Complex projective 3-fold with non-negative canonical Euler–Poincare characteristic

MENG CHEN AND KANG ZUO

Let V be a complex non-singular projective 3-fold of general type with $\chi(\omega_V) \geq 0$ (resp. > 0). We prove that the m-canonical map $\Phi_{|mK_V|}$ is birational onto its image for all $m \geq 14$ (resp. ≥ 8). Known examples show that the lower bound $r_3 = 14$ (resp. $= 8$) is optimal.

1. Introduction

We work over the complex number field $\mathbb{C}$.

In this paper, we study multi-canonical systems $|mK_V|$ on complex projective 3-folds V of general type.

On a smooth complex complete curve C of genus $g(C) \geq 2$, it is well known that the 3-canonical map $\varphi_3 := \Phi_{|3K_C|}$ is always an embedding. On a smooth minimal projective surface S of general type, Bombieri [2] proved that the m-canonical map is a birational morphism for all $m \geq 5$. The work of Tsuji [31], Hacon–McKernan [13] and Takayama [28] says that there exists a universal constant r_3 such that the r_3-canonical map Φ_{r_3} is birational for all smooth projective 3-folds of general type. We note that Tsuji [31] announced a very large r_3. Recently Chen and the first author [5] have given an explicit r_3 (≤ 77).

Under extra assumptions there have been already the following optimal results about r_3 for minimal projective 3-folds X of general type with $\mathbb{Q}$-factorial terminal singularities:

- $\varphi_5 := \Phi_{|5K_X|}$ is birational if either X is Gorenstein (by Chen, Chen and Zhang [4]) or $p_g(X) \geq 4$ (by Chen [8]) or $K_X^3 \gg 0$ (by Todorov [30]). $r_3 = 5$ is optimal.

- $\varphi_8 := \Phi_{|8K_X|}$ is birational if $p_g(X) \geq 2$ (by Chen [8]). $r_3 = 8$ is optimal.

- φ_7 is birational if $q(X) > 0$ (by Chen and Hacon [6]).

- φ_5 is birational if $\chi(\omega_X) \geq 0$ and $q(X) > 0$ (by Chen and Hacon [7]). $r_3 = 5$ is optimal.

Going on the study of 3-folds with $\chi(\omega) \geq 0$, we prove the following.

Theorem 1.1. *Let V be a non-singular projective 3-fold of general type with $\chi(\omega_V) := \sum_{i=0}^{3}(-1)^i h^i(V, \omega_V) \geq 0$, where ω_V is the canonical line bundle of V. Then the m-canonical map φ_m is birational onto its image for all $m \geq 14$.*

The following example of Iano-Fletcher shows that $r_3 = 14$ in Theorem 1.1 is optimal.

Example 1.2. (See [10, No. 19, p. 151]) The canonical hypersurface $X_{28} \subset \mathbb{P}(1, 3, 4, 5, 14)$ has three terminal quotient singularities, $p_g(X) = 1$, $q(X) = h^2(\mathcal{O}_X) = 0$ and $\chi(\omega_X) = -\chi(\mathcal{O}_X) = 0$. A smooth model V of X_{28} has the invariant: $\chi(\omega_V) = 0$ and V is of general type. It is clear that φ_m is birational for all $m \geq 14$ and that φ_{13} is not birational. So the lower bound $r_3 = 14$ in Theorem 1.1 is sharp.

Our method has a direct consequence.

Corollary 1.3. *Let V be a smooth projective 3-fold of general type with $\chi(\omega_V) > 0$. Then the m-canonical map $\Phi_{|mK_V|}$ is birational onto its image for all $m \geq 8$.*

Example 1.4. (See [10, No. 12, p. 151]) Fletcher has a 3-fold $X_{16} \subset \mathbb{P}(1, 1, 2, 3, 8)$ on which the 8-canonical map is birational and the 7-canonical map is not birational. A smooth model V of X_{16} has the invariant $\chi(\omega) = 1$ and V is of general type. V has $r_3(V) = 8$. Thus the lower bound $r_3 = 8$ in Corollary 1.3 is again optimal.

Note that 3-folds of general type with $\chi(\omega) \geq 0$ form an interesting class, around which there have been already some established works:

- Gorenstein minimal 3-folds of general type have $\chi(\omega) > 0$;

- Any 3-fold of general type admitting a generically finite cover over an Abelian variety has $\chi(\omega) > 0$ (see Green–Lazarsfeld [12]);

- Any 3-fold of general type, admitting a representation $\rho : \pi_1(X) \to \mathrm{GL}$ such that the Shafarevich map Sh_ρ is generically finite, has $\chi(\omega) \geq 0$ (see Jost–Zuo [16]);

- Any 3-fold of general type with generically large fundamental group is conjectured to have $\chi(\omega) > 0$ (see Kollár [20, Conjecture 18.12.1]).

It is clear that the key point in studying pluricanonical maps is to compute P_m for $m > 1$. One may do this on a minimal model according to Reid [23]. When $\chi(\omega) = -\chi(\mathcal{O}) \geq 0$, it is clear from Reid's formula that $P_2 > 0$. This makes it possible for us to prove effective birationality of φ_m. But to prove the optimal bound $r_3 = 14$ need more evolved argument, which is exactly what we have done in this paper. The case $\chi(\omega) < 0$ has been treated with quite a different approach very recently by Chen and the first author [5].

2. Notations and set up

Since both the birationality of pluricanonical maps and $\chi(\omega)$ are birationally invariant, we may study a minimal model. By the 3-dimensional Minimal Model Program (MMP) (see [19, 22], for instance), we only have to consider a minimal 3-fold X of general type with $\mathbb{Q}$-factorial terminal singularities. Denote the Cartier index of X by $r := r(X)$ which is the minimal positive integer with rK_X a Cartier divisor, where K_X is a canonical divisor on X. The symbol $\equiv$ stands for the numerical equivalence of divisors, whereas $\sim$ denotes the linear equivalence and $=_{\mathbb{Q}}$ denotes the $\mathbb{Q}$-linear equivalence.

2.1. Set up for $|m_0 K_X|$.

Assume $P_{m_0}(X) := h^0(X, \mathcal{O}_X(m_0 K_X)) \geq 2$ for some positive integer $m_0 > 0$. We study the m_0-canonical map φ_{m_0} which is a rational map.

First we fix an effective Weil divisor $K_{m_0} \sim m_0 K_X$. Take successive blow-ups $\pi : X' \to X$ (along non-singular centers), which exists by Hironaka's big theorem, such that

(i) X' is smooth;

(ii) the movable part of $|m_0 K_{X'}|$ is base point free;

(iii) the support of $\pi^*(K_{m_0})$ is of simple normal crossings.

Denote by g_{m_0} the composition $\varphi_{m_0} \circ \pi$. So $g_{m_0} : X' \longrightarrow W' \subseteq \mathbb{P}^{P_{m_0}(X)-1}$ is a morphism. Let $X' \xrightarrow{f_{m_0}} B \xrightarrow{s} W'$ be the Stein factorization of g_{m_0}. We have the following commutative diagram.

$$
\begin{array}{ccc}
X' & \xrightarrow{\ f_{m_0}\ } & B \\
\pi \downarrow & \searrow^{g_{m_0}} & \downarrow s \\
X & \dashrightarrow & W' \\
& \varphi_{m_0} &
\end{array}
$$

We recall the definition of $\pi^*(K_X)$. One has $r(X)K_{X'} = \pi^*(r(X)K_X) + E_\pi$, where E_π is a sum of effective exceptional divisors. One defines $\pi^*(K_X) := K_{X'} - (1/r(X))E_\pi$. So, whenever we take the round up of $m\pi^*(K_X)$, we always have $\lceil m\pi^*(K_X) \rceil \leq mK_{X'}$ for any integer $m > 0$. We may write $m_0\pi^*(K_X) =_{\mathbb{Q}} M_{m_0} + E'_{m_0}$, where E'_{m_0} is an effective $\mathbb{Q}$-divisor and M_{m_0} is the movable part of $|m_0K_{X'}|$. On the other hand, one has $m_0K_{X'} =_{\mathbb{Q}} \pi^*(m_0K_X) + E_{\pi,m_0} = M_{m_0} + Z_{m_0}$, where Z_{m_0} is the fixed part and E_{π,m_0} an effective $\mathbb{Q}$-divisor which is a $\mathbb{Q}$-sum of distinct exceptional divisors. Clearly $Z_{m_0} = E'_{m_0} + E_{\pi,m_0}$.

If $\dim \varphi_{m_0}(X) = 2$, a general fiber of f_{m_0} is a smooth projective curve of genus at least 2. We say that X is m_0-*canonically fibered by curves*.

If $\dim \varphi_{m_0}(X) = 1$, a general fiber S of f_{m_0} is a smooth projective surface of general type. We say that X is m_0-*canonically fibered by surfaces* with invariants $(c_1^2(S_0), p_g(S))$, where S_0 is the minimal model of S. We may write $M_{m_0} \equiv a_{m_0}S$, where $a_{m_0} \geq P_{m_0}(X) - 1$ by considering the degree of a curve in a projective space.

A generic irreducible element S of $|M_{m_0}|$ means either a general member of $|M_{m_0}|$, whenever $\dim \varphi_{m_0}(X) \geq 2$ or, otherwise, a general fiber of f_{m_0}.

Definition 2.1. Assume that $|M'|$ is movable on V. By abuse of concepts, we also define *a generic irreducible element S'* of an arbitrary linear system $|M'|$ on an arbitrary variety V in a similar way. *A generic irreducible element S'* of $|M'|$ is defined to be a generic irreducible component in a general member of $|M'|$.

3. A technical theorem

Believing that Theorem 2.2 in [8] is quite effective in treating 3-folds X with $p_g(X) \geq 2$, we extend the technique there to build a parallel theorem so as

to study those X with $p_g(X) \leq 1$, but with $P_{m_0}(X) \geq 2$ for some integer $m_0 > 0$.

3.1. Assumptions

We need to make the following assumptions to explain our key method. Keep the same notation as in Section 2.1 above. Let $m > 0$ be certain integer.

(i) Either $m \geq m_0 + 2$ and $p_g(X) > 0$ or $|mK_{X'}|$ separates different irreducible elements S of $|M_{m_0}|$ (namely, $\Phi_{|mK_{X'}|}(S') \neq \Phi_{|mK_{X'}|}(S'')$ for two different irreducible elements S', S'' of $|M_{m_0}|$) and $p_g(X) = 0$.

(ii) Assume that, on the smooth surface S, there is a movable linear system $|G|$ and that C, as a generic irreducible element of $|G|$, is smooth. The linear system $|mK_{X'}|\|_S$ on S (as a sub-linear system of $|mK_{X'}|_S|$) separates different generic irreducible elements of $|G|$. Or sufficiently, the complete linear system

$$|K_S + \ulcorner(m-1)\pi^*(K_X) - S - \frac{1}{p}E'_{m_0}\urcorner|_S|$$

separates different generic irreducible elements of $|G|$.

(iii) There is a rational number $\beta > 0$ such that $\pi^*(K_X)|_S - \beta C$ is numerically equivalent to an effective $\mathbb{Q}$-divisor; **Set**

$$\alpha := \left(m - 1 - \frac{m_0}{p} - \frac{1}{\beta}\right)\xi$$

and $\alpha_0 := \ulcorner\alpha\urcorner$.

(iv) Either the inequality $\alpha > 1$ holds; or C is non-hyperelliptic, $m - 1 - (m_0/p) - (1/\beta) > 0$ and C is an even divisor on S.

(v) Either $\alpha > 2$; or $\alpha_0 \geq 2$ and C is non-hyperelliptic; or C is non-hyperelliptic, $m - 1 - (m_0/p) - (1/\beta) > 0$ and C is an even divisor on S.

Set $\xi := (\pi^*(K_X) \cdot C)_{X'}$ which is a positive rational number and define

$$p := \begin{cases} 1 & \text{if } \dim(B) \geq 2, \\ a_{m_0} (\text{see Section 2.1 for the definition}) & \text{otherwise.} \end{cases}$$

Let $f := f_{m_0} : X' \longrightarrow B$ be an induced fibration by φ_{m_0}.

Theorem 3.1. *Let X be a minimal projective 3-fold of general type with $P_{m_0}(X) \geq 2$ for some integer $m_0 > 0$. Keep the same notation as in Section 2.1 above. Then the inequality*

$$m\xi \geq 2g(C) - 2 + \alpha_0$$

holds under assumptions (iii) *and* (iv). *Furthermore, φ_m of X is birational onto its image under assumptions* (i), (ii), (iii) *and* (v).

Proof. First we shall show that $|mK_{X'}|$ can separate different irreducible elements of $|M_{m_0}|$. When $p_g(X) = 0$, $|mK_{X'}|$ can separate different irreducible elements of $|M_{m_0}|$ by assumption (i). When $p_g(X) > 0$, we consider the sub-system

$$\left| K_{X'} + \ulcorner (m-1)\pi^*(K_X) - \frac{1}{p}E'_{m_0} \urcorner \right| \subset |mK_{X'}|.$$

Let S' and S'' be two different generic irreducible elements of $|M_{m_0}|$. Clearly one has

$$K_{X'} + \ulcorner (m-1)\pi^*(K_X) - \frac{1}{p}E'_{m_0} \urcorner \geq K_{X'} + \ulcorner (m - m_0 - 1)\pi^*(K_X) \urcorner + S \geq S$$

by assumption (i). So $|K_{X'} + \ulcorner (m-1)\pi^*(K_X) - \frac{1}{p}E'_{m_0} \urcorner|$ can separate S' and S'' if either $\dim(B) \geq 2$ (cf. [29, Lemma 2]) or $\dim(B) = 1$ and $g(B) = 0$ ((P2) in [9, Section 2.1]). For the case $\dim(B) = 1$ and $g(B) > 0$, one has $a_{m_0} \geq P_{m_0} \geq 2$. Thus $p \geq 2$. Since

$$m - 1 - \frac{2m_0}{p} \geq 1 + \left(1 - \frac{2}{p}\right)m_0 > 0$$

and then

$$(m-1)\pi^*(K_X) - \frac{2}{p}E'_{m_0} - S' - S'' \equiv \left(m - 1 - \frac{2m_0}{p}\right)\pi^*(K_X)$$

is nef and big, the Kawamata–Viehweg vanishing theorem [17, 32] gives a surjective map:

$$H^0\left(X', K_{X'} + \ulcorner (m-1)\pi^*(K_X) - \frac{2}{p}E'_{m_0} \urcorner\right)$$

$$\longrightarrow H^0\left(S', K_{S'} + \ulcorner (m-1)\pi^*(K_X) - \frac{2}{p}E'_{m_0} \urcorner |_{S'}\right)$$

$$\oplus H^0\left(S'', K_{S''} + \ulcorner (m-1)\pi^*(K_X) - \frac{2}{p}E'_{m_0} \urcorner |_{S''}\right).$$

The last two groups are non-zero because $p_g(X) > 0$ (so $p_g(S')$, $p_g(S'') > 0$) and $m - 1 - m_0 > 0$. Therefore $|K_{X'} + \ulcorner(m-1)\pi^*(K_X) - \frac{2}{p}E'_{m_0}\urcorner|$ can separate S' and S'' and so can $|mK_{X'}|$.

By the birationality principle (P1) and (P2) of [9], it suffices to prove that $|mK_{X'}|\|_S$ on S gives a birational map onto its image. Practically we may study a smaller linear system than $|mK_{X'}|\|_S$ on S. Noting that

$$(m-1)\pi^*(K_X) - \frac{1}{p}E'_{m_0} - S \equiv \left(m - 1 - \frac{m_0}{p}\right)\pi^*(K_X)$$

is nef and big under the assumptions (iv) or (v), the vanishing theorem gives a surjective map

$$H^0\left(X', K_{X'} + \ulcorner(m-1)\pi^*(K_X) - \frac{1}{p}E'_{m_0}\urcorner\right)$$

(3.1)
$$\longrightarrow H^0\left(S, K_S + \ulcorner(m-1)\pi^*(K_X) - S - \frac{1}{p}E'_{m_0}\urcorner|_S\right).$$

Note that $|K_{X'} + \ulcorner(m-1)\pi^*(K_X) - \frac{1}{p}E'_{m_0}\urcorner| \subset |mK_{X'}|$. It suffices to prove that

$$|K_S + \ulcorner(m-1)\pi^*(K_X) - S - \frac{1}{p}E'_{m_0}\urcorner|_S|$$

gives a birational map.

The birationality principle again allows us to study the restriction to curves by assumption (ii). Now consider a generic irreducible element $C \in |G|$. By assumption (iii), there is an effective $\mathbb{Q}$-divisor H on S such that

$$\frac{1}{\beta}\pi^*(K_X)|_S \equiv C + H.$$

By the vanishing theorem, whenever $m - 1 - \frac{m_0}{p} - \frac{1}{\beta} > 0$, we have the surjective map

$$H^0\left(S, K_S + \ulcorner\left((m-1)\pi^*(K_X) - S - \frac{1}{p}E'_{m_0}\right)|_S - H\urcorner\right) \longrightarrow H^0(C, K_C + D),$$

where

$$D := \ulcorner((m-1)\pi^*(K_X) - S - \frac{1}{p}E'_{m_0})|_S - C - H\urcorner|_C$$

is a divisor on C. Noting that

$$\left((m-1)\pi^*(K_X) - S - \frac{1}{p}E'_{m_0}\right)|_S - C - H \equiv \left(m - 1 - \frac{m_0}{p} - \frac{1}{\beta}\right)\pi^*(K_X)|_S$$

and that C is nef on S, we have $\deg(D) \geq \alpha$ and thus $\deg(D) \geq \alpha_0$. Whenever C is non-hyperelliptic, $m - 1 - \frac{m_0}{p} - \frac{1}{\beta} > 0$ and C is an even divisor on S, $\deg(D) \geq 2$ automatically follows and thus $|K_C + D|$ gives a birational map. Whenever $\deg(D) \geq 3$, then

$$|K_S + \ulcorner\left((m-1)\pi^*(K_X) - S - \frac{1}{p}E'_{m_0}\right)|_S - H\urcorner||_C$$

gives a birational map. Since

$$\left|K_S + \ulcorner\left((m-1)\pi^*(K_X) - S - \frac{1}{p}E'_{m_0}\right)|_S - H\urcorner\right|$$

$$\subset \left|K_S + \ulcorner(m-1)\pi^*(K_X) - S - \frac{1}{p}E'_{m_0}\urcorner|_S\right|,$$

the latter linear system gives a birational map. So φ_m of X is birational.

Finally we show the inequality for ξ. Whenever we have $\deg(D) \geq 2$, $|K_C + D|$ is base point free by the curve theory. Denote by $|M_m|$ the movable part of $|mK_{X'}|$ and by $|N_m|$ the movable part of $|K_S + \ulcorner((m-1)\pi^*(K_X) - S - \frac{1}{p}E'_{m_0})|_S - H\urcorner|$. Applying [9, Lemma 2.7] to surjective maps (3.1) and (3.2), we have

$$m\pi^*(K_X)|_S \geq N_m \quad \text{and} \quad (N_m \cdot C)_S \geq 2g(C) - 2 + \deg(D).$$

Note that the above inequality holds without conditions (i) and (ii). We are done. □

Remark 3.2. If we replace M_{m_0} in Theorem 3.1 by any divisor $N_{m_0} \leq M_{m_0}$ with $h^0(X', N_{m_0}) \geq 2$, Theorem 3.1 is still true accordingly. This is clear by the proof. The main point is that it suffices to prove that a sub-linear system of $|mK_{X'}|$ gives a birational map. To avoid frustrating setting up and more complicated notations, we omit the proof in details. The idea is, however, trivially similar.

While applying Theorem 3.1, one has to choose a suitable movable system $|G|$ on S. Then quite a technical problem is to find a suitable β as in Theorem 3.1(iii). The following lemma presents the way for the most difficult case – the rational pencil case.

Lemma 3.3. *Keep the same notation as in Section* 2.1 *and Theorem* 3.1. *Assume $B = \mathbb{P}^1$. Let $f : X' \longrightarrow \mathbb{P}^1$ be an induced fibration of φ_{m_0}. Denote by*

$F := S$ a general fiber of f. Then one can find a sequence of rational numbers $\{\beta_n\}$ with $\lim_{n \mapsto +\infty} \beta_n = p/(m_0 + p)$ such that $\pi^*(K_X)|_F - \beta_n \sigma^*(K_{F_0})$ is $\mathbb{Q}$-linearly equivalent to an effective $\mathbb{Q}$-divisor N_n, where $\sigma : F \longrightarrow F_0$ is the blow down onto the smooth minimal model.

Proof. One has $\mathcal{O}_B(p) \hookrightarrow f_*\omega_{X'}^{m_0}$ and therefore $f_*\omega_{X'/B}^{t_0 p} \hookrightarrow f_*\omega_{X'}^{t_0 p + 2t_0 m_0}$ for any big integer t_0.

For any positive integer k, denote by M_k the movable part of $|kK_{X'}|$. Note that $f_*\omega_{X'/B}^{t_0 p}$ is generated by global sections since it is semi-positive according to Viehweg [33]. So any local section can be extended to a global one. On the other hand, $|t_0 p \sigma^*(K_{F_0})|$ is base point free and is exactly the movable part of $|t_0 p K_F|$ by Bombieri [2] or Reider [25]. Clearly one has the following relation:

$$a_0 \pi^*(K_X)|_F \geq M_{t_0 p + 2t_0 m_0}|_F \geq b_0 \sigma^*(K_{F_0}),$$

where $a_0 := t_0 p + 2t_0 m_0$ and $b_0 := t_0 p$. This means that there is an effective $\mathbb{Q}$-divisor E_0' on F such that

$$a_0 \pi^*(K_X)|_F =_{\mathbb{Q}} b_0 \sigma^*(K_{F_0}) + E_0'.$$

Thus $\pi^*(K_X)|_F =_{\mathbb{Q}} (p/(p + 2m_0))\sigma^*(K_{F_0}) + E_0$ with $E_0 = \frac{1}{a_0}E_0'$.

We consider the case $p \geq 2$.

Assume that we have defined a_n and b_n such that the following is satisfied with $l = n$:

$$a_l \pi^*(K_X)|_F \geq b_l \sigma^*(K_{F_0}).$$

We will define a_{n+1} and b_{n+1} inductively such that the above inequality is satisfied with $l = n + 1$. One may assume from the beginning that $a_n \pi^*(K_X)$ supports on a divisor with normal crossings. Then the Kawamata–Viehweg vanishing theorem implies the surjective map

$$H^0(K_{X'} + \ulcorner a_n \pi^*(K_X) \urcorner + F) \longrightarrow H^0(F, K_F + \ulcorner a_n \pi^*(K_X) \urcorner|_F).$$

One has the relation

$$\begin{aligned}
|K_{X'} + \ulcorner a_n \pi^*(K_X) \urcorner + F|\,|_F &= |K_F + \ulcorner a_n \pi^*(K_X) \urcorner|_F| \\
&\supset |K_F + b_n \sigma^*(K_{F_0})| \\
&\supset |(b_n + 1)\sigma^*(K_{F_0})|.
\end{aligned}$$

Denote by M'_{a_n+1} the movable part of $|(a_n + 1)K_{X'} + F|$. Applying [9, Lemma 2.7] again, one has $M'_{a_n+1}|_F \geq (b_n + 1)\sigma^*(K_{F_0})$. Re-modifying

our original π such that $|M'_{a_n+1}|$ is base point free. In particular, M'_{a_n+1} is nef. Since X is of general type $|mK_X|$ gives a birational map whenever m is big enough. Thus we see that M'_{a_n+1} is big if we fix a very big t_0 in advance.

Now the Kawamata–Viehweg vanishing theorem again gives

$$|K_{X'} + M'_{a_n+1} + F|\|_F = |K_F + M'_{a_n+1}|_F|$$
$$\supset |K_F + (b_n + 1)\sigma^*(K_{F_0})|$$
$$\supset |(b_n + 2)\sigma^*(K_{F_0})|.$$

We may repeat the above procedure inductively. Denote by M'_{a_n+t} the movable part of $|K_{X'} + M'_{a_n+t-1} + F|$ for $t \geq 2$. For the same reason, we may assume $|M'_{a_n+t}|$ to be base point free. Inductively one has:

$$M'_{a_n+t}|_F \geq (b_n + t)\sigma^*(K_{F_0}).$$

Applying the vanishing theorem once more, we have

$$|K_{X'} + M'_{a_n+t} + F|\|_F = |K_F + M'_{a_n+t}|_F|$$
$$\supset |K_F + (b_n + t)\sigma^*(K_{F_0})|$$
$$\supset |(b_n + t + 1)\sigma^*(K_{F_0})|.$$

Take $t = p - 1$. Noting that

$$|K_{X'} + M'_{a_n+p-1} + F| \subset |(a_n + p + m_0)K_{X'}|$$

and applying [9, Lemma 2.7] again, one has

$$a_{n+1}\pi^*(K_X)|_F \geq M_{a_n+p+m_0}|_F \geq M'_{a_n+p}|_F \geq b_{n+1}\sigma^*(K_{F_0}).$$

Here we set $a_{n+1} := a_n + p + m_0$ and $b_{n+1} = b_n + p$. Set $\beta_n = \frac{b_n}{a_n}$. Clearly $\lim_{n\mapsto+\infty} \beta_n = p/(m_0 + p)$.

The case $p = 1$ can be proved similarly, but with a simpler induction. We omit the details. $\square$

The following Lemma is needed in our proof. Though similar one has already been established in several papers of the first author, we include it here for the convenience to readers.

Lemma 3.4. *Keep the same notation as in Section* 2.1. *Let* $f : X' \longrightarrow B$ *be the induced fibration of* φ_{m_0}. *Denote by* $F := S$ *a general fiber of* f. *If* $\dim(B) = 1$ *and* $g(B) > 0$, *then* $\pi^*(K_X)|_F \sim \sigma^*(K_{F_0})$ *where* $\sigma : F \longrightarrow F_0$ *is the blow down onto the smooth minimal model.*

Proof. We shall use the idea of Lemma 14 in Kawamata's paper [18]. By Shokurov's theorem in [26](see also [14]), each fiber of $\pi : X' \longrightarrow X$ is rationally chain connected. Therefore, $f(\pi^{-1}(x))$ is a point for all $x \in X$. Considering the image $G \subset (X \times B)$ of X' via the morphism $(\pi \times f) \circ \triangle_{X'}$, where $\triangle_{X'}$ is the diagonal map $X' \longrightarrow X' \times X'$, one knows that G is a projective variety. Let $g_1 : G \longrightarrow X$ and $g_2 : G \longrightarrow B$ be two projections. Since g_1 is a projective morphism and even a bijective map, g_1 must be both a finite morphism of degree 1 and a birational morphism. Since X is normal, g_1 must be an isomorphism. So f factors as $f_1 \circ \pi$, where $f_1 := g_2 \circ g_1^{-1} : X \to B$ is a well-defined morphisms. In particular, a general fiber F_0 of f_1 must be smooth minimal. So it is clear that $\pi^*(K_X)|_F \sim \sigma^*(K_{F_0})$, where σ is nothing but $\pi|_F$. $\qquad\square$

4. Proof of the main theorem

We begin to prove Theorem 1.1. Let X be a complex minimal projective 3-fold of general type with $\mathbb{Q}$-factorial terminal singularities and $\chi(\omega_X) \geq 0$.

4.1. Reduction to the case $p_g{=}1$

If $p_g(X) > 1$, then φ_8 is birational by [8]. If $q(X) > 0$, then φ_m is birational for all $m \geq 7$ by [6]. Thus we assume, from now on, that $p_g(X) \leq 1$ and $q(X) = 0$. The assumption

$$0 \leq \chi(\omega_X) = -\chi(\mathcal{O}_X) = -1 + q(X) - h^2(\mathcal{O}_X) + p_g(X)$$

implies $p_g(X) \geq h^2(\mathcal{O}_X) + 1 \geq 1$. Then we clearly have $p_g(X) = 1$, $h^2(\mathcal{O}_X) = 0$ and $\chi(\omega_X) = 0$.

According to [4], we only have to study non-Gorenstein minimal 3-folds X. In practice, we may assume the Cartier index $r(X) > 1$.

4.2. Plurigenus

Let X be a minimal 3-fold of general type with terminal singularities. Recall Reid's plurigenus formula ([23, p. 413]):

$$(4.1) \quad P_m(X) = \frac{1}{12}m(m-1)(2m-1)K_X^3 - (2m-1)\chi(\mathcal{O}_X) + \sum_Q R_m(Q),$$

where $m > 1$ is an integer, the correction term

$$R_m(Q) := \frac{r^2 - 1}{12r}(m - \bar{m}) + \sum_{j=0}^{\bar{m}-1} \frac{\overline{bj}(r - \overline{bj})}{2r}$$

and the sum $\sum_Q$ runs through all baskets of singularities Q of type $\frac{1}{r}(a, -a, 1)$ with the positive integer a coprime to r, $0 < a < r$, $0 < b < r$, $ab \equiv 1 \pmod{r}$ and $\bar{m}$ the smallest residue of m mod r. *Reid's result ([23, Theorem 10.2]) says that the above baskets $\{Q\}$ of singularities are in fact virtual (!) though X may have non-quotient terminal singularities. Iano–Fletcher ([15]) actually shows that the set of baskets $\{Q\}$ in Reid's formula is uniquely determined by X.*

Lemma 4.1. *For all basket Q, $R_5(Q) \geq R_4(Q) \geq R_3(Q) \geq R_2(Q)$. In particular,*

$$P_5(X) > P_4(X) > P_3(X) > P_2(X) \geq 1$$

for all 3-fold X with $\chi(\mathcal{O}_X) = 0$.

Proof. Suppose that Q is of type $\frac{1}{r}(a, -a, 1) = \frac{1}{r}(1, -1, b)$ with $r > 1$, a coprime to r, $ab \equiv 1 \pmod{r}$ and $0 < b < r$.

When $r = 2$, one has

$$R_3(Q) = \frac{r^2 - 1}{6r} = \frac{1}{4} = R_2(Q).$$

When $r = 3$, one has

$$R_3(Q) = \frac{r^2 - 1}{4r} = \frac{2}{3} > \frac{1}{3} = \frac{b(r - b)}{2r} = R_2(Q).$$

When $r > 3$, one has $m = \bar{m}$ for $m = 2, 3$ and

$$R_3(Q) = \sum_{j=0}^{2} \frac{\overline{bj}(r - \overline{bj})}{2r} \geq \sum_{j=0}^{1} \frac{\overline{bj}(r - \overline{bj})}{2r} = R_2(Q).$$

If $\chi(\mathcal{O}_X) = 0$, then Reid's formula (4.1) gives

$$P_3(X) = \frac{5}{2}K_X^3 + \sum_Q R_3(Q) > \frac{1}{2}K_X^3 + \sum_Q R_2(Q) = P_2(X) > 0.$$

In particular, $P_3(X) \geq 2$.

Similarly one can verify the inequality $P_5(X) > P_4(X) > P_3(X)$. So $P_5(X) \geq 4$. $\qquad\square$

Now Theorem 1.1 essentially follows from the following theorem by letting $m_0 = 2, 3$.

Theorem 4.2. *Let X be a minimal projective 3-fold of general type with $\mathbb{Q}$-factorial terminal singularities, $p_g(X) = 1$ and $P_{m_0}(X) \geq 2$. Then φ_m is birational onto its image under one of the following conditions:*

(1) *$m \geq 4m_0 + 3$ and $2 \leq m_0 \leq 3$;*

(2) *$m \geq 4m_0 + 2$ and $m_0 \geq 4$;*

(3) *$m \geq 14$, $\chi(\mathcal{O}_X) = 0$ and $m_0 = 3$.*

Proof. Set $d_{m_0} := \dim \varphi_{m_0}(X)$. We discuss according to the value of d_{m_0}. We shall mainly apply Theorem 3.1. Because $p_g(X) > 0$, 3.1(i) is always satisfied whenever $m \geq m_0 + 2$.

Case 1. $d_{m_0} = 3$. Recall that S is a generic irreducible element of $|M_{m_0}|$. We have $p = 1$. On S we take $G := S|_S$. Clearly $|G|$ is base point free and is not composed of a pencil. So a generic irreducible element C of $|G|$ is a smooth curve. Also under the assumption $m \geq m_0 + 2$ one has

$$K_S + \lceil (m-1)\pi^*(K_X) - S - E'_{m_0} \rceil|_S$$
$$\geq (K_{X'}|_S + \lceil (m-1)\pi^*(K_X) - S - E'_{m_0} \rceil|_S) + G \geq G.$$

Thus 3.1(ii) is also satisfied. Because $m_0\pi^*(K_X)|_S \geq C$, we may take $\beta = \frac{1}{m_0}$ and so (iii) is satisfied.

Note that $C^2 \geq 2$ because $|G|$ is not composed of a pencil. So $m_0\pi^*(K_X)|_S \cdot C \geq C^2 \geq 2$, which implies $\xi \geq \frac{2}{m_0}$. Now we take $m \geq 3m_0 + 2$ and run Theorem 3.1. One has $\alpha = (m - 1 - m_0/p - 1/\beta)\xi \geq 2 + 2/m_0 > 2$. This means that φ_m is birational for all $m \geq 3m_0 + 2$. This is not the best. In fact, Theorem 3.1 already gives $\xi \geq (2g(C) + 1)/(3m_0 + 2)$. Note that $2g(C) - 2 = (K_S + C) \cdot C = (K_{X'}|_S + 2C) \cdot C > 4$. One has $\xi \geq 9/(3m_0 + 2)$. Now take $m > \frac{8}{3}m_0 + \frac{13}{9}$. One has $\alpha = (m - 1 - 2m_0)\xi > 2$. Theorem 3.1 says that φ_m is birational whenever $m > \frac{8}{3}m_0 + \frac{13}{9}$. (One may go on optimizing the estimate. We stop here since we have already proved the theorem.)

Case 2. $d_{m_0} = 2$. We have $p = 1$. On S we take $G := S|_S$. Clearly $|G|$ is base point free and is composed of a pencil. So a generic irreducible element C of $|G|$ is a smooth curve. One has $G \equiv tC$ for $t \geq P_{m_0} - 2 \geq 1$.

Also under the assumption $m \geq m_0 + 2$ one has

$$K_S + \ulcorner (m-1)\pi^*(K_X) - S - E'_{m_0} \urcorner|_S$$
$$\geq (K_{X'}|_S + \ulcorner (m-1)\pi^*(K_X) - S - E'_{m_0} \urcorner|_S) + G \geq C.$$

So $|K_S + \ulcorner (m-1)\pi^*(K_X) - S - E'_{m_0} \urcorner|_S|$ can separate different generic irreducible elements of $|G|$ provided that $|G|$ is composed of a rational pencil. When $|G|$ is composed of an irrational pencil, we need the assumption $m \geq 2m_0 + 2$. In fact, we have $S|_S \equiv tC$ with $t \geq 2$ and $m_0 \pi^*(K_X)|_S \equiv tC + E'_{m_0}|_S$. Take two generic irreducible elements C', C'' of $|G|$. Because

$$(m - m_0 - 1)\pi^*(K_X)|_S - C' - C'' - \frac{2}{t}E'_{m_0} \equiv \left(m - m_0 - \frac{2}{t}m_0 - 1 \right)\pi^*(K_X)|_S$$

is nef and big, the Kawamata–Viehweg vanishing theorem gives a surjective map

$$H^0\left(S, K_S + \ulcorner (m - m_0 - 1)\pi^*(K_X)|_S - \frac{2}{t}E'_{m_0}|_S \urcorner \right)$$
$$\longrightarrow H^0(C', K_{C'} + D') \oplus H^0(C'', K_{C''} + D''),$$

where D', D'' are divisors of positive degree. Besides, the last groups are non-zero. Noting that

$$K_S + \ulcorner (m-1)\pi^*(K_X) - S - E'_{m_0} \urcorner|_S \geq K_S + \ulcorner (m - m_0 - 1)\pi^*(K_X)|_S \urcorner$$
$$\geq K_S + \ulcorner (m - m_0 - 1)\pi^*(K_X)|_S - \frac{2}{t}E'_{m_0}|_S \urcorner,$$

$|K_S + \ulcorner (m-1)\pi^*(K_X) - S - E'_{m_0} \urcorner|_S|$ separates C' and C''. So we see that 3.1(ii) is satisfied whenever $m \geq 2m_0 + 2$.

Because $m_0 \pi^*(K_X)|_S \geq C$, we may take $\beta = \frac{1}{m_0}$ and so (iii) is satisfied. If we take a big m such that α is big enough, then Theorem 3.1 gives

$$m\xi \geq 2g(C) - 2 + \left(m - 1 - \frac{m_0}{p} - \frac{1}{\beta} \right),$$

which says $\xi \geq 2/(2m_0 + 1)$. This is only an initial estimate. Take $m > 4m_0 + 2$. Then $\alpha = (m - 2m_0 - 1)\xi > 2$. Theorem 3.1 says that φ_m is birational and that $\xi \geq 5/(4m_0 + 3)$. Take $m = 3m_0 + 2$. Then $\alpha = (m - 2m_0 - 1)\xi > 1$. Theorem 3.1 gives $\xi \geq 4/(3m_0 + 2)$. Finally take $m >$

$\frac{7}{2}m_0 + 2$. Then $\alpha = (m - 2m_0 - 1)\xi > 2$. Theorem 3.1 says that φ_m is birational whenever $m > \frac{7}{2}m_0 + 2$.

Case 3. $d_{m_0} = 1$. We have an induced fibration $f : X' \longrightarrow B$. Denote by $F := S$ a general fiber of f. Note that F is a surface of general type. Denote by $\sigma : F \longrightarrow F_0$ the blowing down onto the minimal model.

Subcase 3.1. $g(B) > 0$. By Lemma 3.4, we have $\pi^*(K_X)|_F \sim \sigma^*(K_{F_0})$. We need to study the condition (ii) in Theorem 3.1. For all $m \geq m_0 + 5$ one has

$$K_F + \ulcorner (m-1)\pi^*(K_X) - F - \frac{1}{p}E'_{m_0}\urcorner|_F \geq K_F + \ulcorner (m - m_0 - 1)\pi^*(K_X)|_F\urcorner$$

$$\geq K_F + (m - m_0 - 1)\pi^*(K_X)|_F$$
$$\geq (m - m_0)\pi^*(K_X)|_F \geq 5\sigma^*(K_{F_0}).$$

So $\Phi_{|K_F + \ulcorner (m-1)\pi^*(K_X) - F - \frac{1}{p}E'_{m_0}\urcorner|_F|}$ already gives a birational map. The proof of Theorem 3.1 tells that we may omit the conditions (iii) $\sim$ (v) in Theorem 3.1. Thus φ_m is birational for all $m \geq m_0 + 5$.

Subcase 3.2. $g(B) = 0$. For a general F the natural map

$$H^0(X', K_{X'} - F) \longrightarrow H^0(X', K_{X'})$$

is a strict inclusion simply because F is movable. This means that $p_g(F) > 0$ since $p_g(X') = p_g(X) > 0$. For our purpose, we classify F into the following three types:

(I) $(K_{F_0}^2, p_g(F)) = (1, 2)$;

(II) $(K_{F_0}^2, p_g(F)) = (2, 3)$;

(III) F does not belong to types (I) and (II).

We first study the type (III) case. By the results of Bombieri [2], Reider [25], Catanese–Ciliberto [3] and Francia [11], $|2\sigma^*(K_{F_0})|$ is always base point free whenever $p_g(F) > 0$. We set $G := 2\sigma^*(K_{F_0})$ to run Theorem 3.1. The inclusion $\mathcal{O}(1) \hookrightarrow f_*\omega_{X'}^{m_0}$ implies the inclusion

$$f_*\omega_{X'/\mathbb{P}^1}^2 \hookrightarrow f_*\omega_{X'}^{4m_0+2}.$$

Viehweg [33] first showed that $f_*\omega_{X'/\mathbb{P}^1}^2$ is semi-positive. So it is also generated by global sections. Thus it is clear that

$$|M_{4m_0+2}||_F \supset |2\sigma^*(K_{F_0})|,$$

where M_{4m_0+2} is the movable part of $|(4m_0 + 2)K_{X'}|$.

(#) So 3.1(ii) is satisfied for all $m \geq 4m_0 + 2$ and for all F with $p_g(F) > 0$, $G \leq 2\sigma^*(K_{F_0})$ and $|G|$ not an irrational pencil.

In type (III) case, G is an even divisor under our setting and is not composed of a pencil of curves. On the other hand, the birationality of $\Phi_{|3K_F|}$ implies that a general $C \in |G|$ is non-hyperelliptic. By Lemma 3.3, we can take a $\beta \mapsto 1/(2m_0 + 2)$ such that $\pi^*(K_X)|_F - \beta C$ is numerically equivalent to an effective $\mathbb{Q}$-divisor. By our definition, one has $p \geq P_{m_0} - 1 \geq 1$. For all $m \geq 3m_0 + 4$, $\alpha = (m - 1 - m_0/p - 1/\beta)\xi > 0$. In general, whenever $m \geq 4m_0 + 2$, Theorem 3.1 asserts that φ_m is birational.

Next we study the type (II) case. We have $(K_{F_0}^2, p_g(F)) = (2, 3)$. By [1, p. 227], we know that $|\sigma^*(K_{F_0})|$ is base point free and $\Phi_{|K_F|}$ is finite of degree 2. We set $G := \sigma^*(K_{F_0})$. Then C, as a generic irreducible element of G, is smooth and of genus 3. We have already showed (statement (#) in type (III) case) that 3.1(ii) is satisfied for all $m \geq 4m_0 + 2$ because $G \leq 2\sigma^*(K_{F_0})$. By Lemma 3.3 we can take a $\beta \mapsto 1/(m_0 + 1)$ such that $\pi^*(K_X)|_F - \beta C$ is numerically equivalent to an effective $\mathbb{Q}$-divisor. So $\xi = \pi^*(K_X)|_F \cdot C \geq \beta C^2 \mapsto 2/(m_0 + 1)$. Taking the limit one has $\xi \geq 2/(m_0 + 1)$. Similarly we have $p \geq 1$. Take $m = 3m_0 + 2$. Then $\alpha = (m - 1 - m_0/p - 1/\beta)\xi \geq 2m_0/(m_0 + 1) > 1$. Theorem 3.1 gives $\xi \geq 6/(3m_0 + 2)$. In order to get the birationality, we need the assumption $m \geq 4m_0 + 2$, under which one has

$$\alpha = \left(m - 1 - \frac{m_0}{p} - \frac{1}{\beta}\right)\xi \geq \frac{12m_0}{3m_0 + 2} > 2.$$

Thus φ_m is birational whenever $m \geq 4m_0 + 2$.

Finally we study the type (I) case. We have $(K_{F_0}^2, p_g(F)) = (1, 2)$. By [1] we know that the movable part of $|K_F|$ has one simple base point. Take $|G|$ to be the movable part of $|K_F|$. Then a generic irreducible element C of $|G|$ is a smooth curve of genus 2. Similarly we need the assumption $m \geq 4m_0 + 2$ to secure the condition 3.1(ii) (by virtue of the statement (#) in the type (III) case because $|G|$ is a rational pencil). By Lemma 3.3 we can take a $\beta \mapsto 1/(m_0 + 1)$ such that $\pi^*(K_X)|_F - \beta C$ is numerically equivalent to an effective $\mathbb{Q}$-divisor. Clearly we have $p \geq 1$. If we take a big m such that α is big enough, Theorem 3.1 gives

$$m\xi \geq 2g(C) - 2 + (m - 2m_0 - 2)\xi$$

which implies $\xi \geq 1/(m_0 + 1)$. Take $m = 4m_0 + 5$. Then $\alpha \geq (2m_0 + 3)/(m_0 + 1) > 2$. Theorem 3.1 gives $\xi \geq 5/(4m_0 + 5)$ and so does the birationality of φ_m. Take $m = 4m_0 + 4$. Then $\alpha \geq (10m_0 + 10)/(4m_0 + 5) > 2$. One has the birationality of φ_m and $\xi \geq 5/(4m_0 + 4)$. Take $m = 4m_0 + 3$. Then $\alpha \geq (10m_0 + 5)/(4m_0 + 4) > 2$ whenever $m_0 \geq 2$. So φ_{4m_0+3} is birational and $\xi \geq 5/(4m_0 + 3)$. We have already proved Theorem 4.2(1).

Assume $m_0 \geq 4$ and take $m = 4m_0 + 2$. Then $\alpha \geq 10m_0/(4m_0 + 3) > 2$. Theorem 3.1 gives the birationality of φ_{4m_0+2}. We have proved Theorem 4.2(2).

The only left case to verify is $m_0 = 3$ and $\chi(\mathcal{O}_X) = 0$ for which we have already the birationality of φ_{4m_0+3}. We have

$$(4.2) \qquad\qquad\qquad \xi \geq \frac{1}{3}$$

as shown above.

> ($**$) Furthermore the assumption $m_0 = 3$, $\chi(\mathcal{O}_X) = 0$ and $\xi > \frac{1}{3}$ gives $\alpha \geq 2m_0\xi > 2$, which means that $\varphi_{4m_0+2} = \varphi_{14}$ is birational by Theorem 3.1.

Claim 4.3. *When $m_0 = 3$ and $\chi(\mathcal{O}_X) = 0$, $\varphi_{4m_0+2} = \varphi_{14}$ is birational.*

To prove the claim, we need to study the 5-canonical system $|5K_{X'}|$. By Lemma 4.1, one has $P_5(X) \geq 4$. Denote $d_5 := \dim \varphi_5(X)$. Keep the same notation as in Section 2.1 and in Theorem 3.1. Recall that M_5 is the movable part of $|5K_{X'}|$. The induced fibration from φ_5 is denoted by $f_5 : X' \longrightarrow W_5$. (In fact, we can take further modifications over X' (still denote by X' the final modification) such that the final X' dominates both φ_3 and φ_5.)

If $d_5 = 3$, then $\dim \varphi_5(F) = 2$ and $\dim \varphi_5(C) = 1$ for a general curve $C \in |G|$ in a general fiber F. This means that the linear system $|M_5||_C$ ($\subset |M_5|_C|$) gives a finite map from C onto a curve and so does $|M_5|_C|$. The Riemann–Roch and Clifford theorem on C says that $5\pi^*(K_X) \cdot C \geq M_5 \cdot C \geq 2$, i.e., $\xi \geq \frac{2}{5} > \frac{1}{3}$. Statement ($**$) tells that φ_{14} is birational.

If $d_5 = 1$, then $|5K_{X'}|$ induces the same fibration $f : X' \longrightarrow W_5 = B$ simply because $5K_{X'} \geq 3K_{X'}$. The typical property here is that $5\pi^*(K_X) \geq 3F$ for a general fiber F. By Lemma 3.3 (just take $m_0' = 5$ and $p' = 3$) we can find a $\beta \mapsto \frac{3}{8}$ such that $\pi^*(K_X)|_F - \beta C$ is numerically equivalent to an effective $\mathbb{Q}$-divisor. Going on the argument just before the claim, we have $m_0 = 3$, $p = 1$, $\beta = \frac{3}{8}$ and $\xi \geq \frac{1}{3}$ as in (4.2). Take $m = 14$. Then $\alpha = (m - 1 - m_0 - \frac{1}{\beta})\xi > 2$. So Theorem 3.1 gives the birationality of φ_{14}.

Finally if $d_5 = 2$, we need to study the image surface W_5' of X' through the morphism $\Phi_{|M_5|}$. In fact, we have the decomposition

$$\Phi_{|M_5|} : X' \xrightarrow{f_5} W_5 \xrightarrow{s_5} W_5' \subset \mathbb{P}^{P_5(X)-1}.$$

There is a very ample divisor H_5 on W_5' such that $M_5 = \Phi_{|M_5|}^*(H_5)$. Furthermore one has $M_5|_{S_5} \equiv a_5\tilde{C}$ for a general member $S_5 \in |M_5|$ and an integer $a_5 \geq \deg s_5 \deg(W_5') \geq \deg(W_5') \geq P_5(X) - 2$, $\tilde{C}$ is a general fiber of f_5. If $a_5 \geq 3$, we may utilize our argument in Case 2 replacing m_0 by $m_0' = 5$. Over there we have shown $\xi \geq 4/(3m_0' + 2) = \frac{4}{17}$. But we can take a better β, namely $\beta = \frac{3}{5}$. We have $p = 1$. Take $m = 12$. Then $\alpha = (12 - 1 - 5 - (5/3))\xi > 1$. Theorem 3.1 gives $\xi \geq \frac{1}{3}$. Take $m = 14$. Then $\alpha = (14 - 1 - 5 - (5/3))\xi \geq \frac{19}{9} > 2$. Theorem 3.1 says that φ_{14} is birational.

So we are left to study what happens if $a_5 = 2$. This means that $\deg(W_5') = 2$, $P_5(X) = 4$ and $\deg(s_5) = 1$. Recall that a degree 2 irreducible surface in $\mathbb{P}^3$ must be one of the following type of surfaces (see, for instance, Reid's lecture notes [24, Exercise 19, p. 30]):

(a) W_5' is the cone $\overline{\mathbb{F}}_2$ by blowing down the unique section with the self-intersection -2 on the Hirzebruch surface $\mathbb{F}_2$ (a ruled surface);

(b) $W_5' = \mathbb{P}^1 \times \mathbb{P}^1$.

We study these two cases separately in the following propositions. $\square$

Proposition 4.4. *For case (a), φ_{14} is birational.*

Proof. We know that $M_5 = g_5^*(H_5')$ for a very ample divisor H_5' on W_5' with $H_5'^2 = 2$ and $g_5 : X' \longrightarrow W_5'$ is the birational morphism. Because W_5' is already normal (which is a cone), by taking further modification to X', we can assume that g_5 factors through the minimal resolution $\mathbb{F}_2$ of W_5'. So we have the map $g_5 : X' \xrightarrow{h_5} \mathbb{F}_2 \xrightarrow{\nu} W_5'$ where h_5 is a fibration and ν is the minimal resolution. Set $\bar{H}_5 = \nu^*(H_5')$. Then $\bar{H}^2 = (H_5')^2 = 2$. Noting that $\bar{H}_5$ is nef and big on $\mathbb{F}_2$, we can write

$$\bar{H} \sim \mu G_0 + nT,$$

where G_0 is the unique section with $G_0^2 = -2$, μ and n are integers and T is the general fiber of the ruling on $\mathbb{F}_e$. The property of $\bar{H}_5$ being nef and big implies $\mu > 0$ and $n \geq 2\mu \geq 2$. Now let $\theta_2 : \mathbb{F}_2 \longrightarrow \mathbb{P}^1$ be the ruling, whose fibers are all smooth rational curves. Set $f_0 := \theta_2 \circ h_5 : X' \longrightarrow \mathbb{P}^1$, which is

a fibration with connected fibers. Denote by $\tilde{F}$ a general fiber of f_0 and by $\tilde{\sigma} : \tilde{F} \longrightarrow \tilde{F}_0$ the contraction onto the minimal model. Clearly $p_g(\tilde{F}) > 0$. We have

$$M_5 \sim g_5^*(H_5') = h_5^*(\bar{H}) \geq 2\tilde{F}.$$

Set $N_5 = 2\tilde{F}$. Replace $|M_5|$ by the sub-linear system $|N_5|$ in Theorem 3.1. We can also study the birationality of φ_m as in Remarked 3.2.

Note that the fibration $f_0 : X' \longrightarrow \mathbb{P}^1$ is the induced fibration by $\Phi_{|2\tilde{F}|}$. We can repeat a similar argument to that in Subcase 3.2 above. We can verify those conditions in Theorem 3.1.

Because $14\,K_{X'} \geq 2\tilde{F}$, 3.1(i) is satisfied. On the other hand, we have $\mathcal{O}(2) \hookrightarrow f_{0*}\omega_{X'}^5$ which implies the inclusion

$$f_{0*}\omega_{X'/\mathbb{P}^1}^2 \hookrightarrow f_{0*}\omega_{X'}^{12}.$$

Similarly Viehweg's semi-positivity and the base point freeness of $|2\sigma^*(K_{\tilde{F}_0})|$ say that $|12\,K_{X'}|_{\tilde{F}}$ can separate different generic irreducible elements of $|2\sigma^*(K_{\tilde{F}_0})|$. So can $|14 K_{X'}|_{\tilde{F}}$. So 3.1(ii) is satisfied.

Now we take $m_0' = 5$ to run Theorem 3.1 with $|M_5|$ replaced by $|N_5|$. One has $p = 2$.

If $\tilde{F}$ is of type (III), we take $\tilde{G} := 2\tilde{\sigma}^*(K_{\tilde{F}_0})$. Take $m = 14$. The proof of Lemma 3.3 says that one can take a $\beta \mapsto \frac{1}{7}$ such that $\pi^*(K_X)|_{\tilde{F}} - 2\beta\tilde{\sigma}^*(K_{\tilde{F}_0})$ is numerically equivalent to an effective $\mathbb{Q}$-divisor. So $\alpha = (m - 1 - m_0'/p - 1/\beta)\tilde{\xi} > 0$. Noting that a generic irreducible element of $|\tilde{G}|$ is non-hyperelliptic and even, φ_{14} is birational by Theorem 3.1.

If $\tilde{F}$ is of type (II), we set $\tilde{G} := \tilde{\sigma}^*(K_{\tilde{F}_0})$. Then $\tilde{C}$, as a generic irreducible element of $|\tilde{G}|$, is smooth and of genus 3. We have already showed above that 3.1(ii) is satisfied for $m = 14$. By Lemma 3.3 we can take a $\beta \mapsto \frac{2}{7}$ such that $\pi^*(K_X)|_{\tilde{F}} - \beta\tilde{C}$ is numerically equivalent to an effective $\mathbb{Q}$-divisor. So $\tilde{\xi} = \pi^*(K_X)|_{\tilde{F}} \cdot \tilde{C} \geq \beta\tilde{C}^2 \mapsto \frac{4}{7}$. Taking the limit one has $\tilde{\xi} \geq \frac{4}{7}$. We have $p = 2$. Take $m = 14$. Then $\alpha = (m - 1 - (m_0'/p) - (1/\beta))\tilde{\xi} \geq 4$. Theorem 3.1 says that φ_{14} is birational.

If $\tilde{F}$ is of type (I), we take $\tilde{G}$ to be the movable part of $|\tilde{\sigma}^*(K_{\tilde{F}_0})|$. Then $\tilde{C}$, as a generic irreducible element of $|\tilde{G}|$, is smooth and of genus 2. We have already showed above that 3.1(ii) is satisfied for $m = 14$. By Lemma 3.3, we can take $\beta = \frac{2}{7}$ such that $\pi^*(K_X)|_{\tilde{F}} - \beta\tilde{\sigma}^*(K_{\tilde{F}_0})$ is numerically equivalent to an effective $\mathbb{Q}$-divisor. So $\tilde{\xi} = \pi^*(K_X)|_{\tilde{F}} \cdot C \geq \beta = \frac{2}{7}$. Take $m = 12$. Then $\alpha = (m - 1 - m_0'/p - 1/\beta)\tilde{\xi} > 1$. Theorem 3.1 gives $\tilde{\xi} \geq \frac{1}{3}$. Take $m = 14$. Then $\alpha = (m - 1 - m_0'/p - 1/\beta)\tilde{\xi} \geq \frac{7}{3} > 2$. Theorem 3.1 says that φ_{14} is birational. $\qquad \square$

Proposition 4.5. *For case* (b), φ_{14} *is birational.*

Proof. Recall that we have two fibrations $f : X' \longrightarrow B$ and $f_5 : X' \longrightarrow W'_5 = \mathbb{P}^1 \times \mathbb{P}^1$. For a general canonical curve C in the a general fiber F of f, we can study $\dim \varphi_5(C)$. If $\dim \varphi_5(C) = 1$, then $\Phi_{|M_5|}$ maps C onto a curve. Clearly $5\pi^*(K_X) \cdot C \geq M_5 \cdot C \geq 2$. So $\xi \geq \frac{2}{5} > \frac{1}{3}$ and φ_{14} is birational according to the statement (**). From now on, we assume that $\dim \varphi_5(C) = 0$ for a general C. We may take further blowing ups to X' so that a general C is simply a generic fiber of f_5.

Because the only very ample divisor H'_5 on W'_5 with ${H'_5}^2 = 2$ is $L_1 + L_2 = q_1^*(\text{point}) + q_2^*(\text{point})$ where q_1, q_2 are projection maps from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^1$. Set $\tilde{f}_i := q_i \circ f_5 : X' \longrightarrow \mathbb{P}^1$, $i = 1, 2$. Then $\tilde{f}_1$ and $\tilde{f}_2$ are two fibrations onto $\mathbb{P}^1$. Let F_1 and F_2 are respectively general fibers of $\tilde{f}_1$ and $\tilde{f}_2$. Then $F_1 \cap F_2$ is simply a general fiber C of f_5. We will prove alternatively that φ_{14} is birational.

Consider the sub-liner system $|K_{X'} + \lceil 8\pi^*(K_X) \rceil + F_1 + F_2| \subset |14K_{X'}|$ which clearly separates different fibers of $\tilde{f}_1$. Take a general F_1 as a fiber of $\tilde{f}_1$. Because $8\pi^*(K_X) + F_2$ is nef and big, the Kawamata–Viehweg vanishing theorem gives a surjective map

$$H^0(K_{X'} + \lceil 8\pi^*(K_X) \rceil + F_2 + F_1) \longrightarrow H^0(F_1, K_{F_1} + \lceil 8\pi^*(K_X) \rceil|_{F_1} + C).$$

We hope to prove that $|K_{F_1} + \lceil 8\pi^*(K_X) \rceil|_{F_1} + C|$ gives a birational map for a general F_1. Note that $|C|$ is a rational pencil on the surface $|F_1|$. Clearly $|K_{F_1} + \lceil 8\pi^*(K_X) \rceil|_{F_1} + C|$ separate different C simply because $p_g(F_1) > 0$ and $\lceil 8\pi^*(K_X) \rceil|_{F_1} \geq 0$. Take a general curve C in the family $|F_2|_{F_1}|$. The vanishing theorem again gives the surjective map

$$H^0(F_1, K_{F_1} + \lceil 8\pi^*(K_X)|_{F_1} \rceil + C) \longrightarrow H^0(C, K_C + D)$$

where $D := (\lceil 8\pi^*(K_X)|_{F_1} \rceil)|_C$ is a divisor of

$$\deg(D) \geq (8\pi^*(K_X)|_{F_1}) \cdot C = 8\pi^*(K_X) \cdot C = 8\xi \geq \frac{8}{3} > 2$$

recalling the inequality (4.2) and that C is also a canonical curve in the surface F. So $\Phi_{|K_C + D|}$ is an embedding and the birationality principle says that φ_{14} is birational. We have completely showed Theorem 4.2. $\square$

Example 4.6. Example 1.2 shows that Theorem 4.2 is optimal for $m_0 = 3$. It is also optimal for $m_0 = 2$. In fact, Fletcher has an example

([10, No. 18, p. 151]): the canonical hypersurface $X_{22} \subset \mathbb{P}(1, 2, 3, 4, 11)$. It is clear $m_0 = 2$ and that φ_{11} is birational, but φ_{10} is not birational.

Recently we were informed of a new construction of a 3-fold Y by E. Stagnaro [27]. Y also has $p_g = 1$, $P_2 = 2$ and φ_{10} is not birational.

By a similar method, we are able to prove Corollary 1.3. To avoid unnecessary redundancy we omit some details.

4.3. A sketch proof of Corollary 1.3

First one has $P_2(X) \geq 4$ by a similar application of Reid's formula (see the proof of Lemma 4.1). So one may take $m_0 = 2$. Set $d_2 := \dim \varphi_2(X)$.

If $d_2 \geq 2$, it is clear that 3.1(i), (ii) are satisfied for $m \geq 8$. Now the situation $d_2 = 3$ follows directly from the argument in Case 1 of Theorem 4.2. In fact, one get the birationality of φ_7. In the situation $d_2 = 2$, one may also follow the argument in Case 2 of Theorem 4.2. The only difference is that we can take $p = 2$ here. So it can be verified that φ_8 is birational.

If $d_2 = 1$, one has an induced fibration $f : X' \longrightarrow B$. When $g(B) > 0$, the argument in Subcase 3.1 of Theorem 4.2 shows that φ_m is birational for all $m \geq 7$. We assume $g(B) = 0$ from now on. Let F be a general fiber of f. If $p_g(F) = 0$, then $q(X) = q(F) = 0$ and the assumption $\chi(\omega_X) > 0$ implies $p_g(X) \geq 2$. So φ_m is birational for all $m \geq 8$ by the main theorem in [8]. We are reduced to study the situation $p_g(F) > 0$. Now we can follow the argument of Subcase 3.2 in Theorem 4.2. The typical property here is the inclusion

$$\mathcal{O}(3) \hookrightarrow f_* \omega_{X'}^2.$$

We can still choose $G \leq 2\sigma^*(K_{F_0})$ on F. We can verify that Theorem 3.1 (i), (ii) are satisfied for all $m \geq 6$. Also we can choose much better p ($p \geq 2$) and β, respectively, than those in Subcase 3.2 in Theorem 4.2. We can verify with less difficulties, case by case (for type (I), (II), (III)) by applying Theorem 3.1, that φ_m is birational for all $m \geq 8$.

In a word, Corollary 1.3 is true.

4.4. Final remarks

Clearly, for the case $p_g = 0$, Theorem 3.1 will generate some new results which improves a corollary of J. Kollár [21] and the first author on the birationality of φ_m, where m is certain function in terms of m_0. We do not include our result here because we are not sure whether that would be optimal.

Acknowledgments

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INSTITUTE OF MATHEMATICS
SCHOOL OF MATHEMATICAL SCIENCES
FUDAN UNIVERSITY
SHANGHAI 200433
CHINA
E-mail address: mchen@fudan.edu.cn

FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
JOHANNES GUTENBERG UNIVERSITAET MAINZ
STAUDINGERWEG 9
D-55128 MAINZ
GERMANY
E-mail address: kzuo@mathematik.uni-mainz.de