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Rigidity of marginally trapped surfaces and the topology of black holes

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In a recent paper the author and Rick Schoen obtained a generalization to higher dimensions of a classical result of Hawking concerning the topology of black holes. It was proved that, apart from certain exceptional circumstances, cross sections of the event horizon, in the stationary case, and "weakly outermost" marginally outer trapped surfaces, in the general case, in black hole spacetimes obeying the dominant energy condition, are of positive Yamabe type. This implies many well-known restrictions on the topology, and is consistent with recent examples of five-dimensional stationary black hole spacetimes with horizon topology $S^2 \times S^1$. In the present paper, we rule out for "outermost" marginally outer trapped surfaces, in particular, for cross sections of the event horizon in stationary black hole spacetimes, the possibility of any such exceptional circumstances (which might have permitted, e.g., toroidal cross sections). This follows from the main result, which is a rigidity result for marginally outer trapped surfaces that are not of positive Yamabe type.

1. Introduction

Some recent developments in physics inspired by string theory, such as the AdS/CFT correspondence and brane world phenomenology, have heightened interest in higher-dimensional gravity. In particular, there has been a considerable amount of recent research devoted to the study of black holes in higher dimensions; for a sample, see [10,12,17], and references cited therein. In [14], Schoen and the author obtained a generalization to higher dimensions of a classical result of Hawking concerning the topology of black holes. We proved that, apart from certain exceptional circumstances, "weakly outermost" marginally outer trapped surfaces, in particular, cross sections of the event horizon in stationary black hole spacetimes, are of positive Yamabe type, i.e., admit metrics of positive scalar curvature, provided the dominant energy condition holds. This implies many well-known restrictions on the topology of the horizon, and is consistent with recent examples [12] of five-dimensional stationary black hole spacetimes with horizon topology $S^2 \times S^1$. In particular, in $(3+1)$ dimensions, the Gauss–Bonnet theorem implies that the horizon is topologically a 2-sphere, and one recovers Hawking's theorem.

If, however, certain quantities vanish on the horizon, e.g., if the horizon is Ricci flat and spacetime is vacuum in its vicinity, then the arguments in [14] do not quite guarantee the conclusion of being positive Yamabe. One of the main aims of the present paper is to rule out the possibility of any exceptions to being positive Yamabe under a natural set of physical circumstances. This will follow as a consequence of a rigidity result for marginally outer trapped surfaces that do not admit metrics of positive scalar curvature. This result may be viewed as a spacetime analog of the rigidity results for area minimizing hypersurfaces in a Riemannian manifold obtained in [8, 9]. The rationale for such a result had been discussed by the author (in the $3+1$ setting) in [13].

Before stating our main results, let us begin with a few definitions, and, in particular, introduce the basic object of study, that of a marginally outer trapped surface. Let Σ^{n-1}, $n \geq 3$, be a compact spacelike submanifold of codimension 2 in a spacetime (time-oriented Lorentzian manifold) (M^{n+1}, g). Under suitable orientation assumptions, Σ admits two smooth nonvanishing future directed null normal vector fields K_+ and K_-. These vector fields are unique up to pointwise scaling. By convention, we refer to K_+ as outward pointing and K_- as inward pointing. Let $\chi_\pm$ denote the null second fundamental form associated to $K_\pm$. Thus, for each $p \in \Sigma$, $\chi_\pm : T_p\Sigma \times T_p\Sigma \to \mathbb{R}$ is the symmetric bilinear form defined by,

$$\chi_\pm(X, Y) = \langle \nabla_X K_\pm, Y \rangle \quad \text{for all } X, Y \in T_p\Sigma, \tag{1.1}$$

where ∇ is the Levi–Civita connection of $g = \langle , \rangle$. Tracing with respect to the induced metric h on Σ, we obtain the null expansion scalars (or null mean curvatures) $\theta_\pm = \operatorname{tr} \chi_\pm = \operatorname{div}_\Sigma K_\pm$. As is well known, the sign of $\theta_\pm$ is invariant under positive rescalings of $K_\pm$. Physically, θ_+ (respectively, θ_-) measures the divergence of the outward pointing (respectively, inward pointing) light rays emanating from Σ. For round spheres in Euclidean slices of Minkowski space, with the obvious choice of inside and outside, one has $\theta_- < 0$ and $\theta_+ > 0$. In fact, this is the case, in general, for large "radial" spheres in asymptotically flat spacelike hypersurfaces. However, in regions of spacetime where the gravitational field is strong, one may have both $\theta_- < 0$ and $\theta_+ < 0$, in which case Σ is called a trapped surface. Under appropriate energy and causality conditions, the occurrence of a trapped surface signals the onset of gravitational collapse [18] and the existence of a black hole [15].

Focusing attention on just the outward null normal, we say that Σ is an *outer trapped surface* if $\theta_+ < 0$, and is a *marginally outer trapped surface* (MOTS) if $\theta_+ = 0$. MOTSs arise in a number of natural situations. For example, compact cross sections of the event horizon in stationary (steady state) black hole spacetimes are MOTSs. For dynamical black hole spacetimes, MOTSs typically occur in the black hole region, i.e., the region inside the event horizon. While there are heuristic arguments for the existence of MOTSs in this situation, based on looking at the boundary of the "trapped region" [15, 22] within a given spacelike slice, a recent result of Eichmair and Schoen [11,19], and of Andersson and Metzger [5] rigorously establishes their existence under natural conditions. MOTSs are the key ingredient behind the development of quasi-local notions of black holes (see [7] and references cited therein). On the more purely mathematical side, there are connections between MOTSs in spacetime and minimal surfaces in Riemannian manifolds. In fact, a MOTS contained in a *totally geodesic* spacelike hypersurface $V^n \subset M^{n+1}$ is simply a minimal hypersurface in V. Despite the absence of a variational characterization of MOTSs like that for minimal surfaces, MOTS have recently been shown to satisfy a number of analogous properties; see, e.g., [2–6, 11, 14, 19], as well as the important earlier work of Schoen and Yau [21]. The rigidity results presented here provide another case in point.

For our main results, we shall only consider spacetimes (M^{n+1}, g) that satisfy the Einstein equations,

$$R_{ab} - \frac{1}{2}Rg_{ab} = T_{ab} \tag{1.2}$$

for which the energy-momentum tensor $\mathcal{T}$ obeys the dominant energy condition, $\mathcal{T}(X, Y) = T_{ab}X^aY^b \geq 0$ for all future pointing causal vectors X, Y.

We now restrict attention to MOTSs contained in a spacelike hypersurface. Thus, let V^n be an n-dimensional, $n \geq 3$, spacelike hypersurface in a spacetime (M^{n+1}, g), and let Σ^{n-1} be a closed hypersurface in V^n. Assume that Σ^{n-1} separates V^n into an "inside" and an "outside." Denote the closure of the outside of V by V_+; hence V_+ is a manifold with boundary $\partial V_+ = \Sigma$.

We adopt the following terminology.

Definition 1.1. Let Σ^{n-1} be a MOTS in a spacelike hypersurface V^n, as above.

(i) We say that Σ is an outermost MOTS in V provided there are no outer trapped or marginally outer trapped surfaces outside of, and homologous to, Σ.

(ii) We say that Σ is a weakly outermost MOTS in V provided there are no outer trapped surfaces outside of, and homologous to, Σ.

Remark. (1) We note that Σ is an outermost MOTS if and only if there are no *weakly outer trapped* surfaces ($\theta_+ \leq 0$) outside of, and homologous to, Σ. The point is, if S is weakly outer trapped then either it is a MOTS or else it can be perturbed, via null mean curvature flow, to an outer trapped surface [5, Lemma 5.2].

(2) By the existence result for MOTSs alluded to above [5, 11, 19], under a natural outer barrier condition (which always holds in the asymptotically flat case), and provided the dimension is not too high, there exists outside of each outer trapped surface a MOTS homologous to it. Hence, under these circumstances, an outermost MOTS, as defined here, is outermost in the conventional sense.

(3) Heuristically, a weakly outermost MOTS Σ is the "outer limit" of outer trapped surfaces in V. Weakly outermost MOTSs were referred to as outer apparent horizons in [14].

One of the main aims of this paper is to present a proof of the following theorem.

Theorem 1.1. *Let (M^{n+1}, g), $n \geq 3$, be a spacetime satisfying the dominant energy condition, and let Σ^{n-1} be an outermost* MOTS *in a spacelike hypersurface V^n. Then Σ^{n-1} is of positive Yamabe type, i.e., admits a metric of positive scalar curvature.*

In fact, we shall prove the following rigidity result, which immediately implies Theorem 1.1.

Theorem 1.2. *Let (M^{n+1}, g), $n \geq 3$, be a spacetime satisfying the dominant energy condition, and let Σ^{n-1} be a weakly outermost* MOTS *in a spacelike hypersurface V^n. If Σ^{n-1} does not admit a metric of positive scalar curvature then there exists a neighborhood $U \approx [0, \epsilon) \times \Sigma$ of Σ in V_+ such that each slice $\Sigma_t = \{t\} \times \Sigma$, $t \in [0, \epsilon)$ is a* MOTS. *In fact, each such slice has vanishing outward null second fundamental form, $\chi_+ = 0$, and is Ricci flat.*

It is also shown that a certain energy-momentum term vanishes along each slice. Theorem 1.2 shall be proved in two stages. The first stage and the main effort of the paper is to prove Theorem 1.2 subject to the additional assumption that V^n has *nonpositive* mean curvature, $\tau \leq 0$;[1] see Theorem 3.1 in Section 3. The second stage uses a "deformation" argument to derive Theorem 1.2 from Theorem 3.1. While Theorem 3.1 is a pure "initial data" result, the proof of Theorem 1.2 makes use of the enveloping spacetime. Theorem 1.1 shows that, for outermost MOTS, the exceptional case in the main result of [14] can be eliminated.

A basic fact about standard $(3+1)$-dimensional black hole spacetimes [15, 22] obeying the null energy condition is that there can be no outer trapped, or even marginally outer trapped, surfaces outside the event horizon. The proof, which relies on the Raychaudhuri equation [15, 22], also works in higher dimensions. Thus, Theorem 1.2 implies the following.

Corollary 1.3. *Cross sections*[2] *of the event horizon in stationary black hole spacetimes obeying the dominant energy condition are of positive Yamabe type.*

In particular, there can be no toroidal horizons. The proof of Theorem 1.2 is presented in Section 3, following some preliminary results, presented in Section 2.

2. Analytic and geometric preliminaries

Let (Σ, h) be a compact Riemannian manifold. We draw together here various facts (all essentially known) about operators $L\colon C^\infty(\Sigma) \to C^\infty(\Sigma)$ of the form

$$L(\phi) = -\triangle\phi + 2\langle X, \nabla\phi\rangle + (Q + \operatorname{div} X - |X|^2)\phi, \tag{2.1}$$

where $Q \in C^\infty(\Sigma)$, X is a smooth vector field on Σ and $\langle\ ,\ \rangle = h$. The stability operator associated with variations in the null expansion, as explicitly introduced in [2], is of this form.

As discussed in [2], although L is not self-adjoint in general, the Krein–Rutman theorem, together with other arguments, implies the following.

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[1] By our sign conventions, the hyperbola $t = -\sqrt{1+x^2}$ in Minkowski 2-space has negative mean curvature.

[2] By *cross section*, we mean smooth compact intersection of the event horizon with a spacelike hypersurface.

Lemma 2.1. *Let $\lambda_1 = \lambda_1(L)$ be the principal eigenvalue of L (eigenvalue with smallest real part). Then the following hold.*

(i) *λ_1 is real and simple. There exists an associated eigenfunction ϕ ($L(\phi) = \lambda_1\phi$) which is strictly positive.*

(ii) *$\lambda_1 \geq 0$ (respectively, $\lambda_1 > 0$) if and only if there exists $\psi \in C^\infty(\Sigma)$, $\psi > 0$, such that $L(\psi) \geq 0$ (respectively, $L(\psi) > 0$).*

We wish to compare L with the "symmetrized" operator $L_0 : C^\infty(\Sigma) \to C^\infty(\Sigma)$, obtained by setting $X = 0$,

$$L_0(\phi) = -\triangle\phi + Q\phi. \tag{2.2}$$

The main argument in [14] shows that if $\lambda_1(L) \geq 0$ then $\lambda_1(L_0) \geq 0$. In fact, as remarked to us by Mars and Simon, a simple tweaking of this argument gives the following.

Lemma 2.2. *The principal eigenvalues $\lambda_1(L)$ of L and $\lambda_1(L_0)$ of L_0 satisfy $\lambda_1(L) \leq \lambda_1(L_0)$.*

Proof. In inequality (2.7) in [14], replace "≥ 0" by "$= \lambda_1\phi$", and proceed. □

A key result in the Schoen–Yau study of manifolds of positive scalar curvature [20] is that a compact stable minimal hypersurface in a manifold of positive scalar curvature admits, itself, a metric of positive scalar curvature. Related results have been obtained in [1, 14], and are proved using a simplification of the original argument of Schoen and Yau due to Cai [8]. These results may be formulated in a slightly more general context, as follows.

Lemma 2.3. *Consider the operator $L_0 = -\triangle + Q$ on (Σ, h), with*

$$Q = \tfrac{1}{2}S - P, \tag{2.3}$$

where S is the scalar curvature of (Σ, h) and $P \geq 0$. If $\lambda_1(L_0) \geq 0$ then Σ admits a metric of positive scalar curvature, unless $\lambda_1(L_0) = 0$, $P \equiv 0$ and (Σ, h) is Ricci flat.

Proof. Let $\phi \in C^\infty(\Sigma)$ be a positive eigenfunction associated to the eigenvalue $\lambda_1 = \lambda_1(L_0)$. The scalar curvature $\tilde{S}$ of Σ in the conformally rescaled metric $\tilde{h} = \phi^{2/(n-2)} h$ is then given by,

$$
\begin{aligned}
\tilde{S} &= \phi^{-n/(n-2)} \left(-2\triangle\phi + S\phi + \frac{n-1}{n-2}\frac{|\nabla\phi|^2}{\phi} \right) \\
&= \phi^{-2/(n-2)} \left(2\lambda_1 + 2P + \frac{n-1}{n-2}\frac{|\nabla\phi|^2}{\phi^2} \right) \qquad (2.4)
\end{aligned}
$$

where the second equation follows from (2.2), (2.3) and the fact that $L_0(\phi) = \lambda_1\phi$. Since all terms in the parentheses above are non-negative, (2.4) implies that $\tilde{S} \geq 0$. If $\tilde{S} > 0$ at some point, then by well-known results [16] one can conformally rescale $\tilde{h}$ to a metric of strictly positive scalar curvature. If, on the other hand, $\tilde{S}$ vanishes identically, then (2.4) implies: $\lambda_1 = 0$, $P \equiv 0$ and ϕ is constant. Equations (2.2) and (2.3) then imply that $S \equiv 0$. By an argument of Bourguignon (see [16]), one can then deform h in the direction of the Ricci tensor of Σ to obtain a metric of positive scalar curvature, unless (Σ, h) is Ricci flat. □

Finally, Lemmas 2.2 and 2.3 combine to give the following.

Lemma 2.4. *Lemma* 2.3 *also holds for the operator L in* (2.1)*, with Q as in* (2.3)*.*

Apart from the conclusion that $\lambda_1(L) = 0$ (if Σ does not admit a metric of positive scalar curvature), this was proved, in a specific context, in [14].

3. Proof of Theorem 1.2

Let the notation and terminology be as in the statement of Theorem 1.2, and the discussion leading up to it. As discussed in the introduction, we begin by proving Theorem 1.2, subject to a restriction on the mean curvature of V^n.

Theorem 3.1. *Let (M^{n+1}, g), $n \geq 3$, be a spacetime satisfying the dominant energy condition, and let V^n be a spacelike hypersurface in M^{n+1} with mean curvature $\tau \leq 0$. Suppose Σ^{n-1} is a weakly outermost* MOTS *in V^n that does not admit a metric of positive scalar curvature. Then there exists a neighborhood $U \approx [0, \epsilon) \times \Sigma$ of Σ in V_+ such that each slice $\Sigma_t = \{t\} \times \Sigma$, $t \in [0, \epsilon)$ is a* MOTS*. In fact, each such slice has vanishing outward null second fundamental form, $\chi_+ = 0$, and is Ricci flat.*

Proof. The first step is to show that a neighborhood of Σ in V_+ is foliated by *constant* null expansion hypersurfaces, with respect to a suitable scaling of the future directed outward null normals.

Let $t \to \Sigma_t$ be a variation of $\Sigma = \Sigma_0$, $-\epsilon < t < \epsilon$, with variation vector field $\mathcal{V} = \frac{\partial}{\partial t}\big|_{t=0} = \phi\nu$, $\phi \in C^\infty(\Sigma)$, where ν is the outward unit normal of Σ in V. Let $\theta(t)$ denote the null expansion of Σ_t with respect to $K_t = Z + \nu_t$, where Z is the future directed timelike unit normal to V and ν_t is the outer unit normal to Σ_t in V. A computation shows [2–4, 9],

$$\text{(3.1)} \qquad \frac{\partial \theta}{\partial t}\bigg|_{t=0} = L(\phi) = -\triangle\phi + 2\langle X, \nabla\phi\rangle + (Q + \operatorname{div} X - |X|^2)\phi,$$

where,

$$\text{(3.2)} \qquad Q = \tfrac{1}{2}S - \mathcal{T}(Z, K) - \tfrac{1}{2}|\chi|^2,$$

S is the scalar curvature of Σ, χ is the null second fundamental form of Σ with respect to $K = \nu + Z$, X is the vector field on Σ defined by $X = \tan(\nabla_\nu Z)$, and $\langle\ ,\ \rangle$ now denotes the induced metric on Σ.

Let λ_1 be the principal eigenvalue of L. As per Lemma 2.1, λ_1 is real, and there is an associated eigenfunction ϕ that is strictly positive. Using ϕ to define our variation, we have from (3.1),

$$\text{(3.3)} \qquad \frac{\partial \theta}{\partial t}\bigg|_{t=0} = \lambda_1\phi.$$

The eigenvalue λ_1 cannot be negative, for otherwise (3.3) would imply that $\frac{\partial\theta}{\partial t} < 0$ on Σ. Since $\theta = 0$ on Σ, this would mean that for $t > 0$ sufficiently small, Σ_t would be outer trapped, contrary to assumption. Thus, $\lambda_1 \geq 0$, and since Σ does not carry a metric of positive scalar curvature, we may apply Lemma 2.4 to L in (3.1), with $P = \mathcal{T}(Z, K) + \frac{1}{2}|\chi|^2 \geq 0$, to conclude that $\lambda_1 = 0$ (and also that $Q = 0$).

For $u \in C^\infty(\Sigma)$, u small, let $\Theta(u)$ denote the null expansion of the hypersurface $\Sigma_u : x \to \exp_x u(x)\nu$ with respect to the (suitably normalized) future directed outward null normal field to Σ_u. Θ has linearization, $\Theta'(0) = L$. We introduce the operator,

$$\text{(3.4)} \qquad \Theta^* : C^\infty(\Sigma) \times \mathbb{R} \to C^\infty(\Sigma) \times \mathbb{R}, \quad \Theta^*(u, k) = \left(\Theta(u) - k, \int_\Sigma u\right).$$

Since, by Lemma 2.1, $\lambda_1 = 0$ is a simple eigenvalue, the kernel of $\Theta'(0) = L$ consists only of constant multiples of the eigenfunction ϕ. We note that $\lambda_1 = 0$ is also a simple eigenvalue for the adjoint L^* of L (with respect to the standard L^2 inner product on Σ), for which there exists a positive eigenfunction ϕ^*. Then the equation $Lu = f$ is solvable if and only if $\int f\phi^* = 0$.

From these facts it follows easily that Θ^* has invertible linearization about $(0,0)$. Thus, by the inverse function theorem, for $s \in \mathbb{R}$ sufficiently small there exists $u(s) \in C^\infty(\Sigma)$ and $k(s) \in \mathbb{R}$ such that

$$\Theta(u(s)) = k(s) \quad \text{and} \quad \int_\Sigma u(s)dA = s. \tag{3.5}$$

By the chain rule, $\Theta'(0)(u'(0)) = L(u'(0)) = k'(0)$. The fact that $k'(0)$ is orthogonal to ϕ^* implies that $k'(0) = 0$. Hence $u'(0) \in \ker \Theta'(0)$. The second equation in (3.5) then implies that $u'(0) = \text{const} \cdot \phi > 0$.

It follows that for s sufficiently small, the hypersurfaces Σ_{u_s} form a smooth foliation of a neighborhood of Σ in V by hypersurfaces of constant null expansion. Thus, one can introduce coordinates (t, x^i) in a neighborhood W of Σ in V, such that, with respect to these coordinates, $W = (-t_0, t_0) \times \Sigma$, and for each $t \in (-t_0, t_0)$, the t- slice $\Sigma_t = \{t\} \times \Sigma$ has constant null expansion $\theta(t)$ with respect to $K|_{\Sigma_t}$, where $K = Z + \nu$, and ν is the outward unit normal field to the Σ_ts in V. In addition, the coordinates (t, x^i) can be chosen so that $\frac{\partial}{\partial t} = \phi\nu$, for some positive function $\phi = \phi(t, x^i)$ on W.

A computation similar to that leading to (3.1) (but where we can no longer assume θ vanishes) shows that the null expansion function $\theta = \theta(t)$ of the foliation obeys the evolution equation,[3]

$$\frac{d\theta}{dt} = \tilde{L}_t(\phi) \tag{3.6}$$

where, for each $t \in (-t_0, t_0)$, $\tilde{L}_t$ is the operator on Σ_t acting on ϕ according to,

$$\begin{aligned} \tilde{L}_t(\phi) &= -\triangle\phi + 2\langle X, \nabla\phi\rangle \\ &\quad + \left(\frac{1}{2}S - \mathcal{T}(Z, K) + \theta\tau - \frac{1}{2}\theta^2 - \frac{1}{2}|\chi|^2 + \operatorname{div} X - |X|^2\right)\phi. \end{aligned} \tag{3.7}$$

It is to be understood that, for each t, the above terms live on Σ_t, e.g., $\triangle = \triangle_t$ is the Laplacian on Σ_t, $S = S_t$ is the scalar curvature of Σ_t, and so on.

The assumption that Σ is weakly outermost, together with the constancy of $\theta(t)$, implies that $\theta(t) \geq 0$ for all $t \in [0, t_0)$. Hence, since $\theta(0) = 0$, to show that $\theta(t) = 0$ for all $t \in [0, t_0)$. It is sufficient to show that $\theta'(t) \leq 0$

[3] Although we have checked this independently, equation (3.6) follows easily from Lemma 3.1 in [3]; see also [4].

for all $t \in [0, t_0)$. Suppose there exists $t \in (0, t_0)$ such that $\theta'(t) > 0$. For this value of t, (3.6) implies $\tilde{L}_t(\phi) > 0$. Then Lemma 2.1 implies that $\lambda_1(\tilde{L}_t) > 0$. Recalling the assumption $\tau \le 0$, we may apply Lemma 2.4 to $\tilde{L}_t$, with $P = \mathcal{T}(Z, K) - \theta\tau + \frac{1}{2}\theta^2 + \frac{1}{2}|\chi|^2 \ge 0$, to conclude that $\Sigma_t \approx \Sigma$ carries a metric of positive scalar curvature, contrary to assumption.

Thus, $\theta(t) = 0$ for all $t \in [0, t_0)$. Since, by (3.6), $\tilde{L}_t(\phi) = \theta' = 0$, Lemma 2.1 implies $\lambda_1(\tilde{L}_t) \ge 0$ for each $t \in [0, t_0)$. Hence, by Lemma 2.4, we have for each $t \in [0, t_0)$, $\chi_t = 0$, Σ_t is Ricci flat and $\mathcal{T}(Z, K)$ vanishes along Σ_t. □

Proof of Theorem 1.2. We now show how Theorem 1.2 can be obtained from Theorem 3.1.

Let the setting be as in the statement of Theorem 1.2. It is straight forward to construct a spacelike hypersurface $\tilde{V}^n$ in M^{n+1} with the following properties: (i) $\tilde{V}$ and V meet tangentially along Σ, (ii) $\tilde{V}$ is in the causal past of V and (iii) $\tilde{V}$ has mean curvature $\tilde{\tau} \le 0$. ($\tilde{V}$ can be constructed from spacelike curves orthogonal to Σ and tangent to V at Σ, having sufficiently large curvature, and bending towards the past.)

The condition that Σ is weakly outermost in V transfers *to a sufficient extent* to $\tilde{V}$, as described in the following claim.

Claim. *For every variation $t \to \Sigma_t$, $-\epsilon < t < \epsilon$, of $\Sigma = \Sigma_0$ in $\tilde{V}$, with variation vector field $\mathcal{V} = \phi\tilde{\nu}$, $\phi > 0$, there exists $t_0 \in (0, \epsilon)$ such that Σ_t is not outer trapped for all $t \in (0, t_0)$.*

Proof of the claim. Suppose, to the contrary, there exists a variation $t \to \Sigma_t$, $0 \le t < \epsilon$, of Σ in $\tilde{V}_+$ (the outside of $\tilde{V}$) and a sequence $t_n \searrow 0$ such that $\Sigma_n := \Sigma_{t_n}$ is outer trapped. Let $\mathcal{H}_n$ be the null hypersurface generated by the future directed outward null geodesics orthogonal to Σ_n. Restricting to a small tubular neighborhood of Σ, for all n sufficiently large, $\mathcal{H}_n$ will be a smooth null hypersurface that meets V in a compact surface $\hat{\Sigma}_n$ outside of, and homologous to, Σ. By Raychaudhuri's equation for a null geodesic congruence [15, 22] and the null energy condition (which is a consequence of the dominant energy condition), the expansion of the null generators of $\mathcal{H}_n$ must be nonincreasing to the future. It follows that, for n large, $\hat{\Sigma}_n$ is outer trapped, contrary to the assumption that Σ is weakly outermost. □

Hence, Σ is weakly outermost in $\tilde{V}$, in the restricted sense of the claim. But this version of weakly outermost is clearly sufficient for the proof of Theorem 3.1. Thus, by this slight modification of Theorem 3.1, there exists a foliation $\{\tilde{\Sigma}_u\}$, $0 \le u \le u_0$, of a neighborhood $\tilde{U}$ of Σ in $\tilde{V}_+$ by MOTS, $\tilde{\theta}_+(u) = 0$. Pushing each $\tilde{\Sigma}_u$ along its future directed outward null normal geodesics into V, we obtain, by taking u_0 smaller if necessary, a smooth foliation $\{\Sigma_u\}$, $0 \le u \le u_0$, of a neighborhood U of Σ in V_+. Moreover, the argument based on Raychaudhuri's equation used in the claim now implies that, for each $u \in (0, u_0)$, Σ_u is weakly outer trapped, i.e., has null expansion $\theta_+(u) \le 0$. If $\theta_+(u) < 0$ at some point, one could perturb Σ_u within V to obtain a strictly outer trapped surface in V homologous to Σ (see the first remark after Definition 1.1). It follows that each Σ_u in the foliation is a MOTS. Moreover, the same argument as that used at the end of the proof of Theorem 3.1 implies that for each $u \in [0, u_0)$, $\chi_u = 0$, Σ_u is Ricci flat and $\mathcal{T}(Z, K)$ vanishes along Σ_u. $\square$

We remark in closing that the curvature estimates of Andersson and Metzger [4] provide criteria for extending the local foliation by MOTS in Theorem 3.1 to a global one.

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