communications in
analysis and geometry
Volume 16, Number 2, 333–393, 2008

On a Gromoll-Meyer type theorem in globally hyperbolic stationary spacetimes

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Following the lines of the celebrated Riemannian result of Gromoll and Meyer, we use infinite dimensional equivariant Morse theory to establish the existence of infinitely many *geometrically distinct* closed geodesics in a class of globally hyperbolic stationary Lorentzian manifolds.

1. Introduction

The question of existence of closed geodesics is one of the most classical themes of Riemannian geometry (see [31]); spectacular contributions to the theory of global geometry have been given in this area by very many authors, including Hadamard, Cartan, Poincaré, Birkhoff, Morse and many others. Variational techniques for establishing the existence and the multiplicity of closed geodesics have been developed and employed by many authors, including among others Ljusternik, Schnirelman, Fet, Klingenberg, Gromoll and Meyer. Closed geodesics are critical points of the geodesic action functional in the space of closed paths, and existence results may be obtained by applying global variational techniques to this variational problem. In particular, Morse theory has been used by Gromoll and Meyer (see [21]) to establish the celebrated result on the existence of infinitely many geometrically distinct closed geodesics in simply connected Riemannian manifolds, whose space of free closed curves has unbounded rational Betti numbers.

As to the existence of closed geodesics in manifolds endowed with a non-positive definite metric, very few results are available in the literature, and basically nothing is known on their multiplicity. An earlier result by Tipler (see [48]) gives the existence of one closed timelike geodesic in compact Lorentzian manifolds that admit a regular covering which has a compact Cauchy surface. More recently, Guediri (see [26, 27]) has extended Tipler's result to the case that the Cauchy surface in the covering is not necessarily compact. In this situation, a closed geodesic is proven to exist in each free timelike homotopy class which is determined by a central deck transformation. It is also proved in [26] that compact flat spacetimes contain a causal (i.e., non-spacelike) closed geodesic, and in [27] the author proves that such spacetimes contain a closed timelike geodesic if and only if the fundamental group of the underlying manifold contains a non-trivial timelike translation. The existence of closed timelike geodesic has been established also by Galloway in [14], where the author proves the existence of a longest closed timelike curve, which is necessarily a geodesic, in each *stable* free timelike homotopy class. Also non-existence results for non-spacelike geodesics are available, see [15, 28].

All these results are based on the notion of Lorentzian distance function (see [6, Chapter 4]). Recall that in Lorentzian geometry *only* non-spacelike geodesics have length extremizing properties, while for spacelike geodesics usual geometrical constructions (curve shortening methods) do not work. The question of existence of closed geodesics of arbitrary causal character has to be studied using the quadratic geodesic action functional in the Hilbert manifold of closed paths of Sobolev regularity H^1; its critical points are typically saddle points. In the Lorentzian (or semi-Riemannian) case, the variational theory associated to the study of the critical points of this quadratic functional is complicated by the fact that, unlike the Riemannian counterpart, the condition of Palais and Smale is never satisfied. Moreover, this functional is not bounded from below, and its critical points always have infinite Morse index. In [2] the authors use an approximation scheme in the theory of Ljusternik and Schnirelman to determine the existence of a critical point of the geodesic action functional in the space of closed H^1 curves in a class of product Lorentzian manifolds, whose metric is of splitting type. Such critical point corresponds to a spacelike closed geodesic; in this situation, thanks to the result of Galloway [14], one has two geometrically distinct closed geodesic, one is timelike and the other is spacelike. Masiello has proved the existence of one (spacelike) closed geodesic in standard stationary Lorentzian manifolds $M = M_0 \times \mathbb{R}$ whose spatial component M_0 is compact. More recently (see [11]), using variational methods the authors have established the existence of a closed geodesic in each free homotopy class corresponding to an element of the fundamental group having finite conjugacy class, in the case of *static* compact Lorentzian manifold. In [44], the author shows that one closed timelike geodesic exists in compact Lorentzian manifolds that are conformally static, provided that that the group of deck transformations of some globally hyperbolic regular covering of the manifold admits a finite conjugacy class containing a closed timelike curve.

In this paper, we develop a Morse theory for closed geodesics in a class of stationary Lorentzian manifolds, obtaining a result of existence of infinitely many distinct closed geodesics analogous to the corresponding result of Gromoll and Meyer in the Riemannian case. More precisely, the result will hold for stationary manifolds whose free loop space has unbounded Betti numbers (relatively to any coefficient field), and that admit a compact Cauchy surface.[1] Let us recall briefly the essential ingredients required in Gromoll and Meyer's theory. One considers the geodesic action functional f on the Hilbert manifold ΛM of all closed paths of Sobolev class H^1 on a compact and simply connected Riemannian manifold (M, g); this functional is bounded from below, it satisfies the Palais–Smale condition and its critical points are exactly the closed geodesics. The compact group $O(2)$ acts equivariantly on ΛM via the operation of $O(2)$ on the parameter circle $\mathbb{S}^1$; the orbits of this action are smooth (compact) submanifolds of ΛM. In particular, the critical points of f are never isolated; nevertheless, using a generalized Morse Lemma for possibly degenerate isolated critical point (see [20]), generalized Morse inequalities can be applied to obtain estimates on the number of critical orbits, provided that these orbits are isolated. Finally, one has to distinguish between critical orbits that correspond to iterates of the same closed geodesic. This is done using an iteration formula for the Morse index (and the nullity) of the n-fold covering of a given closed geodesic, which is obtained from a celebrated result due to Bott (see [9]). Using a Morse index theorem for closed geodesics, the Morse index of the nth iterated of a closed geodesic is proven to be either bounded, or to have linear growth in n. Using this fact one proves that if (M, g) has only a finite number of geometrically distinct closed geodesics, then the rational Betti numbers of ΛM must form a bounded sequence. Restriction to the case of a field of characteristic zero was used by the authors to prove an estimate on the dimension of relative homology spaces of certain fiber bundles; an elementary argument based on the Mayer–Vietoris sequence, discussed in Appendix 7, shows that such restriction is not necessary.[2] Thus, if ΛM has an unbounded sequence of Betti numbers, (M, g) must contain infinitely many geometrically distinct closed geodesics. It is known (see [49]), that the existence of an unbounded sequence of rational Betti numbers of the free loop space of M is equivalent to the fact that the rational cohomology algebra of M is generated by at least two elements. In particular, if M has the same homotopy type of the product of two simply connected compact manifold,

[1] Recall that any two Cauchy surfaces of a globally hyperbolic spacetime are homeomorphic. See Refs. [7, 8] for questions concerning the smoothness of Cauchy surfaces in globally hyperbolic spacetimes.

[2] Extension of the Gromoll and Meyer result to non-zero characteristic seems to have been established in the subsequent literature.

then ΛM has unbounded rational Betti numbers. Ziller [52] has proved that any compact symmetric space of rank greater than 1 has unbounded $\mathbb{Z}_2$-Betti numbers; McCleary and Ziller [35] have later proved that the same conclusion holds for compact, simply connected homogeneous spaces which are not diffeomorphic to a symmetric space of rank 1.

Several extensions of the theory have been developed in the context of Riemannian manifolds (see [3–5, 22–24, 46]), Finsler manifolds ([38]) and, recently, of Riemannian orbifolds (see [25]). Reference [47] is a good survey paper on the classical results of Gromoll–Meyer type.

When passing to the case of Lorentzian metrics, none of the arguments above works. First, the geodesic action functional f is not bounded from below and it does not satisfy the PS condition; besides, the Morse index of each critical point is infinite. In this paper we consider the case of stationary Lorentzian manifolds that admit a complete timelike Killing vector field. Timelike invariance of the metric tensor allows to determine a smooth embedded submanifold $\mathcal{N}$ of ΛM with the following properties:

- $f|_{\mathcal{N}}$ has the same critical points of f;
- $\mathcal{N}$ has the same homotopy type of ΛM;
- f is bounded from below and it satisfies the PS condition on each connected component of $\mathcal{N}$;
- each critical point of $f|_{\mathcal{N}}$ has finite Morse index;
- if a critical point is degenerate for $f|_{\mathcal{N}}$, then it is also degenerate for f.

The abelian group $G = O(2) \times \mathbb{R}$ acts (isometrically) on $\mathcal{N}$, and f is G-invariant. The group $O(2)$ acts on the parameter space $\mathbb{S}^1$ of the curves, and as in the Riemannian case, this action is not smooth, but only continuous. Nevertheless, if γ is a smooth curve, then the orbit $O(2)\gamma$ is smooth, and it is diffeomorphic to $O(2)$ if γ is not constant. In particular, critical orbits are always smooth. The group $\mathbb{R}$ acts by translation along the flow lines of the timelike Killing vector field; obviously, the actions of $O(2)$ and of $\mathbb{R}$ commute. In this situation, we define *geometrically distinct* two closed geodesics that belong to different G-orbits, and that cannot be obtained one from another by iteration. The action of $\mathbb{R}$ is free, the orbit space given by the quotient $\widetilde{\mathcal{N}} = \mathcal{N}/\mathbb{R}$ is a smooth manifold and $\mathcal{N}$ is diffeomorphic to the product $\widetilde{\mathcal{N}} \times \mathbb{R}$. Thus, in order to study multiplicity of distinct closed geodesics, it suffices to study geometrically distinct critical $O(2)$-orbits for the constrained functional $f|_{\widetilde{\mathcal{N}}}$. The central result of this paper, which gives the existence of infinitely many distinct closed geodesics in a class of stationary Lorentzian manifolds, is obtained applying equivariant Morse theory to this setup. Essential tools for the development of the theory are a calculation of the Morse index for each critical point of $f|_{\mathcal{N}}$, and a formula that describes its growth under iterations. The Morse index is given in terms of symplectic invariants of the geodesic, such as the Conley–Zehnder and the Maslov index, and it is computed explicitly in Theorem 5.4, which is a Morse index theorem for possibly degenerate closed Lorentzian geodesics. This result is obtained by purely functional analytical techniques, proving a preliminary result (Theorem 2.6) that gives a method for computing the index of essentially positive symmetric bilinear forms, possibly degenerate, in terms of restrictions to possibly degenerate subspaces. We believe that this result has interest in its own, and that its applicability should go beyond the purposes of the present paper. Using this method, one reduces the computation of the Morse index for periodic geodesics to the Morse index of the corresponding fixed endpoint geodesic, avoiding the usual assumption of *orientability* of the closed geodesic (see [40]). The Morse index theorem is given in terms of a symplectic invariant of the geodesic, called the Maslov index; in order to estimate its growth by iteration, we use a recent formula that gives an estimate on the growth of another symplectic invariant, called the Conley–Zehnder index (Proposition 3.3). For orientation preserving geodesics, the two indices are related by a simple formula, involving a fourfold index, which is called the Hörmander index (Proposition 3.2). Using the growth formula for the Conley–Zehnder index and the (non-trivial) fact that the Morse index is, up to a bounded perturbation, non-decreasing by iteration (Lemma 5.5), we then obtain a superlinear estimate on the growth of the Maslov index of an iterate of a closed geodesic (Proposition 5.7 and Corollary 5.8). As to the nullity of an iterate, the result is totally analogous to the Riemannian case using the linearized Poincaré map (Lemma 5.9). This setup paves the path to an application of infinite dimensional equivariant Morse theory, in the same spirit as Gromoll and Meyer's celebrated result, that gives the existence of infinitely many critical points for the functional $f|_{\widetilde{\mathcal{N}}}$.

We will now give a formal statement of the main result of the paper. Let (M, g) be a globally hyperbolic stationary Lorentzian manifold, and let us assume that M admits a *complete* timelike Killing vector field $\mathcal{Y}$. Denote by $\mathcal{F}_t$, $t \in \mathbb{R}$, the flow of $\mathcal{Y}$; clearly, if γ is a (closed) geodesic in M, then also $\mathcal{F}_t \circ \gamma$ is a (closed) geodesic for all $t \in \mathbb{R}$.

In order to state our main result, we need to give an appropriate notion of geometric equivalence of closed geodesics.

Definition 1.1. Given closed geodesics $\gamma_i : [a_i, b_i] \to M$, $i = 1, 2$, in a stationary Lorentzian manifold (M, g), we will say that they are *geometrically distinct*, if there exists no $t \in \mathbb{R}$ such that the sets $\mathcal{F}_t \circ \gamma_1\big([a_1, b_1]\big)$ and $\gamma_2\big([a_2, b_2]\big)$ coincide.

The main result of this paper is the following:

Theorem 1.2. *Let (M, g) be a simply connected globally hyperbolic stationary Lorentzian manifold having a complete timelike Killing vector field, and having a compact Cauchy surface. Assume that the free loop space ΛM has unbounded Betti numbers with respect to some coefficient field. Then, there are infinitely many geometrically distinct non-trivial (i.e., non-constant) closed geodesics in M.*

Note that, by causality, every closed geodesic in (M, g) is *spacelike*. It should be observed here that, although the notion of geometric equivalence given above depends on the choice of a complete timelike Killing vector field, the property of existence of infinitely many geometrically distinct closed geodesics is intrinsic to (M, g) (see Remark 7.1). It is also interesting to observe that the statement of the Theorem admits a generalization to a class of non-simply connected manifolds (see Remark 7.4).

The paper is organized as follows. Section 2 contains a few basic facts concerning bilinear forms and their index; here we prove the main result concerning the computation of the index of an essentially positive symmetric bilinear form on a real Hilbert space (Theorem 2.6). In Section 3 we recall the notions of Conley–Zehnder index for a continuous symplectic path, and of Maslov index for a continuous Lagrangian path. The central result is an inequality (Corollary 3.7) that provides an estimate on the growth of the Maslov index of the iterate of a periodic solution of a Hamiltonian system. The definition of such index depends on the choice of a periodic symplectic trivialization along the solution of the Hamiltonian. When applied to the case of periodic geodesics on a semi-Riemannian manifold M, under a certain orientability assumption we have a canonical choice of a class of periodic symplectic trivializations along the corresponding periodic solution of the geodesic Hamiltonian in the cotangent bundle TM^* (Section 3.4), and we therefore obtain estimates on the growth of the Maslov index of orientation preserving periodic geodesics. The results in Section 3 are valid for closed geodesics in arbitrary semi-Riemannian manifolds. Section 4 contains some material on the geodesic variational problem in stationary Lorentzian manifolds and on the Palais–Smale condition of the relative action functional.

In Section 5 we prove a general version of the Morse index theorem for closed geodesics in stationary Lorentzian manifold, that holds in the general case of possibly degenerate and non-orientation preserving geodesics (Theorem 5.4). This is obtained as an application of Theorem 2.6, which reduces the periodic case to the case of fixed endpoints geodesics. In Section 5.2 we first show that the Morse index of an N-th iterate is non-decreasing on N, up to adding a bounded sequence. Then, we use the Morse index theorem and the estimates on the growth of the Maslov index to get an estimate on the growth of the Morse index under iteration. The central result (Proposition 5.7, Corollary 5.8), which provides an alternative approach to the iteration theory of Bott [9] also for the Riemannian case, says that the index of an N-th iterate is either bounded or it has linear growth in N, up to adding a bounded sequence. The nullity of an iterate is studied in Section 5.3, and the result is totally analogous to the Riemannian case. Finally, in Section 6, we use equivariant Morse theory for isolated critical $O(2)$-orbits of the action functional f in $\widetilde{\mathcal{N}}$ to prove our main result. We follow closely the original paper by Gromoll and Meyer, but we take advantage of a more recent approach to equivariant Morse theory [12, 50], that simplifies some of the constructions in [21]. The local homological invariant at an isolated critical orbit is defined as the relative homology of the critical sublevel, modulo the sublevel minus the critical orbit. Using excision, this invariant is computed as the relative homology of a fiber bundle over the circle modulo a subbundle; these bundles can be described as associated bundles to the principal fiber bundle $O(2) \to O(2)/\Gamma$, where $\Gamma \subset \mathrm{SO}(2)$ is the stabilizer of the orbit. One of the crucial steps in Gromoll and Meyer construction is an estimate on the dimension of this relative homology (see (6.14)); this estimate is proven in Appendix 7 in the case of homology with coefficients in arbitrary fields using the Mayer–Vietoris sequence in relative homology. This allows a slight generalization of the original result in [21], in that no restriction is posed on the characteristic of the coefficient field.

In order to make the paper essentially self-contained, and to facilitate its reading, we have opted to include in the present version of the manuscript the full statement of some results already appearing in the literature and needed in our theory. Quotations of the original authors and complete bibliographical references are given for the proof of these results.

2. On the index of essentially positive bilinear forms

In this section we will discuss some functional analytical preliminaries needed for the index theorem. The central result is Theorem 2.6, that gives a result concerning the computation of the index of symmetric bilinear forms, possibly degenerate, using restrictions to possibly degenerate subspaces. All vector spaces considered in the entire text are assumed to be real. Given a (normed) vector space X, we will denote by X^* its (topological) dual; throughout this section we will always identify continuous bilinear forms $B : X \times X \to \mathbb{R}$ on a normed space X with the continuous linear map $B : X \to X^*$ given by $x \mapsto B(x, \cdot)$. The B-orthogonal complement of a subspace $S \subset X$ is defined by

$$S^{\perp_B} = \big\{x \in X : B(x, y) = 0, \text{ for all } y \in S\big\};$$

the *kernel* of B is defined by

$$\mathrm{Ker}(B) = X^{\perp_B} = \big\{x \in X : B(x, y) = 0, \text{ for all } y \in X\big\}.$$

We say that B is *non-degenerate* if $\mathrm{Ker}(B) = \{0\}$. A subspace $S \subset X$ is called *isotropic* if $B|_{S\times S} = 0$. Assume now that B is symmetric. We say that B is *positive definite* (resp., *positive semi-definite*) if $B(x, x) > 0$ for all non-zero $x \in X$ (resp., $B(x, x) \geq 0$, for all $x \in X$). Similarly, we say that B is *negative definite* (resp., *negative semi-definite*) if $B(x, x) < 0$ for all non-zero $x \in X$ (resp., $B(x, x) \leq 0$, for all $x \in X$). A subspace $S \subset X$ is called *positive* (resp., *negative*) for B if $B|_{S\times S}$ is positive definite (resp., negative definite). The *index* of B is the (possibly infinite) natural number defined by

$$n_-(B) = \sup\big\{\dim(W) : W \subset X \text{ is a negative subspace for } B\big\}.$$

Given any subspace $Y \subset X$, then:

$$n_+(B|_{Y\times Y}) \leq n_+(B) \leq n_+(B|_{Y\times Y}) + \mathrm{codim}_X(Y), \tag{2.1}$$

$$n_-(B|_{Y\times Y}) \leq n_-(B) \leq n_-(B|_{Y\times Y}) + \mathrm{codim}_X(Y). \tag{2.2}$$

Let Y be a vector space and let $q : X \to Y$ be surjective linear map with $\mathrm{Ker}(q) \subset \mathrm{Ker}(B)$. Then there exists a unique map $\overline{B} : Y \times Y \to \mathbb{R}$ such that

$$\overline{B}\big(q(x_1), q(x_2)\big) = B(x_1, x_2), \quad \text{for all } \ x_1, x_2 \in X; \tag{2.3}$$

the map $\overline{B}$ is a symmetric bilinear form on Y. Moreover:

$$\mathrm{Ker}(\overline{B}) = q\big(\mathrm{Ker}(B)\big) \cong \frac{\mathrm{Ker}(B)}{\mathrm{Ker}(q)}, \tag{2.4}$$

$$n_+(\overline{B}) = n_+(B), \quad n_-(\overline{B}) = n_-(B). \tag{2.5}$$

In particular, if $\mathrm{Ker}(q) = \mathrm{Ker}(B)$ then $\overline{B}$ is non-degenerate. If B is non-degenerate and symmetric, and $S \subset X$ is an isotropic subspace, then

$$\dim(S) \le n_-(B), \quad \dim(S) \le n_+(B). \tag{2.6}$$

Let us now consider a (real) normed space X. If $T : X \to Y$ is a continuous linear map between normed spaces then $T^* : Y^* \to X^*$ denotes the transpose map defined by $T^*(\alpha) = \alpha \circ T$. If $S \subset X$ is a subspace we denote by $S^{\mathrm{o}} \subset X^*$ the *annihilator* of S. If X, Y are Banach spaces and $T : X \to Y$ is a continuous linear map then $\mathrm{Ker}(T^*) = \mathrm{Im}(T)^{\mathrm{o}}$ and $\mathrm{Im}(T^*) \subset \mathrm{Ker}(T)^{\mathrm{o}}$. Moreover, if $\mathrm{Im}(T)$ is closed in Y then $\mathrm{Im}(T^*) = \mathrm{Ker}(T)^{\mathrm{o}}$. Given a closed subspace $S \subset X$, denote by $q : X \to X/S$ the quotient map; then $q^* : (X/S)^* \to X^*$ is injective and its image equals S^{o}. Moreover, if X is reflexive, by identifying X with X^{**} in the usual way, the bi-annihilator $(S^{\mathrm{o}})^{\mathrm{o}}$ equals the closure of S. If $Y \subset X$ is a finite co-dimensional closed subspace and $Z \subset X$ is a subspace with $Y \subset Z$, then Z is also closed in X. If $Y_1 \subset X$ is a closed subspace and $Y_2 \subset X$ is a finite dimensional subspace, then $Y_1 + Y_2$ is closed in X. Assume that X is reflexive, B is a continuous symmetric bilinear form on X and $S \subset X$ is a subspace. If $\mathrm{Im}(B) + S^{\mathrm{o}}$ is closed in X^* then the bi-orthogonal complement of S is given by

$$(S^{\perp_B})^{\perp_B} = \overline{S} + \mathrm{Ker}(B).$$

Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let $B : X \times X \to \mathbb{R}$ be a continuous bilinear form. We say that a continuous linear operator $T : X \to X$ *represents* B with respect to $\langle\cdot,\cdot\rangle$ if

$$B(x, y) = \big\langle T(x), y\big\rangle,$$

for all $x, y \in X$. A continuous bilinear form $B : X \times X \to \mathbb{R}$ on a Banach space X is called *strongly non-degenerate* if the linear map $B : X \to X^*$ is an isomorphism. Obviously if B is strongly non-degenerate then B is non-degenerate. The converse holds if we know that the linear map $B : X \to X^*$ is a Fredholm operator of index zero (for instance, a compact perturbation of an isomorphism).

Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space. A continuous linear operator $P : X \to X$ is called *positive* if the bilinear form $\langle P\cdot,\cdot\rangle$ represented by P is symmetric and positive semi-definite. If $P : X \to X$ is a continuous linear operator, then the bilinear form $\langle P\cdot,\cdot\rangle$ represented by P is an inner product on X that defines the same topology as $\langle\cdot,\cdot\rangle$ if and only if P is a positive isomorphism of X.

If X is a Banach space, B is a continuous bilinear form on X and $S \subset X$ is a closed subspace with $X = S \oplus S^{\perp_B}$, then $B|_{S\times S}$ is non-degenerate. Conversely, if $B|_{S\times S}$ is strongly non-degenerate then $X = S \oplus S^{\perp_B}$.

Definition 2.1. Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let B be a continuous symmetric bilinear form on X. We say that B is *essentially positive* if the operator $T : X \to X$ that represents B is of the form $T = P + K$, with $P : X \to X$ a positive isomorphism and $K : X \to X$ a (symmetric) compact operator.

If $B : X \times X \to \mathbb{R}$ is a continuous symmetric bilinear form then B is essentially positive if and only if there exists an inner product $\langle\cdot,\cdot\rangle_1$ on X and a compact operator $K : X \to X$ such that $B = \langle\cdot,\cdot\rangle_1 + \langle K\cdot,\cdot\rangle$ and such that $\langle\cdot,\cdot\rangle_1$ defines the same topology on X as $\langle\cdot,\cdot\rangle$. If $B : X \times X \to \mathbb{R}$ is an essentially positive symmetric bilinear form, then there exists an inner product $\langle\cdot,\cdot\rangle_1$ on X that defines the same topology as $\langle\cdot,\cdot\rangle$ and such that B is represented with respect to $\langle\cdot,\cdot\rangle_1$ by an operator of the form identity plus compact. Moreover, $\mathrm{Ker}(B)$ is finite dimensional and $n_-(B)$ is finite. In Equation (2.3), if B is essentially positive, then also $\overline{B}$ is essentially positive. If a continuous symmetric bilinear form B on a Hilbert space X is essentially positive then B is non-degenerate if and only if B is strongly non-degenerate. Namely, B is represented by a Fredholm operator of index zero. Restriction to a closed subspace of an essentially positive bilinear form is again essentially positive.

Remark 2.2. If B is a continuous symmetric bilinear form on a Hilbert space X which is essentially positive and if $S \subset X$ is a subspace then

$$(S^{\perp_B})^{\perp_B} = \overline{S} + \mathrm{Ker}(B).$$

Lemma 2.3. *Let X be a Hilbert space and let B be a continuous symmetric bilinear form on X which is essentially positive. If $W \subset X$ is a closed subspace then $W + W^{\perp_B}$ is also closed in X.*

Proof. We can choose the inner product $\langle\cdot,\cdot\rangle$ on X such that B is represented by an operator of the form $T = \mathrm{Id} + K$, with K compact. If W' denotes the orthogonal complement of W with respect to $\langle\cdot,\cdot\rangle$ then $W^{\perp_B} = T^{-1}(W')$. We then have to show that $W + T^{-1}(W')$ is closed in X. Since T is a Fredholm operator, its image is closed in X and so $T : X \to \mathrm{Im}(T)$ is a surjective continuous linear operator between Banach spaces. We have

$\mathrm{Ker}(T) \subset T^{-1}(W') \subset W + T^{-1}(W')$, so that $W + T^{-1}(W')$ is T-saturated;[3] thus $W + T^{-1}(W')$ is closed in X if and only if $T\big[W + T^{-1}(W')\big] = T(W) + \big(W' \cap \mathrm{Im}(T)\big)$ is closed in $\mathrm{Im}(T)$. But

$$T(W) + \big(W' \cap \mathrm{Im}(T)\big) = \big(T(W) + W'\big) \cap \mathrm{Im}(T),$$

and therefore the proof will be completed once we show that $T(W) + W'$ is closed in X. We have

$$T(W) + W' = \big\{x + y + K(x) : x \in W,\ y \in W'\big\} = \mathrm{Im}(\mathrm{Id} + K \circ \pi),$$

where π denotes the orthogonal projection (with respect to $\langle \cdot, \cdot \rangle$) onto W. Since $K \circ \pi$ is compact, $\mathrm{Id} + K \circ \pi$ is a Fredholm operator and hence its image is closed in X. □

Lemma 2.4. *Let X be a Hilbert space and let B be a non-degenerate continuous symmetric bilinear form on X which is essentially positive. If $Z \subset X$ is an isotropic subspace then*

$$n_-(B) = n_-(B|_{Z^{\perp_B} \times Z^{\perp_B}}) + \dim(Z), \tag{2.7}$$

all the terms in the above equality being finite natural numbers.

Proof. Since B is essentially positive, $n_-(B) < +\infty$ and thus $\dim(Z) < +\infty$, by (2.6). This proves that all terms in equality (2.7) are finite natural numbers. Since Z is isotropic, we have $Z \subset Z^{\perp_B}$ and thus we can find a closed subspace $U \subset Z^{\perp_B}$ with $Z^{\perp_B} = Z \oplus U$ (for instance, take U to be the orthogonal complement of Z in $Z^{\perp_B}$ with respect to any Hilbert space inner product). Clearly

$$\mathrm{Ker}(B|_{Z^{\perp_B} \times Z^{\perp_B}}) = Z^{\perp_B} \cap (Z^{\perp_B})^{\perp_B}.$$

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[3] If X, Y are sets and $f : X \to Y$ is a map then a subset $S \subset X$ is called *f-saturated* if $x_1 \in S$, $x_2 \in X$ and $f(x_1) = f(x_2)$ imply $x_2 \in S$. If X, Y are vector spaces, f is linear and $S \subset X$ is a subspace then S is f-saturated if and only if $\mathrm{Ker}(f) \subset S$. Observe that if X, Y are Banach spaces and $f : X \to Y$ is a surjective continuous linear map then, by the open mapping theorem, f is a quotient map in the topological sense; hence a saturated subset $S \subset X$ is open (resp., closed) in X if and only if $f(S)$ is open (resp., closed) in Y. Similarly, a subset $U \subset Y$ is open (resp., closed) in Y if and only if $f^{-1}(U)$ is open (resp., closed) in X.

Now, by Remark 2.2, $(Z^{\perp_B})^{\perp_B} = Z$. We have thus proven that

$$\mathrm{Ker}(B|_{Z^{\perp_B}\times Z^{\perp_B}}) = Z,$$

and thus $B|_{U\times U}$ is non-degenerate. Since $B|_{U\times U}$ is also essentially positive, $B|_{U\times U}$ is actually strongly non-degenerate; thus

$$X = U \oplus U^{\perp_B},$$

and

$$n_-(B) = n_-(B|_{U\times U}) + n_-(B|_{U^{\perp_B}\times U^{\perp_B}})$$

and since Z is isotropic

$$n_-(B|_{U\times U}) = n_-(B|_{Z^{\perp_B}\times Z^{\perp_B}}).$$

To complete the proof, it suffices to show that

$$n_-(B|_{U^{\perp_B}\times U^{\perp_B}}) = \dim(Z).$$

First, we claim that $\dim(U^{\perp_B}) = 2\dim(Z)$. Namely, since $X = U \oplus U^{\perp_B}$, the dimension of $U^{\perp_B}$ equals the co-dimension of U in X. We have

$$U \subset Z^{\perp_B} \subset X;$$

since $Z^{\perp_B} = Z \oplus U$, the co-dimension of U in $Z^{\perp_B}$ equals the dimension of Z. Since $B : X \to X^*$ is an isomorphism and $Z^{\perp_B} = B^{-1}(Z^{\mathrm{o}})$, B induces an isomorphism

$$X/Z^{\perp_B} \xrightarrow{\cong} X^*/Z^{\mathrm{o}};$$

moreover, $X^*/Z^{\mathrm{o}} \cong Z^* \cong Z$. Thus the co-dimension of $Z^{\perp_B}$ in X is equal to the dimension of Z, which proves that $\dim(U^{\perp_B}) = 2\dim(Z)$. To complete the proof, observe that B is non-degenerate on $U^{\perp_B}$ since B is non-degenerate on U. It follows

$$n_+(B|_{U^{\perp_B}\times U^{\perp_B}}) + n_-(B|_{U^{\perp_B}\times U^{\perp_B}}) = \dim(U^{\perp_B}) = 2\dim(Z),$$

and by (2.6)

$$n_+(B|_{U^{\perp_B}\times U^{\perp_B}}) \geq \dim(Z), \quad n_-(B|_{U^{\perp_B}\times U^{\perp_B}}) \geq \dim(Z).$$

This proves that both $n_+(B|_{U^{\perp_B}\times U^{\perp_B}})$ and $n_-(B|_{U^{\perp_B}\times U^{\perp_B}})$ are equal to $\dim(Z)$. $\square$

Lemma 2.5. *Let X be a Hilbert space and let B be a non-degenerate continuous symmetric bilinear form on X which is essentially positive. If $W \subset X$ is a closed subspace then*

$$n_-(B) = n_-(B|_{W\times W}) + n_-(B|_{W^{\perp_B}\times W^{\perp_B}}) + \dim(W \cap W^{\perp_B}),$$

all the terms in the above equality being finite natural numbers.

Proof. Obviously $Z = W \cap W^{\perp_B}$ is an isotropic subspace, and we can apply Lemma 2.4 to obtain

$$n_-(B) = n_-(B|_{Z^{\perp_B}\times Z^{\perp_B}}) + \dim(W \cap W^{\perp_B}).$$

The conclusion will follow once we show that $Z^{\perp_B} = W + W^{\perp_B}$. Using Remark 2.2, we compute

$$(W + W^{\perp_B})^{\perp_B} = W^{\perp_B} \cap (W^{\perp_B})^{\perp_B} = W^{\perp_B} \cap W = Z.$$

Now using Lemma 2.3 and Remark 2.2 we obtain

$$Z^{\perp_B} = \big[(W + W^{\perp_B})^{\perp_B}\big]^{\perp_B} = W + W^{\perp_B}.$$

□

Finally, the central result we aimed at:

Theorem 2.6. *Let X be a Hilbert space and let B be a continuous symmetric essentially positive bilinear form on X. If $W \subset X$ is a closed subspace and S denotes the B-orthogonal space to W, then*

$$n_-(B) = n_-\big(B|_{W\times W}\big) + n_-\big(B|_{S\times S}\big) + \dim(W \cap S) - \dim\big(W \cap \mathrm{Ker}(B)\big),$$

all the terms in the above equality being finite natural numbers.

Proof. Set $N = \mathrm{Ker}(B)$, $Y = X/N$ and denote by $q : X \to Y$ the quotient map. Define $\overline{B}$ as in (2.3); then $\overline{B}$ is a non-degenerate continuous symmetric bilinear form on Y and $\overline{B}$ is essentially positive. We will apply Lemma 2.5 to $\overline{B}$ and to the subspace $q(W)$ of Y; we first check that $q(W)$ is closed in Y. It suffices to observe that $q^{-1}\big(q(W)\big) = W + N$ is closed in X and this follows from the fact that $\dim(N) < +\infty$. Now

$$n_-(\overline{B}) = n_-(\overline{B}|_{q(W)\times q(W)}) + n_-(\overline{B}|_{q(W)^{\perp_B}\times q(W)^{\perp_B}}) + \dim\big(q(W)\cap q(W)^{\perp_B}\big).$$

It is straightforward to verify that

$$q(W)^{\perp_B} = q(W^{\perp_B}).$$

Now using (2.4), (2.5) and considering the surjective linear maps q, $q|_W : W \to q(W)$ and $q|_{W^{\perp_B}} : W^{\perp_B} \to q(W)^{\perp_B}$, we obtain

$$\begin{aligned} n_-(\overline{B}) &= n_-(B), \\ n_-(\overline{B}|_{q(W)\times q(W)}) &= n_-(B|_{W\times W}), \\ n_-(\overline{B}|_{q(W)^{\perp_B}\times q(W)^{\perp_B}}) &= n_-(B|_{W^{\perp_B}\times W^{\perp_B}}). \end{aligned}$$

To complete the proof, we have to show that

$$\dim\big(q(W)\cap q(W^{\perp_B})\big) = \dim(W\cap W^{\perp_B}) - \dim(W\cap N).$$

Keeping in mind that $N \subset W^{\perp_B}$, we compute

$$q^{-1}\big(q(W)\cap q(W^{\perp_B})\big) = (W+N)\cap W^{\perp_B} = (W\cap W^{\perp_B}) + N,$$

so that $q(W)\cap q(W^{\perp_B}) = q\big((W\cap W^{\perp_B})+N\big) \cong \big[(W\cap W^{\perp_B})+N\big]/N$. Then

$$\begin{aligned} \dim\big(q(W)\cap q(W^{\perp_B})\big) &= \dim\big[(W\cap W^{\perp_B})+N\big] - \dim(N) \\ &= \dim(W\cap W^{\perp_B}) - \dim(W\cap N). \end{aligned}$$

This concludes the proof. □

3. On the Maslov index and iteration formulas

In this section we will prove an iteration formula for the Maslov index of a periodic solution of a Hamiltonian system, using a similar formula proved in [19] for the Conley–Zehnder index, and a formula relating the two indices via the Hörmander index. The reader should note that in the literature there are several definitions Maslov index for a continuous Lagrangian path; these definitions differ by a boundary term when the path has endpoints in the Maslov cycle. In Robbin and Salamon [43], the Maslov index is a half integer, obtained as half of the variation of the signature function of certain bilinear forms, whereas in our case we replace half of the signature with the *extended coindex* (i.e., index plus nullity), see formula (3.1). Obviously, the two definitions are totally equivalent; however, the reader should observe that, using our definition, the Maslov index takes integer values, but it fails to have the property that, when one changes the sign of the symplectic form the absolute value of the Maslov index remains constant. The reader should also be aware of the fact that the definition of Maslov index for a semi-Riemannian geodesic adopted here differs slightly from previous definitions, originated from Helfer [29], in that here we also consider the contribution given by the initial endpoint, which is always conjugate. The results in this section (more specifically, in Section 3.4) are valid in any semi-Riemannian manifold (M, g).

3.1. Maslov and Conley–Zehnder index

Let us recall a few definitions from the theory of Maslov index. Let V be a finite dimensional real vector space endowed with a symplectic form ω, and let $\mathrm{Sp}(V,\omega)$ denote the symplectic group of (V,ω); set $\dim(V) = 2n$. Denote by $\mathbf{\Lambda} = \mathbf{\Lambda}(V,\omega)$ the Grassmannian of all n-dimensional subspaces of V, which is a $\frac{1}{2}n(n+1)$-dimensional real-analytic compact manifold. For $L_0 \in \mathbf{\Lambda}$, one has a smooth fibration $\beta_{L_0} : \mathrm{Sp}(V,\omega) \to \mathbf{\Lambda}$ defined by

$$\beta_{L_0}(\Phi) = \Phi[L_0].$$

Let $L_0, L_1 \in \mathbf{\Lambda}$ be transverse Lagrangians; any other Lagrangian L which is transverse to L_1 is the graph of a unique linear map $T : L_0 \to L_1$; we will denote by $\varphi_{L_0,L_1}(L)$ is defined to be the restriction of the bilinear map $\omega(T\cdot,\cdot)$ to $L_0 \times L_0$, which is a symmetric bilinear form on L_0. For $L \in \mathbf{\Lambda}$, we will denote by $\mathbf{\Lambda}_0(L)$ the set of all Lagrangians $L' \in \mathbf{\Lambda}$ that are transverse to L; this is a dense open subset of $\mathbf{\Lambda}$.

Denote by $\pi(\mathbf{\Lambda})$ the *fundamental groupoid* of $\mathbf{\Lambda}$, endowed with the partial operation of concatenation $\diamond$. For all $L_0 \in \mathbf{\Lambda}$, there exists a unique $\mathbb{Z}$-valued groupoid homomorphism μ_{L_0} on $\pi(\mathbf{\Lambda})$ such that

$$\begin{aligned}\mu_{L_0}\big([\gamma]\big) = {} & n_+\big(\varphi_{L_0,L_1}(\gamma(1))\big) + \dim\big(\gamma(1)\cap L_0\big) - n_+\big(\varphi_{L_0,L_1}(\gamma(0))\big)\\ & - \dim\big(\gamma(0)\cap L_0\big)\end{aligned} \tag{3.1}$$

for all continuous curve $\gamma : [0,1] \to \mathbf{\Lambda}_0(L_1)$ and for all $L_1 \in \mathbf{\Lambda}_0(L_0)$. The map $\mu_{L_0} : \pi(\mathbf{\Lambda}) \to \mathbb{Z}$ is called the L_0-*Maslov index*.

Given four Lagrangians $L_0, L_1, L_0', L_1' \in \mathbf{\Lambda}$ and any continuous curve $\gamma : [a,b] \to \mathbf{\Lambda}$ such that $\gamma(a) = L_0'$ and $\gamma(b) = L_1'$, then the value of the quantity $\mathfrak{q}(L_0, L_1; L_0', L_1') = \mu_{L_1}(\gamma) - \mu_{L_0}(\gamma)$ does *not* depend on the choice of γ, and it is called the *Hörmander index* of the quadruple $(L_0, L_1; L_0', L_1')$. Consider the direct sum $V^2 = V \oplus V$, endowed with the symplectic form $\omega^2 = \omega \oplus (-\omega)$, defined by

$$\omega^2\big((v_1, v_2), (w_1, w_2)\big) = \omega(v_1, v_2) - \omega(w_1, w_2), \quad v_1, v_2, w_1, w_2 \in V,$$

and let $\Delta \subset V^2$ denote the diagonal subspace. If $\Phi \in \mathrm{Sp}(V, \omega)$, then the graph of Φ, denoted by $\mathrm{Gr}(\Phi)$, is given by $(\mathrm{Id} \oplus \Phi)[\Delta] \in \mathbf{\Lambda}(V^2, \omega^2)$; in particular $\Delta = \mathrm{Gr}(\mathrm{Id})$ and $\Delta^o = \{(v, -v) : v \in V\} = \mathrm{Gr}(-\mathrm{Id})$ are Lagrangian subspaces of V^2. Given a continuous curve Φ in $\mathrm{Sp}(V, \omega)$, the *Conley–Zehnder index* $\mathfrak{i}_{\mathrm{CZ}}(\Phi)$ *of* Φ is the Δ-Maslov index of the curve $t \mapsto \mathrm{Gr}\big(\Phi(t)\big) \in \mathbf{\Lambda}(V^2, \omega^2)$:

$$\mathfrak{i}_{\mathrm{CZ}}(\Phi) := \mu_\Delta\big(t \mapsto \mathrm{Gr}(\Phi(t))\big).$$

We have the following relation between the Maslov index and the Hörmander index:

Lemma 3.1. *Let $\Phi : [a,b] \to \mathrm{Sp}(V, \omega)$ be a continuous curve and let $L_0, L_1, L_1' \in \mathbf{\Lambda}(V, \omega)$ be fixed. Then*

$$\mu_{L_0}\big(\beta_{L_1} \circ \Phi\big) - \mu_{L_0}\big(\beta_{L_1'} \circ \Phi\big) = \mathfrak{q}\big(L_1, L_1'; \Phi(a)^{-1}(L_0), \Phi(b)^{-1}(L_0)\big).$$

Proof. Using the Maslov index for pairs and the symplectic invariance, we compute as follows:

$$\begin{aligned}\mu_{L_0}\big(\beta_{L_1} \circ \Phi\big) &= \mu\big(\beta_{L_1} \circ \Phi, L_0\big) = \mu\big(L_1, t \mapsto \Phi(t)^{-1}(L_0)\big)\\ &= -\mu_{L_1}\big(t \mapsto \Phi(t)^{-1}(L_0)\big).\end{aligned}$$

Similarly,

$$\mu_{L_0}\big(\beta_{L_1'} \circ \Phi\big) = -\mu_{L_1'}\big(t \mapsto \Phi(t)^{-1}(L_0)\big).$$

The conclusion follows easily from the definition of $\mathfrak{q}$. □

The following relation between the notions of Maslov, Conley–Zehnder and Hörmander index holds:

Proposition 3.2. *Let $\Phi : [a,b] \to \mathrm{Sp}(V,\omega)$ be a continuous curve and L_0, $\ell_0 \in \mathbf{\Lambda}(V,\omega)$ be fixed. Then*

$$\mathrm{i}_{\mathrm{CZ}}(\Phi) + \mu_{L_0}\big(\beta_{\ell_0} \circ \Phi\big) = \mathfrak{q}\big(\Delta, L_0 \oplus \ell_0; \mathrm{Gr}\big(\Phi(a)^{-1}\big), \mathrm{Gr}\big(\Phi(b)^{-1}\big)\big).$$

In particular, if Φ is a loop, then $\mathrm{i}_{\mathrm{CZ}}(\Phi) = -\mu_{L_0}(\beta_{\ell_0} \circ \Phi)$.

Proof. We compute

$$\mathrm{i}_{\mathrm{CZ}}(\Phi) = \mu_\Delta\big(t \mapsto (\mathrm{Id} \oplus \Phi(t))(\Delta)\big)$$

and, using the properties of the Maslov index for pairs of curves,

$$\begin{aligned}\mu_{L_0}\big(\beta_{\ell_0} \circ \Phi\big) &= -\mu_\Delta\big(t \mapsto L_0 \oplus \beta_{\ell_0} \circ \Phi(t)\big)\\ &= -\mu_\Delta\big(t \mapsto (\mathrm{Id} \oplus \Phi(t))(L_0 \oplus \ell_0)\big).\end{aligned}$$

The result follows now easily applying Lemma 3.1 to the curve $t \mapsto \mathrm{Id} \oplus \Phi(t)$ in the symplectic group $\mathrm{Sp}(V^2, \omega^2)$ and to the Lagrangians $\Delta, L_0 \oplus \ell_0 \in \mathbf{\Lambda}(V^2, \omega^2)$. □

3.2. Periodic solutions of Hamiltonian systems

The notion of Conley–Zehnder index is used in the theory of periodic solutions for Hamiltonian systems. Let us recall a few basic facts; let $(\mathcal{M}, \varpi)$ be a $2n$-dimensional symplectic manifold, and let $H : \mathcal{M} \times \mathbb{R} \to \mathbb{R}$ be a (possibly time-dependent) smooth Hamiltonian. Assume that H is T-periodic in time, and that $z : [0,T] \to M$ is a solution of H (i.e., $\dot z = \vec H(z)$ such that $z(0) = z(T)$, where $\vec H$ is the time-dependent Hamiltonian vector field, defined by $\varpi(\vec H, \cdot) = \mathrm{d}H$). Then, the iterates $z^{(N)}$ of z, defined as the concatenation

$$z^{(N)} = \underbrace{z \diamond \cdots \diamond z}_{N\text{-times}} : [0, NT] \longrightarrow \mathcal{M}$$

are also solutions of H. Assume that the following objects are given:

- a *periodic symplectic trivialization* of the tangent bundle of $\mathcal{M}$ along z (i.e., of the pull-back $z^*T\mathcal{M}$), which consists of a smooth family $\Psi = \{\psi_t\}_{t\in[0,T]}$ of symplectomorphisms $\psi_t : T_{z(0)}\mathcal{M} \to T_{z(t)}\mathcal{M}$ with $\psi_0 = \psi_T = \mathrm{Id}$;
- a Lagrangian subspace $L_0 \subset T_{z(0)}\mathcal{M}$.

By a simple orientability argument, periodic symplectic trivializations along periodic solutions always exist. By the periodicity assumption, we have a smooth extension $\mathbb{R} \ni t \mapsto \psi_t$ by setting $\psi_{t+NT} = \psi_t$ for all $t \in [0,T]$. Denote by $\mathcal{F}^H_{t,t'} : \mathcal{M} \to \mathcal{M}$ the flow of $\vec{H}$,[4] i.e., $\mathcal{F}^H_{t,t'}(p) = \gamma(t')$, where γ is the unique integral curve of the time-dependent vector field $\vec{H}$ on $\mathcal{M}$ satisfying $\gamma(t) = p$. It is well known that for all t, t', the $\mathcal{F}^H_{t,t'}$ is a symplectomorphism between open subsets of $\mathcal{M}$. Left composition with ψ_t^{-1} gives a smooth map $\mathbb{R} \ni t \mapsto \Psi(t) = \psi_t^{-1} \circ \mathcal{F}^H_{0,t}\big(z(0)\big)$ of linear symplectomorphisms of $T_{z(0)}\mathcal{M}$; clearly $X(t) = \Psi'(t)\Psi(t)^{-1}$ lies in the Lie algebra $\mathfrak{sp}\big(T_{z(0)}\mathcal{M}, \varpi_{z(0)}\big)$ of the symplectic group $\mathrm{Sp}\big(T_{z(0)}\mathcal{M}, \varpi_{z(0)}\big)$.

The *linearized Hamilton equation along* z is the linear system

$$v'(t) = X(t)v(t), \tag{3.2}$$

in $T_{z(0)}\mathcal{M}$; the fundamental solution of this linear system is a smooth symplectic path $\Phi : \mathbb{R} \to \mathrm{Sp}\big(T_{z(0)}\mathcal{M}, \varpi_{z(0)}\big)$ that satisfies $\Phi(0) = \mathrm{Id}$ and $\Phi' = X\Phi$.

Definition 3.3. The *Conley–Zehnder index* of the solution $z = z^{(1)}$ associated to the symplectic trivialization Ψ, denoted by $\mathfrak{i}_{\mathrm{CZ}}(z,\Psi)$, is the Conley–Zehnder of the path in $\mathrm{Sp}\big(T_{z(0)}\mathcal{M}, \varpi_{z(0)}\big)$ obtained by restriction of the fundamental solution Φ to the interval $[0,T]$. Similarly, the L_0-*Maslov index* of the solution z associated to the symplectic trivialization Ψ, denoted by $\mu_{L_0}(z,\Psi)$, is the L_0-Maslov index of the path in $\mathrm{Sp}\big(T_{z(0)}\mathcal{M}, \varpi_{z(0)}\big)$ given by $[0,T] \ni t \mapsto \Phi(t)[L_0] \in \mathbf{\Lambda}\big(T_{z(0)}\mathcal{M}, \varpi_{z(0)}\big)$.

Remark 3.4. We observe here that both the notions of Conley–Zehnder index and of Maslov index for a periodic solution z of a Hamiltonian system depend on the choice of a symplectic trivialization. More precisely, given two periodic symplectic trivializations $\Psi = \{\psi_t\}_t$, $\widetilde{\Psi} = \{\widetilde{\psi}_t\}_t$ and setting $G_t = \psi_t^{-1} \circ \widetilde{\psi}_t \in \mathrm{Sp}\big(T_{z(0)}\mathcal{M}, \varpi_{z(0)}\big)$, the corresponding paths Φ and $\widetilde{\Phi}$ in $\mathrm{Sp}\big(T_{z(0)}\mathcal{M}, \varpi_{z(0)}\big)$ are related by

$$\Phi(t) = G_t \circ \widetilde{\Phi}(t), \quad \forall\, t \in [0,T].$$

Clearly, $[0,T] \ni t \mapsto G_t$ is a closed path in $\mathrm{Sp}\big(T_{z(0)}\mathcal{M}, \varpi_{z(0)}\big)$ with endpoint in the identity; in this situation, one proves easily[5] that $\mathfrak{i}_{\mathrm{CZ}}(z,\Psi) = \mathfrak{i}_{\mathrm{CZ}}(G) + \mathfrak{i}_{\mathrm{CZ}}(z,\widetilde{\Psi})$. In particular, if the loop G is homotopically trivial, then

[4]For our purposes, we will not be interested in questions of global existence of the flow $\mathcal{F}^H$.

[5]For instance, using the product formula in [19, Lemma 3.3].

$\mathrm{i}_{\mathrm{CZ}}(z,\Psi)=\mathrm{i}_{\mathrm{CZ}}(z,\widetilde{\Psi})$. Similarly, if $G_t[L_0]=L_0$ for all t, and if G is homotopically trivial, then $\mu_{L_0}(z,\Psi)=\mu_{L_0}(z,\widetilde{\Psi})$.

This observation will be used in a situation described in the following Lemma:

Lemma 3.5. *Let $\mathcal{V}$ be a finite dimensional vector space and set $V=\mathcal{V}\oplus\mathcal{V}^*$; V is a symplectic space, endowed with its* canonical symplectic form $\omega\big((v,\alpha),(w,\beta)\big)=\beta(v)-\alpha(w)$, $v,w\in\mathcal{V}$, $\alpha,\beta\in\mathcal{V}^*$. *Given any $\eta\in\mathrm{GL}(\mathcal{V})$, then the linear map*

$$G=\begin{pmatrix}\eta & 0\\ 0 & \eta^{*-1}\end{pmatrix}:V\to V$$

is a symplectomorphism of (V,ω). If $[a,b]\ni t\mapsto G_t\in\mathrm{Sp}(V,\omega)$ is a continuous map of symplectomorphisms of this type with $G_a=G_b=\mathrm{Id}$, then G is homotopically trivial in $\mathrm{Sp}(V,\omega)$.

Proof. The first statement is immediate. In order to prove that G is homotopically trivial, it is not restrictive to assume $\mathcal{V}=\mathbb{R}^n$; identifying $\mathbb{R}^{n*}$ with $\mathbb{R}^n$ via the Euclidean inner product, we will consider the canonical complex structure on $V\cong\mathbb{R}^{2n}$. The thesis is obtained if we prove that, denoting by $G_t=u_tp_t$ the polar decomposition of G_t, with u_t unitary and p_t positive definite, then $t\mapsto u_t$ is homotopically trivial in $\mathrm{U}(n)$. This is equivalent to the fact that the closed in loop $t\mapsto\det(u_t)\in\mathbb{S}^1$ is homotopically trivial in $\mathbb{S}^1$. If $\eta_t=o_tq_t$ is the polar decomposition of η_t, with $o_t\in O(n)$ and q_t positive definite, then the unitary u_t is given by $\begin{pmatrix}o_t & 0\\ 0 & o_t\end{pmatrix}\in\mathrm{U}(n)$, which has constant determinant equal to 1. The conclusion follows easily, recalling that the determinant map $\det:\mathrm{U}(n)\to\mathbb{S}^1$ induces an isomorphism between the fundamental groups. □

3.3. An iteration formula for the Maslov index

Let us recall the following iteration formula for the Conley–Zehnder index, proved in [19]:

Proposition 3.6. *In the notations of Section 3.2, the following inequality holds:*

$$\Big|\mathrm{i}_{\mathrm{CZ}}\big(z^{(N)},\Psi\big)-N\cdot\mathrm{i}_{\mathrm{CZ}}\big(z,\Psi\big)\Big|\le n(N-1). \tag{3.3}$$

In particular, $\big|\mathrm{i}_{\mathrm{CZ}}\big(z^{(N)},\Psi\big)\big|$ *has sublinear growth in* N*. Moreover, if* $\big|\mathrm{i}_{\mathrm{CZ}}\big(z,\Psi\big)\big| > n$*, then* $\mathrm{i}_{\mathrm{CZ}}\big(z^{(N)},\Psi\big)$ *has superlinear growth in* N*.*

Proof. See [19, Corollary 4.4]. Observe that we are using here a slightly different definition of Conley–Zehnder index, and the inequality (3.3) differs by a factor 2 from the corresponding inequality in [19, Corollary 4.4]. □

Let us prove that a similar iteration formula holds for the Maslov index.

Corollary 3.7. *The following inequality holds:*

$$\Big|\mu_{L_0}\big(z^{(N)},\Psi\big) - N\cdot\mu_{L_0}\big(z,\Psi\big)\Big| \le n(7N+5).$$

In particular, $\big|\mu_{L_0}\big(z^{(N)},\Psi\big)\big|$ *has sublinear growth in* N*; moreover, if* $\mu_{L_0}\big(z,\Psi\big) > 7n$*, then* $\mu_{L_0}\big(z^{(N)},\Psi\big)$ *has superlinear growth in* N*.*

Proof. The inequality is obtained easily from (3.3), using Proposition 3.2

$$\begin{aligned}\Big|\mu_{L_0}\big(z^{(N)},\Psi\big) - N\cdot\mu_{L_0}\big(z,\Psi\big)\Big| &\le \Big|\mathrm{i}_{\mathrm{CZ}}\big(z^{(N)},\Psi\big) - N\cdot\mathrm{i}_{\mathrm{CZ}}\big(z,\Psi\big)\Big| \\ &\quad + \big|\mathfrak{q}\big(\Delta, L_0\oplus L_0;\Delta,\mathrm{Gr}\big(\Phi(NT)\big)\big) - N\cdot\mathfrak{q}\big(\Delta, L_0\oplus L_0;\Delta,\mathrm{Gr}\big(\Phi(T)\big)\big)\big| \\ &\le n(N-1) + 6n(N+1) = n(7N+5).\end{aligned}$$

□

3.4. Maslov index of a geodesic and of the corresponding Hamiltonian solution

Let us now define the notion of Maslov index for a closed geodesic γ in a semi-Riemannian manifold (M,g); we will show that when γ is orientation preserving, then its Maslov index coincides with the Maslov index of the corresponding periodic solution of the geodesic Hamiltonian in the cotangent bundle TM^*.

Let us recall the notion of Maslov index for a *fixed endpoint geodesic.* If $\gamma:[0,1]\to M$ is any geodesic, consider a continuous trivialization of TM along γ, i.e., a continuous family of isomorphisms $h_t: T_{\gamma(0)}M \to T_{\gamma(t)}M$, $t\in[0,1]$. Consider the symplectic space $V = T_{\gamma(0)}M \oplus T_{\gamma(0)}M^*$ endowed with its canonical symplectic structure (recall Lemma 3.5), the Lagrangian subspace $L_0 = \{0\}\oplus T_{\gamma(0)}M^*$, and the continuous curve of Lagrangians

$\ell(t) \in \mathbf{\Lambda}(V,\omega)$ given by

$$\ell(t) = \Big\{ \big(h_t^{-1}[J(t)], h_t^*[g(\tfrac{D}{dt}J(t))]\big) : J \text{ Jacobi field along } \gamma, \text{ with } J(0)=0 \Big\}.$$

In the above formula, the metric tensor g is seen as a map $g : T_{\gamma(t)}M \to T_{\gamma(t)}M^*$. The Maslov index of γ, denoted by $\mathrm{i}_M(\gamma)$ is defined as the L_0-Maslov index of the continuous path $[0,1] \ni t \mapsto \ell(t)$.[6] This quantity does not depend on the choice of the trivialization of TM along γ. Let us now consider the case of a closed geodesic, in which case one may study the existence of *periodic* trivializations of TM along γ.

Recall that a closed curve $\gamma : [a,b] \to M$ is said to be *orientation preserving* if for some (and hence for any) continuous trivialization $h_t : T_{\gamma(a)}M \to T_{\gamma(t)}M$, $t \in [a,b]$, of TM along γ, the isomorphism $h_b^{-1} \circ h_a : T_{\gamma(a)}M \to T_{\gamma(a)}M$ is orientation preserving. It is easy to prove that if γ is orientation preserving then there exists a smooth trivialization $h_t : T_{\gamma(a)}M \to T_{\gamma(t)}M$, $t \in [a,b]$, of TM along γ with $h_b^{-1} \circ h_a$ the identity of $T_{\gamma(a)}M$.

Assume that $\gamma : [0,1] \to M$ is a closed geodesic in M, which is orientation preserving. Let $\Gamma : [0,1] \to TM^*$ be the corresponding periodic solution of the geodesic Hamiltonian:

$$H(q,p) = g^{-1}(p,p).$$

Given a smooth periodic trivialization of TM along γ, $h_t : T_{\gamma(0)}M \to T_{\gamma(t)}M$, $t \in [0,1]$, $h_0 = h_1$, then one can define a smooth periodic symplectic trivialization of the tangent bundle $T(TM^*)$ along Γ as follows. Denote by $\pi : TM^* \to M$ the canonical projection; for $p \in TM^*$, denote by $\mathrm{Ver}_p = \mathrm{Ker}(d\pi_p)$ the vertical subspace of $T_p(TM^*)$ and by Hor_p the horizontal subspace of $T_p(TM^*)$ relatively to the Levi–Civita connection ∇. One has a canonical identification $\mathrm{Ver}_p = T_p(T_xM^*) \cong (T_xM)^*$, while the restriction of the differential $d\pi_p$ to Hor_p gives an identification $\mathrm{Hor}_p \cong T_xM$, where $x = \pi(p)$. Since ∇ is torsion free, with these identifications, the *canonical*

[6] A different convention was originally adopted by Helfer [29] in the definition of Maslov index of a semi-Riemannian geodesic. In Helfer's original definition, given a geodesic $\gamma : [0,1] \to M$ with non-conjugate endpoints, $\mathrm{i}_M(\gamma)$ was given by the L_0-Maslov index of the continuous path $[\varepsilon,1] \ni t \mapsto \ell(t)$, where $\varepsilon > 0$ is small enough so that there are no conjugate instants in $]0,\varepsilon]$. This convention was motivated by the necessity of avoiding dealing with curves in the Lagrangian Grassmannian with endpoints in the Maslov cycle. An immediate calculation using (3.1) shows that, if g is Lorentzian, the following simple relation holds: $\mathrm{i}_M\big(\gamma|_{[\varepsilon,1]}\big) = \mathrm{i}_M(\gamma) + 1$.

symplectic form ϖ of TM^* at $p \in TM^*$ becomes the canonical symplectic form of $T_xM \oplus (T_xM)^*$; moreover, for all $t \in [0,1]$ we define an isomorphism

$$\begin{aligned}\psi_t : T_{\Gamma(0)}(TM^*) = \mathrm{Hor}_{\Gamma(0)} \oplus \mathrm{Ver}_{\Gamma(0)} \cong T_{\gamma(0)}M \oplus (T_{\gamma(0)}M)^* \\ \longrightarrow T_{\gamma(t)}M \oplus (T_{\gamma(t)}M)^* \cong \mathrm{Hor}_{\Gamma(t)} \oplus \mathrm{Ver}_{\Gamma(t)} = T_{\Gamma(t)}(TM^*)\end{aligned}$$

by setting

$$\psi_t(v,\alpha) = \big(h_t(v), h_t^{*-1}(\alpha)\big),$$

for all $v \in T_{\gamma(0)}M$ and $\alpha \in (T_{\gamma(0)}M)^*$. This is obviously a symplectomorphism for all t, hence we obtain a smooth periodic symplectic trivialization $\Psi = \{\psi_t\}_{t\in[0,1]}$ of $T(TM^*)$ along Γ. It is immediate to observe that the Maslov index $\mathrm{i}_M(\gamma)$ of the geodesic γ coincides with the L_0-Maslov index $\mu_{L_0}(\Gamma,\Psi)$ of the solution Γ associated to the symplectic trivialization Ψ, where L_0 is the Lagrangian subspace $\{0\} \oplus (T_{\gamma(0)}M)^*$ of $T_{\gamma(0)}M \oplus (T_{\gamma(0)}M)^*$.

Lemma 3.8. *Let γ be an orientation preserving closed geodesic in (M,g), and let Γ be the corresponding periodic solution of the geodesic Hamiltonian in TM^*. The L_0-Maslov index $\mu_{L_0}(\Gamma,\Psi)$, where Ψ is the smooth periodic trivialization of $T(TM^*)$ along Γ constructed from a smooth periodic trivialization $\{h_t\}_{t\in[0,1]}$ of TM along γ, as described above, does not depend on the choice of $\{h_t\}_{t\in[0,1]}$.*

Proof. This is an immediate consequence of Lemma 3.5, observing that two distinct trivializations $\{h_t\}$ and $\{\tilde h_t\}$ of TM along γ, with $\eta_t = \tilde h_t \circ h_t \in \mathrm{GL}\big(T_{\gamma(0)}\big)M$, yield periodic symplectic trivializations $\{\psi_t\}$ and $\{\tilde\psi_t\}$ of $T(TM^*)$ along Γ that *differ* by a loop $\{G_t\}$ in $\mathrm{GL}\big(T_{\Gamma(0)}\big)$ of the form

$$G_t = \begin{pmatrix} \eta_t & 0 \\ 0 & \eta_t^{*-1} \end{pmatrix}.$$

By Lemma 3.5, this loop is contractible in $\mathrm{Sp}\big(T_{\Gamma(0)}(TM^*), \varpi_{\Gamma(0)}\big)$, and clearly $G_t[L_0] = L_0$ for all t, which concludes the proof. □

Using the construction above and Corollary 3.7 we obtain immediately

Corollary 3.9. *Let γ be an orientation preserving closed geodesic in (M,g). Then, denoting by $\gamma^{(N)}$ the Nth iterated of γ, $N \geq 1$, the following inequality* *holds:*

$$\left| \mathrm{i}_M\big(\gamma^{(N)}\big) - N \cdot \mathrm{i}_M(\gamma) \right| \le \dim(M)(7N+5).$$

In particular, $\left|\mathrm{i}_M\big(\gamma^{(N)}\big)\right|$ *has sublinear growth in* N*; moreover, if* $\mathrm{i}_M(\gamma) > 7\dim(M)$*, then* $\mathrm{i}_M\big(\gamma^{(N)}\big)$ *has superlinear growth in* N*.* □

4. The variational setup

Let (M, g) be a stationary Lorentzian manifold, and let $\mathcal{Y} \in \mathfrak{X}(M)$ be a timelike Killing vector field in M. Consider the auxiliary Riemannian metric g_{R} on M, defined by

$$g_{\mathrm{R}}(v, w) = g(v, w) - 2\frac{g(v, \mathcal{Y})g(w, \mathcal{Y})}{g(\mathcal{Y}, \mathcal{Y})}; \tag{4.1}$$

observe that $\mathcal{Y}$ is Killing also relatively to g_{R}. Let $\mathbb{S}^1$ be the unit circle, viewed as the quotient $[0,1]/\{0,1\}$, and denote by $\Lambda M = H^1(\mathbb{S}^1, M)$ the infinite dimensional Hilbert manifold of all loops $\gamma : [0,1] \to M$, i.e., $\gamma(0) = \gamma(1)$, of Sobolev class H^1; if $\Lambda^0 M$ is the set of continuous loops in M endowed with the compact-open topology, the inclusion $\Lambda M \hookrightarrow \Lambda^0 M$ is a homotopy equivalence (this can be proved, for instance, using the results in [42]). Set:

$$\mathcal{N} = \Big\{\gamma \in \Lambda M : g(\dot\gamma, \mathcal{Y}) = c_\gamma \text{ (constant) a.e. on } \mathbb{S}^1\Big\}.$$

For all $\gamma \in \Lambda M$, the tangent space $T_\gamma \Lambda M$ is identified with the space of all sections of the pull-back $\gamma^*(TM)$ (i.e., periodic vector fields along γ) of Sobolev class H^1; this space will be endowed with the Hilbert space inner product:

$$\langle\!\langle V, W \rangle\!\rangle = \int_0^1 \Big[g_{\mathrm{R}}(V, W) + g_{\mathrm{R}}\big(\tfrac{D_R}{dt}V, \tfrac{D_R}{dt}W\big)\Big]\, dt, \tag{4.2}$$

where $\frac{D_R}{dt}$ denotes the covariant differentiation along γ relatively to the Levi–Civita connection of the metric g_{R}.

Recall the definition of the classical geodesic energy functional on ΛM:

$$f(\gamma) = \frac{1}{2}\int_0^1 g(\dot\gamma, \dot\gamma)\, dt.$$

4.1. The constrained variational problem

It is well known that the critical points of f in ΛM are exactly the closed geodesics in M; it is also clear that the set $\mathcal{N}$ contains the closed geodesics in M. It is proven that the equality $g(\dot\gamma,\mathcal{Y}) = c_\gamma$ provides a *natural constraint* for the critical points of the geodesic action functional in a stationary Lorentzian manifold; more precisely:

Proposition 4.1. *The following statements hold:*

(1) *$\mathcal{N}$ is a smooth embedded closed submanifold of ΛM, and for $\gamma \in \mathcal{N}$, the tangent space $T_\gamma\mathcal{N}$ is given by the space of sections V of the pull-back $\gamma^*(TM)$ of Sobolev class H^1, satisfying:*

$$g\big(\tfrac{D}{dt}V,\mathcal{Y}\big) - g\big(V,\tfrac{D}{dt}\mathcal{Y}\big) = C_V \quad \text{(constant) a.e. on } [0,1]; \tag{4.3}$$

(2) *if $\mathcal{Y}$ is complete, then $\mathcal{N}$ is a strong deformation retract of ΛM (hence it is homotopy equivalent to ΛM);*

(3) *a curve $\gamma \in \mathcal{N}$ is a critical point of the restriction of f to $\mathcal{N}$ if and only if γ is a critical point of f in ΛM, i.e., if and only if γ is a closed geodesic in (M,g);*

(4) *if γ is a critical point of f, then the Hessian $H^{f|_\mathcal{N}}$ of the restriction $f|_\mathcal{N}$ at γ is given by the restriction of the index form:*

$$I_\gamma(V,W) = \int_0^1 g\big(\tfrac{D}{dt}V,\tfrac{D}{dt}W\big) + g\big(R_{\gamma(t)}(\dot\gamma,V)\,\dot\gamma,W\big)\,dt$$

to the tangent space $T_\gamma\mathcal{N}$;

(5) *if γ is a critical point of f, then the index form I_γ is essentially positive on $T_\gamma\mathcal{N}$, and in particular the Morse index of $f|_\mathcal{N}$ at γ is finite.*

Proof. See [16, 17, 37, 40]. □

It is clear that f does *not* satisfy the Palais–Smale condition in $\mathcal{N}$; namely, all its critical orbits are non-compact.

Given a closed geodesic γ in (M,g), let us denote by $\mu(\gamma)$ the Morse index of $f|_\mathcal{N}$ at γ, i.e., the index of the restriction of I_γ to $T_\gamma\mathcal{N}$. This index will be computed explicitly using the Morse index theorem (Theorem 5.4) in Section 5. Moreover, let us denote by $\mu_0(\gamma)$ the *extended index* if $f|_\mathcal{N}$

at γ, which is the sum of the index $\mu(\gamma)$ and the *nullity* $n(\gamma)$:

$$n(\gamma) = \dim\bigl[\mathrm{Ker}\bigl(I_\gamma|_{T_\gamma\mathcal{N}\times T_\gamma\mathcal{N}}\bigr)\bigr].$$

We will establish in Lemma 5.1 that $n(\gamma)$ equals the dimension of the space of periodic Jacobi fields along γ.

4.2. The Palais–Smale condition

Let us now assume that (M,g) is a globally hyperbolic stationary Lorentzian manifold, that admits a complete timelike Killing vector field $\mathcal{Y}$. Let us recall that, in this situation, (M,g) is a *standard* stationary manifold (see for instance [10, Theorem 2.3]), i.e., denoting by S a smooth Cauchy surface of M, then M is diffeomorphic to a product $S\times\mathbb{R}$, and the Killing field $\mathcal{Y}$ is the vector field ∂_t which is tangent to the fibers $\{x\}\times\mathbb{R}$. One should observe that such product decomposition of M is *not* canonical; however, all Cauchy surfaces of M are homeomorphic. In particular, M is simply connected if and only if S is, and the inclusion of the free loop space $\Lambda S\hookrightarrow\Lambda M$ is a homotopy equivalence.

The projection onto the second factor $S\times\mathbb{R}\to\mathbb{R}$, that will be denoted by T, is a smooth time function, that satisfies

$$\mathcal{Y}(T) = g\bigl(\nabla T,\mathcal{Y}\bigr)\equiv 1 \tag{4.4}$$

on M. If $\mathbf{L}$ denotes the Lie derivative, from (4.4) it follows that $\mathbf{L}_\mathcal{Y}(dT)$ vanishes identically. For, given an arbitrary smooth vector field X on M:

$$\begin{aligned}\mathbf{L}_\mathcal{Y}(dT)(X) &= \mathcal{Y}\bigl(X(T)\bigr) - dt\bigl([\mathcal{Y},X]\bigr)\\ &= \mathcal{Y}\bigl(X(T)\bigr) - \mathcal{Y}\bigl(X(T)\bigr) + X\bigl(\mathcal{Y}(T)\bigr) = 0.\end{aligned}$$

Since $\mathcal{Y}$ is Killing, then $\mathbf{L}_\mathcal{Y}(g)=0$, and (4.4) implies that the Lie bracket $[\mathcal{Y},\nabla T] = \mathbf{L}_\mathcal{Y}\bigl(g^{-1}dT\bigr)$ also vanishes identically. It follows that the quantity $g(\nabla T,\nabla T)$ is constant along the flow lines of $\mathcal{Y}$:

$$\mathcal{Y}\,g(\nabla T,\nabla T) = 2g\bigl(\nabla_\mathcal{Y}\nabla T,\nabla T\bigr) = 2g\bigl([\mathcal{Y},\nabla T]-\nabla_{\nabla T}\mathcal{Y},\nabla T\bigr) = 0.$$

Lemma 4.2. *The restriction of the functional f to $\mathcal{N}$ is bounded from below; more precisely, $f(\gamma)\ge 0$ for all $\gamma\in\mathcal{N}$, and $f(\gamma)=0$ only if γ is a constant curve.*

Proof. Let $\gamma\in\mathcal{N}$ be fixed, and denote by c_γ the value of the constant $g(\dot\gamma,\mathcal{Y})$. For almost all $t\in[0,1]$, the vector $\dot\gamma-g(\dot\gamma,\nabla T)\,\mathcal{Y}$ is (null or) spacelike, namely, using (4.4), one checks immediately that it is orthogonal to the timelike vector ∇T. Hence

$$\begin{aligned}0\le g\big(\dot\gamma-g(\dot\gamma,\nabla T)\,\mathcal{Y},\dot\gamma-g(\dot\gamma,\nabla T)\,\mathcal{Y}\big)&=g(\dot\gamma,\dot\gamma)-2\,c_\gamma\,g\big(\dot\gamma,\nabla T\big)\\&\quad+g(\dot\gamma,\nabla T)^2g(\mathcal{Y},\mathcal{Y}),\end{aligned}\tag{4.5}$$

and thus

$$g(\dot\gamma,\dot\gamma)\ge 2\,c_\gamma\,g\big(\dot\gamma,\nabla T\big)-g(\dot\gamma,\nabla T)^2g(\mathcal{Y},\mathcal{Y}).$$

Integrating on $[0,1]$, and observing that since γ is closed $\int_0^1 g\big(\dot\gamma,\nabla T\big)\,dt=0$, we get:

$$2\,f(\gamma)\ge-\int_0^1 g(\dot\gamma,\nabla T)^2g(\mathcal{Y},\mathcal{Y})\,dt\ge 0.\tag{4.6}$$

Equality in (4.5) holds only if $\dot\gamma-g(\dot\gamma,\nabla T)\,\mathcal{Y}=0$, while, in the last inequality of (4.6), the equal sign holds only if $g(\dot\gamma,\nabla T)=0$ almost everywhere on $[0,1]$. Hence, $f(\gamma)=0$ only if $\dot\gamma=0$ almost everywhere. □

We will assume in the sequel that the Cauchy surface S is compact; recall that any two Cauchy surfaces of a globally hyperbolic spacetime are homeomorphic.

Lemma 4.3. *The metric g_{R} is complete, and thus ΛM and $\mathcal{N}$ are complete Hilbert manifolds when endowed with the Riemannian structure* (4.2).

Proof. The flow of $\mathcal{Y}$ preserves g_{R}; each orbit of the induced $\mathbb{R}$-action meets the compact subset S, and the conclusion follows easily. □

The flow of the Killing vector field $\mathcal{Y}$ gives an isometric action of $\mathbb{R}$ in ΛM, defined by $\mathbb{R}\times\Lambda M\ni(t,\gamma)\mapsto\mathcal{F}_t\circ\gamma\in\Lambda M$. This action preserves $\mathcal{N}$, and the functional f is invariant by this action; the orbit of a critical point of f consists of a collection of critical points of f with the same Morse index. Such action is obviously free, and the quotient $\widetilde{\mathcal{N}}=\mathcal{N}/\mathbb{R}$ has the structure of a smooth manifold such that the product $\widetilde{\mathcal{N}}\times\mathbb{R}$ is diffeomorphic to $\mathcal{N}$. For $\gamma\in\mathcal{N}$, we will denote by $[\gamma]$ its class in the quotient $\widetilde{\mathcal{N}}$; the tangent space $T_{[\gamma]}\widetilde{\mathcal{N}}$ can be identified with

$$T_{[\gamma]}\widetilde{\mathcal{N}} \cong \frac{T_\gamma\mathcal{N}}{E_\gamma}, \tag{4.7}$$

where E_γ is the 1-dimensional space of vector fields spanned by the restriction of $\mathcal{Y}$ to γ. If S is a Cauchy surface in M, then $\widetilde{\mathcal{N}}$ can also be identified with the set

$$\widetilde{\mathcal{N}} = \big\{\gamma \in \mathcal{N} : \gamma(0) \in S\big\}; \tag{4.8}$$

using this identification, for $\gamma \in \widetilde{\mathcal{N}}$ is given by

$$T_\gamma\widetilde{\mathcal{N}} = \big\{V \in T_\gamma\mathcal{N} : V(0) \in T_{\gamma(0)}S\big\}. \tag{4.9}$$

Obviously, the quotient $\widetilde{\mathcal{N}}$ inherits an isometric action of $O(2)$; it should be observed that, if one uses the identification (4.8), then the action of an element in $O(2)$ is not simply a rotation in the parameter space, but a rotation followed by a translation along the flow of $\mathcal{Y}$.

The function f defines by quotient a smooth function on $\widetilde{\mathcal{N}}$, that will still be denoted by f, and for which the statement of Proposition 4.1 holds *verbatim*. In addition, f satisfies the PS condition on $\widetilde{\mathcal{N}}$.

Proposition 4.4. *$\widetilde{\mathcal{N}}$ is a complete Hilbert manifold, which is homotopically equivalent to $\mathcal{N}$ and to ΛM. The critical points of the functional f in $\widetilde{\mathcal{N}}$ correspond to orbits*

$$[\gamma] = \{\mathcal{F}_t \circ \gamma\}_{t\in\mathbb{R}},$$

where γ is a closed geodesic in M; the Morse index of each critical point $[\gamma]$ of f equals the Morse index of γ, while the nullity of $[\gamma]$ equals $\mathrm{n}(\gamma) - 1$. *Moreover, f satisfies the Palais–Smale condition in $\widetilde{\mathcal{N}}$.*

Proof. Most part of the statement is a direct consequence of the construction of $\widetilde{\mathcal{N}}$. The statement on the Morse index and the nullity of a critical point $[\gamma]$ is obtained easily, observing that the 1-dimensional space E_γ in formula (4.7) is contained in the kernel of the index form I_γ (see Lemma 5.1 below). The Palais–Smale condition is essentially the same as in [36, Lemma 3.2]; we will sketch here a more intrinsic proof along the lines of [10, 17]. Using [17, Section 5] and the compactness of S, for the PS condition it suffices to show that f is *pseudo-coercive on* $\widetilde{\mathcal{N}}$, i.e., that given a sequence $(\gamma_n)_{n\in\mathbb{N}}$ in $\widetilde{\mathcal{N}}$ such that $f(\gamma_n)$ is bounded, then γ_n admits a uniformly convergent subsequence. Using the identification (4.8), let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $\widetilde{\mathcal{N}}$ such that $f(\gamma_n) \le c$ for all n; we claim that the real sequence $c_{\gamma_n} = g(\dot\gamma_n, \mathcal{Y})$ is bounded. Namely, the vector field $\dot\gamma_n - c_{\gamma_n}\nabla T$ along γ_n is a.e. spacelike or null for all n, because it is a.e. orthogonal to the timelike vector field $\mathcal{Y}$. Hence,

$$\int_0^1 g\big(\dot\gamma_n - c_{\gamma_n}\nabla T, \dot\gamma_n - c_{\gamma_n}\nabla T\big)\,dt = 2f(\gamma_n) + c_{\gamma_n}^2 \int_0^1 g(\nabla T, \nabla T)\,dt \ge 0,$$

that gives

$$c_{\gamma_n}^2 \le 2f(\gamma_n)\left(\int_0^1 -g(\nabla T, \nabla T)\,dt\right)^{-1}.$$

Observe that the functions $g(\mathcal{Y}, \mathcal{Y})$ and $g(\nabla T, \nabla T)$ admit minimum and maximum in M, because they are constant along the flow lines of $\mathcal{Y}$, and because S is compact. The claim on the boundedness of c_{γ_n} follows. From this, it follows that the sequence

$$\int_0^1 g_{\mathrm{R}}(\dot\gamma_n, \dot\gamma_n)\,dt = 2f(\gamma_n) - 2c_{\gamma_n}^2 \int_0^1 g(\mathcal{Y}, \mathcal{Y})^{-1}\,dt$$

is bounded. Since g_{R} is complete and S is compact, the theorem of Arzelà–Ascoli implies that, up to subsequences, γ_n is uniformly convergent in M. This concludes the proof. □

From Lemma 4.2 and Proposition 4.4, one obtains the existence of one non-trivial closed geodesic in M, as proved in [36]. Namely, using the theory of Ljusternik and Schnirelman, one shows the existence of a sequence $\big([\gamma_r]\big)_{r\ge1}$ of critical points of $f|_{\widetilde{\mathcal{N}}}$ with $f(\gamma_r) \to \infty$. Thus, these critical points are not constant curves; observe however that the Ljusternik–Schnirelman theory does not give information on whether such curves are geometrically distinct. In the non-simply connected case, the following result follows immediately:

Corollary 4.5. *Let (M, g) be a Lorentzian manifold that admits a complete timelike Killing vector field and a compact Cauchy surface. Then, there is a closed geodesic in each free homotopy class of M.* □

Remark 4.6. The orthogonal group $O(2)$ acts isometrically on ΛM via the operation of $O(2)$ on the parameter circle $\mathbb{S}^1$. It is easy to observe that the stabilizer of each $\gamma \in \Lambda M$ with respect to this action is a finite cyclic subgroup of $\mathrm{SO}(2)$ generated by the rotation of $\frac{2\pi}{N}$, for some $N \ge 1$. A closed

$\gamma \in \Lambda M$ will be called *prime* if its stabilizer in $O(2)$ is trivial, i.e., if γ is not the Nth iterate of some other curve in ΛM with $N > 1$. The functional f defined in ΛM is invariant by the action of $O(2)$; moreover, this action leaves $\mathcal{N}$ invariant, and it commutes with the time translations ($\mathcal{F}_t \circ$). We therefore get an equivariant and isometric action of $O(2)$ on the manifold $\widetilde{\mathcal{N}}$ by $g \cdot [\gamma] = \big[g \cdot \gamma\big]$, $g \in O(2)$. An element $[\gamma] \in \widetilde{\mathcal{N}}$ will be called prime if γ is prime, in which case its orbit $O(2) \cdot [\gamma]$ will contain only prime curves. The existence of infinitely many geometrically distinct (in the sense of the definition given in the Introduction) closed geodesics in M is equivalent to the existence of infinitely many distinct prime critical $O(2)$-orbits of f in $\widetilde{\mathcal{N}}$.

It will be useful to prove the following two results:

Lemma 4.7. *If (M, g) has only a finite number of geometrically distinct closed geodesics, then the critical orbits of f in $\widetilde{\mathcal{N}}$ are isolated.*

Proof. If $\gamma_1, \ldots, \gamma_r$ is a maximal set of pairwise geometrically distinct closed geodesics in M with $\gamma_j(0) \in S$ for all j, then the critical orbits of f in $\widetilde{\mathcal{N}}$ is the countable set formed by all the iterates $O(2)\big[\gamma_j^{(N)}\big]$, $j = 1, \ldots, r$, $N \geq 1$; observe that $f(\gamma_j) > 0$ for all j. Any sequence $k \mapsto \gamma_{j_k}^{(N_k)}$ of pairwise distinct iterates of the γ_js would necessarily have $N_k \to \infty$, hence $f(\gamma_{j_k}^{(N_k)}) \to +\infty$. In particular, no subsequence of such sequence could have a converging subsequence in $\widetilde{\mathcal{N}}$. The group $O(2)$ is compact, and the conclusion follows easily. ☐

Let S be a Cauchy surface in (M, g); we will use the identification (4.8) to prove the existence of a strong deformation retract from the ε-sublevel of f in $\widetilde{\mathcal{N}}$ to the set of constant curves in S.

Lemma 4.8. *For $\varepsilon > 0$ small enough, the closed ε-sublevel of f in $\widetilde{\mathcal{N}}$:*

$$f^\varepsilon = \big\{[\gamma] \in \widetilde{\mathcal{N}} : f(\gamma) \leq \varepsilon\big\}$$

is homotopically equivalent to (the set of constant curves in) S.

Proof. Let us show that the map $f^\varepsilon \ni \gamma \mapsto \gamma(0) \in S$ is a deformation retract. By part (4.1) of Proposition 4.1, it suffices to show that there exists a continuous map $\Phi : f^\varepsilon \times [0,1] \to \Lambda M$ with $\Phi(\gamma, 0) = \gamma$ and $\Phi(\gamma, 1)$ equal to the constant curve $\gamma(0)$. To this aim, consider the auxiliary Riemannian metric h on M defined by $h(v, w) = g(v, w) - 2g(v, \nabla T)g(w, \nabla T)g(\nabla T, \nabla T)^{-1}$. Recalling that the functions $g(\mathcal{Y}, \mathcal{Y})$ and $g(\nabla T, \nabla T)$ admit minimum in

M, set $a_0 = \min\big[-g(\mathcal{Y},\mathcal{Y})\big] > 0$ and $b_0 = \min\big[-g(\nabla T,\nabla T)\big] > 0$. From (4.6), if $[\gamma]\in f^\varepsilon$, then

$$\int_0^1 g(\dot\gamma,\nabla T)^2\,dt \le 2\varepsilon a_0^{-1},$$

and thus

(4.10)
$$\int_0^1 h(\dot\gamma,\dot\gamma)\,dt = \int_0^1 \Big[g(\dot\gamma,\dot\gamma) - 2g(\dot\gamma,\nabla T)^2 g(\nabla T,\nabla T)^{-1}\Big]\,dt \le \varepsilon\left(1+\frac{2}{a_0b_0}\right).$$

Using the Cauchy–Schwartz inequality, we get that the h-length of every curve $\gamma\in f^\varepsilon$ is less than or equal to $\varepsilon\left(1+\frac{2}{a_0b_0}\right)$. Let $\rho_0>0$ be the minimum on the compact manifold S of the radius of injectivity of the Riemannian metric h; choose a positive $\varepsilon<\rho_0\left(\frac{a_0b_0}{a_0b_0+2}\right)$; if γ is a curve in f^ε and $t\in[0,1]$, then the h-distance between $\gamma(t)$ and $\gamma(0)$ is less than ρ_0. The required deformation retract Φ is given by setting $\Phi(\gamma,s)(t) = c(s)$, where $c:[0,1]\to M$ is the unique affinely parameterized minimal h-geodesic from $\gamma(0)$ to $\gamma(t)$. □

5. The Morse index theorem

In this section, we will prove an index theorem for closed geodesics in a stationary Lorentzian manifold with arbitrary endpoints, generalizing the result in [40]. The result is now obtained as a corollary of Theorem 2.6 together with the semi-Riemannian Morse index theorem for fixed endpoints geodesics proved in [18]. An earlier version of the theorem was proven in [40] for the non-degenerate case, under the further assumption that the closed geodesic be orientation preserving. The use of Theorem 2.6 allows to get rid of both these extra assumptions at the same time.

5.1. The index theorem

Let us consider a closed geodesic γ in M; it is easy to check that $T_\gamma\mathcal{N}$ contains the space of all Jacobi fields J along γ such that $J(0)=J(1)$. The following lemma tells us that γ is a non-degenerate critical point of f if and only if it is a non-degenerate critical point of $f|_\mathcal{N}$.

Lemma 5.1. *Let γ be a critical point of $f|_\mathcal{N}$, i.e., a closed geodesic in M. Then, the kernel of the index form I_γ in $T_\gamma\mathcal{N}$ coincides with the Kernel of*

I_γ in $T_\gamma\Lambda M$, and it is given by the space of periodic Jacobi fields along γ:

$$\mathrm{Ker}\big(I_\gamma|_{T_\gamma\mathcal{N}\times T_\gamma\mathcal{N}}\big) = \Big\{J \text{ Jacobi field along } \gamma : J(0)=J(1),\ \tfrac{D}{dt}J(0)=\tfrac{D}{dt}J(1)\Big\}.$$

Moreover, consider the following closed subspace $\mathcal{W}_\gamma \subset T_\gamma\mathcal{N}$:

$$\mathcal{W}_\gamma = \Big\{V\in T_\gamma\mathcal{N} : V(0)=V(1)=0\Big\}.$$

Then, the I_γ-orthogonal space of $\mathcal{W}_\gamma$ in $T_\gamma\mathcal{N}$ is given by

$$\mathcal{S}_\gamma = \Big\{J \text{ Jacobi field along } \gamma : J(0)=J(1)\Big\}.$$

Proof. The statement on the kernel of I_γ is proved readily using the following two facts:

(a) $T_\gamma\Lambda M = T_\gamma\mathcal{N} + \mathfrak{Y}$, where $\mathfrak{Y}$ is the space of vector fields in $T_\gamma\Lambda M$ that are pointwise multiple of the Killing field $\mathcal{Y}$;

(b) $\mathfrak{Y}$ is contained in the I_γ-orthogonal complement of $T_\gamma\mathcal{N}$ in $T_\gamma\Lambda M$.

In order to prove (a), simply observe that for any $W\in T_\gamma\Lambda M$, then the vector field V along γ defined below belongs to $T_\gamma\mathcal{N}$

$$V(t) = W(t) + \lambda_W(t)\cdot\mathcal{Y}\big(\gamma(t)\big),\quad t\in[0,1],$$

where

$$\lambda(t) = \int_0^t \frac{C_W + g\big(W,(D/dt)\mathcal{Y}\big) - g\big((D/dt)W,\mathcal{Y}\big)}{g(\mathcal{Y},\mathcal{Y})}\,ds,$$

and

$$C_W = \left[\int_0^1 \frac{g\big((D/dt)W,\mathcal{Y}\big) - g\big(W,(D/dt)\mathcal{Y}\big)}{g(\mathcal{Y},\mathcal{Y})}\,ds\right]\cdot\left(\int_0^1\frac{ds}{g(\mathcal{Y},\mathcal{Y})}\right)^{-1}.$$

Part (b) is a simple partial integration calculation, which is omitted; similarly, the last part of the statement is obtained by an immediate calculation using the fundamental lemma of calculus of variations. □

Remark 5.2. In the case of a periodic geodesic γ the index form I_γ is always degenerate, being the tangent field $\dot\gamma$ in its kernel. Moreover, also the restriction of the Killing field $\mathcal{Y}$ to γ is a non-trivial Jacobi field in $\mathrm{Ker}\big(I_\gamma|_{T_\gamma\mathcal{N}\times T_\gamma\mathcal{N}}\big)$. Thus, $\dim\big[\mathrm{Ker}\big(I_\gamma|_{T_\gamma\mathcal{N}\times T_\gamma\mathcal{N}}\big)\big]\geq 2$.

Remark 5.3. If S is a Cauchy surface in (M,g), using the identifications (4.8) and (4.9), the null space of the Hessian of $f|_{\widetilde{\mathcal{N}}}$ at $[\gamma]$ is given by the space of periodic Jacobi fields J along γ such that $J(0)\in T_{\gamma(0)}S$. The tangent space $T_{[\gamma]}\big(O(2)[\gamma]\big)$ is given by the space of all constant multiples of the periodic Jacobi vector field J along γ given by $J(t)=\dot\gamma(t)+\alpha\mathcal{Y}\big(\gamma(t)\big)$, where $\alpha\in\mathbb{R}$ is such that $J(0)\in T_{\gamma(0)}S$.

By Lemma 5.1, the nullity $\mathrm{n}(\gamma)$ is equal to the dimension of the space of periodic Jacobi fields J along γ.

Theorem 5.4 (Morse index theorem for closed geodesics with arbitrary endpoints). *Let $\gamma:[0,1]\to M$ be a closed geodesic in M. Then, the Morse index $\mu(\gamma)$ of $f|_{\mathcal{N}}$ at γ is given by*

$$\mu(\gamma)=\mathfrak{i}_M(\gamma)+1+n_-(B_0)-n_1, \tag{5.1}$$

where B_0 is the symmetric bilinear form on the finite dimensional vector space $\mathcal{S}_\gamma$ given by

$$B_0(J_1,J_2)=g\big(\tfrac{D}{dt}J_1(0),J_2(0)\big), \tag{5.2}$$

and n_1 is the dimension of the vector space

$$\mathcal{W}_\gamma\cap\mathrm{Ker}(I_\gamma)=\Big\{J \textit{ Jacobi field along } \gamma : J(0)=J(1)=0,\ \tfrac{D}{dt}J(0)=\tfrac{D}{dt}J(1)\Big\}.$$

Proof. Formula (5.1) follows from Theorem 2.6 applied to the index form I_γ and the closed spaces $\mathcal{W}_\gamma$ and $\mathcal{S}_\gamma$ introduced in Lemma 5.1. One has:

$$n_-\big(I_\gamma|_{\mathcal{W}_\gamma\times\mathcal{W}_\gamma}\big)=\mathfrak{i}_M(\gamma)+1-n_0,$$

where n_0 is the dimension of the vector space:

$$\mathcal{W}_\gamma\cap\mathcal{S}_\gamma=\big\{J \text{ Jacobi field along } \gamma : J(0)=J(1)=0\big\}.$$

Such equality is given by the Morse index theorem for fixed endpoints geodesics, which is proved in [16] in the non-degenerate case and in [18] for the general case.[7] In order to apply the result of [18], one needs to observe that the extended index (i.e., index plus nullity) of I_γ in $\mathcal{W}_\gamma$ is equal to the *spectral flow* of the path of Fredholm symmetric bilinear forms

[7] Recall also that the definition of Maslov index $\mathfrak{i}_M(\gamma)$ employed here differs by 1 from the definition in [18].

$[0,1] \ni s \mapsto I_{\gamma|_{[0,s]}}$ defined on the space of fixed endpoints variational vector fields along $\gamma|_{[0,s]}$. This follows easily from the fact that I_γ is negative semi-definite on the space $\mathfrak{Y}$ defined in the proof of Lemma 5.1. An immediate partial integration argument shows that the restriction of I_γ to $\mathcal{S}_\gamma$ is given by (5.2), and equality (5.1) follows readily. $\square$

Observe that the following inequalities hold:

$$
\begin{gathered}
0 \le n_1 \le \dim(M), \quad 0 \le n_-(B_0) \le \dim(M), \quad 0 \le n(\gamma) \le \dim(M), \\
0 \le n_-(B_0) + n_0 - n_1 \le \dim(M).
\end{gathered}
\tag{5.3}
$$

Inequality (5.3) is obtained easily using (2.1) and (2.2), and observing that $\mathcal{W}_\gamma$ has codimension equal to $\dim(M)$ in $T_\gamma\mathcal{N}$.

5.2. Morse index of an iteration

Throughout this subsection, we will consider a fixed critical point γ of $f|_\mathcal{N}$. Given an integer $N \ge 1$, let us denote by $\gamma^{(N)}$ the *N-iterated of* γ, defined by $\gamma^{(N)}(t) = \widetilde{\gamma}(Nt)$ for all $t \in [0,1]$, where $\widetilde{\gamma} : \mathbb{R} \to M$ is the periodic extension of γ. Observe that $\gamma^{(N)}$ is a critical point of $f|_\mathcal{N}$ for all $N \ge 1$. One of the central results of this paper will be to establish the growth of the sequence $\mu\big(\gamma^{(N)}\big)$ (Proposition 5.7 and Corollary 5.8). The result will be first established for orientation preserving closed geodesics and then extended to the general case using Lemma 5.5 below.

Although it is not clear at all whether the Morse index of a closed geodesic increases by iteration, an argument using a finite codimensional restriction of the index form yields the following interesting consequence.

Lemma 5.5. *There exists a bounded sequence of integers $(d_N)_{N\ge1}$ such that the sequence $N \mapsto \mu\big(\gamma^{(N)}\big) + d_N \in \mathbb{Z}$ is non-decreasing.*

Proof. Let us introduce the following space

$$
\mathcal{W}_\gamma^o = \Big\{ V \in \mathcal{W}_\gamma : g\big(\tfrac{D}{dt}V, \mathcal{Y}\big) - g\big(V, \tfrac{D}{dt}\mathcal{Y}\big) \equiv 0 \Big\}.
\tag{5.4}
$$

Clearly, $\mathcal{W}_\gamma^o$ is a 1-codimensional closed subspace of $\mathcal{W}_\gamma$, being the kernel of the bounded linear functional $\mathcal{W}_\gamma \ni V \mapsto C_V \in \mathbb{R}$ (see (4.3)). Hence, recalling (2.1) and (2.2):

$$n_-\big(I_\gamma|_{\mathcal{W}^o_\gamma\times\mathcal{W}^o_\gamma}\big) \le n_-\big(I_\gamma|_{\mathcal{W}_\gamma\times\mathcal{W}_\gamma}\big) \le n_-\big(I_\gamma|_{\mathcal{W}^o_\gamma\times\mathcal{W}^o_\gamma}\big) + 1.$$

Thus, keeping in mind formula (5.1) and inequality (5.3), in order to prove the lemma it suffices to show that the sequence $\bar\mu\big(\gamma^{(n)}\big)$ is non-decreasing, where

$$\bar\mu(\gamma) = n_-\big(I_\gamma|_{\mathcal{W}^o_\gamma\times\mathcal{W}^o_\gamma}\big). \tag{5.5}$$

To this aim, let $1 \le N \le M$ be given, and consider the map

$$\mathcal{E}_{N,M} : \mathcal{W}^o_{\gamma^{(N)}} \longrightarrow \mathcal{W}^o_{\gamma^{(M)}}$$

defined by $\mathcal{E}_{N,M}(V) = \widetilde V$, where

$$\widetilde V(t) = \begin{cases} V(tM/N) & \text{if } t \in [0, N/M]; \\ 0 & \text{if } t \in \,]N/M, 1]. \end{cases}$$

Obviously, $\mathcal{E}_{N,M}$ is an injective bounded linear map; an immediate computation shows that the following equality holds:

$$I_{\gamma^{(M)}}\big(\mathcal{E}_{N,M}(V), \mathcal{E}_{N,M}(W)\big) = \tfrac{M}{N}\, I_{\gamma^{(N)}}(V, W), \quad \forall\, V, W \in \mathcal{W}^o_{\gamma^{(N)}}. \tag{5.6}$$

Hence, if $\mathcal{V} \subset \mathcal{W}^o_{\gamma^{(N)}}$ is a subspace such that

$$\dim(\mathcal{V}) = n_-\big(I_\gamma|_{\mathcal{W}^o_{\gamma^{(N)}}\times\mathcal{W}^o_{\gamma^{(N)}}}\big)$$

and such that $I_{\gamma^{(N)}}$ is negative definite on $\mathcal{V}$, then $\dim\big(\mathcal{E}_{N,M}(\mathcal{V})\big) = \dim(\mathcal{V})$, and by (5.6) $I_{\gamma^{(M)}}$ is negative definite on $\mathcal{E}_{N,M}(\mathcal{V})$. This shows that $\bar\mu\big(\gamma^{(N)}\big) \le \bar\mu\big(\gamma^{(M)}\big)$ and concludes the proof. □

It will be useful to record here the following relation between the Morse index $\mu(\gamma)$, the Maslov index $\mathrm{i}_M(\gamma)$ and the *restricted Morse index* $\bar\mu(\gamma)$ (see (5.5)) of a closed geodesic γ:

$$\bar\mu(\gamma) \le \mathrm{i}_M(\gamma) \le \mu(\gamma);$$

more precisely

$$\begin{aligned} \mu(\gamma) &= \mathrm{i}_M(\gamma) + A_\gamma, \quad 0 \le A_\gamma \le \dim(M) - 1, \\ \mathrm{i}_M(\gamma) &= \bar\mu(\gamma) + B_\gamma, \quad 0 \le B_\gamma \le 1. \end{aligned} \tag{5.7}$$

Exploiting the same idea in Lemma 5.5, one has the following result on the additivity of the Morse index.

Lemma 5.6. *There exists a bounded sequence $(e_N)_{N\geq 1}$ of non-negative integers such that for all $r, s > 0$, the following inequality holds:*

$$\mu\big(\gamma^{(r+s)}\big) \geq \mu\big(\gamma^{(r)}\big) + \mu\big(\gamma^{(s)}\big) - e_r - e_s. \tag{5.8}$$

Proof. As in the proof of Lemma 5.5, the sequence $\bar\mu\big(\gamma^{(N)}\big)$ satisfies

$$\bar\mu\big(\gamma^{(r+s)}\big) \geq \bar\mu\big(\gamma^{(r)}\big) + \bar\mu\big(\gamma^{(s)}\big);$$

the conclusion follows easily using (5.7) and setting $e_N = A_{\gamma^{(N)}} + B_{\gamma^{(N)}} \leq \dim(M)$. □

Finally, we have our aimed results on the growth of the Maslov index.

Proposition 5.7. *Given any closed geodesic γ in M, the sequence of Morse indices $N \mapsto \mu\big(\gamma^{(N)}\big)$ is either bounded (by a constant depending only on the dimension of M), or it has superlinear growth in N for large N.*

Proof. Assume first that γ is orientation preserving, and that $\mu\big(\gamma^{(N)}\big)$ is not bounded. Let $k_* \in \mathbb{N}$ be the first positive integer such that

$$\mu\big(\gamma^{(k_*)}\big) > 8\dim(M) + 1.$$

Using Theorem 5.4, the non-increasing property of the restricted Morse index proved in Lemma 5.5 and formulas (5.7), for $m \geq k_*$, we compute as follows:

$$\begin{aligned}
\mu\big(\gamma^{(m)}\big) = \bar\mu\big(\gamma^{(m)}\big) + A_{\gamma^{(m)}} + B_{\gamma^{(m)}} &\overset{\text{Lemma 5.5}}{\geq} \bar\mu\big(\gamma^{(\lfloor \frac{m}{k_*}\rfloor k_*)}\big) + A_{\gamma^{(m)}} + B_{\gamma^{(m)}} \\
&\overset{\text{by (5.7)}}{\geq} \mathrm{i}_M\big(\gamma^{(\lfloor \frac{m}{k_*}\rfloor k_*)}\big) - 1 + A_{\gamma^{(m)}} + B_{\gamma^{(m)}} \\
&\overset{\text{Corollary 3.9}}{\geq} \Big(\mathrm{i}_M\big(\gamma^{(k_*)}\big) - 7\dim(M)\Big)\cdot\left\lfloor\frac{m}{k_*}\right\rfloor - 5\dim(M) \\
&\qquad - 1 + A_{\gamma^{(m)}} + B_{\gamma^{(m)}} \\
&\overset{\text{by (5.7)}}{\geq} \left(\frac{\mu\big(\gamma^{(k_*)}\big) - 8\dim(M) - 1}{k_*}\right)\cdot m - \mu\big(\gamma^{(k_*)}\big) + 2\dim(M) \\
&\qquad - 1 + A_{\gamma^{(m)}} + B_{\gamma^{(m)}}.
\end{aligned}$$

Here, $\lfloor\cdot\rfloor$ denotes the integer part function. The conclusion follows, recalling from formulas (5.7), that $A_{\gamma^{(m)}}$ and $B_{\gamma^{(m)}}$ are bounded sequences.

For the general case of possibly non-orientation preserving closed geodesics, observe that the double iterate $\gamma^{(2)}$ of any closed geodesic is orientation preserving. Observe also that, by Lemma 5.5, the sequence $\mu\big(\gamma^{(N)}\big)$ is bounded if and only if $\mu\big(\gamma^{(2N)}\big)$ is bounded. Based on these observations and on Lemma 5.5, establishing the superlinear growth of $\mu\big(\gamma^{(N)}\big)$ in the non-orientable case is obtained by elementary arithmetics from the previous case. □

We will need a slightly refined property on the growth of the Morse index, which is some sort of uniform superlinear growth.

Corollary 5.8. *Let γ be a closed geodesic in M such that $\mu\big(\gamma^{(N)}\big)$ is not bounded. Then, there exist positive constants $\bar\alpha, \bar\beta \in \mathbb{R}$ such that, for s sufficiently large, the following inequality holds:*

$$\mu\big(\gamma^{(r+s)}\big) \geq \mu\big(\gamma^{(r)}\big) + s\,\bar\alpha - \bar\beta, \quad \forall\, r > 0. \tag{5.9}$$

Proof. Let k_* be as in the proof of Proposition 5.7, and set

$$\bar\alpha = \frac{\mu\big(\gamma^{(k_*)}\big) - 8\dim(M) - 1}{k_*}, \quad \bar\beta = \mu\big(\gamma^{(k_*)}\big) + 1.$$

For $s \geq k_*$, inequality (5.9) follows readily from Lemma 5.6 and Proposition 5.7. □

5.3. Nullity of an iteration

The nullity of an iterated closed geodesic γ will be computed using the spectrum of the linearized Poincaré map $\mathfrak{P}_\gamma$ defined below. Given a closed geodesic $\gamma : [0,1] \to M$, denote by $\mathcal{V}$ the space $T_{\gamma(0)}M \oplus T_{\gamma(0)}M^*$, endowed with its canonical symplectic structure, and let $\mathfrak{P}_\gamma : \mathcal{V} \to \mathcal{V}$ be the linear map defined by:

$$\mathfrak{P}_\gamma\big(J(0), g\tfrac{D}{dt}J(0)\big) = \big(J(1), g\tfrac{D}{dt}J(1)\big),$$

where J is a Jacobi field along γ. The map $\mathfrak{P}_\gamma$ is a symplectomorphism of $\mathcal{V}$; denote by $\mathfrak{s}(\mathfrak{P}_\gamma)$ its spectrum. It follows from Lemma 5.1 that Ker $\big(I_\gamma|_{T_\gamma\mathcal{N}\times T_\gamma\mathcal{N}}\big)$ consists of all Jacobi fields J along γ such that $\big(J(0), g(D/dt)\, J(0)\big)$ belongs to the 1-eigenspace of $\mathfrak{P}_\gamma$. The subspace of $\mathcal{V}$ spanned by $\big(\gamma'(0), 0\big)$ and by $\big(\mathcal{Y}(\gamma(0)), g\nabla_{\gamma'(0)}\mathcal{Y}\big)$ is a 2-dimensional isotropic subspace of $\mathrm{Ker}(\mathfrak{P}_\gamma)$. From Proposition 4.4, it follows that $O(2)[\gamma]$ is a non-degenerate critical orbit of f in $\widetilde{\mathcal{N}}$ when $\dim\big[\mathrm{Ker}\big(I_\gamma|_{T_\gamma\mathcal{N}\times T_\gamma\mathcal{N}}\big)\big] = 2$. We have a result on the nullity of an iteration, which is totally analogous to the Riemannian case (see [21, Lemma 2] and [31, Proposition 4.2.6]); its proof, repeated here for the reader's convenience, is purely arithmetical.

Lemma 5.9. *Let γ be a closed geodesic in M and let $\gamma^{(N)}$ denote its Nth iterate, $N \geq 1$. Then, $O(2)\big[\gamma^{(N)}\big]$ is a non-degenerate critical orbit of f in $\widetilde{\mathcal{N}}$ if and only if:*

(a) $O(2)[\gamma]$ *is a non-degenerate critical orbit of f in $\widetilde{\mathcal{N}}$;*

(b) $\mathfrak{s}(\mathfrak{P}_\gamma) \setminus \{1\}$ *does not contain any Nth root of unity.*

Moreover, there exists a sequence $m_1, \ldots, m_s$ of positive integers, $s \leq 2^{\dim(M)}$, and, for each $j \in \{1, \ldots, s\}$, a strictly increasing sequence $q_{j1} < q_{j2} < \cdots < q_{jm} < \cdots$ of positive integers such that the sets $N_j = \big\{m_j q_{ji}, i = 1, 2, \ldots\big\}$ form a partition of $\mathbb{N} \setminus \{0\}$, and such that

$$n\big(\gamma^{m_j q_{ji}}\big) = n\big(\gamma^{m_j}\big), \quad \forall\, i \in \mathbb{N}. \tag{5.10}$$

Proof. The first statement is proved easily observing that $\mathfrak{P}_\gamma^N = \mathfrak{P}_{\gamma^{(N)}}$.

For the second statement, consider all the elements in $\mathfrak{s}(\mathfrak{P}_\gamma)$ of the form $e^{\pm 2\pi(p/q)i}$, with p, q positive integers and relatively prime. Let D the possibly empty set of all these denominators, and for all $E \subset D$ denote by $m(E)$ the least common multiple of all elements of E, setting $m(\emptyset) = 1$. Denote by $m_1, \ldots, m_s$ the set of all pairwise distinct numbers obtained as $m(E)$, for all subsets $E \subset D$, where $m_1 = 1$. Clearly, $s \leq 2^{\dim(M)}$. Finally, for all $j \in \{1, \ldots, s\}$, consider a maximal sequence $\{q_{ji}, i \geq 1\}$ of positive integers such that none of the m_k, with $k \neq j$, divides $m_j q_{ji}$. Then, (5.10) holds; furthermore, every $m \in \mathbb{N} \setminus \{0\}$ can be written as the product $m_j q$, where q is a positive integer, and m_j is some divisor of m among the elements $m_1, \ldots, m_s$. If m_j is the maximum of such divisors, then q must be one of the q_{ji}, for some $i \geq 1$. This concludes the proof. □

Remark 5.10. By Lemma 5.9, we have the following situation. Assuming that there is only a finite number of geometrically distinct closed geodesics in M, it is possible to find a finite number of closed geodesics $\gamma_1, \ldots, \gamma_r$ (possibly not all geometrically distinct) such that any closed geodesics γ in M is geometrically equivalent to some iterate $\gamma_{i_0}^{(N)}$ of one of the γ_is, and it has the same nullity as γ_{i_0}.

6. Equivariant Morse theory for the action functional

6.1. Abstract Morse relations

Given sequences $(\mu_k)_{k\geq 0}$ and $(\beta_k)_{k\geq 0}$ in $\mathbb{N}\bigcup\{+\infty\}$, we will say that the sequence of pairs $(\mu_k,\beta_k)_{k\geq 0}$ satisfies the *Morse relations* if there exists a formal power series $Q(t)=\sum_{k\geq 0} q_k\, t^k$ with coefficients in $\mathbb{N}\bigcup\{+\infty\}$ such that

$$\sum_{k\geq 0}\mu_k\, t^k=\sum_{k\geq 0}\beta_k\, t^k+(1+t)Q(t).$$

This condition implies (and, in fact it is equivalent to if all μ_ks are finite) the familiar set of inequalities:

$$\begin{aligned}
\mu_0 &\geq \beta_0,\\
\mu_1-\mu_0 &\geq \beta_1-\beta_0\\
\mu_2-\mu_1+\mu_0 &\geq \beta_2-\beta_1+\beta_0,\\
&\vdots\\
\mu_k-\mu_{k-1}+\cdots+(-1)^k\mu_0 &\geq \beta_k-\beta_{k-1}+\cdots+(-1)^k\beta_0,\\
&\cdots
\end{aligned}$$

that are called the *strong Morse inequalities.* In turn, these inequalities imply the *weak Morse inequalities*:

$$\mu_k\geq\beta_k,\quad \forall\, k\geq 0. \tag{6.1}$$

Given a pair $Y\subset X$ of topological space and a coefficient field $\mathbb{K}$, let us denote by $H_k(X,Y;\mathbb{K})$ the kth relative homology vector space with coefficients in $\mathbb{K}$, and by $\beta_k(X,Y;\mathbb{K})=\dim\big(H_k(X,Y;\mathbb{K})\big)$ the kth Betti number of the pair. We set $H_k(X;\mathbb{K})=H_k(X,\emptyset;\mathbb{K})$ and $\beta_k(X;\mathbb{K})=\beta_k(X,\emptyset;\mathbb{K})$. Using standard homological techniques, one proves the following:

Proposition 6.1. *Let $\mathbb{K}$ be a field, and let $(X_n)_{n\geq 0}$ be a filtration of a topological space X; assume that every compact subset of X is contained in some X_n. Setting*

$$\mu_k=\sum_{n=0}^{\infty}\beta_k(X_{n+1},X_n;\mathbb{K}),\quad k\geq 0,$$

and $\beta_k=\beta_k(X,X_0;\mathbb{K})$, then the sequence $(\mu_k,\beta_k)_{k\geq 0}$ satisfies the Morse relations.

6.2. Homological invariants at isolated critical points and critical orbits

Let us recall here a few basic facts on the homological invariants associated to isolated critical points and group orbits; the basic references are [12,20,21, 24,50]. Let $\mathcal{M}$ be a smooth Hilbert manifold and let $\mathfrak{f} : \mathcal{M} \to \mathbb{R}$ be a smooth function; for $d \in \mathbb{R}$, denote by $\mathfrak{f}^d$ the closed sublevel $\big\{x \in \mathcal{M} : \mathfrak{f}(x) \le d\big\}$. Let $p \in \mathcal{M}$ be a critical point of $\mathfrak{f}$, and assume that the Hessian $\mathrm{H}^{\mathfrak{f}}(p)$ of $\mathfrak{f}$ at p is represented by a compact perturbation of the identity of $T_p\mathcal{M}$. A generalized Morse lemma for this situation [20, Lemma 1] says that there exists a smooth local parametrization of $\mathcal{M}$ around p, $\Phi : U \to V$, where U is an open neighborhood of $0 \in T_p\mathcal{M} \cong \mathrm{Ker}\big(\mathrm{H}^{\mathfrak{f}}(p)\big)^{\perp} \oplus \mathrm{Ker}\big(\mathrm{H}^{\mathfrak{f}}(p)\big)$, V is an open neighborhood of p, with $\Phi(0) = p$, and there exists an orthogonal projection P on $\mathrm{Ker}\big(\mathrm{H}^{\mathfrak{f}}(p)\big)^{\perp}$ such that $\mathfrak{f} \circ \Phi(x, y) = \|Px\|^2 - \|(1-P)x\|^2 + \mathfrak{f}_0(y)$, where $\mathfrak{f}_0 : U \cap \mathrm{Ker}\big(\mathrm{H}^{\mathfrak{f}}(p)\big) \to \mathbb{R}$ is a smooth function having 0 as an isolated completely degenerate critical point. Using this decomposition of $\mathfrak{f}$, a homological invariant $\mathfrak{H}(\mathfrak{f}, p; \mathbb{K})$ of $\mathfrak{f}$ at p is defined by:

$$\mathfrak{H}(\mathfrak{f}, p; \mathbb{K}) = H_*(W_p, W_p^-; \mathbb{K}),$$

where $\mathbb{K}$ is any coefficient field, and (W, W^-) is a pair of topological spaces constructed in [20] and called *admissible pair* (a *GM-pair* in the language of [50]). Let us describe briefly such construction. Denote by $\eta : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ the flow of $-\nabla\mathfrak{f}$ and set $\mathfrak{f}(p) = c$; an admissible pair (W_p, W_p^-) is characterized by the following properties (see [50, Definition 2.3]):

(1) W_p is a closed neighborhood of p that contains a unique critical point of $\mathfrak{f}$ and such that:
   (a) if $t_1 < t_2$ and $\eta(t_i, x) \in W$ for $i = 1, 2$, then $\eta(t, x) \in W$ for all $t \in [t_1, t_2]$;
   (b) there exists an $\varepsilon > 0$ such that $\mathfrak{f}$ has no critical value in $[c - \varepsilon, c[$ and such that $W \cap \mathfrak{f}^{c-\varepsilon} = \emptyset$;

(2) $W^- = \big\{x \in W : \eta(x, t) \in W,\ \forall\, t > 0\big\}$ is closed in W;

(3) W^- is a (piecewise smooth) hypersurface of $\mathcal{M}$ which is transversal to $\nabla\mathfrak{f}$.

By [50, Theorem 2.1], if (W_p, W_p^-) is an admissible pair, then

$$H_*(W_p, W_p^-; \mathbb{K}) = H_*(\mathfrak{f}^c, \mathfrak{f}^c \setminus \{p\}; \mathbb{K});$$

furthermore, by excision, if U is any open subset of $\mathcal{M}$ containing p, then

$$\mathfrak{H}_*(\mathfrak{f}, p; \mathbb{K}) = H_*\big(U \cap \mathfrak{f}^c, U \cap (\mathfrak{f}^c) \setminus \{p\}); \mathbb{K}\big).$$

If $\mathcal{M}$ is complete, c is the only critical value of $\mathfrak{f}$ in $[c-\varepsilon, c+\varepsilon]$, and $p_1, \ldots, p_r$ are the critical points of $\mathfrak{f}$ in $\mathfrak{f}^{-1}(c)$, then the relative homology $H_*\big(\mathfrak{f}^{c+\varepsilon}, \mathfrak{f}^{c-\varepsilon}; \mathbb{K}\big)$ can be computed as:

$$H_*\big(\mathfrak{f}^{c+\varepsilon}, \mathfrak{f}^{c-\varepsilon}; \mathbb{K}\big) = \bigoplus_{i=1}^{r} \mathfrak{H}_*(\mathfrak{f}, p_i; \mathbb{K}).$$

Another homological invariant $\mathfrak{H}^o(\mathfrak{f}, p; \mathbb{K})$ is defined by setting

$$\mathfrak{H}^o(\mathfrak{f}, p; \mathbb{K}) = \mathfrak{H}(\mathfrak{f}_0, p; \mathbb{K}),$$

where $\mathfrak{f}_0$ is the degenerate component of $\mathfrak{f}$ described above. Among the main results of [20], the celebrated *shifting theorem* gives a relation between $\mathfrak{H}(\mathfrak{f}, p; \mathbb{K})$ and $\mathfrak{H}^o(\mathfrak{f}, p; \mathbb{K})$. The shifting theorem states that if $\mu(p)$ is the Morse index of $\mathfrak{f}$ at p, then:

$$\mathfrak{H}_{k+\mu(p)}(\mathfrak{f}, p; \mathbb{K}) = \mathfrak{H}^o_k(\mathfrak{f}, p; \mathbb{K}), \quad \forall\, k \in \mathbb{Z}. \tag{6.2}$$

The homological invariant $\mathfrak{H}$, as well as $\mathfrak{H}^o$, is of finite type, i.e., $\mathfrak{H}_k$ is finite-dimensional for all k and $\mathfrak{H}_k = \{0\}$ except for a finite number of k's. Moreover, the homological invariant $\mathfrak{H}^o$ has the following localization property.

Lemma 6.2. *Let $\mathcal{M}$ be a smooth Hilbert manifold, $\mathfrak{f} : \mathcal{M} \to R$ be a smooth map, $p \in \mathcal{M}$ an isolated critical point of $\mathfrak{f}$ such that the Hessian $\mathrm{H}^{\mathfrak{f}}(p)$ is represented by compact perturbation of the identity. Let $\widehat{\mathcal{M}}$ be a smooth closed submanifold of $\mathcal{M}$ containing p such that $\nabla \mathfrak{f}_q \in T_q\widehat{\mathcal{M}}$ for all $q \in \widehat{M}$, and such that the null space of the Hessian $\mathrm{H}^{\mathfrak{f}}(p)$ is contained in $T_p\widehat{M}$. Then, $\mathfrak{H}^o(\mathfrak{f}, p) = \mathfrak{H}^o\big(\mathfrak{f}|_{\widehat{\mathcal{M}}}, p\big)$.*

Proof. See [20, Lemma 7, p. 368–369]. □

Consider now the case of a compact Lie group G acting by isometries on $\mathcal{M}$, and let $\mathfrak{f} : \mathcal{M} \to \mathbb{R}$ be a G-invariant smooth function satisfying the Palais–Smale condition. If p is a critical point of $\mathfrak{f}$, denote by Gp its G-orbit, which consists of critical points of $\mathfrak{f}$. If such critical orbit is isolated, i.e., if there exists an open neighborhood of Gp that does not contain critical points of $\mathfrak{f}$ outside Gp, then one defines a homological invariant at the critical orbit Gp by setting

$$\mathfrak{H}(\mathfrak{f}, Gp; \mathbb{K}) = H_*(\mathfrak{f}^c, \mathfrak{f}^c \setminus Gp; \mathbb{K}),$$

where $c = \mathfrak{f}(p)$. Again, by excision, if U is any open subset of $\mathcal{M}$ containing Gp, then

$$\mathfrak{H}_*(\mathfrak{f}, Gp; \mathbb{K}) = H_*\big(U \cap \mathfrak{f}^c, (U \cap \mathfrak{f}^c) \setminus Gp); \mathbb{K}\big).$$

If $\mathcal{M}$ is complete, c is the unique critical value of $\mathfrak{f}$ in $[c-\varepsilon, c+\varepsilon]$, and the critical set of $\mathfrak{f}$ at c consists of a finite number of isolated critical orbits Gp_1, $\ldots$, Gp_r, then by [43, Theorem 2.1], the relative homology $H_*(\mathfrak{f}^{c+\varepsilon}, \mathfrak{f}^{c-\varepsilon}; \mathbb{K})$ can be computed as:

$$H_*(\mathfrak{f}^{c+\varepsilon}, \mathfrak{f}^{c-\varepsilon}; \mathbb{K}) = \bigoplus_{i=1}^{r} \mathfrak{H}_*(\mathfrak{f}, Gp_i; \mathbb{K}). \tag{6.3}$$

6.3. Local homological invariants at critical $O(2)$-orbits in $\widetilde{\mathcal{N}}$

Let us now consider the Hilbert manifold $\widetilde{\mathcal{N}}$ (4.8) and the geodesic action functional $f : \widetilde{\mathcal{N}} \to \mathbb{R}$. Consider a non-constant critical point $[\gamma]$ of f and assume that the critical orbit $O(2)[\gamma]$ is isolated. Recalling that the Hessian of f at each critical orbit is a Fredholm form which is a compact perturbation of the identity (part (4.1) of Proposition 4.1), the completeness of $\widetilde{\mathcal{N}}$ and the Palais–Smale condition (Proposition 4.4), the construction of the local homological invariant at the critical orbit $O(2)[\gamma]$ can be performed as follows. Denote by $\Gamma \subset \mathrm{SO}(2)$ the stabilizer of γ, which is a finite cyclic group; observe that the quotient $O(2)/\Gamma \cong O(2)[\gamma]$ is diffeomorphic to the union of two copies of the circle and denote by $\nu\big(O(2)[\gamma]\big) \subset T\widetilde{\mathcal{N}}$ the normal bundle of $O(2)[\gamma]$ in $\widetilde{\mathcal{N}}$. Denote by EXP the exponential map of $\widetilde{\mathcal{N}}$ relatively to the metric (4.2), and let $r > 0$ be chosen small enough so that EXP gives a diffeomorphism between

$$\mathcal{A}_r = \Big\{ v \in \nu\big(O(2)[\gamma]\big) : \|v\| < r \Big\}$$

and an open subset $\mathcal{D}$ of $\widetilde{\mathcal{N}}$ containing $O(2)[\gamma]$. For $u \in O(2)[\gamma]$, set

$$\mathcal{D}^u = \mathrm{EXP}_u\big(\mathcal{A}_r \cap T_u\widetilde{\mathcal{N}}\big);$$

$\mathcal{D}$ is a normal disc bundle over $O(2)[\gamma]$ whose fiber at u is $\mathcal{D}^u$. Observe that, since $O(2)$ acts by isometries on $\widetilde{\mathcal{N}}$, then for all $g \in O(2)$ and all $u \in O(2)[\gamma]$,

$g\mathcal{D}^u = \mathcal{D}^{gu}$. In particular, the restriction of the $O(2)$-action gives an action of Γ on each fiber $\mathcal{D}^u$.

Consider the principal fiber bundle $O(2) \mapsto O(2)/\Gamma \cong O(2)[\gamma]$; we claim that the bundle $\mathcal{D}$ can be described as the fiberwise product:[8]

$$\mathcal{D} \cong O(2) \times_\Gamma \mathcal{D}^\gamma. \tag{6.4}$$

Namely, consider the local diffeomorphism $\psi : O(2) \times \mathcal{D}^\gamma \to \mathcal{D}$

$$O(2) \times \mathcal{D}^\gamma \ni (g, \sigma) \longmapsto g\sigma \in \mathcal{D};$$

assuming $\psi(g,\sigma) = \psi(g',\sigma')$ gives $g^{-1}g' = h \in \Gamma$ and $h\sigma = \sigma'$, thus $(g',\sigma') = (gh^{-1}, h\sigma)$ and ψ passes to the quotient giving a diffeomorphism $\bar{\psi} : O(2) \times_\Gamma \mathcal{D}^\gamma \to \mathcal{D}$, and (6.4) is proved. As observed above, by excision, the local homological invariant $\mathfrak{H}\big(f, O(2)[\gamma]; \mathbb{K}\big)$ can be computed as:

$$\mathfrak{H}_*\big(f, O(2)[\gamma]; \mathbb{K}\big) = H_*\big(f^c \cap \mathcal{D}, (f^c \setminus O(2)[\gamma]) \cap \mathcal{D}; \mathbb{K}\big),$$

where $c = f(\gamma)$. Since f is $O(2)$-invariant, with this construction, we have $g(f^c \cap \mathcal{D}^\gamma) = f^c \cap \mathcal{D}^{g\gamma}$ for all $g \in O(2)$; in particular, $f^c \cap \mathcal{D}^\gamma$ is Γ-invariant, and we have two fiber bundles over $O(2)$

$$\begin{aligned} f^c \cap \mathcal{D} = O(2) \times_\Gamma (f^c \cap \mathcal{D}^\gamma), \quad (f^c \cap \mathcal{D}) \setminus O(2)[\gamma] = O(2) \\ \times_\Gamma \big((f^c \cap \mathcal{D}^\gamma) \setminus \{[\gamma]\}\big). \end{aligned} \tag{6.5}$$

If c is the only critical value of f in $[c-\varepsilon, c+\varepsilon]$, and $O(2)\big[\gamma_1\big], \ldots, O(2)\big[\gamma_1\big]$ are the critical orbits of f in $f^{-1}(c)$, then by (6.3) the relative homology $H_*\big(f^{c+\varepsilon}, f^{c-\varepsilon}; \mathbb{K}\big)$ is given by

$$H_*\big(f^{c+\varepsilon}, f^{c-\varepsilon}; \mathbb{K}\big) = \bigoplus_{i=1}^r \mathfrak{H}_*\big(f, O(2)\big[\gamma_i\big]; \mathbb{K}\big). \tag{6.6}$$

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[8]Recall that given a $\mathcal{G}$-principal fiber bundle $\mathcal{P} \to \mathcal{X}$ over the manifold $\mathcal{X}$, and given a topological space $\mathcal{Y}$ endowed with a left $\mathcal{G}$-action, the fiberwise product $\mathcal{P} \times_\mathcal{G} \mathcal{Y}$ is a fiber bundle over $\mathcal{X}$ whose fiber at $x \in \mathcal{X}$ is the quotient of the product $\mathcal{P}_x \times \mathcal{Y}$ by the left action of $\mathcal{G}$ given by:

$$\mathcal{G} \times (\mathcal{P}_x \times \mathcal{Y}) \ni \big(g, (p, y)\big) = (pg^{-1}, gy) \in \mathcal{P}_x \times \mathcal{Y}.$$

Since the right action of $\mathcal{G}$ on $\mathcal{P}_x$ is free and transitive, then each fiber of $\mathcal{P} \times_\mathcal{G} \mathcal{Y}$ is homeomorphic to $\mathcal{Y}$. Fiberwise products are examples of associated bundles to principal bundles.

Remark 6.3. The restriction $f|_{\mathcal{D}^\gamma}$ of f to the disc $\mathcal{D}^\gamma$ has an isolated critical point at $[\gamma]$. By (4.1) of Proposition 4.1, the Hessian $\mathrm{H}^{f|_{\mathcal{D}^\gamma}}$ at $[\gamma]$ of the restriction $f|_{\mathcal{D}^\gamma}$ is essentially positive. We can therefore define the local homological invariant $\mathfrak{H}(f|_{\mathcal{D}^\gamma},[\gamma];\mathbb{K})$ as the relative homology

$$\mathfrak{H}(f|_{\mathcal{D}^\gamma},[\gamma];\mathbb{K}) = H_*(f^c\cap\mathcal{D}^\gamma,(f^c\cap\mathcal{D}^\gamma)\setminus\{[\gamma]\};\mathbb{K}).$$

Observe also that the Morse index of $[\gamma]$ as a critical point of the restriction $f|_{\mathcal{D}^\gamma}$ equals the Morse index of f at $[\gamma]$; the dimension of the kernel of $\mathrm{H}^{f|_{\mathcal{D}^\gamma}}$ at $[\gamma]$ equals the dimension of the kernel of H^f at $[\gamma]$ minus one.

For all $k\ge 0$, set

$$B_k(\gamma;\mathbb{K}) = \dim\big[\mathfrak{H}_k(f,O(2)[\gamma];\mathbb{K})\big],\quad C_k(\gamma;\mathbb{K}) = \dim\big[\mathfrak{H}_k(f|_{\mathcal{D}^\gamma},[\gamma];\mathbb{K})\big]$$
$$\text{and}\quad C_k^o(\gamma,\mathbb{K}) = \dim\big[\mathfrak{H}_k^o(f|_{\mathcal{D}^\gamma},[\gamma];\mathbb{K})\big].$$

Our construction of the local homological invariants does not clarify that, in fact, the invariants C_k and C_k^o do not depend on the metric structure of $\widetilde{\mathcal{N}}$; observe that in Proposition 6.8 we will need to employ different Riemannian structures on $\widetilde{\mathcal{N}}$. In order to prove the independence on the metric, we will now establish that $C_k(\gamma;\mathbb{K})$ and $C_k^o(\gamma,\mathbb{K})$ can be computed by considering restrictions of f to any hypersurface Σ of $\widetilde{\mathcal{N}}$ through $[\gamma]$ which is transversal to the orbit $O(2)[\gamma]$:

Lemma 6.4. *Let $O(2)[\gamma]$ be an isolated critical orbit of f in $\widetilde{\mathcal{N}}$, with $f(\gamma)=c$, and let Σ be any smooth hypersurface of $\widetilde{\mathcal{N}}$ with $[\gamma]\in\Sigma$ and with $T_{[\gamma]}\widetilde{\mathcal{N}} = T_{[\gamma]}\Sigma\oplus T_{[\gamma]}\big(O(2)[\gamma]\big)$. Then, $[\gamma]$ is an isolated critical point of $f|_\Sigma$, and*

$$\mathfrak{H}_*(f|_{\mathcal{D}^\gamma},[\gamma];\mathbb{K}) \cong H_*\big(\Sigma\cap f^c,(\Sigma\cap f^c)\setminus\{[\gamma]\};\mathbb{K}\big).$$

Moreover, the Morse indexes and the nullities of $[\gamma]$ as a critical points of $f|_{\mathcal{D}^\gamma}$ and of $f|_\Sigma$ coincide, respectively.

Proof. Let Σ be as above; the entire result will follow from the existence of an f-invariant diffeomorphism ψ from (a small neighborhood of $[\gamma]$ in) $\mathcal{D}^\gamma$ onto (a small neighborhood of $[\gamma]$ in) Σ with $\psi\big([\gamma]\big)=[\gamma]$. Consider the smooth map $\Sigma\times O(2)\ni(u,g)\mapsto gu\in\widetilde{\mathcal{N}}$; the assumption of transversality of Σ to the orbit $O(2)[\gamma]$ implies that the differential of this map at the point $\big([\gamma],1\big)$ is an isomorphism, and hence the map restricts to a diffeomorphism from a neighborhood of $\big([\gamma],1\big)$ to a neighborhood of $[\gamma]$ in $\widetilde{\mathcal{N}}$. Since $\mathcal{D}^\gamma$ is also transversal to $O(2)[\gamma]$, a neighborhood of $[\gamma]$ in $\mathcal{D}^\gamma$ is diffeomorphic, via this map, to the graph of a smooth function $\varphi : \widetilde{\Sigma} \to O(2)$, where $\widetilde{\Sigma}$ is a neighborhood of $[\gamma]$ in Σ and $\varphi\big([\gamma]\big) = 1$. The required f-invariant diffeomorphism ψ is given by $\widetilde{\Sigma} \ni u \mapsto \varphi(u)u \in \mathcal{D}^\gamma$. □

Corollary 6.5. *Under the assumptions of Lemma* 6.4*:*

$$\mathfrak{H}^o_*\big(f|_{\mathcal{D}^\gamma}, [\gamma]; \mathbb{K}\big) \cong \mathfrak{H}^o_*\big(f|_\Sigma, [\gamma]; \mathbb{K}\big).$$

Proof. Follows immediately from Lemma 6.4 and the shifting theorem (6.2). □

Remark 6.6. More generally, from the proof of Lemma 6.4 we get that if Σ is any hypersurface of $\widetilde{\mathcal{N}}$ as in the statement, all the properties of $f|_{\mathcal{D}^\gamma}$ discussed in Remark 6.3 also hold for the restriction $f|_\Sigma$. Under the circumstance that Σ is a hypersurface through $[\gamma]$ in $\widetilde{\mathcal{N}}$ that is orthogonal (relatively to an arbitrary Riemannian metric on $\widetilde{\mathcal{N}}$) to the critical orbit $O(2)[\gamma]$ at $[\gamma]$, then the null space of the Hessian of $f|_\Sigma$ at $[\gamma]$ is the intersection of the null space of the Hessian of $f|_{\widetilde{\mathcal{N}}}$ at γ and $T_{[\gamma]}\Sigma$. This follows easily from the observation that $T_{[\gamma]}\big(O(2)[\gamma]\big)$ is contained in the kernel of the Hessian of $f|_{\widetilde{\mathcal{N}}}$ at γ.

Finally, the key result of this subsection is to show that the local homological invariants at $[\gamma]$ coincide with the invariants at the iterate $\big[\gamma^{(N)}\big]$ when γ and $\gamma^{(N)}$ have the same nullity (Proposition 6.8). It will therefore be necessary to study the *N-times iteration map* $\mathfrak{N} : \widetilde{\mathcal{N}} \to \widetilde{\mathcal{N}}$, defined by $\mathfrak{N}\big([\gamma]\big) = \big[\gamma^{(N)}\big]$.

Lemma 6.7. $\mathfrak{N}$ *is a smooth embedding.*

Proof. We use the following criterion, which is proved easily. Let A, B be Banach manifolds and let $A' \subset A$ be an embedded submanifold. Let $\mathfrak{g} : A' \to B$ and $\mathfrak{h} : B \to A$ be smooth maps such that $\mathfrak{h} \circ \mathfrak{g}$ is the inclusion of A' into A. Then $\mathfrak{g}$ is a smooth embedding. In order to prove the lemma, the criterion is used in the following setup. The manifolds A and B are the sets of all curves $\sigma : [0, 1] \to M$ of Sobolev class H^1, with $\sigma(0) \in S$, and satisfying $g(\dot\sigma, \mathcal{Y})$ constant almost everywhere (the existence of a Hilbert manifold structure of this set is proved exactly as for $\widetilde{\mathcal{N}}$). The submanifold A' is $\widetilde{\mathcal{N}}$, which corresponds to the subset of A consisting of closed curves. The map $\mathfrak{g}$ is the N-times iteration map $\mathfrak{N}$, and the map $\mathfrak{h}$ is defined by $\mathfrak{h}(\sigma) = \tilde\sigma$ and $\tilde\sigma(t) = \sigma(t/N)$, for all $t \in [0, 1]$. □

The differential $\mathrm{d}\mathfrak{N}_{[\gamma]}$ at $[\gamma]$ is the N-times iteration map for vector fields along γ. Let us now prove the following central result.

Proposition 6.8. *Let γ be a closed geodesic in M, let $N \geq 1$ be fixed, and assume that $O(2)\big[\gamma^{(N)}\big]$ is an isolated critical orbit of f in $\widetilde{\mathcal{N}}$. Then, $O(2)[\gamma]$ is an isolated critical orbit of f in $\widetilde{\mathcal{N}}$, and if $\mathrm{n}(\gamma) = \mathrm{n}\big(\gamma^{(N)}\big)$, one has $C^o_k(\gamma;\mathbb{K}) = C^o_k\big(\gamma^{(N)};\mathbb{K}\big)$ for all k.*

Proof. The idea of the proof is analogous to that of [21, Theorem 3] and [24, Proposition 3.6]; several adaptations are needed due to the fact that we are dealing with different metric structures in the manifold M: the Lorentzian structure g and the Riemannian structure g_{R} (recall (4.1)) employed in the definition of the Hilbert structure of $\widetilde{\mathcal{N}}$.

Consider a modified Riemannian structure on $\widetilde{\mathcal{N}}$ induced by the inner product (compare with (4.2)) on each tangent space $T_\gamma\mathcal{N}$ given by

$$\langle\!\langle V, W\rangle\!\rangle_N = \int_0^1 \Big[N^2 g_{\mathrm{R}}(V,W) + g_{\mathrm{R}}\big(\tfrac{D_R}{dt}V, \tfrac{D_R}{dt}W\big)\Big]\,dt. \tag{6.7}$$

Consider the N-times iteration map $\mathfrak{N} : \big(\widetilde{\mathcal{N}}, \langle\!\langle\cdot,\cdot\rangle\!\rangle\big) \to \big(\widetilde{\mathcal{N}}, \langle\!\langle\cdot,\cdot\rangle\!\rangle_N\big)$, which is an embedding onto a smooth submanifold $\mathfrak{N}(\widetilde{\mathcal{N}})$ of $\widetilde{\mathcal{N}}$ by Lemma 6.7, and it preserves the metric up to a factor N^2. We claim that, at the points in the image of the map $\mathfrak{N}$, the gradient $\nabla^N f$ of the functional $f|_{\widetilde{\mathcal{N}}}$ relatively to the metric $\langle\!\langle\cdot,\cdot\rangle\!\rangle_N$ is tangent to the image of $\mathfrak{N}$. The set of points in the image of $\mathfrak{N}$ where this situation occurs is closed, and so, by a density argument, it suffices to prove the claim at those points $\sigma^{(N)} = \mathfrak{N}(\sigma)$ in the image of $\mathfrak{N}$ that are curves of class C^2. Given one such point $\sigma^{(N)}$, using the fundamental theorem of calculus of variations, one sees that the gradient $\nabla^N f(\sigma^{(N)})$ of f at $\sigma^{(N)}$ is the unique periodic vector field X along $\sigma^{(N)}$ that solves the differential equation

$$\tfrac{D_R^2}{dt^2}X - N^2X - 2\frac{g(D_R^2/dt^2X - N^2X, \mathcal{Y})}{g(\mathcal{Y},\mathcal{Y})}\,\mathcal{Y} = \tfrac{D}{dt}\tfrac{d}{dt}\sigma^{(N)}. \tag{6.8}$$

Now, if X_* is the vector field along σ which is the unique periodic solution of:

$$\tfrac{D_R^2}{dt^2}X_* - X_* - 2\frac{g(D_R^2/dt^2X_* - X_*, \mathcal{Y})}{g(\mathcal{Y},\mathcal{Y})}\,\mathcal{Y} = \tfrac{D}{dt}\tfrac{d}{dt}\sigma,$$

i.e., X_* is the gradient of f relatively to the metric $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ at σ, then the iterate $X_*^{(N)} = d\mathfrak{N}_\sigma(X_*)$ satisfies (6.8), which proves the claim.[9]

[9] Observe that $d\mathfrak{N}_\sigma\big(\mathcal{Y}|_\sigma\big) = \mathcal{Y}|_{\sigma^{(N)}}$.

Let $\Gamma \subset \mathrm{SO}(2)$ be the stabilizer of γ; consider a normal disc bundle $\mathcal{D} = O(2)\mathcal{D}^\gamma \cong O(2) \times_\Gamma \mathcal{D}^\gamma$ of the critical orbit $O(2)[\gamma]$ as described in Subsection 6.2. The image $\mathfrak{N}(\mathcal{D}^\gamma)$ is a smooth embedded submanifold of $\widetilde{\mathcal{N}}$ containing $\gamma^{(N)}$; since $\mathfrak{N} : \mathcal{D}^\gamma \to \mathfrak{N}(\mathcal{D}^\gamma)$ is a diffeomorphism and $f \circ \mathfrak{N} = N^2 f$, then

$$\mathfrak{H}^o_*\big(f|_{\mathfrak{N}(\mathcal{D}^\gamma)}, \gamma^{(N)}; \mathbb{K}\big) = \mathfrak{H}^o_*\big(f|_{\mathcal{D}^\gamma}, \gamma; \mathbb{K}\big). \tag{6.9}$$

In order to conclude the proof, we will now determine a hypersurface Σ in $\widetilde{\mathcal{N}}$ through $\gamma^{(N)}$ which is transversal at $\big[\gamma^{(N)}\big]$ to the orbit $O(2)\big[\gamma^{(N)}\big]$ and satisfying the following two properties:

(a) $\mathfrak{N}\big(\mathcal{D}^\gamma\big) \subset \Sigma$;

(b) the gradient $\nabla^N\big(f|_\Sigma\big)$ at the points of $\mathfrak{N}(\mathcal{D}^\gamma)$ is tangent to $\mathfrak{N}(\mathcal{D}^\gamma)$;

(c) the null space of the Hessian $\mathrm{H}^{f|_\Sigma}$ at $\big[\gamma^{(N)}\big]$ is contained in $T_{[\gamma^{(N)}]}\mathfrak{N}(\mathcal{D}^\gamma)$.

By Corollary 6.5 it will follow that

$$C^o_k(\gamma^{(N)}; \mathbb{K}) = \dim\Big[H_k\big(\Sigma \cap f^d, (\Sigma \cap f^d) \setminus \{[\gamma^{(N)}]\}; \mathbb{K}\big)\Big], \quad \forall\, k \geq 0, \tag{6.10}$$

where $d = f\big(\gamma^{(N)}\big) = cN^2$ and $c = f(\gamma)$. Moreover, using Lemma 6.2, properties (a), (b) and (c) will imply that

$$H_*\big(\Sigma \cap f^d, (\Sigma \cap f^d) \setminus \{[\gamma^{(N)}]\}; \mathbb{K}\big) \cong \mathfrak{H}^o_*\big(f|_{\mathfrak{N}(\mathcal{D}^\gamma)}, \gamma^{(N)}; \mathbb{K}\big). \tag{6.11}$$

The thesis will follow then from (6.9), (6.10) and (6.11).

For the construction of the desired Σ, consider be the normal bundle $\nu\big(\mathfrak{N}(\mathcal{D})\big)$ of the submanifold $\mathfrak{N}(\mathcal{D})$ in $\widetilde{\mathcal{N}}$ relatively to the metric $\langle\cdot,\cdot\rangle_N$. Let $\widetilde{\mathrm{EXP}}$ be the exponential map of $\widetilde{\mathcal{N}}$ relatively to the metric $\langle\cdot,\cdot\rangle_N$; define Σ to be the image under $\widetilde{\mathrm{EXP}}$ of a small neighborhood U of the zero section of the bundle $\nu\big(\mathfrak{N}(\mathcal{D})\big)|_{\mathfrak{N}(\mathcal{D}^\gamma)}$, i.e., the restriction to $\mathfrak{N}(\mathcal{D}^\gamma)$ of the normal bundle of $\mathfrak{N}(\mathcal{D})$. Since $\mathcal{D}^\gamma$ is a hypersurface in $\mathcal{D}$, if U is sufficiently small, then Σ is a hypersurface in $\widetilde{\mathcal{N}}$; clearly, $\mathfrak{N}(\mathcal{D}^\gamma) \subset \Sigma$.

The image $\mathfrak{N}\big(\mathrm{SO}(2)[\gamma]\big)$ coincides with the orbit $\mathrm{SO}(2)\big[\gamma^{(N)}\big]$; this is easily seen observing that the map $\mathrm{SO}(2) \ni g \mapsto g^N \in \mathrm{SO}(2)$ is surjective. Since $\mathcal{D}^\gamma$ is orthogonal to $O(2)[\gamma]$ and $\mathfrak{N}$ is metric preserving up to a constant factor, it follows that Σ is orthogonal to $O(2)\big[\gamma^{(N)}\big]$ at $\big[\gamma^{(N)}\big]$ (observe that $\big[\gamma^{(N)}\big]$ belongs to the connected component $\mathrm{SO}(2)\big[\gamma^{(N)}\big]$ of $O(2)\big[\gamma^{(N)}\big]$) relatively to the metric $\langle\cdot,\cdot\rangle_N$.

For $u \in \mathfrak{N}(\mathcal{D}^\gamma)$, the tangent space $T_u\Sigma$ is given by the orthogonal direct sum (see Lemma 6.9 below)

$$T_u\Sigma = T_u\big(\mathfrak{N}(\mathcal{D}^\gamma)\big) \overset{\perp}{\oplus} T_u\big(\mathfrak{N}(\mathcal{D})\big)^\perp. \tag{6.12}$$

From the first part of the proof we know that at the points $u \in \mathfrak{N}(\mathcal{D}^\gamma)$, the gradient $\nabla^N f(u)$ is tangent to $\mathfrak{N}(\mathcal{D})$; from (6.12), the orthogonal projection of $\nabla^N f(u)$ onto $T_u\Sigma$, which is the gradient of $f|_\Sigma$ at u, must be tangent $\mathfrak{N}(\mathcal{D}^\gamma)$. Property (b) is thus satisfied.

Finally, we claim that the differential $d\mathfrak{N}_{[\gamma]}$ of $\mathfrak{N}$ at $[\gamma]$ carries the null space of the Hessian of $f|_{\mathcal{D}^\gamma}$ at $[\gamma]$ (injectively) into the null space of the Hessian of $f|_\Sigma$ at $[\gamma]$. Namely, recall from Remark 6.6 that the null space of the Hessian of $f|_{\mathcal{D}^\gamma}$ (resp., of $f|_\Sigma$) at $[\gamma]$ (resp., at $[\gamma^{(N)}]$) consists of all periodic Jacobi fields that are orthogonal to the critical orbit $O(2)[\gamma]$ (resp., $O(2)[\gamma^{(N)}]$). Thus, the proof of the claim follows easily observing that the map $\mathrm{d}\mathfrak{N}_{[\gamma]}$:

- carries periodic Jacobi fields along γ to periodic Jacobi fields along $\gamma^{(N)}$;
- carries $T_{[\gamma]}\big(O(2)[\gamma]\big)$ isomorphically onto $T_{[\gamma^{(N)}]}\big(O(2)[\gamma^{(N)}]\big)$;
- preserves orthogonality.

The null spaces of the two Hessians have the same dimension, because of our assumption on the nullity of $[\gamma]$ and of $[\gamma^{(N)}]$ (recall from Proposition 4.4 and Remarks 6.3, 6.6 that these two spaces have dimensions $n(\gamma)-2$ and $n(\gamma^{(N)})-2$, respectively). This implies that the null space of $\mathrm{H}^{f|_\Sigma}$ at $[\gamma^{(N)}]$ is in the image of $d\mathfrak{N}_{[\gamma]}$, and hence it is contained in $T_{[\gamma^{(N)}]}\mathfrak{N}(\mathcal{D}^\gamma)$, which gives property (c). This concludes the proof. □

Lemma 6.9 below has been used in the proof of Proposition 6.8 to the following setup: $A = \widetilde{\mathcal{N}}$, $B = \mathfrak{N}(\mathcal{D}^\gamma)$ and $E = \nu\big(\mathfrak{N}(\mathcal{D})\big)|_{\mathfrak{N}(\mathcal{D}^\gamma)}$.

Lemma 6.9. *Let A be a Hilbert manifold and $B \subset A$ a submanifold. Let $\nu(B) \subset TA$ be the normal bundle of B in A and let $E \subset \nu(B)$ be a sub-bundle. Let $U \subset \nu(B)$ be a small open subset containing the zero section and set $\Sigma = \exp(U \cap E)$. Then, B is a submanifold of Σ, and for all $b \in B$, the tangent space $T_b\Sigma$ is the orthogonal direct sum $T_bB \oplus E_b$.*

Proof. B is the image of the zero section $\mathbf{0}$ of E. At each point $0_b \in \mathbf{0}$, $b \in B$, there is a canonical isomorphism $T_{0_b}E \cong T_bB \oplus E_b$, where T_bB is identified with the tangent space at 0_b of $\mathbf{0}$. Using this identification, the differential

$d(\exp|_{\mathbf{0}})(0_b) : T_bB \oplus \{0\} \to T_bB$ of the restriction of exp to $\mathbf{0}$ at 0_b is the identity. Moreover, the restriction of $d\exp(0_b)$ to $\{0\} \oplus E_b$ coincides with the differential $d\exp_b(0_b)$, which is the identity. Thus, $d\exp(0_b)$ carries $T_{0_b}E$ isomorphically onto $T_bB \oplus E_b$, and the conclusion follows. □

6.4. Equivariant Morse theory for closed geodesics

As observed in Remark 4.6, in order to prove the theorem we need to show the existence of infinitely many distinct prime critical $O(2)$-orbits of the functional f in $\widetilde{\mathcal{N}}$; this will be obtained by contradiction, showing that assuming the existence of only a finite number of geometrically distinct closed geodesics will yield a uniform upper bound on the Betti numbers of ΛM.

Let us assume that there is only a finite number of geometrically distinct critical orbits, and hence by Lemma 4.7, the critical orbits of f in $\widetilde{\mathcal{N}}$ are isolated. If $0 \le a < b$ are regular values of f, and if $O(2)\big[\gamma_1\big], \ldots, O(2)\big[\gamma_r\big]$ are all the critical orbits of f in $f^{-1}\big([a,b]\big)$, then, using (6.6) and the fact that the β_k are subadditive functions, one has the Morse inequalities:

$$\beta_k(f^b, f^a; \mathbb{K}) \le \sum_{j=1}^{r} B_k(\gamma_j; \mathbb{K}). \tag{6.13}$$

In particular, since $\mathfrak{H}$ is of finite type, i.e., $B_k(\gamma;\mathbb{K})$ is finite for all k and $B_k(\gamma;\mathbb{K}) = 0$ except for a finite number of k's, then $\beta_k(f^b, f^a; \mathbb{K}) < +\infty$ for all a, b and k.

Using the relative Mayer–Vietoris sequence to the pair of bundles (6.5) over $O(2)$, which is homeomorphic to the disjoint union of two copies of the circle, one proves that the following inequality:

$$B_k(\gamma; \mathbb{K}) \le 2\big(C_k(\gamma, \mathbb{K}) + C_{k-1}(\gamma, \mathbb{K})\big), \tag{6.14}$$

holds for all $k \ge 1$. The details of this computation will be given in Appendix 7; it should be observed that in [21, 24] the inequality is stated only in the case of a field $\mathbb{K}$ of characteristic zero.

By the shifting theorem (see (6.2)), inequalities (6.14) become

$$B_k(\gamma; \mathbb{K}) \le 2\big(C^o_{k-\mu(\gamma)}(\gamma; \mathbb{K}) + C^o_{k-\mu(\gamma)-1}(\gamma; \mathbb{K})\big). \tag{6.15}$$

Proposition 6.10. *Let γ be a closed geodesic in M. If all the critical orbits $O(2)\big[\gamma^{(N)}\big]$ of f in $\widetilde{\mathcal{N}}$ are isolated, then the double sequence $(k, N) \mapsto$*

$C^o_k\big(\gamma^{(N)};\mathbb{K}\big)$ *is uniformly bounded:*

$$C^o_k\big(\gamma^{(N)};\mathbb{K}\big) \le B, \quad \forall\, k, N \in \mathbb{N}\setminus\{0\}. \tag{6.16}$$

Moreover, there exists a k_0 *such that* $C^o_k\big(\gamma^{(N)};\mathbb{K}\big) = 0$ *for all* $k > k_0$ *and all* $N \ge 1$.

Proof. Inequality (6.16) follows readily from Lemma 5.9 (see Remark 5.10) and Proposition 6.8. For a fixed N, the existence of k_0 as above is guaranteed by the fact that the invariant $\mathfrak{H}^o$ is of finite type. Again, independence on N is obtained easily from Lemma 5.9 and Proposition 6.8. □

Corollary 6.11. *Under the assumptions of Proposition* 6.10*, the following inequality holds:*

$$B_k\big(\gamma^{(N)};\mathbb{K}\big) \le 4B, \quad \forall\, N \ge 1. \tag{6.17}$$

Moreover, for $k > k_0 + 8\dim(M) + 2$*, the number of iterates* $\gamma^{(N)}$ *of* γ *such that* $B_k\big(\gamma^{(N)};\mathbb{K}\big) \ne 0$ *is bounded by a constant* C *which does not depend on* k.

Proof. Inequality (6.17) follows from (6.15) and (6.16). Moreover, using (6.15) and Proposition 6.10 we get that $B_k\big(\gamma^{(N)};\mathbb{K}\big) \ne 0$ only if

$$k - k_0 - 1 \le \mu\big(\gamma^{(N)}\big) \le k. \tag{6.18}$$

If the sequence $\mu\big(\gamma^{(N)}\big)$ is bounded, then by our assumption on k and Proposition 5.7, no iterate $\gamma^{(N)}$ of γ satisfies (6.18). Assume that $\mu\big(\gamma^{(N)}\big)$ is not bounded, and let k_* be as in the proof of Proposition 5.7. Let $\bar k \ge k_*$ be the smallest integer for which $\mu\big(\gamma^{(\bar k)}\big) \ge k - k_0 - 1$; we need to estimate the number of positive integers s such that $\mu\big(\gamma^{(\bar k + s)}\big) \le k$. If $s \ge k_*$, then by Corollary 5.8

$$k_0 + 1 \ge \mu\big(\gamma^{(\bar k + s)}\big) - \mu\big(\gamma^{(\bar k)}\big) \ge \bar\alpha\, s - \bar\beta,$$

where $\bar\alpha, \bar\beta > 0$. Thus, the number of iterates $\gamma^{(N)}$ such that $B_k\big(\gamma^{(N)};\mathbb{K}\big) \ne 0$ is bounded by the constant

$$\max\left\{k_*, \frac{k_0 + 1 + \bar\beta}{\bar\alpha}\right\}.$$

□

Proposition 6.12. *Let (M,g) be a Lorentzian manifold that has a complete timelike Killing vector field and a compact Cauchy surface. If there is only a finite number of geometrically distinct non-trivial closed geodesics in M, then the Betti numbers $\beta_k(\Lambda M;\mathbb{K})$ form a bounded sequence for k large enough.*

Proof. Since ΛM is homotopically equivalent to $\widetilde{\mathcal{N}}$, $\beta_k(\Lambda M;\mathbb{K}) = \beta_k(\widetilde{\mathcal{N}};\mathbb{K})$ for all $k \geq 0$. Denote by $\gamma_1,\ldots,\gamma_r$ a maximal family of pairwise geometrically distinct non trivial closed geodesic in M, and let $0 = c_0 < c_1 < \cdots < c_n < \cdots$ be the critical values of f in $\widetilde{\mathcal{N}}$ corresponding to the critical orbits $O(2)\big[\gamma_i^{(N)}\big]$, $N \geq 1$, $i = 1,\ldots,r$. By Lemma 4.7, these critical orbits are isolated, and each fixed sublevel f^b of f in $\widetilde{\mathcal{N}}$ contains only a finite number of them. By Corollary 6.11, the sequence $(i,k,N) \mapsto B_k\big(\gamma_i^{(N)};\mathbb{K}\big)$ takes a finite number of values, and we can define $\widehat{B} = \max_{i,k,N} B_k\big(\gamma_i^{(N)};\mathbb{K}\big)$.

For each geodesic γ_i, choose numbers $k_0^{(i)}$ and $C^{(i)}$ as in Proposition 6.10 and Corollary 6.11; set $\hat{k}_0 = \max_i k_0^{(i)}$ and $\widehat{C} = \max_i C^{(i)}$.

By Corollary 6.11, for all $k > \hat{k}_0 + 8\dim(M) + 2$, the constant $\widehat{C}$ is an upper bound for the number of orbits $O(2)\big[\gamma_i^{(N)}\big]$ with $B_k\big(\gamma_i^{(N)};\mathbb{K}\big) \neq 0$. Using the Morse inequalities (6.13), we have for all regular values a, b of f in $\widetilde{\mathcal{N}}$, with $0 < a < b$, and for all $k > \hat{k}_0 + 8\dim(M) + 2$ the following inequality holds:

$$\beta_k(f^b, f^a;\mathbb{K}) \leq 4\widehat{B}\widehat{C}. \tag{6.19}$$

By Lemma 4.8, there exists an $\varepsilon \in \,]0, c_1[$ such that the sublevel f^ε is homotopically equivalent to a Cauchy surface S of M. For all $n \geq 1$, set $d_n = 1/2(c_n + c_{n+1})$, $d_0 = \varepsilon$, and for all $n \geq 0$ set $X_n = f^{d_n}$; each d_n is a regular value of f in $\widetilde{\mathcal{N}}$, and the X_n form a filtration of $\widetilde{\mathcal{N}}$ as in Proposition 6.1. Since X_0 is homotopically equivalent to S, which is a finite dimensional compact manifold, for k large enough, $\beta_k(\widetilde{\mathcal{N}}, X_0;\mathbb{K}) = \beta_k(\widetilde{\mathcal{N}};\mathbb{K})$. We claim that, for $k > \hat{k}_0 + 8\dim(M) + 2$, the number of indices n such that $\beta_k(X_{n+1}, X_n;\mathbb{K}) \neq 0$ is bounded by a constant N_0 that does not depend on k. Namely, arguing as in the proof of Corollary 6.11, one proves easily that such constant N_0 can be taken equal to $\sum_{i=1}^r C^{(i)}$.

Now, using (6.19), it follows that

$$\sum_{n=0}^{\infty} \beta_k(X_{n+1}, X_n;\mathbb{K}) \leq 4\widehat{B}\widehat{C}N_0,$$

for all $k > \hat{k}_0 + 8\dim(M) + 2$. Using Proposition 6.1 (and the weak Morse inequalities (6.1)), we get

$$\beta_k(\Lambda M; \mathbb{K}) = \beta_k(\widetilde{\mathcal{N}}; \mathbb{K}) = \beta_k(\widetilde{\mathcal{N}}, X_0; \mathbb{K}) \le 4\widehat{B}\widehat{C}N_0$$

for k large enough, which concludes the proof. □

Remark 6.13. Observe that in Proposition 6.12 we have not used any assumption on the topology of M. Examples of non-simply connected spaces M satisfying $\beta_k(\Lambda M; \mathbb{K}) = +\infty$ for some small value of k but for which $\beta_k(\Lambda M; \mathbb{K})$ is bounded for k large can be obtained as follows. Consider a standard stationary Lorentzian manifold $M = S \times \mathbb{R}$, where S is a compact connected manifold whose universal cover is contractible.[10] The free loop space ΛM of M is homotopically equivalent to the free loop space ΛS of S. Given $p \in S$, denote by $\Omega_p S$ the loop space of S based at p; the map $\Lambda S \ni \gamma \mapsto \gamma(0) \in S$ is a fibration, whose fiber at p is $\Omega_p S$. The space $\Omega_p S$ has infinitely many connected component ($\pi_1(S)$ must be infinite), each of which is contractible, by the assumption on the universal cover of S. It follows that each connected component of ΛS is homotopically equivalent to S, and therefore, given any coefficient field $\mathbb{K}$, $\beta_k(\Lambda S, \mathbb{K}) = +\infty$ for some $k \in \{0, \dots, \dim(S)\}$, while $\beta_k(\Lambda S; \mathbb{K}) = 0$ for all $k > \dim(S)$.

We are now in the position of finalizing the proof of our main result.

Proof of the main theorem. Assume that (M, g) is a simply connected stationary globally hyperbolic spacetime, having a compact Cauchy surface S and a complete timelike Killing vector field $\mathcal{Y}$. Then, S is simply connected, and by [45] the Betti numbers of the free loop space of S (or, equivalently, of M) are finite. Then, by Proposition 6.12, the finiteness of the number of geometrically distinct closed geodesics in M implies that the Betti numbers of ΛM form a bounded sequence. The thesis follows. □

7. Final remarks

A few observations on the result presented in the paper and its proof are in order.

Remark 7.1. As to the notion of geometric equivalence for closed geodesics given in the Introduction, and based on the choice of some complete timelike

[10]This example has been suggested by Prof. Gudlaugur Thorbergsson.

Killing vector field, we observe that the property of existence of infinitely many geometrically distinct closed geodesic is independent on such choice. This can be seen using the following construction. Assume that $S \subset M$ is a Cauchy surface of (M, g); given a complete timelike Killing vector field $\mathcal{Y}$, one can define a diffeomorphism $\mathbb{P}_{\mathcal{Y}} : \widetilde{\mathcal{N}} \to \Lambda S$ by considering *projections* onto S along the flow lines of $\mathcal{Y}$ (note that also the definition of $\widetilde{\mathcal{N}}$ employs the given vector field $\mathcal{Y}$). More precisely, given $\gamma \in \widetilde{\mathcal{N}}$, the curve $x = \mathbb{P}_{\mathcal{Y}}(\gamma)$ is defined by $x(t) = \mathcal{F}_{h_\gamma(t)}\big(\gamma(t)\big)$, where $\mathcal{F}$ is the flow of $\mathcal{Y}$ and $h_\gamma : [0,1] \to \mathbb{R}$ is uniquely defined by the property that $\mathcal{F}_{h_\gamma(t)}\big(\gamma(t)\big) \in S$. By an elementary ODE argument, it is easy to see that $\mathbb{P}_{\mathcal{Y}}$ is indeed a bijection, by proving that, given $x \in \Lambda S$, there exists a unique closed curve γ with $\gamma(0) = x(0)$ such that $\mathbb{P}_{\mathcal{Y}}(\gamma) = x$ and such that $g(\dot\gamma, \mathcal{Y})$ is constant. The smoothness of $\mathbb{P}_{\mathcal{Y}}$ is obtained by standard smooth dependence results for ODE's. The map $\mathbb{P}_{\mathcal{Y}}$ is $O(2)$-equivariant; thus, geometrically distinct closed geodesics in M correspond to distinct critical $O(2)$-orbits of the functional $\mathfrak{f}_{\mathcal{Y}} = f \circ \mathbb{P}_{\mathcal{Y}}^{-1} : \Lambda S \to \mathbb{R}$ (this is precisely the variational problem considered in [36]). Given two complete timelike Killing vector field $\mathcal{Y}_1$ and $\mathcal{Y}_2$ in M, the number of critical $O(2)$-orbits of the functionals $\mathfrak{f}_1$ and $\mathfrak{f}_2 = \mathfrak{f}_1 \circ \mathbb{P}_{\mathcal{Y}_1} \circ \mathbb{P}_{\mathcal{Y}_2}^{-1}$ on ΛS coincide, which proves that the number of geometrically distinct closed geodesic in (M, g) is an intrinsic notion.

Remark 7.2. Under the assumptions of our main result, if, in addition, the Killing vector field $\mathcal{Y}$ is irrotational, i.e., if the orthogonal distribution $\mathcal{Y}^\perp$ is integrable, then the proof of our result is immediate. Namely, in this situation, a maximal integrable submanifold S of $\mathcal{Y}^\perp$ is a compact *totally geodesic* Cauchy surface in (M, g). Thus, infinitely many closed geodesics in M can be obtained applying the classical Gromoll and Meyer result to the Riemannian manifold $(S, g|_S)$.

Remark 7.3. It must be emphasized that the estimates on the Conley–Zehnder index and the Maslov index discussed in Section 3 are very far from being sharp, and they only serve the purposes of the present paper. An intense literature on the iteration formulas in the context of periodic solutions of Hamiltonians on symplectic manifolds has been produced in the last decade (see, for instance, [13, 33, 34] and the references therein). On the other hand, the naive approach discussed in Section 3 seems to simplify significantly the approach using Bott's deep results in [9] on the Morse index of an iteration, even in the Riemannian case. Sharper growth estimates on the iterates of closed geodesics have been proved recently in [30].

Remark 7.4. As to the assumption that M be simply connected, one should note that the central result in Proposition 6.12 does not use this. The simple connectedness hypotheses are used in the final argument to guarantee the finiteness of the dimensions of *all* the homology spaces of the free loop space of M, by a result on spectral sequences due to Serre [45]. Observe that Proposition 6.12 does not give any information on the dimension of the homology spaces $\beta_k(\Lambda M;\mathbb{K})$ for $k = 0, \ldots, \hat{k}_0 + 8\dim(M) + 2$. As already observed in Remark 6.13, if M is not simply connected, then ΛM (and $\widetilde{\mathcal{N}}$) is not connected, and it might be the case that $\beta_k(\Lambda M;\mathbb{K}) = +\infty$ for small values of k even if (M,g) has only a finite number of geometrically distinct non-trivial closed geodesics. This might happen when there is a non-trivial closed geodesic whose iterates have bounded Morse indexes. Thus, one can state the main result of the paper in the following slightly more general form:

Theorem 7.5. *Let (M,g) be a globally hyperbolic stationary Lorentzian manifold having a complete timelike Killing vector field and having a compact Cauchy surface. Assume that the free loop space ΛM has Betti numbers β_k with respect to some coefficient field that satisfy*

$$\limsup_{k\to\infty} \beta_k = +\infty.$$

Then, there are infinitely many geometrically distinct non-trivial closed geodesics in M.

Examples of non-simply connected stationary spacetimes to which the theorem above applies can be constructed by considering standard stationary manifolds $M = S_0 \times \mathbb{R}$, where $S_0 = S \times P$ is a compact manifold given by the product of a non-simply connected manifold S as in Remark 6.13 and a compact manifold P whose free loop space has non-vanishing Betti numbers in arbitrarily large dimension with respect to a field of characteristic zero (for instance, $P = S^n$, $n \geq 2$).

Remark 7.6. Although it is clear how to produce examples of non-trivial closed geodesics all of whose iterates have null Morse index (any minimum of f in a non-trivial free homotopy class of M), it would be extremely interesting to produce Lorentzian examples having bounded, but non-zero, Morse indexes. The homology generated by the iterates of such closed geodesics might be richer than the homology of the free loop space, as described in [5] for the Riemannian case.

Remark 7.7. The proof of the main result of the paper can be simplified significantly under the further assumption that all the critical orbits of the geodesic action functional are non-degenerate. In analogy with the Riemannian case, we will call *bumpy* a Lorentzian metric for which such non-degeneracy assumption is satisfied. It is an interesting open problem to establish if, as in the Riemannian case (see [1, 32, 51]), bumpy metrics are *generic* in the space of (stationary) Lorentzian metrics of a given manifold. More generally, it would be interesting to determine which properties of the Lorentzian geodesic flow are generic.

Remark 7.8. Extensions of the result of existence of multiple closed geodesics in Lorentzian geometry are possible and indeed desirable in more general classes of manifolds. The non-simply connected case can be studied following the lines of the corresponding results in Riemannian geometry, as in [3, 4]. Finally, we observe that in view to applications to General Relativity, it would be interesting to establish multiplicity results for (causal) geodesics satisfying more general boundary conditions. A particularly interesting case is that of causal geodesics whose spatial component is periodic. In the stationary case such geodesics have endpoints related by a global isometry of the spacetime, and an analysis of this case might be based on a variational setup as in [22–24, 46].

Appendix A. An estimate on the relative homology of fiber bundles over $\mathbb{S}^1$

In this short appendix we will prove a result on the relative homology of fiber bundles over the circle with coefficients in an arbitrary field $\mathbb{K}$ that will allow a slight generalization of the result of Gromoll and Mayer.

Proposition A.1. *Let $\mathbb{K}$ be a field, and let $\pi : E \to \mathbb{S}^1$ be a fiber bundle with typical fiber E_0. Let $E' \subset E$ and $E'_0 \subset E_0$ be subsets such that for all $p \in \mathbb{S}^1$ there exists a trivialization $\phi_p : \pi^{-1}\big(\mathbb{S}^1 \setminus \{p\}\big) \to \big(\mathbb{S}^1 \setminus \{p\}\big) \times E_0$ whose restriction to $\pi^{-1}\big(\mathbb{S}^1 \setminus \{p\}\big) \cap E'$ gives a homeomorphism with $\big(\mathbb{S}^1 \setminus \{p\}\big) \times E'_0$. Then, for all $k \geq 0$, the following inequality holds:*

$$\dim\big(H_k(E, E'; \mathbb{K})\big) \leq \dim\big(H_k(E_0, E'_0; \mathbb{K})\big) + \dim\big(H_{k-1}(E_0, E'_0; \mathbb{K})\big).$$

Proof. Consider two distinct points $p_1, p_2 \in \mathbb{S}^1$ and set:

$$X_i = \pi^{-1}\big(\mathbb{S}^1 \setminus \{p_i\}\big), \quad A_i = X_i \cap E', \qquad i = 1, 2$$

such that $E = X_1 \cup X_2$ and $E' = A_1 \cup A_2$. The pairs (X_1, X_2) and (A_1, A_2) are excisive couples for E and E', respectively, since X_i is open in X and A_i is open in A, $i = 1, 2$. Hence, there is an exact sequence (Mayer–Vietoris, see, for instance, [39, § 8.1]):

$$\cdots \longrightarrow H_k(X_1 \cap X_2, A_1 \cap A_2; \mathbb{K}) \xrightarrow{\alpha_1^k \oplus \alpha_2^k} H_k(X_1, A_1; \mathbb{K}) \oplus H_k(X_2, A_2; \mathbb{K})$$
$$\longrightarrow H_k(E, E'; \mathbb{K}) \longrightarrow H_{k-1}(X_1 \cap X_2, A_1 \cap A_2; \mathbb{K}) \xrightarrow{\alpha_1^{k-1} \oplus \alpha_2^{k-1}} \cdots$$

Clearly,

$$X_1 \cap X_2 = \pi^{-1}\big(\mathbb{S}^1 \setminus \{p_1, p_2\}\big), \quad A_1 \cap A_2 = \pi^{-1}\big(\mathbb{S}^1 \setminus \{p_1, p_2\}\big) \cap E'.$$

We will determine an estimate on the size of the image and the kernel of the map:

$$\alpha_1^j : H_j(X_1 \cap X_2, A_1 \cap A_2; \mathbb{K}) \longrightarrow H_j(X_1, A_1; \mathbb{K}),$$

$j \geq 0$, that is induced by the inclusion $\mathfrak{i}_1 : (X_1 \cap X_2, A_1 \cap A_2) \to (X_1, A_1)$. Choose a trivialization $\phi : \pi^{-1}\big(\mathbb{S}^1 \setminus \{p_1\}\big) \to \big(\mathbb{S}^1 \setminus \{p_1\}\big) \times E_0$ compatible with E' as in the assumptions, and denote by $\widetilde{\phi}$ the restriction of ϕ to $\pi^{-1}\big(\mathbb{S}^1 \setminus \{p_1, p_2\}\big)$. We have induced isomorphisms:

$$H_j\Big(\pi^{-1}\big(\mathbb{S}^1 \setminus \{p_1\}\big), \pi^{-1}\big(\mathbb{S}^1 \setminus \{p_1\}\big) \cap E'; \mathbb{K}\Big)$$
$$\Big\downarrow \phi_*$$
$$H_j\big((\mathbb{S}^1 \setminus \{p_i\}) \times E_0, (\mathbb{S}^1 \setminus \{p_1\}) \times E_0'; \mathbb{K}\big) \cong H_j(E_0, E_0'; \mathbb{K}),$$

$$H_j\Big(\pi^{-1}\big(\mathbb{S}^1 \setminus \{p_1, p_2\}\big), \pi^{-1}\big(\mathbb{S}^1 \setminus \{p_1, p_2\}\big) \cap E'; \mathbb{K}\Big)$$
$$\Big\downarrow \widetilde{\phi}_*$$
$$H_j\big((\mathbb{S}^1 \setminus \{p_1, p_2\}) \times E_0, (\mathbb{S}^1 \setminus \{p_1, p_2\}) \times E_0'; \mathbb{K}\big)$$

$$\cong H_j(E_0, E_0'; \mathbb{K}) \oplus H_j(E_0, E_0'; \mathbb{K}).$$

It is immediate to verify that the map

$$\phi_* \circ \alpha_1^j \circ \widetilde{\phi}_*^{-1} : H_j(E_0, E_0'; \mathbb{K}) \oplus H_j(E_0, E_0'; \mathbb{K}) \to H_j(E_0, E_0'; \mathbb{K})$$

is the sum $(x,y)\mapsto x+y$, which is surjective. It follows that the dimension of the image of the map $\alpha_1^j\oplus\alpha_2^j$ is greater than or equal to $\dim\big(H_j(E_0,E_0';\mathbb{K})\big)$, while the kernel of $\alpha_1^j\oplus\alpha_2^j$ has dimension less than or equal to $\dim\big(H_j(E_0,E_0';\mathbb{K})\big)$. From the Mayer–Vietoris sequence, we now pass to the short exact sequence

$$0\to V_k\to H_k(E,E';\mathbb{K})\to \mathrm{Ker}(\alpha_1^{k-1}\oplus\alpha_2^{k-1})\to 0,$$

where

$$V_k=\big(H_k(E_0,E_0';\mathbb{K})\oplus H_k(E_0,E_0';\mathbb{K})\big)/\mathrm{Im}(\alpha_1^k\oplus\alpha_2^k),$$

obtaining

$$\begin{aligned}\dim\big(H_k(E,E';\mathbb{K}) &= \dim(V_k)+\dim\big(\mathrm{Ker}(\alpha_1^{k-1}\oplus\alpha_2^{k-1})\big)\\ &\le \dim\big(H_k(E_0,E_0';\mathbb{K})\big)+\dim\big(H_{k-1}(E_0,E_0';\mathbb{K})\big). \qquad \square\end{aligned}$$

An example where Proposition A.1 applies is given by considering fiber bundles E that are *associated bundles* $P\times_G E_0$ of a G-principal fiber bundle P over $\mathbb{S}^1$, where E_0 is a G-space (i.e., a topological space endowed with a continuous left G-action), $E_0'\subset E_0$ is a G-subspace of E_0, and $E'=P\times_G E_0'$ (see [41, Chapter 1]). This is the situation in which Proposition A.1 is used in the present paper (recall the definitions of the pair of bundles (6.5)).

Acknowledgment

F. M. and P. P. are partially sponsored by CNPq and Fapesp. P. P. is grateful to G. Thorbergsson for suggesting the example in Remark 6.13.

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Received March 26, 2007