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# Locally symmetric connections on complex surfaces and some equations of Monge-Ampère type

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We study locally symmetric connections induced by transversal bundles on non-degenerate complex surfaces. Each of such surfaces together with its transversal bundle can be described locally by a solution to some partial differential equation of Monge-Ampère type.

## 1. Introduction

Let $M$ be an $n$-dimensional connected complex manifold and $f : M \to \mathbf{C}^{n+1}$ a holomorphic immersion. Let $\mathcal{N}$ be a $\mathcal{C}^\infty$ transversal bundle, that is,

$$\mathcal{N} = \bigcup_{p \in M} \mathcal{N}_p,$$

where $\mathcal{N}_p$ is a complex vector subspace of $\mathbf{C}^{n+1} \cong \mathbf{R}^{2n+2}$ such that $f_*\big(T_pM\big) \oplus \mathcal{N}_p = \mathbf{C}^{n+1}$. The $\mathcal{C}^\infty$ class means that for any $q \in M$ there exists a neighbourhood $U$ of $q$ such that $\mathcal{N}\big|_U$ is spanned over $\mathbf{C}$ by a vector field $\xi$ on $\mathbf{C}^{n+1}$ defined along $f$:

$$M \supset U \ni p \mapsto \xi_p \in T_{f(p)}\mathbf{C}^{n+1} \cong \mathbf{C}^{n+1},$$

which is of class $\mathcal{C}^\infty$ and not necessarily holomorphic.

The connection $\nabla$, the $\mathbf{C}$-bilinear symmetric affine fundamental form $h = h_1 + ih_2$, the affine shape operator $S$ and the transversal connection form $\tau = \mu + i\nu$ which are induced on $U$ by $f$ and $\xi$ are defined by the following Gauss and Weingarten formulae:

$$D_X f_*Y = f_*\nabla_X Y + h_1(X,Y)\xi + h_2(X,Y)J\xi \tag{1.1}$$

$$D_X \xi = -f_*SX + \mu(X)\xi + \nu(X)J\xi \tag{1.2}$$

(see, e.g., [4, 5]). Here $D$ denotes the standard connection on $\mathbf{C}^{n+1}$. The manifold $M$ is regarded as a $2n$-dimensional real manifold with the complex structure $J$. To simplify notation, we use the same letter $J$ for the complex structure in $\mathbf{C}^{n+1} \cong \mathbf{R}^{2n+2}$. The identification of $\mathbf{C}^k$ with $\mathbf{R}^{2k}$ is given by: $(z^1 + iz^2, \dots, z^{2k-1} + iz^{2k}) \mapsto (z^1, z^2, \dots, z^{2k-1}, z^{2k})$.

If we replace vector field $\xi$ by $\widetilde{\xi} = \varphi\xi + \vartheta J\xi + f_* Z$, then we obtain $\widetilde{\nabla}$, $\widetilde{h}$, $\widetilde{S}$ and $\widetilde{\tau}$:

$$\widetilde{\nabla}_X Y = \nabla_X Y - \frac{1}{\varphi + i\vartheta} h(X,Y)Z, \tag{1.3}$$

$$\widetilde{h} = \frac{1}{\varphi + i\vartheta}\, h, \tag{1.4}$$

$$\widetilde{S}X = (\varphi + i\vartheta)SX - \nabla_X Z + \widetilde{\tau}(X)Z, \tag{1.5}$$

$$\widetilde{\tau}(X) = \frac{X(\varphi) + iX(\vartheta)}{\varphi + i\vartheta} + \frac{1}{\varphi + i\vartheta}\, h(X,Z) + \tau(X). \tag{1.6}$$

Let $\xi$ and $\widetilde{\xi}$ be two local sections of $\mathcal{N}$. Then we have $\widetilde{\xi} = \big(\varphi + i\vartheta\big)\,\xi$ and $Z = 0$, therefore $\widetilde{\nabla}_X Y = \nabla_X Y$ for any $\mathcal{C}^\infty$ vector fields $X, Y$ on $U$. It follows that $\nabla$ depends on $\mathcal{N}$ only and can be defined on the whole of $M$. Moreover, the complex rank of affine fundamental form $h$ does not depend on the transversal bundle $\mathcal{N}$. We call it *the type number of* $f$. This type number is constant on a dense open subset $M'$ of $M$ [6]. The immersion is called *non-degenerate*, if $h$ is non-degenerate ($\forall\, X \neq 0\ \exists\, Y : h(X,Y) \neq 0$).

The induced connection $\nabla$, he affine fundamental form $h$, the shape operator $S$ and the transversal connection form $\tau$ satisfy the following fundamental equations.

Gauss equation:

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY. \tag{1.7}$$

Codazzi equation for $h$:

$$(\nabla_X h)(Y,Z) + \tau(X)\, h(Y,Z) = (\nabla_Y h)(X,Z) + \tau(Y)\, h(X,Z). \tag{1.8}$$

Codazzi equation for $S$:

$$(\nabla_X S)(Y) - \tau(X)\, SY = (\nabla_Y S)(X) - \tau(Y)\, SX. \tag{1.9}$$

Ricci equation:

$$h(X,SY) - h(SX,Y) = 2\, d\tau(X,Y). \tag{1.10}$$

If $f$ is non-degenerate, there exists some canonical transversal bundle — the *complex affine normal bundle* – which is the bundle of affine normal complex lines. According to the definition given by F. Dillen, L. Vrancken and L. Verstraelen in [1] (see also [2]), the affine normal complex line of $M$ at $p$ is a complex line in $T_{f(p)}\mathbf{C}^{n+1}$ determined by the complex affine normal vector $\xi_p$. The complex affine normal vector field $\xi$ is a local vector field on $\mathbf{C}^{n+1}$ defined along $f$ satisfying the following two conditions:

$$^{\mathbf{C}}H_\xi = 1 \quad \text{and} \quad \tau = 0.$$

Here $\tau$ is a transversal connection form and

$$^{\mathbf{C}}H_\xi := \left|\det\big[h(X_k, X_l)\big]_{k,l=1}^{n}\right|^2,$$

where $X_1, \ldots, X_n$ is a local complex basis of $TM$ such that

$$\left|{}^{\mathbf{C}}\omega(f_*X_1, \ldots, f_*X_n, \xi)\right| = 1.$$

The symbol $^{\mathbf{C}}\omega$ denotes the complex volume form on $\mathbf{C}^{n+1}$ such that $^{\mathbf{C}}\omega(e_1, \ldots, e_{n+1}) = 1$ for the standard basis $e_1, \ldots, e_{n+1}$ of $\mathbf{C}^{n+1}$. Actually, if for $k = 1, \ldots, n+1$, $Y_k = \sum_{l=1}^{n+1} Y_k^l e_l$, then $^{\mathbf{C}}\omega(Y_1, \ldots, Y_{n+1}) = \det\big[Y_k^l\big]_{k,l=1}^{n+1}$.

If $\xi$ and $\widetilde{\xi}$ are complex affine normal vector fields defined on the same open domain $U$, then there exists a real number $\theta$ such that $\widetilde{\xi} = e^{i\theta}\,\xi$ [1]. It follows that for any $p \in M$ the affine normal complex line at $p$ is uniquely determined.

The aim of this paper is to give a local description of some of those immersions and transversal bundles for which the induced connection $\nabla$ is locally symmetric. The local symmetry of $\nabla$ is equivalent to the condition $\nabla R = 0$, where $R$ is the curvature tensor of $\nabla$ [3]. Here we consider the case of a non-degenerate immersion, $\dim_{\mathbf{C}} M = 2$ and we shall study connections of ranks 1 and 2. By the *rank* of locally symmetric connection we mean, following [8], the (complex) dimension of the subspace

$$\text{(1.11)} \qquad \mathrm{im} R_x := \mathrm{span}_{\mathbf{R}}\{R(X,Y)Z : X, Y, Z \in T_xM\}.$$

For any $x \in M$, (1.11) is a complex subspace of $T_xM$. If $\nabla R = 0$ and $M$ is connected, then $\dim \mathrm{im} R_x$ does not depend on $x$. We shall also use the subspace

$$\ker R_x := \bigcap_{X,Y\in T_xM} \ker R(X,Y). \tag{1.12}$$

Let us denote by $r$ the type number of $f$ on $M'$ and by $\pi$ the projection of $\mathbf{C}^{n+1}$ onto $\mathbf{C}^{r+1}$ parallel to $\mathbf{C}^{n-r} \cong f_*(\ker h)$. The following theorem, which in particular gives the full classification of locally symmetric hypersurfaces with $r > 2$, has been proved by B. Opozda in [5].

**Theorem 1.1.** *Let $f : M \to \mathbf{C}^{n+1}$ be a complex hypersurface endowed with a complex transversal vector bundle $\mathcal{N}$ inducing a non-flat locally symmetric connection $\nabla$. Then around every point $x \in M'$ there is an open neighbourhood $U$ of $x$ of the form $N' \times N^0$, where $N^0$ endowed with $\nabla$ restricted to $N^0$ is affine isomorphic by $f$ to an open subset of $\mathbf{C}^{n-r}$ and $N'$ is immersed by $f$ into $\mathbf{C}^{r+1}$ as a non-degenerate hypersurface. If $r > 1$, then the bundle $\pi(\mathcal{N})|_U$ is holomorphic and induces a locally symmetric connection $\nabla'$ on $U$ as well as on $N'$. If $r > 2$, then $\nabla'$ is flat or $f(N')$ is an open part of a central quadric in $\mathbf{C}^{r+1}$. If $r > 1$ and $\nabla$ is affine Kähler* [i.e., $R(JX,JY) = R(X,Y)$ for any $X, Y$]*, then $\nabla'$ is flat.*

A local description of complex hypersurfaces with type number one endowed with transversal bundles inducing locally symmetric connections is given in [9].

In the present paper we associate with any locally symmetric complex surface some partial differential equation such that this surface is locally equivalent to the graph of a solution $F$ to this equation and the transversal bundle is also determined by this solution. It is known that the real equation $F_{xx}F_{yy} - F_{xy}F_{xy} = \kappa(1 + {F_x}^2 + {F_y}^2)^2$ describes the Euclidean surfaces with constant Gauss curvature $\kappa$. Similar description we obtain for complex locally symmetric surfaces in the case of $\dim \operatorname{im} R = 2$, because in this case $\nabla$ turns out to be metrizable in the sense that there exists non-degenerate, $\mathbf{C}$-bilinear, symmetric $g$ such that $\nabla g = 0$. The local symmetry implies that the complex sectional curvature of $M$ is then constant. The connection $\nabla$ is induced by the transversal bundle which is perpendicular to $f_*(TM)$ with respect to some $\mathbf{C}$-bilinear metric $G$ in $\mathbf{C}^3$. The surface $(M, g)$ is isometrically immersed in $(\mathbf{C}^3, G)$.

In the case of $\dim \operatorname{im} R = 1$ the equation has the form $F_{zz}F_{ww} - F_{zw}F_{zw} = \Phi(F_z)$ with some arbitrary holomorphic function $\Phi$, which is also associated with the given surface. A local section of the transversal bundle may be expressed in terms of $F_z$ and $\eta$, where $\eta$ is a holomorphic function such that

$\Phi = \frac{\eta}{\eta''}$. If $\mathcal{N}$ is the complex affine normal bundle, then $\Phi$ is more strictly determined and the right-hand side of the equation has the form $\left(1 + {F_z}^2\right)^2$.

## 2. Locally symmetric connections of rank 1 on surfaces — a class of examples

Let $V$ be an open subset of $\mathbf{C}$. Let $\eta : V \to \mathbf{C}$ be a holomorphic function such that $\forall \zeta \in V : \eta(\zeta) \neq 0$ and $\forall \zeta \in V : \eta''(\zeta) \neq 0$. Let the holomorphic function $F = F^1 + iF^2 : U \to \mathbf{C}$ of two variables $z = z^1 + iz^2$, $w = w^1 + iw^2$ satisfy the following partial differential equation:

$$F_{zz}F_{ww} - F_{zw}F_{zw} = \frac{\eta(F_z)}{\eta''(F_z)}. \tag{2.1}$$

Let $e_1, e_2, e_3$ be the standard basis of $\mathbf{C}^3$. As a local basis of $TM$ over $\mathbf{C}$ we shall use the vector fields $\frac{\partial}{\partial z^1}$ and $\frac{\partial}{\partial w^1}$. For $\alpha, \beta \in \mathbf{R}$ we have $(\alpha + i\beta)\frac{\partial}{\partial z^1} = \alpha\frac{\partial}{\partial z^1} + \beta\frac{\partial}{\partial z^2}$ and likewise for the $w$-variables.

**Proposition 2.1.** *The transversal vector field*

$$\xi = f_*(-T_0) + e_3 \tag{2.2}$$

*with*

$$T_0 = \frac{\eta'(F_z)}{\eta(F_z)}\frac{\partial}{\partial z^1} \tag{2.3}$$

*induces on the surface*

$$f : U \ni (z, w) \mapsto (z, w, F(z, w)) \in \mathbf{C}^3 \tag{2.4}$$

*a real holomorphic, locally symmetric connection of rank 1.*

*Proof.* A connection $\nabla$ is real holomorphic if and only if its curvature tensor $R$ satisfies the condition $R(JX, Y) = JR(X, Y)$ for all $X$, $Y$ [4]. From the Cauchy–Riemann equations for the holomorphic function $\eta'(F_z)/\eta(F_z)$ it follows easily that $\nabla_{JY}T_0 = J\nabla_Y T_0$ for any $Y$. Consequently $D_{JY}\xi = JD_Y\xi$ for any $Y$, which is equivalent to the condition that $\xi$ is real holomorphic and $S$, $\tau$ are $\mathbf{C}$-linear (see [4]). Therefore $R(JX, Y)Z = JR(X, Y)Z$ for any $X$ by the Gauss equation.

Using the Gauss and Weingarten formulae for the immersion (2.4), which we identify with $(z^1, z^2, w^1, w^2) \mapsto (z^1, z^2, w^1, w^2, F^1(z, w), F^2(z, w))$, and the transversal field (2.2), where $e_3$ when looked at as an element of $\mathbf{R}^6$ is equal to $(0,0,0,0,1,0)$, we easily obtain

$$
\begin{aligned}
&h\left(\frac{\partial}{\partial z^i},\frac{\partial}{\partial z^j}\right)=\frac{\partial^2F^1}{\partial z^i\partial z^j}+i\frac{\partial^2F^2}{\partial z^i\partial z^j},\\
(2.5)\qquad &h\left(\frac{\partial}{\partial z^i},\frac{\partial}{\partial w^j}\right)=\frac{\partial^2F^1}{\partial z^i\partial w^j}+i\frac{\partial^2F^2}{\partial z^i\partial w^j},\\
&h\left(\frac{\partial}{\partial w^i},\frac{\partial}{\partial w^j}\right)=\frac{\partial^2F^1}{\partial w^i\partial w^j}+i\frac{\partial^2F^2}{\partial w^i\partial w^j},\\
(2.6)\qquad &\nabla_X\frac{\partial}{\partial z^1}=X(F_z)T_0,\quad \nabla_Y\frac{\partial}{\partial w^1}=Y(F_w)T_0.\\
(2.7)\qquad &SX=\nabla_XT_0,\\
(2.8)\qquad &\tau(X)=-h(X,T_0).
\end{aligned}
$$

Here for a complex valued function $f=f_1+if_2$ by $X(f)$ we mean $X(f_1)+iX(f_2)$ and for a holomorphic function $F$ we have

$$
(2.9)\qquad \frac{\partial F}{\partial z}=\frac{\partial F^1}{\partial z^1}+i\frac{\partial F^2}{\partial z^1},\quad \frac{\partial F}{\partial w}=\frac{\partial F^1}{\partial w^1}+i\frac{\partial F^2}{\partial w^1}.
$$

Using (2.3), (2.6) and (2.7), we obtain

$$
(2.10)\qquad SX=X(F_z)\,\frac{\eta''(F_z)}{\eta(F_z)}\,\frac{\partial}{\partial z^1}.
$$

From (1.7), (2.1) and (2.5), it follows that

$$
\begin{aligned}
R\left(\frac{\partial}{\partial z^1},\frac{\partial}{\partial w^1}\right)\frac{\partial}{\partial z^1}&=F_{wz}S\frac{\partial}{\partial z^1}-F_{zz}S\frac{\partial}{\partial w^1}\\
&=(F_{wz}F_{zz}-F_{zz}F_{wz})\,\frac{\eta''(F_z)}{\eta(F_z)}\,\frac{\partial}{\partial z^1}=0,\\
R\left(\frac{\partial}{\partial z^1},\frac{\partial}{\partial w^1}\right)\frac{\partial}{\partial w^1}&=F_{ww}S\frac{\partial}{\partial z^1}-F_{zw}S\frac{\partial}{\partial w^1}\\
&=(F_{ww}F_{zz}-F_{zw}F_{wz})\,\frac{\eta''(F_z)}{\eta(F_z)}\,\frac{\partial}{\partial z^1}=\frac{\partial}{\partial z^1},
\end{aligned}
$$

and it is easy to check that $\nabla R=0$. $\square$

## 3. The classification theorem

Here and subsequently, $\mathcal{A}^{\rightarrow}$ denotes the linear part of an affine map $\mathcal{A}$. The symbol dim stands for the complex dimension $\dim_{\mathbf{C}}$.

**Theorem 3.1.** *Let $M$ be a two-dimensional complex manifold and $f: M \rightarrow \mathbf{C}^3$ a non-degenerate holomorphic immersion. Assume that $M$ is endowed with $\mathcal{C}^\infty$ transversal bundle $\mathcal{N}$ inducing on $M$ a non-flat locally symmetric connection $\nabla$. Let $R$ be the curvature tensor of $\nabla$.*

*Then for any $m_0 \in M$ there exist a neighbourhood $U$ of $m_0$, a complex chart $\varphi : U \rightarrow \mathbf{C}^2$, an affine complex isomorphism $\mathcal{A}$ of $\mathbf{C}^3$ and a holomorphic function $F$ of two variables such that*

(i) $\mathcal{A} \circ f \circ \varphi^{-1}(z, w) = (z, w, F(z, w))$,

(ii) $\xi = (\mathcal{A} \circ f)_*(-T_0) + e_3$, *with some vector field $T_0$ on $U$, is a local section of $\mathcal{A}^{\rightarrow}\mathcal{N}$.*

*Moreover, $\mathcal{A}$ and $\varphi$ may be chosen in such a way that $F$ and $T_0$ satisfy the following conditions:*

(iii) *If* $\dim \operatorname{im} R = 1$, *then $T_0$ is described by* (2.3) *with some holomorphic function $\eta$ of one variable and $F$ satisfies the differential equation* (2.1).

(iv) *If* $\dim \operatorname{im} R = 1$ *and $\mathcal{N}$ is the complex affine normal bundle, then*

$$T_0 = \frac{F_z}{1 + {F_z}^2} \frac{\partial}{\partial z^1} \tag{3.1}$$

*and $F$ satisfies the differential equation*

$$F_{zz} F_{ww} - F_{zw} F_{zw} = \left(1 + {F_z}^2\right)^2. \tag{3.2}$$

(v) *If* $\dim \operatorname{im} R = 2$, *then $\mathcal{N}$ is the complex affine normal bundle,*

$$T_0 = \frac{F_z}{1 + {F_z}^2 + {F_w}^2} \frac{\partial}{\partial z^1} + \frac{F_w}{1 + {F_z}^2 + {F_w}^2} \frac{\partial}{\partial w^1} \tag{3.3}$$

*and $F$ satisfies the differential equation:*

$$F_{zz} F_{ww} - F_{zw} F_{zw} = \left(1 + {F_z}^2 + {F_w}^2\right)^2. \tag{3.4}$$

*Proof.* Let $m_0 \in M$. Since the immersion $f$ is locally a graph, we may choose a complex chart $\varphi_1$ on some neighbourhood $U$ of $m_0$ and a complex isomorphism $\mathcal{A}_1$ of $\mathbf{C}^3$ such that

$$\mathcal{A}_1 \circ f \circ \varphi_1^{-1}(z,w) = (z, w, F(z,w)) \tag{3.5}$$

with a holomorphic function $F$ of two variables $z, w$ and such that

$$\mathcal{A}_1^{\rightarrow}\mathcal{N}_{m_0} = \mathbf{C}e_3. \tag{3.6}$$

We may assume that $\left(\frac{\partial}{\partial z^1}\right)_{m_0} \notin \ker R_{m_0}$ and $\left(\frac{\partial}{\partial w^1}\right)_{m_0} \notin \ker R_{m_0}$, for if not, we replace $\varphi_1$ by $\psi \circ \varphi_1$ and $\mathcal{A}_1$ by $\mathcal{A}_2 \circ \mathcal{A}_1$, where $\psi(z,w) = (\alpha z + \beta w, \gamma z + \delta w)$ and $\mathcal{A}_2(z,w,u) = (\alpha z + \beta w, \gamma z + \delta w, u)$ with some appropriate complex constants $\alpha, \beta, \gamma, \delta$. We can also assume, by decreasing $U$ if necessary, that the condition $\frac{\partial}{\partial z^1} \notin \ker R$ and $\frac{\partial}{\partial w^1} \notin \ker R$ is satisfied on the whole of $U$.

The pair $(\mathcal{A}_1 \circ f, \mathcal{A}_1^{\rightarrow}\mathcal{N})$ induces on $M$ the same connection $\nabla$ as the pair $(f, \mathcal{N})$ and $\mathcal{A}_1 \circ f$ is also a non-degenerate immersion.

Let $\widehat{\xi} : U \to \mathbf{C}^3$ be a local section of $\mathcal{A}_1^{\rightarrow}\mathcal{N}$. Since $e_3$ is transversal to $(\mathcal{A}_1 \circ f)_*(T_{m_0}M)$, on some neighbourhood $U'$ of $m_0$ we have a decomposition $\widehat{\xi} = (\mathcal{A}_1 \circ f)_*(-T_1) + \lambda e_3$ where $\lambda$ is a complex valued function such that $\forall x \in U' : \lambda(x) \neq 0$. Dividing $\widehat{\xi}$ by $\lambda$ we obtain the section $\xi = -(\mathcal{A}_1 \circ f)_*(T_0) + e_3$ of $\mathcal{A}_1^{\rightarrow}\mathcal{N}$. From (3.6) it follows that $T_{0m_0} = 0$. From the Gauss and Weingarten formulae we obtain (2.5) to (2.8) for $\nabla$, $h$, $S$ and $\tau$ induced by $(\mathcal{A}_1 \circ f, \xi)$.

Locally symmetric connection is semi-symmetric, which means that $R(X,Y) \cdot R = 0$ for any $X$, $Y$; here $R(X,Y)$ acts on $R$ as a derivation. Therefore for any $m \in U$ we can apply to $h_m$, $S_m$ and $R_m$ the following algebraic lemma [5].

**Lemma O1** *Let $\mathcal{V}$ be a complex vector space, $\dim_{\mathbf{C}} \mathcal{V} > 1$, endowed with a $\mathbf{C}$-bilinear symmetric non-degenerate form $h$. Let $R$ be a tensor of type $(1,3)$ on $\mathcal{V}$ and $S$ an $\mathbf{R}$-linear endomorphism of $\mathcal{V}$ satisfying the Gauss equation*

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY.$$

*If for every $X \in \mathcal{V}$, $R(X,JX) \cdot R = 0$, then $S$ is complex [$\mathbf{C}$-linear].*

The following two lemmas are consequences of the $\mathbf{C}$-linearity of $S$.

**Lemma 3.2.** *If $\nabla R = 0$, then $\nabla$ is a real holomorphic connection, that is, $R(X,Y)$ is $\mathbf{C}$-linear in $X$ and $Y$.*

*Proof.* The claimed $\mathbf{C}$-linearity of $R$ follows from the Gauss equation (1.7). □

**Lemma 3.3.** *$\xi$ is a holomorphic section of $\mathcal{A}_1^{\to}\mathcal{N}$ and $T_0$ is a holomorphic vector field.*

*Proof.* From (2.8) and from the $\mathbf{C}$-bilinearity of $h$ it follows that $\tau$ is $\mathbf{C}$-linear. We have now

$$\begin{aligned} D_{JX}\xi &= -(A_1 \circ f)_*(SJX) + \tau(JX)\xi \\ &= -(A_1 \circ f)_*(JSX) + \tau(X)J\xi = J\, D_X\xi, \end{aligned}$$

therefore $\xi$ is holomorphic. From (2.7) we obtain $\nabla_{JX}T_0 = SJX = JSX = J\nabla_X T_0$ and the lemma follows. □

**Lemma 3.4.** *There exist a neighbourhood $U'$ of $m_0$ and a holomorphic function $H = A + iB : U' \to \mathbf{C}$ such that $\tau = dA + idB$.*

*Proof.* We use here a part of another lemma from [5].

**Lemma O2** *Let $\mathcal{V}$ be a complex vector space endowed with a $\mathbf{C}$-bilinear symmetric non-degenerate form $h$. Let $R$ be a tensor of type $(1,3)$ on $\mathcal{V}$ and $S$ an $\mathbf{R}$-linear endomorphism of $\mathcal{V}$ satisfying the Gauss equation. If $\dim_{\mathbf{C}} \mathcal{V} > 2$, then $R \cdot R = 0$ if and only if $S = \lambda\, \mathrm{id}_{\mathcal{V}}$ for some $\lambda \in \mathbf{C}$. If $\dim_{\mathbf{C}} \mathcal{V} = 2$, then $R \cdot R = 0$ if and only if $h(X, SY) = h(SY, X)$ for every $X, Y \in \mathcal{V}$.*

From the Ricci equation (1.10) it follows that $d\tau = 0$, which implies $d\mu = 0$ and $d\nu = 0$ on $U$. Hence there exist a neighbourhood $U'$ of $m_0$ and real functions $A$ and $B$ on $U'$ such that $\mu = dA$ and $\nu = dB$. Since $\tau$ is $\mathbf{C}$-linear (Equation (2.8)), $A + iB$ is holomorphic. □

We first consider the case $\dim \operatorname{im} R = 1$.

**Lemma 3.5.** $\dim \ker R = 1$.

*Proof.* We fix a point $x \in U$, where $U$ is the domain of the chart $\varphi$. Let $X_1, X_2$ be a basis of $T_xM$ over $\mathbf{C}$. Since $SJ = JS$, $R$ is $\mathbf{C}$-linear with respect to any variable. Therefore $Z \in \ker R_x$ if and only if $R(X_1, X_2)Z = 0$. Since the type number of the immersion is greater than 1, $\operatorname{im} R_x = \operatorname{im}_{\mathbf{C}} S_x$ [5]. For the complex $S$ we have $\operatorname{im}_{\mathbf{C}} S_x = \operatorname{im} S_x$. By assumption, $\dim \operatorname{im} R_x = 1$, therefore $\dim \operatorname{im} S_x = 1$. Hence $SX_1$ and $SX_2$ are linearly dependent over $\mathbf{C}$. There exist complex numbers $\alpha$, $\beta$, $(\alpha, \beta) \neq (0,0)$, such that $\alpha SX_1 + \beta SX_2 = 0$. From the non-degeneracy of $h$ it follows that there exists a solution $\gamma$, $\delta$ of the system of linear equations:

$$\begin{aligned} h(X_1, X_1)\gamma + h(X_1, X_2)\delta &= -\beta, \\ h(X_2, X_1)\gamma + h(X_2, X_2)\delta &= \alpha. \end{aligned} \tag{3.7}$$

Of course $(\gamma, \delta) \neq (0, 0)$. Therefore $Z_0 := \gamma X_1 + \delta X_2$ is non-zero. Using (1.7) and the system (3.7) it is easy to check that $R(X_1, X_2)Z_0 = 0$. Since $\nabla$ is non-flat, $\ker R_x = \mathbf{C}Z_0$. □

**Lemma 3.6.** *Let $Z_0$ be a non-zero vector from $\ker R_x$. Then for any $X, Y \in T_xM$, $R(R(X, Z_0)X, Z_0)Y = 0$.*

*Proof.* Since $\nabla$ is semi-symmetric, we have

$$\begin{aligned} 0 &= (R(X, Z_0) \cdot R)(X, Z_0)Y \\ &= R(X, Z_0)(R(X, Z_0)Y) - R(R(X, Z_0)X, Z_0)Y \\ &\quad - R(X, R(X, Z_0)Z_0)Y - R(X, Z_0)(R(X, Z_0)Y) \\ &= -R(R(X, Z_0)X, Z_0)Y - R(X, R(X, Z_0)Z_0)Y \\ &= -R(R(X, Z_0)X, Z_0)Y. \end{aligned}$$

□

**Lemma 3.7.** (a) *If $R(W_1, W_2)Y = 0$ for any $Y \in T_xM$, then $W_1$ and $W_2$ are linearly dependent over $\mathbf{C}$.*

(b) *There exists $X \in T_xM$ such that $R(X, Z_0)X \neq 0$.*

(c) $\operatorname{im} R_x = \ker R_x$.

*Proof.* (a) Suppose, contrary to our claim, that $W_1$ and $W_2$ are linearly independent over $\mathbf{C}$. Then $T_xM$ is generated by $W_1$ and $W_2$ and $R(W_1, W_2)Y = 0$ implies $Y \in \ker R_x$. But this contradicts the assumption of (a), because $\ker R_x$ is a proper subset of $T_xM$.

(b) Suppose the assertion is false. Then $R(X, Z_0)X = 0$ and $R(X, Z_0)Z_0 = 0$ for any $X \in T_xM$. Using (1.7) we obtain

$$h(Z_0, Z_0)SX - h(X, Z_0)SZ_0 = 0, \tag{3.8}$$

$$h(Z_0, X)SX - h(X, X)SZ_0 = 0. \tag{3.9}$$

Subtracting (3.9) multiplied by $h(X, Z_0)$ from (3.8) multiplied by $h(X, X)$ yields

$$\begin{vmatrix} h(Z_0, Z_0) & h(Z_0, X) \\ h(X, Z_0) & h(X, X) \end{vmatrix} SX = 0. \tag{3.10}$$

It follows that if $X$ and $Z_0$ are $\mathbf{C}$-linearly independent, then $SX = 0$. We choose $Z_1$ such that $Z_0, Z_1$ is a $\mathbf{C}$-basis of $T_xM$. By this, $SZ_1 = 0$ and $S(Z_0 + Z_1) = 0$. Consequently, $SZ_0 = 0$ and $S = 0$, which contradicts the fact that $\dim \mathrm{im} S = \dim \mathrm{im} R = 1$.

(c) According to (b) we may choose $X_0 \in T_xM$ such that $R(X_0, Z_0)X_0 \neq 0$. By Lemma 3.6 and (a), $R(X_0, Z_0)X_0$ and $Z_0$ are $\mathbf{C}$-linearly dependent. Since $R(X_0, Z_0)X_0 \neq 0$, there exists $\lambda \in \mathbf{C}$ such that $Z_0 = \lambda R(X_0, Z_0)X_0$. Hence $Z_0 \in \mathrm{im}\, R_x$ and $\ker R_x = \mathbf{C}Z_0 \subset \mathrm{im}\, R_x$. By assumption, $\dim_{\mathbf{C}} \mathrm{im} R = 1$, and the lemma follows. □

**Lemma 3.8.**

$$\nabla_X \left( R\left(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial w^1}\right) T_0 \right) = h(X, T_0)\, R\left(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial w^1}\right) T_0. \tag{3.11}$$

*Proof.* From $\nabla R = 0$ it follows that

$$\begin{aligned} 0 &= (\nabla_X R)\left(\left(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial w^1}\right) T_0\right) \\ &= \nabla_X \left( R\left(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial w^1}\right) T_0 \right) - R\left(\nabla_X \frac{\partial}{\partial z^1}, \frac{\partial}{\partial w^1}\right) T_0 \\ &\quad - R\left(\frac{\partial}{\partial z^1}, \nabla_X \frac{\partial}{\partial w^1}\right) T_0 - R\left(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial w^1}\right) (\nabla_X T_0). \end{aligned}$$

The last term vanishes, because $\nabla_X T_0 = SX$ and $\mathrm{im} S = \mathrm{im} R = \ker R$ by Lemma 3.7 (c). From Lemma 3.3, it follows that there exist holomorphic functions $\psi_1$ and $\psi_2$ such that

$$T_0 = \psi_1 \frac{\partial}{\partial z^1} + \psi_2 \frac{\partial}{\partial w^1}. \tag{3.12}$$

By the $\mathbf{C}$-bilinearity and anti-symmetry of $R(\cdot,\cdot)$, we have from (2.5), (2.6) and (3.12)

$$\begin{aligned} & R\left(\nabla_X \frac{\partial}{\partial z^1}, \frac{\partial}{\partial w^1}\right) T_0 + R\left(\frac{\partial}{\partial z^1}, \nabla_X \frac{\partial}{\partial w^1}\right) T_0 \\ &\quad = h\left(X, \frac{\partial}{\partial z^1}\right) R\left(T_0, \frac{\partial}{\partial w^1}\right) T_0 + h\left(X, \frac{\partial}{\partial w^1}\right) R\left(\frac{\partial}{\partial z^1}, T_0\right) T_0 \\ &\quad = \left(\psi_1 h\left(X, \frac{\partial}{\partial z^1}\right) + \psi_2 h\left(X, \frac{\partial}{\partial w^1}\right)\right) R\left(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial w^1}\right) T_0 \\ &\quad = h\left(X, \psi_1 \frac{\partial}{\partial z^1} + \psi_2 \frac{\partial}{\partial w^1}\right) R\left(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial w^1}\right) T_0 = h(X, T_0) R\left(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial w^1}\right) T_0. \end{aligned}$$

□

Let $U'$ and $H$ be as in Lemma 3.4. We may assume that $U'$ is connected. From now on we shall write $U$ instead of $U'$.

**Lemma 3.9.** $\nabla_X\left(e^H R\left(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial w^1}\right) T_0\right) = 0$ *for any* $X \in TM|_U$.

*Proof.* It suffices to use (2.8) and Lemma 3.8. □

**Lemma 3.10.** *If* $(T_0)_{m_0} \in \ker R_{m_0}$, *then* $(T_0)_m \in \ker R_m$ *for any* $m \in U$.

*Proof.* Assume that the vector field $W$ on $U$ has the property $\nabla_X W = 0$ for any $X$. Any two points $x$ and $y$ of $U$ we can connect with some curve $\gamma$. The coordinates of $W_{\gamma(t)}$ in the basis of $T_{\gamma(t)}M$ obtained from a basis of $T_xM$ by parallel displacement along $\gamma$ do not depend on $t$. It follows that if $W_x = 0$ at some $x \in U$, then $W \equiv 0$ on $U$. Now let $W = e^H R\left(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial w^1}\right) T_0$. By assumption, $W_{m_0} = 0$, therefore $W \equiv 0$ and $R\left(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial w^1}\right) T_0 = e^{-H} W \equiv 0$. □

**Lemma 3.11.** *For any* $m \in U$, $\psi_1(m) = 0$ *if and only if* $\psi_2(m) = 0$.

*Proof.* To obtain a contradiction, suppose for example that $\psi_1(m) = 0$ and $\psi_2(m) \neq 0$. Then $\frac{\partial}{\partial w^1}|_m = \frac{1}{\psi_2(m)} T_{0m}$, which contradicts the assumption that $\frac{\partial}{\partial w^1} \notin \ker R$. □

**Lemma 3.12.** *There exists an open dense subset* $U_2$ *of* $U$ *such that* $\psi_1 \neq 0$ *everywhere on* $U_2$.

*Proof.* Suppose that $\psi_1 \equiv 0$ on some open, non-empty subset $V$ of $U$. Then, by Lemma 3.11, $\psi_2 \equiv 0$ on $V$ and consequently $T_0 \equiv 0$ on $V$. This contradicts the fact that $\dim \mathrm{im} S = 1$, because $\nabla_X T_0 = SX$. □

**Lemma 3.13.** *There exists a constant* $C \neq 0$ *such that* $\psi_2 = C\psi_1$ *on* $U$.

*Proof.* Let $U_3$ be a connected, open, non-empty subset of $U_2$ and let $X \in TM|_{U_3}$. From the equality $\mathrm{im}\, S = \mathrm{im}\, R = \ker R$ and from $T_0 \in \ker R$ it follows that $X(\psi_1)\frac{\partial}{\partial z^1} + X(\psi_2)\frac{\partial}{\partial w^1} \in \ker R$, because

$$\begin{aligned} X(\psi_1)\frac{\partial}{\partial z^1} + X(\psi_2)\frac{\partial}{\partial w^1} &= \nabla_X T_0 - \psi_1 \nabla_X \frac{\partial}{\partial z^1} - \psi_2 \nabla_X \frac{\partial}{\partial w^1} \\ &= SX - \psi_1 h\left(X, \frac{\partial}{\partial z^1}\right) T_0 - \psi_2 h\left(X, \frac{\partial}{\partial w^1}\right) T_0. \end{aligned}$$

Since $T_0 \in \ker R$ and $\dim \ker R = 1$, the tangent vectors $\psi_1 \frac{\partial}{\partial z^1} + \psi_2 \frac{\partial}{\partial w^1}$ and $X(\psi_1)\frac{\partial}{\partial z^1} + X(\psi_2)\frac{\partial}{\partial w^1}$ are linearly dependent over $\mathbf{C}$. Consequently

$$X\left(\frac{\psi_2}{\psi_1}\right) = \frac{1}{(\psi_1)^2} \cdot \begin{vmatrix} \psi_1 & \psi_2 \\ X(\psi_1) & X(\psi_2) \end{vmatrix} = 0. \tag{3.13}$$

It follows that $X\left(\frac{\psi_2}{\psi_1}\right) = 0$ for any $m \in U_3$, for any $X \in T_mM$. Since $U_3$ is connected, there exists a constant $C$ such that $\frac{\psi_2}{\psi_1} = C$ on $U_3$. The constant $C \neq 0$, for if not, then $\psi_2 \equiv 0$. Now $\psi_2 - C\psi_1$ is a holomorphic function defined on the connected subset $U$ of $M$ and equal to zero on an open, non-empty set $U_3$. From the identity principle for holomorphic functions it follows that $\psi_2 - C\psi_1 \equiv 0$ on $U$.

Let $(\widetilde{z}, \widetilde{w}) = \varphi_3(z,w) := (z, -Cz + w)$ and $\mathcal{A}_3(z,w,u) = (z, -Cz + w, u)$. Then $\mathcal{A}_3 \circ \mathcal{A}_1 \circ f \circ (\varphi_3 \circ \varphi_1)^{-1}(\widetilde{z}, \widetilde{w}) = \mathcal{A}_3 \circ \mathcal{A}_1 \circ f \circ \varphi_1^{-1}(\widetilde{z}, C\widetilde{z} + \widetilde{w}) = \mathcal{A}_3(\widetilde{z}, C\widetilde{z} + \widetilde{w}, F(\widetilde{z}, C\widetilde{z} + \widetilde{w})) = (\widetilde{z}, \widetilde{w}, F(\widetilde{z}, C\widetilde{z} + \widetilde{w})) = (\widetilde{z}, \widetilde{w}, \widetilde{F}(\widetilde{z}, \widetilde{w}))$,

$$\mathcal{A}_3\xi = -(\mathcal{A}_3 \circ \mathcal{A}_1 \circ f)_*(T_0) + \mathcal{A}_3 e_3 = -(\mathcal{A}_3 \circ \mathcal{A}_1 \circ f)_*(T_0) + e_3,$$

$\frac{\partial}{\partial \widetilde{z}^1} = \frac{\partial}{\partial z^1} + C\frac{\partial}{\partial w^1}$ and $\frac{\partial}{\partial \widetilde{w}^1} = \frac{\partial}{\partial w^1}$. Using the new coordinates we can rewrite $T_0$ as

$$T_0 = \psi_1 \left(\frac{\partial}{\partial z^1} + C\frac{\partial}{\partial w^1}\right) = \psi_1 \frac{\partial}{\partial \widetilde{z}^1} = \alpha(\widetilde{z}, \widetilde{w})\frac{\partial}{\partial \widetilde{z}^1} \tag{3.14}$$

where $\alpha = \psi_1 \circ \varphi_1^{-1} \circ \varphi_3^{-1}$. From now on we write $z, w, F$ instead of $\widetilde{z}, \widetilde{w}, \widetilde{F}$. Then the formulae (2.6) to (2.8) hold. $\square$

**Lemma 3.14.** *There exist an open neighbourhood $U'$ of $m_0$ and a holomorphic function $g$ of one variable such that $\alpha(z,w) = g\big(F_z(z,w)\big)$ on $\varphi_3 \circ \varphi_1(U')$.*

*Proof.* An easy computation shows that

$$\nabla_X T_0 = X(\psi_1)\frac{\partial}{\partial z^1} + \psi_1^2\, h\left(X, \frac{\partial}{\partial z^1}\right)\frac{\partial}{\partial z^1} \tag{3.15}$$

and

$$R\left(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial w^1}\right)\frac{\partial}{\partial z^1} = \left(\frac{\partial^2 F}{\partial w \partial z}\frac{\partial \alpha}{\partial z} - \frac{\partial^2 F}{\partial z \partial z}\frac{\partial \alpha}{\partial w}\right)\frac{\partial}{\partial z^1}. \tag{3.16}$$

By Lemma 3.12 and by (3.14) $\alpha \neq 0$ on some dense open subset $\widetilde{U}$ of $\varphi_3 \circ \varphi_1(U)$. For any $(z,w) \in \widetilde{U}$ we can write

$$\left(\frac{\partial^2 F}{\partial w \partial z}\frac{\partial \alpha}{\partial z} - \frac{\partial^2 F}{\partial z \partial z}\frac{\partial \alpha}{\partial w}\right)\frac{\partial}{\partial z^1} = R\left(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial w^1}\right)\left(\frac{1}{\alpha(z,w)}T_0\right) = 0. \tag{3.17}$$

Hence

$$\frac{\partial^2 F}{\partial w \partial z}\frac{\partial \alpha}{\partial z} - \frac{\partial^2 F}{\partial z \partial z}\frac{\partial \alpha}{\partial w} = 0 \tag{3.18}$$

on $\widetilde{U}$ and, by continuity, on $\varphi_3 \circ \varphi_1(U)$. Furthermore, $\frac{\partial^2 F}{\partial z \partial z} \neq 0$ or $\frac{\partial^2 F}{\partial w \partial z} \neq 0$ for any point $m \in M$, since otherwise $\frac{\partial}{\partial z^1} \in \ker h$ which contradicts the non-degeneracy of $f$. Therefore $\Psi := \frac{\partial F}{\partial z}$ satisfies the assumptions of the following lemma. Applying Lemma 3.15 to $\Psi$ and $\Lambda := \alpha$ completes the proof of Lemma 3.14. □

**Lemma 3.15.** *Let $V$ be an open subset of $\mathbf{C}^2$. Let $\Psi : V \to \mathbf{C}$ be a holomorphic function of two variables such that for any $(z,w) \in V$, $\frac{\partial \Psi}{\partial z}(z,w) \neq 0$ or $\frac{\partial \Psi}{\partial w}(z,w) \neq 0$. Then $\Lambda : V \to \mathbf{C}$ satisfies the equation*

$$\frac{\partial \Psi}{\partial z}\frac{\partial \Lambda}{\partial w} - \frac{\partial \Psi}{\partial w}\frac{\partial \Lambda}{\partial z} = 0 \tag{3.19}$$

*if and only if for any $(z_0,w_0) \in V$ there exist an open neighbourhood $V'$ of $(z_0,w_0)$ and a holomorphic function $g : \Psi(V') \to \mathbf{C}$ of one variable such that $\Lambda\big|_{V'} = g \circ \Psi\big|_{V'}$.*

*Proof.* is similar to that of constant-rank mapping theorem. Let $\Lambda$ satisfy (3.19). It follows that rank of the holomorphic mapping

$$V \ni (z,w) \mapsto \Big(\Psi(z,w), \Lambda(z,w)\Big) \in \mathbf{C}^2$$

is equal to 1 on $V$. Let $(z_0,w_0) \in V$. Without loss of generality we can assume that $\frac{\partial \Psi}{\partial z}(z_0,w_0) \neq 0$. Then there exists a neighbourhood $V'$ of $(z_0,w_0)$ such that $\Phi : V' \ni (z,w) \mapsto (\Psi(z,w), w) \in \Phi(V') \subset \mathbf{C}^2$ is biholomorphic. We may also assume that $\Phi(V')$ is a product of two open discs $D_1 \subset \mathbf{C}$ and $D_2 \subset \mathbf{C}$. Let $\widetilde{g}(u,v) := \Lambda(\Phi^{-1}(u,v))$. Rank of the mapping

$$(\Psi, \Lambda) \circ \Phi^{-1} : \Phi(V') \ni (u,v) \mapsto \big(u, \widetilde{g}(u,v)\big) \in \mathbf{C}^2$$

is also equal to 1, therefore $\frac{\partial \widetilde{g}}{\partial v}(u,v) = 0$ for any $(u,v) \in D_1 \times D_2$. Let $u \in D_1$. Since the function $D_2 \ni v \mapsto \widetilde{g}(u,v) \in \mathbf{C}$ is a constant one, we may define $g(u) := \widetilde{g}(u,v)$ with an arbitrary $v \in D_2$. We have then $(\Psi, \Lambda) \circ \Phi^{-1}(u,v) = (u, g(u))$. If we take $(u,v) = \Phi(z,w) = (\Psi(z,w), w)$, the assertion follows.

Conversely, let $\Lambda = g \circ \Psi$ on some open set $V'$. Applying the chain rule we obtain $\frac{\partial \Lambda}{\partial z}(z,w) = g'(\Psi(z,w))\frac{\partial \Psi}{\partial z}(z,w)$ and $\frac{\partial \Lambda}{\partial w}(z,w) = g'(\Psi(z,w))\frac{\partial \Psi}{\partial w}(z,w)$. Multiplying the first equation by $\frac{\partial \Psi}{\partial w}(z,w)$, the second by $\frac{\partial \Psi}{\partial z}(z,w)$ and subtracting we obtain (3.19). □

Let $(z_0, w_0) = \varphi_3 \circ \varphi_1(m_0)$. We can decrease the neighbourhood $U'$ of $m_0$ so as to obtain a connected, simply connected open neighbourhood $F_z(U')$ of $\zeta_0 := F_z(z_0, w_0)$. The holomorphic function $\zeta \mapsto \int_{\gamma(\zeta_0,\zeta)} g(\sigma)d\sigma$, where $\gamma(\zeta_0, \zeta)$ denotes a path joining $\zeta_0$ with $\zeta$, is then well defined on $F_z(U')$. Let

$$\eta(\zeta) := e^{\int_{\gamma(\zeta_0,\zeta)} g(\sigma)d\sigma}. \tag{3.20}$$

We have then $g(\zeta) = \frac{\eta'(\zeta)}{\eta(\zeta)}$ and

$$SX = \nabla_X T_0 = \frac{\eta''(F_z)}{\eta(F_z)} X(F_z) \frac{\partial}{\partial z^1}. \tag{3.21}$$

Since $\dim \operatorname{im} S = 1$, $\eta''(F_z) \neq 0$ everywhere on $U'$.

**Lemma 3.16.** *$F$ satisfies the differential equation*

$$F_{zz}F_{ww} - F_{zw}F_{zw} = \kappa \frac{\eta(F_z)}{\eta''(F_z)} \tag{3.22}$$

*where $\kappa \in \mathbf{C} \setminus \{0\}$.*

*Proof.* Using the Gauss equation and (3.21) we obtain

$$R\left(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial w^1}\right)\frac{\partial}{\partial w^1} = (F_{zz}F_{ww} - F_{zw}F_{wz})\frac{\eta''(F_z)}{\eta(F_z)}\frac{\partial}{\partial z^1} =: \Phi \frac{\partial}{\partial z^1}. \tag{3.23}$$

From $\nabla R = 0$ it follows that for any $X \in TM|_{U'}$

$$\begin{aligned} 0 = (\nabla_X R)\left(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial w^1}\right)\frac{\partial}{\partial w^1} &= \nabla_X\left(\Phi\frac{\partial}{\partial z^1}\right) - R\left(\nabla_X \frac{\partial}{\partial z^1}, \frac{\partial}{\partial w^1}\right)\frac{\partial}{\partial w^1} \\ &- R\left(\frac{\partial}{\partial z^1}, \nabla_X \frac{\partial}{\partial w^1}\right)\frac{\partial}{\partial w^1} - R\left(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial w^1}\right)\left(\nabla_X \frac{\partial}{\partial w^1}\right). \end{aligned}$$

The last two terms vanish, because $\frac{\partial}{\partial z^1}$ and $\nabla_X \frac{\partial}{\partial w^1}$ are linearly dependent and $\nabla_X \frac{\partial}{\partial w^1} = X(F_w)T_0$ with $T_0 \in \ker R$. Hence

$$0 = X(\Phi)\frac{\partial}{\partial z^1} + \Phi X(F_z)T_0 - X(F_z)R\left(T_0, \frac{\partial}{\partial w^1}\right)\frac{\partial}{\partial w^1} = X(\Phi)\frac{\partial}{\partial z^1}.$$

Since $U'$ is connected, $\Phi = \mathrm{const} =: \kappa$. From $\eta'' \neq 0$ and from the non-degeneracy of $f$ it follows that $\kappa \neq 0$. □

Let $\beta$ be a complex number such that $\beta^2 = \kappa$. Let $(\widetilde{z}, \widetilde{w}) = \varphi_4(z,w) = (z, \beta w)$, $\mathcal{A}_4(z,w,u) = (z, \beta w, u)$, $\varphi = \varphi_4 \circ \varphi_3 \circ \varphi_1$ and $\mathcal{A} = \mathcal{A}_4 \circ \mathcal{A}_3 \circ \mathcal{A}_1$. It is easy to check that $\mathcal{A} \circ f \circ \varphi^{-1}(z,w) = (z, w, \widehat{F}(z,w))$ where $\widehat{F}(z,w) = F(z, \frac{1}{\beta}\, w)$ satisfies the differential equation (2.1). Since at the corresponding points $\widehat{F}_{\widetilde{z}} = F_z$ and $\frac{\partial}{\partial \widetilde{z}^1} = \frac{\partial}{\partial z^1}$, we have a local section of $\mathcal{A}^{\to}\mathcal{N}$ as claimed.

Having $\mathcal{A}$, $\varphi$, $F$, $\eta$ and $\xi$ which satisfy (i), (ii) and (iii) of Theorem 3.1, we consider now the particular case when $\mathcal{N}$ is the complex affine normal bundle of the immersion $f$.

**Lemma 3.17.** *$\eta^3 \cdot \eta'' = c$ with some $c \in \mathbf{C} \setminus \{0\}$.*

*Proof.* The transversal field

$$\xi_{eq} = \eta(F_z)\,\xi = -(\mathcal{A} \circ f)_*\left(\eta'(F_z)\frac{\partial}{\partial z^1}\right) + \eta(F_z)e_3 \tag{3.24}$$

is the equiaffine section of the bundle $\mathcal{A}^{\to}\mathcal{N}$, which is the complex affine normal bundle for $\mathcal{A} \circ f$. Therefore, there exists a complex number $b$ such that $\widehat{\xi} = b\xi_{eq}$ is the complex affine normal vector field for $\mathcal{A} \circ f$. Let $Z = \frac{1}{b\eta(F_z)}\,\frac{\partial}{\partial z^1}$, $W = \frac{\partial}{\partial w^1}$. We have then ${}^{\mathbf{C}}\omega\left((\mathcal{A} \circ f)_*(Z), (\mathcal{A} \circ f)_*(W), \widehat{\xi}\right) = 1$. By the definition of the affine normal vector field,

$$\left|\det\begin{pmatrix} \widehat{h}(Z,Z) & \widehat{h}(Z,W) \\ \widehat{h}(W,Z) & \widehat{h}(W,W) \end{pmatrix}\right| = 1, \tag{3.25}$$

where $\widehat{h} = \frac{1}{b\eta(F_z)}\,h$ is the affine fundamental form induced by $\widehat{\xi}$. Using (2.5) we obtain

$$\left|\left(\frac{1}{b\,\eta(F_z)}\right)^4 (F_{zz}F_{ww} - F_{zw}F_{zw})\right| = 1, \tag{3.26}$$

which together with (2.1) implies

$$\left|(\eta(\zeta))^3 \cdot \eta''(\zeta)\right| = \left|\frac{1}{b}\right|^4 = \text{const} \tag{3.27}$$

for $\zeta \in F_z(U')$. According to the maximum principle, if for a holomorphic function $\mathcal{F} : \Omega \to \mathbf{C}$, where $\Omega \subset \mathbf{C}$ is an open and connected set, the function $|\mathcal{F}|$ has a local maximum at some point of $\Omega$, then $\mathcal{F}$ must be constant on $\Omega$. From (3.27) it follows that $|\eta^3 \cdot \eta''|$ has a local maximum at any $\zeta \in F_z(U')$. Therefore $\eta^3 \cdot \eta'' = \text{const}$. □

**Lemma 3.18.** *$\eta(\zeta) = \sqrt{A\zeta^2 + B\zeta + C}$, where $A \in \mathbf{C} \setminus \{0\}$, $B, C \in \mathbf{C}$, $AC - \frac{B^2}{4} = c$ and $\sqrt{\cdot}$ is some holomorphic branch of the square root defined on some neighbourhood of the non-zero complex number $A\zeta_0^2 + B\zeta_0 + C$, $\zeta_0 = F_z(z_0, w_0)$.*

*Proof.* From $T_{0m_0} = 0$ and (2.3) it follows that $\eta'(\zeta_0) = 0$. Let

$$E(\zeta) := \left(\eta'(\zeta)\right)^2 + \frac{c}{(\eta(\zeta))^2} \tag{3.28}$$

for $\zeta \in V$, where $V$ is some sufficiently small, connected neighbourhood of $\zeta_0$. Using Lemma 3.17 we obtain $E'(\zeta) = \frac{2\eta'(\zeta)}{(\eta(\zeta))^3}\Big((\eta(\zeta))^3 \cdot \eta''(\zeta) - c\Big) = 0$, therefore $E(\zeta) = E(\zeta_0) = \frac{c}{(\eta(\zeta_0))^2}$ for $\zeta \in V$. It follows that

$$\left(\eta'(\zeta)\right)^2 = \frac{c}{(\eta(\zeta_0))^2} - \frac{c}{(\eta(\zeta))^2}. \tag{3.29}$$

We consider now the function $\psi(\zeta) := (\eta(\zeta))^2$. Using (3.29) and Lemma 3.17 we obtain

$$\begin{aligned}\psi''(\zeta) &= 2\left(\eta'(\zeta)\right)^2 + 2\,\eta(\zeta)\cdot\eta''(\zeta)\\ &= \frac{2c}{(\eta(\zeta_0))^2} - \frac{2c}{(\eta(\zeta))^2} + 2\,\eta(\zeta)\cdot\eta''(\zeta) = \frac{2c}{(\eta(\zeta_0))^2}.\end{aligned}$$

It follows that

$$\psi(\zeta) = \frac{c}{(\eta(\zeta_0))^2}\zeta^2 + B\zeta + C. \tag{3.30}$$

Since $\eta'(\zeta_0) = 0$ implies $\psi'(\zeta_0) = 0$, we have $B = -\frac{2c}{(\eta(\zeta_0))^2}\zeta_0$. Computing $\psi(\zeta_0)$ we obtain $C = (\eta(\zeta_0))^2 + \frac{c\zeta_0^2}{(\eta(\zeta_0))^2}$ and

$$\psi(\zeta) = \frac{c}{(\eta(\zeta_0))^2}\,(\zeta - \zeta_0)^2 + (\eta(\zeta_0))^2\,. \tag{3.31}$$

Since $\eta(\zeta_0) \neq 0$, there exists a holomorphic branch $\sqrt{\cdot}$ of square root on some neighbourhood of $(\eta(\zeta_0))^2$. But $\eta$ is also holomorphic, therefore we may conclude, replacing $\sqrt{\cdot}$ by $-\sqrt{\cdot}$ if necessary, that $\eta(\zeta) = \sqrt{\psi(\zeta)}$. □

**Lemma 3.19.** *There exist an affine isomorphism $\mathcal{A}_8$ of $\mathbf{C}^3$ and a local diffeomorphism $\varphi_8$, $(\widetilde{z}, \widetilde{w}) = \varphi_8(z, w)$ such that*

$$\mathcal{A}_8 \circ \mathcal{A} \circ f \circ \varphi^{-1} \circ \varphi_8^{-1}(\widetilde{z}, \widetilde{w}) = (\widetilde{z}, \widetilde{w}, \widetilde{F}(\widetilde{z}, \widetilde{w})).$$

*$\widetilde{F}$ satisfies the differential equation*

$$\widetilde{F}_{\widetilde{z}\widetilde{z}}\widetilde{F}_{\widetilde{w}\widetilde{w}} - \widetilde{F}_{\widetilde{z}\widetilde{w}}\widetilde{F}_{\widetilde{z}\widetilde{w}} = \left(1 + \widetilde{F}_{\widetilde{z}}{}^2\right)^2 \tag{3.32}$$

*and $-\mathcal{A}_8 \circ \mathcal{A} \circ f_*(T_0) + e_3$ with*

$$T_0 = \frac{\widetilde{F}_{\widetilde{z}}}{1 + \widetilde{F}_{\widetilde{z}}{}^2}\frac{\partial}{\partial \widetilde{z}^1}$$

*is a local section of $\mathcal{A}_8^{\rightarrow}\mathcal{A}^{\rightarrow}\mathcal{N}$.*

*Proof.* For $\eta$ as in Lemma 3.18 we have

$$\frac{\eta(\zeta)}{\eta''(\zeta)} = \frac{1}{c}\,(\eta(\zeta))^4 = \frac{AC - (B^2/4)}{A^2}$$
$$\left[\left(\frac{A}{\sqrt{AC - (B^2/4)}}\,\zeta + \frac{(B/2)}{\sqrt{AC - (B^2/4)}}\right)^2 + 1\right]^2.$$

Let $\varphi_8(z, w) = \left(\frac{\sqrt{AC - (B^2/4)}}{A} z, w\right)$, $\mathcal{A}_8(z, w, u) = \left(\frac{\sqrt{AC - (B^2/4)}}{A} z, w, u + \frac{B}{2A} z\right)$ and

$$\widetilde{F}(\widetilde{z}, \widetilde{w}) = F\left(\frac{A}{\sqrt{AC - (B^2/4)}}\widetilde{z}, \widetilde{w}\right) + \frac{(B/2)}{\sqrt{AC - (B^2/4)}}\widetilde{z}.$$

It is easy to check that at the corresponding points

$$\frac{A}{\sqrt{AC-(B^2/4)}}F_z + \frac{(B/2)}{\sqrt{AC-(B^2/4)}} = \widetilde{F}_{\widetilde{z}} \tag{3.33}$$

and

$$F_{zz}F_{ww} - F_{zw}F_{zw} = \frac{AC-(B^2/4)}{A^2}\left(\widetilde{F}_{\widetilde{z}\widetilde{z}}\widetilde{F}_{\widetilde{w}\widetilde{w}} - \widetilde{F}_{\widetilde{z}\widetilde{w}}\widetilde{F}_{\widetilde{z}\widetilde{w}}\right), \tag{3.34}$$

therefore $\widetilde{F}$ satisfies Equation (3.32). Since $\mathcal{A}_8\, e_3 = e_3$, we do not have to change $T_0$ but it should be described in the new coordinates. We have at the corresponding points

$$\begin{aligned}\frac{\eta'(F_z)}{\eta(F_z)} &= \frac{AF_z + (B/2)}{\frac{1}{A}\left(AF_z + (B/2)\right)^2 + C - (B^2/4A)} \\ &= \frac{\widetilde{F}_{\widetilde{z}}\sqrt{AC-(B^2/4)}}{\frac{AC-(B^2/4)}{A}\left(\widetilde{F}_{\widetilde{z}}^{\,2}+1\right)} = \frac{A}{\sqrt{AC-(B^2/4)}}\,\frac{\widetilde{F}_{\widetilde{z}}}{\widetilde{F}_{\widetilde{z}}^{\,2}+1},\end{aligned}$$

$$\frac{\partial}{\partial z^1} = \frac{\sqrt{AC-(B^2/4)}}{A}\,\frac{\partial}{\partial \widetilde{z}^1} \tag{3.35}$$

and the lemma follows. □

We now turn to the case $\dim \operatorname{im} R = 2$. The shape operator $S$ is then invertible. We first show that there exists a $\mathbf{C}$-bilinear, complex valued non-degenerate symmetric holomorphic tensor field $g$ such that $\nabla g = 0$. Let

$$g(X,Y) := e^{2H}\, h(S^{-1}X, Y), \tag{3.36}$$

where $H$ is a holomorphic function as in Lemma 3.4 and $h$, $S$, $\tau$ are induced by the pair $(\mathcal{A}_1 \circ f, \xi)$, or, equivalently, by $(f, (\mathcal{A}_1^{\rightarrow})^{-1}\xi)$ on some neighbourhood of $m_0$. Since $H$ is defined up to a constant, we may assume that $H_{m_0} = 0$. Since $S$ is $\mathbf{C}$-linear and $h$ $\mathbf{C}$-bilinear, $g$ is $\mathbf{C}$-bilinear. It is non-degenerate because $h$ is non-degenerate and $S_x$ is an isomorphism at any $x$. According to Lemma O2, $h(S^{-1}X,Y) = h(S^{-1}X, SS^{-1}Y) = h(SS^{-1}X, S^{-1}Y) = h(X, S^{-1}Y) = h(S^{-1}Y, X)$, therefore $g$ is symmetric.

We fix now some basis $Z, W$ of $T_xM$ and define $\alpha : T_xM \to T_xM$ and $L : T_xM \times T_xM \to \mathbf{C}$:

$$\alpha(Y) := h(W, Y)Z - h(Z, Y)W, \tag{3.37}$$

$$L(Y, U) := \det \begin{pmatrix} h(Z, Y) & h(Z, U) \\ h(W, Y) & h(W, U) \end{pmatrix}. \tag{3.38}$$

**Lemma 3.20.** (i) *$\alpha$ is a $\mathbf{C}$-linear isomorphism.*

(ii) *$L$ is $\mathbf{C}$-bilinear and anti-symmetric.*

(iii) *$L(Z, W) \neq 0$.*

(iv) *$\alpha \circ \alpha = -L(Z, W)\, \mathrm{id}_{T_xM}$.*

(v) *$h(Y, \alpha(U)) = -h(U, \alpha(Y))$ for any $Y, U \in T_xM$.*

(vi) *$L(\alpha(Y), U) = L(Z, W)\, h(Y, U)$ for any $Y, U \in T_xM$.*

*Proof.* (i) and (ii) are obvious, (iii) follows from the non-degeneracy of $h$. To prove (iv) we need only to compute $\alpha \circ \alpha(Z)$ and $\alpha \circ \alpha(W)$. An easy computation shows that $h(Y, \alpha(U)) + h(U, \alpha(Y)) = 0$. For (vi), it suffices to take as $(Y, U)$ the pairs of basis vectors, to use the definition of $\alpha$ and only the anti-symmetry of $L$. □

In the following lemmas we will need the assumption that $\nabla R = 0$.

**Lemma 3.21.** *For any $X, U$*

$$L(Z, W)\ (\nabla_X S)\, U = (\nabla_X h)\, (W, \alpha(U))\, SZ - (\nabla_X h)\, (Z, \alpha(U))\, SW.$$

*Proof.* From the Gauss equation (1.7) it follows that

$$\begin{aligned}(\nabla_X R)\, (Z, W)Y = {}& (\nabla_X h)\, (W, Y)\, SZ - (\nabla_X h)\, (Z, Y)\, SW \\ & + \ h(W, Y)\ (\nabla_X S)\, Z - h(Z, Y)\ (\nabla_X S)\, W.\end{aligned} \tag{3.39}$$

If $\nabla R = 0$, then

$$\begin{aligned}& - (\nabla_X h)\, (W, Y)\, SZ + (\nabla_X h)\, (Z, Y)\, SW \\ & = h(W, Y)\ (\nabla_X S)\, Z - h(Z, Y)\ (\nabla_X S)\, W = (\nabla_X S)\, (\alpha(Y))\, .\end{aligned}$$

We take now $Y = \alpha(U)$ and use Lemma 3.20(iv). □

**Lemma 3.22.** *For any $X, U, Y$*

$$(\nabla_X h)(U, \alpha(Y)) - (\nabla_X h)(Y, \alpha(U)) = h(X, T_0)\Big[L(U, Y) + h(U, \alpha(Y))\Big].$$

*Proof.* Since both sides are **C**-bilinear and anti-symmetric with respect to $Y, U$ (see Lemma 3.20(ii) and (v)), it suffices to prove the formula for $U = Z$ and $Y = W$. If we apply Lemma 3.21 to $X = Z$, $U = W$, next to $X = W$, $U = Z$ and subtract the formulae, then we obtain

$$\begin{aligned} L(Z, W)&\Big[(\nabla_Z S) W - (\nabla_W S) Z\Big] \\ &= \Big[(\nabla_Z h)(W, \alpha(W)) - (\nabla_W h)(W, \alpha(Z))\Big] SZ \\ &- \Big[(\nabla_Z h)(Z, \alpha(W)) - (\nabla_W h)(Z, \alpha(Z))\Big] SW. \end{aligned}$$

From the Codazzi equation (1.9) and (2.8) it follows that

$$(\nabla_Z S) W - (\nabla_W S) Z = h(W, T_0)\, SZ - h(Z, T_0) SW.$$

Since $S$ is invertible, $SZ$ and $SW$ are linearly independent over **C**, therefore

$$\begin{aligned} L(Z, W) h(W, T_0) &= (\nabla_Z h)(W, \alpha(W)) - (\nabla_W h)(W, \alpha(Z)), \\ L(Z, W) h(Z, T_0) &= (\nabla_Z h)(Z, \alpha(W)) - (\nabla_W h)(Z, \alpha(Z)). \end{aligned}$$

Using the Codazzi equation (1.8), (2.8) and Lemma 3.20(v), we obtain

$$\begin{aligned} (\nabla_Z h)(W, \alpha(W)) &= (\nabla_W h)(Z, \alpha(W)) - h(W, T_0)\, h(Z, \alpha(W)), \\ (\nabla_W h)(Z, \alpha(Z)) &= (\nabla_Z h)(W, \alpha(Z)) + h(Z, T_0)\, h(Z, \alpha(W)). \end{aligned}$$

It follows that

$$\begin{aligned} (\nabla_W h)(Z, \alpha(W)) - (\nabla_W h)(W, \alpha(Z)) &= h(W, T_0)\Big[L(Z, W) + h(Z, \alpha(W))\Big], \\ (\nabla_Z h)(Z, \alpha(W)) - (\nabla_Z h)(W, \alpha(Z)) &= h(Z, T_0)\Big[L(Z, W) + h(Z, \alpha(W))\Big]. \end{aligned}$$

Since the **C**-linear mappings

$$X \mapsto (\nabla_X h)(Z, \alpha(W)) - (\nabla_X h)(W, \alpha(Z))$$

and

$$X \mapsto h(X, T_0)\Big[L(Z, W) + h(Z, \alpha(W))\Big]$$

have the same values on the basis vectors $Z, W$, they are equal and the lemma follows. $\square$

**Lemma 3.23.**

$$(\nabla_X h)\,(\alpha(U),\alpha(Y)) = L(Z,W)\Big[-(\nabla_X h)\,(U,Y) + 2h(X,T_0)h(U,Y)\Big].$$

*Proof.* We apply Lemma 3.22 to $\alpha(U)$ and $Y$, then we use Lemma 3.20(iv), (v) and (vi). □

**Lemma 3.24.**

$$L(Z,W)\,h\Big((\nabla_X S)\,U,Y\Big) = -(\nabla_X h)\,\Big(\alpha(SY),\alpha(U)\Big).$$

*Proof.* Using Lemmas 3.21 and O2 we obtain

$$\begin{aligned}
&L(Z,W)\,h\Big((\nabla_X S)\,U,Y\Big)\\
&\quad= (\nabla_X h)\,(W,\alpha(U))\,h(SZ,Y) - (\nabla_X h)\,(Z,\alpha(U))\,h(SW,Y)\\
&\quad= (\nabla_X h)\,(W,\alpha(U))\,h(Z,SY) - (\nabla_X h)\,(Z,\alpha(U))\,h(W,SY)\\
&\quad= (\nabla_X h)\,\Big(h(Z,SY)W - h(W,SY)Z,\alpha(U)\Big)\\
&\quad= -(\nabla_X h)\,\Big(\alpha(SY),\alpha(U)\Big).
\end{aligned}$$

□

**Lemma 3.25.**

$$h\Big((\nabla_X S)\,U,Y\Big) = (\nabla_X h)\,(SY,U) - 2h(X,T_0)\,h(SY,U).$$

*Proof.* From Lemmas 3.23 and 3.24 we have

$$L(Z,W)h\Big((\nabla_X S)\,U,Y\Big) = L(Z,W)\Big[(\nabla_X h)\,(SY,U) - 2h(X,T_0)h(SY,U)\Big].$$

Since $L(Z,W)\neq 0$, the lemma follows. □

**Lemma 3.26.** $\nabla g = 0$.

*Proof.* It suffices to check that $(\nabla_X g)\,(SU,SY) = 0$ for any $X,U,Y$. We have

$$\begin{aligned}
&(\nabla_X g)\,(SU,SY)\\
&\quad= X\Big(g(SU,SY)\Big) - g\Big(\nabla_X(SU),SY\Big) - g\Big(SU,\nabla_X(SY)\Big)\\
&\quad= X\Big(e^{2H}\,h(U,SY)\Big) - e^{2H}\,h(\nabla_X(SU),Y) - e^{2H}\,h(U,\nabla_X(SY))
\end{aligned}$$

$$
\begin{aligned}
&= 2\,dH(X)\,e^{2H}h(U,SY) + e^{2H}\,(\nabla_X h)\,(U,SY) + e^{2H}h(\nabla_X U,SY) \\
&\quad + e^{2H}h(U,\nabla_X(SY)) - e^{2H}h(\nabla_X(SU),Y) - e^{2H}h(U,\nabla_X(SY)) \\
&= -2h(X,T_0)\,e^{2H}\,h(U,SY) + e^{2H}\,(\nabla_X h)\,(U,SY) \\
&\quad + e^{2H}h(S\,(\nabla_X U)\,,Y) - e^{2H}h(\nabla_X(SU),Y) \\
&= e^{2H}\left[-2h(X,T_0)\,h(U,SY) + (\nabla_X h)\,(U,SY) - h\Big((\nabla_X S)\,U,Y\Big)\right]
\end{aligned}
$$

which is equal to zero by symmetry of $h$ and $\nabla_X h$ and by Lemma 3.25.

Let $x \in U$ and let for $X, Y \in T_x M$

$$
\begin{aligned}
G_x\Big((\mathcal{A}_1 \circ f)_* X, (\mathcal{A}_1 \circ f)_* Y\Big) &:= g(X,Y), \\
G_x\Big((\mathcal{A}_1 \circ f)_* X, \xi_x\Big) := 0, \quad G_x(\xi_x,\xi_x) &:= e^{2H}.
\end{aligned}
$$

□

**Lemma 3.27.** *$DG = 0$.*

*Proof.* From $\nabla g = 0$ it follows easily that $(D_X G)((\mathcal{A}_1 \circ f)_* Y, (\mathcal{A}_1 \circ f)_* U) = 0$ for any $X, Y, U$. By definition of $g$, $g(SX,Y) - e^{2H}\,h(X,Y) = 0$, which implies $(D_X G)((\mathcal{A}_1 \circ f)_* Y, \xi) = 0$. Finally, $(D_X G)(\xi,\xi) = 0$ because $dH = \tau$. □

In that way we have defined a symmetric, $\mathbf{C}$-bilinear mapping $G : \mathbf{C}^3 \times \mathbf{C}^3 \to \mathbf{C}$. It is easy to check that $G$ is non-degenerate.

**Remark 3.28.** By the formula (3.36) we have defined the metric tensor $g$ only locally. Let $\widetilde{h}$, $\widetilde{S}$ and $\widetilde{\tau}$ be induced by $\widetilde{\mathcal{A}}_1 \circ f$ and a local section $\widetilde{\xi} = (\widetilde{\mathcal{A}}_1 \circ f)_*(-\widetilde{T}_0) + e_3$ of $\widetilde{\mathcal{A}}_1^{\rightarrow}\mathcal{N}$. Since $(\mathcal{A}_1^{\rightarrow})^{-1}\xi$ and $(\widetilde{\mathcal{A}}_1^{\rightarrow})^{-1}\widetilde{\xi}$ are local holomorphic sections of $\mathcal{N}$, there exists a holomorphic function $\phi$ such that $(\widetilde{\mathcal{A}}_1^{\rightarrow})^{-1}\widetilde{\xi} = \phi\,(\mathcal{A}_1^{\rightarrow})^{-1}\xi$ on some neighbourhood $U$ of $m_0$. From (1.4), (1.5) and (1.6) we obtain $\widetilde{h} = \frac{1}{\phi}\,h$, $\widetilde{S} = \phi S$ and $d\,\widetilde{H} = d\,H + d\,\log\phi$, where log is some holomorphic branch of logarithm in the neighbourhood of $m_0$. If $U$ is connected, then we have $\widetilde{H} = H + \log\phi + C$, $C \in \mathbf{C}$, and $\widetilde{g} = e^{2C}\,g$.

**Remark 3.29.** In [7], B. Opozda has shown that the Ricci tensor Ric of a locally symmetric torsion-free connection of rank 2 on a 2-dimensional real manifold is symmetric and non-degenerate, hence $\nabla$ is the Levi–Civita connection for the metric tensor $g :=$ Ric. Following this, we could in the complex case instead of Ric consider, defined in [5], the complex Ricci tensor $\mathrm{ric}(X,Y) = \frac{1}{2}\Big[\mathrm{Ric}(X,Y) - i\,\mathrm{Ric}(X,JY)\Big]$ which for a holomorphic connection $\nabla$ is equal to $\mathrm{tr}_{\mathbf{C}}\{V \mapsto R(V,X)Y\}$. In the case of induced connection we obtain $\mathrm{ric}(X,Y) = h(X,Y)\,\mathrm{tr}_{\mathbf{C}}S - h(SX,Y)$, where $h$, $S$ are induced by $f$ and some local section of $\mathcal{N}$. The right-hand side does not depend on the particular section, but on $f$ and $\mathcal{N}$ only. From Lemma O2 it follows that it is symmetric. Let $X \in T_mM$ and let $\mathrm{ric}(X,Y) = 0$ for any $Y \in T_mM$. Then $h(\mathrm{tr}_{\mathbf{C}}S\,X - SX, Y) = 0$ for any $Y \in T_mM$ and from the non-degeneracy of $h$ it follows that $SX = \mathrm{tr}_{\mathbf{C}}S\,X$, which for 2-dimensional vector space $T_mM$ and invertible $S$ implies $X = 0$. Therefore ric is non-degenerate. From $\nabla R = 0$ and $\nabla J = 0$ it follows that $\nabla\mathrm{ric} = 0$. In this way we can on the whole of $M$ define a $\mathbf{C}$-bilinear metric tensor $\widehat{g} := \mathrm{ric}$ such that $\nabla\widehat{g} = 0$. According to the complex version of the Cartan–Norden theorem, there exists a $\mathbf{C}$-bilinear, non-degenerate symmetric $\widehat{G} : \mathbf{C}^3 \times \mathbf{C}^3 \to \mathbf{C}$ such that for $X, Y \in TM$,

(3.40)
$$\widehat{G}((\mathcal{A}_1 \circ f)_*(X), (\mathcal{A}_1 \circ f)_*(Y)) = \widehat{g}(X,Y) \quad \text{and} \quad \widehat{G}((\mathcal{A}_1 \circ f)_*(X), \xi) = 0$$

for any local section of $\mathcal{A}_1^{\rightarrow}\mathcal{N}$. These conditions together with the non-degeneracy of $\widehat{g}$ are sufficient to prove the following Lemma 3.30, but in Lemma 3.31 we need not only the formula $\widehat{g}(SX,Y) = C_1 e^{2H} h(X,Y)$, which may occur in the proof of the Cartan–Norden theorem, or which we may derive using (3.40), but also there should be $C_1 = 1$, because we use (3.36). To this aim we should locally modify $\widehat{g}$.

From $H_{m_0} = 0$ and $T_{m_0} = 0$ it follows that $G(e_3, e_3) = G(\xi_{m_0}, \xi_{m_0}) = e^{2H_{m_0}} = 1$. There exists a complex linear isomorphism $\mathcal{A}_5$ of $\mathbf{C}^3$ such that $\mathcal{A}_5 e_1, \mathcal{A}_5 e_2, \mathcal{A}_5 e_3$ is a $G$-orthonormal basis of $\mathbf{C}^3$ and $\mathcal{A}_5 e_3 = e_3$. Let $\mathcal{A}_6 := \mathcal{A}_5^{-1}$. We have $\mathcal{A}_6 e_3 = \mathcal{A}_6\mathcal{A}_5 e_3 = e_3$. For the given $\mathcal{A}_5$ and $\mathcal{A}_6$ it is easy to find $\varphi_6$ and $\widehat{F}$ such that

$$\mathcal{A}_6 \circ \mathcal{A}_1 \circ f \circ \varphi_1^{-1} \circ \varphi_6^{-1}(z,w) = (z, w, \widehat{F}(z,w)). \tag{3.41}$$

From $\mathcal{A}_6 e_3 = e_3$ it follows that $\mathcal{A}_6\xi = -(\mathcal{A}_6 \circ \mathcal{A}_1 \circ f)_*(T_0) + e_3$. We can look at $\nabla$, $h$, $S$, $\tau$ as at objects induced by $(\mathcal{A}_6 \circ \mathcal{A}_1 \circ f, \mathcal{A}_6\xi)$. The new function $\widehat{F}$ from now on we shall denote by $F$.

**Lemma 3.30.**

$$T_0 = \frac{F_z}{1 + {F_z}^2 + {F_w}^2}\,\frac{\partial}{\partial z^1} + \frac{F_w}{1 + {F_z}^2 + {F_w}^2}\,\frac{\partial}{\partial w^1}.$$

*Proof.* From (3.41) we obtain $(\mathcal{A}_6 \circ \mathcal{A}_1 \circ f)_*\left(\frac{\partial}{\partial z^1}\right) = e_1 + F_z\,e_3$, $(\mathcal{A}_6 \circ \mathcal{A}_1 \circ f)_*\left(\frac{\partial}{\partial w^1}\right) = e_2 + F_w\,e_3$, and consequently $(\mathcal{A}_1 \circ f)_*\left(\frac{\partial}{\partial z^1}\right) = \mathcal{A}_5\,e_1 + F_z\,\mathcal{A}_5\,e_3$,

$(\mathcal{A}_1 \circ f)_* \left(\frac{\partial}{\partial w^1}\right) = \mathcal{A}_5\, e_2 + F_w\, \mathcal{A}_5\, e_3$. Hence

$$g\left(T_0, \frac{\partial}{\partial z^1}\right) = G\left(\xi + (\mathcal{A}_1 \circ f)_*(T_0), (\mathcal{A}_1 \circ f)_* \left(\frac{\partial}{\partial z^1}\right)\right)$$
$$= G(\mathcal{A}_5 e_3, \mathcal{A}_5 e_1 + F_z \mathcal{A}_5 e_3) = F_z, \tag{3.42}$$
$$g\left(T_0, \frac{\partial}{\partial w^1}\right) = G\left(\xi + (\mathcal{A}_1 \circ f)_*(T_0), (\mathcal{A}_1 \circ f)_* \left(\frac{\partial}{\partial w^1}\right)\right)$$
$$= G(\mathcal{A}_5 e_3, \mathcal{A}_5 e_2 + F_w \mathcal{A}_5 e_3) = F_w, \tag{3.43}$$
$$g\left(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^1}\right) = G\left((\mathcal{A}_1 \circ f)_* \left(\frac{\partial}{\partial z^1}\right), (\mathcal{A}_1 \circ f)_* \left(\frac{\partial}{\partial z^1}\right)\right)$$
$$= G(\mathcal{A}_5 e_1 + F_z \mathcal{A}_5 e_3, \mathcal{A}_5 e_1 + F_z \mathcal{A}_5 e_3) = 1 + {F_z}^2, \tag{3.44}$$
$$g\left(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial w^1}\right) = G\left((\mathcal{A}_1 \circ f)_* \left(\frac{\partial}{\partial z^1}\right), (\mathcal{A}_1 \circ f)_* \left(\frac{\partial}{\partial w^1}\right)\right)$$
$$= G(\mathcal{A}_5 e_1 + F_z \mathcal{A}_5 e_3, \mathcal{A}_5 e_2 + F_w \mathcal{A}_5 e_3) = F_z\, F_w, \tag{3.45}$$
$$g\left(\frac{\partial}{\partial w^1}, \frac{\partial}{\partial w^1}\right) = G\left((\mathcal{A}_1 \circ f)_* \left(\frac{\partial}{\partial w^1}\right), (\mathcal{A}_1 \circ f)_* \left(\frac{\partial}{\partial w^1}\right)\right)$$
$$= G(\mathcal{A}_5 e_2 + F_w \mathcal{A}_5 e_3, \mathcal{A}_5 e_2 + F_w \mathcal{A}_5 e_3) = 1 + {F_w}^2. \tag{3.46}$$

Let $T_0 = a\frac{\partial}{\partial z^1} + b\frac{\partial}{\partial w^1}$. From (3.42) to (3.46) we obtain the following system of linear equations:

$$(1 + {F_z}^2)\, a + F_z\, F_w\, b = F_z,$$
$$F_z\, F_w\, a + (1 + {F_w}^2)\, b = F_w.$$

It remains to find the solution $a, b$ and the lemma follows. We use here the fact that $g$ is non-degenerate, which implies

$$g\left(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^1}\right) g\left(\frac{\partial}{\partial w^1}, \frac{\partial}{\partial w^1}\right) - g\left(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial w^1}\right) g\left(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial w^1}\right) \neq 0.$$

Let $Z, W$ be a basis of $T_xM$. For $\mathbf{C}$-bilinear $g$ and a holomorphic connection $\nabla$ such that $\nabla g = 0$

$$\kappa := \frac{g(R(Z,W)W, Z)}{g(Z,Z)g(W,W) - g(Z,W)g(W,Z)}$$

is a complex valued analogue of the sectional curvature of 2-dimensional real manifold. It is easy to check that $\kappa$ does not depend on the choice of the basis and depends on $x$ only. $\square$

**Lemma 3.31.** *If* $\dim_{\mathbf{C}} M = 2$, $\nabla R = 0$, *g is a* **C***-bilinear metric tensor on* $U \subset M$ *such that* $\nabla g = 0$ *and U is connected, then* $\kappa = \text{const}$.

*Proof.* We take a local basis $E, F$ such that $g(E,E) = g(F,F) = 1$ and $g(E,F) = 0$. Since $\nabla g = 0$, there exists a complex valued 1-form $\omega$ such that $\nabla_X E = \omega(X)\, F$ and $\nabla_X F = -\omega(X)\, E$. As $\nabla g = 0$ and $\nabla R = 0$ we have

$$\begin{aligned} X(\kappa) &= X(g(R(E,F)F,E)) \\ &= g(R(\nabla_X E,F)F + R(E,\nabla_X F)F + R(E,F)\nabla_X F, E) \\ &\quad + g(R(E,F)F,\nabla_X E) = 0 \end{aligned}$$

because $R(X,Y) = -R(Y,X)$ and $g(R(K,L)M,N) = -g(R(K,L)N,M)$. □

**Lemma 3.32.** *F satisfies the differential equation*

$$F_{zz}F_{ww} - F_{zw}F_{zw} = \kappa(1 + {F_z}^2 + {F_w}^2)^2.$$

*Proof.* Let $H := -\frac{1}{2}\log(1 + {F_z}^2 + {F_w}^2)$, where log is a holomorphic branch of logarithm, defined in the neighbourhood of 1. Then

$$\begin{aligned} dH(X) &= -\frac{F_z X(F_z) + F_w X(F_w)}{1 + {F_z}^2 + {F_w}^2} = -\frac{F_z h\left(X, \partial/\partial z^1\right) + F_w h\left(X, \partial/\partial w^1\right)}{1 + {F_z}^2 + {F_w}^2} \\ &= -h(X, T_0) = \tau(X). \end{aligned}$$

Since $T_{0m_0} = 0$, from Lemma 3.30 we obtain $F_z(z_0,w_0) = 0$ and $F_w(z_0, w_0) = 0$, where $(z_0,w_0) = \varphi_6 \circ \varphi_1(m_0)$. Therefore $H_{m_0} = 0$. It follows that we may use $H$ to define $g$. From (1.7), (2.5) and (3.36), we obtain

$$g\left(R\left(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial w^1}\right)\frac{\partial}{\partial w^1}, \frac{\partial}{\partial z^1}\right) = e^{2H}\left(F_{zz}F_{ww} - F_{zw}F_{zw}\right). \tag{3.47}$$

By (3.44) to (3.46) we have

$$\begin{aligned} &g\left(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^1}\right) g\left(\frac{\partial}{\partial w^1}, \frac{\partial}{\partial w^1}\right) - g\left(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial w^1}\right) g\left(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial w^1}\right) \\ &\quad = 1 + {F_z}^2 + {F_w}^2. \end{aligned}$$

It follows that $e^{2H}\left(F_{zz}F_{ww} - F_{zw}F_{zw}\right) = \kappa(1 + {F_z}^2 + {F_w}^2)$, hence

$$F_{zz}F_{ww} - F_{zw}F_{zw} = \kappa e^{-2H}\left(1 + {F_z}^2 + {F_w}^2\right) = \kappa(1 + {F_z}^2 + {F_w}^2)^2.$$

To prove Lemma 3.32 one can also directly compute $SX$ as $\nabla_X T_0$, then using the Gauss equation compute $R$ and using (3.44)–(3.46) express $\kappa$ by the derivatives of $F$. □

Let $\beta$ be a complex number such that $\beta^2 = \kappa$. Let $\mathcal{A}_7(z,w,u) := (\beta z, \beta w, \beta u)$, $\varphi_7(z,w) := (\beta z, \beta w)$, $\mathcal{A} := \mathcal{A}_7 \circ \mathcal{A}_6 \circ \mathcal{A}_1$ and $\varphi = \varphi_7 \circ \varphi_6 \circ \varphi_1$. It is easy to check that $\mathcal{A} \circ f \circ \varphi^{-1}(\widetilde{z}, \widetilde{w}) = \left(\widetilde{z}, \widetilde{w}, \beta F\left(\frac{1}{\beta}\widetilde{z}, \frac{1}{\beta}\widetilde{w}\right)\right)$. As a local section of $\mathcal{A}^{\rightarrow}\mathcal{N}$ we take $\frac{1}{\beta}\mathcal{A}_7 \circ \mathcal{A}_6 \xi = -(\mathcal{A} \circ f)_*\left(\frac{1}{\beta}T_0\right) + e_3 =: -(\mathcal{A} \circ f)_*(\widetilde{T_0}) + e_3$. Let $\widetilde{F}(\widetilde{z}, \widetilde{w}) := \beta F\left(\frac{1}{\beta}\widetilde{z}, \frac{1}{\beta}\widetilde{w}\right)$, then $\widetilde{F}$ satisfies the differential equation

$$\widetilde{F}_{\widetilde{z}\widetilde{z}}\widetilde{F}_{\widetilde{w}\widetilde{w}} - \widetilde{F}_{\widetilde{z}\widetilde{w}}\widetilde{F}_{\widetilde{z}\widetilde{w}} = (1 + \widetilde{F}_{\widetilde{z}}{}^2 + \widetilde{F}_{\widetilde{w}}{}^2)^2.$$

From $\widetilde{F}_{\widetilde{z}}(\widetilde{z}, \widetilde{w}) = F_z\left(\frac{1}{\beta}\widetilde{z}, \frac{1}{\beta}\widetilde{w}\right)$ and $\widetilde{F}_{\widetilde{w}}(\widetilde{z}, \widetilde{w}) = F_w\left(\frac{1}{\beta}\widetilde{z}, \frac{1}{\beta}\widetilde{w}\right)$ it follows that $\widetilde{F}_{\widetilde{z}}(\varphi(m)) = F_z(\varphi_6 \circ \varphi_1(m))$ and $\widetilde{F}_{\widetilde{w}}(\varphi(m)) = F_w(\varphi_6 \circ \varphi_1(m))$, therefore

$$\begin{aligned}\widetilde{T_0} &= \frac{F_z}{1 + F_z{}^2 + F_w{}^2}\frac{1}{\beta}\frac{\partial}{\partial z^1} + \frac{F_w}{1 + F_z{}^2 + F_w{}^2}\frac{1}{\beta}\frac{\partial}{\partial w^1} \\ &= \frac{\widetilde{F}_{\widetilde{z}}}{1 + \widetilde{F}_{\widetilde{z}}{}^2 + \widetilde{F}_{\widetilde{w}}^2}\frac{\partial}{\partial \widetilde{z}^1} + \frac{\widetilde{F}_{\widetilde{w}}}{1 + \widetilde{F}_{\widetilde{z}}{}^2 + \widetilde{F}_{\widetilde{w}}^2}\frac{\partial}{\partial \widetilde{w}^1}.\end{aligned}$$

It is easy to check that the equiaffine section of $\mathcal{A}^{\rightarrow}\mathcal{N}$

$$\begin{aligned}\xi_{\text{eq}} = -\mathcal{A} \circ f_*&\left(\frac{F_z}{\sqrt{F_z{}^2 + F_w{}^2 + 1}}\frac{\partial}{\partial z^1} + \frac{F_w}{\sqrt{F_z{}^2 + F_w{}^2 + 1}}\frac{\partial}{\partial w^1}\right) \\ &+ \sqrt{F_z{}^2 + F_w{}^2 + 1}\, e_5\end{aligned}$$

is the complex affine normal vector field.

## 4. Examples

1. Let $4AC - B^2 = 1$, $F(z,w) = Az^2 + Bzw + Cw^2 + Kz + Lw$, $\eta(\zeta) = e^\zeta$. We have then

$$F_{zz}F_{ww} - F_{zw}F_{zw} = 1,$$
$$T_0 = \frac{\partial}{\partial z^1}, \quad \xi = (-1, 0, -2Az - Bw - K + 1).$$

   Note that at no point $m_0$, $T_{0m_0} = 0$, but in Proposition 2.1 we do not need such point. The equiaffine section of $\mathcal{A}\mathcal{N}$ is

$$\xi_{\text{eq}} = \eta(F_z)\xi = (-e^{2Az+Bw+K}, 0, e^{2Az+Bw+K}(-2Az - Bw - K + 1)).$$

2. Let $4AC - B^2 = -1, \quad F(z,w) = Az^2 + Bzw + Cw^2 + Kz + Lw,$ $\eta(\zeta) = \sin\zeta$. Then

$$F_{zz}F_{ww} - F_{zw}F_{zw} = -1,$$
$$T_0 = \cot(2Az + Bw + K)\,\frac{\partial}{\partial z^1},$$
$$\xi = (-\cot(2Az + Bw + K), 0, -(2Az + Bw + K)\cot(2Az + Bw + K) + 1),$$
$$\xi_{\text{eq}} = (-\cos(2Az + Bw + K), 0, -(2Az + Bw + K)\cos(2Az + Bw + K) + \sin(2Az + Bw + K)).$$

3. Let $F(z,w) = z^2\, e^{iw}$, $\eta(\zeta) = \frac{1}{2}\,\zeta^2$. Then $F_{zz}F_{ww} - F_{zw}F_{zw} = \frac{1}{2}{F_z}^2$,

$$T_0 = \frac{1}{z}\, e^{-iw}\,\frac{\partial}{\partial z^1},\ \xi = \left(-\frac{1}{z}\, e^{-iw}, 0, -1\right),\ \xi_{\text{eq}} = \left(-2z\, e^{iw}, 0, -2z^2\, e^{2iw}\right).$$

4. *Warped helicoid.* An example of locally symmetric complex surface with $\nabla$ induced by the complex affine normal vector field is a warped helicoid (see [2]). Under a suitable parametrization it can be described by a solution $F$ of the differential equation $F_{zz}F_{ww} - F_{zw}F_{zw} = (1 + {F_z}^2)^2$ which we obtain, taking in (2.1) $\eta(\zeta) = \sqrt{\zeta^2 + 1}$. In this case the solution is $F(z,w) = \big(z - f_1(iw)\big)\tan(iw) + f_2(iw)$, where $f_1$ and $f_2$ are holomorphic functions of one variable. The surface $(z,w) \to (z,w,F(z,w))$ is a warped helicoid, because $z_1 := z$, $z_2 := F(z,w)$, $z_3 := iw$ satisfy the equation $\big(z_1 - f_1(z_3)\big)\sin z_3 = \big(z_2 - f_2(z_3)\big)\cos z_3$, and $(z,w,F(z,w)) = \mathcal{B}(z_1,z_2,z_3)$ where $\mathcal{B} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -i \\ 0 & 1 & 0 \end{pmatrix}$ is an equiaffine transformation. We obtain $T_0 = \sin(iw)\cos(iw)\,\frac{\partial}{\partial z^1}$, $\xi = (-\sin(iw)\cos(iw), 0, \cos^2(iw))$ and $\xi_{\text{eq}} = (-\sin(iw), 0, \cos(iw))$.

5. As an example of locally symmetric complex surface of rank 2 let us take $F(z,w) = \sqrt{1 - z^2 - w^2}$. Then $T_0 = -\sqrt{1 - z^2 - w^2}\,\big(z\frac{\partial}{\partial z^1} + w\frac{\partial}{\partial w^1}\big)$, $\xi = \sqrt{1 - z^2 - w^2}(z, w, \sqrt{1 - z^2 - w^2})$ and $\xi_{\text{eq}} = (z, w, \sqrt{1 - z^2 - w^2}) = (z, w, F(z,w))$.

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## References

[1] F. Dillen, L. Vrancken and L. Verstraelen, *Complex affine differential geometry*, Preprint, Reihe Mathematik, Fachbereich 3 T.U. Berlin, No. 153 (1986).

[2] F. Dillen, *Locally symmetric complex affine surfaces*, J. Geom. **33** (1988), 27–38.

[3] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, vol. I, John Wiley and Sons, New York, 1963.

[4] K. Nomizu and T. Sasaki, *Affine differential geometry*, Cambridge University Press, 1994.

[5] B. Opozda, *On some properties of the curvature and Ricci tensors in complex affine geometry*, Geom. Dedicata **55** (1995), 141–163.

[6] B. Opozda, *Equivalence theorems for complex affine hypersurfaces*, Result. Math. **27** (1995), 316–327.

[7] B. Opozda, *Some relations between Riemannian and affine geometry*, Geom. Dedicata **47** (1993), 225–236.

[8] B. Opozda, *Locally symmetric connections on surfaces*, Result. Math. **20** (1991), 725–743.

[9] M. Robaszewska, *Locally symmetric connections on complex hypersurfaces with type number 1*, Geom. Dedicata **114** (2005), 13–47.

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