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On the rational homotopy type of a moduli space of vector bundles over a curve

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We study the rational homotopy of the moduli space $\mathcal{N}_X$ that parametrizes the isomorphism classes of all stable vector bundles of rank two and fixed determinant of odd degree over a compact connected Riemann surface X of genus g, with $g \geq 2$. The symplectic group $\mathrm{Aut}(H_1(X,\mathbb{Z})) \cong \mathrm{Sp}(2g,\mathbb{Z})$ has a natural action on the rational homotopy groups $\pi_n(\mathcal{N}_X)\otimes_{\mathbb{Z}}\mathbb{Q}$. We prove that this action extends to an action of $\mathrm{Sp}(2g,\mathbb{C})$ on $\pi_n(\mathcal{N}_X)\otimes_{\mathbb{Z}}\mathbb{C}$. We also show that $\pi_n(\mathcal{N}_X)\otimes_{\mathbb{Z}}\mathbb{C}$ is a non-trivial representation of $\mathrm{Sp}(2g,\mathbb{C}) \cong \mathrm{Aut}(H_1(X,\mathbb{C}))$ for all $n \geq 2g-1$. In particular, $\mathcal{N}_X$ is a rationally hyperbolic space. In the special case where $g = 2$, for each $n \in \mathbb{N}$, we compute the leading $\mathrm{Sp}(2g,\mathbb{C})$ representation occurring in $\pi_n(\mathcal{N}_X) \otimes_{\mathbb{Z}} \mathbb{C}$.

1. Introduction

Moduli spaces of vector bundles over curves have been studied from various points of view. The aim here is to initiate investigations of their rational homotopy groups.

Let X be an irreducible smooth projective curve, defined over $\mathbb{C}$, of genus g, with $g \geq 2$. Fix a holomorphic line bundle L_0 over X of degree 1, and consider the moduli space $\mathcal{N}_X$ of stable vector bundles $E \to X$ of rank two with $\bigwedge^2 E \cong L_0$. This moduli space $\mathcal{N}_X$ is an irreducible smooth complex projective variety of complex dimension $3g-3$ (see [19]).

The mapping class group of X acts in a natural way on the cohomology algebra $H^*(\mathcal{N}_X, \mathbb{Q})$ of $\mathcal{N}_X$. This action actually factors through an action of the symplectic group $\mathrm{Aut}(H_1(X,\mathbb{Z})) \cong \mathrm{Sp}(2g,\mathbb{Z})$, which is a quotient of the mapping class group. Moreover, the descended action of $\mathrm{Aut}(H_1(X,\mathbb{Z}))$ on $H^*(\mathcal{N}_X,\mathbb{Q})$ extends to an action of $\mathrm{Aut}(H_1(X,\mathbb{C})) \cong \mathrm{Sp}(2g,\mathbb{C})$ on $H^*(\mathcal{N}_X,\mathbb{C})$. On the other hand, using the fact that $\mathcal{N}_X$ is simply connected, the mapping class group acts naturally on the homotopy groups $\pi_*(\mathcal{N}_X)$. Therefore, the mapping class group acts on $\pi_*(\mathcal{N}_X)\otimes_{\mathbb{Z}}\mathbb{Q}$.

Fix a symplectic basis of $H_1(X,\mathbb{Z})$. Using this basis $\mathrm{Aut}(H_1(X,\mathbb{Z}))$ (respectively, $\mathrm{Aut}(H_1(X,\mathbb{C}))$) gets identified with $\mathrm{Sp}(2g,\mathbb{Z})$ (respectively, $\mathrm{Sp}(2g,\mathbb{C})$).

Our first main result is the following (see Theorem 5.2).

Theorem 1.1. *The action of the mapping class group on the rational homotopy groups $\pi_*(\mathcal{N}_X)\otimes_{\mathbb{Z}}\mathbb{Q}$ factors through an action of the symplectic group $\mathrm{Sp}(2g,\mathbb{Z})$. This descended action of $\mathrm{Sp}(2g,\mathbb{Z})$ extends to an action of $\mathrm{Sp}(2g,\mathbb{C})$ on $\pi_*(\mathcal{N}_X)\otimes_{\mathbb{Z}}\mathbb{C}$.*

We shall prove this theorem in Section 5 using the formality of $\mathcal{N}_X$ and endowing the minimal model of $\mathcal{N}_X$ with an action of $\mathrm{Sp}(2g,\mathbb{C})$.

In Sections 6 and 7, we study the $\mathrm{Sp}(2g,\mathbb{C})$ representations $\pi_n(\mathcal{N}_X)\otimes_{\mathbb{Z}}\mathbb{C}$. In the special case of $g=2$, we compute the leading representation for each $n\geq 2$ (Theorem 6.3).

In the general case where $g\geq 2$, we find some non-trivial irreducible $\mathrm{Sp}(2g,\mathbb{C})$ representations contained in $\pi_n(\mathcal{N}_X)\otimes_{\mathbb{Z}}\mathbb{C}$ for each $n\geq 2g$ (see Theorem 6.3 for the case of $g=2$ and Theorem 7.1 for $g>2$). We have the following result.

Theorem 1.2. *Take any integer n with $n\geq 2g$. The $\mathrm{Sp}(2g,\mathbb{C})$ module $\pi_n(\mathcal{N}_X)\otimes_{\mathbb{Z}}\mathbb{C}$ is non-trivial. So the action of $\mathrm{Sp}(2g,\mathbb{Z})$ on the rational homotopy groups $\pi_n(\mathcal{N}_X)\otimes_{\mathbb{Z}}\mathbb{Q}$ is non-trivial, and the action of the mapping class group on $\pi_n(\mathcal{N}_X)$ is non-trivial.*

A simply connected finite CW complex Z (e.g., a compact one-connected manifold) is said to be *rationally elliptic* if the total dimension of the rational homotopy groups is finite, or in other words,

$$\sum_{n\in\mathbb{N}}\dim\pi_n(Z)\otimes_{\mathbb{Z}}\mathbb{Q}\,<\,\infty\,.$$

Otherwise, Z is called *rationally hyperbolic* (see [6]). If Z is rationally elliptic of dimension N, then $\pi_n(Z)\otimes_{\mathbb{Z}}\mathbb{Q}=0$ for all $n\geq 2N$ (equivalently, $\pi_n(Z)$ are torsion for $n\geq 2N$). On the other hand, if Z is rationally hyperbolic of dimension N, then

$$f(k)=\sum_{i=1}^{N-1}\dim\pi_{k+i}(Z)\otimes_{\mathbb{Z}}\mathbb{Q}$$

grows faster than any polynomial in k. This dichotomy is discussed in [6].

A byproduct of Theorem 1.2 is the following corollary.

Corollary 1.3. *The moduli space $\mathcal{N}_X$ is rationally hyperbolic for all $g \geq 2$.*

2. The moduli space $\mathcal{N}_X$

Let X be an irreducible smooth complex projective curve of genus $g \geq 2$. Fix a holomorphic line bundle L_0 over X of degree 1. Let $\mathcal{N}_X$ denote the moduli space of stable vector bundles over X of rank two and $\det(E) = \bigwedge^2 E = L_0$. This moduli space $\mathcal{N}_X$ is an irreducible smooth complex projective variety of complex dimension $3g-3$ (see [19]). In particular, it is a compact connected C^∞ (real) manifold of dimension $6g-6$. The complex structure of X endows $\mathcal{N}_X$ with a natural Kähler structure [1].

If we take any holomorphic line bundle L_1 over X of odd degree $2d+1$, then there is a holomorphic line bundle μ over X of degree d such that $L_1 \cong L_0 \otimes \mu^2$. Therefore, the map defined by $E \longmapsto E \otimes \mu$ is an algebraic isomorphism from $\mathcal{N}_X$ to the moduli space of stable vector bundles of rank two over X with determinant L_1. In particular, the isomorphism class of the variety $\mathcal{N}_X$ is independent of the choice of the line bundle L_0.

The diffeomorphism class of the real manifold $\mathcal{N}_X$ is independent of the complex structure of X. This can be seen as follows. Fix a point $x_0 \in X$, and set $X' = X \setminus \{x_0\}$ to be the complement. Choosing a point $x' \in X'$, consider the subset

$$\mathrm{Hom}^0(\pi_1(X', x')\,, \mathrm{SU}(2)) \subset \mathrm{Hom}(\pi_1(X', x')\,, \mathrm{SU}(2))$$

parametrizing all homomorphisms from the fundamental group $\pi_1(X', x')$ to $\mathrm{SU}(2)$ satisfying the condition that the image of the conjugacy class in $\pi_1(X', x')$ corresponding to the free homotopy class of oriented loops in X' around x_0 (with anticlockwise orientation) is $-\mathrm{Id}$. Let

$$\frac{\mathrm{Hom}^0(\pi_1(X', x')\,, \mathrm{SU}(2))}{\mathrm{SU}(2)} \tag{2.1}$$

be the quotient space for the adjoint action of $\mathrm{SU}(2)$ on itself. It is easy to see that $\mathcal{R}_g$ is a connected compact C^∞ manifold of dimension $6g-6$ (see [16]). Given any homomorphism $\rho \in \mathrm{Hom}^0(\pi_1(X', x'), \mathrm{SU}(2))$, the corresponding flat vector bundle over the Riemann surface X' extends to X as a holomorphic vector bundle with a logarithmic connection which has residue $-\frac{1}{2}\mathrm{Id}$ at x_0 (see [4]). The underlying holomorphic vector bundle E_ρ on X is stable and $\det(E_\rho) = \mathcal{O}_X(x_0)$. Sending any ρ to E_ρ we obtain a diffeomorphism of $\mathcal{R}_g$ with $\mathcal{N}_X$ for $L_0 = \mathcal{O}_X(x_0)$ (cf. [14]). Therefore, the diffeomorphism class of the real manifold $\mathcal{N}_X$ is independent of the complex structure of X (it depends only on the genus of X).

It is easy to see that the manifold $\mathcal{R}_g$ in Equation (2.1) is simply connected [16, Corollary 2]. By [1, Theorem 9.10], the cohomology ring $H^*(\mathcal{N}_X, \mathbb{Z})$ is torsion-free, and by [1, Proposition 9.13], we have $H^2(\mathcal{N}_X, \mathbb{Z}) = \mathbb{Z}$. Consequently, the variety $\mathcal{N}_X$ has a natural polarization. Henceforth, we shall denote by α the natural positive (i.e., ample) generator of $H^2(\mathcal{N}_X, \mathbb{Z})$.

Next we will describe an action of the mapping class group on the cohomology of $\mathcal{N}_X$. For that purpose, consider the moduli space $\mathcal{M}_g^1$ parametrizing all isomorphism classes of one-pointed compact Riemann surfaces (Y, y) of genus g with $\mathrm{Aut}(Y, y) = e$ (i.e., Y does not have any non-trivial automorphism that fixes the marked point $y \in Y$). This moduli space is a smooth irreducible quasiprojective variety of dimension $3g - 2$ defined over the field $\mathbb{C}$. Given any (Y, y), there is a natural choice of a holomorphic line bundle of degree 1 over Y, namely $\mathcal{O}_Y(y)$. There is a universal family of Riemann surfaces

$$p : \mathcal{C}_g \longrightarrow \mathcal{M}_g^1 \tag{2.2}$$

and a holomorphic section $h : \mathcal{M}_g^1 \longrightarrow \mathcal{C}_g$ giving the marked point. Let

$$P : \widetilde{\mathcal{N}} \longrightarrow \mathcal{M}_g^1 \tag{2.3}$$

be the family of moduli spaces of stable vector bundles of rank two with fixed determinant corresponding to the family of Riemann surfaces in Equation (2.2). For any one-pointed Riemann surface $\underline{x} = (Y, y) \in \mathcal{M}_g^1$, the fiber $P^{-1}(\underline{x})$ is the moduli space $\mathcal{N}_Y$ parametrizing all stable vector bundles over Y of rank two and determinant $\mathcal{O}_Y(y)$.

Fix a base point $\underline{x}_0 = (X, x_0) \in \mathcal{M}_g^1$ of the moduli space. Let $G_{\mathbb{Z}}$ (respectively, $G_{\mathbb{C}}$) denote the group of all automorphisms of $H^1(X, \mathbb{Z})$ (respectively, $H^1(X, \mathbb{C})$) preserving the symplectic pairing given by the cup product. Choosing a symplectic basis of $H^1(X, \mathbb{Z})$, the groups $G_{\mathbb{Z}}$ and $G_{\mathbb{C}}$ get identified with $\mathrm{Sp}(2g, \mathbb{Z})$ and $\mathrm{Sp}(2g, \mathbb{C})$ respectively.

Convention. In the sequel, we will interchange $G_{\mathbb{Z}}$ (respectively, $G_{\mathbb{C}}$) and $\mathrm{Sp}(2g, \mathbb{Z})$ (respectively, $\mathrm{Sp}(2g, \mathbb{C})$).

Consider the local system $R^1 p_* \underline{\underline{\mathbb{Z}}}$ on $\mathcal{M}_g^1$, where p is the projection in Equation (2.2), and $\underline{\underline{\mathbb{Z}}}$ is the constant local system on $\mathcal{C}_g$ with stalk $\mathbb{Z}$. Using its monodromy, the group $G_{\mathbb{Z}}$ is a quotient of the fundamental group

$$\Gamma_g^1 := \pi_1(\mathcal{M}_g^1, \underline{x}_0) .$$

This group Γ_g^1 is known as the *mapping class group*, and the kernel of the projection of Γ_g^1 to $G_{\mathbb{Z}}$ is known as the *Torelli group.*

Actually, the mapping class group has a natural action on the moduli space $\mathcal{N}_X = P^{-1}(\underline{x}_0)$. To see this action, note that using the earlier mentioned identification $\mathcal{N}_X = \mathcal{R}_g$ (defined in Equation (2.1)), the fiber bundle P in Equation (2.3) has a natural flat connection (this flat connection is not holomorphic). The monodromy of this flat connection gives an action of $\Gamma_g^1 = \pi_1(\mathcal{M}_g^1, \underline{x}_0)$ on $\mathcal{N}_X$; more details can be found in [2].

The action of Γ_g^1 on $H^i(\mathcal{N}_X, \mathbb{Z})$ induced by the above action of Γ_g^1 on $\mathcal{N}_X$ evidently coincides with the monodromy representation of the local system $R^iP_*\underline{\mathbb{Z}}$ on $\mathcal{M}_g^1$, where $\underline{\mathbb{Z}}$ is the constant local system on $\widetilde{\mathcal{N}}$ with stalk $\mathbb{Z}$.

Proposition 2.1. *The action of the mapping class group on the cohomology algebra $H^*(\mathcal{N}_X, \mathbb{Q})$ factors through an action of the symplectic group $G_{\mathbb{Z}} =$ $\mathrm{Sp}(2g, \mathbb{Z})$. Moreover, this action of $\mathrm{Sp}(2g, \mathbb{Z})$ on $H^i(\mathcal{N}_X, \mathbb{Q})$ extends to an action of $G_{\mathbb{C}} = \mathrm{Sp}(2g, \mathbb{C})$ on $H^i(\mathcal{N}_X, \mathbb{C})$.*

Proof. The cohomology algebra $H^*(\mathcal{N}_X, \mathbb{Q})$ is generated by the Künneth components of the second Chern class of the adjoint bundle of a universal vector bundle over $X \times \mathcal{N}_X$ (see [1, 18] and also Section 3). Note that although there is no unique universal bundle over $X \times \mathcal{N}_X$, any two universal bundles differ by tensoring with a line bundle pulled back from $\mathcal{N}_X$. Therefore, the universal adjoint bundle is unique. Consequently, the local system $\bigoplus_{i\geq 0} R^iP_*\underline{\mathbb{C}}$ on $\mathcal{M}_g^1$, where $\underline{\mathbb{C}}$ is the constant local system on $\widetilde{\mathcal{N}}$ with stalk $\mathbb{C}$, is a quotient of some local system on $\mathcal{M}_g^1$ of the form

$$\mathcal{W} := \bigoplus_{j=1}^{\ell} \left(\left(\bigoplus_{i=0}^{2} R^i p_* \underline{\underline{\mathbb{C}}} \right)^{\oplus a_j} \right)^{\otimes b_j},$$

where $\ell, a_j, b_j \in \mathbb{N}$, the map p is the projection in Equation (2.2) and $\underline{\underline{\mathbb{C}}}$ is the constant local system on $\mathcal{C}_g$ with stalk $\mathbb{C}$. In other words, we have a surjective homomorphism of local systems

$$\mathcal{W} \longrightarrow \bigoplus_{i\geq 0} R^iP_*\underline{\mathbb{C}} \longrightarrow 0. \tag{2.4}$$

Both $R^0p_*\underline{\underline{\mathbb{C}}}$ and $R^2p_*\underline{\underline{\mathbb{C}}}$ are constant local systems on $\mathcal{M}_g^1$, and the monodromy of the local system $R^1p_*\underline{\underline{\mathbb{C}}}$, by definition, factors through $G_{\mathbb{Z}}$. Consequently, the monodromy representation

$$\Gamma_g^1 \longrightarrow \mathrm{Aut}(\mathcal{W}_{\underline{x}_0}) \tag{2.5}$$

of the mapping class group for the local system $\mathcal{W}$ on $\mathcal{M}_g^1$ factors through $G_{\mathbb{Z}}$. Hence, the Torelli group is in the kernel of the monodromy representation

$$\Gamma_g^1 \longrightarrow \prod_{i\geq 0} \mathrm{Aut}((R^i P_* \underline{\mathbb{C}})_{\underline{x}_0}) \tag{2.6}$$

of the mapping class group for the quotient local system in Equation (2.4). Therefore, the homomorphism in Equation (2.6) factors through the quotient $G_{\mathbb{Z}}$ of Γ_g^1.

To prove that the action of $\mathrm{Sp}(2g, \mathbb{Z})$ on $H^i(\mathcal{N}_X, \mathbb{Q})$ extends to an action of $G_{\mathbb{C}} = \mathrm{Sp}(2g, \mathbb{C})$ on $H^i(\mathcal{N}_X, \mathbb{C})$, first note that the monodromy representation

$$G_{\mathbb{Z}} \longrightarrow \mathrm{Aut}(\mathcal{W}_{\underline{x}_0})$$

in Equation (2.5) extends to a homomorphism from $\mathrm{Sp}(2g, \mathbb{C})$. The kernel of the surjective homomorphism

$$\mathcal{W}_{\underline{x}_0} \longrightarrow \bigoplus_{i\geq 0} (R^i P_* \underline{\mathbb{C}})_{\underline{x}_0} \tag{2.7}$$

obtained from Equation (2.4) is preserved by $G_{\mathbb{Z}}$. On the other hand, $\mathrm{Sp}(2g, \mathbb{Z})$ is Zariski dense in $\mathrm{Sp}(2g, \mathbb{C})$ (see [3]). Hence the kernel of the homomorphism in Equation (2.7) is preserved by the action of $\mathrm{Sp}(2g, \mathbb{C})$ on $\mathcal{W}_{\underline{x}_0}$. Consequently, the action of $\mathrm{Sp}(2g, \mathbb{C})$ on $\mathcal{W}_{\underline{x}_0}$ induces an action of $\mathrm{Sp}(2g, \mathbb{C})$ on the quotient in Equation (2.7). This completes the proof of the proposition. □

3. Cohomology ring of $\mathcal{N}_X$

Let us recall the known description of the cohomology ring $H^*(\mathcal{N}_X, \mathbb{Q})$ of the moduli space $\mathcal{N}_X$ (see [9, 11, 20]). Consider a universal bundle $\mathcal{U} \to X \times \mathcal{N}_X$. Let $\mathrm{End}_0(\mathcal{U}) \to X \times \mathcal{N}_X$ be the adjoint vector bundle (we recall that $\mathrm{End}_0(\mathcal{U}) \subset \mathrm{End}(\mathcal{U})$ is subbundle of rank three given by the trace-free endomorphisms of the fibers of $\mathcal{U}$). The Künneth decomposition of the second Chern class $c_2(\mathrm{End}_0(\mathcal{U})) \in H^4(X \times \mathcal{N}_X, \mathbb{Z})$ can be written as

$$c_2(\mathrm{End}_0\,(\mathcal{U})) = 2[X] \otimes \alpha + 4\,\psi - 1 \otimes \beta\,, \tag{3.1}$$

where $\beta \in H^4(\mathcal{N}_X, \mathbb{Z})$, $[X] \in H^2(X, \mathbb{Z})$ denotes the fundamental class of the Riemann surface X, $\alpha \in H^2(\mathcal{N}_X, \mathbb{Z})$ as before is the positive generator of $H^2(\mathcal{N}_X, \mathbb{Z}) = \mathbb{Z}$, and $\psi \in H^1(X, \mathbb{Z}) \otimes_{\mathbb{Z}} H^3(\mathcal{N}_X, \mathbb{Z})$. Let $\{c_1, \ldots, c_{2g}\}$

be a symplectic basis of $H^1(X,\mathbb{Z})$, which means that $c_i \cup c_{i+g} = [X]$ for all $1 \le i \le g$, and $c_j \cup c_k = 0$ for all j,k with $|j-k| \ne g$. It is known that $\psi = \sum_{i=1}^{2g} c_i \otimes \gamma_i$, where $\{\gamma_1,\dots,\gamma_{2g}\}$ is a basis for $H^3(\mathcal{N}_X,\mathbb{Z})$; see [12]. In other words, ψ gives an isomorphism

$$H^1(X,\mathbb{Z}) \;=\; H^1(X,\mathbb{Z})^* \;\longrightarrow\; H^3(\mathcal{N},\mathbb{Z}). \tag{3.2}$$

The elements α,β and γ_i, $1 \le i \le 2g$, together generate $H^*(\mathcal{N}_X,\mathbb{Q})$ as an algebra [18, 1, 21]. We can rephrase this as saying that there exists an epimorphism

$$F\colon \bigwedge(\alpha,\gamma_1,\dots,\gamma_{2g},\beta) := \mathbb{Q}[\alpha,\beta]\otimes\wedge(\gamma_1,\dots,\gamma_{2g}) \twoheadrightarrow H^*(\mathcal{N}_X,\mathbb{Q}), \tag{3.3}$$

where $\deg(\alpha)=2$, $\deg(\beta)=4$ and $\deg(\gamma_i)=3$, $1\le i\le 2g$. Here $\bigwedge$ means the free graded algebra generated by the given elements, which is the tensor product of the symmetric algebra on the even-degree elements and the exterior algebra on the odd-degree elements.

We shall denote by W the standard $\mathbb{Q}$ representation of $G_{\mathbb{Z}} = \mathrm{Sp}(2g,\mathbb{Z})$, so

$$H^1(X,\mathbb{Q}) \cong W\,.$$

We noted in Section 2 that the monodromy action of Γ_g^1 on $H^*(\mathcal{N}_X,\mathbb{Q})$ factors through an action of $\mathrm{Sp}(2g,\mathbb{Z})$. It is easy to see that this action fixes both α and β, and furthermore, the isomorphism in Equation (3.2) is $\mathrm{Sp}(2g,\mathbb{Z})$-equivariant. Therefore,

$$H^3(\mathcal{N}_X,\mathbb{Q}) \cong H^1(X,\mathbb{Q})^* \cong W^* \cong W$$

as $\mathrm{Sp}(2g,\mathbb{Z})$ representations.

Let

$$H_I^*(\mathcal{N}_X,\mathbb{Q}) \;\subset\; H^*(\mathcal{N}_X,\mathbb{Q})$$

be the subalgebra fixed pointwise by the action of $\mathrm{Sp}(2g,\mathbb{Z})$. The epimorphism in Equation (3.3) is $\mathrm{Sp}(2g,\mathbb{Z})$-equivariant, and $\mathrm{Sp}(2g,\mathbb{Z})$ is Zariski dense in the reductive group $\mathrm{Sp}(2g,\mathbb{C})$ [3]. Using these we conclude that the invariant part $H_I^*(\mathcal{N}_X,\mathbb{Q})$ is generated by α, β and $\gamma = -2\sum_{i=1}^{g}\gamma_i\gamma_{i+g}$ (the factor of -2 is for convenience, to be in accordance with the existing literature). Then the epimorphism F in Equation (3.3) gives an epimorphism

$$\mathbb{Q}[\alpha,\beta,\gamma] \;\twoheadrightarrow\; H_I^*(\mathcal{N}_X,\mathbb{Q})\,, \tag{3.4}$$

where $\deg(\alpha) = 2$, $\deg(\beta) = 4$ and $\deg(\gamma) = 6$. Hence we may write

$$H_I^*(\mathcal{N}_X, \mathbb{Q}) = \frac{\mathbb{Q}[\alpha, \beta, \gamma]}{I_g}, \tag{3.5}$$

where I_g is an ideal of relations satisfied by α, β and γ.

For each $0 \leq k \leq g$, the primitive component of $\wedge^k W$ is defined as

$$\wedge_0^k W = \ker(\gamma^{g-k+1} \colon \wedge^k W \longrightarrow \wedge^{2g-k+2} W).$$

The spaces $\wedge_0^k W$ are irreducible $\mathrm{Sp}(2g, \mathbb{Z})$ representations.

The descriptions of the ideal I_g and the cohomology ring $H^*(\mathcal{N}_X, \mathbb{Q})$ are given in the following proposition.

Proposition 3.1 [9, 20]. *Define $q_0^1 = 1$, $q_0^2 = 0$, $q_0^3 = 0$ and then recursively, for all $r \geq 1$,*

$$\begin{aligned} q_{r+1}^1 &= \alpha q_r^1 + r^2 q_r^2 , \\ q_{r+1}^2 &= \beta q_r^1 + \frac{2r}{r+1} q_r^3 , \\ q_{r+1}^3 &= \gamma q_r^1 . \end{aligned}$$

Then $I_g = (q_g^1, q_g^2, q_g^3) \subset \mathbb{Q}[\alpha, \beta, \gamma]$, for all $g \geq 1$. Note that $\deg(q_g^1) = 2g$, $\deg(q_g^2) = 2g + 2$ *and* $\deg(q_g^3) = 2g + 4$. *Moreover the* $\mathrm{Sp}(2g, \mathbb{Z})$ *decomposition of $H^*(\mathcal{N}_X, \mathbb{Q})$ is*

$$H^*(\mathcal{N}_X, \mathbb{Q}) = \bigoplus_{k=0}^{g-1} \wedge_0^k W \otimes \frac{\mathbb{Q}[\alpha, \beta, \gamma]}{I_{g-k}} . \tag{3.6}$$

Lemma 3.2. *The vector space*

$$E = \langle q_g^1 \rangle \oplus \langle q_g^2 \rangle \oplus (q_{g-1}^1 \cdot W) \oplus (q_{g-2}^1 \cdot \wedge_0^2 W) \oplus \cdots \oplus (q_1^1 \cdot \wedge_0^{g-1} W) \oplus \wedge_0^g W ,$$

realized as a subspace of $\mathbb{A} := \bigwedge(\alpha, \gamma_1, \ldots, \gamma_{2g}, \beta) = \mathbb{Q}[\alpha , \beta] \otimes \wedge(\gamma_1, \ldots, \gamma_{2g})$ using the identification $W = \langle \gamma_1, \ldots, \gamma_{2g} \rangle$, generates the ideal $\mathrm{kernel}(F)$ *of the map F in Equation* (3.3).

Proof. Clearly we have $E \subset \mathrm{kernel}(F)$. We will prove the reverse inclusion

$$\mathrm{kernel}(F) \subset I(E) ,$$

where $I(E)$ is the ideal generated by E in $\mathbb{A}$.

By Proposition 3.1, kernel(F) is generated by $q^i_{g-k} \cdot \wedge^k_0 W$, where $i \in [1,3]$ and $k \in [0,g]$. Note that since $q_0^2 = 0$ and $q_0^3 = 0$, it suffices to prove the following two:

(1) $q^2_{g-k} \cdot \wedge^k_0 W \subset I(E)$ for $1 \leq k \leq g-1$, and

(2) $q^3_{g-k} \cdot \wedge^k_0 W \subset I(E)$ for $0 \leq k \leq g-1$.

We shall use the following inclusions:

$$\gamma \cdot \wedge^j_0 W \subset I(\wedge^{j+1}_0 W), \quad 0 \leq j \leq g-1\,, \tag{3.7}$$

$$\wedge^{j+1}_0 W \subset I(\wedge^j_0 W), \quad 0 \leq j \leq g-1\,. \tag{3.8}$$

For proving Equation (3.7), first note that $\wedge^j_0 W$ is an irreducible $\mathrm{Sp}(2g,\mathbb{Z})$ representation, so it is enough to see that there is a non-zero element in $\gamma \cdot \wedge^j_0 W$ which lies in $I(\wedge^{j+1}_0 W)$. Consider $\gamma_1 \cdots \gamma_j \in \wedge^j_0 W$. Then

$$\gamma \cdot \gamma_1 \cdots \gamma_j = -\,2 \sum_{i=j+1}^{g} \gamma_1 \cdots \gamma_j \gamma_{j+1} \gamma_{j+1+g}\,,$$

and $\gamma_1 \cdots \gamma_{j+1} \in \wedge^{j+1}_0 W$. Therefore $\gamma \cdot \gamma_1 \cdots \gamma_j \in I(\wedge^{j+1}_0 W)$, as required.

To prove Equation (3.8), we first note that $\wedge^{j+1}_0 W$ is an irreducible $\mathrm{Sp}(2g,\mathbb{Z})$ representation and $\gamma_1 \cdots \gamma_{j+1} \in \wedge^{j+1}_0 W$. Clearly we have $\gamma_1 \cdots \gamma_j \in \wedge^j_0 W$. Hence it follows that $\gamma_1 \cdots \gamma_j \gamma_{j+1} \in I(\wedge^j_0 W)$. This gives the required inclusion.

Using Equation (3.7), we have that

$$q^3_{g-k} \cdot \wedge^k_0 W = q^1_{g-k-1} \gamma \cdot \wedge^k_0 W \subset I(q^1_{g-k-1} \cdot \wedge^{k+1}_0 W) \subset I(E)$$

for all $0 \leq k \leq g-1$. Also, using Equation (3.8) we have

$$\begin{aligned} q^2_{g-k} \cdot \wedge^k_0 W \quad &= \quad \frac{1}{(g-k)^2} (q^1_{g-k+1} - \alpha q^1_{g-k}) \cdot \wedge^k_0 W \\ &\subset \quad I(q^1_{g-k+1} \cdot \wedge^{k-1}_0 W \oplus q^1_{g-k} \cdot \wedge^k_0 W) \subset I(E)\,, \end{aligned}$$

for all $1 \leq k \leq g-1$. □

Remark 3.3. The subspace E in Lemma 3.2 is minimal in the sense that no proper subspace of E generates kernel(F).

4. Minimal models

Let us recall some definitions and results about minimal models (see [5, 8] for more details). Let (A, d) be a *differential graded algebra* (in the sequel, we shall just say a differential algebra). This means that A is a graded (in non-negative degrees) commutative algebra over a field K, of characteristic zero, and $d\colon A^n \longrightarrow A^{n+1}$ is a differential which satisfies the derivation condition which says that

$$d(a \cdot b) = (da) \cdot b + (-1)^{\deg(a)} a \cdot (db)\,,$$

where $\deg(a)$ is the degree of a. Throughout this article we shall assume that $K = \mathbb{C}$, the field of complex numbers.

Morphisms between differential algebras are required to be degree-preserving algebra maps that commute with the differentials. Given a differential algebra (A, d), we denote by $H^*(A, d)$ its cohomology. We say that A is *connected* if $H^0(A, d) = \mathbb{C}$, and *one-connected* if, in addition, $H^1(A, d) = 0$.

A differential algebra (A, d) is said to be *minimal* if the following two hold:

(i) A is free as a graded algebra, that is, $A = \bigwedge V$, where $V = \oplus_{i>0} V^i$ is a graded vector space, and

(ii) there exists a collection of generators $\{a_\tau\}_{\tau \in I}$ of the algebra A, where I is some well-ordered index set, such that $\deg(a_\mu) \leq \deg(a_\tau)$ if $\mu < \tau$ and each da_τ is expressed in terms of preceding a_μ, $\mu < \tau$.

As before, $\bigwedge V$ is the tensor product of the symmetric algebra on the even degree part of V with the exterior algebra on the odd degree part of V

For notational convenience, we shall use the dot "$\cdot$" to denote the product operation on $\bigwedge V$.

For any n, define $V^{\leq n} := \bigoplus_{i \leq n} V^i$. So $\bigwedge V^{\leq n} = \bigwedge(\bigoplus_{i\leq n} V^i)$ is the subalgebra generated by elements of degrees at most n. For any m, let $(\bigwedge V)^m$ denote the subspace of $\bigwedge V$ spanned by all elements of total degree m. Finally, for $k \geq 1$, let $\bigwedge^{\geq k} V$ denote the ideal formed by elements which are products of at least k generators. In other words,

$$\bigwedge^{+} V := \left(\bigwedge V\right)^{>0} = \bigoplus_{m>0} \left(\bigwedge V\right)^{m},$$

and

$$\bigwedge^{\geq k} = \overbrace{\bigwedge^{+} V \cdots \bigwedge^{+} V}^{k\text{-times}}.$$

Note that the condition (ii) in the definition of minimality implies that $d\colon V \longrightarrow \bigwedge^{\geq 2} V$, and hence $d\colon \bigwedge^{i} V \longrightarrow \bigwedge^{\geq (i+1)} V$, for all $i \geq 1$. Notations like $(\bigwedge^{\geq i} V^{<j})^n$ have natural meaning.

Given a differential algebra (A, d), we shall say that $(\bigwedge V, d)$ is a *minimal model* of (A, d) if $(\bigwedge V, d)$ is minimal and there exists a morphism of differential graded algebras $\rho\colon (\bigwedge V, d) \longrightarrow (A, d)$ such that the induced homomorphism of cohomologies

$$\rho^* : H^*\left(\bigwedge V, d\right) \longrightarrow H^*(A, d)$$

is an isomorphism. Such a homomorphism ρ is called a *quasi-isomorphism.* Any one-connected differential algebra (A, d) has a minimal model unique up to an isomorphism [5, 8].

A *minimal model* of a connected differentiable manifold M is a minimal model $(\bigwedge V, d)$ for the de Rham complex $(\Omega^*(M, \mathbb{C}), d)$ of complex C^∞ differential forms on M. If M is simply connected, then the dual of the complex homotopy vector space $\pi_i(M) \otimes_{\mathbb{Z}} \mathbb{C}$ is isomorphic to V^i for any $i > 0$ (see [8]).

A minimal model $(\bigwedge V, d)$ is said to be *formal* if there is a morphism of differential algebras

$$\psi : \left(\bigwedge V, d\right) \longrightarrow (H^*\left(\bigwedge V, d\right), 0)$$

which induces the identity map on cohomology. This means that $(\bigwedge V, d)$ is the minimal model of the algebra $(H^*(\bigwedge V, d), 0)$ with zero differential.

We shall say that a connected differentiable manifold M is *formal* if its minimal model is formal, or equivalently, the two differential algebras $(\Omega^*(M, \mathbb{C}), d)$ and $(H^*(M, \mathbb{C})\,, 0)$ have the same minimal model. Therefore, if M is formal and simply connected, then the complex homotopy groups $\pi_i(M) \otimes_{\mathbb{Z}} \mathbb{C}$ are obtained by computing the minimal model of $(H^*(M, \mathbb{C})\,, 0)$.

The main result of [5] gives the following strong topological restriction on the rational homotopy type of Kähler manifolds.

Theorem 4.1 [5]**.** *Let M be a compact connected Kähler manifold. Then M is formal.*

Therefore, the minimal model of a compact connected Kähler manifold M can be obtained from the minimal model of its cohomology algebra $(H^*(M,\mathbb{C}),0)$. Moreover, if the Kähler manifold M is simply connected, this process will also give us the complex homotopy group $\pi_i(M)\otimes_{\mathbb{Z}}\mathbb{C}$ of M.

We will briefly review a construction of the minimal model of a differential algebra (A,d). For simplicity, we shall assume that (A,d) is one-connected. We need to find a graded vector space $V=\oplus_{n\geq 1}V^n$, a differential d, with

$$d|_{V^n}\colon V^n \longrightarrow \bigwedge^{\geq 2} V^{\leq(n-1)},$$

and a graded linear map

$$\rho=\sum \rho_n\colon V=\oplus V^n \longrightarrow A=\oplus A^n$$

such that the induced homomorphism $\rho\colon \bigwedge V\longrightarrow A$ respects the differentials, which means that $\rho\circ d=d\circ\rho$, and furthermore, the map on cohomology

$$\rho^*\colon H^*\left(\bigwedge V,d\right)\longrightarrow H^*(A,d)$$

is an isomorphism.

We shall construct V^n, ρ_n and $d|_{V^n}$, where $n\geq 1$, using induction on n. They will satisfy the following conditions:

(i) $\rho_n\colon V^n\longrightarrow A^n$;

(ii) $d_n=d|_{V^n}\colon V^n\longrightarrow \bigwedge^{\geq 2}V^{\leq(n-1)}$;

(iii) $\rho_{\leq(n-1)}\circ d_n=d\circ\rho_n$ on V^n, where $\rho_{\leq(n-1)}\colon \bigwedge V^{\leq(n-1)}\longrightarrow A$ is induced by the map ρ_i, $i\leq n-1$;

(iv) $\rho^*_{\leq n}\colon H^i(\bigwedge V^{\leq n},d)\xrightarrow{\simeq} H^i(A,d)$ is an isomorphism for $i\leq n$;

(v) $\rho^*_{\leq n}\colon H^{n+1}(\bigwedge V^{\leq n},d)\hookrightarrow H^{n+1}(A,d)$ is an injection.

From these conditions it follows that the map $\rho\colon(\bigwedge V,d)\longrightarrow(A,d)$, constructed using ρ_n on each subspace V^n, is a quasi-isomorphism. Given any i, we evidently have $(\bigwedge V)^k=(\bigwedge V^{\leq(i+1)})^k$ for all $k\leq i+1$. So $H^i(\bigwedge V,d)\cong H^i(\bigwedge V^{\leq(i+1)},d)$. The composition

$$\left(\bigwedge V^{\leq(i+1)},d\right)\hookrightarrow\left(\bigwedge V,d\right)\xrightarrow{\rho}(A,d)$$

equals $\rho_{\leq(i+1)}$. Hence

$$\rho^* = \rho^*_{\leq(i+1)} : H^i\left(\bigwedge V, d\right) \cong H^i\left(\bigwedge V^{\leq(i+1)}, d\right) \xrightarrow{\simeq} H^i(A, d)$$

is an isomorphism. This proves that $(\bigwedge V, d)$ is a minimal model for (A, d).

To construct V^n, ρ_n and $d|_{V^n}$, we start with $V^1 = 0$. All conditions (i)–(v) hold trivially, since $H^1(A, d) = 0$.

Now assume that conditions (i)–(v) are satisfied for all $j \in [1, n-1]$ with $n-1 \geq 1$; let us see that we can find V^n, ρ_n and d_n also fulfilling these conditions. Take

$$\begin{aligned} V^n &= C^n \oplus N^n, \\ C^n &= \operatorname{coker}\left(\rho^*_{\leq(n-1)} \colon H^n\left(\bigwedge V^{\leq(n-1)}, d\right) \hookrightarrow H^n(A, d)\right), \\ N^n &= \ker\left(\rho^*_{\leq(n-1)} \colon H^{n+1}\left(\bigwedge V^{\leq(n-1)}, d\right) \longrightarrow H^{n+1}(A, d)\right). \end{aligned}$$

Define $\rho_n \colon V^n \longrightarrow A^n$ as follows. First, we introduce the notation

$$\begin{aligned} Z^n(A, d) &= \ker(d \colon A^n \longrightarrow A^{n+1}), \\ B^n(A, d) &= \operatorname{im}(d \colon A^{n-1} \longrightarrow A^n), \end{aligned}$$

for the spaces of cocycles and coboundaries, respectively. Let $\imath_1 \colon C^n \longrightarrow H^n(A, d)$ be a linear map which is a splitting of the projection $H^n(A, d) \twoheadrightarrow C^n$. Also, let $\imath_2 : H^n(A, d) \longrightarrow Z^n(A, d)$ be a splitting of the projection $Z^n(A, d) \twoheadrightarrow H^n(A, d)$. Let $\imath_3 \colon Z^n(A, d) \hookrightarrow A^n$ be the inclusion map. Then define

$$\rho_n|_{C^n} = \imath_3 \circ \imath_2 \circ \imath_1.$$

To define $\rho_n|_{N^n}$, let $\jmath_1 \colon N^n \hookrightarrow H^{n+1}(\bigwedge V^{\leq(n-1)}, d)$ be the inclusion. Take a splitting $\jmath_2 \colon H^{n+1}(\bigwedge V^{\leq(n-1)}, d) \longrightarrow Z^{n+1}(\bigwedge V^{\leq(n-1)}, d)$ of the obvious projection. Then $\rho_{\leq(n-1)} \circ \jmath_2 \circ \jmath_1$ has image in $B^{n+1}(A, d) \subset A^{n+1}$. Take a splitting of the map $d \colon A^n \twoheadrightarrow B^{n+1}(A, d)$, say

$$\varrho : B^{n+1}(A, d) \longrightarrow A^n,$$

and finally define

$$\rho_n|_{N^n} = \varrho \circ \rho_{\leq(n-1)} \circ \jmath_2 \circ \jmath_1.$$

Now, define d_n as follows. On C^n, we set $d_n|_{C^n} = 0$. On N^n, we put $d_n|_{N^n} = \jmath_3 \circ \jmath_2 \circ \jmath_1$, where

$$\jmath_3 : Z^{n+1}\left(\bigwedge V^{\leq(n-1)}, d\right) \hookrightarrow \left(\bigwedge V^{\leq(n-1)}\right)^{n+1} = \left(\bigwedge^{\geq 2} V^{\leq(n-1)}\right)^{n+1}$$

is the inclusion. Clearly condition (ii) holds.

To check condition (iii), we need to verify that $\rho_{\leq(n-1)} \circ d_n = d \circ \rho_n$. On C^n, we have $\rho_{\leq(n-1)} \circ d_n = 0$ and $d \circ \rho_n = d \circ \imath_3 \circ \imath_2 \circ \imath_1 = 0$. On N^n, we have

$$\rho_{\leq(n-1)} \circ d_n = \rho_{\leq(n-1)} \circ \jmath_3 \circ \jmath_2 \circ \jmath_1 = d \circ \varrho \circ \rho_{\leq(n-1)} \circ \jmath_2 \circ \jmath_1 = d \circ \rho_n$$

as $d \circ \varrho = \mathrm{Id}$. Therefore, condition (iii) holds.

Consider the inclusion $j\colon (\bigwedge V^{\leq(n-1)}, d) \hookrightarrow (\bigwedge V^{\leq n}, d)$ and the cokernel

$$B = \left(\bigwedge V^{\leq n}\right) / \left(\bigwedge V^{\leq(n-1)}\right).$$

Then (B, d) is a graded differential algebra, and $B^i = 0$ for all $i < n$, and also, $B^n = V^n = C^n \oplus N^n$. We have $(\bigwedge V^{\leq n})^{n+1} = (\bigwedge V^{\leq(n-1)})^{n+1}$ as $V^1 = 0$, and hence $B^{n+1} = 0$. Therefore,

$$j^* : H^k\left(\bigwedge V^{\leq(n-1)}, d\right) \longrightarrow H^k\left(\bigwedge V^{\leq n}, d\right)$$

is an isomorphism for all $k < n$. As $\rho^*_{\leq n} \circ j^* = \rho^*_{\leq(n-1)}$, we have that

$$\rho^*_{\leq n} : H^k\left(\bigwedge V^{\leq n}, d\right) \longrightarrow H^k(A, d)$$

is an isomorphism for all $k < n$.

To deal with the cases where $k = n, n+1$, consider the long-exact sequence associated to $\bigwedge V^{\leq(n-1)} \hookrightarrow \bigwedge V^{\leq n} \longrightarrow B$,

(4.1)

$$\begin{array}{lllll} 0 \to & H^n\left(\bigwedge V^{\leq(n-1)}, d\right) & \xrightarrow{j^*} & H^n\left(\bigwedge V^{\leq n}, d\right) & \to\ H^n(B,d) = B^n \\ & & & & \quad = C^n \oplus N^n \\ \xrightarrow{\partial^*} & H^{n+1}\left(\bigwedge V^{\leq(n-1)}, d\right) & \to & H^{n+1}\left(\bigwedge V^{\leq n}, d\right) & \to\ 0. \end{array}$$

For $x = u + w \in B^n = C^n \oplus N^n$, we have $\partial^*(x) = [d(u+w)] = [\jmath_3 \circ \jmath_2 \circ \jmath_1(w)] = \jmath_1(w)$.

Therefore the exact sequence Equation (4.1) splits into two short exact sequences:

$$0 \longrightarrow H^n\left(\bigwedge V^{\le(n-1)}, d\right) \longrightarrow H^n\left(\bigwedge V^{\le n}, d\right) \longrightarrow C^n \longrightarrow 0 \tag{4.2}$$

and

$$0 \longrightarrow N^n \xrightarrow{\jmath_1} H^{n+1}\left(\bigwedge V^{\le(n-1)}, d\right) \longrightarrow H^{n+1}\left(\bigwedge V^{\le n}, d\right) \longrightarrow 0. \tag{4.3}$$

From Equation (4.2), we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^n\left(\bigwedge V^{\le(n-1)}, d\right) & \longrightarrow & H^n\left(\bigwedge V^{\le n}, d\right) & \longrightarrow & C^n & \longrightarrow & 0 \\ & & \| & & \rho^*_{\le n}\downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^n\left(\bigwedge V^{\le(n-1)}, d\right) & \xrightarrow{\rho^*_{\le(n-1)}} & H^n(A,d) & \longrightarrow & C^n & \longrightarrow & 0. \end{array}$$

We note that the right vertical arrow is the identity map. Indeed, it sends $u \in C^n$ to the class of $\rho_n(u) = \imath_3(\imath_2(\imath_1(u)))$ in

$$\operatorname{coker}\left(\rho^*_{\le(n-1)}\colon H^n\left(\bigwedge V^{\le(n-1)}, d\right) \longrightarrow H^n(A,d)\right),$$

which is u itself. Thus the middle vertical arrow is an isomorphism, proving condition (iv) for $k = n$.

In Equation (4.3), the homomorphism $\jmath_1$ is the inclusion of

$$N^n = \ker\left(\rho^*_{\le(n-1)}\colon H^{n+1}\left(\bigwedge V^{\le(n-1)}, d\right) \longrightarrow H^{n+1}(A,d)\right)$$

in $H^{n+1}\left(\bigwedge V^{\le(n-1)}, d\right)$. So $\rho^*_{\le(n-1)}$ induces an inclusion

$$H^{n+1}\frac{\left(\bigwedge V^{\le(n-1)}, d\right)}{N^n} \cong H^{n+1}\left(\bigwedge V^{\le n}, d\right) \hookrightarrow H^{n+1}(A,d)\,.$$

This map actually coincides with $\rho^*_{\le n}$, since $(\bigwedge V^{\le(n-1)})^{n+1} = (\bigwedge V^{\le n})^{n+1}$. This proves that condition (v) holds.

Remark 4.2. Note that $d\colon N^n \longrightarrow (\bigwedge V)^{n+1}$ is always injective, and $d|_{C^n} = 0$ for all n.

Remark 4.3. If $(A,0)$ is a differential algebra with zero differential, then the minimal model $\rho\colon (\bigwedge V, d) \longrightarrow (A,0)$ constructed before has the property that $\rho(N^n) = 0$ for all n.

5. Minimal models and G-actions

Let G be a reductive complex Lie group. An *action* of the group G on a differential algebra (A, d) is a representation $r : G \longrightarrow \mathrm{GL}(A)$ such that

- $r(z)(A^n) = A^n$ for all $n \geq 0$ and $z \in G$,
- $r(z)(v_1 \cdot v_2) = r(z)(v_1) \cdot r(z)(v_2)$ for all $v_1, v_2 \in A$ and
- $r(z)(dv) = d(r(z)(v))$ for all $z \in G$ and $v \in A$.

If G acts on (A, d), then we say that (A, d) is a *G-differential algebra.*

A *G-minimal differential algebra* is a minimal differential algebra $(\bigwedge V, d)$ on which G acts satisfying the condition that each graded vector space V^n, $n \geq 0$, is preserved by the action of G. A *G-minimal model* of a G-differential algebra (A, d) is a G-minimal differential algebra $(\bigwedge V, d)$ such that there is a G-equivariant map

$$\rho : \left(\bigwedge V, d\right) \longrightarrow (A, d)$$

which is a quasi-isomorphism.

Note that a G-minimal model is in particular a minimal model.

Proposition 5.1. *Let (A, d) be a one-connected G-differential algebra. Then there exists a G-minimal model $(\bigwedge V, d)$ of (A, d).*

Proof. The construction of a minimal model in Section 4 works in the context of G-differential algebras. All we need is to substitute the vector spaces V^n in the construction by G representations. We note that the reductivity of the group G ensures that any short exact sequence of G modules splits. □

Let (X, x_0) be a one-pointed compact connected Riemann surface of genus $g \geq 2$, and, as in Section 2, let $\mathcal{N}_X$ be the moduli space of stable vector bundles over X of rank two and fixed determinant $\mathcal{O}_X(x_0)$. Then the mapping class group acts on the cohomology ring $H^*(\mathcal{N}_X, \mathbb{Q})$, with the action factoring through an action of $\mathrm{Sp}(2g, \mathbb{Z})$; moreover, this action extends to an action of $\mathrm{Sp}(2g, \mathbb{C})$ on $H^*(\mathcal{N}_X, \mathbb{C})$ (Proposition 2.1). We will now show that a similar result holds for the rational homotopy groups.

Theorem 5.2. *The mapping class group acts on the homotopy groups $\pi_*(\mathcal{N}_X)$. The induced action on the rational homotopy groups $\pi_*(\mathcal{N}_X) \otimes_{\mathbb{Z}} \mathbb{Q}$ factors through an action of the symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$. This action extends uniquely to an action of $\mathrm{Sp}(2g, \mathbb{C})$ on $\pi_*(\mathcal{N}_X) \otimes_{\mathbb{Z}} \mathbb{C}$.*

Let $(\bigwedge V, d)$ *be the* $\mathrm{Sp}(2g, \mathbb{C})$*-minimal model, provided by Proposition* 5.1, *for the one-connected* $\mathrm{Sp}(2g, \mathbb{C})$*-differential algebra* $(H^*(\mathcal{N}_X, \mathbb{C}), 0)$*. Then*

$$V^n \cong (\pi_n(\mathcal{N}_X) \otimes_{\mathbb{Z}} \mathbb{C})^*$$

as $\mathrm{Sp}(2g, \mathbb{C})$*-modules.*

Proof. First note that the formality of $\mathcal{N}_X$ (Theorem 4.1) means that $(\bigwedge V, d)$ is also the minimal model of $\mathcal{N}_X$.

Now, let $\eta \in \Gamma_g^1$, where Γ_g^1 is the mapping class group of (X, x_0). As we noted prior to Proposition 2.1, the element η acts on $\mathcal{N}_X$ by a diffeomorphism $f_\eta \colon \mathcal{N}_X \longrightarrow \mathcal{N}_X$. Hence we have an action on the free homotopy groups of $\mathcal{N}_X$. As $\mathcal{N}_X$ is simply connected, the free homotopy groups of $\mathcal{N}_X$ coincide with the homotopy groups of $\mathcal{N}_X$. So we have an induced map

$$\rho(\eta) : \pi_*(\mathcal{N}_X) \longrightarrow \pi_*(\mathcal{N}_X).$$

The diffeomorphism f_η induces a map on differential forms,

$$f_\eta^* : (\Omega^*(\mathcal{N}_X, \mathbb{C}), d) \longrightarrow (\Omega^*(\mathcal{N}_X, \mathbb{C}), d), \tag{5.1}$$

which lifts to a map on the minimal model

$$\widehat{f}_\eta^* : \left(\bigwedge V, d\right) \longrightarrow \left(\bigwedge V, d\right). \tag{5.2}$$

Such a lift is not unique; it is only unique up to homotopy of maps of differential algebras [5]. However, the induced map on the indecomposables,

$$\widetilde{f}_\eta^* \colon V = \frac{\bigwedge V}{\bigwedge^{\geq 2} V} \longrightarrow V = \frac{\bigwedge V}{\bigwedge^{\geq 2} V}, \tag{5.3}$$

is unique [15, Proposition 2.12], and moreover, it coincides with the dual of the map

$$\rho(\eta) \otimes \mathbb{C} : \pi_*(\mathcal{N}_X) \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow \pi_*(\mathcal{N}_X) \otimes_{\mathbb{Z}} \mathbb{C},$$

under the isomorphism of vector spaces $V^n \cong (\pi_n(\mathcal{N}_X) \otimes_{\mathbb{Z}} \mathbb{C})^*$ (see [5, p. 259]).

Let

$$\pi : \Gamma_g^1 \longrightarrow \mathrm{Aut}(H_1(X,\mathbb{Z})) = \mathrm{Sp}(2g,\mathbb{Z}) \tag{5.4}$$

be the natural projection of the mapping class group onto the symplectic group. Then the automorphism of cohomology

$$\overline{f}_\eta^* : H^*(\mathcal{N}_X,\mathbb{C}) \longrightarrow H^*(\mathcal{N}_X,\mathbb{C})$$

induced by f_η^* in Equation (5.1) coincides with the action of $\pi(\eta)$ on the cohomology, where π is the earlier projection. The map $\widehat{f}_\eta^*$ in equation (5.2) evidently induces the previous automorphism $\overline{f}_\eta^*$.

The minimal model $(\bigwedge V, d)$ has an action of $\mathrm{Sp}(2g,\mathbb{Z})$. Indeed, by Proposition 5.1, the group $\mathrm{Sp}(2g,\mathbb{C})$ acts on $(\bigwedge V, d)$ and this restricts to an action of $\mathrm{Sp}(2g,\mathbb{Z}) \subset \mathrm{Sp}(2g,\mathbb{C})$. The homomorphism $\overline{f}_\eta^*$ is induced by the action of $\pi(\eta)$ on $(\bigwedge V, d)$, where π is the projection in Equation (5.4). Therefore the map

$$\widetilde{f}_\eta^* : V^n \longrightarrow V^n$$

defined in Equation (5.3) coincides with the action of $\pi(\eta)$ on V^n. Hence, under the isomorphism $V^n \cong (\pi_n(\mathcal{N}_X)\otimes_{\mathbb{Z}}\mathbb{C})^*$, the actions of $\pi(\eta)$ and $(\rho(\eta)\otimes\mathbb{C})^*$ coincide.

If $\eta \in \Gamma_g^1$ belongs to the Torelli group, then $\rho(\eta)\otimes\mathbb{C}$ must be the identity map of $\pi_n(\mathcal{N}_X)\otimes_{\mathbb{Z}}\mathbb{C}$, and hence $\rho(\eta)\otimes\mathbb{Q} = \mathrm{Id}$ on $\pi_n(\mathcal{N}_X)\otimes_{\mathbb{Z}}\mathbb{Q}$. This proves that the action of the mapping class group on $\pi_*(\mathcal{N}_X)\otimes_{\mathbb{Z}}\mathbb{Q}$ factors through an action of $\mathrm{Sp}(2g,\mathbb{Z})$. Moreover, this action coincides with the restriction of the $\mathrm{Sp}(2g,\mathbb{C})$ action on V^n to the subgroup $\mathrm{Sp}(2g,\mathbb{Z}) \subset \mathrm{Sp}(2g,\mathbb{C})$ under the isomorphism $(\pi_n(\mathcal{N}_X)\otimes_{\mathbb{Z}}\mathbb{C})^* \cong V^n$. So the action of $\mathrm{Sp}(2g,\mathbb{Z})$ on $\pi_*(\mathcal{N}_X)\otimes_{\mathbb{Z}}\mathbb{C}$ extends to an action of $\mathrm{Sp}(2g,\mathbb{C})$. Since $\mathrm{Sp}(2g,\mathbb{Z})$ is Zariski dense in $\mathrm{Sp}(2g,\mathbb{C})$ [3], the extension is unique. This completes the proof of the theorem. □

Let $G = \mathrm{Sp}(2g,\mathbb{C})$, and let $(\bigwedge V, d)$ be the G-minimal model of $(H^*(\mathcal{N}_X,\mathbb{C}),0)$. Then we may decompose V^n into irreducible G representations. Let $\{\Gamma_i\}_{i\in\Lambda}$ be a complete set of irreducible G representations, where Λ parametrizes the isomorphism classes of irreducible G representations. So

$$V^n = \bigoplus_{i\in\Lambda} a_{i,n}\Gamma_i$$

for some set of integers $a_{i,n} \geq 0$.

6. The minimal model of $\mathcal{N}_X$ for $g = 2$

In this section we will assume that X is a compact connected Riemann surface of genus two. In this case, the moduli space $\mathcal{N}_X$, whose dimension is now three, can be described explicitly [17, 13]. It turns out to be isomorphic to the intersection of two quadrics in $\mathbb{P}^5$.

The integral cohomology ring of $\mathcal{N}_X$ has no torsion [16, Section 10]. Let $h \in H^2(\mathcal{N}_X, \mathbb{Z})$ be the hyperplane class. Note that, by the Lefschetz hyperplane theorem, $H^2(\mathcal{N}_X, \mathbb{Z}) = H^2(\mathbb{P}^5, \mathbb{Z}) = \mathbb{Z}$. So $h = \alpha$, the generator of the ample cone. The intersection of two quadrics in $\mathbb{P}^5$ contains many lines $\mathbb{P}^1 \subset \mathcal{N}_X \subset \mathbb{P}^5$. Let $l \in H^4(\mathcal{N}_X, \mathbb{Z})$ be the Poincaré dual of such a line. Then $h \cup l = [\mathcal{N}_X]$, so $H^4(\mathcal{N}_X, \mathbb{Z}) \cong \mathbb{Z}$ is generated by l. We have $h^3 = h \cup h \cup h = 4[\mathcal{N}_X]$, as the degree of $\mathcal{N}_X \subset \mathbb{P}^5$ is 4. Therefore, we conclude that $h \cup h = 4l$. Finally, $H^3(\mathcal{N}_X, \mathbb{Z}) \cong H^1(X, \mathbb{Z})^*$, so $H^3(\mathcal{N}_X, \mathbb{Z}) \cong W_0$, the standard $\mathrm{Sp}(4, \mathbb{Z})$ representation $W_0 = \mathbb{Z}^4$. Moreover, the pairing

$$H^3(\mathcal{N}_X, \mathbb{Z}) \otimes_{\mathbb{Z}} H^3(\mathcal{N}_X, \mathbb{Z}) \longrightarrow H^6(\mathcal{N}_X, \mathbb{Z}) \cong \mathbb{Z}$$

is perfect (Poincaré duality) and $\mathrm{Sp}(4, \mathbb{Z})$-equivariant, so it is equivalent to the standard symplectic form on W_0. The conclusion is that

$$\begin{aligned}
H^0(\mathcal{N}_X, \mathbb{Z}) &= \langle 1 \rangle\,, \\
H^1(\mathcal{N}_X, \mathbb{Z}) &= 0\,, \\
H^2(\mathcal{N}_X, \mathbb{Z}) &= \langle h \rangle\,, \\
H^3(\mathcal{N}_X, \mathbb{Z}) &\cong W_0\,, \\
H^4(\mathcal{N}_X, \mathbb{Z}) &= \langle l \rangle\,, \\
H^5(\mathcal{N}_X, \mathbb{Z}) &= 0\,, \\
H^6(\mathcal{N}_X, \mathbb{Z}) &= \langle [\mathcal{N}_X] \rangle\,.
\end{aligned}$$

This can also be seen by using Proposition 3.1, at least for rational coefficients. Since $I_1 = (\alpha, \beta, \gamma)$ and $I_2 = (\alpha^2 + \beta, \alpha\beta + \gamma, \alpha\gamma)$, Proposition 3.1 says that

$$H^*(\mathcal{N}_X, \mathbb{Q}) = \frac{\mathbb{Q}[\alpha, \beta, \gamma]}{I_2} \oplus \left(W \otimes \frac{\mathbb{Q}[\alpha, \beta, \gamma]}{I_1} \right) \cong \frac{\mathbb{Q}[\alpha]}{(\alpha^4)} \oplus W\,,$$

where $\beta = -\alpha^2$ and $\gamma = \alpha^3$ in this ring, and $\gamma_i \cup \gamma_j = -\frac{1}{4}(\gamma_i \cdot \gamma_j)\alpha^3$, for any $\gamma_i, \gamma_j \in W$. Note that $\beta = -4l$. Note that $W = W_0 \otimes_{\mathbb{Z}} \mathbb{Q}$ is the standard $\mathbb{Q}$ representation of $\mathrm{Sp}(4, \mathbb{Z})$.

Now we pass on to compute the minimal model $(\bigwedge V, d)$ of $\mathcal{N}_X$ by computing the minimal model of its cohomology algebra $H^*(\mathcal{N}_X, \mathbb{C})$. This is possible because $\mathcal{N}_X$ is formal by Theorem 4.1. By Proposition 5.1, $(\bigwedge V, d)$ is a G minimal model for $G = \mathrm{Sp}(4, \mathbb{C})$.

The irreducible representations of $\mathrm{Sp}(4, \mathbb{C})$ are labeled by pairs (a, b), $a, b \geq 0$, such that the corresponding representation $\Gamma_{a,b}$ has highest weight $aL_1 + b(L_1 + L_2) = (a + b)L_1 + bL_2$, where L_1 and L_2 are the orthogonal generators (with respect to the Killing form) of the weight lattice; see [7, Part III, Section 16].

The standard representation $W_c = W \otimes_{\mathbb{Q}} \mathbb{C} = \mathbb{C}^4$ of $\mathrm{Sp}(4, \mathbb{C})$ is $W_c = \Gamma_{1,0}$, whereas the irreducible $\mathrm{Sp}(4, \mathbb{C})$ representation $\wedge_0^2 W_c$ is $\Gamma_{0,1}$. Some easy cases are dealt with in [7, Part III, Section 16],

$$
\begin{aligned}
&\wedge^2 \Gamma_{1,0} = \wedge_0^2 W_c \oplus \mathbb{C} = \Gamma_{0,1} \oplus \Gamma_{0,0} \,, \\
&\mathrm{Sym}^a \Gamma_{1,0} = \Gamma_{a,0} \,, \\
&\Gamma_{0,1} \otimes \Gamma_{1,0} = W_c \otimes \wedge_0^2 W_c = \Gamma_{1,1} \oplus \Gamma_{1,0} \,.
\end{aligned}
$$

We define a partial order in the set of weights of $\mathrm{Sp}(4, \mathbb{C})$ as follows:

$$
(a, b) \leq (c, d) \Longleftrightarrow \begin{cases} a + b \leq c + d, \\ a + 2b \leq c + 2d. \end{cases}
$$

This corresponds to the fact that the weights of the representation $\Gamma_{a,b}$ are a subset of the convex hull of the weights of $\Gamma_{c,d}$. Otherwise said, $(a, b) \geq 0$ means that the highest weight $(a + b)L_1 + bL_2$ is a linear combination with non-negative coefficients of the positive roots (see [7]). (We point out that this order is defined in [10, p. 47] with the difference that in [10], $(a, b) \geq 0$ means that $(a + b)L_1 + bL_2$ is a linear combination with non-negative *integer* coefficients of the positive roots. This is equivalent to $a + b \geq 0$, $a + 2b \geq 0$ and $a + 2b \equiv 0 \pmod 2$.)

In particular, for representations Γ_{a_1,b_1} and Γ_{a_2,b_2}, the sub-representations $\Gamma_{c,d}$ of the tensor product $\Gamma_{a_1,b_1} \otimes \Gamma_{a_2,b_2}$ satisfy the condition

$$
(c, d) \leq (a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2) \,,
$$

and furthermore, there is exactly one sub-representation (the Cartan component) satisfying the equality. Note that this says in particular that $\Gamma_{a,b} \subset W_c^{\otimes a} \otimes (\wedge_0^2 W_c)^{\otimes b}$ appears with multiplicity 1.

We compute the $\mathrm{Sp}(4, \mathbb{C})$-minimal model $(\bigwedge V, d)$ following the mechanism laid out in Section 4 and Proposition 5.1.

Proposition 6.1. *Let* $(\bigwedge V, d)$ *be the minimal model of* $\mathcal{N}_X$ *for a curve* X *of genus* $g = 2$. *Then we have, as* $\mathrm{Sp}(4, \mathbb{Z})$*-representations,*

$$\begin{aligned}
V^2 &= \Gamma_{0,0} \\
V^3 &= \Gamma_{1,0} \\
V^4 &= \Gamma_{1,0} \\
V^5 &= \Gamma_{0,1} \oplus \Gamma_{0,0} \\
V^6 &= \Gamma_{2,0} \oplus \Gamma_{0,1} \\
V^7 &= \Gamma_{1,1} \oplus \Gamma_{2,0} \oplus \Gamma_{1,0} \,.
\end{aligned}$$

Proof. Abbreviating $H^*(\mathcal{N}_X)$ for $H^*(\mathcal{N}_X, \mathbb{C})$, we have

$$V^2 = C^2 = H^2(\mathcal{N}_X) = \langle h \rangle \cong \Gamma_{0,0} \,.$$

Recall that $d|_{C^n} = 0$, for any n.

In the next step, we have $V^3 = C^3 \oplus N^3$, with

$$\begin{aligned}
C^3 &= \mathrm{coker}\left(H^3 \left(\bigwedge V^{\leq 2} \right) = 0 \longrightarrow H^3(\mathcal{N}_X) \right) = H^3(\mathcal{N}_X) \cong W_c \cong \Gamma_{1,0} \,, \\
N^3 &= \ker \left(H^4 \left(\bigwedge V^{\leq 2} \right) = \langle h^2 \rangle \longrightarrow H^4(\mathcal{N}_X) = \langle h^2 \rangle \right) = 0.
\end{aligned}$$

For $n = 4$, we have $V^4 = C^4 \oplus N^4$, with

$$\begin{aligned}
C^4 &= \mathrm{coker}\left(H^4 \left(\bigwedge V^{\leq 3} \right) = \langle h^2 \rangle \longrightarrow H^4(\mathcal{N}_X) \langle h^2 \rangle \right) = 0, \\
N^4 &= \ker \left(H^5 \left(\bigwedge V^{\leq 3} \right) \longrightarrow H^5(\mathcal{N}_X) = 0 \right) = H^5 \left(\bigwedge V^{\leq 3} \right) \\
&= V^3 \cdot V^2 \cong \Gamma_{1,0} \otimes \Gamma_{0,0} = \Gamma_{1,0}.
\end{aligned}$$

The differential $d\colon N^4 \longrightarrow V^3 \cdot V^2 \subset \bigwedge V$ is an isomorphism.

We continue with $V^5 = C^5 \oplus N^5$, where

$$\begin{aligned}
C^5 &= \mathrm{coker}\left(H^5 \left(\bigwedge V^{\leq 4} \right) \longrightarrow H^5(\mathcal{N}_X) = 0 \right) = 0 \,, \\
N^5 &= \ker \left(H^6 \left(\bigwedge V^{\leq 4} \right) = \wedge^2 V^3 \oplus \langle h^3 \rangle \longrightarrow H^6(\mathcal{N}_X) = \langle h^3 \rangle \right) \\
&\cong \wedge^2 V^3 \cong \wedge^2 \Gamma_{1,0} = \Gamma_{0,1} \oplus \Gamma_{0,0} \,.
\end{aligned}$$

The differential $d : N^5 \longrightarrow \wedge^2 V^3 \oplus \langle h^3 \rangle$ is an isomorphism of N^5 with the kernel of the map $\wedge^2 V^3 \oplus \langle h^3 \rangle \longrightarrow \langle h^3 \rangle$. This map is the sum of a multiple of the intersection product $\wedge^2 V^3 \longrightarrow \mathbb{C} \cong \langle h^3 \rangle$ in the first summand, and the identity in the second summand.

For $n = 6$, we have $V^6 = C^6 \oplus N^6$. Now

$$C^6 = \operatorname{coker}\left(H^6\left(\bigwedge V^{\le 5}\right) \twoheadrightarrow H^6(\mathcal{N}_X) = \langle h^3\rangle\right) = 0\,,$$

since $h^3 \in H^6(\bigwedge V^{\le 5})$. Moreover $C^k = 0$ for $k > 6$ since $H^k(\mathcal{N}_X) = 0$. Also for all $k \ge 6$, we have $N^k = H^{k+1}(\bigwedge V^{\le (k-1)})$, since $H^{k+1}(\mathcal{N}_X) = 0$. Now

$$\begin{aligned}\left(\bigwedge V^{\le 5}\right)^6 &= (V^3 \cdot V^3) \oplus (V^2 \cdot V^2 \cdot V^2) \oplus (V^4 \cdot V^2)\,,\\ \left(\bigwedge V^{\le 5}\right)^7 &= (V^4 \cdot V^3) \oplus (V^3 \cdot V^2 \cdot V^2) \oplus (V^5 \cdot V^2)\,.\end{aligned}$$

The space of coboundaries is $B^7(\bigwedge V^{\le 5}) = d(V^4 \cdot V^2) = V^3 \cdot V^2 \cdot V^2$. The differential d maps $(V^4 \cdot V^3) \oplus (V^5 \cdot V^2)$ onto $\wedge^2 V^3 \cdot V^2 \oplus \langle h^4\rangle$, and it has kernel isomorphic to $\ker(V^4 \cdot V^3 \longrightarrow \langle h^4\rangle)$. But

$$V^4 \cdot V^3 = V^4 \otimes V^3 \cong \Gamma_{1,0} \otimes \Gamma_{1,0} = \mathrm{Sym}^2\Gamma_{1,0} \oplus \wedge^2\Gamma_{1,0} \cong \Gamma_{2,0} \oplus \Gamma_{0,1} \oplus \Gamma_{0,0}\,.$$

So the conclusion is that

$$N^6 = H^7\left(\bigwedge V^{\le 5}\right) = \frac{Z^7\left(\bigwedge V^{\le 5}\right)}{B^7\left(\bigwedge V^{\le 5}\right)} \cong \Gamma_{2,0} \oplus \Gamma_{0,1}\,,$$

and the differential $d\colon N^6 \longrightarrow (V^4 \cdot V^3) \oplus (V^5 \cdot V^2)$ is the sum of the two maps $d\colon N^6 = \Gamma_{2,0} \oplus \Gamma_{0,1} \longrightarrow V^4 \cdot V^3 = \Gamma_{2,0} \oplus \Gamma_{0,1} \oplus \Gamma_{0,0}$ which is injective, and $d\colon N^6 = \Gamma_{2,0} \oplus \Gamma_{0,1} \longrightarrow V^5 \cdot V^2 = \Gamma_{0,1} \oplus \Gamma_{0,0}$ mapping onto the $\Gamma_{0,1}$ summand.

The next case is $V^7 = C^7 \oplus N^7 = N^7$. Then

$$\begin{aligned}\left(\bigwedge V^{\le 6}\right)^7 &= (V^5 \cdot V^2) \oplus (V^4 \cdot V^3) \oplus (V^3 \cdot V^2 \cdot V^2),\\ \left(\bigwedge V^{\le 6}\right)^8 &= (V^6 \cdot V^2) \oplus (V^5 \cdot V^3) \oplus (V^4 \cdot V^4) \oplus (V^4 \cdot V^2 \cdot V^2)\\ &\qquad \oplus (V^3 \cdot V^3 \cdot V^2) \oplus \langle h^4\rangle.\end{aligned}$$

The space of coboundaries is $B^8(\bigwedge V^{\le 6}) = (V^3 \cdot V^3 \cdot V^2) \oplus \langle h^4\rangle$, from our knowledge of d on both V^5 and V^4. To compute

$$\begin{aligned}N^7 = H^8\left(\bigwedge V^{\le 7}\right) &= \frac{Z^8(\bigwedge V^{\le 6})}{B^8(\bigwedge V^{\le 6})} =\\ &= \ker\left(d\colon (V^6 \cdot V^2) \oplus (V^5 \cdot V^3) \oplus (V^4 \cdot V^4) \oplus (V^4 \cdot V^2 \cdot V^2) \longrightarrow \bigwedge V\right),\end{aligned}$$

we look at each summand,

$$\begin{aligned}
&d\colon V^5\cdot V^3 \longrightarrow \wedge^3 V^3 \oplus V^3\cdot (V^2)^3,\\
&d\colon V^4\cdot V^2\cdot V^2 \xrightarrow{\cong} V^3\cdot (V^2)^3,\\
&d\colon V^6\cdot V^2 \hookrightarrow (V^4\cdot V^3\cdot V^2)\oplus (V^5\cdot V^2\cdot V^2)\\
&d\colon V^4\cdot V^4 \hookrightarrow V^4\cdot V^3\cdot V^2.
\end{aligned}$$

So $N^7 = K_1\oplus K_2$, where $K_1 = \ker((V^5\cdot V^3)\oplus (V^4\cdot V^2\cdot V^2)\longrightarrow \bigwedge V)$ and $K_2=\ker((V^6\cdot V^2)\oplus(V^4\cdot V^4)\longrightarrow \bigwedge V)$. Clearly, $K_1\cong \ker(V^5\otimes V^3\longrightarrow \wedge^3V^3)$, but $V^5\otimes V^3\cong(\Gamma_{0,1}\oplus\Gamma_{0,0})\otimes\Gamma_{1,0}=\Gamma_{1,1}\oplus\Gamma_{1,0}\oplus\Gamma_{1,0}$ and $\wedge^3V^3\cong V^3\cong\Gamma_{1,0}$, so $K_1\cong\Gamma_{1,1}\oplus\Gamma_{1,0}$. On the other hand, d maps $V^4\cdot V^4 = \mathrm{Sym}^2V^4\cong\Gamma_{2,0}$ to the corresponding summand in $V^4\cdot V^3\cdot V^2\cong\Gamma_{1,0}\otimes\Gamma_{1,0}=\Gamma_{2,0}\oplus\Gamma_{0,1}\oplus\Gamma_{0,0}$, and d maps $V^6\cdot V^2\cong\Gamma_{2,0}\oplus\Gamma_{0,1}$ injectively to $(V^4\cdot V^3\cdot V^2)\oplus(V^5\cdot V^2\cdot V^2)\cong(\Gamma_{2,0}\oplus\Gamma_{0,1}\oplus\Gamma_{0,0})\oplus(\Gamma_{0,1}\oplus\Gamma_{0,0})$. Thus $K_2\cong\Gamma_{2,0}$. This concludes that $N^7=\Gamma_{1,1}\oplus\Gamma_{2,0}\oplus\Gamma_{1,0}$, and the proof of the proposition is complete. □

We may carry on the process as long as we want, but the calculations get more involved, since we must keep track of the irreducible summands of $\bigwedge V$ onto which $d|_{V_n}$ maps for each n. It is easier to find the "leading representation". We need a preliminary result.

Lemma 6.2. *For any* $\mathrm{Sp}(4,\mathbb{C})$*-irreducible representation* $\Gamma_{a,b}$*,* $a,b\geq 0$*, if* $\Gamma_{a,b}\subset V^n$*, then* $n\geq n(a,b)$*, where*

$$n(a,b)=\begin{cases} 2a+4b+1, & \text{if } b\geq 1 \text{ or } (a,b)=(1,0),\\ 2a+2, & \text{if } b=0 \text{ and } a\neq 1.\end{cases}$$

Proof. We shall prove this by induction on n. By Proposition 6.1, the result is true for $1\leq n\leq 7$. So suppose $n\geq 8$. Let $U\subset V^n$ be a sub-representation with $U\cong\Gamma_{a,b}$. We want to prove that $n(a,b)\leq n$, so we may assume that $(a,b)\neq(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$ and $(2,0)$. As $H^n(\mathcal{N}_X)=0$, we have that $C^n=0$. So $V^n=N^n$, and in particular $d\colon V^n\longrightarrow(\bigwedge V)^{n+1}$ is injective. Hence

$$U\cong d(U)\subset d(V^n)\subset\left(\bigwedge\nolimits^{\geq 2}V\right)^{n+1}=\sum_{n_1+\ldots+n_r=n+1,\ r\geq 2} V^{n_1}\cdots V^{n_r}.$$

The projection of $d(U)$ to some of these summands must be non-zero. Hence there exists

$$U'\subset V^{n_1}\cdots V^{n_r}\subset V^{n_1}\otimes\cdots\otimes V^{n_r},$$

for some $r \geq 2$, with $n_1 + \cdots + n_r = n+1$, $U' \cong U \cong \Gamma_{a,b}$. Note that all $n_i < n$, because $n_i \geq 2$, $1 \leq i \leq r$. Decomposing each V^{n_i} into $\mathrm{Sp}(4,\mathbb{C})$-irreducible representations, there must be $(a_1,b_1),\ldots,(a_r,b_r)$ such that

$$\Gamma_{a,b} \subset \Gamma_{a_1,b_1} \otimes \cdots \otimes \Gamma_{a_r,b_r}\,, \tag{6.1}$$

with $\Gamma_{a_i,b_i} \subset V^{n_i}$. Applying the induction hypothesis it follows that $n_i \geq n(a_i,b_i)$ for all $1 \leq i \leq r$. We note that Equation (6.1) implies that $(a,b) \leq (a_1,b_1) + \cdots + (a_r,b_r)$, that is,

$$\begin{cases} a+b \leq \sum (a_i+b_i), \\ a+2b \leq \sum (a_i+2b_i). \end{cases} \tag{6.2}$$

If $a, b \geq 1$, we have

$$\begin{aligned} n+1 &= n_1 + \cdots + n_r \\ &\geq n(a_1,b_1) + \cdots + n(a_r,b_r) \\ &\geq \sum_{i=1}^{r} (2a_i + 4b_i + 1) \\ &\geq 2a + 4b + r \\ &\geq 2a + 4b + 2 \\ &= n(a,b) + 1, \end{aligned} \tag{6.3}$$

using Equation (6.2). So $n \geq n(a,b)$ in this case.

If $b = 0$, $a \geq 3$, then Equation (6.3) proves that $n+1 \geq n(a,b)$. If there is equality, then $r = 2$, $n_i = n(a_i,b_i) = 2a_i + 4b_i + 1$ and $a = a + 2b = \sum (a_i + 2b_i)$, for all i. Since $a = a + b \leq \sum (a_i + b_i)$, we get $\sum (a_i + 2b_i) = a \leq \sum (a_i + b_i)$, so $b_i = 0$ for all i. As also $n_i = n(a_i, 0) = 2a_i + 1$, we have that $a_i = 1$, $n_i = 3$. But then $a = 2$ which is a case treated before. $\square$

Theorem 6.3. *Let $n \geq 4$. The decomposition of V^n into a direct sum of irreducible $\mathrm{Sp}(4,\mathbb{C})$-representations is as follows:*

(i) *If $n = 2m$ is even, then*

$$V^n = \Gamma_{m-1,0} \oplus \left(\bigoplus_{(a,b) < (m-1,0)} n_{ab} \Gamma_{a,b} \right),$$

with $n_{ab} \geq 0$.

(ii) *If* $n = 2m + 1$ *is odd, then*

$$V^n = \Gamma_{m-2,1} \oplus \left(\bigoplus_{(a,b)<(m-2,1)} n_{ab}\Gamma_{a,b} \right),$$

with $n_{ab} \geq 0$.

Proof. (i) Let $n = 2m \geq 4$. By Lemma 6.2, if $\Gamma_{a,b} \subset V^n$ then $2a + 4b + 1 \leq n = 2m$. This implies that

$$(a, b) \leq (m - 1, 0),$$

since $a + 2b \leq m - 1$ and $a + b \leq a + 2b \leq m - 1$. So the leading representation in V^n is $\Gamma_{m-1,0}$. We will show that it actually appears and that its multiplicity is one.

To see that there is $\Gamma_{m-1,0} \subset V^n$, we shall prove by induction on m that there is a sub-representation $U_{m-1} \subset V^n$, $U_{m-1} \cong \Gamma_{m-1,0}$, such that $d(U_{m-1}) \subset V^{n-2} \cdot V^3$. By Proposition 6.1, this is true for $m = 2, 3$. Assume that it is true for $m - 1 \geq 3$ and let us prove it for m. So $U_{m-2} \subset V^{2m-2}$ and $d(U_{m-2}) \subset V^{2m-4} \cdot V^3$. Then $d\colon U_{m-2} \cdot V^3 \longrightarrow V^{2m-4} \cdot V^3 \cdot V^3 \subset \bigwedge V$. But

$$U_{m-2} \cdot V^3 \cong \Gamma_{m-2,0} \otimes \Gamma_{1,0}$$

contains a sub-representation $\widetilde{U}_{m-1} \subset U_{m-2} \cdot V^3$ such that $\widetilde{U}_{m-1} \cong \Gamma_{m-1,0}$. On the other hand,

$$V^{2m-4} \cdot V^3 \cdot V^3 = V^{2m-4} \otimes \wedge^2 V^3 \cong V^{2m-4} \otimes \Gamma_{0,1} \,.$$

Decomposing V^{2m-4} into irreducible representations $\Gamma_{c,d}$, and noting that $(c, d) \leq (m - 3, 0)$ by induction hypothesis, we see that if $\Gamma_{a,b} \subset V^{2m-4} \cdot V^3 \cdot V^3$ then $\Gamma_{a,b} \subset \Gamma_{c,d} \otimes \Gamma_{0,1}$ for some $(c, d) \leq (m - 3, 0)$. Thus

$$(a, b) \leq (c, d) + (0, 1) = (c, d + 1) \leq (m - 3, 1) < (m - 1, 0).$$

As a consequence, $\Gamma_{m-1,0} \not\subset V^{2m-4} \cdot V^3 \cdot V^3$, and so $d(\widetilde{U}_{m-1}) = 0$. This implies that $\widetilde{U}_{m-1} \subset Z^{n+1}(\bigwedge V, d) = B^{n+1}(\bigwedge V, d)$, since $H^{n+1}(\bigwedge V, d) = 0$. There must exist a sub-representation $U_{m-1} \subset (\bigwedge V)^n$ with $d(U_{m-1}) = \widetilde{U}_{m-1}$. As d maps $(\bigwedge^{\geq i} V)^n \longrightarrow (\bigwedge^{\geq (i+1)} V)^{n+1}$, it cannot be $U_{m-1} \subset (\bigwedge^2 V)^n$, so the projection of U_{m-1} by $p\colon (\wedge V)^n \longrightarrow V^n$ is a sub-representation isomorphic to $\Gamma_{m-1,0}$. (Here we are allowed to substitute V^n by $U_{m-1} \oplus p(U_{m-1})^\perp$, where $p(U_{m-1})^\perp$ is a $\mathrm{Sp}(4, \mathbb{C})$-invariant complement of $p(U_{m-1}) \subset V^n$; this yields an isomorphic minimal model and ensures that $d(U_{m-1}) \subset V^{2n-2} \cdot V^3$).

Now let us compute the multiplicity of $\Gamma_{m-1,0}$ in V^n. The argument of the proof of Lemma 6.2 implies that the multiplicity of $\Gamma_{m-1,0}$ in V^n is at most the sum of the multiplicities of $\Gamma_{m-1,0}$ in $V^{n_1} \cdots V^{n_r}$, for the different possibilities $n_1 + \cdots + n_r = n+1$, $r \geq 2$. Let $(a,b) = (m-1,0)$. As in the proof of Lemma 6.2, for any sub-representation $\Gamma_{m-1,0}$ there are $(a_1,b_1), \ldots, (a_r,b_r)$ such that

$$
\begin{aligned}
2m+1 &= n+1 = n_1 + \cdots + n_r \\
&\geq n(a_1,b_1) + \cdots + n(a_r,b_r) \\
&\geq \textstyle\sum_{i=1}^r (2a_i + 4b_i + 1) \\
&\geq 2a + 4b + r = 2m - 2 + r.
\end{aligned} \tag{6.4}
$$

In particular $r \leq 3$. If $r = 3$, then $n_i = n(a_i,b_i) = 2a_i + 4b_i + 1$ for all i, and $a = a + 2b = \sum (a_i + 2b_i)$. Since $a = a + b \leq \sum (a_i + b_i)$, we get $b_i = 0$ for all i. This implies that $a_i = 1$ and $n_i = 3$. But then $a = m - 1 = 3$ and

$$\Gamma_{3,0} \not\subset V^3 \cdot V^3 \cdot V^3 = \wedge^3 V^3 \cong \wedge^3 W_c \cong W_c \cong \Gamma_{1,0}\,.$$

If $r = 2$, then $2a + 4b + 1 \geq \sum (2a_i + 4b_i) \geq 2a + 4b$. So $\sum (a_i + 2b_i) = a + 2b = a$. As before, this implies that $b_i = 0$ for all i. At most one of the a_is is bigger than 1, so we can put $(a_1,b_1) = (m-2,0)$, $(a_2,b_2) = (1,0)$. This corresponds to the summand $\Gamma_{m-2,0} \otimes \Gamma_{1,0} \subset V^{2m-4} \cdot V^3$. This representation contains $\Gamma_{m-1,0}$ with multiplicity one.

Since we know that the multiplicity of $\Gamma_{m-1,0}$ in V^n is non-zero, we conclude that it is exactly one.

(ii) Let $n = 2m + 1 \geq 5$. By Proposition 6.1, the result holds for $m = 2, 3$, so assume that $m \geq 4$.

If $\Gamma_{a,b} \subset V^n$, then by Lemma 6.2, we have that $2a + 4b + 1 \leq n = 2m + 1$, so $a + 2b \leq m$. This implies that

$$(a,b) \leq (m-2,1),$$

since if $b \geq 1$ then $a + b \leq a + 2b - 1 \leq m - 1$; and if $b = 0$ then Lemma 6.2 says that $2a + 2 \leq n = 2m + 1$, so $a + b = a \leq m - 1$. So the leading representation in V^n is $\Gamma_{m-2,1}$. We will show that it actually appears with multiplicity one.

As in the previous case, one can see using induction on m that there is a sub-representation $U_{m-1} \subset V^{2m+1}$, with $U_{m-1} \cong \Gamma_{m-2,1}$, such that $d(U_{m-1}) \subset V^{2m-1} \cdot V^3$.

To compute the multiplicity of $\Gamma_{m-2,1}$ in V^n, let us find the multiplicity of $\Gamma_{m-2,1}$ in $V^{n_1} \cdots V^{n_r}$, for $n_1 + \cdots + n_r = n+1$, $r \geq 2$. As $n = 2m+1 = n(m-2,1)$, there must be equality in Equation (6.3) for $(a,b) = (m-2,1)$, which means that $r=2$, $n_i = n(a_i, b_i) = 2a_i + 4b_i + 1$ and $\sum(a_i + 2b_i) = a + 2b = m$. Since $m - 1 = a + b \leq \sum(a_i + b_i)$, we have $\sum(a_i + 2b_i) = m \leq \sum(a_i + b_i) + 1$, so $\sum b_i \leq 1$. As least one b_i is zero, say $b_2 = 0$. Then $a_2 = 1$, $n_2 = 3$. Also $m - 1 \leq \sum(a_i + b_i) \leq a_1 + 2$ and $m = a_1 + 2b_1 + 1$, implying that $(a_1, b) = (m-3, 1)$ or $(m-1, 0)$ and $n_1 = 2a_1 + 4b_1 + 1 = 2m - 1$. By induction hypothesis, $\Gamma_{m-1,0} \not\subset V^{2m-1}$, so the second case is ruled out. The multiplicity of $\Gamma_{m-2,1}$ in $\Gamma_{m-3,1} \otimes \Gamma_{1,0} \subset V^{2m-1} \cdot V^3$ is 1. This proves that the multiplicity of $\Gamma_{m-2,1}$ in V^n is one. $\square$

7. Sub-representations in the minimal model of $\mathcal{N}_X$ for $g > 2$

Suppose now that X is a smooth irreducible projective complex curve of genus $g > 2$. The action of $\mathrm{Sp}(2g, \mathbb{C})$ on the cohomology algebra $H^*(\mathcal{N}_X, \mathbb{C})$ of the moduli space $\mathcal{N}_X$ gives an action of $\mathrm{Sp}(2g, \mathbb{C})$ on the minimal model $(\bigwedge V, d)$ of $\mathcal{N}_X$, by Proposition 5.1. By Theorem 5.2, the action of $\mathrm{Sp}(2g, \mathbb{C})$ on the minimal model $(\bigwedge V, d)$ is compatible with the action of $\mathrm{Sp}(2g, \mathbb{C})$ on the complex homotopy groups $\pi_*(\mathcal{N}_X) \otimes_{\mathbb{Z}} \mathbb{C}$.

The isomorphism classes of irreducible $\mathrm{Sp}(2g, \mathbb{C})$-representations are labeled by g-tuples $(a_1, \ldots, a_g) \in (\mathbb{Z}_{\geq 0})^g$ (see [7, Part III, Section 17]). The representation corresponding to $(a_1, \ldots, a_g)$ is denoted by

$$\Gamma_{(a_1, \ldots, a_g)} = \Gamma_{a_1 \mathbf{e}_1 + \cdots + a_g \mathbf{e}_g},$$

where $\mathbf{e}_i = (0, \ldots, 1, \ldots, 0)$, with 1 in the ith position and 0 elsewhere. The $\mathrm{Sp}(2g, \mathbb{C})$-module $\Gamma_{(a_1, \ldots, a_g)}$ is characterized by its highest weight $(a_1 + a_2 + \cdots + a_g)L_1 + (a_2 + \cdots + a_g)L_2 + \cdots + a_g L_g$, where $\{L_1, \ldots, L_g\}$ is the standard basis for the weight lattice.

Let $W_c = \mathbb{C}^{2g}$ be the standard representation of $\mathrm{Sp}(2g, \mathbb{C})$. Then $W_c = \Gamma_{\mathbf{e}_1}$, and $\Gamma_{\mathbf{e}_k} = \wedge_0^k W_c$ is the complexification of the representation $\wedge_0^k W$ introduced in Section 3.

We shall use two well-known facts: (i) the representation $\Gamma_{(a_1, \ldots, a_g)} \otimes \Gamma_{(b_1, \ldots, b_g)}$ contains $\Gamma_{(a_1 + b_1, \ldots, a_g + b_g)}$ (actually this is the highest weight representation appearing with multiplicity one); and (ii) the representation $\Gamma_{(k-2, 0, \ldots, 0)} \otimes \Gamma_{(0,1,\ldots,0)}$ does not contain $\Gamma_{(k,0,\ldots 0)}$ (this holds because the weight kL_1 does not appear in the tensor product), and the representation $\Gamma_{(k-2,1,0,\ldots,0)} \otimes \Gamma_{(0,1,0,\ldots,0)}$ does not contain $\Gamma_{(k,1,0,\ldots,0)}$.

Theorem 7.1. *Let X be a complex smooth projective irreducible curve of genus $g > 2$. Let $(\bigwedge V, d)$ be the minimal model of the moduli space $\mathcal{N}_X$. Then, as* $\mathrm{Sp}(2g, \mathbb{C})$ *representations, we have*

$$\begin{aligned}
&V^2 = \Gamma_0\,, \\
&V^3 = \Gamma_{\mathbf{e}_1}\,, \\
&V^4 = \Gamma_0\,, \\
&V^n = 0, && 5 \leq n \leq 2g-2, \\
&V^{2g-1} = \Gamma_0\,, \\
&V^{2g} = \Gamma_{\mathbf{e}_1}\,, \\
&V^{2g+1} = \Gamma_{\mathbf{e}_2} \oplus \Gamma_0\,.
\end{aligned}$$

Moreover, for $n \geq 2g+2$, we have the following.

(i) *If $n = 2(g+k-1)$ with $k \geq 2$, then V^n contains $\Gamma_{k\mathbf{e}_1}$.*

(ii) *If $n = 2(g+k)+1$ with $k \geq 1$, then V^n contains $\Gamma_{k\mathbf{e}_1+\mathbf{e}_2}$.*

Proof. Clearly, $V^2 = C^2 = \langle \alpha \rangle \cong \Gamma_0$,

$$V^3 = C^3 = H^3(\mathcal{N}_X) = W_c \cong \Gamma_{\mathbf{e}_1}$$

and $V^4 = C^4 = \langle \beta \rangle \cong \Gamma_0$. Now

$$\bigwedge V^{\leq 4} = \bigwedge(\alpha, \gamma_1, \ldots, \gamma_{2g}, \beta) = \mathbb{A}_c\,,$$

where $\mathbb{A}_c = \mathbb{A} \otimes_{\mathbb{Q}} \mathbb{C}$ is the complexification of the rational vector space defined in Lemma 3.2. So the natural homomorphism $\bigwedge V^{\leq 4} \longrightarrow H^*(\mathcal{N}_X)$ is surjective. This implies that $C^n = 0$ and

$$V^n = N^n = \ker\left(H^{n+1}\left(\bigwedge V^{<n}\right) \longrightarrow H^{n+1}(\mathcal{N}_X)\right)$$

for all $n > 4$. Since $F_c : \mathbb{A}_c \longrightarrow H^*(\mathcal{N}_X)$ is the complexification of the map F in Equation (3.3), its kernel, kernel(F_c), has the lowest degree element q_g^1, which is of degree $2g$. So $V^n = N^n = 0$ for all $5 \leq n \leq 2g-2$. For $n = 2g-1$, we have

$$\begin{aligned}
V^{2g-1} &= \ker\left(H^{2g}\left(\bigwedge V^{<(2g-1)}\right) = \mathbb{A}_c^{2g} \longrightarrow H^{n+1}(\mathcal{N}_X)\right) = (\mathrm{kernel}(F_c))^{2g} \\
&= \langle q_g^1 \rangle \cong \Gamma_0
\end{aligned}$$

with $d\colon V^{2g-1} \xrightarrow{\simeq} \langle q_g^1 \rangle \subset \bigwedge V$.

For $n=2g$, we have

$$H^{2g+1}\left(\bigwedge V^{\leq(2g-1)}\right)=H^{2g+1}\left(\bigwedge V^{<(2g-1)}\right)=\mathbb{A}_c^{2g+1},$$

since $(\bigwedge V^{\leq(2g-1)})^{2g+1}=\mathbb{A}_c^{2g+1}\oplus(V^{2g-1}\cdot V^2)$ and the non-zero elements in $V^{2g-1}\cdot V^2$ are not closed. So

$$V^{2g}=(\mathrm{kernel}(F_c))^{2g+1}=q_{g-1}^1\cdot W_c\cong\Gamma_{\mathbf{e}_1},$$

and $d\colon V^{2g}\xrightarrow{\simeq}q_{g-1}^1\cdot V^3\subset\bigwedge V$.

For $n=2g+1$, we have

$$\left(\bigwedge V^{\leq 2g}\right)^{2g+1}=\mathbb{A}_c^{2g+1}\oplus(V^{2g-1}\cdot V^2),$$
$$\left(\bigwedge V^{\leq 2g}\right)^{2g+2}=\mathbb{A}_c^{2g+2}\oplus(V^{2g-1}\cdot V^3)\oplus(V^{2g}\cdot V^2),$$

with $d\colon V^{2g-1}\cdot V^3\xrightarrow{\simeq}q_g^1\cdot V^3$ and $d\colon V^{2g}\cdot V^2\xrightarrow{\simeq}\alpha\,q_{g-1}^1\cdot V^3$. But $\alpha\,q_{g-1}^1$ and q_g^1 are linearly independent, so we have

$$H^{2g+2}\left(\bigwedge V^{\leq 2g}\right)=\frac{Z^{2g+2}\left(\bigwedge V^{\leq 2g}\right)}{B^{2g+2}\left(\bigwedge V^{\leq 2g}\right)}=\frac{\mathbb{A}_c^{2g+2}}{\langle\alpha\,q_g^1\rangle}.$$

This gives

$$V^{2g+1}=\ker\left(\frac{\mathbb{A}_c^{2g+2}}{\langle\alpha q_g^1\rangle}\longrightarrow H^{2g+2}(\mathcal{N}_X)\right)=\langle q_g^2\rangle\oplus q_{g-2}^1\cdot\wedge_0^2W_c\cong\Gamma_0\oplus\Gamma_{\mathbf{e}_2},$$

which follows easily using Lemma 3.2. Note that the differential d maps the summand $\Gamma_{\mathbf{e}_2}$ to $q_{g-2}^1\cdot\wedge_0^2V^3$.

We now proceed to prove the second part of the theorem.

Proof of (i). Let us prove by induction on $k\geq 1$ that there exists a sub-representation $U_k\subset V^{2g+2k-2}$ with $U_k\cong\Gamma_{k\,\mathbf{e}_1}$ such that $d(U_k)\subset U_{k-1}\cdot V^3\subset V^{2g+2k-4}\cdot V^3$, where $U_0:=\langle q_{g-1}^1\rangle\subset V^{2g-2}$.

If $k=1$, then

$$V^n=V^{2g}=\Gamma_{\mathbf{e}_1}$$

with $d\colon V^{2g}\longrightarrow q_{g-1}^1\cdot V^3\subset\bigwedge V$.

Now assume that there exists a sub-representation $U_{k-1}\subset V^{2g+2k-4}$ with $U_{k-1}\cong\Gamma_{(k-1)\mathbf{e}_1}$ such that $d(U_{k-1})\subset U_{k-2}\cdot V^3\subset V^{2g+2k-6}\cdot V^3$. Then we have

$$d(U_{k-1} \cdot V^3) \subset U_{k-2} \cdot V^3 \cdot V^3 .$$

On one hand,

$$\begin{aligned} &U_{k-1} \cdot V^3 = U_{k-1} \otimes V^3 \subset V^{2g+2k-4} \otimes V^3 \ , \\ &U_{k-1} \otimes V^3 \cong \Gamma_{(k-1)\mathbf{e}_1} \otimes \Gamma_{\mathbf{e}_1} \ ; \end{aligned}$$

so there exists $\widetilde{U}_k \subset U_{k-1} \cdot V^3$ with $\widetilde{U}_k \cong \Gamma_{k\,\mathbf{e}_1}$ such that $\widetilde{U}_k \subset U_{k-1} \cdot V^3$. On the other hand,

$$U_{k-2} \cdot V^3 \cdot V^3 \subset U_{k-2} \otimes \wedge^2 V^3 \cong \Gamma_{(k-2)\mathbf{e}_1} \otimes \Gamma_{\mathbf{e}_2}$$

does not contain $\Gamma_{k\,\mathbf{e}_1}$. So $d(\widetilde{U}_k) = 0$, or in other words, $\widetilde{U}_k \subset Z^{2g+2k-1}(\bigwedge V)$.

By Remark 4.3, we have $\rho(V^n) = 0$ for all $n \geq 5$, where $\rho\colon (\bigwedge V, d) \longrightarrow (H^*(\mathcal{N}_X), 0)$ is the minimal model. As $U_{k-1} \subset V^{2g+2k-4}$, we have $\rho(U_{k-1}) = 0$. Hence $\rho(\widetilde{U}_k) = 0$, or in other words, $\widetilde{U}_k \subset B^{2g+2k-1}(\bigwedge V)$. This is only possible if there exists $U_k \subset V^{2g+2k-2}$ with $U_k \cong \Gamma_{k\,\mathbf{e}_1}$ and $d\colon U_k \xrightarrow{\cong} \widetilde{U}_k \subset U_{k-1} \cdot V^3$. Therefore, the proof of statement (i) is complete by induction.

Proof of (ii). We will show using induction on $k \geq 1$ that there exists a sub-representation

$$U_k \subset V^{2g+2k+1}$$

with $U_k \cong \Gamma_{k\,\mathbf{e}_1 + \mathbf{e}_2}$ such that $d(U_k) \subset U_{k-1} \cdot V^3 \subset V^{2g+2k-1} \cdot V^3$, where $U_0 \subset V^{2g+1}$ is the sub-representation of $V^{2g+1} = \Gamma_0 \oplus \Gamma_{\mathbf{e}_2}$ isomorphic to $\Gamma_{\mathbf{e}_2}$.

For $k = 1$, note that $d\colon U_0 \to q^1_{g-2} \cdot \wedge^2_0 V^3$. So $d\colon U_0 \cdot V^3 \to q^1_{g-2} \cdot \wedge^2_0 V^3 \cdot V^3$, where

$$\wedge^2_0 V^3 \cdot V^3 \cong \wedge^3_0 V^3 \oplus \gamma \cdot V^3 \cong \wedge^3_0 W_c \oplus W_c = \Gamma_{\mathbf{e}_3} \oplus \Gamma_{\mathbf{e}_1} .$$

As $U_0 \cong \Gamma_{\mathbf{e}_2}$, we conclude that

$$U_0 \cdot V^3 \cong U_0 \otimes V^3 \cong \Gamma_{\mathbf{e}_2} \otimes \Gamma_{\mathbf{e}_1}$$

contains a sub-representation $\widetilde{U}_1 \subset U_0 \cdot V^3$ with $\widetilde{U}_1 \cong \Gamma_{\mathbf{e}_1 + \mathbf{e}_2}$ and $d(\widetilde{U}_1) = 0$. Working as in the proof of (i), this yields that there exists $U_1 \subset V^{2g+3}$ with $U_1 \cong \Gamma_{\mathbf{e}_1 + \mathbf{e}_2}$ and $d(U_1) = \widetilde{U}_1 \subset U_0 \cdot V^3$.

Now assume that $k \geq 2$ and that there exists a sub-representation $U_{k-1} \subset V^{2g+2k-1}$ with $U_{k-1} \cong \Gamma_{(k-1)\mathbf{e}_1 + \mathbf{e}_2}$ such that $d(U_{k-1}) \subset U_{k-2} \cdot V^3 \subset$

$V^{2g+2k-3} \cdot V^3$. Then we have $d(U_{k-1} \cdot V^3) \subset U_{k-2} \cdot V^3 \cdot V^3$. On one hand,

$$U_{k-1} \cdot V^3 = U_{k-1} \otimes V^3 \cong \Gamma_{(k-1)\mathbf{e}_1+\mathbf{e}_2} \otimes \Gamma_{\mathbf{e}_1};$$

so there exists $\widetilde{U}_k \subset U_{k-1} \cdot V^3$ with $\widetilde{U}_k \cong \Gamma_{k\,\mathbf{e}_1+\mathbf{e}_2}$ such that $\widetilde{U}_k \subset U_{k-1} \cdot V^3$. On the other hand,

$$U_{k-2} \cdot V^3 \cdot V^3 \;\subset\; U_{k-2} \otimes \wedge^2 V^3 \cong \Gamma_{(k-2)\mathbf{e}_1+\mathbf{e}_2} \otimes \Gamma_{\mathbf{e}_2}$$

does not contain $\Gamma_{k\,\mathbf{e}_1+\mathbf{e}_2}$. Therefore, $d(\widetilde{U}_k) = 0$. Thus there exists $U_k \subset V^{2g+2k-2}$ with $U_k \cong \Gamma_{k\,\mathbf{e}_1+\mathbf{e}_2}$ and $d\colon U_k \xrightarrow{\simeq} \widetilde{U}_k \subset U_{k-1} \cdot V^3$. This completes the proof of the theorem. $\square$

From Theorem 6.3 and 7.1 it follows that for each $g \geq 2$, the rational homotopy group $\pi_n(\mathcal{N}_X) \otimes_{\mathbb{Z}} \mathbb{C}$ is non-zero for infinitely many n. As noted in the Introduction, this means that the moduli space $\mathcal{N}_X$ is rationally hyperbolic for all $g \geq 2$. Therefore,

$$f(k) = \sum_{i=1}^{\dim_{\mathbb{R}} \mathcal{N}_X - 1} \dim \pi_{k+i}(\mathcal{N}_X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

grows faster than any polynomial in k.

Remark 7.2. Let X be a smooth irreducible projective complex curve of genus $g \geq 2$. Whereas the minimal model of $(H^*(\mathcal{N}_X, \mathbb{C}), 0)$ has infinitely many $n \in \mathbb{N}$ for which $V^n \neq 0$, the minimal model of the algebra $(H_I^*(\mathcal{N}_X, \mathbb{C}), 0)$ has a very different behavior. Actually, from Equation (3.5) we find that the minimal model of $(H_I^*(\mathcal{N}_X, \mathbb{C}), 0)$ is

$$\left(\bigwedge(\alpha, \beta, \gamma, f_1, f_2, f_3), d\right), \quad df_1 = q_g^1,\; df_2 = q_g^2,\; df_3 = q_g^3,$$

where $\deg(\alpha) = 2$, $\deg(\beta) = 4$, $\deg(\gamma) = 6$, $\deg(f_1) = 2g-1$, $\deg(f_2) = 2g+1$ and $\deg(f_3) = 2g+3$.

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