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A formula relating entropy monotonicity to Harnack inequalities

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1. Introduction

In [1], Perelman considered the functional

$$\mathcal{W}(g, f, \tau) = \int_X \big(\tau(|\nabla f|^2 + R) + f - (n+1)\big)\, u\, dV$$

for $\tau > 0$ and smooth functions f on a closed $(n+1)$-dimensional Riemannian manifold (X, g) where

$$u = \frac{\mathrm{e}^{-f}}{(4\pi\tau)^{(n+1)/2}}$$

and defined an associated entropy by

$$\mu(g, \tau) = \inf \left\{ \mathcal{W}(g, f, \tau),\ \int_X u\, dV = 1 \right\}.$$

His ingenious realization was that when $\tau(t) > 0$ satisfies $\frac{\partial \tau}{\partial t} = -1$, $(X, g(t))$ evolves by the Ricci flow

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$$

and f satisfies the equation

$$\frac{\partial f}{\partial t} + \Delta f + R = |\nabla f|^2 + \frac{n+1}{2\tau}$$

which preserves the condition

$$\int_X u\, dV = 1$$

then

$$\frac{d}{dt} \mathcal{W}\left(g(t), f(t), \tau(t)\right) = 2\tau \int_X \left| R_{ij} + \nabla_i \nabla_j f - \frac{g_{ij}}{2\tau} \right|^2 u\, dV.$$

This implies, in particular, that

$$\frac{d}{dt}\mu(g(t),\tau(t)) \geq 0$$

with equality exactly for homothetically shrinking solutions of Ricci flow.

An important consequence of this entropy formula is a lower volume ratio bound for solutions of Ricci flow on a closed manifold for a finite time interval $[0,T)$, asserting the existence of a constant $\kappa > 0$, only depending on n, T and $g(0)$, such that the inequality

$$\frac{V_t(B_r^t(x_0))}{r^{n+1}} \geq \kappa$$

holds for all $t \in [0,T)$ and $r \in [0,\sqrt{T})$ for balls $B_r^t(x_0)$ (with respect to $g(t)$) in which the inequality $r^2|Rm| \leq 1$ for the Riemann tensor of $g(t)$ is satisfied.

This lower volume ratio bound rules out certain collapsed metrics as rescaling limits near singularities of Ricci flow such as products of Euclidean spaces with the so-called cigar soliton solution of Ricci flow given by $X = \mathbb{R}^2$ with the metric

$$ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}.$$

In this paper, we aim at adapting Perelman's entropy formula to the situation where a family of bounded open regions $(\Omega_t)_{t\in[0,T)}$ in $\mathbb{R}^{n+1}$ with smooth boundary hypersurfaces $M_t = \partial\Omega_t$ is evolving with smooth normal speed

$$\beta_{M_t} = -\frac{\partial x}{\partial t} \cdot \nu.$$

Here, x denotes the embedding map of M_t and ν is the normal pointing out of Ω_t.

For open subsets $\Omega \subset \mathbb{R}^{n+1}$, smooth functions $f : \bar{\Omega} \to \mathbb{R}$ and $\beta : \partial\Omega \to \mathbb{R}$ and $\tau > 0$, we consider the quantity

$$\mathcal{W}_\beta(\Omega, f, \tau) = \int_\Omega \big(\tau|\nabla f|^2 + f - (n+1)\big)\, u\, dx + 2\tau \int_{\partial\Omega} \beta u\, dS$$

with

$$u = \frac{\mathrm{e}^{-f}}{(4\pi\tau)^{(n+1)/2}}$$

and the associated entropy

$$\mu_\beta(\Omega, \tau) = \inf\left\{\mathcal{W}_\beta(\Omega, f, \tau), \int_\Omega u\,dx = 1\right\}.$$

We then derive a formula which states that if (Ω_t) evolves as above, the condition $\tau(t) > 0$ satisfies $\frac{\partial \tau}{\partial t} = -1$, f satisfies the evolution equation

$$\frac{\partial f}{\partial t} + \Delta f = |\nabla f|^2 + \frac{n+1}{2\tau}$$

in Ω_t with Neumann boundary condition

$$\nabla f \cdot \nu = \beta$$

on $M_t = \partial \Omega_t$ and if we introduce a family of diffeomorphisms $\varphi_t : \bar{\Omega} \to \bar{\Omega}_t$ with $x = \varphi_t(q)$, $q \in \bar{\Omega}$ obeying

$$\frac{\partial x}{\partial t} = -\nabla f(x, t)$$

then

$$\frac{d}{dt}\mathcal{W}_\beta(\Omega_t, f(t), \tau(t)) = 2\tau \int_{\Omega_t} \left|\nabla_i \nabla_j f - \frac{\delta_{ij}}{2\tau}\right|^2 u\,dx - \int_{M_t} \nabla W \cdot \nu\,dS$$

where $W = \tau(2\Delta f - |\nabla f|^2) + f - (n+1)$.

For evolving bounded regions Ω_t inside a fixed Riemannian manifold (X, g) or inside a Ricci flow solution, one can derive analoguous versions of this formula.

The main observation in this paper is that this can be converted to

$$\begin{aligned}
&\frac{d}{dt}\mathcal{W}_\beta(\Omega_t, f(t), \tau(t)) \\
&\quad = 2\tau \int_{\Omega_t} \left|\nabla_i \nabla_j f - \frac{\delta_{ij}}{2\tau}\right|^2 u\,dx \\
&\qquad + 2\tau \int_{M_t} \left(\frac{\partial \beta}{\partial t} - 2\,\nabla^M \beta \cdot \nabla^M f + A(\nabla^M f, \nabla^M f) - \frac{\beta}{2\tau}\right) u\,dS
\end{aligned}$$

where A denotes the second fundamental form of M_t.

For functions β for which the hypersurface integral is non-negative, the inequality

$$\frac{d}{dt}\mathcal{W}_\beta(\Omega_t, f(t), \tau(t)) \geq 2\tau \int_{\Omega_t} \left|\nabla_i \nabla_j f - \frac{\delta_{ij}}{2\tau}\right|^2 u\, dx$$

results. When $\beta = 0$, i.e. for a fixed bounded region Ω with smooth boundary inside a fixed manifold of non-negative Ricci curvature, Ni [2] has previously obtained this inequality under the additional assumption of convexity of Ω.

This inequality implies, as in Perelman's situation,

$$\frac{d}{dt}\mu_\beta(\Omega_t, \tau(t)) \geq 0$$

and therefore also the following localized lower volume ratio bound:

There is a constant $\kappa > 0$ depending only on $n, \Omega_0, T, \sup_{M_0}|\beta|$ and c_1 such that

$$\frac{V(\Omega_t \cap B_r(x_0))}{r^{n+1}} \geq \kappa$$

holds for all $t \in [0, T)$ and $r \in (0, \sqrt{T}]$ in balls $B_r(x_0) \subset \mathbb{R}^{n+1}$ satisfying the conditions $V(\Omega_t \cap B_{r/2}(x_0)) > 0$ and

$$\frac{V(\Omega_t \cap B_r(x_0)) + r^2 \int_{M_t \cap B_r(x_0)} |\beta|\, dS}{V(\Omega_t \cap B_{r/2}(x_0))} \leq c_1.$$

Since this statement is scaling invariant for suitably homogeneous β, it is also valid on any smooth limit of suitably rescaled solutions of the flow consisting of smooth, compact embedded hypersurfaces, but now for all radii $r > 0$ as long as the other conditions still hold for the balls $B_r(x_0)$ which we consider.

In the important case of mean curvature flow, that is where β_{M_t} is the mean curvature H_{M_t} of the hypersurfaces M_t, the expression

$$Z(\nabla^M f) \equiv \frac{\partial H}{\partial t} - 2\,\nabla^M H \cdot \nabla^M f + A(\nabla^M f, \nabla^M f)$$

is the central quantity in Hamilton's Harnack inequality for convex solutions of the mean curvature flow [3]. Note that the right-hand side of the above identity vanishes on homothetically shrinking solutions and for $f = |x|^2/4\tau$. This motivates the following conjecture:

Conjecture. *In the case of mean curvature flow in* $\mathbb{R}^{n+1}$ *for compact embedded hypersurfaces* M_t*, the inequality*

$$\int_{\Omega_t} \left|\nabla_i \nabla_j f - \frac{\delta_{ij}}{2\tau}\right|^2 u\,dx + \int_{M_t} \left(Z(\nabla^M f) - \frac{H}{2\tau}\right) u\,dS \geq 0$$

holds for $f = f(t)$ *as above and for* $\tau(t) > 0$ *where* $\frac{\partial \tau}{\partial t} = -1$.

As we shall see, the validity of this conjecture in particular implies the above lower volume ratio bound in the case of mean curvature flow. A direct calculation shows that regions bounded by certain eternal solutions of mean curvature flow, such as the product of $\mathbb{R}^{n-1}$ with the grim reaper curve given by $y = -\log\cos x + t$, do not satisfy the lower volume bound statement for large r and, hence, should the conjecture hold, cannot occur as rescaling limits in this situation. Similarly, certain stationary (zero mean curvature) hypersurfaces would then be ruled out as rescaling limits such as for instance the catenoid minimal surface in $\mathbb{R}^3$ and two parallel hyperplanes. In the positive mean curvature case, White [4] has previously shown that certain solutions of mean curvature flow, in particular the grim reaper hypersurface, cannot occur as rescaling limits.

The embeddedness assumption for the hypersurfaces M_t is essential. Angenent [5] has shown that solutions of the curve-shortening flow with self-intersections have the grim reaper curve as rescaling limit near singularities.

This paper is organized as follows. In Section 2, we define entropies for open subsets Ω of complete (possibly non-compact) Riemannian manifolds with respect to a given smooth function β defined on $\partial\Omega$ and establish some of their properties.

In Section 3, we derive the entropy formula involving the Harnack expression for evolving domains in $\mathbb{R}^{n+1}$. All of the calculations go through with necessary modifications such as adding Ricci and scalar curvature terms in the appropriate places in the case of a fixed ambient manifold or a background Ricci flow solution. However, at the moment we do not see how they might lead to interesting consequences.

In Section 4, we state our conjecture and show several consequences it would lead to, such as a lower local volume ratio bound and non-existence of certain degenerate rescaling limits.

In Appendix A, we give some explicit examples of entropy functionals and their values in $\mathbb{R}^{n+1}$.

In the paper, a version of the logarithmic Sobolev inequality on bounded open sets Ω in complete Riemannian manifolds is used. In Appendix B, we provide a proof based on the standard Sobolev inequality, essentially following Gross [6].

In Appendix C, we give a derivation of a Harnack-type evolution equation associated with solutions of a backward heat equation. This equation is one of the central results in [1] and also one of the main ingredients in the proof of our entropy formula. Details of this calculation first appeared in [7] and [2].

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2. Entropy-type functionals for domains in Riemannian manifolds

For open subsets Ω of an $(n+1)$-dimensional complete (possibly non-compact) Riemannian manifold (X,g), functions $f:\bar{\Omega}\to\mathbb{R}$ and $\beta:\partial\Omega\to\mathbb{R}$ and $\tau>0$ we consider the quantity

$$\mathcal{W}_\beta(\Omega,g,f,\tau)=\int_\Omega \big(\tau(|\nabla f|^2+R)+f-(n+1)\big)\,u\,dV+2\tau\int_{\partial\Omega}\beta u\,dS$$

where

$$u=\frac{\mathrm{e}^{-f}}{(4\pi\tau)^{(n+1)/2}}.$$

The scalar curvature R, the expression $|\nabla f|^2$ and the volume and area elements dV and dS are taken with respect to the metric g. We then define an associated entropy by

$$\mu_\beta(\Omega,g,\tau)=\inf\left\{\mathcal{W}_\beta(\Omega,g,f,\tau)\,,\int_\Omega u\,dV=1\right\}.$$

For $\beta=0$ and $\Omega=X$, $\mathcal{W}_\beta(\Omega,g,f,\tau)$ and $\mu_\beta(\Omega,g,\tau)$ reduce to Perelman's functional $\mathcal{W}(g,f,\tau)$ and his entropy quantity $\mu(g,\tau)$. We therefore write $\mathcal{W}$ for $\mathcal{W}_0$ and μ for μ_0. We use $n+1$ instead of n as we will later be interested mainly in the hypersurface $\partial\Omega$ which we prefer to be n-dimensional.

When we do not intend to vary the ambient metric, we consider

$$\mathcal{W}_\beta(\Omega, f, \tau) = \int_\Omega \left(\tau|\nabla f|^2 + f - (n+1)\right) u\, dV + 2\tau \int_{\partial\Omega} \beta u\, dS$$

with infimum $\mu_\beta(\Omega, \tau)$.

We shall only consider sets with smooth boundaries and smooth functions f and β although the above expressions also make sense for more general sets and functions. In case Ω is unbounded, we require suitable integrability conditions on f and β. The function β could be the restriction to $\partial\Omega$ of a function on X or be defined only on $\partial\Omega$. An important example of the latter is $\beta = H$, where H is the mean curvature of $\partial\Omega$ with respect to the outer unit normal.

In this section, we derive several basic properties for these entropies. Some specific examples including the calculations of entropy values for some natural choices of sets in $\mathbb{R}^{n+1}$ are discussed in Appendix A.

Proposition 2.1. *Suppose that Ω is bounded with smooth boundary and that β is smooth. Then for any $\tau > 0$ we have*

$$\mu_\beta(\Omega, g, \tau) \geq -c(n, \Omega, g)\left(1 + \log(1+\tau) + \tau \sup_{\partial\Omega} |\beta|(1 + \sup_{\partial\Omega} |\beta|)\right).$$

The same lower bound holds for $\mu_\beta(\Omega, \tau)$.

Remark 2.2. The lower bound for $\mu_\beta(\Omega, g, \tau)$ and for $\mu_\beta(\Omega, \tau)$ follows from the logarithmic Sobolev inquality for Ω which in turn can be derived from the standard Sobolev inequality (see Appendix B). The constant $c(n, \Omega, g)$ thus depends on the constant in the Sobolev inequality and the $L^1(\partial\Omega)$-trace inequality for $C^1(\bar{\Omega})$-functions, the latter controlling the boundary integral. The ambient metric enters via bounds for the Riemann curvature tensor on $\bar{\Omega}$ and the explicit bound for the $\sup_\Omega |R|$-term arising from the functional. Proposition 2.1 holds for more general sets such as bounded sets of finite perimeter and for bounded β.

Proof of Proposition 2.1. We give the proof only for $\mu_\beta(\Omega, g, \tau)$. For $\mu_\beta(\Omega, \tau)$ simply set the scalar curvature term to zero. We essentially modify the arguments in [7] and [2].

Setting $u=\varphi^2$ and using the condition $\int_\Omega u\,dV=1$, we obtain

$$\begin{aligned}\mathcal{W}_\beta(\Omega,g,f,\tau) &= \int_\Omega \left(\tau(4|\nabla\varphi|^2+R\varphi^2)-\varphi^2\log\varphi^2\right)\,dV \\ &\quad + 2\tau\int_{\partial\Omega}\beta\varphi^2\,dS - c(n)(1+\log\tau)\end{aligned} \tag{2.1}$$

with $\int_\Omega \varphi^2\,dV=1$. The trace inequality

$$\int_{\partial\Omega}\varphi^2\,dS \le c_2\int_\Omega \left(|\nabla\varphi^2|+\varphi^2\right)dV$$

with $c_2=c_2(\Omega,g)$ in conjunction with Young's inequality yields

$$\left|2\tau\int_{\partial\Omega}\beta\varphi^2\,dS\right| \le \int_\Omega 2\tau|\nabla\varphi|^2\,dV + c_3\,\tau\,\sup_{\partial\Omega}|\beta|\,(1+\sup_{\partial\Omega}|\beta|)$$

where c_3 depends on c_2. Here, we have used again the condition $\int_\Omega\varphi^2\,dV=1$. Combining this with (2.1) yields

$$\begin{aligned}\mathcal{W}_\beta(\Omega,g,f,\tau) \ge& \int_\Omega \left(2\tau|\nabla\varphi|^2-\varphi^2\log\varphi^2\right)\,dV \\ & -c_4\left(1+\log\tau+\tau\left(\sup_\Omega|R|+\sup_{\partial\Omega}|\beta|(1+\sup_{\partial\Omega}|\beta|)\right)\right)\end{aligned} \tag{2.2}$$

where c_4 depends on the previous constants. Scaling the metric gives

$$\begin{aligned}\int_\Omega\!\left(2\tau|\nabla\varphi|^2-\varphi^2\log\varphi^2\right)dV &= \int_\Omega\!\left(|\nabla\varphi_\tau|_\tau^2-\varphi_\tau^2\log\varphi_\tau^2\right)dV_\tau \\ &\quad - c(n)(1+\log\tau)\end{aligned} \tag{2.3}$$

and

$$\int_\Omega \varphi_\tau^2\,dV_\tau = 1$$

where $\varphi_\tau=(2\tau)^{(n+1)/4}\varphi$ and dV_τ and $|\nabla\varphi_\tau|_\tau^2$ are taken with respect to $g_\tau=(2\tau)^{-1}g$.

By scaling the standard Sobolev inequality

$$\left(\int_\Omega |\psi|^{(n+1)/n}\, dV\right)^{n/(n+1)} \leq c_S(\Omega, g) \int_\Omega \left(|\nabla\psi| + |\psi|\right)\, dV$$

we see that the Sobolev constant $c_S(\Omega, g_\tau)$ can be estimated by $c_S(\Omega, g)$ $(1+\sqrt{\tau})$. Therefore, by the logarithmic Sobolev inequality applied in Ω with respect to the metric g_τ (see Appendix B)

$$\int_\Omega \left(|\nabla\varphi_\tau|_\tau^2 - \varphi_\tau^2 \log\varphi_\tau^2\right)\, dV_\tau \geq -c(n)\left(1 + \log c_S(\Omega, g) + \log(1+\tau)\right).$$

Combining this inequality with (2.2) and (2.3), we arrive at

$$\mathcal{W}_\beta(\Omega, g, f, \tau) \geq -c_5 \left(1 + \log(1+\tau) + \tau \sup_{\partial\Omega} |\beta|(1 + \sup_{\partial\Omega}|\beta|)\right)$$

with $c_5 = c_5(n, \Omega, g)$ and for f satisfying $\int_\Omega u\, dV = 1$. This gives the desired lower bound for $\mu_\beta(\Omega, g, \tau)$. □

Proposition 2.3. *Let Ω be bounded with smooth boundary and assume β to be smooth. Then for every $\tau > 0$ there exists a unique smooth minimizer for $\mu_\beta(\Omega, g, \tau)$ and $\mu_\beta(\Omega, \tau)$. The minimizer depends smoothly on Ω, g, β and τ.*

Proof. We consider only $\mu_\beta(\Omega, g, \tau)$ again. The argument is analogous as in [8]. The necessary semicontinuity and coercivity in $W^{1,2}(\Omega)$ for the transformed functional

$$\begin{aligned}\mathcal{E}(\varphi) = &\int_\Omega \left(\tau(4|\nabla\varphi|^2 + R\varphi^2) - \varphi^2 \log\varphi^2\right)\, dV \\ &+ 2\tau \int_{\partial\Omega} \beta\varphi^2\, dS - c(n)(1+\log\tau)\end{aligned}$$

for $u = \varphi^2$ subject to the condition $\int_\Omega \varphi^2\, dV = 1$ follow from similar arguments as in the proof of the lower bound for $\mu_\beta(\Omega, g, \tau)$ given above. The uniqueness and smooth dependence on the data is standard. □

The quantity

$$W = W(f) = \tau(2\Delta f - |\nabla f|^2 + R) + f - (n+1)$$

is featured in Ch. 9 of [1] and in [2]. It arises naturally in the Euler–Lagrange equation for the functional $\mathcal{W}_\beta(\Omega, g, f, \tau)$.

Proposition 2.4. *The minimizer f_{min} for the functional $\mathcal{W}_\beta(\Omega, g, f, \tau)$ subject to the constraint $\int_\Omega u\, dV = 1$ satisfies the Euler–Lagrange equation*

$$W(f_{min}) = \mu_\beta(\Omega, g, \tau)$$

in Ω and the natural boundary condition

$$\langle \nabla f_{min}, \nu \rangle = \beta$$

on $\partial\Omega$. Here $\langle \cdot, \cdot \rangle$ refers to the metric g. For the minimizer of $\mathcal{W}_\beta(\Omega, f, \tau)$, we have instead

$$W(f_{min}) = \mu_\beta(\Omega, \tau)$$

where

$$W(f) = \tau(2\Delta f - |\nabla f|^2) + f - (n+1).$$

Proof. Standard computation using Lagrange multipliers. □

Remark 2.5. The Euler–Lagrange equation for the transformed functional

$$\begin{aligned}\mathcal{E}(\varphi) = &\int_\Omega \big(\tau(4|\nabla\varphi|^2 + R\varphi^2) - \varphi^2 \log \varphi^2\big)\, dV \\ &+ 2\tau \int_{\partial\Omega} \beta\varphi^2\, dS - c(n)(1 + \log \tau)\end{aligned}$$

for $\varphi^2 = u$ subject to the condition $\int_\Omega \varphi^2\, dV = 1$ is

$$-4\tau\, \Delta\varphi - 2\varphi \log \varphi + \tau R\varphi = \mu(\Omega, g, \tau) + (n+1)\left(1 + \frac{1}{2}\log(4\pi\tau)\right)\varphi$$

in Ω with boundary condition $2\langle \nabla\varphi, \nu\rangle = -\beta\varphi$ on $\partial\Omega$.

Proposition 2.6. *For any smooth enough function $f : \bar{\Omega} \to \mathbb{R}$ satisfying*

$$\langle \nabla f, \nu \rangle = \beta$$

on $\partial\Omega$ with respect to the outer unit normal ν we have

$$\mathcal{W}_\beta(\Omega, g, f, \tau) = \int_\Omega W u\, dV$$

with $W = W(f) = \tau(2\Delta f - |\nabla f|^2 + R) + f - (n+1)$ and

$$\mathcal{W}_\beta(\Omega, f, \tau) = \int_\Omega W u\, dV$$

for $W = W(f) = \tau(2\Delta f - |\nabla f|^2) + f - (n+1)$.

Proof. The boundary condition implies $\langle \nabla u, \nu\rangle = -\beta u$ on $\partial\Omega$ and hence

$$\mathcal{W}_\beta(\Omega, g, f, \tau) = \int_\Omega \big(\tau(|\nabla f|^2 + R) + f - (n+1)\big)\, u\, dV - 2\tau \int_\Omega \Delta u\, dV$$

by the divergence theorem. Since

$$\Delta u = u(|\nabla f|^2 - \Delta f).$$

the claim follows. □

For the next statement we do not require Ω to be bounded.

Proposition 2.7. *Suppose that* $\mu_\beta(\Omega, g, r^2) \geq -c_0$ *or* $\mu_\beta(\Omega, r^2) \geq -c_0$. *Let* $B_r(x_0) \subset (X, g)$ *satisfy* $V(\Omega \cap B_{r/2}(x_0)) > 0$,

$$\frac{V(\Omega \cap B_r(x_0)) + r^2 \int_{\partial\Omega \cap B_r(x_0)} |\beta|\, dS}{V(\Omega \cap B_{r/2}(x_0))} \leq c_1$$

and $r^2|Rm| \leq c_2$ *in* $\Omega \cap B_r(x_0)$ *for the Riemann tensor of* g. *Then*

$$\frac{V(\Omega \cap B_r(x_0))}{r^{n+1}} \geq \kappa > 0$$

with $\kappa = \kappa(n, c_0, c_1, c_2)$.

Remark 2.8. We will actually prove that $\mu_\beta(\Omega, g, r^2)$ and $\mu_\beta(\Omega, r^2)$ are bounded above by the expression

$$\log \frac{V(\Omega \cap B_r(x_0))}{r^{n+1}} + c\, \frac{V(\Omega \cap B_r(x_0)) + r^2 \int_{\partial\Omega \cap B_r(x_0)} |\beta|\, dS}{V(\Omega \cap B_{r/2}(x_0))}$$

with $c = c(n, r^2|Rm|)$. From this, the claim follows immediately.

Proof of Proposition 2.7. In the case $\Omega = X$ and $\beta = 0$, the proof is sketched in Ch. 3 of [1] (see [7] and [2] for more details). We proceed along similar lines.

If we set $e^{-f} = a\zeta$, the normalization condition for f becomes

$$a = \frac{(4\pi r^2)^{(n+1)/2}}{\int_\Omega \zeta \, dV}.$$

The functional $\mathcal{W}_\beta(\Omega, f, r^2)$ can then be expressed as

$$\frac{a}{(4\pi r^2)^{(n+1)/2}} \int_\Omega \left(4r^2 \frac{|\nabla \zeta|^2}{\zeta} + r^2 R\, \zeta - \zeta \log(a\zeta) \right) dV \\ - (n+1) + 2r^2 \frac{\int_{\partial\Omega} \beta\zeta \, dS}{\int_\Omega \zeta \, dV}.$$

By approximation, we may substitute functions $\zeta \in C_0^2(X)$ into this expression. We choose ζ as a cut-off function for $B_{r/2}(x_0)$, that is ζ satisfies $\chi_{B_{r/2}(x_0)} \leq \zeta \leq \chi_{B_r(x_0)}$ as well as

$$4r^2 \frac{|\nabla\zeta|^2}{\zeta} \leq 8r^2 \sup |\nabla^2 \zeta| \leq c$$

where c is a constant which depends on $r^2 \sup_{\Omega \cap B_r(x_0)} |Rm|$ and is therefore bounded by c_2. Since

$$\int_\Omega \zeta \, dV \geq V(\Omega \cap B_{r/2}(x_0)) > 0$$

we can thus estimate

$$\frac{1}{(4\pi r^2)^{(n+1)/2}} \int_\Omega 4r^2 a \frac{|\nabla\zeta|^2}{\zeta} \, dV \leq c \frac{V(\Omega \cap \operatorname{spt} \zeta)}{\int_\Omega \zeta \, dV} \leq c \frac{V(\Omega \cap B_r(x_0))}{V(\Omega \cap B_{r/2}(x_0))}.$$

Jensen's inequality now implies

$$- \frac{1}{(4\pi r^2)^{(n+1)/2}} \int_\Omega a\zeta \log(a\zeta) \, dV \\ \leq - \frac{1}{(4\pi r^2)^{(n+1)/2}} \int_\Omega a\zeta \, dV \log \left(\frac{1}{V(\Omega \cap \operatorname{spt}\zeta)} \int_\Omega a\zeta \, dV \right).$$

Since $\operatorname{spt} \zeta = \overline{B_r(x_0)}$ and in view of the normalization condition the right-hand side equals

$$\log \left(\frac{V(\Omega \cap B_r(x_0))}{(4\pi r^2)^{(n+1)/2}} \right).$$

The scalar curvature integral is estimated using the boundedness assumption on the Riemann tensor in $\Omega \cap B_r(x_0)$. This yields the upper bound for $\mu(\Omega, g, r^2)$ and $\mu(\Omega, r^2)$ stated in Remark 2.8. $\square$

Remark 2.9. In [1], Perelman ruled out the occurrence of collapsed metrics as rescaling limits of compact, finite time solutions of Ricci flow. A metric g on X is called *collapsed* if there exists a sequence of balls $B_{r_k}(x_k) \subset (X, g)$ satisfying $r_k^2|Rm| \leq 1$ in $B_{r_k}(x_k)$ for which

$$\frac{V(B_{r_k}(x_k))}{r_k^{n+1}} \longrightarrow 0.$$

An important example of a collapsed metric is the so-called *cigar soliton* solution of the Ricci flow given by $X = \mathbb{R}^2$ endowed with the metric

$$ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}.$$

On collapsed metrics we have $\inf_{\tau>0} \mu_\beta(g, \tau) = -\infty$ by the proposition.

The following reformulation of Proposition 2.7 links a kind of volume collapsing behaviour of subsets of (X, g) to a property of the entropy $\mu_\beta(\Omega, g, \tau)$.

Corollary 2.10. *If for some fixed constants c_1 and c_2 we can find a sequence of balls $B_{r_k}(x_k) \subset (X, g)$ such that $V(\Omega \cap B_{r_k/2}(x_k)) > 0$,*

$$\frac{V(\Omega \cap B_{r_k}(x_k)) + r_k^2 \int_{\partial\Omega \cap B_{r_k}(x_k)} |\beta| \, dS}{V(\Omega \cap B_{r_k/2}(x_k))} \leq c_1,$$

$r_k^2|Rm| \leq c_2$ in $\Omega \cap B_{r_k}(x_k)$ and

$$\frac{V(\Omega \cap B_{r_k}(x_k))}{r_k^{n+1}} \longrightarrow 0$$

then $\inf_{\tau>0} \mu_\beta(\Omega, g, \tau) = -\infty$ *and* $\inf_{\tau>0} \mu_\beta(\Omega, \tau) = -\infty$.

For compact Ω, we can of course always find such a sequence of balls with radii tending to infinity. In the case of non-compact regions, the sitation is more interesting. Examples are the following regions in $X = \mathbb{R}^{n+1}$:

(1) The slab

$$\Omega = \{x \in \mathbb{R}^{n+1},\ -d < x_{n+1} < d\}$$

for some $d > 0$. On the hypersurface $M = \partial\Omega$ we have $H = 0$. The enclosed region Ω satisfies $V(\Omega \cap B_{r/2}) > 0$ and

$$\frac{V(\Omega \cap B_r)}{V(\Omega \cap B_{r/2})} \le c(n, d)$$

for all balls $B_r = B_r(0)$. Moreover,

$$\lim_{r\to\infty} \frac{V(\Omega \cap B_r)}{r^{n+1}} = 0.$$

(2) The "smaller" of the two regions bounded by the catenoid minimal surface $M = \partial\Omega$ in $\mathbb{R}^3$ given by

$$\Omega = \{x = (\hat{x}, x_3) \in \mathbb{R}^3,\ |\hat{x}| > 1,\ |x_3| < \cosh^{-1}|\hat{x}|\}.$$

Note that $H = 0$ on $\partial\Omega$. One checks that there is a constant c_1 such that for all $r \ge 2$

$$\frac{V(\Omega \cap B_r)}{V(\Omega \cap B_{r/2})} \le c_1$$

and

$$V(\Omega \cap B_r) \le c_1 r^2 \log(1 + r)$$

so that

$$\lim_{r\to\infty} \frac{V(\Omega \cap B_r)}{r^3} = 0.$$

(3) The translating solution of mean curvature flow corresponding to the grim reaper hypersurface $M = \partial\Omega$ where $\Omega = \mathbb{R}^{n-1} \times G$ with

$$G = \left\{(x_n, x_{n+1}) \in \mathbb{R}^2,\ -\pi/2 < x_n < \pi/2,\ x_{n+1} > -\log\cos x_n\right\}.$$

An explicit calculation shows that the mean curvature satisfies $H(x) = \mathrm{e}^{-x_{n+1}}$ for any $x \in M = \partial\Omega$. Therefore, one checks directly that there is a sequence of balls $B_{r_k}(x_k)$ with $r_k \to \infty$ satisfying

$V(\Omega \cap B_{r_k/2}(x_k)) > 0$,

$$\frac{V(\Omega \cap B_{r_k}(x_k))}{V(\Omega \cap B_{r_k/2}(x_k))} \leq c(n),$$
$$\frac{r_k^2 \int_{\partial\Omega \cap B_{r_k}(x_k)} H\,dS}{V(\Omega \cap B_{r_k/2}(x_k))} \leq 1$$

and

$$\frac{V(\Omega \cap B_{r_k}(x_k))}{r_k^{n+1}} \to 0.$$

3. An entropy-type formula for evolving domains in $\mathbb{R}^{n+1}$

In this section, we restrict ourselves to domain evolution in $\mathbb{R}^{n+1}$. All the calculations go through for fixed Riemannian manifolds or Ricci flow solutions as ambient space if we add Ricci and scalar curvature terms in the appropriate places. However, in this case, the formulas do not immediately seem to lead to any interesting consequences.

We evolve bounded open subsets $(\Omega_t)_{t\in[0,T)}$ with smooth boundary hypersurfaces $(M_t)_{t\in[0,T)}$ in $\mathbb{R}^{n+1}$. More precisely, $\bar{\Omega}_t = \phi_t(\bar{\Omega})$ with $M_t = \partial\Omega_t = \phi_t(\partial\Omega)$ where $\phi_t = \phi(\cdot, t) : \bar{\Omega} \to \mathbb{R}^{n+1}, t \in [0, T)$ is a smooth one-parameter family of diffeomorphisms. We will often abbreviate

$$x = \phi(p, t)$$

for $p \in \bar{\Omega}$. The normal speed of M_t with respect to the inward pointing normal $-\nu$ is defined by

$$\beta = \beta_{M_t} = -\frac{\partial x}{\partial t} \cdot \nu$$

for $x \in M_t$ or expressed in terms of the embedding map $\phi(\cdot, t)$ by

$$\beta(p, t) = -\frac{\partial \phi}{\partial t}(p, t) \cdot \nu(\phi(p, t))$$

for $p \in \partial\Omega$. We assume the function β to be smooth. If, for instance, $\beta = H$, the mean curvature of M_t, this describes mean curvature flow up to diffeomorphisms tangential to M_t.

Let us assume more specifically that the family of subsets $(\Omega_t)_{t\in(0,T)}$ evolves by the equation

$$\frac{\partial x}{\partial t} = -\nabla f(x,t) \tag{3.1}$$

for $x \in \Omega_t$. This flow is compatible with the evolution of the boundaries $M_t = \partial\Omega_t$ with normal speed β if f satisfies the condition $\nabla f \cdot \nu = \beta$ on M_t. Suppose $f(t)$ satisfies the equation

$$\left(\frac{\partial}{\partial t} + \Delta\right) f = |\nabla f|^2 + \frac{n+1}{2\tau} \tag{3.2}$$

in Ω_t for $t \in (0,T)$. The total time derivative of f is given by

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \nabla f \cdot \frac{\partial x}{\partial t} = \frac{\partial f}{\partial t} - |\nabla f|^2. \tag{3.3}$$

Hence, (3.2) can also be written as

$$\left(\frac{d}{dt} + \Delta\right) f = \frac{n+1}{2\tau}. \tag{3.4}$$

If $\tau(t) > 0$ evolves by $\frac{\partial\tau}{\partial t} = -1$, then (3.2) is equivalent to the equation

$$\left(\frac{\partial}{\partial t} + \Delta\right) u = 0 \tag{3.5}$$

for

$$u = \frac{\mathrm{e}^{-f}}{(4\pi\tau)^{(n+1)/2}}.$$

The above equations are more precisely expressed in terms of the pull back of the function f via the diffeomorphisms evolving Ω_t. In fact, if we set $x = \phi(q,t)$ where $\phi_t = \phi(\cdot,t) : \Omega \to \Omega_t$, the pulled back function given by

$$\tilde{f}(q,t) = f(\phi(q,t),t)$$

satisfies

$$\frac{df}{dt}(x,t) = \frac{\partial \tilde{f}}{\partial t}(q,t).$$

Analogously to Ch. 9 in [1] (see also [2]) the function $W = \tau(2\Delta f - |\nabla f|^2) + f - (n+1)$ satisfies a nice evolution equation:

Proposition 3.1. *Let $(\Omega_t)_{t\in(0,T)}$ be a family of subsets evolving by (3.1) that is according to the negative gradient of functions $f(t)$ satisfying equation (3.2). Suppose also that $\tau(t) > 0$ evolves by $\frac{\partial\tau}{\partial t} = -1$ for $t \in (0,T)$. Then, the function*

$$W = \tau(2\Delta f - |\nabla f|^2) + f - (n+1)$$

satisfies the evolution equation

$$\left(\frac{d}{dt} + \Delta\right) W = 2\tau \left|\nabla_i\nabla_j f - \frac{\delta_{ij}}{2\tau}\right|^2 + \nabla W \cdot \nabla f$$

in Ω_t.

Proof. We use Perelman's identity

$$\left(\frac{\partial}{\partial t} + \Delta\right) W = 2\tau \left|\nabla_i\nabla_j f - \frac{\delta_{ij}}{2\tau}\right|^2 + 2\,\nabla W \cdot \nabla f$$

from Ch. 9 in [1]. A derivation of this can be found in [7] and in [2]. In our evolving coordinates $x = \phi(q,t)$ we change to total time derivatives for W via

$$\frac{dW}{dt} = \frac{\partial W}{\partial t} - \nabla W \cdot \nabla f,$$

which yields the result. For the convenience of the reader, we repeat the details of the calculation discussed in [2] for the expression $\left(\frac{d}{dt} + \Delta\right) W$ on evolving sets $\Omega_t \subset \mathbb{R}^{n+1}$ in Appendix C. □

Proposition 3.2. *Suppose the conditions of the previous proposition hold. Then*

$$\frac{d}{dt}\int_{\Omega_t} u\,dx = 0$$

for all $t \in (0,T)$. If f satisfies additionally $\nabla f \cdot \nu = \beta$ on $M_t = \partial\Omega_t$ then

$$\begin{aligned} \frac{d}{dt}\mathcal{W}_\beta(\Omega_t, f(t), \tau(t)) &= 2\tau \int_{\Omega_t} \left|\nabla_i\nabla_j f - \frac{\delta_{ij}}{2\tau}\right|^2 u\,dx \\ &\quad - \int_{M_t} \nabla W \cdot \nu\, u\,dS \end{aligned} \tag{3.6}$$

where $W = \tau(2\Delta f - |\nabla f|^2) + f - (n+1)$.

Proof. In view of the family of diffeomorphisms generated by

$$\frac{\partial x}{\partial t} = -\nabla f = \frac{1}{u}\nabla u,$$

the volume element dx on the evolving sets Ω_t changes by

$$\frac{d}{dt}\,dx = -\Delta f\,dx.$$

Since also

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{|\nabla u|^2}{u}$$

and

$$\Delta u = (|\nabla f|^2 - \Delta f)\,u,$$

we obtain in Ω_t

$$\frac{d}{dt}\,(u\,dx) = \left(\frac{\partial u}{\partial t} + \Delta u\right) dx = 0 \tag{3.7}$$

by Equation (3.5). Thus

$$\frac{d}{dt}\int_{\Omega_t} u\,dx = 0.$$

Combining the Neumann boundary condition, Proposition 2.6, identity (3.7) and the evolution equation for W in Proposition 3.1 we then calculate

$$\begin{aligned}
&\frac{d}{dt}\mathcal{W}_\beta(\Omega_t, f(t), \tau(t))\\
&\quad= \frac{d}{dt}\int_{\Omega_t} W u\,dx = \int_{\Omega_t}\left(\frac{d}{dt} + \Delta\right) W\,u\,dx - \int_{\Omega_t} \Delta W\,u\,dx\\
&\quad= \int_{\Omega_t} 2\tau\left|\nabla_i\nabla_j f - \frac{\delta_{ij}}{2\tau}\right|^2 u\,dx - \int_{\Omega_t} (\nabla W\cdot\nabla u + \Delta W\,u)\;dx
\end{aligned}$$

where we again used $\nabla u = -u\nabla f$. The last integral equals

$$-\int_{\Omega_t} \operatorname{div}(\nabla W u)\;dx.$$

The result then follows by applying the divergence theorem. □

Remark 3.3. For a fixed domain Ω (i.e. when $\beta = 0$) inside a Riemannian manifold of non-negative Ricci curvature, the inequality

$$\frac{d}{dt}\mathcal{W}(\Omega, f(t), \tau(t)) \geq -\int_{M_t} \nabla W \cdot \nu \, u \, dS$$

for a solution f of the above backward heat equation discussed in [2]. Ni then shows that

$$-\langle \nabla W, \nu \rangle = 2\tau A(\nabla^M f, \nabla^M f)$$

and is therefore non-negative for a convex boundary (see below for a generalization of the corresponding calculation to evolving domains), thus obtaining

$$\frac{d}{dt}\mathcal{W}(\Omega, f(t), \tau(t)) \geq 0.$$

When examining the integrand $-\nabla W \cdot \nu$ of the above boundary integral more closely, an interesting relation with the expression in Hamilton's Harnack inequality for the mean curvature of a hypersurface evolving by mean curvature flow emerges. To appreciate this, one should first note that the hypersurfaces M_t evolve by the equation

$$\frac{\partial x}{\partial t} = -\beta \nu - \nabla^M f \tag{3.8}$$

due to the Neumann boundary condition for f where ∇^M denotes the tangential gradient on the hypersurfaces M_t.

Proposition 3.4. *Under the above conditions on (M_t) and $f(t)$ the quantity W satisfies the identity*

$$-\nabla W \cdot \nu = 2\tau \left(\frac{\partial \beta}{\partial t} - 2\, \nabla^M \beta \cdot \nabla^M f + A(\nabla^M f, \nabla^M f) - \frac{\beta}{2\tau} \right) \tag{3.9}$$

for all $t < T$, $a \geq T$ and $\tau = a - t$ where A denotes the second fundamental form of M_t. This implies the inequality

$$\begin{aligned} &\frac{d}{dt}\mathcal{W}_\beta(\Omega_t, f(t), \tau(t)) \\ &\quad \geq 2\tau \int_{M_t} \left(\frac{\partial \beta}{\partial t} - 2\, \nabla^M \beta \cdot \nabla^M f + A(\nabla^M f, \nabla^M f) - \frac{\beta}{2\tau} \right) u \, dS \end{aligned}$$

Proof. In view of Equation (3.4) we have

$$W = -\tau\left(2\frac{df}{dt} + |\nabla f|^2\right) + f.$$

We now calculate similarly as in Appendix C

$$\frac{d}{dt}\nabla f = \nabla^2 f\,(\nabla f,\,\cdot\,) + \nabla\frac{df}{dt}.$$

A calculation as for instance in [9] using the evolution Equation (3.8) for the hypersurfaces M_t yields

$$\frac{d\nu}{dt} = \nabla^M\beta - A(\nabla^M f,\,\cdot\,)$$

for the outward unit normal field on M_t. The second term arises from the definition of A in terms of tangential derivatives of ν. Combining these and differentiating the identity $\beta = \nabla f\cdot\nu$ yields

$$\frac{d\beta}{dt} = \nabla^2 f\,(\nabla f,\nu) + \nabla\frac{df}{dt}\cdot\nu + \nabla^M\beta\cdot\nabla^M f - A(\nabla^M f, \nabla^M f).$$

Since

$$\nabla W\cdot\nu = -\tau\left(2\nabla\frac{df}{dt}\cdot\nu + \nabla|\nabla f|^2\cdot\nu\right) + \nabla f\cdot\nu$$

and $\nabla|\nabla f|^2\cdot\nu = 2\,\nabla^2 f\,(\nabla f,\nu)$ we obtain the result by observing

$$\frac{d\beta}{dt} = \frac{\partial\beta}{\partial t} - \nabla^M\beta\cdot\nabla^M f$$

in view of (3.8). The integral inequality then follows from Proposition 3.2. $\square$

Remark 3.5. Let f^{t_0} be the minimizer for $\mu_\beta(\Omega_{t_0}, \tau(t_0))$. Since $W(f^{t_0}) \equiv$ constant (see Proposition 2.4) we have

$$\int_{M_{t_0}} \nabla W\cdot\nu\, u\, dS = 0$$

at time t_0. However, even if we assume that $f(t)$ for $t < t_0$ satisfies the "end" condition $f(t_0) = f^{t_0}$ we cannot conclude that

$$\lim_{t\to t_0}\int_{M_t} \nabla W\cdot\nu\, u\, dS = 0$$

and that therefore (note that $\mathcal{W}_\beta$ is differentiable at t_0)

$$\frac{d}{dt}_{|t_0}\mathcal{W}_\beta(\Omega_t, f(t), \tau(t)) \geq 0.$$

The problem occurs since ∇W involves third derivatives of f which do not behave continuously on the boundary for $t \to t_0$ unless we impose some kind of higher order compatibility condition on the "end" data f^{t_0} on $M_{t_0} = \partial\Omega_{t_0}$.

4. A conjectured Harnack-type inequality for mean curvature flow and its consequences

For $\beta = H$, the expression

$$Z(\nabla^M f) \equiv \frac{\partial H}{\partial t} - 2\,\nabla^M H \cdot \nabla^M f + A(\nabla^M f, \nabla^M f)$$

in Proposition 3.4 is the central quantity in Hamilton's Harnack inequality for convex solutions of the mean curvature flow (see [3]). Hamilton showed that $Z(V)$ vanishes on translating solutions of mean curvature flow for some vector field V which is tangential to the hypersurfaces M_t. His Harnack inequality states that

$$Z(V) + \frac{H}{2t} \geq 0$$

holds for any tangential vector field V on a convex solution of mean curvature flow for $t > 0$ with equality for a suitable vector field on a homothetically expanding solution. We observe that on homothetically shrinking solutions that is where

$$H = \frac{x \cdot \nu}{2\tau}$$

the identity

$$2\tau Z(V) - H = 0$$

holds for $V = \nabla^M f$ where $f = |x|^2/4\tau$.

Because of the term $-H$, we cannot expect this expression to be non-negative for a general solution and for a general V. Certainly, it is negative on translating solutions for a suitable V. For non-compact solutions, our calculations do not lead to the integral inequality in Proposition 3.2 since the integral expressions are usually not well-defined in this case as will see a little later in the case of translating solutions. The above considerations motivate the following.

Conjecture. *Let $(M_t)_{t<T}$ be a family of compact embedded hypersurfaces evolving by their mean curvature. Let $\tau(t) > 0$ with $\frac{\partial\tau}{\partial t} = -1$. Then the inequality*

$$\int_{\Omega_t} \left|\nabla_i\nabla_j f - \frac{\delta_{ij}}{2\tau}\right|^2 u\,dx + \int_{M_t} \left(Z(\nabla^M f) - \frac{H}{2\tau}\right) u\,dS \geq 0 \qquad (C)$$

holds if f satisfies

$$\left(\frac{\partial}{\partial t} + \Delta\right) f = |\nabla f|^2 + \frac{n+1}{2\tau}$$

in Ω_t for $t < T$ with the boundary condition $\nabla f \cdot \nu = H$ and the domains evolve by a family of diffeomorphisms generated by $-\nabla f$.

Let us give two explicit examples of mean curvature flow solutions which illustrate the situation: first, note that the evolution equation for the hypersurfaces M_t in the case $\beta = H$ is

$$\frac{\partial x}{\partial t} = -H\nu - \nabla^M f,$$

which is mean curvature flow up to tangential diffeomorphisms.

If Ω_t is the interior of a homothetically shrinking solution of mean curvature flow, that is up to translation in time,

$$\Omega_t = \sqrt{2\tau}\,\Omega_0$$

for $\tau = T - t$, then $f = |x|^2/4(T-t) + c$ with c chosen such that $\int_{\Omega_t} u\,dx = 1$ is a solution of Equation (3.2). The Neumann boundary condition above becomes simply

$$H = \frac{x\cdot\nu}{2\tau}.$$

In this situation,

$$\frac{\partial H}{\partial t} - 2\,\nabla^M H \cdot \nabla^M f + A(\nabla^M f, \nabla^M f) - \frac{H}{2\tau} = 0$$

so $\nabla W \cdot \nu = 0$.

For translating solutions of mean curvature flow, the quantity $-\nabla W \cdot \nu$ is negative for positive H. However, our rate of change formula for $\mathcal{W}_H$ is not well defined in this case as the entropy calculations are not justified in this situation.

Indeed, if Ω_t is the interior of a translating solution of mean curvature flow, that is up to rotation in $\mathbb{R}^{n+1}$,

$$\Omega_t = \Omega + te_{n+1}$$

for some fixed set Ω and for all $t \in \mathbb{R}$ then

$$f = -x_{n+1} + \tau - \log(4\pi\tau)^{(n+1)/2}$$

solves the boundary value problem. The Neumann boundary condition on M_t in this case becomes $H = -\nu_{n+1}$.

We note that M_t and Ω_t are necessarily unbounded since compact solutions cannot exist for all $t \in \mathbb{R}$ by comparison with spheres shrinking to points in finite time. Moreover, the function u featuring in the integrand of the entropy functional as well as in the normalization condition required for the entropy is given by

$$u = \frac{\mathrm{e}^{-f}}{(4\pi\tau)^{(n+1)/2}} = \mathrm{e}^{x_{n+1}-\tau}$$

in our example. In view of the comparison principle for mean curvature flow applied to M_t and hyperplanes $\{x \in \mathbb{R}^{n+1},\ x_{n+1} = a\}$, which are stationary solutions of mean curvature flow, the sets Ω_t have an unbounded intersection with the upper half space $\{x \in \mathbb{R}^{n+1},\ x_{n+1} > 0\}$ for every $t \in \mathbb{R}$. Therefore, the function u cannot satisfy the normalization condition $\int_{\Omega_t} u\,dx = 1$ in Ω_t.

There are a number of important consequences of inequality (C) especially for the open problem of no local volume collapse for mean curvature flow solutions (an analogue of Perelman's no local collapsing for Ricci flow solutions) and consequently non-existence of certain degenerate rescaling limits. This should provide sufficient motivation for settling the conjecture.

Proposition 4.1. *Suppose that the conjectured inequality (C) holds. Then*

$$\frac{d}{dt}\mathcal{W}_H(\Omega_t, f(t), \tau(t)) \geq 0$$

holds for $t < T$ where $f(t)$ and $\tau(t)$ are as above. In particular, the entropy is monotonic that is

$$\mu_H(\Omega_{t_1}, a - t_1) \leq \mu_H(\Omega_{t_2}, a - t_2)$$

for $0 \leq t_1 \leq t_2 < T$ and any $a > t_2$.

Proof. The first inequality follows directly from Propositions 3.2 and 3.4 applied to $\beta = H$ and from (C). To derive the second inequality, we let f^{t_0} for $t_0 < T$ be the minimizer for $\mu_H(\Omega_{t_0}, \tau(t_0))$ and let $f(t)$ in addition to solving the equation

$$\left(\frac{\partial}{\partial t} + \Delta\right) f = |\nabla f|^2 + \frac{n+1}{2\tau}$$

in Ω_t and the boundary condition $\nabla f \cdot \nu = H$ on $\partial\Omega_t$ for $t < t_0$ obey the "end" condition $f(t_0) = f^{t_0}$. Since

$$\frac{d}{dt}\mathcal{W}_H(\Omega_t, f(t), \tau(t)) \geq 0$$

we have

$$\begin{aligned}\mathcal{W}_H(\Omega_t, f(t), \tau(t)) &\leq \mathcal{W}_H(\Omega_{t_0}, f(t_0), \tau(t_0)) \\ &= \mathcal{W}_H(\Omega_{t_0}, f^{t_0}, \tau(t_0)) = \mu_H(\Omega_{t_0}, \tau(t_0)).\end{aligned}$$

Taking the infimum on the left-hand side over all functions satisfying the normalization condition

$$\int_{\Omega_t} \frac{\mathrm{e}^{-f}}{(4\pi\tau)^{(n+1)/2}}\, dV = 1,$$

we obtain the desired inequality for the entropies at t and at t_0. Since t and t_0 were arbitrary we are done. □

Corollary 4.2. *Let $(M_t)_{t\in[0,T)}$ be a solution of mean curvature flow consisting of smooth, compact, embedded hypersurfaces which enclose bounded regions $(\Omega_t)_{t\in[0,T)}$ in $\mathbb{R}^{n+1}$. Suppose furthermore that inequality (C) holds. Then for every $r > 0$ and every $t \in [0, T)$*

$$\mu_H(\Omega_t, r^2) \geq \mu_H(\Omega_0, t + r^2).$$

Since $T < \infty$ we have for every $t \in [0, T)$ and $r \in (0, \sqrt{T}]$

$$\mu_H(\Omega_t, r^2) \geq -c_0$$

where c_0 depends only on n, Ω_0, T and $\sup_{M_0} |H|$. In particular, there is a constant $\kappa > 0$ depending only on $n, \Omega_0, T, \sup_{M_0} |H|$ and c_1 such *that the inequality*

$$\frac{V(\Omega_t \cap B_r(x_0))}{r^{n+1}} \geq \kappa$$

holds for all $t \in [0,T)$ *and* $r \in (0,\sqrt{T}]$ *in balls* $B_r(x_0)$ *satisfying the conditions* $V(\Omega_t \cap B_{r/2}(x_0)) > 0$ *and*

$$\frac{V(\Omega_t \cap B_r(x_0)) + r^2 \int_{M_t \cap B_r(x_0)} |H|\, dS}{V(\Omega_t \cap B_{r/2}(x_0))} \leq c_1.$$

Proof. By the above Proposition applied with $a = r^2 + t$, $t_1 = 0$ and $t_2 = t$ and Proposition 2.1 applied to Ω_0, we have

$$\mu_H(\Omega_t, r^2) \geq \mu_H(\Omega_0, t + r^2) \geq -c(n, T, \sup_{M_0} |H|, \Omega_0)$$

for all $r \leq \sqrt{T}$ and $t < T$. The lower volume ratio bounds then follow from Proposition 2.7 applied to Ω_t. $\square$

For $\lambda_j \searrow 0$, $t_j \nearrow T$ and $x_j \to x \in \mathbb{R}^{n+1}$, we define a sequence (Ω_s^j) of rescaled and translated

$$\Omega_s^j = \frac{1}{\lambda_j}\left(\Omega_{\lambda_j^2 s + t_j} - x_j\right)$$

where $s \in (-\lambda_j^{-2} t_j, \lambda^{-2}(T - t_j)) \equiv (a_j, b_j)$.

Definition 4.3. *Let* $(M_t)_{t\in[0,T)}$ *be a compact, smooth, embedded solution of mean curvature flow enclosing bounded regions* $(\Omega_t)_{t\in[0,T)}$ *in* $\mathbb{R}^{n+1}$. *We call a smooth, embedded solution* $(M_s')_{s\in(-\infty,b)}$ *of mean curvature flow enclosing (not necessarily bounded) regions* $(\Omega_s')_{s\in(-\infty,b)}$ a rescaling limit *of* $(M_t)_{t\in(0,T)}$ *at* (x,T) *if there are sequences* $\lambda_j \searrow 0$, $t_j \nearrow T$ *and* $x_j \to x \in \mathbb{R}^{n+1}$ *such that*

$$(\Omega_s^j)_{s\in(a_j,b_j)} \to (\Omega_s')_{s\in(-\infty,b)}$$

smoothly in compact subsets in space-time (that is in particular, the hypersurfaces $M_s^j = \partial\Omega_s^j$ *converge smoothly).*

Remark 4.4. For a solution $(M_t)_{t\in[0,T)}$ which becomes singular for $t \nearrow T$, that is $\sup_{t<T} \sup_{M_t} |A|^2 = \infty$ for the second fundamental form A on M_t, one can always find a rescaling limit for a suitable choice of sequences (x_j) in $\mathbb{R}^{n+1}$ and $(\lambda_j) \searrow 0$ (for example the reciprocal of the maximum of $|A|$ at an appropriately chosen sequence of times $t_j \nearrow T$). The smooth convergence

follows from standard a priori estimates for mean curvature flow (see for instance [10]).

Rescaling limits are the so-called *ancient solutions*, which means that they have existed forever. Examples of ancient solutions are all homothetically shrinking solutions of mean curvature flow such as the shrinking spheres given by $M'_s = \partial B_{\sqrt{-2ns}}$ for $s \in (-\infty, 0)$.

If the solution $(M_t)_{t\in(0,T)}$ has a so-called *type II-singularity*, that is

$$\sup_{t<T}\left((T-t)\sup_{M_t}|A|^2\right) = \infty,$$

then by a rescaling process described in [11] one can even find a limit flow which is an *eternal* solution, that is $b = \infty$. Examples of eternal solutions are all *stationary* solutions, that is solutions with $\Omega'_s = \Omega$ for all $s \in \mathbb{R}$. In this case, the hypersurface $M = \partial\Omega$ is minimal that it satisfies $H = 0$. Other eternal solutions are *translating* solutions of mean curvature flow for which $\Omega'_s = \Omega + s\omega$ for $s \in \mathbb{R}$ where $\Omega \subset \mathbb{R}^{n+1}$ and ω is a fixed unit vector in $\mathbb{R}^{n+1}$. The corresponding hypersurfaces $M = \partial\Omega$ satisfy the equation $H + \nu\cdot\omega = 0$.

The statement of Corollary 4.2 is invariant under scaling and translation. Hence, the rescaled solution $(M^j_s)_{s\in(a_j,b_j)}$ satisfies

$$\frac{V(\Omega^j_s \cap B_r(x_0))}{r^{n+1}} \geq \kappa > 0$$

for all $s \in (a_j, b_j)$ and $r \in (0, \sqrt{T}/\lambda_j)$ in balls with $V(\Omega^j_s \cap B_{r/2}(x_0)) > 0$ and

$$\frac{V(\Omega^j_s \cap B_r(x_0)) + r^2\int_{M^j_s\cap B_r(x_0)}|H|\,dS}{V(\Omega^j_s \cap B_{r/2}(x_0))} \leq c_1.$$

The constant $\kappa = \kappa(n, \Omega_0, T, \sup_{M_0}|H|, c_1)$ is the same as for the unscaled solution. As a consequence, we obtain a lower volume ratio bound for rescaling limits, but without the radius restriction.

Corollary 4.5. *Let $(M_t)_{t\in[0,T)}$ be a solution of mean curvature flow consisting of compact smooth, embedded hypersurfaces which enclose bounded regions $(\Omega_t)_{t\in[0,T)}$ in $\mathbb{R}^{n+1}$. Suppose furthermore that inequality (C) holds. Then there is a constant $\kappa > 0$ depending only on $n, \Omega_0, T, \sup_{M_0}|H|$ and c_1 such that any rescaling limit $(M'_s)_{s\in(-\infty,b)}$ of $(M_t)_{t\in[0,T)}$ with limiting* *enclosed regions* $(\Omega'_s)_{s\in(-\infty,b)}$ *satisfies*

$$\frac{V(\Omega'_s \cap B_r(x_0))}{r^{n+1}} \geq \kappa$$

for every $s \in (-\infty, b)$ *and* $r > 0$ *in balls* $B_r(x_0)$ *with* $V(\Omega'_s \cap B_{r/2}(x_0)) > 0$ *and*

$$\frac{V(\Omega'_s \cap B_r(x_0)) + r^2 \int_{M'_s \cap B_r(x_0)} |H|\, dS}{V(\Omega'_s \cap B_{r/2}(x_0))} \leq c_1.$$

This Corollary rules out certain solutions of mean curvature flow as rescaling limits under the assumption that our conjecture is valid.

Corollary 4.6. *If the conditions of the above corollary are satisfied then the following eternal solutions of mean curvature flow cannot occur as rescaling limits of a compact, smooth embedded mean-convex solution* $(M_t)_{t\in[0,T)}$ *of mean curvature flow which encloses bounded regions* $(\Omega_t)_{t\in[0,T)}$ *in* $\mathbb{R}^{n+1}$*:*

(1) *The stationary solution corresponding to a pair of parallel hyperplanes that is given by* $\Omega'_s = \Omega$ *for all* $s \in \mathbb{R}$ *where*

$$\Omega = \{x \in \mathbb{R}^{n+1},\, -d < x_{n+1} < d\}$$

for some $d > 0$.

(2) *The stationary solution of mean curvature flow corresponding to the catenoid minimal surface* $M = \partial\Omega$ *in* $\mathbb{R}^3$ *given by*

$$\Omega = \{x = (\hat{x}, x_3) \in \mathbb{R}^3,\, |\hat{x}| > 1,\, |x_3| < \cosh^{-1} |\hat{x}|\}.$$

(3) *The translating solution corresponding to the grim reaper hypersurface* $M = \partial\Omega$ *where* $\Omega = \mathbb{R}^{n-1} \times G$ *with*

$$G = \left\{(x_n, x_{n+1}) \in \mathbb{R}^2,\, -\pi/2 < x_n < \pi/2,\, x_{n+1} > -\log\cos x_n\right\}.$$

Proof. All three examples admit sequences of balls for radii increasing to infinity for which the volume ratio tends to zero while the other quantities are controlled. This was discussed in Corollary 2.10. □

Remark 4.7. (1) In the special situation where the original solution (M_t) is mean convex, that is $H > 0$ for M_0 and subsequently for all M_t by the maximum principle, White [4] ruled out the grim reaper hypersurface as a rescaling limit using techniques from minimal surface theory and geometric measure theory. His methods extend also to non-smooth limit flows of generalized mean curvature flow solutions in the mean-convex case.

(2) In view of Corollary 2.1, the first two examples satisfy $\inf_{\tau>0}\mu(\Omega,\tau) = -\infty$ and the third one satisfies $\inf_{\tau>0}\mu_H(\Omega,\tau) = -\infty$.

(3) The embeddedness assumption on the hypersurfaces M_t is essential. In [5], it is proved that rescaling limits of non-embedded planar curves near singularities are given by the grim reaper curve $\Gamma = \partial G$ defined above.

(4) Some other translating solutions can occur as rescaling limits such as, for instance, a rotationally symmetric translating bowl (see, for instance, [12]). The region bounded by this translating bowl opens up quadratically so one can show that it satisfies the conclusions of the above corollary.

(5) For the shrinking solution $\Omega'_s = B_{\sqrt{-2ns}}$, there is no lower bound of the form

$$\frac{V(\Omega'_s \cap B_r)}{r^{n+1}} \geq \kappa > 0$$

with a fixed κ for all $s < 0$ and all $r > 0$ since the balls Ω'_s shrink to the origin for $s \nearrow 0$. This does not contradict Corollary 4.5 though as κ depends on c_1 and in this case c_1 behaves like $-c(n)r^2s^{-1}$ since for $s \in [-1/(2n), 0)$ and $r \geq 1$

$$\frac{r^2\int_{M'_s\cap B_r} H\,dS}{V(\Omega'_s \cap B_{r/2})} = \frac{\int_{M'_s} H\,dS}{V(\Omega'_s)} = -c(n)\frac{r^2}{s}.$$

Appendix A. Some basic properties of entropies in $\mathbb{R}^{n+1}$

Here, we discuss some explicit examples of entropies in $\mathbb{R}^{n+1}$.

(1) When $\beta = 0$ and $\Omega = \mathbb{R}^{n+1}$ we have (see [1])

$$\mathcal{W}(\mathbb{R}^{n+1}, f, \tau) = \int_{\mathbb{R}^{n+1}} \big(\tau|\nabla f|^2 + f - (n+1)\big)\,u\,dx \geq 0$$

for all f satisfying

$$\int_{\mathbb{R}^{n+1}} u\,dx = 1$$

with equality when $f(x) = \frac{|x|^2}{4\tau}$. In particular, therefore,

$$\mu(\mathbb{R}^{n+1}, \tau) = 0$$

for all $\tau > 0$.

This identity is equivalent to the Gaussian logarithmic Sobolev inequality as discussed by Gross [6]. Scaling by $x = \sqrt{2\tau}y$, setting $f = \frac{|y|^2}{2} - \log\varphi^2$ as in [1] and using the identity

$$\int_{\mathbb{R}^{n+1}} \left(|y|^2 - (n+1)\right)\gamma_{n+1}\,dy = -\int_{\mathbb{R}^{n+1}} \operatorname{div}\left(y\gamma_{n+1}\right)\,dy = 0$$

for the Gaussian

$$\gamma_{n+1}(y) = \frac{\mathrm{e}^{-(|y|^2/2)}}{(2\pi)^{n+1}}$$

we obtain its standard form

$$\int_{\mathbb{R}^{n+1}} \varphi^2 \log\varphi\;\gamma_{n+1}\,dy \le \frac{1}{2}\int_{\mathbb{R}^{n+1}} |\nabla\varphi|^2\;\gamma_{n+1}\,dy$$

for all φ satisfying

$$\int_{\mathbb{R}^{n+1}} \varphi^2\gamma_{n+1}\,dx = 1.$$

(2) For $x \in \Omega \subset \mathbb{R}^{n+1}$, we set $x = \lambda y + x_0$ where $\lambda > 0$ and $x_0 \in \mathbb{R}^{n+1}$. We then obtain

$$\begin{aligned}\mathcal{W}_\beta(\Omega, f, \tau) = {}& \mathcal{W}(\lambda^{-1}(\Omega - x_0), f(\lambda\;\cdot + x_0), \lambda^{-2}\tau) \\ &+ 2(\lambda^{-2}\tau)\int_{\lambda^{-1}(\partial\Omega - x_0)} \lambda\beta(\lambda y + x_0)\frac{\mathrm{e}^{-f(\lambda y + x_0)}}{(4\pi\lambda^{-2}\tau)^{(n+1)/2}}\,dS(y)\end{aligned}$$

and

$$1 = \int_\Omega u(x)\,dx = \int_{\lambda^{-1}(\Omega - x_0)} u(\lambda y + x_0)\,dy.$$

Therefore

$$\mu_\beta(\Omega, \tau) = \mu_{\lambda\beta(\lambda\,\cdot + x_0)}(\lambda^{-1}(\Omega - x_0), \lambda^{-2}\tau).$$

Suppose that $\beta : \mathbb{R}^{n+1} \to \mathbb{R}$ satisfies

$$\beta\left(\frac{x - x_0}{\lambda}\right) = \lambda\beta(x)$$

for $x, x_0 \in \mathbb{R}^{n+1}$ and $\lambda > 0$ or that $\beta = \beta_{\partial\Omega}$ is a geometric quantity which behaves like

$$\beta(y) = \lambda\beta(x)$$

where $x = \lambda y + x_0 \in \partial\Omega$ for $y \in \frac{1}{\lambda}(\partial\Omega - x_0)$ such as for example the mean curvature of $\partial\Omega$. Then

$$\mu_\beta(\Omega, \tau) = \mu_\beta(\lambda^{-1}(\Omega - x_0), \lambda^{-2}\tau).$$

For $x_0 = 0, \lambda = \sqrt{2\tau}, \Omega$ replaced by $\sqrt{2\tau}\,\Omega$ and such functions β this yields

$$\mu_\beta(\sqrt{2\tau}\,\Omega, \tau) = \mu_\beta(\Omega, 1/2).$$

(3) If $x_0 \in \Omega$ then

$$\lambda^{-1}(\Omega - x_0) \to \mathbb{R}^{n+1}.$$

Using this, the scaling identity for μ_β with $\lambda = \sqrt{2\tau}$ as well as the identity $\mu(\mathbb{R}^{n+1}, 1/2) = 0$ we expect that

$$\mu_\beta(\Omega, \tau) \to 0$$

for $\tau \to 0$. This should follow along the same lines as in [2].

(4) A natural example is

$$\beta = \frac{x \cdot \nu}{2\tau}$$

where ν is the unit outward pointing normal to $\partial\Omega$. By the above scaling property we have

$$\mu_{\frac{x\cdot\nu}{2\tau}}(\sqrt{2\tau}\,\Omega, \tau) = \mu_{y\cdot\nu}(\Omega, 1/2)$$

where $x = \sqrt{2\tau} y$ and $y \in \Omega$.

An example of a function f on $\Omega \subset \mathbb{R}^{n+1}$ satisfying the normalization condition

$$\int_\Omega \frac{\mathrm{e}^{-f}}{(4\pi\tau)^{(n+1)/2}}\,dx = 1$$

is

$$f = \frac{|x|^2}{4\tau} - \log c$$

where

$$\frac{1}{c} = \int_\Omega \frac{\mathrm{e}^{-(|x|^2/4\tau)}}{(4\pi\tau)^{(n+1)/2}}\,dx.$$

For this f and $\beta = \frac{x\cdot\nu}{2\tau}$, one calculates

$$\mathcal{W}_\beta(\Omega, f, \tau) = c\left(\int_\Omega \left(\frac{|x|^2}{2\tau} - (n+1)\right)\frac{\mathrm{e}^{-(|x|^2/4\tau)}}{(4\pi\tau)^{(n+1)/2}}\,dx + \int_{\partial\Omega} x\cdot\nu \frac{\mathrm{e}^{-(|x|^2/4\tau)}}{(4\pi\tau)^{(n+1)/2}}\,dS\right) + \log c.$$

Since

$$\operatorname{div}\left(x\mathrm{e}^{-\frac{|x|^2}{4\tau}}\right) = -\left(\frac{|x|^2}{2\tau} - (n+1)\right)\mathrm{e}^{-\frac{|x|^2}{4\tau}}$$

the previous identity

$$\mathcal{W}_\beta(\Omega, f, \tau) = \log c$$

by the divergence theorem.

Note that for $\Omega = \mathbb{R}^{n+1}$ we have $c = 1$ and hence $\mathcal{W}_\beta(\Omega, f, \tau) = \mathcal{W}(\Omega, f, \tau) = 0$ for $f = \frac{|x|^2}{4\tau}$.

For the half-space $H_a = \{x \in \mathbb{R}^{n+1}, x_{n+1} < a\}$, $a \in \mathbb{R}$ and $\beta = \frac{x\cdot\nu}{2\tau}$ we calculate

$$\frac{1}{c} = \int_{-\infty}^{\frac{a}{\sqrt{2\tau}}} \mathrm{e}^{-\frac{z^2}{2}}\,dz.$$

This implies that $\mathcal{W}_{\frac{x\cdot\nu}{2\tau}}(H_a, f, \tau) \to -\infty$ for $a \to -\infty$ as well as $\lim_{\tau\to 0} \mathcal{W}_{\frac{x\cdot\nu}{2\tau}}(H_a, f, \tau) = 0$ and $\lim_{\tau\to 0} \mathcal{W}_{\frac{x\cdot\nu}{2\tau}}(H_a, f, \tau) = -\log 2 < 0$ for fixed $a \in \mathbb{R}$.

By the scaling and translation property above we have

$$\mathcal{W}_{\frac{x\cdot\nu}{2\tau}}\left(\Omega, \frac{|x|^2}{4\tau} - \log c, \tau\right) = \mathcal{W}_{y\cdot\nu}\left(\frac{1}{\sqrt{2\tau}}\Omega, \frac{|y|^2}{2} - \log c, \frac{1}{2}\right) = \log c$$

for $x = \sqrt{2\tau}y \in \Omega$ with the condition

$$\int_{(1/\sqrt{2}\tau)\Omega} \gamma_{n+1}\,dy = \frac{1}{c}.$$

If the $(n+1)$-dimensional volume of a set Ω inside large balls grows like

$$V(\Omega \cap B_R) \le cR^p$$

for $R \ge R_0$ and $p < n+1$, one checks that

$$\int_{(1/\sqrt{2}\tau)\Omega} \gamma_{n+1}\,dy \longrightarrow 0$$

for $\tau \to \infty$. Therefore, as $\tau \to \infty$ we have $c \to \infty$ and hence

$$\mu_{\frac{x\cdot\nu}{2\tau}}(\Omega, \tau) \to -\infty.$$

Such sets Ω include, for instance, all bounded sets but also unbounded sets which lie in a slab in $\mathbb{R}^{n+1}$. In the latter case the volume in balls grows like R^n.

Appendix B. Sobolev and logarithmic Sobolev inequalities

For the convenience of the reader who is unfamiliar with logarithmic Sobolev inequalities we show how these can be derived from the standard Sobolev inequality. We essentially follow a proof given in [6].

Theorem (Logarithmic Sobolev inequality)**.** *For any open subset Ω of a Riemannian manifold (X, g) which satisfies the Sobolev inequality*

$$\left(\int_\Omega |\psi|^{\frac{n+1}{n}}\, dV\right)^{\frac{n}{n+1}} \le c_S(\Omega, g) \int_\Omega (|\nabla\psi| + |\psi|)\, dV$$

for all $\psi \in C^1(\bar{\Omega})$ there also holds a logarithmic Sobolev inequality of the form

$$\int_\Omega \left(\epsilon\, |\nabla\varphi|^2 - \varphi^2 \log \varphi^2\right)\, dV \ge -c(n)(1 + \log c_S(\Omega, g)) - \frac{1}{\epsilon}$$

for functions φ satisfying $\int_\Omega \varphi^2\, dV = 1$ and every $\epsilon > 0$.

Proof. By a standard approximation argument it will be sufficient to prove the theorem for non-negative functions. We abbreviate

$$\|\psi\|_p \equiv \left(\int_\Omega \psi^p\, dV\right)^{\frac{1}{p}}$$

for $p > 0$. The interpolation inequality for L^p-norms says for functions ψ satisfying $\|\psi\|_1 = 1$ that

$$\|\psi\|_q \le \|\psi\|_{\frac{n}{n-1}}^{n-\frac{n}{q}}$$

for $1 \le q \le \frac{n}{n-1}$. Since for $q = 1$ we have equality, differentiation with respect to q at $q = 1$ preserves the inequality and leads to

$$\int_\Omega \psi \log \psi\, dV \le n \log \|\psi\|_{\frac{n}{n-1}}.$$

In view of the Sobolev inequality

$$\|\psi\|_{\frac{n}{n-1}} \leq c_S(\Omega, g)\left(\|\nabla\psi\|_1 + 1\right)$$

for such functions this yields

$$\begin{aligned}\int_\Omega \psi \log \psi \, dV &\leq n \log\left(c_S(\Omega, g)\left(\|\nabla\psi\|_1 + 1\right)\right) \\ &= n \log\left(\frac{1}{n}\left(\|\nabla\psi\|_1 + 1\right)\right) + n \log(n c_S(\Omega, g)).\end{aligned}$$

The inequality $\log x \leq x - 1$ implies

$$\int_\Omega \psi \log \psi \, dV \leq \|\nabla\psi\|_1 + c(n)(1 + \log c_S(\Omega, g)).$$

Setting $\psi = \varphi^2$ with $\int_\Omega \varphi^2 \, dV = 1$ gives

$$\int_\Omega \varphi^2 \log \varphi^2 \, dV \leq \int_\Omega |\nabla\varphi^2| \, dV + c(n)(1 + \log c_S(\Omega, g)).$$

Using Young's inequality, we finally arrive

$$\int_\Omega \varphi^2 \log \varphi^2 \, dV \leq \epsilon \int_\Omega |\nabla\varphi|^2 \, dV + \frac{1}{\epsilon} + c(n)(1 + \log c_S(\Omega, g))$$

where we again used $\int_\Omega \varphi^2 \, dV = 1$. □

Appendix C. Proof of the evolution equation for W

For the convenience of the reader we give a detailed proof of the evolution equation of Proposition 3.1 in Section 3. In Section 3, we merely modified the appropriate formulas discussed in [1] and [2] by transforming to total time derivatives.

Let us briefly recall the set-up given in Section 3 in the case of evolving domains in $\mathbb{R}^{n+1}$. We consider a family of subsets $(\Omega_t)_{t\in(0,T)}$ in $\mathbb{R}^{n+1}$ which evolve by the equation

$$\frac{\partial x}{\partial t} = -\nabla f(x, t) \tag{C.1}$$

for $x \in \Omega_t$ where $f(t)$ satisfies the equation

$$\left(\frac{\partial}{\partial t} + \Delta\right) f = |\nabla f|^2 + \frac{n+1}{2\tau} \tag{C.2}$$

in Ω_t for $t \in (0, T)$. The total time derivative of f is given by

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \left\langle \nabla f, \frac{\partial x}{\partial t} \right\rangle = \frac{\partial f}{\partial t} - |\nabla f|^2 \tag{C.3}$$

and so (C.2) can also be written as

$$\left(\frac{d}{dt} + \Delta \right) f = \frac{n+1}{2\tau}. \tag{C.4}$$

We also assume that $\tau(t) > 0$ evolves by $\frac{\partial \tau}{\partial t} = -1$.

Proposition. *In the above setting, the function* $W = \tau(2\Delta f - |\nabla f|^2) + f - (n+1)$ *satisfies the evolution equation*

$$\left(\frac{d}{dt} + \Delta \right) W = 2\tau \left| \nabla_i \nabla_j f - \frac{1}{2\tau}\, \delta_{ij} \right|^2 + \nabla W \cdot \nabla f.$$

Proof. We adapt the computation in [2] to the case of domains evolving by (C.1) (the different sign in Ni's Lemma 2.2 stems from the fact that he considers the forward heat equation by interchanging the roles of τ and t.) In a general Riemannian manifold (X, g), an additional Ricci term arises when we interchange third derivatives of f. In the Ricci flow case, this expression is balanced by terms coming from the time derivative of the metric. Details of the latter can be found in [7].

If we write above $x = \phi(q, t)$ where $\phi_t = \phi(\cdot, t) : \Omega \to \Omega_t$ are the diffeomorphisms evolving Ω_t, the pulled back function f given by

$$\tilde{f}(q, t) = f(\phi(q, t), t)$$

satisfies

$$\frac{df}{dt}(x, t) = \frac{\partial \tilde{f}}{\partial t}(q, t).$$

The evolution Equation (C.1) written in terms of $\tilde{f}(q, t) = f(\phi(q, t), t)$ looks like

$$\frac{\partial \phi}{\partial t}(q, t) = -\tilde{\nabla} \tilde{f}(q, t)$$

where $\tilde{\nabla}$ is the gradient with respect to the pull-back of the Euclidean metric under ϕ_t on $\Omega \subset \mathbb{R}^{n+1}$ given by

$$g_{ij}(q, t) = \frac{\partial \phi}{\partial q_i}(q, t) \cdot \frac{\partial \phi}{\partial q_j}(q, t).$$

In these coordinates we have

$$|\nabla f|^2 = g^{ij}\frac{\partial \tilde f}{\partial q_i}\frac{\partial \tilde f}{\partial q_j}.$$

One now calculates

$$\frac{\partial}{\partial t} g_{ij} = -2\tilde\nabla_i\tilde\nabla_j f,$$

and the inverse metric satisfies

$$\frac{\partial}{\partial t} g^{ij} = 2\tilde\nabla^i\tilde\nabla^j f.$$

Furthermore, one computes for the Christoffel symbols of the g_{ij}

$$g^{ij}\frac{\partial}{\partial t}\Gamma^k_{ij} = \tilde\nabla^k\tilde\Delta\tilde f. \tag{C.5}$$

One then checks from this and $\Delta f(x,t) = \tilde\Delta\tilde f(q,t)$ with

$$\tilde\Delta\tilde f = g^{ij}\left(\frac{\partial^2 \tilde f}{\partial q_i\partial q_j} - \Gamma^k_{ij}\frac{\partial \tilde f}{\partial q_k}\right)$$

that the identities

$$\frac{d}{dt}|\nabla f|^2 = 2\nabla_i\nabla_j f\nabla_i f\nabla_j f + 2\nabla f\cdot\nabla\frac{df}{dt} \tag{C.6}$$

and

$$\frac{d}{dt}\Delta f = \Delta\frac{df}{dt} + 2|\nabla^2 f|^2 + \nabla f\cdot\nabla\Delta f \tag{C.7}$$

hold. We now follow [2] exactly, except for working with $\frac{df}{dt}$ instead of $\frac{\partial f}{\partial t} - |\nabla f|^2$. The latter of the above identities in combination with (C.4) and the relation $\frac{\partial\tau}{\partial t} = -1$ implies

$$\left(\frac{d}{dt} + \Delta\right)\frac{df}{dt} = \frac{n+1}{2\tau^2} - 2|\nabla^2 f|^2 - \nabla f\cdot\nabla\Delta f. \tag{C.8}$$

Combining (C.6) and (C.4) with the Bochner identity

$$\Delta|\nabla f|^2 = 2|\nabla^2 f|^2 + 2\nabla f\cdot\nabla\Delta f$$

we find

$$\left(\frac{d}{dt}+\Delta\right)|\nabla f|^2 = 2|\nabla^2 f|^2 + 2\nabla_i\nabla_j f\nabla_i f\nabla_j f. \tag{C.9}$$

To break up the calculation for W, we rewrite $W = \tau(2\Delta f - |\nabla f|^2) + f - (n+1)$ using (C.4) as

$$W = \tau w + f$$

where

$$w = -2\frac{df}{dt} - |\nabla f|^2 = 2\Delta f - |\nabla f|^2 - \frac{n+1}{\tau}. \tag{C.10}$$

From (C.8) and (C.9) we calculate

$$\left(\frac{d}{dt}+\Delta\right) w = 2|\nabla^2 f|^2 - \frac{n+1}{\tau^2} - 2\nabla_i\nabla_j f\nabla_i f\nabla_j f + 2\nabla f\cdot\nabla\Delta f.$$

Since

$$-2\nabla_i\nabla_j f\nabla_i f\nabla_j f + 2\nabla f\cdot\nabla\Delta f = \nabla f\cdot\nabla w$$

we thus arrive at

$$\left(\frac{d}{dt}+\Delta\right) w = 2|\nabla^2 f|^2 - \frac{n+1}{\tau^2} + \nabla f\cdot\nabla w.$$

Using again $\frac{\partial\tau}{\partial t} = -1$ we now compute

$$\begin{aligned}\left(\frac{d}{dt}+\Delta\right) W &= \left(\frac{d}{dt}+\Delta\right)(\tau w + f)\\ &= -w + 2\tau|\nabla^2 f|^2 - \frac{n+1}{\tau} + \nabla f\cdot\nabla(\tau w) + \frac{n+1}{2\tau}\\ &= \nabla f\cdot\nabla W - w - |\nabla f|^2 + 2\tau|\nabla^2 f|^2 - \frac{n+1}{2\tau}.\end{aligned}$$

Substituting the identities

$$2\tau|\nabla^2 f|^2 = 2\tau\left|\nabla_i\nabla_j f - \frac{\delta_{ij}}{2\tau}\right|^2 + 2\Delta f - \frac{n+1}{2\tau}$$

and

$$w = 2\Delta f - |\nabla f|^2 - \frac{n+1}{\tau}$$

yields the desired evolution equation for W. □

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