescribed boundary map $\text{tr}_{b\tilde{X}^j} \tilde{\psi}_i$ (the trace of $\tilde{\psi}_i$ on $b\tilde{X}^j$) there exists a unique map $\tilde{\theta} : \tilde{X}^j \rightarrow \tilde{Y}$ which minimizes the energy among all finite-energy maps $\tilde{X}^j \rightarrow \tilde{Y}$ having the same trace on $b\tilde{X}^j$ as $\tilde{\psi}_i$, and that minimizer $\tilde{\theta}$ is *locally E-minimizing* in \tilde{B}^j . In the first place, \tilde{B}^j is noncompact and covered by the open stars \tilde{U} in a triangulation of \tilde{B}^j . Every map $\tilde{\beta} \in \mathcal{E}_{\text{loc}}(\tilde{B}^j, \tilde{Y})$ such that $\tilde{\beta} = \tilde{\theta}$ on $\tilde{B}^j \setminus \tilde{U}$ extends by $\tilde{\theta}|_{b\tilde{X}^j} = \tilde{\psi}_i|_{b\tilde{X}^j}$ to a map $\tilde{\xi} \in \mathcal{E}(\tilde{X}^j, \tilde{Y})$, for if \tilde{C} denotes the closure of \tilde{U} then the restrictions $\tilde{\theta}|_{\tilde{X}^j \setminus \tilde{C}}$ and $\tilde{\beta}$ of $\tilde{\xi}$ to the open sets $\tilde{X}^j \setminus \tilde{C}$ and \tilde{B}^j , respectively, covering \tilde{X}^j , have locally finite energy, and so property (i) preceding Equation (2.21) is fulfilled. Secondly, $E(\tilde{\beta}) = E(\tilde{\xi}) \geq E(\tilde{\theta})$, and hence $E(\tilde{\beta}|\tilde{U}) \geq E(\tilde{\theta}|\tilde{U})$ by Lemma 2.3 because $\tilde{\mu}(\partial\tilde{U}) = 0$ and $\tilde{\beta} = \tilde{\theta}$ in $\tilde{B}^j \setminus \tilde{U}$. Having thus verified that $\tilde{\theta}$ is *locally E-minimizing* in \tilde{B}^j , we infer from [9, Theorem 10.1] that $\tilde{\theta}$ is continuous in \tilde{B}^j (after being redefined on a $\tilde{\mu}$ -nullset), and hence *harmonic* in \tilde{B}^j ; cf. Remark 2.6. Furthermore, $\tilde{\theta}$ is *continuous* on \tilde{X}^j , again by [13, Theorem 1(a)], because $\tilde{\psi}_i$ is continuous, in particular on $b\tilde{X}^j$.

Much as in [28, p. 63] we define the following extension $\tilde{\varphi}_i^j$ of $\tilde{\theta}$ to all of \tilde{X} :

$$(3.6) \quad \tilde{\varphi}_i^j = \begin{cases} \chi_*(\gamma) \circ \tilde{\theta} \circ \gamma^{-1} & \text{in } \gamma(\tilde{B}^j) \text{ for any } \gamma \in \Gamma_X \\ \tilde{\psi}_i & \text{elsewhere in } \tilde{X}. \end{cases}$$

Thus $\tilde{\varphi}_i^j$ agrees with $\tilde{\psi}_i$ outside $p_X^{-1}(B^j) = \bigcup_{\gamma \in \Gamma_X} \gamma(\tilde{B}^j)$. Clearly, $\tilde{\varphi}_i^j$ is *continuous* (even on $\gamma(b\tilde{X}^j)$, $\gamma \in \Gamma_X$), and each $\tilde{\varphi}_i^j$ is of class $\mathcal{E}_{\text{loc}}(\tilde{X}, \tilde{Y})$, like $\tilde{\psi}_i$. Furthermore, $\tilde{\varphi}_i^j : \tilde{X} \rightarrow \tilde{Y}$ is χ_* -equivariant (cf. (3.2)) along with $\tilde{\psi}_i$; and $\tilde{\varphi}_i^j$ is *Hölder continuous and harmonic* in $\gamma(\tilde{B}^j)$, along with $\tilde{\theta}$ and $\tilde{\theta} \circ \gamma^{-1}$.

The projection $\varphi_i^j \in \mathcal{E}(X, Y)$ of $\tilde{\varphi}_i^j \in \mathcal{E}_{\text{loc}}(\tilde{X}, \tilde{Y})$ agrees with ψ_i outside $B^j = p_X(\tilde{B}^j) = p_X(\gamma(\tilde{B}^j))$. The energy $E(\varphi_i^j)$ satisfies

$$(3.7) \quad E_0 \leq E(\varphi_i^j) \leq E(\psi_i).$$

This is because $E(\tilde{\theta}) \leq E(\tilde{\psi}_i|\tilde{X}^j)$ by definition of $\tilde{\theta}$, and hence by (3.6) and $\gamma = \text{id}$, followed by projection: $E(\varphi_i^j|X^j) \leq E(\psi_i|X^j)$, where $X^j := p_X(\tilde{X}^j)$. Since $\mu(X^j \setminus B^j) = 0$ it follows by Lemma 2.3(a) that $E(\varphi_i^j|B^j) \leq E(\psi_i|B^j)$. For each j the sequence of χ_* -equivariant maps $\tilde{\varphi}_1^j, \tilde{\varphi}_2^j, \dots$ is therefore *minimizing*, along with the sequence $(\tilde{\psi}_i)$.

The desired modification $\tilde{\varphi}_i$ of $\tilde{\psi}_i$ is now defined (as in [25, p. 650]) by having for its value at $\tilde{x} \in \tilde{X}$ the average of the L^2 -maps $\tilde{\varphi}_i^1, \dots, \tilde{\varphi}_i^k$ from (3.6) with respect to the probability measure $\nu_{\tilde{x}}$ on $\{1, \dots, k\}$ given by

$$(3.8) \quad \nu_{\tilde{x}}(\{j\}) = \tilde{\eta}^j(\tilde{x}), \quad j \in \{1, \dots, k\},$$

with $\tilde{\eta}^j(\tilde{x})$ as defined in the first paragraph of the present Step 1. In terms of [25, Lemma 2.5.1] this means that, for each $\tilde{x} \in \tilde{X}$,

$$(3.9) \quad \tilde{\varphi}_i(\tilde{x}) \text{ minimizes } \sum_{j=1}^k \tilde{\eta}^j(\tilde{x}) d_{\tilde{Y}}^2(\tilde{\varphi}_i^j(\tilde{x}), \tilde{y}) \text{ as a function of } \tilde{y} \in \tilde{Y}.$$

It follows that $\tilde{y} = \tilde{\varphi}_i(\gamma(\tilde{x}))$ minimizes

$$\begin{aligned} \sum_{j=1}^k \tilde{\eta}^j(\gamma(\tilde{x})) d_{\tilde{Y}}^2(\tilde{\varphi}_i^j(\gamma(\tilde{x})), \tilde{y}) &= \sum_{j=1}^k \tilde{\eta}^j(\tilde{x}) d_{\tilde{Y}}^2(\chi_*(\gamma)(\tilde{\varphi}_i^j(\tilde{x})), \tilde{y}) \\ &= \sum_{j=1}^k \tilde{\eta}^j(\tilde{x}) d_{\tilde{Y}}^2(\tilde{\varphi}_i^j(\tilde{x}), (\chi_*(\gamma))^{-1}(\tilde{y})), \end{aligned}$$

which is also minimized by $(\chi_*(\gamma))^{-1}(\tilde{y}) = \tilde{\varphi}_i(\tilde{x})$, that is, by $\tilde{y} = (\chi_*(\gamma))(\tilde{\varphi}_i(\tilde{x}))$. Consequently, $\tilde{\varphi}_i(\gamma(\tilde{x})) = (\chi_*(\gamma))(\tilde{\varphi}_i(\tilde{x}))$; so (3.2) holds (with $\tilde{\varphi}$ replaced by $\tilde{\varphi}_i$).

Step 2. The sequence of maps $\tilde{\varphi}_i : \tilde{X} \rightarrow \tilde{Y}$ is uniformly Hölder equicontinuous. Fix $\delta_0 > 0$ independently of j so that $\delta_0 \leq r_X$ from (2.2) and that

$$(3.10) \quad \begin{aligned} \{x \in X : d_X(x, \text{supp } \eta^j) \leq \delta_0\} &\subset Z^j, \\ W^j := \{x \in X : d_X(x, Z^j) \leq \delta_0\} &\subset B^j \end{aligned}$$

for $j \in \{1, \dots, k\}$; cf. the first paragraph of Step 1. Then Z^j and W^j are compact subsets of B^j . Because \tilde{Y} is simply connected it follows by

Proposition 3.1 with X replaced by \tilde{B}^j that the harmonic maps $\tilde{\varphi}_i^j : \tilde{X} \rightarrow \tilde{Y}$ from (3.6) are uniformly Hölder equicontinuous on the compact set $\tilde{B}^j \cap p_X^{-1}(W^j)$ and even in $\tilde{W}^j := p_X^{-1}(W^j) = \bigcup_{\gamma \in \Gamma_X} \gamma(\tilde{B}^j \cap p_X^{-1}(W^j))$ because $\tilde{\varphi}_i^j$ is χ_* -equivariant, that γ and $\chi_*(\gamma)$ are isometries, and that two points $\tilde{x}, \tilde{y} \in \tilde{W}^j$ with $d_{\tilde{X}}(\tilde{x}, \tilde{y}) \leq \varrho$ lie in the same set $\gamma(\tilde{B}^j \cap p_X^{-1}(W^j))$ when we take $\varrho \leq \delta_0$; cf. (2.2) and subsequent text. Consequently,

$$(3.11) \quad d_{\tilde{Y}}(\tilde{\varphi}_i^j(\tilde{x}), \tilde{\varphi}_i^j(\tilde{y})) \leq C \sqrt{E(\psi_i)} d_{\tilde{X}}^\alpha(\tilde{x}, \tilde{y}) \quad \text{for } \tilde{x}, \tilde{y} \in \tilde{W}^j, d_{\tilde{X}}(\tilde{x}, \tilde{y}) \leq \varrho$$

in view of (3.7) and (3.10), with C, α, ϱ depending on X, Λ_X only. In particular, C, α, ϱ in (3.11) may be taken independently of $i \in \{1, 2, \dots\}$ and $j \in \{1, \dots, k\}$. We may clearly take $\alpha \leq 1$. Note that (3.11) replaces in our setting the Lipschitz continuity established in [25, Theorem 2.6.4].

Continuing much as in [25, p. 652 (mid)], consider points $\tilde{x} \in \tilde{Z}^j \cap \tilde{Z}^l$ ($j, l \in \{1, \dots, k\}$) and $\tilde{y} \in \tilde{X}$ with $d_{\tilde{X}}(\tilde{x}, \tilde{y}) \leq \delta$ for a suitable δ with $0 < \delta \leq \varrho \leq \delta_0$ (δ to be determined in the paragraph containing (3.17) and (3.18) below). Then $\tilde{y} \in \tilde{W}^j \cap \tilde{W}^l$. By (3.11) and the triangle inequality we obtain

$$d_{\tilde{Y}}(\tilde{\varphi}_i^j(\tilde{y}), \tilde{\varphi}_i^l(\tilde{y})) \geq d_{\tilde{Y}}(\tilde{\varphi}_i^j(\tilde{x}), \tilde{\varphi}_i^l(\tilde{x})) - 2C \sqrt{E(\psi_i)} \delta^\alpha,$$

and hence by integration with respect to \tilde{y} (again for $\tilde{x} \in \tilde{Z}^j \cap \tilde{Z}^l \subset p_X^{-1}(B^j \cap B^l)$):

$$(3.12) \quad C_1 \int_{B_{\tilde{X}}(\tilde{x}, \delta)} d_{\tilde{Y}}^2(\tilde{\varphi}_i^j(\tilde{y}), \tilde{\varphi}_i^l(\tilde{y})) d\tilde{\mu}(\tilde{y}) \geq \delta^m (d_{\tilde{Y}}(\tilde{\varphi}_i^j(\tilde{x}), \tilde{\varphi}_i^l(\tilde{x})) - 2C \sqrt{E(\psi_i)} \delta^\alpha)^2$$

provided that $d_{\tilde{Y}}(\tilde{\varphi}_i^j(\tilde{x}), \tilde{\varphi}_i^l(\tilde{x})) \geq 2C \sqrt{E(\psi_i)} \delta^\alpha$. (We have applied [9, Lemma 4.4] about ball volumes.) This inequality (3.12) replaces [25, (2.6xvii)] in our setting.

Because $x := p_X(\tilde{x}) \in B^j \cap B^l$, we have $B^j \cup B^l \in S$ for some star S of (X, T) , cf. the beginning of the proof. Denote by $\tilde{S}, \tilde{B}^j, \tilde{B}^l$ the component of $p_X^{-1}(S), p_X^{-1}(B^j), p_X^{-1}(B^l)$, respectively, containing the given point \tilde{x} . This definition of \tilde{B}^j (and \tilde{B}^l) agrees with the previous one (in the paragraph following (3.5)); and we have $\tilde{B}^j \cup \tilde{B}^l \in \tilde{S}$. Choose a compact “fundamental domain” $\tilde{F} \subset \tilde{X}$ for p_X relative to T' (cf. the paragraph preceding the one containing (3.3)) so that \tilde{F} contains \tilde{S} ; this is possible because \tilde{S} is contained in some ball in \tilde{X} of radius r_X , so no two simplexes of \tilde{S} are congruent modulo Γ_X ; cf. text following (2.2). The function $d_{\tilde{Y}}(\tilde{\varphi}_i^j, \tilde{\varphi}_i^l)$ considered on \tilde{F} equals zero outside $\tilde{B}^j \cup \tilde{B}^l \in \tilde{S}$ because $\tilde{\varphi}_i^j = \psi_i$ in $\tilde{F} \setminus \tilde{B}^j$ whereas $\tilde{\varphi}_i^l = \psi_i$

in $\tilde{F} \setminus \tilde{B}^l$. It follows that

$$(3.13) \quad \int_{\tilde{S}} d_{\tilde{Y}}^2(\tilde{\varphi}_i^j, \tilde{\varphi}_i^l) d\tilde{\mu} \leq C_2 \int_{\tilde{S}} |\nabla d_{\tilde{Y}}(\tilde{\varphi}_i^j, \tilde{\varphi}_i^l)|^2 d\tilde{\mu}$$

by the Poincaré inequality (4.9) below (for the present \tilde{S}). Now apply the energy convexity inequality [25, (2.2vi); 9, (11.2)] (cf. Equation (3.33) below), with $t = \frac{1}{2}$ to the χ_* -equivariant maps $\tilde{\varphi}_i^j, \tilde{\varphi}_i^l$, and their χ_* -equivariant midpoint map \tilde{w}_i^{jl} with projection w_i^{jl} , all three maps $\tilde{X} \rightarrow \tilde{Y}$ being restricted to the m -simplexes \tilde{s}_ν of \tilde{F} . By adding over $\nu = 1, \dots, n$ we obtain in view of (3.3), (3.4) and (3.7) the bound

$$(3.14) \quad \int_{\tilde{S}} |\nabla d_{\tilde{Y}}(\tilde{\varphi}_i^j, \tilde{\varphi}_i^l)|^2 d\tilde{\mu} \leq 4(E(\psi_i) - E(w_i^{jl}))$$

for the integral on the right-hand side of (3.13) because $\tilde{S} \subset \tilde{F}$ and $\tilde{\varphi}_i^j = \tilde{\varphi}_i^l$ in $\tilde{F} \setminus \tilde{S}$, hence $\nabla d_{\tilde{Y}}(\tilde{\varphi}_i^j, \tilde{\varphi}_i^l) = 0$ $\tilde{\mu}$ -a.e. in $\tilde{F} \setminus \tilde{S}$ by [9, Remark 5.2]. We have $B_{\tilde{X}}(\tilde{x}, \delta) \subset B_{\tilde{X}}(\tilde{x}, \delta_0) \subset \tilde{W}^j \cap \tilde{W}^l$. By definition of \tilde{B}^j, \tilde{B}^l it follows that $B_{\tilde{X}}(\tilde{x}, \delta_0) \subset \tilde{B}^j \cap \tilde{B}^l \subset \tilde{B}^j \cup \tilde{B}^l$. We may therefore combine (3.12), (3.13), and (3.14), to get

$$(3.15) \quad d_i \leq 2C\sqrt{E(\psi_i)}\delta^\alpha + 2C_1^{1/2}C_2^{1/2}\delta^{-m/2}\sqrt{E(\psi_i) - E(w_i^{jl})}$$

in terms of the notation

$$(3.16) \quad d_i = \sup\{d_{\tilde{Y}}(\tilde{\varphi}_i^j(\tilde{x}), \tilde{\varphi}_i^l(\tilde{x})) : \tilde{x} \in \tilde{Z}^j \cap \tilde{Z}^l; j, l \in \{1, \dots, k\}\}.$$

Note that the points \tilde{x} where $d_{\tilde{Y}}(\tilde{\varphi}_i^j(\tilde{x}), \tilde{\varphi}_i^l(\tilde{x})) < 2C\sqrt{E(\psi_i)}\delta^\alpha$, left out of consideration after (3.12), trivially do not violate (3.15). If $d_i \geq 4C\sqrt{E(\psi_i)}\varrho^\alpha$ take the above $\delta = \varrho$; then $\frac{1}{2}d_i \leq d_i - 2C\sqrt{E(\psi_i)}\varrho^\alpha$, and so by (3.15)

$$(3.17) \quad d_i \leq C_3\sqrt{E(\psi_i) - E(w_i^{jl})}$$

with $C_3 = 4C_1^{1/2}C_2^{1/2}\varrho^{-m/2}$. If instead $d_i < 4C\sqrt{E(\psi_i)}\varrho^\alpha$, take $\delta^\alpha = d_i/(4C\sqrt{E(\psi_i)}) (< \varrho^\alpha)$; this implies $d_i = 2d_i - 4C\sqrt{E(\psi_i)}\delta^\alpha$, and so by (3.15),

$$(3.18) \quad \begin{aligned} d_i^{m/(2\alpha)+1} &= d_i^{m/(2\alpha)}d_i = (4C\sqrt{E(\psi_i)}\delta^\alpha)^{m/(2\alpha)}(2d_i - 4C\sqrt{E(\psi_i)}\delta^\alpha) \\ &\leq C_4\sqrt{E(\psi_i)}^{m/(2\alpha)}\sqrt{E(\psi_i) - E(w_i^{jl})} \end{aligned}$$

with $C_4 = (4C)^{m/(2\alpha)} 4C_1^{1/2} C_2^{1/2}$ (note that δ drops out here). In any case,

$$(3.19) \quad d_i \leq C_5 \sqrt{E(\psi_i)}^{m/(m+2\alpha)} \sqrt{E(\psi_i) - E(w_i^{jl})}^{2\alpha/(m+2\alpha)} \leq C_5 \sqrt{E(\psi_i)}$$

(with $C_5 = \max\{C_3, C_4^{2\alpha/(m+2\alpha)}\}$), having inserted $E(\psi_i) \geq E(\psi_i) - E(w_i^{jl})$ for the latter inequality. Since \tilde{w}_i^{jl} is χ_* -equivariant (along with $\tilde{\varphi}_i^j$ and $\tilde{\varphi}_i^l$) we have $E(w_i^{jl}) \geq E_0$, and since (ψ_i) is minimizing we obtain from (3.19) $\lim_i d_i = 0$, thus recovering [25, (2.6xv)] in our setting.

In view an application in Section 5 we continue without taking this and other minimization properties into account until later. We proceed to derive [25, (2.6xiv)] from (3.16). Recall from the first paragraph of Step 1 that $\text{supp } \eta^j \subset Z^j$ and hence $\text{supp } \tilde{\eta}^j \subset \tilde{Z}^j$. For each $\tilde{x} \in \tilde{X}$ and $i \in \{1, 2, \dots\}$ we find from (3.9) that (with an apparently different notation)

$$(3.20) \quad \tilde{\varphi}_i(\tilde{x}) \text{ minimizes } \sum_{j \in J(\tilde{x})} \tilde{\eta}^j(\tilde{x}) d_{\tilde{Y}}^2(\tilde{\varphi}_i^j(\tilde{x}), \tilde{y}) \quad \text{for } \tilde{y} \in \tilde{Y},$$

where $J(\tilde{x}) := \{j \in \{1, \dots, k\} : \tilde{x} \in \tilde{Z}^j\}$ (note that $\tilde{\eta}^j(\tilde{x}) = 0$ for $\tilde{x} \notin \tilde{Z}^j$). The points $\tilde{\varphi}_i^j(\tilde{x})$ with $j \in J(\tilde{x})$ form a set of diameter $\leq d_i$, by (3.16). These points $\tilde{\varphi}_i^j(\tilde{x})$ therefore lie in each ball $B_{\tilde{Y}}(\tilde{\varphi}_i^l(\tilde{x}), d_i)$ for which $\tilde{x} \in \tilde{Z}^l$. Every ball in \tilde{Y} being convex, it follows by [25, Proposition 2.5.4] (or see [9, Lemma 10.4]) that the average $\tilde{\varphi}_i(\tilde{x})$ from (3.20) likewise lies in $B_{\tilde{Y}}(\tilde{\varphi}_i^l(\tilde{x}), d_i)$ when $\tilde{x} \in \tilde{Z}^l$. Replacing again l by j we conclude that

$$(3.21) \quad \sup\{d_{\tilde{Y}}(\tilde{\varphi}_i^j(\tilde{x}), \tilde{\varphi}_i(\tilde{x})) : \tilde{x} \in \tilde{Z}^j, j \in \{1, \dots, k\}\} \leq d_i.$$

Because $d_i \rightarrow 0$ as $i \rightarrow \infty$ we thus recover [25, (2.6xiv)] as well, in our setting.

Returning to the sequence of maps $\tilde{\varphi}_i$ defined in (3.9) we apply the inequality in [25, Proposition 2.5.2] to the measures $\nu = \nu_{\tilde{x}}$ and $\nu' = \nu_{\tilde{y}}$ on $\{1, \dots, k\}$ (cf. (3.8)) and the maps f, h given by $f(j) = \tilde{\varphi}_i^j(\tilde{x}), h(j) = \tilde{\varphi}_i^j(\tilde{y})$, whereby $j \in \{1, \dots, k\}$ and $\tilde{x}, \tilde{y} \in \tilde{X}$. Neglecting the negative terms on the right of the quoted inequality we obtain (by a slight deviation from the argument in [25, pp. 650–652])

$$(3.22) \quad P^2 \leq Q^2 + 2PR \leq Q^2 + \tau P^2 + \tau^{-1} R^2 \quad \text{for } 0 < \tau \leq 1,$$

where P, Q, R are defined as follows, as functions of $\tilde{x}, \tilde{y} \in \tilde{X}$ with $d_{\tilde{X}}(\tilde{x}, \tilde{y}) \leq \varrho$:

$$\begin{aligned}
 P &= d_{\tilde{Y}}(\tilde{\varphi}_i(\tilde{x}), \tilde{\varphi}_i(\tilde{y})) \quad (\text{to be estimated}), \\
 Q &= \left(\sum_{j=1}^k \tilde{\eta}^j(\tilde{x}) d_{\tilde{Y}}^2(\tilde{\varphi}_i^j(\tilde{x}), \tilde{\varphi}_i^j(\tilde{y})) \right)^{1/2} \leq \sup_{j \in J(\tilde{x})} d_{\tilde{Y}}(\tilde{\varphi}_i^j(\tilde{x}), \tilde{\varphi}_i^j(\tilde{y})) \\
 &\leq C \sqrt{E(\psi_i)} d_{\tilde{X}}^\alpha(\tilde{x}, \tilde{y}), \\
 R &= \sum_{j=1}^k d_{\tilde{Y}}(\tilde{\varphi}_i^j(\tilde{y}), \tilde{\varphi}_i(\tilde{y})) |\tilde{\eta}^j(\tilde{y}) - \tilde{\eta}^j(\tilde{x})| \leq C_6 d_i d_{\tilde{X}}(\tilde{x}, \tilde{y}).
 \end{aligned}$$

The estimates of Q and R are uniform in i, j and in $\tilde{x}, \tilde{y} \in \tilde{X}$ with $d_{\tilde{X}}(\tilde{x}, \tilde{y}) \leq \varrho$. The former estimate of Q follows from $\sum_j \tilde{\eta}^j = 1$. For the latter estimate of Q note that, for any $\tilde{x} \in \tilde{X}$ and any $j \in J(\tilde{x})$, we have $x \in Z^j$ for $x := p_X(\tilde{x})$. It follows by (3.10) that $y \in W^j$ for $y := p_X(\tilde{y})$ because $d_X(x, y) \leq d_{\tilde{X}}(\tilde{x}, \tilde{y}) \leq \varrho \leq \delta_0$. Consequently, $\tilde{x}, \tilde{y} \in \tilde{W}^j$, and (3.11) applies as stated. Finally, the estimate of R holds because $\tilde{\eta}^j = \eta^j \circ p_X \in \text{Lip}(\tilde{X})$ and $d_{\tilde{Y}}(\tilde{\varphi}_i^j(\tilde{y}), \tilde{\varphi}_i(\tilde{y})) \leq d_i$ for $\tilde{y} \in \tilde{Z}^j$, by (3.21); and since $|\tilde{\eta}^j(\tilde{y}) - \tilde{\eta}^j(\tilde{x})| = |\eta^j(y) - \eta^j(x)|$ and $d_{\tilde{X}}(\tilde{x}, \tilde{y}) = d_X(x, y)$, we may take $C_6 = \sum_1^k \|\eta^j\|_{\text{Lip}}$.

Because $\alpha, \tau \leq 1$ and $d_{\tilde{X}}(\tilde{x}, \tilde{y}) \leq \varrho$ it follows from (3.19), (3.22), and the above estimates of Q and R that

$$\begin{aligned}
 (1 - \tau) d_{\tilde{Y}}^2(\tilde{\varphi}_i(\tilde{x}), \tilde{\varphi}_i(\tilde{y})) &\leq Q^2 + \tau^{-1} R^2 \\
 (3.23) \qquad \qquad \qquad &\leq (C^2 E(\psi_i) + \tau^{-1} C_6^2 d_i^2 \delta_0^{2-2\alpha}) d_{\tilde{X}}^{2\alpha}(\tilde{x}, \tilde{y}) \\
 &\leq C_7 \tau^{-1} E(\psi_i) d_{\tilde{X}}^{2\alpha}(\tilde{x}, \tilde{y})
 \end{aligned}$$

for $\tilde{x}, \tilde{y} \in \tilde{X}$ with $d_{\tilde{X}}(\tilde{x}, \tilde{y}) \leq \varrho \leq \delta_0$ because $1 \leq \tau^{-1}$ and $d_{\tilde{X}}(\tilde{x}, \tilde{y}) \leq \delta_0^{1-\alpha} d_{\tilde{X}}^\alpha(\tilde{x}, \tilde{y})$. (By the latter estimate (3.19) we may take $C_7 = C^2 + C_5^2 C_6^2 \delta_0^{2-2\alpha}$). Taking now $\tau = \frac{1}{2}$ in (3.23) and recalling that the convergent sequences $(E(\psi_i))$ and (d_i^2) are bounded, we conclude that the $\tilde{\varphi}_i$ indeed are uniformly Hölder equicontinuous. It follows that the $\tilde{\varphi}_i$ are χ_* -equivariant (see the end of Step 1), and their projections $\varphi_i \in \mathcal{H}$ are likewise (uniformly) Hölder equicontinuous.

Step 3. The χ_ -equivariant sequence $(\tilde{\varphi}_i)$ is minimizing.* For $x, y \in X$ with $d_X(x, y) \leq \varrho$ choose $\tilde{x} \in p_X^{-1}(x), \tilde{y} \in p_X^{-1}(y)$ so that $d_{\tilde{X}}(\tilde{x}, \tilde{y}) = d_X(x, y) \leq \varrho$ (noting that $\varrho \leq \delta_0 \leq r_X$, cf. the paragraph containing (2.2)). Inserting the definitions of Q and R in the first inequality (3.23) we obtain for $0 <$

$\tau < 1$

$$(3.24) \quad (1 - \tau) d_{\tilde{Y}}^2(\tilde{\varphi}_i(\tilde{x}), \tilde{\varphi}_i(\tilde{y})) \leq \sum_{j=1}^k \tilde{\eta}^j(\tilde{x}) d_{\tilde{Y}}^2(\tilde{\varphi}_i^j(\tilde{x}), \tilde{\varphi}_i^j(\tilde{y})) + \tau^{-1} \left(\sum_{j \in J(\tilde{y})} d_{\tilde{Y}}^2(\tilde{\varphi}_i^j(\tilde{y}), \tilde{\varphi}_i(\tilde{y})) \right) \left(\sum_{j \in J(\tilde{y})} |\tilde{\eta}^j(\tilde{y}) - \tilde{\eta}^j(\tilde{x})|^2 \right)$$

by Cauchy’s inequality. Note at this point that, in the definition of R , it suffices to add over $j \in J(\tilde{y})$ (see text following (3.20)); for if $j \neq J(\tilde{y})$ then $\tilde{y} \notin \tilde{Z}^j$, hence $\tilde{y}, \tilde{x} \notin \text{supp } \tilde{\eta}^j$, and so $\tilde{\eta}^j(\tilde{y}) = \tilde{\eta}^j(\tilde{x}) = 0$. Consider an m -simplex \tilde{s} ($= \tilde{s}_\nu$) of the “fundamental domain” $\tilde{F} \subset \tilde{X}$. After integrating (3.24) over $\tilde{y} \in B_{\tilde{s}}(\tilde{x}, \varepsilon)$ for $\varepsilon \leq \rho$, and next over $\tilde{x} \in \tilde{s}$, dividing by $c_m \varepsilon^{m+2}$, and letting $\varepsilon \rightarrow 0$ whilst invoking (2.18), (2.21), and (2.23) (applied with \tilde{X} replaced by \tilde{s}) we obtain from (3.24)

$$(3.25) \quad (1 - \tau)E(\tilde{\varphi}_i|\tilde{s}) \leq \sum_{j=1}^k \int_{\tilde{s}} e(\tilde{\varphi}_i^j) \tilde{\eta}^j d\tilde{\mu} + C_8 \tau^{-1} d_i^2$$

with $C_8 = k \sum_1^k \int_s |\nabla \eta^j|^2 d\mu$, noting moreover that $d_{\tilde{Y}}(\tilde{\varphi}_i^j, \tilde{\varphi}_i) \leq d_i$ on \tilde{Z}^j according to (3.21).

Because the $\tilde{\eta}^j$ form a partition of unity on \tilde{X} , (3.25) may be rewritten as follows after summation over the m -simplexes \tilde{s} of \tilde{F} followed by projection, invoking (3.3):

$$(3.26) \quad (1 - \tau)E(\varphi_i) \leq E(\psi_i) + \sum_{j=1}^k \int_X (e(\varphi_i^j) - e(\psi_i)) \eta^j d\mu + C_9 \tau^{-1} d_i^2,$$

where we may take C_9 equal to C_8 times the number of m -simplexes of X .

Still following [25] we proceed to show that the term in (3.26) with summation over j approaches 0 as $i \rightarrow \infty$ because $(\varphi_i^j)_i$ and (ψ_i) are E -minimizing sequences. Applying quadrilateral comparison [25, (2.1iv)] with $t = \frac{1}{2}$ and $\alpha = 0$ to the χ_* -equivariant midpoint map $\tilde{w}_i^j = \frac{1}{2}\tilde{\varphi}_i^j + \frac{1}{2}\tilde{\psi}_i : s^\circ \rightarrow \tilde{Y}$ we obtain after projection in view of the uniform equicontinuity of the χ_* -equivariant maps $\tilde{\varphi}_i^j, \tilde{\psi}_i$, and \tilde{w}_i^j :

$$d_Y^2(w_i^j(x), w_i^j(y)) \leq \frac{1}{2}d_Y^2(\varphi_i^j(x), \varphi_i^j(y)) + \frac{1}{2}d_Y^2(\psi_i(x), \psi_i(y)) - \frac{1}{4}(d_Y(\varphi_i^j(x), \varphi_i^j(y)) - d_Y(\psi_i(x), \psi_i(y)))^2$$

for $d_{\tilde{X}}(\tilde{x}, \tilde{y})$ sufficiently small that $d_{\tilde{Y}}(\tilde{w}_i^j(\tilde{x}), \tilde{w}_i^j(\tilde{y})) \leq r_Y$, etc.; cf. (2.2) and subsequent text. It follows by (2.18), (2.19), for any open m -simplex s° of X and any $f \in C_c(s^\circ, [0, 1])$ that (up to a dimensional constant factor)

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \frac{m+2}{4} \iint_{\substack{x, y \in s^\circ \\ d_X(x, y) \leq \varepsilon}} \left(\frac{d_Y(\varphi_i^j(x), \varphi_i^j(y))}{\varepsilon} - \frac{d_Y(\psi_i(x), \psi_i(y))}{\varepsilon} \right)^2 f(x) \frac{d\mu(x) d\mu(y)}{\varepsilon^m} \\ (3.27) \qquad \leq \int_{s^\circ} \left(\frac{1}{2} e(\varphi_i^j) + \frac{1}{2} e(\psi_i) - e(w_i^j) \right) f \, d\mu. \end{aligned}$$

According to [25, p. 648 (top)] the left-hand member of (3.27) equals

$$(3.28) \qquad \int_{S^{m-1}} \int_{s^\circ} (|(\varphi_i^j)_*(\omega)| - |(\psi_i)_*(\omega)|)^2 f \, d\mu \, d\sigma(\omega),$$

where for example $|(\psi_i)_*(\omega)|^2$ denotes the *directional energy density* of $\psi_i \in \mathcal{E}(s^\circ, Y)$ (cf. [25, Theorem 1.9.6 and eq. (1.10v)]) in the direction corresponding to $\omega \in S^{m-1}$ (the unit sphere in \mathbb{R}^m with surface measure σ). On the other hand, by [25, (1.10ii)] applied to the maps φ_i^j and ψ_i , and the function $\eta^j f$ (in place of f):

$$\begin{aligned} (3.29) \qquad \int_{s^\circ} (e(\varphi_i^j) - e(\psi_i)) \eta^j f \, d\mu \\ = \int_{s^\circ} \int_{S^{m-1}} (|(\varphi_i^j)_*(\omega)|^2 - |(\psi_i)_*(\omega)|^2) \, d\sigma(\omega) \eta^j f \, d\mu. \end{aligned}$$

By the Cauchy–Schwarz inequality applied to the product

$$(|(\varphi_i^j)_*(\omega)| + |(\psi_i)_*(\omega)|) \eta^j \sqrt{f} \times (|(\varphi_i^j)_*(\omega)| - |(\psi_i)_*(\omega)|) \sqrt{f},$$

the square of the right-hand member of (3.29) is no bigger than

$$\begin{aligned} \int_{s^\circ} \int_{S^{m-1}} (|(\varphi_i^j)_*(\omega)| + |(\psi_i)_*(\omega)|)^2 \, d\sigma(\omega) (\eta^j)^2 f \, d\mu \times \\ (3.30) \qquad \int_{s^\circ} \int_{S^{m-1}} (|(\varphi_i^j)_*(\omega)| - |(\psi_i)_*(\omega)|)^2 \, d\sigma(\omega) f \, d\mu \\ \leq 2(E(\varphi_i^j) + E(\psi_i)) \int_{s^\circ} \left(\frac{1}{2} e(\varphi_i^j) + \frac{1}{2} e(\psi_i) - e(w_i^j) \right) f \, d\mu \end{aligned}$$

in view of (3.27) and (3.28), inserting $(\eta^j)^2 f \leq 1$, using $(a + b)^2 \leq 2a^2 + 2b^2$, and again invoking [25, 1.10ii]. In the limit as $f \rightarrow 1$ pointwise through

an increasing sequence we altogether obtain (up to a dimensional constant factor) by dominated convergence and by (2.22), (3.7), (3.29), and (3.30),

$$(3.31) \quad \int_{s^\circ} (e(\varphi_i^j) - e(\psi_i)) \eta^j d\mu \leq 2\sqrt{E(\psi_i)} \sqrt{E(\psi_i) - E(w_i^j)}.$$

Because the midpoint map $\tilde{w}_i^j = \frac{1}{2}\tilde{\varphi}_i^j + \frac{1}{2}\tilde{\psi}_i$ is χ_* -equivariant along with $\tilde{\varphi}_i^j$ and $\tilde{\psi}_i$, its projection w_i^j belongs to \mathcal{H} and so has energy $\geq E_0$. Adding over all m -simplexes s of X we conclude that the sum term in (3.26) indeed has the limit 0 as $i \rightarrow \infty$. By (3.26) we therefore obtain

$$(3.32) \quad \limsup_{i \rightarrow \infty} [(1 - \tau)E(\varphi_i)] \leq E_0,$$

and hence $\limsup_i E(\varphi_i) \leq E_0$, by letting $\tau \rightarrow 0$. In the opposite direction, $E_0 \leq E(\varphi_i)$ by definition of E_0 . Consequently, $\lim_i E(\varphi_i) = E_0$, and the sequence $(\tilde{\varphi}_i)$ is indeed minimizing; cf. (3.4) and (3.5).

Step 4. Existence and harmonicity of E -minimizers. Using the above minimizing sequence of χ_* -equivariant maps $\tilde{\varphi}_i : \tilde{X} \rightarrow \tilde{Y}$ we commence as indicated in the proof of [25, Theorem 2.7.1]. Since $(\tilde{\varphi}_i)$ is uniformly Hölder equicontinuous, so is the sequence of projected maps $\varphi_i : X \rightarrow Y$. By Ascoli's theorem, a subsequence, again denoted (φ_i) , converges uniformly to a continuous map $\varphi : X \rightarrow Y$. Fix $i \in \{1, 2, \dots\}$ so that $d_Y(\varphi(x), \varphi_i(x)) \leq r_Y$ for all $x \in X$, with r_Y from the paragraph containing (2.2). Also fix $x_0 \in X$ and $\tilde{x}_0 \in p_X^{-1}(x_0)$. Let $\tilde{\varphi} : \tilde{X} \rightarrow \tilde{Y}$ denote the unique lift of $\varphi : X \rightarrow Y$ such that $d_{\tilde{Y}}(\tilde{\varphi}(\tilde{x}_0), \tilde{\varphi}_i(\tilde{x}_0)) = d_Y(\varphi(x_0), \varphi_i(x_0)) (\leq r_Y)$. Because $\tilde{\varphi}$ and $\tilde{\varphi}_i$ are continuous and Γ_Y is discrete, it follows that $d_{\tilde{Y}}(\tilde{\varphi}(\tilde{x}), \tilde{\varphi}_i(\tilde{x})) \leq r_Y$ for every $\tilde{x} \in \tilde{X}$. We show that $\tilde{\varphi}$ is χ_* -equivariant. For any $\gamma \in \Gamma_X$ the continuous map $\chi_*(\gamma)^{-1} \circ \tilde{\varphi} \circ \gamma$ from \tilde{X} to \tilde{Y} is likewise a lift of φ because

$$p_Y \circ (\chi_*(\gamma)^{-1} \circ \tilde{\varphi} \circ \gamma) = p_Y \circ \tilde{\varphi} \circ \gamma = \varphi \circ p_X \circ \gamma = \varphi \circ p_X;$$

cf. (2.3). Because $\tilde{\varphi}_i$ is χ_* -equivariant (see end of Step 2), we have

$$d_{\tilde{Y}}((\chi_*(\gamma)^{-1} \circ \tilde{\varphi} \circ \gamma)(\tilde{x}), \tilde{\varphi}_i(\tilde{x})) = d_{\tilde{Y}}(\tilde{\varphi}(\gamma(\tilde{x})), \tilde{\varphi}_i(\gamma(\tilde{x}))) \leq r_Y$$

for any $\tilde{x} \in \tilde{X}$, noting that $\chi_*(\gamma)$ and γ are isometries of \tilde{Y} and \tilde{X} , respectively. By uniqueness of $\tilde{\varphi}$ we have $\chi_*(\gamma)^{-1} \circ \tilde{\varphi} \circ \gamma = \tilde{\varphi}$, and so indeed $\tilde{\varphi}$ is χ_* -equivariant. Its projection φ therefore belongs to \mathcal{H} . From the uniform convergence $\varphi_i \rightarrow \varphi$ we infer that $D^2(\varphi_i, \varphi) = \int_X d_Y^2(\varphi_i(x), \varphi(x)) d\mu(x) \rightarrow 0$ because $\mu(X) < \infty$. This means that $\varphi_i \rightarrow \varphi$ in $L^2(X, Y)$ (cf. the paragraph

following the one containing (2.17)), and consequently

$$E(\varphi) \leq \liminf_{i \rightarrow \infty} E(\varphi_i) = E_0$$

by [9, Lemma 9.1]. Thus φ is an E -minimizer relative to \mathcal{H} .

For the proof that every E -minimizer φ relative to \mathcal{H} is harmonic in X in the sense of (c), and hence of (a) and (b) in Remark 2.6, cf. Lemma 2.7, fix a point $p \in X$. Since Y is complete and has nonpositive curvature, $\varphi(p)$ has a convex open neighbourhood V in Y such that V is a CAT(0) space, [2, p. 159 f]. Thus any two points y, z of V can be joined by a unique constant-speed geodesic $[0, 1] \ni t \mapsto \zeta(y, z; t) \in V$, [2, pp. 4, 160]; and ζ depends continuously on $(y, z, t) \in V \times V \times [0, 1]$, [2, p. 158 ff]. For a given neighbourhood N of p in X choose a connected open set $U \Subset X$ with $p \in U \subset N$ so that $\varphi(U) \Subset V$. Consider any continuous map $\psi \in \mathcal{E}(X, Y)$ such that $\psi = \varphi$ in $X \setminus U$ and $\psi(U) \subset V$. We prove that $\psi \in \mathcal{H}$, hence $E(\psi) \geq E(\varphi)$, and so, by varying p and N , that φ satisfies (c) in Remark 2.6. Define $\Psi : X \times [0, 1] \rightarrow Y$ by

$$\Psi(x, t) = \begin{cases} \zeta(\varphi(x), \psi(x); t) & \text{for } x \in U, t \in [0, 1] \\ \varphi(x) = \psi(x) & \text{for } x \in X \setminus U, t \in [0, 1]. \end{cases}$$

For $U \ni x \rightarrow x_0 \in \partial U$ and $[0, 1] \ni t \rightarrow t_0 \in [0, 1]$ we have $\varphi(x) \rightarrow \varphi(x_0) \in V$ and $\psi(x) \rightarrow \psi(x_0) = \varphi(x_0)$, and hence $\zeta(\varphi(x), \psi(x); t) \rightarrow \zeta(\varphi(x_0), \varphi(x_0); t_0) = \varphi(x_0)$, that is, $\Psi(x, t) \rightarrow \Psi(x_0, t_0)$. Thus Ψ is continuous both relative to \bar{U} and to $X \setminus U$, and hence on all of X . Clearly $\Psi(x, 0) = \varphi(x)$, $\Psi(x, 1) = \psi(x)$; so indeed $\psi \in \mathcal{H}$, and φ is harmonic according to Remark 2.6. It may be added that every minimizing χ_* -equivariant map $\tilde{\varphi} : \tilde{X} \rightarrow \tilde{Y}$ likewise is harmonic, harmonicity being a local property by Lemma 2.7.

Step 5. The uniqueness assertion. We begin by proving that any two χ_* -equivariant energy minimizers $\tilde{\varphi}_0, \tilde{\varphi}_1$ have constant distance $d_{\tilde{Y}}(\tilde{\varphi}_0(\cdot), \tilde{\varphi}_1(\cdot))$. First note that the restriction of the distance function $d_{\tilde{Y}}(\tilde{\varphi}_0, \tilde{\varphi}_1)$ to each m -simplex \tilde{s} ($= \tilde{s}_\nu$) of the “fundamental domain” \tilde{F} is of class $\mathcal{E}(\tilde{s}, \mathbb{R}) = W^{1,2}(\tilde{s})$; see for example [14, p. 252 (bottom)]. For any $t \in [0, 1]$ the restriction of the map $\tilde{\varphi}_t := (1 - t)\tilde{\varphi}_0 + t\tilde{\varphi}_1$ (with the usual abuse of notation) to each \tilde{s} is of class $\mathcal{E}(\tilde{s}, \tilde{Y})$ by the energy convexity property (3.33)

$$E(\tilde{\varphi}_t|\tilde{s}) \leq (1 - t)E(\tilde{\varphi}_0|\tilde{s}) + tE(\tilde{\varphi}_1|\tilde{s}) - t(1 - t) \int_{\tilde{s}} |\nabla d_{\tilde{Y}}(\tilde{\varphi}_0, \tilde{\varphi}_1)|^2 d\tilde{\mu}$$

(valid for any two maps $\tilde{\varphi}_0, \tilde{\varphi}_1 \in \mathcal{E}(\tilde{s}, \tilde{Y})$), cf. [9, eq. (11.2); 25, 2.2iv]. Also, each $\tilde{\varphi}_t$ is χ_* -equivariant, $\gamma \in \Gamma_X$ and $\chi_*(\gamma) \in \Gamma_Y$ being isometries. For the

projection $\varphi_t \in \mathcal{H}$ of $\tilde{\varphi}_t$ we therefore have from (3.4) after summation over all m -simplexes \tilde{s} of \tilde{F}

$$E_0 \leq E(\varphi_t) \leq E_0 - t(1-t) \int_{\tilde{F}} |\nabla d_{\tilde{Y}}(\tilde{\varphi}_0, \tilde{\varphi}_1)|^2 d\tilde{\mu} \leq E_0,$$

and hence $\int_{\tilde{F}} |\nabla d_{\tilde{Y}}(\tilde{\varphi}_0, \tilde{\varphi}_1)|^2 d\tilde{\mu} = 0$, by taking $0 < t < 1$. It follows that $d_{\tilde{Y}}(\tilde{\varphi}_0, \tilde{\varphi}_1)$ is constant in any \tilde{s} . Because every open star \tilde{S} of \tilde{X} is contained in some open ball of radius r_X (cf. the beginning of the proof of the present Theorem 3.2), \tilde{F} can be chosen so as to contain \tilde{S} , cf. text between (3.12) and (3.13)); hence $d_{\tilde{Y}}(\tilde{\varphi}_0, \tilde{\varphi}_1)$ equals a constant \tilde{d} in \tilde{S} . Thus $d_{\tilde{Y}}(\tilde{\varphi}_0, \tilde{\varphi}_1)$ is locally constant in \tilde{X} , and therefore constant in all of \tilde{X} (connected). By projection, this leads to Hartman’s uniqueness result [21] in the present setting, as stated after the formulation of Theorem 3.2: If $\tilde{d} < r_Y$ then $\tilde{d} = d_{\tilde{Y}}(\tilde{\varphi}_0(\tilde{x}), \tilde{\varphi}_1(\tilde{x})) = d_Y(\varphi_0(x), \varphi_1(x))$ by (2.2) and subsequent text, and the geodesic segment $t \mapsto \varphi_t(x)$ therefore becomes minimizing. Finally, if $\varphi_0(x) = \varphi_1(x)$ at some point x then $\tilde{d} = 0$ and consequently $\varphi_0 = \varphi_1$. \square

Remark 3.3. Under the hypotheses on X and Y in Theorem 3.2, Hölder continuity of a harmonic map $\varphi : X \rightarrow Y$ (cf. Proposition 3.1, now with X compact) holds in the global uniform sense that there are constants $C, \alpha > 0$ depending only on (X, Λ_X) such that

$$(3.34) \quad d_Y(\varphi(x), \varphi(y)) \leq C \sqrt{E(\varphi)} d_X^\alpha(x, y), \quad x, y \in X.$$

With the extra requirement that $d_X(x, y) \leq \varrho$ for a suitable constant $\varrho > 0$, this holds according to Proposition 3.1, X being compact. Since (X, d_X) is a geodesic space [9, Proposition 4.1(b)], we may join any two points x and y of X by a geodesic η in X of length $d_X(x, y) \leq$ the diameter of (X, d_X) , and it suffices to apply the version of (3.34) for $d_X(x, y) \leq \varrho$ after first subdividing η into a fixed number of arcs of length not exceeding ϱ (cf. the analogous paragraph following the one containing (2.12)).

Remark 3.4. We shall need in Section 5 that, for variable Riemannian metric g on X , the constants $r_X, C, \alpha, \varrho, \delta_0$, and C_1 through C_9 in the proof of Theorem 3.2 can be chosen so as to depend only on X, Λ_X (as already noted concerning C, α, ϱ in Proposition 3.1). If r_X^e denotes a value of r_X relative to the Euclidean Riemannian metric g^e on X (see [9, p. 49]), then we may take $r_X = \Lambda_X^{-1} r_X^e$ for the constant r_X relative to a given Riemannian metric g on X with ellipticity constant Λ_X ; cf. [9, (4.1)]. This is because

the intrinsic Riemannian distances d_X and d_X^e satisfy

$$(3.35) \quad \Lambda_X^{-1}d_X^e \leq d_X \leq \Lambda_X d_X^e,$$

cf. [9, p. 52 (bottom)]. It follows that $B_X(x, \Lambda_X^{-1}r_X^e) \subset B_X^e(x, r_X^e)$ for $x \in X$; and since $p_X|_{B_X^e(x, r_X^e)}$ is an isometry, by (2.2), we infer that $p_X|_{B_X(x, \Lambda_X^{-1}r_X^e)}$ is injective and hence likewise an isometry; cf. [31, Proposition 3.5.7]. In this way, r_X depends on X, Λ_X only. In view of (3.35) the same holds for δ_0 , cf. (3.10). It is easily verified that C_1 through C_9 likewise depend on (X, Λ_X) only.

4. The Dirichlet homotopy problem

We continue considering a compact admissible Riemannian polyhedron (X, g) , but now with $bX \neq \emptyset$; and a compact geodesic space Y of nonpositive curvature. We therefore have the equivalent definitions (a), (b), and (c) of harmonicity of a continuous map $\varphi : X \rightarrow Y$, formulated in Remark 2.6 above, cf. Lemma 2.7.

Let Ω denote a nonvoid connected open subset of X , and denote by $\partial\Omega$ the topological boundary and by $\bar{\Omega}$ the closure of Ω in X . Note that X is a harmonic space in the sense of Brelot [4, Part IV; 5, §3.1]; see [9, Theorem 7.1]. We require throughout that Ω is *regular* in the sense of potential theory and that $\Omega \cap bX$ is not dense in bX . Recall that Ω is said to be regular if $\partial\Omega \neq \emptyset$ and every continuous function $\partial\Omega \rightarrow \mathbb{R}$ has a (necessarily unique) continuous extension $\bar{\Omega} \rightarrow \mathbb{R}$ which is harmonic in Ω . Equivalently, every point $x \in \partial\Omega$ shall be regular, that is, $X \setminus \Omega$ shall be nonthin at x in the sense of Brelot [4, p. 137], or see [1, Theorem 7.5.1; 5, §6.3]. In particular, $X \setminus bX$ is always regular. The boundary of a regular domain can, however, be rather “wild”; cf. [14, Example 2].

We proceed to show that the Dirichlet homotopy problem for harmonic maps has a unique solution. For any map $\chi \in C(X, Y)$ (i.e., $\chi : X \rightarrow Y$ is continuous) write

$$C_\chi(\Omega, Y) = \{\varphi \in C(X, Y) : \varphi = \chi \text{ on } X \setminus \Omega\}.$$

Denote by $\mathcal{H}_\chi(\Omega, Y)$ the class of all maps in $C_\chi(\Omega, Y)$ which are homotopic with χ , *the homotopy being through maps of class $C_\chi(\Omega, Y)$* . Denote by \mathcal{H} the homotopy class of all maps in $C(X, Y)$ which are *freely homotopic* to χ ; then $\mathcal{H}_\chi(\Omega, Y) \subset \mathcal{H}$.

Theorem 4.1. *Let Ω be a connected regular open subset of a compact admissible Riemannian polyhedron (X, g) with $bX \neq \emptyset$, and suppose that $\Omega \cap bX$ is not dense in bX . Let (Y, d_Y) be a compact geodesic space of non-positive curvature; and let a continuous map $\chi : X \rightarrow Y$ of finite energy be given. Then there exists in $\mathcal{H}_\chi(\Omega, Y)$ a unique map φ_0 of least energy, and $\varphi_0|_\Omega$ is harmonic and (locally uniformly) Hölder continuous.*

In earlier work on the Dirichlet problem for harmonic maps (for example in [13, 25]), the present regular domain Ω was replaced by the particular regular domain $\Omega = X \setminus bX$. For that case, Theorem 4.1 can be established alternatively by modifying the proof given below, notably by using [13, Theorem 1] (as in the proof of Theorem 3.2 above) in place of [14, Theorem 1], needed in Step 2 of the present proof.

Proof of Theorem 4.1. Again, we pass to the universal covers \tilde{X} and \tilde{Y} . In view of [42, Theorem 35, p. 317] we may assume that every simplex of X has diameter $\leq r_X$ from (2.2), and from (2.8) in the proof of Proposition 2.2. According to [18, Theorem 1 and Corollary 3] the connected open set $\Omega \subset X$ is an admissible Riemannian polyhedron (along with X) with a triangulation T_Ω such that every simplex of (Ω, T_Ω) is a subset of some simplex of X , and hence has diameter $\leq r_X$. Define $\tilde{\Omega} = p_X^{-1}(\Omega)$.

Henceforth, until Step 4 below, we suppose that Ω is not dense in X .

Step 1. A quasi-equivariant minimizer $\tilde{\varphi}_0$. Fix a base point $x_0 \in X \setminus \Omega$ and a point $\tilde{x}_0 \in p_X^{-1}(x_0)$. For any $\varphi \in \mathcal{H}_\chi(\Omega, Y)$ there exists a continuous map $\Phi : X \times [0, 1] \rightarrow Y$ such that $\Phi(x, 0) = \chi(x)$ and $\Phi(x, 1) = \varphi(x)$ for $x \in X$, and that $\Phi(x, t) = \chi(x)$ for $x \in X \setminus \Omega$ and $t \in [0, 1]$. Because $X \times [0, 1]$ has the same properties as listed in Section 2 for X itself, allowing lifting of continuous maps into Y , there exists a lift $\tilde{\Phi} : \tilde{X} \times [0, 1] \rightarrow \tilde{Y}$. For each $t \in [0, 1]$ the restriction $\tilde{\Phi}(\cdot, t) : \tilde{X} \rightarrow \tilde{Y}$ is a lift of $\Phi(\cdot, t) : X \rightarrow Y$. In particular, $\tilde{\chi} := \tilde{\Phi}(\cdot, 0)$ is a lift of χ . For $\tilde{x} \in \tilde{X} \setminus \tilde{\Omega}$ we have $x := p_X(\tilde{x}) \in X \setminus \Omega$, and hence $\tilde{\Phi}(\tilde{x}, t) \in p_Y^{-1}(\Phi(x, t)) = p_Y^{-1}(\chi(x))$ for $t \in [0, 1]$. Because Γ_Y is discrete and $\tilde{\Phi}(\tilde{x}, \cdot)$ is continuous it follows that $\tilde{\Phi}(\tilde{x}, t)$ is independent of $t \in [0, 1]$ for each $\tilde{x} \in \tilde{X} \setminus \tilde{\Omega}$. Hence $\tilde{\varphi}(\tilde{x}) := \tilde{\Phi}(\tilde{x}, 1) = \tilde{\Phi}(\tilde{x}, 0) = \tilde{\chi}(\tilde{x})$ for $\tilde{x} \in \tilde{X} \setminus \tilde{\Omega}$, in particular $\tilde{\varphi}(\tilde{x}_0) = \tilde{\chi}(\tilde{x}_0)$. Since $\varphi \in \mathcal{H}_\chi(\Omega, Y) \subset \mathcal{H}$ there is accordingly a unique homomorphism $\chi_* : \Gamma_X \rightarrow \Gamma_Y$ such that $\tilde{\varphi}$ is χ_* -equivariant, as in (3.2). Denote by \mathcal{A} the class of all (quasicontinuous) maps $\tilde{\varphi} \in \mathcal{E}_{\text{loc}}(\tilde{X}, \tilde{Y})$ which are quasi-equivariant in the sense that (3.2) holds q.e.:

$$(4.1) \quad \tilde{\varphi} \circ \gamma = \chi_*(\gamma) \circ \tilde{\varphi} \quad \text{q.e. in } \tilde{X} \text{ for all } \gamma \in \Gamma_X.$$

Denote

$$\mathcal{A}_{\tilde{\chi}} = \{\tilde{\varphi} \in \mathcal{A} : \tilde{\varphi} = \tilde{\chi} \text{ q.e. in } \tilde{X} \setminus \tilde{\Omega}\}.$$

Then $\tilde{\chi} \in \mathcal{A}_{\tilde{\chi}}$; in particular $\mathcal{A}_{\tilde{\chi}} \neq \emptyset$.

Every map $\tilde{\varphi} \in \mathcal{A}$ has a quasicontinuous *projection* $\varphi \in \mathcal{E}(X, Y)$. Indeed, (3.2) holds in $\tilde{X} \setminus \tilde{P}$ for some polar set $\tilde{P} \subset \tilde{X}$. Replacing \tilde{P} by the countable union of polar sets $\gamma(\tilde{P})$, $\gamma \in \Gamma_X$, we arrange that \tilde{P} becomes Γ_X -invariant. For any $x \in X$ off the polar set $p_X(\tilde{P})$ define $\varphi(x) = (p_Y \circ \tilde{\varphi})(\tilde{x})$ for $\tilde{x} \in p_X^{-1}(x) \subset \tilde{X} \setminus \tilde{P}$; this does not depend on the choice of \tilde{x} on account of (3.2). Finally, $\varphi \in \mathcal{E}_{\text{loc}}(X, Y) = \mathcal{E}(X, Y)$, and φ is quasicontinuous along with $\tilde{\varphi}$, both because p_X and p_Y are local isometries.

Every *continuous* map $\tilde{\varphi} \in \mathcal{A}$ is χ_* -equivariant in the sense of (3.2), that is, (4.1) holds *everywhere* in \tilde{X} . If even $\tilde{\varphi} \in \mathcal{A}_{\tilde{\chi}}$ then the continuous projection $\varphi : X \rightarrow Y$ is of class $C_\chi(\Omega, Y)$, and even of class $\mathcal{H}_\chi(\Omega, Y)$, as shown by geodesic homotopy; see the text following (3.2), and note that we now have $\Phi(x, t) = \chi(x)$ for $x \in X \setminus \Omega$.

For a given triangulation of X , and hence of \tilde{X} , consider a compact “fundamental domain” $\tilde{F} \subset \tilde{X}$ of Γ_X as in the paragraph containing (3.3) and its predecessor. Thus \tilde{F} is a union of m -simplexes \tilde{s}_ν projected by p_X isometrically on the m -simplexes s_ν of X , $\nu \in \{1, \dots, n\}$. For $\varepsilon \leq r_X$ we have $B_X(x, \varepsilon) = p_X(B_{\tilde{X}}(\tilde{x}, \varepsilon))$ for all $\tilde{x} \in p_X^{-1}(x)$ according to (2.2). Since p_Y is nonexpanding, the projection φ of a map $\tilde{\varphi} \in \mathcal{A}$ therefore satisfies $\varphi|_{s_\nu} \in L^2(s_\nu, Y)$ and hence $\varphi \in L^2(X, Y)$. Furthermore, $e_\varepsilon(\varphi) \circ p_X \leq e_\varepsilon(\tilde{\varphi})$, and hence $e(\varphi) \circ p_X \leq e(\tilde{\varphi})$ and $E(\varphi) \leq E(\tilde{\varphi}|\tilde{F}) = \sum_{\nu=1}^n E(\tilde{\varphi}|\tilde{s}_\nu) = \int_{\tilde{F}} e(\tilde{\varphi}) d\tilde{\mu} (< \infty)$; cf. Lemma 2.3(b). If $\tilde{\varphi}$ is continuous then so is φ , and the sign \leq can here be replaced in the last two cases by equality (as in (3.3)), p_Y being locally isometric.

There exists a constant c_1 such that the following Poincaré-style inequality holds (cf. [6, Proposition 2.2]):

$$(4.2) \quad \int_{\tilde{F}} (u - \bar{u})^2 d\tilde{\mu} \leq c_1 \int_{\tilde{F}} |\nabla u|^2 d\tilde{\mu}$$

for any function $u \in W_{\text{loc}}^{1,2}(\tilde{X})$ which is essentially Γ_X -invariant: $u \circ \gamma = u$ μ -a.e. in \tilde{X} for every $\gamma \in \Gamma_X$. Here $\bar{u} = \tilde{\mu}(\tilde{F})^{-1} \int_{\tilde{F}} u d\tilde{\mu}$ denotes the average of u over \tilde{F} . We prove (4.2) in the equivalent projected form where \tilde{F} and $\tilde{\mu}$ are replaced by X and μ , respectively, and u is replaced by any function $u \in W^{1,2}(X)$. The result may then be viewed as a particular case of [9, Theorem 5.1] (where X is noncompact), but we give a simpler, direct proof. We may assume that $\bar{u} = 0$; otherwise replace u by $u - \bar{u}$. If there were

no such constant c_1 there would be a sequence of functions $(u_n) \subset W^{1,2}(X)$ such that

$$\int_X u_n d\mu = 0, \quad \int_X u_n^2 d\mu = 1, \quad \lim_{n \rightarrow \infty} \int_X |\nabla u_n|^2 d\mu = 0.$$

After passing to a subsequence we may assume that u_n converges in $L^2(X, \mu)$ and pointwise μ -a.e. in X to some $u \in L^2(X, \mu)$ with $\int_X |\nabla u|^2 d\mu = 0$; this follows from [25, § 1.6], or from [9, Lemmas 9.1 and 9.2] applied to $Y = \mathbb{R}$ with X replaced by any $(m - 1)$ -simplex s of X ; cf. [9, Corollary 9.2]. Consequently, u equals a constant c_s μ -a.e. on each m -simplex s . For two m -simplexes s, s' such that $s \cap s'$ is an $(m - 1)$ -simplex we have $c_s = c_{s'}$ in view of [25, Theorem 1.12.2]. For arbitrary m -simplexes s, s' of the admissible polyhedron X there is a chain s_0, \dots, s_p of m -simplexes of X such that $s_{j-1} \cap s_j$ is an $(m - 1)$ -simplex of X for $j \in \{1, \dots, p\}$, and that $s_0 = s, s_p = s'$; see [18, Lemma 5]. We conclude that $c_s = c_{s'}$, and so there is a constant c such that $u = c$ μ -a.e. in s for all m -simplexes s of X . Consequently, $u_n \rightarrow c$ in $L^2(X, \mu)$, and pointwise μ -a.e. in X . The constant c equals 0 because $0 = \int_X u_n d\mu \rightarrow c \mu(X)$, but on the other hand $c = 1/\sqrt{\mu(X)}$ because $1 = \int_X u_n^2 d\mu \rightarrow c^2 \mu(X)$. This contradiction establishes (4.2).

Write

$$(4.3) \quad E_0 = \inf\{E(\tilde{\varphi}|\tilde{F}) : \tilde{\varphi} \in \mathcal{A}_{\tilde{X}}\},$$

and consider a corresponding minimizing sequence $(\tilde{\varphi}_i) \subset \mathcal{A}_{\tilde{X}}$. The mid-point map of two maps $\tilde{\varphi}, \tilde{\varphi}' \in \mathcal{A}_{\tilde{X}}$ is likewise of class $\mathcal{A}_{\tilde{X}}$. Furthermore, the distance function $d_{\tilde{\varphi}}(\tilde{\varphi}, \tilde{\varphi}') \in W^{1,2}(\tilde{X})$ (see for example [14, paragraph preceding (2.4)]) is essentially Γ_X -invariant in view of (4.1). Now apply [25, Theorem 2.2] or [14, (3.3), (3.5)] (involving quadrilateral comparison) to each of the essentially Γ_X -invariant functions $\tilde{u}_{ij} := d_{\tilde{\varphi}}(\tilde{\varphi}_i, \tilde{\varphi}_j) \in W_{\text{loc}}^{1,2}(\tilde{X})$ considered on each m -simplex \tilde{s}_ν . After summation over ν , this leads to

$$(4.4) \quad \lim_{i,j \rightarrow \infty} \int_{\tilde{F}} |\nabla \tilde{u}_{ij}|^2 d\tilde{\mu} = 0.$$

Because \tilde{u}_{ij} is Γ_X -invariant the integrals in (4.4) and in (4.5) (below) do not depend on the particular choice of \tilde{F} . Denoting by a_{ij} the mean value of $\tilde{u}_{ij}|\tilde{F}$ with respect to $\tilde{\mu}|\tilde{F}$, we obtain from (4.2) and (4.4)

$$(4.5) \quad \lim_{i,j \rightarrow \infty} \int_{\tilde{F}} (\tilde{u}_{ij} - a_{ij})^2 d\tilde{\mu} = 0.$$

Note that $\tilde{u}_{ij} = 0$ q.e. (in particular $\tilde{\mu}$ -a.e.) in $\tilde{F} \setminus \tilde{\Omega}$, and that $\tilde{\mu}(\tilde{F} \setminus \tilde{\Omega}) > 0$ (under our temporary hypothesis that $\tilde{\Omega} \neq X$). It follows from (4.5) (with integration over $\tilde{F} \setminus \tilde{\Omega}$ only) that $a_{ij} \rightarrow 0$ as $i, j \rightarrow \infty$. Inserting this in (4.5) as it stands yields

$$\int_{\tilde{F}} \tilde{u}_{ij}^2 d\tilde{\mu} \leq 2 \int_{\tilde{F}} (\tilde{u}_{ij} - a_{ij})^2 d\tilde{\mu} + 2a_{ij}^2 \int_{\tilde{F}} d\tilde{\mu} \rightarrow 0 \quad \text{as } i, j \rightarrow \infty$$

since $\tilde{u}_{ij}^2 \leq 2(\tilde{u}_{ij} - a_{ij})^2 + 2a_{ij}^2$. Clearly, this applies with \tilde{F} replaced by any m -simplex of \tilde{F} , or even of \tilde{X} , hence also with \tilde{F} replaced by any closed star \tilde{S} in \tilde{X} . The Cauchy sequence $(\tilde{\varphi}_i)$ in $L^2(\tilde{S}, \tilde{Y})$ therefore converges to a map $\tilde{\varphi}_0 \in L^2(\tilde{S}, \tilde{Y})$, i.e.,

$$\int_{\tilde{S}} d_{\tilde{Y}}^2(\tilde{\varphi}_i, \tilde{\varphi}_0) d\tilde{\mu} \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

The sequence $(E(\tilde{\varphi}_i))$ is bounded because $E(\tilde{\varphi}_i|\tilde{s}) \leq E(\tilde{\varphi}_i|\tilde{F}) \rightarrow E_0$ for any m -simplex \tilde{s} of \tilde{S} and any choice of a “fundamental domain” \tilde{F} containing \tilde{s} . It follows by Lemma 2.5 with X replaced by \tilde{S} (compact and admissible) that $\tilde{\varphi}_0 \in \mathcal{E}(\tilde{S}, \tilde{Y})$, and after passing to a subsequence, still denoted $(\tilde{\varphi}_i)$, that

$$(4.6) \quad \frac{1}{k} \sum_{i=1}^k d_{\tilde{Y}}(\tilde{\varphi}_i, \tilde{\varphi}_0) \rightarrow 0 \quad \text{q.e. in } \tilde{S} \text{ as } k \rightarrow \infty$$

when also $\tilde{\varphi}_0$ is taken to be quasicontinuous in \tilde{S} . As the stars \tilde{S} vary, the corresponding maps $\tilde{\varphi}_0$ (after being redefined on a μ -nullset) become restrictions of a single quasicontinuous map $\tilde{\varphi}_0 \in \mathcal{E}_{\text{loc}}(\tilde{X}, \tilde{Y})$, and (4.6) holds q.e. in \tilde{X} . Since \tilde{X} has a countable base, countably many stars \tilde{S} cover \tilde{X} , and (4.6) therefore holds q.e. in \tilde{X} as $k \rightarrow \infty$.

We show that $\tilde{\varphi}_0 \in \mathcal{A}_{\tilde{X}}$. Because each $\tilde{\varphi}_i \in \mathcal{A}$ we have for $\gamma \in \Gamma_X$ by the triangle inequality, interpolating the point $\tilde{\varphi}_i \circ \gamma = \chi_*(\gamma) \circ \tilde{\varphi}_i$, summing over $i \in \{1, \dots, k\}$, and dividing by k :

$$(4.7) \quad \begin{aligned} d_{\tilde{Y}}(\tilde{\varphi}_0 \circ \gamma, \chi_*(\gamma) \circ \tilde{\varphi}_0) &\leq \frac{1}{k} \sum_{i=1}^k d_{\tilde{Y}}(\tilde{\varphi}_0 \circ \gamma, \tilde{\varphi}_i \circ \gamma) \\ &\quad + \frac{1}{k} \sum_{i=1}^k d_{\tilde{Y}}(\chi_*(\gamma) \circ \tilde{\varphi}_i, \chi_*(\gamma) \circ \tilde{\varphi}_0) \\ &\rightarrow 0 \text{ q.e. in } \tilde{X} \text{ as } k \rightarrow \infty, \end{aligned}$$

according to (4.6). Here we use that Γ_Y is countable and that $\chi_*(\gamma) \in \Gamma_Y$ is an isometry of \tilde{Y} . From (4.7) it follows that $d_{\tilde{Y}}(\tilde{\varphi}_0 \circ \gamma, \chi_*(\gamma) \circ \tilde{\varphi}_0) = 0$, that is,

$$(4.8) \quad \chi_*(\gamma) \circ \tilde{\varphi}_0 = \tilde{\varphi}_0 \circ \gamma \quad \text{q.e. in } \tilde{X}$$

for every $\gamma \in \Gamma_X$. Because $\tilde{\varphi}_0 \in \mathcal{E}_{\text{loc}}(\tilde{X}, \tilde{Y})$ we conclude from (4.8) that $\tilde{\varphi}_0 \in \mathcal{A}$. Inserting $\tilde{\varphi}_i = \tilde{\chi}$ q.e. in $\tilde{X} \setminus \tilde{\Omega}$ in (4.6) (with \tilde{S} replaced by \tilde{X}) we obtain $d_{\tilde{Y}}(\tilde{\chi}, \tilde{\varphi}_0) = 0$ q.e. in $\tilde{X} \setminus \tilde{\Omega}$. Thus $\tilde{\varphi}_0 = \tilde{\chi}$ q.e. in $\tilde{X} \setminus \tilde{\Omega}$, and so altogether $\tilde{\varphi}_0 \in \mathcal{A}_{\tilde{\chi}}$.

Finally, by definition of E_0 ,

$$\begin{aligned} E_0 \leq E(\tilde{\varphi}_0|\tilde{F}) &= \sum_{\nu=1}^n E(\tilde{\varphi}_0|\tilde{s}_\nu) \leq \sum_{\nu=1}^n \liminf_{i \rightarrow \infty} E(\tilde{\varphi}_i|\tilde{s}_\nu) \\ &\leq \liminf_{i \rightarrow \infty} \sum_{\nu=1}^n E(\tilde{\varphi}_i|\tilde{s}_\nu) = \lim_{i \rightarrow \infty} E(\tilde{\varphi}_i|\tilde{F}) = E_0, \end{aligned}$$

the second inequality by [25, Theorem 1.6.1] or [9, Lemma 9.1] (on lower semicontinuity of energy), applied to the m -simplexes \tilde{s}_ν of \tilde{F} . Consequently, $E(\tilde{\varphi}_0|\tilde{F}) = E_0$, and so $\tilde{\varphi}_0 \in \mathcal{A}_{\tilde{\chi}}$ *minimizes the energy of restrictions of all maps of class $\mathcal{A}_{\tilde{\chi}}$ to any prescribed “fundamental domain” \tilde{F} for Γ_X , as in the paragraph preceding the one containing (3.3).*

Step 2. The remaining properties of $\tilde{\varphi}_0$ and its projection φ_0 . Fix a point $\tilde{p} \in \tilde{\Omega} \cup \partial\tilde{\Omega}$. By suitably subdividing the given triangulation of X and hence of \tilde{X} we obtain by [37, Lemma 1, p. 144] a triangulation T of X in which $p := p_X(\tilde{p})$ is a vertex, and by [42, Theorem 35] that every open star is contained in some ball in X of radius r_X . It follows that \tilde{p} is a vertex of the induced triangulation \tilde{T} of \tilde{X} , and the open star \tilde{S} of \tilde{p} in (\tilde{X}, \tilde{T}) is contained in $B_{\tilde{X}}(\tilde{p}, r_X)$ because $B_{\tilde{X}}(\tilde{p}, r_X)$ projects isometrically on $B_X(p, r_X)$ (cf. (2.2)) and hence $p_X(\tilde{S}) = S$. It follows that $\tilde{S} \subset \tilde{F}$ for a suitable “fundamental domain” \tilde{F} in (\tilde{X}, \tilde{T}) . According to [18, Theorem 1 and Corollary 3], \tilde{S} is admissible, along with \tilde{X} . Furthermore, \tilde{S} satisfies the Poincaré inequality

$$(4.9) \quad \int_{\tilde{S}} \tilde{u}^2 d\tilde{\mu} \leq c_2 \int_{\tilde{S}} |\nabla \tilde{u}|^2 d\tilde{\mu} \quad \text{for all } \tilde{u} \in W_0^{1,2}(\tilde{S}),$$

for some constant c_2 ; see [9, Remark 5.5, p. 76] or [13, Lemma 1(c)]. In a triangulation of \tilde{S} denote by $\tilde{\omega} \Subset \tilde{S}$ an open star containing \tilde{p} . (Here $\tilde{\omega}$ replaces the disc D in [28, p. 63], or the star B^j in the paragraph containing

(3.6) above, both for the free homotopy problem.) Then $\tilde{F} \cap \gamma(\tilde{\omega}) = \emptyset$ for all $\gamma \in \Gamma_X \setminus \{\text{id}\}$, cf. (2.2) and subsequent text. Given a (quasicontinuous) map $\tilde{\theta} \in \mathcal{E}(\tilde{S}, \tilde{Y})$ such that $\tilde{\theta} = \tilde{\varphi}_0$ in $\tilde{S} \setminus (\tilde{\omega} \cap \tilde{\Omega})$ (that is, in $\tilde{S} \setminus \tilde{\omega}$ and in $\tilde{S} \setminus \tilde{\Omega}$), define $\tilde{\theta}' : \tilde{X} \rightarrow \tilde{Y}$ in analogy with (3.6) by

$$(4.10) \quad \tilde{\theta}' = \begin{cases} \chi_*(\gamma) \circ \tilde{\theta} \circ \gamma^{-1} & \text{in } \gamma(\tilde{\omega}) \text{ for any } \gamma \in \Gamma_X \\ \tilde{\varphi}_0 & \text{elsewhere in } \tilde{X}. \end{cases}$$

In particular, $\tilde{\theta}' = \tilde{\varphi}_0$ in $\tilde{F} \setminus \tilde{S}$ because \tilde{F} is “almost fundamental” and hence $\tilde{F} \setminus \tilde{S} \subset \tilde{F} \setminus \tilde{\omega} = \tilde{F} \setminus \bigcup_{\gamma \in \Gamma_X} \gamma(\tilde{\omega})$. For every $\gamma \in \Gamma_X$ we have

$$(4.11) \quad \tilde{\theta}' = \chi_*(\gamma) \circ \tilde{\theta} \circ \gamma^{-1} \quad \text{q.e. in } \gamma(\tilde{S}).$$

Indeed, this holds (even everywhere) in $\gamma(\tilde{\omega})$ by the former equation of (4.10). And (4.11) also holds q.e. in $\gamma(\tilde{S} \setminus \tilde{\omega})$ because we have, q.e. in that set,

$$\tilde{\theta}' = \tilde{\varphi}_0 = \chi_*(\gamma) \circ \tilde{\varphi}_0 \circ \gamma^{-1} = \chi_*(\gamma) \circ \tilde{\theta} \circ \gamma^{-1}$$

according to the latter equation of (4.10) followed by (4.8), and next by $\tilde{\varphi}_0 = \tilde{\theta}$ in $\tilde{S} \setminus \tilde{\omega}$, whence $\tilde{\varphi}_0 \circ \gamma^{-1} = \tilde{\theta} \circ \gamma^{-1}$ in $\gamma(\tilde{S} \setminus \tilde{\omega})$. Explicitly, the latter equation of (4.10) reads

$$(4.12) \quad \tilde{\theta}' = \tilde{\varphi}_0 \quad \text{in } \tilde{R} := \tilde{X} \setminus \bigcup_{\gamma \in \Gamma_X} \gamma(\tilde{\omega}).$$

The sets $\gamma(\tilde{S})$ ($\gamma \in \Gamma_X$) together with the interior \tilde{R}° of \tilde{R} form a countable open cover of \tilde{X} because $\tilde{\omega} \Subset \tilde{S}$. From (4.11) and (4.12) it follows that $\tilde{\theta}'$ is quasicontinuous and of locally finite energy in \tilde{X} because $\tilde{\varphi}_0$ and each $\chi_*(\gamma) \circ \tilde{\theta} \circ \gamma^{-1}$ have these two properties in $\tilde{R}^\circ \subset \tilde{X}$ and in $\gamma(\tilde{S})$, respectively, cf. (i) preceding Equation (2.21). Similarly, $\tilde{\theta}' = \tilde{\varphi}_0 = \tilde{\chi}$ q.e. in $\tilde{X} \setminus \tilde{\Omega}$. Altogether, this shows that $\tilde{\theta}' \in \mathcal{A}_{\tilde{X}}$.

In what follows we assume for convenience that $\tilde{\omega} \cap \tilde{\Omega}$ is *connected* (otherwise consider each component of $\tilde{\omega} \cap \tilde{\Omega}$ separately). In the display (4.13) below, the first equation holds because $\tilde{\theta} = \tilde{\theta}'$ in $\tilde{\omega}$ by (4.10) with $\gamma = \text{id}$. The second equation in (4.13) holds by Lemma 2.3(b) (applied with \tilde{X} replaced by \tilde{F} , or rather by each m -simplex of \tilde{F}) because $\tilde{\mu}(\partial \tilde{S}) = 0$. The inequality $E(\tilde{\theta}'|\tilde{F}) \geq E(\tilde{\varphi}_0|\tilde{F})$ holds because $\tilde{\varphi}_0, \tilde{\theta}' \in \mathcal{A}_{\tilde{X}}$ and that $\tilde{\varphi}_0$ is energy minimizing relative to $\mathcal{A}_{\tilde{X}}$; see the end of Step 1. Again by Lemma 2.3(b) we similarly have $e(\tilde{\theta}'|\tilde{F}) = e(\tilde{\varphi}_0|\tilde{F})$ μ -a.e. in $\tilde{F} \setminus \tilde{S}$ because $\tilde{\theta}' = \tilde{\varphi}_0$ in $\tilde{F} \setminus \tilde{S}$,

as noted after (4.10). Altogether we thus obtain

$$\begin{aligned}
 E(\tilde{\theta}) &= E(\tilde{\theta}'|\tilde{S}) = E(\tilde{\theta}'|\tilde{F}) - \int_{\tilde{F}\setminus\tilde{S}} e(\tilde{\theta}'|\tilde{F}) d\tilde{\mu} \\
 (4.13) \qquad &\geq E(\tilde{\varphi}_0|\tilde{F}) - \int_{\tilde{F}\setminus\tilde{S}} e(\tilde{\varphi}_0|\tilde{F}) d\tilde{\mu} = E(\tilde{\varphi}_0|\tilde{S})
 \end{aligned}$$

again because $\tilde{\mu}(\partial\tilde{S}) = 0$. In view of the property of $\tilde{\theta}$ imposed in the lines preceding (4.10) we conclude that $\tilde{\varphi}_0|_{\tilde{S}}$ is the unique variational solution to the ordinary Dirichlet problem in \tilde{S} , minimizing the energy of maps $\tilde{\theta}$ of class

$$\mathcal{E}_{\tilde{\chi}|\tilde{S}}(\tilde{\omega} \cap \tilde{\Omega}, \tilde{Y}) := \{\tilde{\theta} \in \mathcal{E}(\tilde{S}, \tilde{Y}) : \tilde{\theta} = \tilde{\chi} \text{ in } \tilde{S} \setminus (\tilde{\omega} \cap \tilde{\Omega})\};$$

cf. [14, Theorem 1], applied with $X, Y, \Omega, \varphi, \chi$ replaced by $\tilde{S}, \tilde{Y}, \tilde{\omega} \cap \tilde{\Omega}, \tilde{\varphi}_0|_{\tilde{S}}, \tilde{\chi}|_{\tilde{S}}$, noting that \tilde{S} is admissible and satisfies the Poincaré inequality (4.9), and that $\tilde{\omega} \cap \tilde{\Omega} \Subset \tilde{S}$.

Every component of $\tilde{\omega} \cap \tilde{\Omega}$ is regular. We shall, however, only need that every point $\tilde{x} \in \tilde{\omega} \cap \partial\tilde{\Omega}$ is regular for $\tilde{\omega} \cap \tilde{\Omega}$, that is, $\tilde{X} \setminus (\tilde{\omega} \cap \tilde{\Omega})$ is non-thin at \tilde{x} ; and that holds because even $\tilde{X} \setminus \tilde{\Omega}$ is non-thin at \tilde{x} , noting that $X \setminus \Omega$ is non-thin at $x := p_X(\tilde{x})$ because Ω is regular, that p_X is a local isometry, and that thinness is a local property.

We have chosen $\tilde{p} \in \tilde{\Omega} \cup \partial\tilde{\Omega}$. Suppose first that $\tilde{p} \in \partial\tilde{\Omega}$. Because we have seen that the points of $\tilde{\omega} \cap \partial\tilde{\Omega}$ are regular for $\tilde{\omega} \cap \tilde{\Omega}$, and because $\tilde{\chi}$ is continuous, it follows from [14, Theorem 1] or [16, Theorem 4] (applied as above) that a version of $\tilde{\varphi}_0$ is continuous up to the boundary at the portion $\tilde{\omega} \cap \partial\tilde{\Omega} \ni \tilde{p}$ of the boundary of $\tilde{\omega} \cap \tilde{\Omega}$, and hence up to the entire boundary $\partial\tilde{\Omega}$, by varying \tilde{p} (and $\tilde{\omega}$).

If instead $\tilde{p} \in \Omega$, and hence $\tilde{p} \in \tilde{S} \cap \tilde{\Omega}$, choose the open star $\tilde{\omega} \ni \tilde{p}$, now in a triangulation of $\tilde{S} \cap \tilde{\Omega}$, and so that $\tilde{\omega} \Subset \tilde{S} \cap \tilde{\Omega}$. Then $\tilde{\varphi}_0$ is, in particular, locally energy minimizing in $\tilde{\omega} \cap \tilde{\Omega} = \tilde{\omega}$ (cf. the paragraph preceding the one containing (3.6)). It follows by [11, Theorem 1] or [17, Theorem 1] (applied as above, noting that now $\tilde{\Omega} \cap \tilde{\omega} = \tilde{\omega}$) that a version of $\tilde{\varphi}_0$ is (locally uniformly) Hölder continuous in $\tilde{\omega}$, and hence in $\tilde{\Omega}$, by varying \tilde{p} .

Because $\tilde{\varphi}_0 = \tilde{\chi}$ q.e. off $\tilde{\Omega}$, and $\tilde{\chi}$ is continuous, we altogether conclude that $\tilde{\varphi}_0$ ($\in \mathcal{A}_{\tilde{\chi}}$) has a continuous version satisfying (3.2) and (4.1) everywhere in \tilde{X} (with $\tilde{\varphi}$ replaced by $\tilde{\varphi}_0$), so $\tilde{\varphi}_0$ is χ_* -equivariant. It follows that $\tilde{\varphi}_0$ has a continuous projection $\varphi_0 : X \rightarrow Y$, freely homotopic with χ , as shown in the paragraph containing (3.2). The homotopy Φ constructed there has $\Phi(x, t) = \chi(x)$ for $x \in X \setminus \Omega$, and φ_0 is therefore homotopic with χ through maps of class $C_\chi(\Omega, Y)$, so $\varphi_0 \in \mathcal{H}_\chi(\Omega, Y)$.

To show that $E(\varphi_0) \leq E(\varphi)$ for every $\varphi \in \mathcal{H}_\chi(\Omega, Y)$, note first that $E(\tilde{\varphi}_0|\tilde{F}) \leq E(\tilde{\varphi}|\tilde{F})$ by (4.3) for any $\tilde{\varphi} \in \mathcal{A}_{\tilde{\chi}}$ (see the end of Step 1). For continuous $\tilde{\varphi} \in \mathcal{A}_{\tilde{\chi}}$ we have shown in (3.3) that $E(\tilde{\varphi}|\tilde{F}) = E(\varphi)$, where now $\varphi \in \mathcal{H}_\chi(\Omega, Y)$ denotes the projection of $\tilde{\varphi}$ (see the paragraph containing (3.2)). As shown in the beginning of Step 1, every map $\varphi \in \mathcal{H}_\chi(\Omega, Y)$ has a χ_* -equivariant lift $\tilde{\varphi}$ which equals $\tilde{\chi}$ in $\tilde{X} \setminus \tilde{\Omega}$, and so $\tilde{\varphi} \in \mathcal{A}_{\tilde{\chi}}$. Consequently, $E(\varphi_0) = E(\tilde{\varphi}_0|\tilde{F}) \leq E(\tilde{\varphi}|\tilde{F}) = E(\varphi)$, so φ_0 minimizes energy within $\mathcal{H}_\chi(\Omega, Y)$. Furthermore, φ_0 is (locally uniformly) Hölder continuous in Ω because $\tilde{\varphi}_0$ is so in $\tilde{\Omega}$ and because p_X and p_Y are local isometries.

The proof that φ_0 is harmonic in Ω (cf. Remark 2.6) is the same as in Step 4 of the proof of Theorem 3.2, except that X, φ shall now be replaced by Ω, φ_0 , and any subset of X by its intersection with Ω (though the homotopy Ψ is still defined on $X \times [0, 1]$, and the set $X \setminus \bar{U}$ shall remain unaltered).

Step 3. Uniqueness. Consider two solutions φ_0, φ_1 to the Dirichlet homotopy problem, that is, $\varphi_0, \varphi_1 \in \mathcal{H}_\chi(\Omega, Y)$ have (the same) minimal energy. Then φ_0, φ_1 lift to continuous energy minimizing χ_* -equivariant maps $\tilde{\varphi}_0, \tilde{\varphi}_1$. Step 5 of the proof of Theorem 3.2 carries over (with \mathcal{H} replaced by $\mathcal{H}_\chi(\Omega, Y)$). Thus $d_{\tilde{\varphi}}(\tilde{\varphi}_0, \tilde{\varphi}_1)$ is a constant \tilde{d} , which equals 0 because $\tilde{\varphi}_0 = \tilde{\varphi}_1$ on $\tilde{X} \setminus \tilde{\Omega} \neq \emptyset$. Consequently, $\tilde{\varphi}_0 = \tilde{\varphi}_1$, and so indeed $\varphi_0 = \varphi_1$, by projection. Thus $\varphi = \varphi_0$ is the unique solution to our Dirichlet homotopy problem for the present case $\bar{\Omega} \neq X$.

Step 4. The general case $\bar{\Omega} \cap b\tilde{X} \neq bX$, in particular $bX \neq \emptyset$. Here $\Omega \subset X \setminus \sigma$ for some nonvoid relatively open subset σ of bX . By subdivision of the triangulation of X we arrange that σ contains an $(m - 1)$ -simplex τ of bX . (To justify this, write $\sigma = U \cap bX$ for some open subset U of X . Choose a point $a \in \sigma$ and an open subset V of $X \setminus \{a\}$ so that $U \cup V = X$. According to [42, Theorem 35] there exists a subdivision of the triangulation of X such that an open star S containing a is a subset of U , the alternative $S \subset V$ being ruled out since $a \notin V$. Because $a \in bX$, there exists an $(m - 1)$ -simplex τ of S contained in bX , and hence $\tau \subset U \cap bX = \sigma$.)

Consider the disjoint union X^* of X and another copy X' of X , though glued together along τ . Then X^* is a compact admissible Riemannian polyhedron (possibly with boundary), and X and X' are compact subpolyhedra of X^* .⁷ Denote by $\iota : X^* \rightarrow X^*$ the natural involution of X^* , interchanging X and X' . For any map $\varphi \in \mathcal{H}_\chi(\Omega, Y)$ denote φ^* the extension of

⁷For the easy proof that X^* is admissible, apply [18, Lemma 5] as in [13, p. 773, third and fourth paragraphs]. Here Z_+ and Z_- correspond to the present X and its stated copy, and hence Z is the present X^* ; furthermore, $Z_0 := Z_+ \cap \tau = Z_- \cap \tau$ corresponds to the present τ .

φ by $\chi \circ \iota|_{X^* \setminus X}$; then $\varphi^* \in \mathcal{E}(X^*, Y)$ and $E(\varphi^*) = E(\varphi) + E(\chi \circ \iota|_{X'}) = E(\varphi) + E(\chi)$; cf. [9, Theorem 9.1 and Remark 9.1]. In particular, $\chi^* \in \mathcal{E}(X^*, Y)$ is the symmetric extension of χ to X^* , and clearly $\varphi^* \in \mathcal{H}_{\chi^*}(\Omega, Y)$. Conversely, for any $\varphi^* \in \mathcal{H}_{\chi^*}(\Omega, Y)$, the restriction $\varphi := \varphi^*|_X$ is of class $\mathcal{H}_\chi(\Omega, Y)$. Furthermore, φ is then an energy minimizer relative to $\mathcal{H}_\chi(\Omega, Y)$ if and only if φ^* is the unique energy minimizer φ_0^* relative to $\mathcal{H}_{\chi^*}(\Omega, Y)$. Note at this point that $\Omega \subset X$ is a connected regular open subset also of X^* , but Ω is not dense in X^* ; thus Theorem 4.1 applies with X, χ replaced by X^*, χ^* . We conclude that the restriction φ_0 of φ_0^* to X is the unique energy minimizer relative to $\mathcal{H}_\chi(\Omega, Y)$, and $\varphi_0|_\Omega = \varphi_0^*|_\Omega$ is indeed harmonic and (locally uniformly) Hölder continuous, by Theorem 4.1 applied to X^*, χ^*, Ω . □

5. The case of Riemannian manifold targets

In this final section we specialize the target by replacing the metric space (Y, d_Y) with a (smooth) Riemannian *manifold* (Y, h) without boundary (in particular, Y is locally compact and carries an intrinsic Riemannian distance d_Y associated with the Riemannian metric h). On the other hand, we no longer require that the Riemannian metric g on the compact admissible m -dimensional source polyhedron X be simplex-wise smooth, but merely *measurable*, again with elliptic bounds $\Lambda_X, \Lambda_X^{-1}$, that is,

$$\Lambda_X^{-2} \leq \sum_{i,j=1}^m g_{ij}(x)\xi^i\xi^j = \sum_{i,j=1}^m g^{ij}(x)\xi_i\xi_j \leq \Lambda_X^2$$

for a.e. $x \in X$ and for every m -tuple (ξ^1, \dots, ξ^m) , respectively (ξ_1, \dots, ξ_m) , in \mathbb{R}^m with Euclidean norm $|\xi| = 1$. Measurability and the term “almost everywhere (a.e.)” refer to a fixed Riemannian volume measure on X , for example the measure μ^e corresponding to the Euclidean Riemannian metric g^e on X (cf. [9, p. 52]). We then dispose of an alternative concept of energy, of a more explicit character and defined for example in terms of an isometric C^1 -embedding of (Y, h) as a Riemannian submanifold of some \mathbb{R}^N , as in Nash’s theorem: *A map $\varphi : X \rightarrow Y$ is said to be of class $\mathcal{E}(X, Y)$ if the components $\varphi^1, \dots, \varphi^N$ of $\varphi : X \rightarrow Y \hookrightarrow \mathbb{R}^N$ are of class $W^{1,2}(X)$.* In the affirmative case the energy density of φ is defined by

$$(5.1) \quad e(\varphi) = \sum_{\nu=1}^N |\nabla \varphi^\nu|^2 = \sum_{\nu=1}^N g^{ij} \partial_i \varphi^\nu \partial_j \varphi^\nu \in L^1(X, \mu)$$

(using the standard summation convention); and the energy is defined by $E(\varphi) = \int_X e(\varphi) d\mu (< \infty)$. Here μ denotes the Riemannian volume measure on X corresponding to g , that is, $d\mu = \sqrt{\det g} d\mu^e$ in terms of the covariant components g_{ij} ($i, j = 1, \dots, m$) of g and their determinant. For an equivalent, covariant definition of $E(\varphi)$, see [9, Definition 9.2 and Lemma 9.3]. If g is simplex-wise smooth and if $\varphi(X)$ has compact closure in Y , then the energy density $e(\varphi)$ from (5.1) equals a.e. the energy density $e(\varphi)$ defined by (2.21) (i.e., as the weak limit of the ε -approximate energy density $e_\varepsilon(\varphi)$ from (2.18) as $\varepsilon \rightarrow 0$), provided that $E(\varphi) < \infty$ in the one sense or the other [9, Theorem 9.2(b)]. The hypothesis $\varphi(X) \Subset Y$ is of course fulfilled if Y is compact, or if X is compact and φ is continuous. Every map $X \rightarrow Y$ of finite energy in the present sense has a *quasicontinuous version* (because every function of class $W^{1,2}(X)$ has so [9, p. 106 f]).

Recall that a complete Riemannian manifold (Y, h) with intrinsic Riemannian distance function d_Y is a geodesic space, and (Y, d_Y) has non-positive Alexandrov curvature if and only if (Y, h) has *nonpositive sectional curvature* [2, p. 173].

Propositions 2.1 and 2.2, as well as Lemma 2.4, do not involve energy of maps, and hence carry over right away to the present setting without requiring that the Riemannian metric g on X be simplex-wise smooth. For the proof of Lemma 2.5 with the present energy concept (5.1), use [9, Remark 9.6, p. 165 f] concerning lower semicontinuity of energy.

The proof of Lemma 2.7 (the three equivalent definitions of harmonicity of a map) reduces to the case where Y is simply connected, in the same way as described in the latter part of the proof of the lemma. For the simply connected case, discussed in the former part of the proof, the *energy convexity property* in the present setting can be found in [13, Lemma 3 and eq. (10.1)]; furthermore, one shall replace [13, Theorem 1] by [13, Theorem 4] in order to establish the present version of Lemma 2.7 above. (Similarly, in footnote 5 to Remark 2.8, the reference to [13, Theorem 3] shall be replaced by [13, Theorem 5] in order to recover Remark 2.8 in the present setting.)

For Proposition 3.1 one begins by using [9, Proposition 12.1] to recover [9, Lemma 10.2(b)]. Next, use [11, Theorem 3 and Corollary 1], the latter about the weak Poincaré inequality (see [9, Remark 9.6 (p. 165)]), leading to (3.1), from where the above proof of Proposition 3.1 carries over to the present setting.

Finally, note that Lemma 2.3 now becomes obvious, and that Lemma 2.3 (b) remains valid for measurable A , even if $\mu(\partial A) > 0$, because this reduces to the case $Y = \mathbb{R}$ of functions of class $W_{\text{loc}}^{1,2}(X)$ in view of (5.1); cf. [9, Remark 5.2, p. 65]. For that reason the hypothesis $\mu(\partial\Omega) = 0$ in

Theorem 4.1 can be omitted in the present setting of maps into a Riemannian manifold, see Theorem 5.2 below.

Proceeding to the solution of our two homotopy problems in the present setting of manifold target, we shall need the following lemma about approximating a measurable Riemannian metric on X by simplex-wise smooth ones:

Lemma 5.1. *Let X be a compact admissible polyhedron, and let g be a measurable Riemannian metric on X with elliptic bounds $\Lambda_X, \Lambda_X^{-1}$. Consider a sequence (g_n) of simplex-wise smooth Riemannian metrics on X such that $g_n \rightarrow g$ pointwise a.e. as $n \rightarrow \infty$, and furthermore that $\Lambda_X, \Lambda_X^{-1}$ are elliptic bounds for each g_n as well. For any continuous map $\varphi \in \mathcal{E}(X, Y)$ (in the present sense) and any function $f \in C(X)$ we then have*

$$\int_X f e(\varphi) d\mu = \lim_{n \rightarrow \infty} \int_X f e_n(\varphi) d\mu,$$

where $e_n(\varphi)$ denotes the energy density of φ relative to g_n , cf. (5.1); and μ_n denotes the volume measure on X relative to g_n ; that is, $d\mu_n = \sqrt{\det g_n} d\mu^e = \sqrt{\det g_n / \det g} d\mu$.

Proof. A sequence (g_n) as in Lemma 5.1 is constructed in [13, p. 791, last paragraph]. In view of (5.1),

$$\begin{aligned} \int_X f e_n(\varphi) d\mu_n &= \sum_{\nu=1}^N \int_X f g_n^{ij} \partial_i \varphi^\nu \partial_j \varphi^\nu \sqrt{\det g_n / \det g} d\mu \\ &\rightarrow \sum_{\nu=1}^N \int_X f g^{ij} \partial_i \varphi^\nu \partial_j \varphi^\nu d\mu = \int_X f e(\varphi) d\mu \end{aligned}$$

as $n \rightarrow \infty$. By dominated convergence we have in fact (when $f \geq 0$)

$$f \sum_{\nu=1}^N g_n^{ij} \partial_i \varphi^\nu \partial_j \varphi^\nu \leq \Lambda_X^2 f \sum_{\nu=1}^N \sum_{i=1}^m |\partial_i \varphi^\nu|^2 \in L^1(X, \mu),$$

and $\sqrt{\det g_n} \leq \Lambda_X^m, \sqrt{\det g} \geq \Lambda_X^{-m}$. □

For the proof of Theorem 3.2 in the present setting of manifold target, refer in Step 2 for energy convexity to [13, Lemma 3] instead of [9, (11.2); 25, 2.2iv]; and to [14, Theorem 3] instead of [14, Theorem 1]. Inserting $\tau = \frac{1}{2}$ in the last estimate in (3.23) leads after projection to Hölder equicontinuity

of the φ_i :

$$(5.2) \quad d_Y(\varphi_i(x), \varphi_i(y)) \leq C_{10} \sqrt{E(\psi_i)} d_X^\alpha(x, y) \quad \text{for } x, y \in X$$

(with $C_{10} = 2\sqrt{C_7}$), first relative to g_n (simplex-wise smooth) and next relative to the given measurable Riemannian metric g in view of Lemma 3.1, noting that C_{10} is independent of n , by Remark 3.4.

In Step 3, again first for g_n and next for g , form in (3.31) the sum over the m -simplexes s of X and over $j \in \{1, \dots, k\}$, and insert the resulting inequality in (3.26). Likewise, insert the former estimate (3.19) in the remainder term in (3.26). Together, this leads to

$$(5.3) \quad \begin{aligned} (1 - \tau)E(\varphi_i) &\leq E(\psi_i) + C_{11}\sqrt{E(\psi_i)} \sum_{j=1}^k \sqrt{E(\psi_i) - E(w_i^j)} \\ &+ C_5^2 C_9 \tau^{-1} E(\psi_i)^{m/(m+2\alpha)} (E(\psi_i) - E(w_i^{j_l}))^{2\alpha/(m+2\alpha)} \end{aligned}$$

(where C_{11} equals twice the number of m -simplexes in X). The midpoint maps $\tilde{w}_i^j = \frac{1}{2}\tilde{\varphi}_i^j + \frac{1}{2}\tilde{\psi}_i$ and $\tilde{w}_i^{j_l} = \frac{1}{2}\tilde{\varphi}_i^j + \frac{1}{2}\tilde{\varphi}_i^{l_l}$ being χ_* -equivariant (along with $\tilde{\varphi}_i^j$ and $\tilde{\psi}_i$, respectively $\tilde{\varphi}_i^{l_l}$), their projections w_i^j and $w_i^{j_l}$ are of class \mathcal{H} , and of energy $\geq E_0$. Inserting this in (5.3) and making $i \rightarrow \infty$, whereby $E(\psi_i) - E_0 \rightarrow 0$, we recover (3.32) and conclude that the sequence $(\tilde{\varphi}_i^j)$ indeed is minimizing. Step 4 (harmonicity) carries over right away to the present setting, using [9, Remark 9.6] in place of Lemma 9.1 there.

In the proof of Theorem 4.1 (in the fifth paragraph of Step 1) the inequalities $e(\varphi) \circ p_X \leq e(\tilde{\varphi})$ and hence $E(\varphi) \leq E(\tilde{\varphi}|F)$, etc., clearly remain in force in the present setting, with equality if $\tilde{\varphi}$ is continuous. At the end of Step 1, [9, Lemma 9.1] (lower semicontinuity of energy) carries over in view of [9, Remark 9.6]. In the case $\tilde{p} \in \partial\tilde{\Omega}$ replace [14, Theorem 1] and [16, Theorem 4] by [14, Theorem 3]. In the case $\tilde{p} \in \tilde{\Omega}$ replace [11, Theorem 1] and [17, Theorem 1] by [14, Theorem 3]. The rest of the proof of Theorem 4.1 carries over right away, again in view of [9, Lemma 9.1] (lower semicontinuity) and [13, Lemma 3] (energy convexity).

By way of summary, we have obtained the following combined version of Theorems 3.2 and 4.1 in the present setting of a manifold codomain, whereby g is no longer required to be simplex-wise smooth:

Theorem 5.2. *Suppose that the admissible Riemannian polyhedron X is compact and has measurable Riemannian metric g with elliptic bounds.*

Furthermore, suppose that the smooth Riemannian manifold (Y, h) without boundary is compact and has nonpositive sectional curvature. The conclusions of Theorems 3.2 and 4.1 then hold true in terms of the energy concept (5.1).

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