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Coassociative 4-folds with conical singularities

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This article studies the deformation theory of coassociative 4-folds N with conical singularites in a G_2 manifold. We describe three moduli spaces: first we consider deformations with the same singularities as N, then allow for changes in the singularities and, finally, include variations of the ambient G_2 structure. We show that the moduli space, in each case, is locally homeomorphic to the kernel of a smooth map between smooth manifolds and determine a lower bound for its expected dimension. Further, by relaxing the condition on the G_2 structure, we prove a generic smoothness result for the second and third moduli spaces.

1. Introduction

Coassociative 4-folds were introduced by Harvey and Lawson in [3] as examples of *calibrated* 4-dimensional submanifolds of $\mathbb{R}^7$ which are linked to the exceptional Lie group G_2. Moreover, the coassociative condition is shown to be equivalent to the vanishing of a certain differential 3-form on the submanifold. The definition of coassociative submanifolds, both in $\mathbb{R}^7$ and in more general 7-manifolds, is given in § 2.

Manifolds with singularities modelled on cones have been studied by a number of authors, but most relevantly by Lockhart and McOwen [15], in their work on analysis on noncompact manifolds, and by Joyce [6–10], who considered *special Lagrangian* submanifolds with conical singularities. The latter's work will be discussed in § 1.1. Coassociative 4-folds with conical singularities specifically appear in the work of Kovalev [11] on coassociative fibrations of G_2 manifolds. The definition of this class of coassociative 4-folds can be found in § 3.

1.1. Motivation

McLean [17] showed that a *compact* coassociative 4-fold admits a smooth moduli space of deformations with dimension b^2_+. This result was extended by the author [16] to the situation where the coassociative 4-fold is asymptotic to a cone at infinity — a natural progression from the compact case.

The moduli space is shown to be smooth if the rate at which the 4-fold converges to the cone at infinity is generic and above a critical value. These studies motivate the work presented here, as a submanifold with conical singularities can be considered as another generalization from the compact to the noncompact category.

Another primary impetus for our research is the work on special Lagrangian m-folds with conical singularities by Joyce in the series of papers [6–10]. Special Lagrangian m-folds are calibrated submanifolds of Calabi–Yau manifolds, and can be characterized by the vanishing of differential forms on them. Joyce shows that the moduli space of deformations, for a fixed Calabi–Yau structure, is locally homeomorphic to the kernel of a smooth map between smooth manifolds. Further, he proves a generic smoothness result for the moduli space of deformations in *almost* Calabi–Yau manifolds. These results are mimicked in this article, but the proofs are made more difficult essentially because Joyce is able to restrict his attention to analysis using functions, whereas we are forced to deal with differential forms. This may appear to be a minor detail, but it proves to be surprisingly challenging. Another major difference is that we make a detailed calculation of the expected dimension of the moduli space, which requires analytic and topological considerations specific to the coassociative scenario.

We also study coassociative 4-folds with conical singularities because understanding their deformations should be a useful step towards attempting to prove a 7-dimensional analogue of the SYZ conjecture — a proposed geometric explanation of Mirror Symmetry in String Theory.

1.2. Summary

We begin, in § 2, by discussing the notions of G_2 structures on 7-manifolds, G_2 manifolds and coassociative 4-folds. In § 3, we define a distinguished class of singular manifolds which we call *CS manifolds*. CS manifolds have conical singularities and their nonsingular part is a *noncompact* Riemannian manifold. We also define what we mean by tangent cones at conical singularities and CS coassociative 4-folds, that is, coassociative 4-folds with conical singularities.

In order that we may employ various analytic techniques in the course of our study, we choose to use *weighted Banach spaces* of forms on the nonsingular part of a CS manifold. These spaces are described in § 4. We then focus, in § 5, on a linear, elliptic, first-order differential operator acting between weighted Banach spaces on a 4-dimensional CS manifold. The Fredholm

and index theory of this operator is discussed using the theory developed in [15].

In § 6, we stratify the types of deformations allowed into three problems, each with an associated nonlinear first-order differential operator whose kernel gives a local description of the moduli space. For Problem 1, we force deformations of the CS coassociative 4-fold N to have the singularities at the same points as N with the same tangent cones. In Problem 2, we allow the singular points of the deformations, and the tangent cones at these points, to differ from those of N. Finally, in Problem 3, we consider deformations of N as in Problem 2 but which now can be coassociative under a deformation of the G_2 structure within a prescribed family. The main result for each problem, given in § 7, is that the moduli space of deformations near N is homeomorphic to the kernel of a smooth map between smooth manifolds near zero. In each case, the map in question can be considered as a projection from the *infinitesimal deformation space* onto the *obstruction space*. Thus, when there are no obstructions, the moduli space is a smooth manifold. Furthermore, using the material in § 5 helps to provide a lower bound on the expected dimension of the moduli space in § 8.

Finally, in § 9, we show that, for a suitable larger category of G_2 structures on the ambient 7-manifold, there is always a family of nearby G_2 structures such that the third moduli space is smooth and that, for generic fixed elements of this family, the second moduli space of deformations is smooth as well.

Notes.

(a) Manifolds are taken to be nonsingular and submanifolds to be embedded, for convenience, unless stated otherwise.

(b) We use the convention that the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$.

2. Coassociative 4-folds

In this section, we present the basic definitions we need for our study.

The key to defining coassociative 4-folds lies with the introduction of a distinguished 3-form on $\mathbb{R}^7$.

Definition 2.1. Let $(x_1, \ldots, x_7)$ be coordinates on $\mathbb{R}^7$ and write $d\mathbf{x}_{ij\cdots k}$ for the form $dx_i \wedge dx_j \wedge \cdots \wedge dx_k$. Define a 3-form φ_0 by:

$$\varphi_0 = d\mathbf{x}_{123} + d\mathbf{x}_{145} + d\mathbf{x}_{167} + d\mathbf{x}_{246} - d\mathbf{x}_{257} - d\mathbf{x}_{347} - d\mathbf{x}_{356}.$$

The 4-form $*\varphi_0$, where φ_0 and $*\varphi_0$ are related by the Hodge star, is given by:

$$*\varphi_0 = d\mathbf{x}_{4567} + d\mathbf{x}_{2367} + d\mathbf{x}_{2345} + d\mathbf{x}_{1357} - d\mathbf{x}_{1346} - d\mathbf{x}_{1256} - d\mathbf{x}_{1247}.$$

Our choice of expression for φ_0 follows that of [5, Chapter 10]. This form is sometimes known as the G_2 3-form because the Lie group G_2 is the subgroup of $\mathrm{GL}(7,\mathbb{R})$ preserving φ_0.

Definition 2.2. A 4-dimensional submanifold N of $\mathbb{R}^7$ is coassociative if and only if $\varphi_0|_N \equiv 0$ and $*\varphi_0|_N > 0$.

This definition is not standard but is equivalent to the usual definition in the language of *calibrated geometry* by [3, Proposition IV.4.5 and Theorem IV.4.6].

Remark. The condition $\varphi_0|_N \equiv 0$ forces $*\varphi_0$ to be a nonvanishing 4-form on N. Thus, the positivity of $*\varphi_0|_N$ is equivalent to a choice of orientation on N.

So that we may describe coassociative submanifolds of more general 7-manifolds, we make two definitions following [2, p. 7] and [5, p. 243].

Definition 2.3. Let M be an oriented 7-manifold. For each $x \in M$, there exists an orientation preserving isomorphism $\iota_x : T_xM \to \mathbb{R}^7$. Since dim $\mathrm{G}_2 = 14$, dim $\mathrm{GL}_+(T_xM) = 49$ and $\dim \Lambda^3 T_x^*M = 35$, the $\mathrm{GL}_+(T_xM)$ orbit of $\iota_x^*(\varphi_0)$ in $\Lambda^3 T_x^*M$, denoted by $\Lambda^3_+ T_x^*M$, is open. A 3-form φ on M is *definite* if $\varphi|_{T_xM} \in \Lambda^3_+ T_x^*M$ for all $x \in M$. Denote the bundle of definite 3-forms by $\Lambda^3_+ T^*M$. It is a bundle with fibre $\mathrm{GL}_+(7,\mathbb{R})/\mathrm{G}_2$ which is *not* a vector subbundle of $\Lambda^3 T^*M$.

Essentially, a definite 3-form is identified with the G_2 3-form on $\mathbb{R}^7$ at each point in M. Therefore, to each definite 3-form φ, we can uniquely associate a 4-form $*\varphi$ and a metric g on M such that the triple $(\varphi, *\varphi, g)$ corresponds to $(\varphi_0, *\varphi_0, g_0)$ at each point. This leads us to our next definition.

Definition 2.4. Let M be an oriented 7-manifold, let φ be a definite 3-form on M and let g be the metric associated with φ. We call (φ, g) a G_2 *structure* on M. If φ is closed (or coclosed) then (φ, g) is a *closed* (or *coclosed*) G_2 structure. A closed and coclosed G_2 structure is called *torsion-free*.

Our choice of notation here agrees with [2].

Remarks.

(a) There is a 1-1 correspondence between pairs (φ, g) and principal G_2 subbundles of the frame bundle.

(b) By [21, Lemma 11.5], (φ, g) is a torsion-free G_2 structure if and only if the holonomy of g is contained in G_2.

Definition 2.5. Let M be an oriented 7-manifold endowed with a G_2 structure (φ, g), denoted (M, φ, g). We say that (M, φ, g) is a *φ-closed*, or *φ-coclosed*, 7-manifold if (φ, g) is a closed, respectively coclosed, G_2 structure. If (φ, g) is torsion-free, we call (M, φ, g) a G_2 *manifold*.

We are now able to extend our definition of coassociative 4-folds.

Definition 2.6. A 4-dimensional submanifold N of (M, φ, g) is coassociative if and only if $\varphi|_N \equiv 0$ and $*\varphi|_N > 0$.

We end this section with a result, [17, cf. Proposition 4.2], that is invaluable in describing the deformation theory of coassociative 4-folds.

Proposition 2.7. *Let N be a coassociative 4-fold in (M, φ, g). There is an isometric isomorphism between the normal bundle $\nu(N)$ of N in M and $\Lambda^2_+ T^*N$ given by $\jmath_N : v \mapsto (v \cdot \varphi)|_{TN}$.*

3. Conical singularities

Here we first give the general definition of a manifold with conical singularities, then specialise to coassociative 4-folds. The second definition relies on the introduction of a suitable local coordinate system on the ambient 7-manifold.

3.1. CS manifolds

Definition 3.1. Let N be a connected Hausdorff topological space and let $z_1, \ldots, z_s \in N$ be distinct points. Suppose that $\hat{N} = N \setminus \{z_1, \ldots, z_s\}$ has the structure of a (nonsingular) n-dimensional Riemannian manifold, with Riemannian metric g, compatible with its topology. Then N is a *manifold with conical singularities* (at $z_1, \ldots, z_s$ with rate λ) if there exist constants $\epsilon > 0$ and $\lambda > 1$, a compact $(n-1)$-dimensional Riemannian manifold

(Σ_i, h_i), an open set $U_i \ni z_i$ in N with $U_i \cap U_j = \emptyset$ for $j \neq i$ and a diffeomorphism $\Psi_i\colon (0,\epsilon) \times \Sigma_i \to U_i \setminus \{z_i\} \subseteq \hat{N}$, for $i = 1, \ldots, s$, such that

$$\left|\nabla_i^j\big(\Psi_i^*(g) - g_i\big)\right| = O\big(r_i^{\lambda-1-j}\big) \text{ for } j \in \mathbb{N} \quad \text{as } r_i \longrightarrow 0, \tag{3.1}$$

where r_i is the coordinate on $(0,\infty)$ on the cone $C_i = (0,\infty) \times \Sigma_i$, $g_i = dr_i^2 + r_i^2 h_i$ is the conical metric on C_i, ∇_i is the Levi–Civita connection derived from g_i and $|\cdot|$ is calculated using g_i.

We call C_i the *cone* at the singularity z_i and let the *ends* $\hat{N}_\infty$ of $\hat{N}$ be the disjoint union $\hat{N}_\infty = \bigsqcup_{i=1}^s U_i \setminus \{z_i\}$.

We say that N is *CS* or a *CS manifold* (with rate λ) if it is a manifold with conical singularities which have rate λ and it is compact as a topological space. In these circumstances it may be written as the disjoint union $N = K \sqcup \bigsqcup_{i=1}^s U_i$, where K is compact as it is closed in N.

The condition $\lambda > 1$ guarantees that the metric on $\hat{N}$ genuinely tends to the conical metric on C_i near the singularities, as is evident from (3.1). Since N is Hausdorff, the set $U_i \setminus \{z_i\}$ is open in $\hat{N}$ for all i. Moreover, the condition that the U_i are disjoint is easily satisfied since, if $i \neq j$, z_i and z_j may be separated by disjoint open sets, and there are only a finite number of singularities.

Remark. If N is a CS manifold, $\hat{N}$ is a *noncompact* manifold.

Definition 3.2. Let N be a CS manifold. Using the notation of Definition 3.1, a *radius function* on $\hat{N}$ is a smooth map $\rho\colon \hat{N} \to (0,1]$ such that there exist constants $0 < c_1 < 1 < c_2$ with $c_1 r_i < \Psi_i^*(\rho) < c_2 r_i$ on $(0,\epsilon) \times \Sigma_i$, for all i.

If N is CS, we may construct a radius function on $\hat{N}$ as follows. Let $\rho(x) = 1$ for all $x \in \hat{N} \setminus \hat{N}_\infty$. Define $\rho_i\colon \Psi_i\big((0,\epsilon/2) \times \Sigma_i\big) \to (0,1)$ to be equal to r_i/ϵ for $i = 1, \ldots, s$ and then define ρ by interpolating smoothly between its definition on $\hat{N} \setminus \hat{N}_\infty$ and ρ_i on each of the disjoint sets $\Psi_i\big((\epsilon/2,\epsilon) \times \Sigma_i\big)$.

3.2. CS coassociative 4-folds

Let $B(0;\eta)$ denote the open ball about 0 in $\mathbb{R}^7$ with radius $\eta > 0$. We define a preferred choice of local coordinates on a G_2 manifold near a finite set of points, which is an analogue of the local coordinate system for almost Calabi–Yau manifolds used by Joyce [6, Definition 3.6].

Definition 3.3. Let (M, φ, g) be a G_2 manifold and let $z_1, \ldots, z_s$ be distinct points in M. There exist a constant $\eta \in (0,1)$, an open set $V_i \ni z_i$ in M with $V_i \cap V_j = \emptyset$ for $j \neq i$, and a diffeomorphism $\chi_i : B(0;\eta) \subseteq \mathbb{R}^7 \to V_i$ with $\chi_i(0) = z_i$, for $i = 1, \ldots, s$, such that $\zeta_i = d\chi_i|_0 \colon \mathbb{R}^7 \to T_{z_i}M$ is an isomorphism identifying the standard G_2 structure (φ_0, g_0) on $\mathbb{R}^7$ with the pair $(\varphi|_{T_{z_i}M}, g|_{T_{z_i}M})$. We call the set $\{\chi_i : B(0;\eta) \to V_i : i = 1, \ldots, s\}$ a G_2 *coordinate system near* $z_1, \ldots, z_s$.

We say that two G_2 coordinate systems near $z_1, \ldots, z_s$, with maps χ_i and $\tilde{\chi}_i$ for $i = 1, \ldots, s$ respectively, are *equivalent* if $d\tilde{\chi}_i|_0 = d\chi_i|_0 = \zeta_i$ for all i.

Although the family of G_2 coordinate systems near $z_1, \ldots, z_s$ is clearly infinite-dimensional, there are only finitely many equivalence classes, given by the number of possible sets $\{\zeta_1, \ldots, \zeta_s\}$. Moreover, the family of choices for each ζ_i is isomorphic to G_2.

Note. Definition 3.3 does not require the G_2 structure (φ, g) to be torsion-free.

Definition 3.4. Let (M, φ, g) be a G_2 manifold, let $N \subseteq M$ be compact and connected and let $z_1, \ldots, z_s \in N$ be distinct points. Let $\{\chi_i : B(0;\eta) \to V_i : i = 1, \ldots, s\}$ be a G_2 coordinate system near $z_1, \ldots, z_s$ as in Definition 3.3. We say that N is a 4-fold in M with *conical singularities at* $z_1, \ldots, z_s$ *with rate* λ, denoted a *CS 4-fold*, if $\hat{N} = N \setminus \{z_1, \ldots, z_s\}$ is a (nonsingular) 4-dimensional submanifold of M and there exist constants $0 < \epsilon < \eta$ and $\lambda > 1$, a compact 3-dimensional Riemannian submanifold (Σ_i, h_i) of $\mathcal{S}^6 \subseteq \mathbb{R}^7$, where h_i is the restriction of the round metric on $\mathcal{S}^6$ to Σ_i, an open set $U_i \ni z_i$ in N with $U_i \subseteq V_i$ and a smooth map $\Phi_i : (0,\epsilon) \times \Sigma_i \to B(0;\eta) \subseteq \mathbb{R}^7$, for $i = 1, \ldots, s$, such that $\Psi_i = \chi_i \circ \Phi_i : (0,\epsilon) \times \Sigma_i \to U_i \setminus \{z_i\}$ is a diffeomorphism, and Φ_i satisfies

$$\left|\nabla_i^j \big(\Phi_i(r_i, \sigma_i) - \iota_i(r_i, \sigma_i)\big)\right| = O\big(r_i^{\lambda - j}\big) \text{ for } j \in \mathbb{N} \quad \text{as } r_i \longrightarrow 0, \tag{3.2}$$

where $\iota_i(r_i, \sigma_i) = r_i \sigma_i \in B(0;\eta)$, ∇_i is the Levi–Civita connection of the cone metric $g_i = dr_i^2 + r_i^2 h_i$ on $C_i = (0,\infty) \times \Sigma_i$ coupled with partial differentiation on $\mathbb{R}^7$ and $|\cdot|$ is calculated with respect to g_i.

We call C_i the *cone* at the singularity z_i and Σ_i the *link* of the cone C_i. We may write N as the disjoint union $N = K \sqcup \bigsqcup_{i=1}^s U_i$, where K is compact.

If $\hat{N}$ is coassociative in M, we say that N is a *CS coassociative 4-fold*.

Suppose N is a CS 4-fold with rate λ in M and use the notation of Definition 3.4. The induced metric on $\hat{N}$, $g|_{\hat{N}}$, makes $\hat{N}$ into a Riemannian manifold. Moreover, it is clear from (3.2) that the maps Ψ_i satisfy (3.1) in Definition 3.1 with the same constant λ. Thus, N may be considered as a CS manifold with rate λ.

We now show that, if $\lambda \in (1,2)$, Definition 3.4 is independent of the choice of G_2 coordinate system near the singularities, up to equivalence. Suppose we have two equivalent coordinate systems defined using maps χ_i and $\tilde{\chi}_i$. These maps must agree up to second order since the zero and first-order behaviour of each is prescribed, as stated in Definition 3.3. Therefore, the transformed maps $\tilde{\Phi}_i$ corresponding to $\tilde{\chi}_i$, such that $\tilde{\Psi}_i = \tilde{\chi}_i \circ \tilde{\Phi}_i = \chi_i \circ \Phi_i = \Psi_i$, are defined by $\tilde{\Phi}_i = (\tilde{\chi}_i^{-1} \circ \chi_i) \circ \Phi_i$. Hence

$$\left|\nabla_i^j\big(\tilde{\Phi}_i(r_i,\sigma_i) - \Phi_i(r_i,\sigma_i)\big)\right| = O\big(r_i^{2-j}\big) \quad \text{for } j \in \mathbb{N} \quad \text{as } r_i \longrightarrow 0,$$

where ∇_i and $|\cdot|$ are calculated as in Definition 3.4. Thus, if $\lambda < 2$, the terms generated by the transformation of the G_2 coordinate system neither dominate nor are of equal magnitude to the $O\big(r_i^{\lambda-j}\big)$ terms given in (3.2).

We now make a definition which also depends only on equivalence classes of G_2 coordinate systems near the singularities.

Definition 3.5. Let N be a CS 4-fold at $z_1,\ldots,z_s$ in a G_2 manifold (M,φ,g). Use the notation of Definitions 3.3 and 3.4. For $i = 1,\ldots,s$ define a cone $\hat{C}_i$ in $T_{z_i}M$ by $\hat{C}_i = (\zeta_i \circ \iota_i)(C_i)$. We call $\hat{C}_i$ the *tangent cone* at z_i.

One can show that $\hat{C}_i$ is a tangent cone to N at z_i in the sense of *geometric measure theory* (see, for example, [4, p. 233]). We also have a straightforward result related to the tangent cones at singular points of CS coassociative 4-folds.

Proposition 3.6. *Let N be a CS coassociative 4-fold at $z_1,\ldots,z_s$ in a G_2 manifold (M,φ,g). The tangent cones at $z_1,\ldots,z_s$ are coassociative.*

Proof. Use the notation of Definitions 3.3 and 3.4. It is enough to show that $\iota_i(C_i)$ is coassociative in $\mathbb{R}^7$ for all i, since $\zeta_i\colon \mathbb{R}^7 \to T_{z_i}M$ is an isomorphism identifying (φ_0, g_0) with $(\varphi|_{T_{z_i}M}, g|_{T_{z_i}M})$. This is equivalent to the condition $\iota_i^*(\varphi_0) \equiv 0$ for $i = 1,\ldots,s$.

Note that $\varphi|_{\hat{N}} \equiv 0$ implies that, for all i, $\varphi|_{U_i \setminus \{z_i\}} \equiv 0$. Hence, $\Psi_i^*(\varphi) = \Phi_i^*\big(\chi_i^*(\varphi)\big)$ vanishes on C_i for all i. Using (3.2),

$$\big|\Phi_i^*\big(\chi_i^*(\varphi)\big) - \iota_i^*\big(\chi_i^*(\varphi)\big)\big| = O\big(r_i^{\lambda-1}\big) \text{ as } r_i \longrightarrow 0$$

for all i. Moreover,

$$\big|\iota_i^*\big(\chi_i^*(\varphi)\big) - \iota_i^*\big(\varphi_0\big)\big| = O(r_i) \text{ as } r_i \longrightarrow 0$$

since

$$\chi_i^*(\varphi) = \varphi_0 + O(r_i) \quad \text{and} \quad |\nabla \iota_i| = O(1) \text{ as } r_i \longrightarrow 0.$$

Therefore, because $\lambda > 1$,

$$|\iota_i^*(\varphi_0)| \longrightarrow 0 \text{ as } r_i \longrightarrow 0$$

for all i. As $T_{r_i\sigma_i}\iota_i(C_i) = T_{\sigma_i}\iota_i(C_i)$ for all $(r_i, \sigma_i) \in C_i$, $|\iota_i^*(\varphi_0)|$ is independent of r_i and thus vanishes for all i as required. □

4. Weighted Banach spaces

For this section, let N be an n-dimensional CS manifold and let $\hat{N}$ be its nonsingular part, as in Definition 3.1. Let g be the metric and let ∇ be the Levi–Civita connection on $\hat{N}$. Moreover, let ρ be a radius function on $\hat{N}$ as in Definition 3.2. We define *weighted* Banach spaces of forms on $\hat{N}$ as in [1, § 1], as well as the usual "unweighted" spaces.

We begin with Sobolev spaces.

Definition 4.1. Let $p \geq 1$ and let $k, m \in \mathbb{N}$ with $m \leq n$. The *Sobolev space* $L_k^p(\Lambda^m T^*\hat{N})$ is the set of m-forms ξ on $\hat{N}$ which are k times weakly differentiable and such that the norm

$$\|\xi\|_{L_k^p} = \left(\sum_{j=0}^{k} \int_{\hat{N}} |\nabla^j \xi|^p \, dV_g\right)^{1/p}$$

is finite. The normed vector space $L_k^p(\Lambda^m T^*\hat{N})$ is a Banach space for all $p \geq 1$ and $L_k^2(\Lambda^m T^*\hat{N})$ is a Hilbert space.

We introduce the space of m-forms

$$L_{k,\,\mathrm{loc}}^p(\Lambda^m T^*\hat{N}) = \{\xi\colon f\xi \in L_k^p(\Lambda^m T^*\hat{N}) \text{ for all } f \in C_{\mathrm{cs}}^\infty(\hat{N})\}$$

where $C_{\mathrm{cs}}^\infty(\hat{N})$ is the space of smooth functions on $\hat{N}$ with compact support.

Let $\mu \in \mathbb{R}$. The *weighted Sobolev space* $L^p_{k,\mu}(\Lambda^m T^*\hat{N})$ is the subspace of $L^p_{k,\mathrm{loc}}(\Lambda^m T^*\hat{N})$ such that the following norm is finite:

$$\|\xi\|_{L^p_{k,\mu}} = \left(\sum_{j=0}^{k} \int_{\hat{N}} |\rho^{j-\mu}\nabla^j \xi|^p \rho^{-n}\, dV_g\right)^{1/p}.$$

Then $L^p_{k,\mu}(\Lambda^m T^*\hat{N})$ is a Banach space and $L^2_{k,\mu}(\Lambda^m T^*\hat{N})$ is a Hilbert space.

Note. By comparing the respective norms, it is clear that $L^p = L^p_{0,-n/p}$. In particular,

$$L^2(\Lambda^m T^*\hat{N}) = L^2_{0,-\frac{n}{2}}(\Lambda^m T^*\hat{N}). \tag{4.1}$$

For the following two definitions, of weighted C^k and Hölder spaces, we take $C^k_{\mathrm{loc}}(\Lambda^m T^*\hat{N})$ to be the vector space of k times continuously differentiable m-forms. In the definition of Hölder spaces, we refer to the usual normed vector space $C^k(\Lambda^m T^*\hat{N})$ of k times continuously differentiable m-forms such that the following norm is finite:

$$\|\xi\|_{C^k} = \sum_{j=0}^{k} \sup_{\hat{N}} |\nabla^j \xi|.$$

Definition 4.2. Let $\mu \in \mathbb{R}$ and let $k, m \in \mathbb{N}$ with $m \leq n$. The *weighted C^k-space* $C^k_\mu(\Lambda^m T^*\hat{N})$ is the subspace of $C^k_{\mathrm{loc}}(\Lambda^m T^*\hat{N})$ such that the norm

$$\|\xi\|_{C^k_\mu} = \sum_{j=0}^{k} \sup_{\hat{N}} |\rho^{j-\mu}\nabla^j \xi|$$

is finite. We also define $C^\infty_\mu(\Lambda^m T^*\hat{N}) = \bigcap_{k\geq 0} C^k_\mu(\Lambda^m T^*\hat{N})$. Then $C^k_\mu(\Lambda^m T^*\hat{N})$ is a Banach space, but in general $C^\infty_\mu(\Lambda^m T^*\hat{N})$ is not.

Definition 4.3. Let V be a vector bundle on $\hat{N}$ endowed with Euclidean metrics on its fibres and a connection preserving these metrics. Let $d(x, y)$ be the geodesic distance between points $x, y \in \hat{N}$, let $a \in (0, 1)$ and let $k, m \in \mathbb{N}$

with $m \leq n$. Let

$$H = \{(x,y) \in \hat{N} \times \hat{N} \colon x \neq y,\ c_1\rho(x) \leq \rho(y) \leq c_2\rho(x) \text{ and } \\ \text{there exists a geodesic in } \hat{N} \text{ of length } d(x,y) \text{ from } x \text{ to } y\},$$

where $0 < c_1 < 1 < c_2$ are constant. A section s of V is *Hölder continuous* (with *exponent* a) if

$$[s]^a = \sup_{(x,y)\in H} \frac{|s(x) - s(y)|_V}{d(x,y)^a} < \infty.$$

We understand the quantity $|s(x) - s(y)|_V$ as follows. Given $(x,y) \in H$, there exists a geodesic γ of length $d(x,y)$ connecting x and y. Parallel translation along γ using the connection on V identifies the fibres over x and y and the metrics on them. Thus, with this identification, $|s(x) - s(y)|_V$ is well defined.

The *Hölder space* $C^{k,a}(\Lambda^m T^*\hat{N})$ is the set of $\xi \in C^k(\Lambda^m T^*\hat{N})$ such that $\nabla^k\xi$ is Hölder continuous (with exponent a) and the norm

$$\|\xi\|_{C^{k,a}} = \|\xi\|_{C^k} + [\nabla^k\xi]^a$$

is finite. The normed vector space $C^{k,a}(\Lambda^m T^*\hat{N})$ is a Banach space.

We also introduce the notation

$$C^{k,a}_{\mathrm{loc}}(\Lambda^m T^*\hat{N}) \\ = \{\xi \in C^k_{\mathrm{loc}}(\Lambda^m T^*\hat{N}) \colon f\xi \in C^{k,a}(\Lambda^m T^*\hat{N}) \text{ for all } f \in C^\infty_{\mathrm{cs}}(\hat{N})\}.$$

Let $\mu \in \mathbb{R}$. The *weighted Hölder space* $C^{k,a}_\mu(\Lambda^m T^*\hat{N})$ is the subspace of $C^{k,a}_{\mathrm{loc}}(\Lambda^m T^*\hat{N})$ such that the norm

$$\|\xi\|_{C^{k,a}_\mu} = \|\xi\|_{C^k_\mu} + [\xi]^{k,a}_\mu$$

is finite, where

$$[\xi]^{k,a}_\mu = [\rho^{k+a-\mu}\nabla^k\xi]^a.$$

Then $C^{k,a}_\mu(\Lambda^m T^*\hat{N})$ is a Banach space. It is clear that we have an embedding $C^{k,a}_\mu(\Lambda^m T^*\hat{N}) \hookrightarrow C^l_\mu(\Lambda^m T^*\hat{N})$ whenever $l \leq k$.

Remark. The set H in Definition 4.3 is introduced so that $[\xi]^{k,a}_\mu$ is well defined.

We shall need the analogue of the Sobolev embedding theorem for weighted spaces, which is adapted from [15, Lemma 7.2] and [1, Theorem 1.2].

Theorem 4.4 (Weighted Sobolev Embedding Theorem). *Let $p, q \geq 1$, $a \in (0,1)$, $\mu, \nu \in \mathbb{R}$ and $k, l, m \in \mathbb{N}$ with $m \leq n$.*

(a) *If $k \geq l$, $k - \frac{n}{p} \geq l - \frac{n}{q}$ and either $p \leq q$ and $\mu \geq \nu$, or $p > q$ and $\mu > \nu$, there is a continuous embedding $L^p_{k,\mu}(\Lambda^m T^*\hat{N}) \hookrightarrow L^q_{l,\nu}(\Lambda^m T^*\hat{N})$.*

(b) *If $k - \frac{n}{p} \geq l + a$, there is a continuous embedding $L^p_{k,\mu}(\Lambda^m T^*\hat{N}) \hookrightarrow C^{l,a}_{\mu}(\Lambda^m T^*\hat{N})$.*

Finally, we shall also require an implicit function theorem for Banach spaces, which follows immediately from [13, Chapter 6, Theorem 2.1].

Theorem 4.5 (Implicit Function Theorem). *Let X and Y be Banach spaces and let $W \subseteq X$ be an open neighbourhood of 0. Let $\mathcal{G}\colon W \to Y$ be a C^k map ($k \geq 1$) such that $\mathcal{G}(0) = 0$. Suppose further that $d\mathcal{G}|_0\colon X \to Y$ is surjective with kernel K such that $X = K \oplus A$ for some closed subspace A of X. There exist open sets $V \subseteq K$ and $V' \subseteq A$, both containing 0, with $V \times V' \subseteq W$, and a unique C^k map $\mathcal{V}\colon V \to V'$ such that*

$$\operatorname{Ker} \mathcal{G} \cap (V \times V') = \{(x, \mathcal{V}(x)) \,:\, x \in V\}$$

in $X = K \oplus A$.

5. The operator $d + d^*$

In this section we let N be a *4-dimensional* CS manifold and let $\hat{N}$ be as in Definition 3.1. We also let ρ be a radius function on $\hat{N}$ as in Definition 3.2. An essential part of our study is the use of the Fredholm and index theory for the elliptic operator $d + d^*$ acting from $\Lambda^2_+ T^*\hat{N} \oplus \Lambda^4 T^*\hat{N}$ to $\Lambda^3 T^*\hat{N}$, where we identify the image of d and d^* acting on self-dual 2-forms. We therefore consider

$$d + d^*\colon L^p_{k+1,\mu}(\Lambda^2_+ T^*\hat{N} \oplus \Lambda^4 T^*\hat{M}) \longrightarrow L^p_{k,\mu-1}(\Lambda^3 T^*\hat{N}), \tag{5.1}$$

where $p \geq 2$, $k \in \mathbb{N}$ and $\mu \in \mathbb{R}$.

5.1. Fredholm theory

Our first result follows from [15, Theorems 1.1 and 6.1].

Proposition 5.1. *There exists a countable discrete set $\mathcal{D} \subseteq \mathbb{R}$ such that* (5.1) *is Fredholm if and only if $\mu \notin \mathcal{D}$.*

Moreover, we can give an explicit description of $\mathcal{D}$ by similar arguments to [18, Lemma 6.1.2] and [16, pp. 13–14], which are both for *asymptotically conical* (AC) manifolds. The argument is long and technical, but the idea is that to use the formalism of [15] one needs to consider manifolds with *cylindrical* ends, so we associate to our operator (5.1) another which acts on cylinders. We then push down to a map acting on forms on the cross-sections of the cylinders and it is the Fredholm theory of this map which determines the set $\mathcal{D}$.

Recall the notation of Definition 3.1. If (r_i, σ_i) are the coordinates on $(0, \epsilon) \times \Sigma_i$, let $t_i = -\log r_i$, recalling that $\epsilon < \eta < 1$. Thus (t_i, σ_i) are *cylindrical* coordinates on $(T, \infty) \times \Sigma_i$, where $T = -\log \epsilon > 0$. Transform the metric on $\hat{N}$ to a conformally equivalent metric which is asymptotically cylindrical on the ends $\hat{N}_\infty \cong \bigsqcup_{i=1}^s (T, \infty) \times \Sigma_i$ of $\hat{N}$; that is, the metric on $\hat{N}$ is asymptotic to $dt_i^2 + h_i$ on the ith end. With respect to this new metric, the asymptotically cylindrical operator corresponding to $d + d^*$ acts as

$$\rho^{-m} d\rho^m + \rho^{-m+2} d^* \rho^m$$

on m-forms on $\hat{N}$. We refer the interested reader to [14, Proposition and Definition 4.4] to explain the introduction of these powers of ρ. Since ρ is asymptotic to $r_i = e^{-t_i}$, we have a cylindrical operator $(d + d^*)_\infty$ associated to $d + d^*$ which acts on m-forms on each component of $\hat{N}_\infty \cong \bigsqcup_{i=1}^s (T, \infty) \times \Sigma_i$ by

$$(d + d^*)_\infty = e^{mt_i}(d + e^{-2t_i} d^*) e^{-mt_i}.$$

Notice that an m-form ξ on $\hat{N}_\infty$ can be written on the ith end as

$$\xi(t_i, \sigma_i) = \Omega_i(t_i, \sigma_i) + dt \wedge \omega_i(t_i, \sigma_i),$$

where for each $t_i \in (T, \infty)$, $\Omega_i(t_i, \sigma_i)$ and $\omega_i(t_i, \sigma_i)$ are m- and $(m-1)$-forms on Σ_i respectively. Thus, if $p_i \colon (0, \infty) \times \Sigma_i \to \Sigma_i$ is the natural projection map,

$$\Lambda^m T^* \hat{N}_\infty \cong \bigsqcup_{i=1}^s \big(p_i^*(\Lambda^m T^* \Sigma_i) \oplus p_i^*(\Lambda^{m-1} T^* \Sigma_i) \big).$$

Hence, $(d+d^*)_\infty$ maps sections of $p_i^*(\Lambda^2 T^*\Sigma_i) \oplus p_i^*(\Lambda^{\mathrm{odd}} T^*\Sigma_i)$ to sections of $p_i^*(\Lambda^{\mathrm{odd}} T^*\Sigma_i) \oplus p_i^*(\Lambda^{\mathrm{even}} T^*\Sigma_i)$. Moreover, this action is given by:

$$\begin{aligned}(d+d^*)_\infty &\begin{pmatrix} \alpha_i(t_i,\sigma_i) \\ \gamma_i(t_i,\sigma_i)+\beta_i(t_i,\sigma_i)\end{pmatrix} \\ &= \begin{pmatrix} d_i+d_i^* & -\frac{\partial}{\partial t_i}+3-m \\ \frac{\partial}{\partial t_i}-m & -(d_i+d_i^*)\end{pmatrix}\begin{pmatrix} \alpha_i(t_i,\sigma_i) \\ \gamma_i(t_i,\sigma_i)+\beta_i(t_i,\sigma_i)\end{pmatrix},\end{aligned} \tag{5.2}$$

where d_i and d_i^* are the exterior derivative and its adjoint on Σ_i, and m denotes the operator which multiplies m-forms by a factor m.

However, we wish to restrict attention to forms on Σ_i which correspond, via p_i^*, to self-dual 2-forms on $\hat{N}_\infty$ rather than general 2-forms. So, we define $S_i \subseteq \Lambda^2 T^*\Sigma_i \oplus \Lambda^{\mathrm{odd}} T^*\Sigma_i$ by

$$S_i = \{(\alpha_i, *_i\alpha_i+\beta_i)\colon \alpha_i \in \Lambda^2 T^*\Sigma_i,\ \beta_i \in \Lambda^3 T^*\Sigma_i\},$$

where $*_i$ is the Hodge star on Σ_i. Then $\bigsqcup_{i=1}^s p_i^*(S_i) \cong \Lambda^2_+ T^*\hat{N}_\infty \oplus \Lambda^4 T^*\hat{N}_\infty$.

For $w \in \mathbb{C}$, define a map $(d+d^*)_\infty(w)$ which acts on sections of $S_i \otimes \mathbb{C}$ by:

$$(d+d^*)_\infty(w) = \begin{pmatrix} d_i+d_i^* & w+3-m \\ -w-m & -(d_i+d_i^*)\end{pmatrix}. \tag{5.3}$$

Notice that we have formally substituted w for $-\frac{\partial}{\partial t_i}$ in (5.2).

Let

$$T_i = \{(*_i\alpha_i+\beta_i, \alpha_i)\colon \alpha_i \in \Lambda^2 T^*\Sigma_i,\ \beta_i \in \Lambda^3 T^*\Sigma_i\},$$

so that $\bigsqcup_{i=1}^s p_i^*(T_i) \cong \Lambda^3 T^*\hat{N}_\infty$. Define $\mathcal{C} \subseteq \mathbb{C}$ as the set of w for which

$$(d+d^*)_\infty(w)\colon L^p_{k+1}(S_i\otimes\mathbb{C}) \to L^p_k(T_i\otimes\mathbb{C}),$$

where we include a projection to T_i, is *not* an isomorphism for any i. By the proof of [15, Theorem 1.1], $\mathcal{D} = \{\operatorname{Re} w : w \in \mathcal{C}\}$. By [18, Lemma 6.1.13], the corresponding sets $\mathcal{C}(\Delta^m)$, where Δ^m is the Laplacian on m-forms, are all real for an AC manifold. Since the same will be true for the CS case, we deduce that $\mathcal{C} \subseteq \mathbb{R}$. Hence $\mathcal{C} = \mathcal{D}$.

The symbol, hence the index ind_w, of $(d+d^*)_\infty(w)$ is independent of w. Furthermore, $(d+d^*)_\infty(w)$ is an isomorphism for generic values of w since

$\mathcal{D}$ is countable and discrete. Therefore $\mathrm{ind}_w = 0$ for all $w \in \mathbb{C}$; that is,

$$\dim \operatorname{Ker}(d+d^*)_\infty(w) = \dim \operatorname{Coker}(d+d^*)_\infty(w),$$

so that (5.3) is not an isomorphism precisely when it is not injective.

The condition $(d+d^*)_\infty(w) = 0$, using (5.3), corresponds to the existence of $\alpha_i \in C^\infty(\Lambda^2 T^*\Sigma_i)$ and $\beta_i \in C^\infty(\Lambda^3 T^*\Sigma_i)$, for some i, satisfying

$$d_i\alpha_i = -w\beta_i \quad \text{and} \quad d_i *_i \alpha_i + d_i^*\beta_i = -(w+2)\alpha_i. \tag{5.4}$$

Before we state our proposition, we make a few observations concerning $\mathcal{D}$.

Notes.

(a) Clearly, (5.4) implies that $d_i d_i^* \beta_i = \Delta_i \beta_i = w(w+2)\beta_i$. Since eigenvalues of the Laplacian on Σ_i must necessarily be positive, $\beta_i = 0$ if $w \in (-2, 0)$.

(b) If $w = 0$ and we take $\alpha_i = 0$, (5.4) forces β_i to be coclosed. As there are nontrivial coclosed 3-forms on Σ_i, $(d+d^*)_\infty(0)$ is not injective, so $0 \in \mathcal{D}$.

(c) Suppose that $w = -2 \in \mathcal{D}$. Then (5.4) gives $[\beta_i] = 0$ in $H^3_{\mathrm{dR}}(\Sigma_i)$. Using (a), β_i is harmonic so, by Hodge theory, $\beta_i = 0$. Therefore $-2 \in \mathcal{D}$ if and only if there exists a nonzero closed and coclosed 2-form on Σ_i for some i.

Proposition 5.2. *Use the notation of Definition* 3.1 *and the discussion above. For* $i = 1, \ldots, s$ *let*

$$\begin{aligned} D(\mu, i) = \{(\alpha_i, \beta_i) \in C^\infty(\Lambda^2 T^*\Sigma_i \oplus \Lambda^3 T^*\Sigma_i) \colon d_i\alpha_i = -\mu\beta_i, \\ d_i *_i \alpha_i + d_i^*\beta_i = -(\mu+2)\alpha_i\}. \end{aligned}$$

The set $\mathcal{D}$ *of real numbers* μ *such that* (5.1) *is not Fredholm is given by:*

$$\mathcal{D} = \bigcup_{i=1}^{s} \{\mu \in \mathbb{R} \colon D(\mu, i) \neq 0\}.$$

Remark. A perhaps more illuminating way to characterise $D(\mu, i)$ is by:

$$(\alpha_i, \beta_i) \in D(\mu, i) \iff \xi_i = (r_i^{\mu+2}\alpha_i + r_i^{\mu+1} dr_i \wedge *\alpha_i, r_i^{\mu+3} dr_i \wedge \beta_i)$$
$$\text{is an } O(r_i^{\mu}) \text{ solution of } (d + d^*)\xi_i = 0 \text{ on } C_i \text{ as } r_i \longrightarrow 0,$$

using the notation of Definition 3.1.

Lockhart and McOwen [15, §10] study the Laplacian on m-forms on a manifold with a conical singularity. From this work, which can easily be extended to manifolds with more than one singularity, we can make an important observation about the set $\mathcal{D}$.

Proposition 5.3. *In the notation of Proposition* 5.2, $\mathcal{D} \cap (-2, -1] = \emptyset$.

Proof. Let $\mu \in (-2, -1]$ and let

$$\Delta^m \colon L^p_{k+1,\,\mu}(\Lambda^m T^* \hat{N}) \longrightarrow L^p_{k-1,\,\mu-2}(\Lambda^m T^* \hat{N})$$

be the Laplacian on m-forms and denote the set of μ such that it is not Fredholm by $\mathcal{D}(\Delta^m)$. Since $\mu > -2$ and $p \geq 2$, we see that $L^p_{k+1,\,\mu} \hookrightarrow L^2_{0,\,-2} = L^2$ by Theorem 4.4 and (4.1).

Applying [15, Theorem 10.2] to Δ^2 and Δ^4, we see that

$$\mathcal{D}(\Delta^2) \cap (-2, -1] = \mathcal{D}(\Delta^4) \cap (-2, -1] = \emptyset.$$

Note that our rate μ is related to the weighting factor ν in [15, §10] by $\mu = -\nu - 2$. Since it is clear that $\mathcal{D} \subseteq \big(\mathcal{D}(\Delta^2) \cup \mathcal{D}(\Delta^4)\big)$, the result follows. $\square$

5.2. Index theory

We begin with some definitions following [15].

Definition 5.4. Use the notation of §5.1. Let $\mu \in \mathcal{D}$. Define $d(\mu)$ to be the dimension of the vector space of solutions of $(d + d^*)_\infty \xi = 0$ of the form

$$\xi(t_i, \sigma_i) = e^{-\mu t_i} q_i\,(t_i, \sigma_i)$$

on $(T, \infty) \times \Sigma_i$, for $i = 1, \ldots, s$, where $q_i\,(t_i, \sigma_i)$ is a polynomial in t_i with coefficients in $C^\infty(S_i \otimes \mathbb{C})$.

The next result is immediate from [15, Theorem 1.2].

Theorem 5.5. *Use the notation of Proposition* 5.1 *and Definition* 5.4. *Let* $\lambda, \lambda' \notin \mathcal{D}$ *with* $\lambda' \leq \lambda$. *If* $\operatorname{ind}_\mu(d+d^*)$ *denotes the index of* (5.1) *for* $\mu \notin \mathcal{D}$,

$$\operatorname{ind}_{\lambda'}(d+d^*) - \operatorname{ind}_\lambda(d+d^*) = \sum_{\mu \in \mathcal{D} \cap (\lambda', \lambda)} d(\mu).$$

We make an observation which shall be useful in later sections.

Proposition 5.6. *Let* $\lambda, \lambda' \in \mathbb{R}$ *such that* $\lambda' \leq \lambda$ *and* $[\lambda', \lambda] \cap \mathcal{D} = \emptyset$. *The kernels, and cokernels, of* (5.1) *when* $\mu = \lambda$ *and* $\mu = \lambda'$ *are equal.*

Proof. Denote the dimensions of the kernel and cokernel of (5.1), for $\mu \notin \mathcal{D}$, by $k(\mu)$ and $c(\mu)$ respectively. Since $[\lambda', \lambda] \cap \mathcal{D} = \emptyset$, $k(\lambda) - c(\lambda) = k(\lambda') - c(\lambda')$, so

$$k(\lambda) - k(\lambda') = c(\lambda) - c(\lambda'). \tag{5.5}$$

We know that $k(\lambda) \leq k(\lambda')$ because $L^p_{k+1,\lambda} \hookrightarrow L^p_{k+1,\lambda'}$ by Theorem 4.4 as $\lambda \geq \lambda'$. Similarly, since $c(\mu)$ is equal to the dimension of the kernel of the formal adjoint operator acting on a Sobolev space with weight $-3-\mu = -4-(\mu-1)$, $c(\lambda) \geq c(\lambda')$. Noting that the right-hand side of (5.5) is non-negative and the left-hand side is less than or equal to zero, we conclude that both must be zero. The result follows from the fact that the kernel of $d+d^*$ in $L^p_{k+1,\lambda}$ is contained in the kernel of $d+d^*$ in $L^p_{k+1,\lambda'}$, and vice versa for the cokernels. □

We conclude by relating $d(\mu)$ and $D(\mu, i)$ in a result analogous to [16, Proposition 5.4], and which can be proved in exactly the same manner.

Proposition 5.7. *Using the notation of Proposition* 5.2 *and Definition* 5.4, $d(\mu) = \sum_{i=1}^s \dim D(\mu, i)$ *for* $\mu \in \mathcal{D}$.

6. The deformation problems

We have a common notation for the next three sections. Let N be a CS coassociative 4-fold at $z_1, \ldots, z_s$ with rate λ in a G_2 manifold (M, φ, g). Suppose $\lambda \in (1,2) \setminus \mathcal{D}$, where $\mathcal{D}$ is given by Proposition 5.2, and the cone at z_i is C_i with link Σ_i. We shall then use the notation of Definitions 3.4 and 3.5. In particular, we let $\{\chi_i \colon B(0;\eta) \to V_i \colon i = 1, \ldots, s\}$, with $d\chi_i|_0 = \zeta_i$ for all i, be the G_2 coordinate system near $z_1, \ldots, z_s$ used to define N and we let $\hat{C}_i$ be the tangent cone at z_i. We also write $N = K \sqcup \bigsqcup_{i=1}^s U_i$, where

K is compact and U_i is an open set in N containing z_i. Moreover, we let ρ be a radius function on $\hat{N}$ as in Definition 3.2 and choose the smooth maps $\Phi_i\colon (0,\epsilon)\times\Sigma_i \to B(0;\eta)$ uniquely by imposing the condition

$$\Phi_i(r_i,\sigma_i) - \iota_i(r_i,\sigma_i) \in \big(T_{r_i\sigma_i}\iota_i(C_i)\big)^{\perp} \text{ for all } (r_i,\sigma_i)\in(0,\epsilon)\times\Sigma_i, \tag{6.1}$$

where $\iota_i(r_i,\sigma_i) = r_i\sigma_i$ is the inclusion map on C_i. The condition on Φ_i can always be achieved by making ϵ and the sets U_i smaller. We shall endeavour to remind the reader when we use specific elements of this notation.

In this section, we consider deformations of N which are CS coassociative 4-folds at s points with rate λ in (M,φ,g) with the same cones at the singularities as N, but the singularities need not be at the same points, nor have identical tangent cone. We also, eventually, allow variations in the G_2 structure on the ambient 7-manifold M, and include deformations of N which may be coassociative with respect to a nearby G_2 structure.

We tackle this deformation theory by splitting it into three moduli space problems, each one allowing for more possible deformations of N. We do this to help the reader because each time the notation becomes more cluttered and the analysis more complicated, but most of the important ideas are clear from the solution of the first problem.

6.1. Problem 1: fixed singularities and G_2 structure

The first problem we study is where the deformations of N have identical singular points to N with the same rate, cones and tangent cones, and the G_2 structure on M is fixed. We give a formal definition for this moduli space.

Definition 6.1. The *moduli space of deformations* $\mathcal{M}_1(N,\lambda)$ for Problem 1 is the set of N' in (M,φ,g) which are CS coassociative 4-folds at $z_1,\dots,z_s$ with rate λ, having cone C_i and tangent cone $\hat{C}_i$ at z_i for all i, such that there exists a diffeomorphism $h\colon M\to M$, isotopic to the identity, such that $h(z_i)=z_i$ for $i=1,\dots,s$, $h|_N\colon N\to N'$ is a homeomorphism and $h|_{\hat{N}}\colon \hat{N}\to N'\setminus\{z_1,\dots,z_s\}$ is a diffeomorphism.

We begin our formulation of a local description of $\mathcal{M}_1(N,\lambda)$ with a result which is immediate from the proof of [12, Chapter IV, Theorem 9] since M is a Riemannian manifold.

Theorem 6.2. *Let P be a closed embedded submanifold of M. There exist an open subset V of the normal bundle $\nu(P)$ of P in M, containing the zero*

section, and an open set S in M containing P, such that the exponential map $\exp|_V : V \to S$ *is a diffeomorphism.*

Note. The proof of this result relies entirely on the observation that $\exp|_{\nu(P)}$ is a local isomorphism upon the zero section.

This information provides us with a useful corollary to Theorem 6.2.

Corollary 6.3. *Let $P_i = \iota_i\big((0,\epsilon)\times\Sigma_i\big)$, $Q_i = \Phi_i\big((0,\epsilon)\times\Sigma_i\big)$ and define $n_i\colon \nu(P_i) \to \mathbb{R}^7$ by $n_i(r_i\sigma_i, v) = v + \Phi_i(r_i,\sigma_i)$. For all i, there exist an open subset $\hat{V}_i$ of $\nu(P_i)$ in $\mathbb{R}^7$, containing the zero section, and an open set $\hat{S}_i$ in $B(0;\eta) \subseteq \mathbb{R}^7$ containing Q_i such that $n_i|_{\hat{V}_i} : \hat{V}_i \to \hat{S}_i$ is a diffeomorphism. Moreover, $\hat{V}_i$ and $\hat{S}_i$ can be chosen such that $P_i \subseteq \hat{S}_i$ and $\hat{V}_i$ is an open set in C_1^1.*

Proof. Note that n_i takes the zero section of $\nu(P_i)$ to Q_i. By the definition of Φ_i, n_i is a local isomorphism upon the zero section. Thus, the proof of Theorem 6.2 gives open sets $\hat{V}_i$ and $\hat{S}_i$ such that $n_i|_{\hat{V}_i} : \hat{V}_i \to \hat{S}_i$ is a diffeomorphism. We can ensure that $\hat{S}_i$ lies in $B(0;\eta)$ by making $\hat{V}_i$ smaller if necessary.

Furthermore, since $\Phi_i - \iota_i$ is orthogonal to P_i by (6.1), it can be identified with a small section of $\nu(P_i)$. Hence P_i lies in $\hat{S}_i$ as long as $\hat{S}_i$ grows at $O(r_i)$ as $r_i \to 0$. As we can form $\hat{S}_i$ and $\hat{V}_i$ in a translation equivariant way because we are working on a portion of the cone C_i, we can construct our sets with this decay rate as $r_i \to 0$ and such that they do not collapse as $r_i \to \epsilon$. Thus, we can ensure that $\hat{V}_i$ is an open set in C_1^1 as claimed. $\square$

The reason for using Corollary 6.3, rather than simply considering exponential normal deformations as given by Theorem 6.2, is that it gives us control of the behaviour of the sets $\hat{V}_i$ and $\hat{S}_i$ near 0. This is essential in making our analysis valid.

By Proposition 2.7 and Theorem 6.2, we can think of nearby deformations of N as graphs of small self-dual 2-forms. The next proposition gives a formal backing to this concept.

Proposition 6.4. *There exist an open set $\hat{U} \subseteq \Lambda^2_+ T^*\hat{N}$ containing the zero section, an open set $\hat{T} \subseteq M$ containing $\hat{N}$ and a diffeomorphism $\delta\colon \hat{U} \to \hat{T}$ which takes the zero section to $\hat{N}$. Moreover, $\hat{U}$ can be chosen to be an open set in C_1^1 and δ is compatible with the identifications $(0,\epsilon)\times\Sigma_i \overset{\Psi_i}{\cong} U_i \setminus \{z_i\}$ and the isomorphism $\nu(\hat{N}) \overset{\mathcal{J}}{\cong} \Lambda^2_+ T^*\hat{N}$ given by Proposition 2.7.*

Proof. Use the notation from the start of this section and of Corollary 6.3. Define $\hat{T}_i = \chi_i(\hat{S}_i)$. Then $\hat{T}_i$ is an open set in M such that $U_i \setminus \{z_i\} \subseteq \hat{T}_i \subseteq V_i$, since $Q_i \subseteq \hat{S}_i$ by Corollary 6.3 and $\chi_i(Q_i) = U_i \setminus \{z_i\}$.

Consider the bundle $(\Lambda^2_+)_{\chi_i^*(g)}T^*\big((0,\epsilon)\times\Sigma_i\big)$, where the notation $(\Lambda^2_+)_h$ indicates that the Hodge star is calculated using the metric h, and we identify $(0,\epsilon)\times\Sigma_i$ with $P_i \subseteq \mathbb{R}^7$ using the inclusion map ι_i. Then

$$\begin{aligned} \jmath_i\colon \nu(P_i) &\longrightarrow (\Lambda^2_+)_{\chi_i^*(g)}T^*P_i \\ v|_{r_i\sigma_i} &\longmapsto \big(v|_{r_i\sigma_i}\cdot\chi_i^*(\varphi)|_{\Phi_i(r_i,\sigma_i)}\big)\big|_{T_{r_i\sigma_i}P_i} \end{aligned}$$

is an isomorphism because $U_i \setminus \{z_i\}$ is coassociative and thus P_i is, with respect to $\big(\chi_i^*(\varphi), \chi_i^*(g)\big)$, and hence we may apply Proposition 2.7. Note also that

$$\Psi_i^*\colon (\Lambda^2_+)_g T^*(U_i\setminus\{z_i\}) \longrightarrow (\Lambda^2_+)_{\chi_i^*(g)}T^*\big((0,\epsilon)\times\Sigma_i\big)$$

is clearly a diffeomorphism. Therefore, let $\hat{U}_i \subseteq (\Lambda^2_+)_g T^*(U_i\setminus\{z_i\})$ be such that $\Psi_i^*(\hat{U}_i) = \jmath_i(\hat{V}_i)$. Note that $\hat{U}_i$ is an open set in C_1^1 since $\hat{V}_i$ is by Corollary 6.3.

Define a diffeomorphism $\delta_i\colon \hat{U}_i \to \hat{T}_i$ such that the following commutes:

$$\begin{tikzcd} \hat{U}_i \arrow[r, "\Psi_i^*"] \arrow[dd, "\delta_i"'] & \jmath_i(\hat{V}_i) \arrow[d, "\jmath_i^{-1}"] \\ & \hat{V}_i \arrow[d, "n_i"] \\ \hat{T}_i & \hat{S}_i. \arrow[l, "\chi_i"'] \end{tikzcd} \tag{6.2}$$

Interpolating smoothly over K, we extend $\bigcup_{i=1}^s \hat{U}_i$ and $\bigcup_{i=1}^s \hat{T}_i$ to $\hat{U}$ and $\hat{T}$ as required and extend the diffeomorphisms δ_i smoothly to a diffeomorphism $\delta\colon \hat{U} \to \hat{T}$ such that δ acts as the identity on $\hat{N}$, which is identified with the zero section in $\Lambda^2_+T^*\hat{N}$.

Note that we have a splitting $T\hat{U}|_{(x,0)} = T_x\hat{N} \oplus \Lambda^2_+T^*_x\hat{N}$ for all $x \in \hat{N}$. Thus we can consider $d\delta$ at $\hat{N}$ as a map from $T\hat{N} \oplus \Lambda^2_+T^*\hat{N}$ to $T\hat{N} \oplus \nu(\hat{N}) \cong TM|_{\hat{N}}$. Hence, we require in our extension of δ from δ_i to ensure that, in matrix notation,

$$d\delta|_{\hat{N}} = \begin{pmatrix} I & A \\ 0 & \jmath^{-1} \end{pmatrix}, \tag{6.3}$$

where I is the identity and A is arbitrary. This can be achieved because of the definition of δ_i.

The compatibility of δ with $\jmath$ and Ψ_i mentioned in the statement of the proposition, is given by (6.2) and the behaviour of $d\delta|_{\hat{N}}$ stipulated in (6.3). $\square$

Note. The open set $\hat{U}$ will be used frequently throughout the rest of the article.

We now define our deformation map for Problem 1. Let

$$C^k_{\mathrm{loc}}(\hat{U}) = \{\alpha \in C^k_{\mathrm{loc}}(\Lambda^2_+ T^*\hat{N}) \colon \Gamma_\alpha \subseteq \hat{U}\},$$

where $\hat{U}$ is given in Proposition 6.4 and Γ_α is the graph of α. We also adopt similar notation to define subsets of the spaces of forms described in § 4, though we must be careful that the forms are continuous so that their graphs are well-defined. Moreover, we notice that the subset will be open whenever the Banach space embeds continuously in C^1_1.

Definition 6.5. Use the notation of Proposition 6.4. Let Γ_α be the graph of $\alpha \in C^1_{\mathrm{loc}}(\hat{U})$ and let $\pi_\alpha \colon \hat{N} \to \Gamma_\alpha$ be given by $\pi_\alpha(x) = (x, \alpha(x))$. Let $f_\alpha = \delta \circ \pi_\alpha$ and let $\hat{N}_\alpha = f_\alpha(\hat{N}) \subseteq \hat{T}$. Define a map F_1 from $C^1_{\mathrm{loc}}(\hat{U})$ to $C^0_{\mathrm{loc}}(\Lambda^3 T^*\hat{N})$ by:

$$F_1(\alpha) = f_\alpha^* \left(\varphi|_{\hat{N}_\alpha} \right).$$

By Proposition 2.6, $\operatorname{Ker} F_1$ is the set of $\alpha \in C^1_{\mathrm{loc}}(\hat{U})$ such that $\hat{N}_\alpha$ is coassociative. Using [17, p. 731], which we are allowed to do by our choice of δ, the linearization of F_1 at 0 acts on $\alpha \in C^1_{\mathrm{loc}}(\Lambda^2_+ T^*\hat{N})$ as

$$dF_1|_0(\alpha) = L_1(\alpha) = d\alpha.$$

Remark. The operator L_1 is *not* elliptic.

We want to study CS coassociative deformations $N_\alpha = \hat{N}_\alpha \sqcup \{z_1, \ldots, z_s\}$ with singularities at the same points as N with the same tangent cones, so we restrict our choices of forms $\alpha \in C^1_{\mathrm{loc}}(\hat{U})$. Since $\hat{N}_\alpha$ is supposed to be smooth and nonsingular, $\alpha \in C^\infty(\hat{U})$.

As N_α is a CS coassociative 4-fold, there exist smooth maps $(\Phi_\alpha)_i \colon (0, \epsilon) \times \Sigma_i \to B(0; \eta)$ satisfying (3.2) such that $(\Psi_\alpha)_i = \chi_i \circ (\Phi_\alpha)_i$ is a diffeomorphism onto an open subset of $\hat{N}_\alpha$ for all i as in Definition 3.4. We are free to use χ_i because the tangent cones at the singularities of N_α must be

the same as for N, so any G_2 coordinate system near the singularities used to define N_α must be equivalent to the one given by χ_i for $i = 1, \dots, s$. Choose $(\Phi_\alpha)_i$ uniquely so as to satisfy a similar orthogonality condition to (6.1).

Use the notation of Corollary 6.3 and the proof of Proposition 6.4. Recall that we identify $(0,\epsilon) \times \Sigma_i$ with P_i using ι_i. By (6.1), $\Phi_i - \iota_i$ can be identified using $\jmath_i$ with the graph of $\beta_i \in C^\infty\big((\Lambda^2_+)_{\chi_i^*(g)} T^*\big((0,\epsilon) \times \Sigma_i\big)\big)$. Thus, if ∇_i is the Levi–Civita connection of the cone metric on C_i,

$$|\nabla_i^j \beta_i| = O(r_i^{\lambda - j}) \qquad \text{for } j \in \mathbb{N} \quad \text{as } r_i \longrightarrow 0 \tag{6.4}$$

by (3.2), and therefore $\beta_i \in C^\infty_\lambda\big((\Lambda^2_+)_{\chi_i^*(g)} T^*\big((0,\epsilon) \times \Sigma_i\big)\big)$.

We may similarly deduce, by the definition of δ, Φ_i and $(\Phi_\alpha)_i$, that $(\Phi_\alpha)_i - \iota_i = \big((\Phi_\alpha)_i - \Phi_i\big) + (\Phi_i - \iota_i)$ corresponds to the graph of $\Psi_i^*(\alpha) + \beta_i$ on $(0,\epsilon) \times \Sigma_i$, recalling that

$$\Psi_i^* \colon \Lambda^2_+ T^*(U_i \setminus \{z_i\}) \to (\Lambda^2_+)_{\chi_i^*(g)} T^*\big((0,\epsilon) \times \Sigma_i\big)$$

is a diffeomorphism for all i. Since N_α has the same types of singularities as N, both β_i and $\Psi_i^*(\alpha) + \beta_i$ lie in $C^\infty_\lambda\big((\Lambda^2_+)_{\chi_i^*(g)} T^*\big((0,\epsilon) \times \Sigma_i\big)\big)$ for each i. Thus α must lie in $C^\infty_\lambda(\Lambda^2_+ T^* \hat{N})$.

We conclude that $\hat{N}_\alpha$ is a deformation of $\hat{N}$ in $\hat{T}$ with the same conical singularities if and only if $\alpha \in C^\infty_\lambda(\hat{U})$. We state this as a proposition.

Proposition 6.6. *In the notation of Definitions* 6.1 *and* 6.5, $\mathcal{M}_1(N,\lambda)$ *is locally homeomorphic to* $\operatorname{Ker} F_1 = \{\alpha \in C^\infty_\lambda(\hat{U}) \colon F_1(\alpha) = 0\}$.

We define an associated map G_1 to F_1 which is a nonlinear elliptic operator at zero. This will be useful to prove regularity results.

Definition 6.7. Define $G_1 \colon C^1_{\mathrm{loc}}(\hat{U}) \times C^1_{\mathrm{loc}}(\Lambda^4 T^* \hat{N}) \to C^0_{\mathrm{loc}}(\Lambda^3 T^* \hat{N})$ by:

$$G_1(\alpha, \beta) = F_1(\alpha) + d^* \beta.$$

This is a nonlinear first-order *elliptic* operator at $(0,0)$ since its linearization there acts on $(\alpha, \beta) \in C^1_{\mathrm{loc}}(\Lambda^2_+ T^* \hat{N} \oplus \Lambda^4 T^* \hat{N})$ as

$$dG_1|_{(0,0)}(\alpha, \beta) = d\alpha + d^* \beta.$$

Note. If $G_1(\alpha, \beta) = 0$ and $\beta \in C^\infty_\lambda(\Lambda^4 T^* \hat{N})$, $*\beta$ is a harmonic function which decays with order $O(\rho^\lambda)$ as $\rho \to 0$. Since $\lambda > 1$, $*\beta \to 0$ as $\rho \to 0$ and hence, by the maximum principle for harmonic functions, it must be 0.

From the note, we deduce the following.

Proposition 6.8. $\operatorname{Ker} F_1 \cong \{(\alpha,\beta) \in C^\infty_\lambda(\hat{U}) \times C^\infty_\lambda(\Lambda^4 T^*\hat{N}) \colon G_1(\alpha,\beta) = 0\}$.

We conclude this section by stating and proving two results on *regularity* which are analogous to [16, Proposition 4.3] and [16, Proposition 4.14], respectively. The proof of the first is long and technical, but works by relating F_1 to functions on the cones C_i, then using scale equivariance properties of these new functions to prove regularity results using their restrictions to the links Σ_i.

Proposition 6.9. *The map F_1 given in Definition* 6.5 *can be written as*

$$F_1(\alpha)(x) = d\alpha(x) + P_{F_1}\big(x, \alpha(x), \nabla\alpha(x)\big) \tag{6.5}$$

for $x \in \hat{N}$, where $P_{F_1}\colon \{(x,y,z)\colon (x,y) \in \hat{U},\ z \in T^*_x\hat{N} \otimes \Lambda^2_+ T^*_x\hat{N}\} \to \Lambda^3 T^*\hat{N}$ is a smooth map such that $P_{F_1}(x,y,z) \in \Lambda^3 T^*_x\hat{N}$. Denote $P_{F_1}(x,\alpha(x), \nabla\alpha(x))$ by $P_{F_1}(\alpha)(x)$ for convenience.

Let $\alpha \in C^1_{\mathrm{loc}}(\hat{U})$ with $\|\alpha\|_{C^1_1}$ sufficiently small. If $\alpha \in C^\infty_\lambda(\hat{U})$, $P_{F_1}(\alpha) \in C^\infty_{2\lambda-2}(\Lambda^3 T^*\hat{N})$. Moreover, for each $k \in \mathbb{N}$, if $\alpha \in C^{k+1}_\lambda(\hat{U})$, then $P_{F_1}(\alpha) \in C^k_{2\lambda-2}(\Lambda^3 T^*\hat{N})$ and there exists a constant $c_k > 0$ such that

$$\|P_{F_1}(\alpha)\|_{C^k_{2\lambda-2}} \le c_k \|\alpha\|^2_{C^{k+1}_\lambda}. \tag{6.6}$$

Note. For each $k \in \mathbb{N}$, $C^k_{2\lambda-2} \hookrightarrow C^k_{\lambda-1}$ as $\lambda > 1$.

Proof. First, by the definition of F_1, $F_1(\alpha)(x)$ relates to the tangent space to Γ_α at $\pi_\alpha(x)$. Note that $T_{\pi_\alpha(x)}\Gamma_\alpha$ depends on both $\alpha(x)$ and $\nabla\alpha(x)$ and hence so must $F_1(\alpha)(x)$. We may then define P_{F_1} by (6.5) such that it is a smooth function of its arguments as claimed.

Recall the notation from the start of this section and that $(0,\epsilon) \times \Sigma_i \overset{\iota_i}{\cong} P_i \subseteq \mathbb{R}^7$, where ι_i is the inclusion map. We argued before Proposition 6.6 that we may identify $\Phi_i - \iota_i$ on $(0,\epsilon) \times \Sigma_i$ with

$$\beta_i \in C^\infty_\lambda\big((\Lambda^2_+)_{\chi_i^*(g)} T^*\big((0,\epsilon) \times \Sigma_i\big)\big)$$

for $i = 1, \ldots, s$. Recall that

$$\Psi_i^* \colon \Lambda^2_+ T^*(U_i \setminus \{z_i\}) \longrightarrow (\Lambda^2_+)_{\chi_i^*(g)} T^*\big((0,\epsilon) \times \Sigma_i\big)$$

is a diffeomorphism. Let $\alpha \in C^1_\lambda(\hat{U})$, $\alpha_i = \alpha|_{U_i \setminus \{z_i\}}$ and $\gamma_i = \Psi_i^*(\alpha_i)$.

For each i, define a function $F_{C_i}(\gamma_i+\beta_i)$ on $(0,\epsilon)\times\Sigma_i$ by

$$F_{C_i}(\gamma_i+\beta_i)(r_i,\sigma_i)=F_1(\alpha_i)\big(\Psi_i(r_i,\sigma_i)\big). \tag{6.7}$$

Further, define a smooth function P_{C_i} by an equation analogous to (6.5):

$$\begin{aligned} F_{C_i}(\gamma_i+\beta_i)(r_i,\sigma_i) &= d(\gamma_i+\beta_i)(r_i,\sigma_i)\\ &\quad + P_{C_i}\big((r_i,\sigma_i),(\gamma_i+\beta_i)(r_i,\sigma_i),\nabla(\gamma_i+\beta_i)(r_i,\sigma_i)\big). \end{aligned} \tag{6.8}$$

We notice that F_{C_i} and P_{C_i} are only dependent on the cone C_i and, rather trivially, on ϵ. Therefore, because of this fact and our choice of δ in Proposition 6.4, these functions have scale equivariance properties. We may therefore derive equations and inequalities on $\{\epsilon\}\times\Sigma_i$ and deduce the result on all of $(0,\epsilon)\times\Sigma_i$ by introducing appropriate scaling factors of r_i.

Now, since $\alpha=0$ corresponds to our coassociative 4-fold $\hat{N}$, $F_1(0)=0$. Adopting similar notation for $P_{C_i}(\beta_i)$ as for $P_{F_1}(\alpha_i)$, we see that

$$F_{C_i}(\beta_i)=d\beta_i+P_{C_i}(\beta_i)=0, \tag{6.9}$$

by (6.7). Using (6.5) to (6.9), we deduce that

$$\begin{aligned} P_{F_1}(\alpha_i) &= d\beta_i+P_{C_i}(\gamma_i+\beta_i)=d\beta_i+P_{C_i}(\gamma_i+\beta_i)-\big(d\beta_i+P_{C_i}(\beta_i)\big)\\ &= P_{C_i}(\gamma_i+\beta_i)-P_{C_i}(\beta_i). \end{aligned} \tag{6.10}$$

We then calculate

$$\begin{aligned} P_{C_i}(\gamma_i+\beta_i)-P_{C_i}(\beta_i) &= \int_0^1 \frac{d}{dt}\,P_{C_i}(t\gamma_i+\beta_i)\,dt\\ &= \int_0^1 \gamma_i\cdot\frac{\partial P_{C_i}}{\partial y}(t\gamma_i+\beta_i)+\nabla_i\gamma_i\cdot\frac{\partial P_{C_i}}{\partial z}(t\gamma_i+\beta_i)\,dt, \end{aligned} \tag{6.11}$$

recalling that P_{C_i} is a function of three variables x, y and z and that ∇_i is the Levi–Civita connection of the cone metric g_i on C_i. Using Taylor's Theorem,

$$\begin{aligned} &P_{C_i}(\gamma_i+\beta_i)\\ &\quad = P_{C_i}(\beta_i)+\gamma_i\cdot\frac{\partial P_{C_i}}{\partial y}(\beta_i)+\nabla_i\gamma_i\cdot\frac{\partial P_{C_i}}{\partial z}(\beta_i)+O(r_i^{-2}|\gamma_i|^2+|\nabla_i\gamma_i|^2) \end{aligned} \tag{6.12}$$

when $|\gamma_i|$ and $|\nabla_i\gamma_i|$ are small.

Since $dF_1|_0(\alpha_i)=d\alpha_i$, $dF_{C_i}|_{\beta_i}(\gamma_i+\beta_i)=d\gamma_i$ and hence $dP_{C_i}|_{\beta_i}=0$. Thus, the first derivatives of P_{C_i} with respect to y and z must vanish at

β_i by (6.12). Therefore, given small $\nu > 0$, there exists a constant $A_0 > 0$ such that

$$\begin{aligned} &\left|\frac{\partial P_{C_i}}{\partial y}(t\gamma_i + \beta_i)\right| \leq A_0(r_i^{-2}|\gamma_i| + r_i^{-1}|\nabla_i\gamma_i|), \text{ and} \\ &\left|\frac{\partial P_{C_i}}{\partial z}(t\gamma_i + \beta_i)\right| \leq A_0(r_i^{-1}|\gamma_i| + |\nabla_i\gamma_i|) \end{aligned} \tag{6.13}$$

for $t \in [0,1]$ whenever

$$r_i^{-1}|\gamma_i|, \quad r_i^{-1}|\beta_i|, \quad |\nabla_i\gamma_i| \quad \text{and} \quad |\nabla_i\beta_i| \leq \nu. \tag{6.14}$$

We determine the appropriate factors of r_i by considering the scaling properties of P_{C_i} and γ_i under changes in r_i.

By (6.4), $r_i^{-1}|\beta_i|$ and $|\nabla_i\beta_i|$ tend to zero as $r_i \to 0$. We can thus ensure that (6.14) is satisfied by the β_i components by making ϵ smaller. Hence, (6.14) holds if $\|\gamma_i\|_{C_1^1} \leq \nu$. Therefore, putting estimates (6.13) in (6.11) and using (6.10),

$$|P_{F_1}(\alpha_i)| = |P_{C_i}(\gamma_i + \beta_i) - P_{C_i}(\beta_i)| \leq A_0(r_i^{-1}|\gamma_i| + |\nabla_i\gamma_i|)^2 \tag{6.15}$$

whenever $\|\gamma_i\|_{C_1^1} \leq \nu$. As $r_i \to 0$ the terms in the bracket on the right-hand side of (6.15) are of order $O(r_i^{\lambda-1})$ by (6.4). Thus, $|P_{F_1}(\alpha_i)|$ is of order $O(r_i^{2\lambda-2})$ as $r_i \to 0$ for $i = 1, \ldots, s$. We deduce that $|P_{F_1}(\alpha)|$ is of order $O(\rho^{2\lambda-2})$ as $\rho \to 0$ if $\|\alpha\|_{C_1^1}$ sufficiently small. Moreover, there exists a constant c_0 such that

$$\sup_{\hat{N}} |\rho^{2-2\lambda} P_{F_1}(\alpha)| \leq c_0 \left(\sum_{j=0}^{1} \sup_{\hat{N}} |\rho^{j-\lambda}\nabla^j\alpha| \right)^2;$$

that is, (6.6) holds for $k = 0$.

We now consider the first derivative of P_{F_1} and, in doing so, provide the method to tackle higher derivatives as well. Suppose $\alpha \in C_\lambda^2(\hat{U})$ and $\|\alpha\|_{C_1^1}$ is sufficiently small that $\|\gamma_i\|_{C_1^1} \leq \nu$ for all i. From (6.11) we calculate

$$\begin{aligned} &\nabla_i\big(P_{C_i}(\gamma_i + \beta_i) - P_{C_i}(\beta_i)\big) \\ &\quad = \int_0^1 \nabla_i\left(\gamma_i \cdot \frac{\partial P_{C_i}}{\partial y}(t\gamma_i + \beta_i) + \nabla_i\gamma_i \cdot \frac{\partial P_{C_i}}{\partial z}(t\gamma_i + \beta_i)\right) dt \end{aligned}$$

$$= \int_0^1 \nabla_i \gamma_i \cdot \frac{\partial P_{C_i}}{\partial y} + \gamma_i \cdot \left(\nabla_i (t\gamma_i + \beta_i) \cdot \frac{\partial^2 P_{C_i}}{\partial y^2} + \nabla_i^2 (t\gamma_i + \beta_i) \cdot \frac{\partial^2 P_{C_i}}{\partial y \partial z} \right)$$
$$+ \nabla_i^2 \gamma_i \cdot \frac{\partial P_{C_i}}{\partial z} + \nabla_i \gamma_i \cdot \left(\nabla_i (t\gamma_i + \beta_i) \cdot \frac{\partial^2 P_{C_i}}{\partial z \partial y} + \nabla_i^2 (t\gamma_i + \beta_i) \cdot \frac{\partial^2 P_{C_i}}{\partial z^2} \right) dt.$$

There exists a constant $A_1 > 0$ such that (6.13) holds with A_0 replaced by A_1 and, for $t \in [0,1]$,

$$\left| \frac{\partial^2 P_{C_i}}{\partial y^2} (t\gamma_i + \beta_i) \right|, \quad \left| \frac{\partial^2 P_{C_i}}{\partial y \partial z} (t\gamma_i + \beta_i) \right| \quad \text{and} \quad \left| \frac{\partial^2 P_{C_i}}{\partial z^2} (t\gamma_i + \beta_i) \right| \leq A_1,$$

since the second derivatives of P_{C_i} are continuous functions defined on the closed bounded set given by $\|\gamma_i\|_{C_1^1} \leq \nu$. We deduce that

$$\left| \nabla \big(P_{F_1}(\alpha_i) \big) \right| = \left| \nabla_i \big(P_{C_i}(\gamma_i + \beta_i) - P_{C_i}(\beta_i) \big) \right| \leq A_1 r_i \left(\sum_{j=0}^{2} r_i^{j-2} |\nabla_i^j \gamma_i| \right)^2.$$

Therefore, $\left| \nabla \big(P_{F_1}(\alpha_i) \big) \right|$ is of order $O(r_i^{2\lambda-3})$ as $r_i \to 0$, and (6.6) holds for $k = 1$.

In general, if $\alpha \in C_\lambda^{k+1}(\hat{U})$ with $\|\alpha\|_{C_1^1}$ sufficiently small, we have

$$\left| \nabla^k \big(P_{F_1}(\alpha_i) \big) \right| \leq A_k r_i^k \left(\sum_{j=0}^{k+1} r_i^{j-(k+1)} |\nabla_i^j \gamma_i| \right)^2$$

for some $A_k > 0$. The result follows. □

We now consider the regularity of solutions to the nonlinear elliptic equation $G_1(\alpha, \beta) = 0$ near $(0,0)$. Our result is proved using a standard "bootstrap" argument.

Proposition 6.10. *Let $(\alpha, \beta) \in L^p_{k+1,\lambda}(\hat{U}) \times L^p_{k+1,\lambda}(\Lambda^4 T^* \hat{N})$ for some $p > 4$ and $k \geq 2$. If $G_1(\alpha, \beta) = 0$ and $\|\alpha\|_{C_1^1}$ is sufficiently small, where G_1 is given by Definition 6.7, then $(\alpha, \beta) \in C_\lambda^\infty(\hat{U}) \times C_\lambda^\infty(\Lambda^4 T^* \hat{N})$.*

Notes.

(a) The conditions $p > 4$ and $k \geq 2$ ensure, by Theorem 4.4, that $L^p_{k+1,\lambda} \hookrightarrow C^2_\lambda \hookrightarrow C^1_1$. We need to control at least the first two derivatives of forms in the kernel of G_1 since, although G_1 is a first-order operator, the proof uses a related second-order elliptic operator.

(b) Since $\hat{U}$ is an open subset of C^1_1 and we are free to make $\hat{U}$ smaller if necessary, we can always ensure that $\alpha \in L^p_{k+1,\,\lambda}(\hat{U})$ has small enough C^1_1 norm for Proposition 6.10 to apply.

Proof. Since F_1 smoothly depends on α and $\nabla\alpha$, G_1 is a smooth function of $\alpha, \beta, \nabla\alpha$ and $\nabla\beta$. We apply [20, Theorem 6.8.1], which is a general regularity result for C^1_{loc} solutions of nonlinear elliptic equations, to conclude that α and β are smooth. However, we want more than this: the derivatives of α and β must decay at the required rates.

Recall the note after Definition 6.7, which shows that $G_1(\alpha,\beta) = 0$ implies that $\beta = 0$. Thus $\beta \in C^\infty_\lambda(\Lambda^4 T^*\hat{N})$ trivially.

For the following argument, we find it useful to work with weighted Hölder spaces. By Theorem 4.4, $\alpha \in C^{k,\,a}_\lambda(\hat{U})$ with $a = 1 - 4/p \in (0,1)$ since $p > 4$. We also know that $d^*\big(G_1(\alpha,\beta)\big) = d^*\big(F_1(\alpha)\big) = 0$. Hence, if $\pi_{\Lambda^2_+}$ is the projection from 2-forms to their self-dual part,

$$\tilde{F}_1(\alpha) = \pi_{\Lambda^2_+}\Big(d^*\big(F_1(\alpha)\big)\Big) = 0$$

is a nonlinear elliptic equation at 0, since $dF_1|_0(\alpha) = d\alpha$.

As $\tilde{F}_1(\alpha)$ is linear in the second derivative of α, we see that

$$\tilde{F}_1(\alpha)(x) = R\big(x, \alpha(x), \nabla\alpha(x)\big)\nabla^2\alpha(x) + E\big(x, \alpha(x), \nabla\alpha(x)\big),$$

where $R\big(x, \alpha(x), \nabla\alpha(x)\big)$ and $E\big(x, \alpha(x), \nabla\alpha(x)\big)$ are smooth functions of their arguments. Define

$$S_\alpha(\gamma)(x) = R\big(x, \alpha(x), \nabla\alpha(x)\big)\nabla^2\gamma(x)$$

for $\gamma \in C^2_{\mathrm{loc}}(\Lambda^2_+ T^*\hat{N})$. Then S_α is a smooth, *linear*, *elliptic*, second-order operator, whose coefficients depend on x, $\alpha(x)$ and $\nabla\alpha(x)$. These coefficients therefore lie in $C^{k-1,\,a}_{\mathrm{loc}}$. We also notice that

$$S_\alpha(\alpha)(x) = -E\big(x, \alpha(x), \nabla\alpha(x)\big) \in C^{k-2,\,a}_{2\lambda-3}(\Lambda^2_+ T^*\hat{N}) \subseteq C^{k-2,\,a}_{\lambda-2}(\Lambda^2_+ T^*\hat{N}),$$

using the fact that $d^*\big(P_F(\alpha)\big) \in C^{k-2,\,a}_{2\lambda-3}$ by Proposition 6.9 and $\lambda > 1$. However, $E\big(x, \alpha(x), \nabla\alpha(x)\big)$ only depends on α and $\nabla\alpha$ and is at worst quadratic in these quantities by Proposition 6.9, so it must in fact lie in $C^{k-1,\,a}_{\lambda-2}$ since we are given control on the decay of the first k derivatives of α near the singularities of N.

In [19], a regularity result is given for linear elliptic operators acting between weighted Hölder spaces, whose coefficients locally lie in a suitable

Hölder space. This result can also be found in [16, Theorem 4.12]. We deduce that, if $\gamma \in C^2_\lambda$ and $S_\alpha(\gamma) \in C^{k-1,\,a}_{\lambda-2}$, then $\gamma \in C^{k+1,\,a}_\lambda$. Since $k \geq 2$, α and $S_\alpha(\alpha)$ satisfy these conditions by the discussion above. Thus $\alpha \in C^{k+1,\,a}_\lambda$ only knowing a priori that $\alpha \in C^{k,\,a}_\lambda$. We proceed by induction to show that $\alpha \in C^{l,\,a}_\lambda$ for all $l \geq k$. $\square$

6.2. Problem 2: moving singularities and fixed G_2 structure

For this problem, we again consider deformations of N in (M, φ, g) which are CS coassociative 4-folds at s points with the same rate and cones at the singularities, but now we allow the singular points and tangent cones at those points to differ from those of N. However, we still assume that the G_2 structure on M is fixed. We again define the moduli space formally and recall the notation introduced at the start of the section.

Definition 6.11. The *moduli space of deformations* $\mathcal{M}_2(N, \lambda)$ for Problem 2 is the set of N' in (M, φ, g) which are CS coassociative 4-folds at $z'_1, \ldots, z'_s$ with rate λ, having cone C_i and tangent cone $\hat{C}'_i$ at z'_i for all i, such that there exists a diffeomorphism $h : M \to M$, isotopic to the identity, such that $h(z_i) = z'_i$ for $i = 1, \ldots, s$, $h|_N \colon N \to N'$ is a homeomorphism and $h|_{\hat{N}} \colon \hat{N} \to N' \setminus \{z'_1, \ldots, z'_s\}$ is a diffeomorphism.

Here it is more difficult to create a local description of the moduli space which is compatible with our analytic framework. What one would consider "intuitive" approaches do not, as far as the author is aware, bear fruit. We therefore follow what is, at first sight, a slightly indirect route.

Let $\{B_i \colon i = 1, \ldots, s\}$ be a collection of pairwise disjoint open sets in M such that $z_i \in B_i$. Let $B = \prod_{i=1}^s B_i$. For each $\mathbf{z}' = (z'_1, \ldots, z'_s) \in B$, we have a family $I(\mathbf{z}')$ of choices of s-tuples $\boldsymbol{\zeta}' = (\zeta'_1, \ldots, \zeta'_s)$ of isomorphisms $\zeta'_i \colon \mathbb{R}^7 \to T_{z'_i}M$ identifying (φ_0, g_0) with $(\varphi|_{T_{z'_i}M}, g|_{T_{z'_i}M})$. Clearly, for each $\mathbf{z}' \in B$, $I(\mathbf{z}') \cong \mathrm{G}_2^s$. We thus make the following definition.

Definition 6.12. The *translation space* is

$$\mathcal{T} = \{(\mathbf{z}', \boldsymbol{\zeta}') \colon \mathbf{z}' \in B,\ \boldsymbol{\zeta}' \in I(\mathbf{z}')\}.$$

It is a principal G_2^s bundle over B and hence is a smooth manifold.

We now deal with the issue that the cone $\iota_i(C_i)$ in $\mathbb{R}^7$ may have G_2 symmetries, which means that different choices of $\boldsymbol{\zeta}'$ could give the same set of corresponding tangent cones. Let H_i denote the Lie subgroup of G_2

preserving $\iota_i(C_i)$ in $\mathbb{R}^7$ and let $\mathrm{H} = \prod_{i=1}^s \mathrm{H}_i \subseteq \mathrm{G}_2^s$. Then H acts freely on $\mathcal{T}$ by

$$(\mathbf{z}', \boldsymbol{\zeta}') \longmapsto \big(\mathbf{z}', (\zeta_1' \circ A_1^{-1}, \ldots, \zeta_s' \circ A_s^{-1})\big),$$

where $(A_1, \ldots, A_s) \in \mathrm{H}$. Thus there exists an H-orbit through $(\mathbf{z}, \boldsymbol{\zeta})$ in $\mathcal{T}$, where

$$\mathbf{z} = (z_1, \ldots, z_s) \quad \text{and} \quad \boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_s).$$

Define $\hat{\mathcal{T}}$ to be a small open ball in $\mathbb{R}^n$ containing 0, where $n = \dim \mathcal{T} - \dim \mathrm{H}$, and let $h_{\hat{\mathcal{T}}} \colon \hat{\mathcal{T}} \to \mathcal{T}$ be an embedding with $h_{\hat{\mathcal{T}}}(0) = (\mathbf{z}, \boldsymbol{\zeta})$ such that $h_{\hat{\mathcal{T}}}(\hat{\mathcal{T}})$ is transverse to the H-orbit through $(\mathbf{z}, \boldsymbol{\zeta})$. Write $h_{\hat{\mathcal{T}}}(t) = \big(\mathbf{z}(t), \boldsymbol{\zeta}(t)\big)$ for $t \in \hat{\mathcal{T}}$, with $\mathbf{z}(0) = \mathbf{z}$ and $\boldsymbol{\zeta}(0) = \boldsymbol{\zeta}$.

We shall use the notation from this definition throughout the rest of the article.

Notes.

(a) If $t, t' \in \hat{\mathcal{T}}$, with $t \neq t'$, are such that $\mathbf{z}(t) = \mathbf{z}(t')$, the s-tuples of tangent cones, $\{\hat{C}_1(t), \ldots, \hat{C}_s(t)\}$ and $\{\hat{C}_1(t'), \ldots, \hat{C}_s(t')\}$, are distinct.

(b) $\hat{\mathcal{T}}$ is an open ball in $\mathbb{R}^n \cong T_0\hat{\mathcal{T}}$ and hence can be considered as an open subset of $T_0\hat{\mathcal{T}}$.

We use $\hat{\mathcal{T}}$ to extend N to a family of nearby CS 4-folds and provide an analogue to Proposition 6.4 for Problem 2. In defining N, we chose a G_2 coordinate system $\{\chi_i \colon B(0;\eta) \to V_i \colon i = 1, \ldots, s\}$ with $d\chi_i|_0 = \zeta_i$ for $i = 1, \ldots, s$. Extend this to a smooth family of G_2 coordinate systems

$$\Big\{\{\chi_i(t) : B(0;\eta) \to V_i(t) : i = 1, \ldots, s\} : t \in \hat{\mathcal{T}}\Big\},$$

where $V_i(t)$ is an open set in M containing $z_i(t)$, $\chi_i(t)(0) = z_i(t)$, $d\chi_i(t)|_0 = \zeta_i(t)$, $\chi_i(0) = \chi_i$ and $V_i(0) = V_i$ for $i = 1, \ldots, s$.

Proposition 6.13. *Use the notation of Proposition* 6.4 *and Definition* 6.12.

(a) *There exists a family $\mathcal{N} = \{N(t) \colon t \in \hat{\mathcal{T}}\}$ of CS 4-folds in M, with $N(0) = N$, such that $N(t)$ has a singularity at $z_i(t)$, for $i = 1, \ldots, s$, with rate λ, cone C_i and tangent cone $\hat{C}_i(t) = \big(\zeta_i(t) \circ \iota_i\big)(C_i)$.*

(b) *Let $\hat{N}(t) = N(t) \setminus \{z_1(t), \ldots, z_s(t)\}$ and write $N(t) = K(t) \sqcup \bigsqcup_{i=1}^s U_i(t)$, where $K(t)$ is compact and $U_i(t) \setminus \{z_i(t)\} \cong (0, \epsilon) \times \Sigma_i$ for all i, in the obvious way, ensuring that $K(0) = K$ and $U_i(0) = U_i$.*

For $t \in \hat{\mathcal{T}}$, there exist open sets $\hat{T}(t) \subseteq M$ containing $\hat{N}(t)$ and diffeomorphisms $\delta(t)\colon \hat{U} \to \hat{T}(t)$ taking the zero section to $\hat{N}(t)$, varying smoothly in t, with $\hat{T}(0) = \hat{T}$ and $\delta(0) = \delta$. Moreover, $\delta(t)$ is compatible with the identifications $U_i(t) \setminus \{z_i(t)\} \cong (0, \epsilon) \times \Sigma_i$ for all i.

Remark. $\mathcal{N}$ does *not* necessarily consist of CS *coassociative* 4-folds and $\delta(t)$ is not required to be compatible with the isomorphism $\nu(\hat{N}) \cong \Lambda^2_+ T^* \hat{N}$ for $t \neq 0$.

Proof. Use the notation from Corollary 6.3 and the proof of Proposition 6.4. For $t \in \hat{\mathcal{T}}$, define $\hat{T}_i(t) = \chi_i(t)(\hat{S}_i)$ and

$$U_i(t) = \Big(\chi_i(t) \circ \Phi_i\big((0, \epsilon) \times \Sigma_i\big)\Big) \cup \{z_i(t)\}$$

for $i = 1, \ldots, s$. Then $\hat{T}_i(t)$ contains $U_i(t) \setminus \{z_i(t)\}$. Define a diffeomorphism $\delta_i(t)$ such that the following diagram commutes:

$$\begin{tikzcd}
\hat{U}_i \arrow[r, "\Psi_i^*"] \arrow[d, "\delta_i(t)"'] & \jmath_i(\hat{V}_i) \arrow[d, "\jmath_i^{-1}"] \\
 & \hat{V}_i \arrow[d, "n_i"] \\
\hat{T}_i(t) & \hat{S}_i. \arrow[l, "\chi_i(t)"']
\end{tikzcd} \tag{6.16}$$

Interpolating smoothly over K, we extend $\bigcup_{i=1}^s \hat{T}_i(t)$ to $\hat{T}(t)$ and $\delta_i(t)$ to $\delta(t)$.

Let $e(t) = \delta(t)|_{\hat{N}}$ and define $\hat{N}(t) = e(t)(\hat{N})$. Then $e(t)\colon \hat{N} \to \hat{N}(t)$ is a diffeomorphism for all $t \in \hat{\mathcal{T}}$ and $e(0)$ is the identity. Let $N(t) = \hat{N}(t) \cup \{z_1(t), \ldots, z_s(t)\}$. We then have a family $\mathcal{N} = \{N(t)\colon t \in \hat{\mathcal{T}}\}$ as claimed. Note that $K(t) = e(t)(K)$.

By the construction of $\delta(t)$ and the family $\mathcal{N}$, it is clear that the proposition is proved, where the compatibility conditions on $\delta(t)$ are given by (6.16). $\square$

We can now define the deformation map for Problem 2.

Definition 6.14. Use the notation of Definition 6.5 and Proposition 6.13. For $t \in \hat{\mathcal{T}}$, let $f_\alpha(t) = \delta(t) \circ \pi_\alpha$, $\hat{N}_\alpha(t) = f_\alpha(t)(\hat{N})$ and $N_\alpha(t) = \hat{N}_\alpha(t) \cup$

$\{z_1(t), \ldots, z_s(t)\}$. Define F_2 from $C^1_{\mathrm{loc}}(\hat{U}) \times \hat{\mathcal{T}}$ to $C^0_{\mathrm{loc}}(\Lambda^3 T^* \hat{N})$ by:

$$F_2(\alpha, t) = f_\alpha(t)^* \left(\varphi|_{\hat{N}_\alpha(t)} \right).$$

The linearization of F_2 at $(0,0)$ acts as

$$dF_2|_{(0,0)} \colon (\alpha, t) \longmapsto d\alpha + L_2(t),$$

where $\alpha \in C^1_{\mathrm{loc}}(\Lambda^2_+ T^* \hat{N})$, $t \in T_0 \hat{\mathcal{T}}$ and L_2 is a linear map into the space of smooth exact 3-forms on $\hat{N}$ since φ is exact near $\hat{N}$.

Remark. By construction $F_2(\alpha, 0) = F_1(\alpha)$, where F_1 is given in Definition 6.5.

Clearly, $\operatorname{Ker} F_2$ is the set of $\alpha \in C^1_{\mathrm{loc}}(\hat{U})$ and $t \in \hat{\mathcal{T}}$ such that $\hat{N}_\alpha(t)$ is coassociative. However, we have not yet encoded the information that $N_\alpha(t)$ is CS with *rate* λ. This is the subject of the next proposition.

Proposition 6.15. *In the notation of Definitions* 6.11 *and* 6.14, $\mathcal{M}_2(N, \lambda)$ *is locally homeomorphic to* $\operatorname{Ker} F_2 = \{(\alpha, t) \in C^\infty_\lambda(\hat{U}) \times \hat{\mathcal{T}} \colon F_2(\alpha, t) = 0\}$.

Proof. For each $t \in \hat{\mathcal{T}}$, we are in the situation of Problem 1 in the sense that we want coassociative deformations $\hat{N}_\alpha(t)$ of $\hat{N}(t)$, defined by a self-dual 2-form α, which have the same singular points, cones and tangent cones as $\hat{N}(t)$. It is thus clear that $\alpha \in C^\infty_\lambda(\hat{U})$ by Proposition 6.6. □

We now introduce an associated map G_2 to F_2.

Definition 6.16. Use the notation of Definition 6.14. Define $G_2 \colon C^1_{\mathrm{loc}}(\hat{U}) \times C^1_{\mathrm{loc}}(\Lambda^4 T^* \hat{N}) \times \hat{\mathcal{T}} \to C^0_{\mathrm{loc}}(\Lambda^3 T^* \hat{N})$ by

$$G_2(\alpha, \beta, t) = F_2(\alpha, t) + d^* \beta,$$

so that $dG_2|_{(0,0,0)} \colon (\alpha, \beta, t) \longmapsto d\alpha + d^*\beta + L_2(t)$.

We then have an analogous result to Proposition 6.8, which follows in exactly the same fashion because $F_2(\alpha, t)$ is exact.

Proposition 6.17.

$$\operatorname{Ker} F_2 \cong \{(\alpha, \beta, t) \in C^\infty_\lambda(\hat{U}) \times C^\infty_\lambda(\Lambda^4 T^* \hat{N}) \times \hat{\mathcal{T}} \colon G_2(\alpha, \beta, t) = 0\}.$$

The next result studies the regularity of the kernel of G_2 near $(0,0,0)$ and is the analogue of Proposition 6.10.

Proposition 6.18. *Let* $(\alpha,\beta,t) \in L^p_{k+1,\lambda}(\hat{U}) \times L^p_{k+1,\lambda}(\Lambda^4 T^*\hat{N}) \times \hat{\mathcal{T}}$, *where* $p > 4$ *and* $k \geq 2$. *If* $G_2(\alpha,\beta,t) = 0$, *where* G_2 *is given in Definition* 6.16, *and* $\|\alpha\|_{C^1_1}$ *and* t *are sufficiently small, then* $(\alpha,\beta) \in C^\infty_\lambda(\hat{U}) \times C^\infty_\lambda(\Lambda^4 T^*\hat{N})$.

Note. We can always ensure that t is sufficiently small by shrinking $\hat{\mathcal{T}}$.

Proof. Recall the proof of Proposition 6.10 and Definition 6.16.

First, $d\big(G_2(\alpha,\beta,t)\big) = \Delta\beta = 0$ since $F_2(\alpha,t)$ is exact. Thus, $\beta = 0$ by the maximum principle, so it is trivially smooth.

Note that $d^*\big(G_2(\alpha,\beta,t)\big) = d^*\big(F_2(\alpha,t)\big) = 0$. Hence,

$$\tilde{F}_2(\alpha,t) = \pi_{\Lambda^2_+}\Big(d^*\big(F_2(\alpha,t)\big)\Big) = 0$$

is a nonlinear equation on α which is linear in $\nabla^2\alpha$. We can thus write

$$\tilde{F}_2(\alpha,t)(x) = R_t\big(x,\alpha(x),\nabla\alpha(x)\big)\nabla^2\alpha(x) + E_t\big(x,\alpha(x),\nabla\alpha(x)\big),$$

where R_t and E_t are smooth functions of their arguments. If we define

$$S_{(\alpha,t)}(\gamma)(x) = R_t\big(x,\alpha(x),\nabla\alpha(x)\big)\nabla^2\gamma(x),$$

then $S_{(\alpha,t)}$ is a linear differential operator acting on $\gamma \in C^2_{\mathrm{loc}}(\Lambda^2_+ T^*\hat{N})$, with coefficients which lie in $C^{k-1,a}_{\mathrm{loc}}$. The ellipticity of $S_\alpha = S_{(\alpha,0)}$ results from the coassociativity of $\hat{N}$. Ellipticity is an *open* condition so, although $\hat{N}(t)$ is not necessarily coassociative, the fact that it is "close" to being coassociative means that $S_{(\alpha,t)}$ is elliptic, as long as t is sufficiently small.

Since $F_2(\alpha,t)$ depends smoothly on t and $\hat{N}(t)$ is asymptotically coassociative near the singular points, we can apply the elliptic regularity theory from [19] to $S_{(\alpha,t)}$ as in the proof of Proposition 6.10. Moreover, as $E_0 = E$ maps into $C^{k-1,a}_{\lambda-2}$ and F_2 varies smoothly with t, E_t maps into $C^{k-1,a}_{\lambda-2}$ for t sufficiently small. We can thus follow the proof of Proposition 6.10 to give the result. $\square$

6.3. Problem 3: moving singularities and varying G_2 structure

For our final problem, we consider CS deformations N' of N as in Problem 2, except now we allow N' to be coassociative under a deformation of the G_2 structure on the ambient 7-manifold M.

Before we can define the moduli space, we need to choose a suitable family of nearby G_2 structures to (φ, g) on M. To achieve this end, we discuss the third cohomology of a tubular neighbourhood of N. Recall the notation from the start of this section and denote by $H^m_{\mathrm{cs}}(\hat{N})$ the mth compactly supported cohomology of $\hat{N}$; that is, the quotient of closed compactly supported m-forms by the derivatives of compactly supported $(m-1)$-forms.

Proposition 6.19. *Use the notation of Proposition* 6.4. *Let*

$$T = \hat{T} \cup \bigcup_{i=1}^{s} V_i \supseteq N.$$

By making $\hat{T}$ and V_i, for $i = 1, \ldots, s$, smaller if necessary, T retracts onto N. There exists an isomorphism $\Xi\colon H^3_{\mathrm{dR}}(T) \to H^3_{\mathrm{cs}}(\hat{N})$.

Proof. Let $[\xi] \in H^3_{\mathrm{dR}}(T)$ and use the notation from the start of this section. Since V_i retracts onto $\{z_i\}$ for $i = 1, \ldots, s$, ξ can be chosen such that $\xi|_{V_i} = 0$. Therefore, $\xi|_{U_i \setminus \{z_i\}} = 0$ which implies that the support of $\xi|_{\hat{N}}$ is contained in K, which is compact. Hence $[\xi|_{\hat{N}}]$ is a well-defined element of $H^3_{\mathrm{cs}}(\hat{N})$.

Define Ξ by $[\xi] \mapsto [\xi|_{\hat{N}}]$. We show that Ξ is well defined. Suppose that $\xi' = \xi + dv$, for $v \in C^\infty(\Lambda^2 T^*T)$, such that $\xi'|_{V_i} = 0$ for all i. Then $dv|_{V_i} = 0$ for all i. Since V_i retracts onto $\{z_i\}$, we can choose v such that $v|_{V_i} = 0$ without affecting dv by smoothly interpolating over $\hat{T}$. Thus, $v|_{\hat{N}}$ is compactly supported and $\xi|_{\hat{N}} + d(v|_{\hat{N}}) = \xi'|_{\hat{N}}$. Hence Ξ is well-defined and injective.

Any closed form on $\hat{N}$ with support in K can be extended smoothly to a closed form on T which vanishes on V_i for all i. Thus, any cohomology class in $H^3_{\mathrm{cs}}(\hat{N})$ has a representative γ that can be lifted to a form ξ on T such that $\Xi([\xi]) = [\gamma]$, which implies that Ξ is surjective. $\square$

Notes. The reason for this result is 2-fold.

(a) The condition $\Xi([\varphi|_T]) = 0$ in $H^3_{\mathrm{cs}}(\hat{N})$ is implied by the coassociativity of $\hat{N}$ and it forces $[\varphi|_{\hat{N}}] = 0$ in $H^3_{\mathrm{cs}}(\hat{N})$. This is stronger than the seemingly more natural condition of $[\varphi|_{\hat{N}}] = 0$ in $H^3_{\mathrm{dR}}(\hat{N})$, which would be the correct requirement if $\hat{N}$ were compact by the work of McLean [17].

(b) If a G_2 structure (φ', g') on M is such that $\Xi([\varphi'|_T]) \neq 0$, then $\varphi'|_{\hat{N}'} \neq 0$ for any deformation $\hat{N}'$ of $\hat{N}$ in T, so there are *no* coassociative deformations.

From these observations, it is clear that Proposition 6.19 allows us to define a suitable distinguished family of "nearby" G_2 structures to (φ, g).

Definition 6.20. Let $\hat{\mathcal{F}}$ be a small open ball about 0 in $\mathbb{R}^m$ for some m and let Ξ be given by Proposition 6.19. Let

$$\mathcal{F} = \{(\varphi^f, g^f)\colon f \in \hat{\mathcal{F}}\}$$

be a smooth family of torsion-free G_2 structures, with $(\varphi^0, g^0) = (\varphi, g)$, such that $\Xi([\varphi^f|_T]) = 0$ in $H^3_{\mathrm{cs}}(\hat{N})$ and the map $h_{\hat{\mathcal{F}}}\colon \hat{\mathcal{F}} \to \mathcal{F}$ given by $h_{\hat{\mathcal{F}}}(f) = (\varphi^f, g^f)$ is an embedding.

This is again notation which we shall use for the rest of the paper.

Note. $\hat{\mathcal{F}}$ can be considered as an open subset of $T_0\hat{\mathcal{F}}$.

We are now able to define the moduli space for Problem 3.

Definition 6.21. The *moduli space of deformations* $\mathcal{M}_3(N, \lambda)$ for Problem 3 is the set of pairs (N', f) of $f \in \hat{\mathcal{F}}$ and N' in (M, φ^f, g^f) which are CS coassociative 4-folds at $z'_1, \ldots, z'_s$ with rate λ, having cone C_i and tangent cone $\hat{C}'_i$ at z'_i for all i, such that there exists a diffeomorphism $h\colon M \to M$, isotopic to the identity, such that $h(z_i) = z'_i$ for $i = 1, \ldots, s$, $h|_N\colon N \to N'$ is a homeomorphism and $h|_{\hat{N}}\colon \hat{N} \to N' \setminus \{z'_1, \ldots, z'_s\}$ is a diffeomorphism.

We have a projection map $\pi_{\hat{\mathcal{F}}}\colon \mathcal{M}_3(N, \lambda) \to \hat{\mathcal{F}}$, with $\pi_{\hat{\mathcal{F}}}(N', f) = f$, whose fibres $\pi^{-1}_{\hat{\mathcal{F}}}(f)$ are equal to the moduli space for Problem 2 defined using the G_2 structure (φ^f, g^f).

To incorporate the variation of the singular points and tangent ones, one might naively take the product of the translation space $\mathcal{T}$ with $\mathcal{F}$, then embed $\hat{\mathcal{T}} \times \hat{\mathcal{F}}$ into it in the obvious way. Unfortunately, the changing G_2 structure means we must adapt $\mathcal{T}$ before we proceed. This is tackled in the next definition.

Definition 6.22. Use the notation of Definitions 6.12 and 6.20. For $f \in \hat{\mathcal{F}}$ and $\mathbf{z}' \in B$, let $I^f(\mathbf{z}')$ denote the family of choices of s-tuples $\boldsymbol{\zeta}' = (\zeta'_1, \ldots, \zeta'_s)$ of isomorphisms $\zeta'_i\colon \mathbb{R}^7 \to T_{z'_i}M$ identifying (φ_0, g_0) with $(\varphi^f|_{T_{z'_i}M}, g^f|_{T_{z'_i}M})$.

The *translation space* corresponding to $\hat{\mathcal{F}}$ is

$$\mathcal{T}^{\hat{\mathcal{F}}} = \{(\mathbf{z}', \boldsymbol{\zeta}', f)\colon \mathbf{z}' \in B,\ f \in \hat{\mathcal{F}},\ \boldsymbol{\zeta}' \in I^f(\mathbf{z}')\}.$$

It is a principal G_2^s bundle over $B \times \hat{\mathcal{F}}$.

There is a natural free action of H on $\mathcal{T}^{\hat{\mathcal{F}}}$ and hence an H-orbit through $(\mathbf{z}, \boldsymbol{\zeta}, 0)$. Therefore, we may embed $\hat{\mathcal{T}} \times \hat{\mathcal{F}}$ into $\mathcal{T}^{\hat{\mathcal{F}}}$ by $h_{\hat{\mathcal{T}}\times\hat{\mathcal{F}}}\colon (t, f) \mapsto \big(\mathbf{z}(t, f), \boldsymbol{\zeta}(t, f), f\big)$ such that $h_{\hat{\mathcal{T}}\times\hat{\mathcal{F}}}(\hat{\mathcal{T}} \times \hat{\mathcal{F}})$ is transverse to this H-orbit, $h_{\hat{\mathcal{T}}\times\hat{\mathcal{F}}}(t, 0) = h_{\hat{\mathcal{T}}}(t)$ for all t and $\mathbf{z}(0, f) = \mathbf{z}$ for all f.

Use the notation introduced before Proposition 6.13. Extend the G_2 coordinate system used to define N to a smooth family of G_2 coordinate systems

$$\Big\{\{\chi_i(t, f)\colon B(0; \eta) \longrightarrow V_i(t, f)\colon\ i = 1, \ldots, s\}\colon (t, f) \in \hat{\mathcal{T}} \times \hat{\mathcal{F}}\Big\}$$

such that $V_i(t, f)$ is an open set in M containing $z_i(t, f)$, $\chi_i(t, f)(0) = z_i(t, f)$, $d\chi_i(t, f)|_0 = \zeta_i(t, f)$, $\chi_i(t, 0) = \chi_i(t)$, $V_i(0, f) = V_i$ and $V_i(t, 0) = V_i(t)$ for $i = 1, \ldots, s$. This extends the family of G_2 coordinate systems introduced for Problem 2. Furthermore, the conditions $z_i(0, f) = z_i$ and $V_i(0, f) = V_i$ are natural because $(0, f)$ corresponds to the moduli space for Problem 1 in (M, φ^f, g^f). We cannot expect $\zeta_i(0, f)$ to equal ζ_i because they identify different G_2 structures with the standard G_2 structure on $\mathbb{R}^7$. This trend continues in the statement of the next result, which is the analogue of Proposition 6.13.

Proposition 6.23. *Use the notation from the start of this section, from Propositions* 6.4 *and* 6.13*, and from Definition* 6.22*.*

(a) *There exists a family $\mathcal{N}^{\hat{\mathcal{F}}} = \{N(t, f)\colon (t, f) \in \hat{\mathcal{T}} \times \hat{\mathcal{F}}\}$ of CS 4-folds in M, with $N(0, f) = N$ and $N(t, 0) = N(t)$, such that $N(t, f)$ has a singularity at $z_i(t, f)$, for $i = 1, \ldots, s$, with rate λ, cone C_i and tangent cone $\hat{C}_i(t, f) = \big(\zeta_i(t, f) \circ \iota_i\big)(C_i)$.*

(b) *Let $\hat{N}(t, f) = N(t, f) \setminus \{z_1(t, f), \ldots, z_s(t, f)\}$ and write $N(t, f) = K(t, f) \sqcup \bigsqcup_{i=1}^s U_i(t, f)$, where $K(t, f)$ is compact and $U_i(t, f) \setminus \{z_i(t, f)\} \cong (0, \epsilon) \times \Sigma_i$ for all i, in the obvious way, ensuring that $K(0, f) = K$, $K(t, 0) = K(t)$, $U_i(0, f) = U_i$ and $U_i(t, 0) = U_i(t)$.*

For $(t, f) \in \hat{\mathcal{T}} \times \hat{\mathcal{F}}$, there exist open sets $\hat{T}(t, f) \subseteq M$ containing $\hat{N}(t, f)$ and diffeomorphisms $\delta(t, f)\colon \hat{U} \to \hat{T}(t, f)$ taking the zero section to $\hat{N}(t, f)$, varying smoothly in t and f, with $\hat{T}(0, f) = \hat{T}$,

$\hat{T}(t,0) = \hat{T}(t)$ and $\delta(t,0) = \delta(t)$. Moreover, $\delta(t,f)$ is compatible with the identifications $U_i(t,f) \setminus \{z_i(t,f)\} \cong (0,\epsilon) \times \Sigma_i$ for $i = 1,\dots,s$.

The proof is almost identical to that of Proposition 6.13 so we omit it. The compatibility conditions on $\delta(t,f)$ are given by commutative diagrams like (6.16).

Remark. $\delta(t,f)$ is not required to be compatible with the isomorphism $\nu(\hat{N}) \cong \Lambda^2_+ T^*\hat{N}$ for $(t,f) \neq (0,0)$.

We proceed by defining our final deformation map.

Definition 6.24. Use the notation of Definition 6.5 and Proposition 6.23. For $(t,f) \in \hat{\mathcal{T}} \times \hat{\mathcal{F}}$, let $f_\alpha(t,f) = \delta(t,f) \circ \pi_\alpha$, $\hat{N}_\alpha(t,f) = f_\alpha(t,f)(\hat{N})$ and $N_\alpha(t,f) = \hat{N}_\alpha(t,f) \cup \{z_1(t,f),\dots,z_s(t,f)\}$. Define F_3 on $C^1_{\mathrm{loc}}(\hat{U}) \times \hat{\mathcal{T}} \times \hat{\mathcal{F}}$ by:

$$F_3(\alpha,t,f) = f_\alpha(t,f)^* \left(\varphi^f|_{\hat{N}_\alpha(t,f)}\right) \in C^0_{\mathrm{loc}}(\Lambda^3 T^*\hat{N}).$$

The linearization of F_3 at $(0,0,0)$ acts as

$$dF_3|_{(0,0,0)}\colon (\alpha,t,f) \longmapsto d\alpha + L_2(t) + L_3(f),$$

where $\alpha \in C^1_{\mathrm{loc}}(\Lambda^2_+ T^*\hat{N})$, $(t,f) \in T_0\hat{\mathcal{T}} \oplus T_0\hat{\mathcal{F}}$, L_2 is given in Definition 6.14 and L_3 is a linear map into the space of smooth exact 3-forms on $\hat{N}$ by the condition imposed on φ^f in Definition 6.20.

Note. $F_3(\alpha,t,0) = F_2(\alpha,t)$ as given in Definition 6.14.

Clearly, $\operatorname{Ker} F_3$ corresponds to $\hat{N}_\alpha(t,f)$ on which φ^f vanishes, and we have the analogue of Proposition 6.15 by considering its proof.

Proposition 6.25. *In the notation of Definitions* 6.20 *and* 6.24, $\mathcal{M}_3(N,\lambda)$ *is locally homeomorphic to* $\operatorname{Ker} F_3 = \{(\alpha,t,f) \in C^\infty_\lambda(\hat{U}) \times \hat{\mathcal{T}} \times \hat{\mathcal{F}}\colon F_3(\alpha,t,f) = 0\}$.

We again associate a map to our deformation map.

Definition 6.26. Use the notation of Definition 6.24. Define $G_3\colon C^1_{\mathrm{loc}}(\hat{U}) \times C^1_{\mathrm{loc}}(\Lambda^4 T^*\hat{N}) \times \hat{\mathcal{T}} \times \hat{\mathcal{F}} \to C^0_{\mathrm{loc}}(\Lambda^3 T^*\hat{N})$ by

$$G_3(\alpha,\beta,t,f) = F_3(\alpha,t,f) + d^*\beta,$$

so that $dG_3|_{(0,0,0,0)}\colon (\alpha,\beta,t,f) \longmapsto d\alpha + d^*\beta + L_2(t) + L_3(f)$.

The next result is analogous to Propositions 6.8 and 6.17 and may be immediately deduced from the exactness of $F_3(\alpha, t, f)$, which follows from the choice of φ^f in Definition 6.20.

Proposition 6.27.

$$\operatorname{Ker} F_3 \cong \{(\alpha, \beta, t, f) \in C^\infty_\lambda(\hat{U}) \times C^\infty_\lambda(\Lambda^4 T^* \hat{N}) \times \hat{\mathcal{T}} \times \hat{\mathcal{F}} \colon G_3(\alpha, \beta, t, f) = 0\}.$$

The argument used to prove the regularity result Proposition 6.18 is easily generalized to the map G_3, so we end the section with the following.

Proposition 6.28. *Let* $(\alpha, \beta, t, f) \in L^p_{k+1,\lambda}(\hat{U}) \times L^p_{k+1,\lambda}(\Lambda^4 T^* \hat{N}) \times \hat{\mathcal{T}} \times \hat{\mathcal{F}}$, *where* $p > 4$ *and* $k \geq 2$. *If* $G_3(\alpha, \beta, t, f) = 0$, *where* G_3 *is given in Definition* 6.26, *and* $\|\alpha\|_{C^1_1}$, t *and* f *are sufficiently small,* $(\alpha, \beta) \in C^\infty_\lambda(\hat{U}) \times C^\infty_\lambda(\Lambda^4 T^* \hat{N})$.

Note. We are free to make $\hat{\mathcal{F}}$ smaller, so we can make f as small as necessary.

7. The deformation and obstruction spaces

The scheme we use in this section, for each of our deformation problems in turn, is the following. First we get an analytic description of the infinitesimal deformation and obstruction spaces: the deformation space is the kernel of the linearization of the deformation map at zero, and the obstruction space is a distinguished subspace of the cokernel of the linearization, both in suitable weighted Banach spaces. We then use the implicit function theorem to show that the set of self-dual 2-forms whose image under the deformation map lies in the obstruction space is locally diffeomorphic to the infinitesimal deformation space. Finally, since the moduli space is locally characterized as the kernel of the deformation map, the projection from this set of self-dual 2-forms to the obstruction space is a smooth map between smooth manifolds whose fibre at zero is locally homeomorphic to the moduli space.

We recollect the common notation introduced in § 6, particularly at the start. In addition, fix some $p > 4$ and integer $k \geq 2$ so that we can use the regularity results in § 6 for the kernels of the deformation maps in $L^p_{k+1,\lambda}$.

7.1. Problem 1

Recall the maps F_1 and G_1 given in Definitions 6.5 and 6.7, respectively. Their kernels, which are isomorphic, give a local description for the moduli

space $\mathcal{M}_1(N,\lambda)$ by Propositions 6.6 and 6.8. Therefore the kernels of $dF_1|_0$ and $dG_1|_{(0,0)}$ describe the infinitesimal deformations.

Definition 7.1. The *infinitesimal deformation space* for Problem 1 is

$$\mathcal{I}_1(N,\lambda) = \{(\alpha,\beta) \in L^p_{k+1,\,\lambda}(\Lambda^2_+ T^*\hat{N} \oplus \Lambda^4 T^*\hat{N})\colon d\alpha + d^*\beta = 0\}.$$

Therefore, $\mathcal{I}_1(N,\lambda)$ is finite-dimensional.

To see that this is a reasonable choice, Proposition 6.10 gives that

$$\begin{aligned}\mathcal{I}_1(N,\lambda) &= \{(\alpha,\beta) \in C^\infty_\lambda(\Lambda^2_+ T^*\hat{N} \oplus \Lambda^4 T^*\hat{N})\colon d\alpha + d^*\beta = 0\}\\ &\cong \{\alpha \in C^\infty_\lambda(\Lambda^2_+ T^*\hat{N})\colon d\alpha = 0\},\end{aligned}$$

where the equivalence follows from the maximum principle.

We turn to possible obstructions and start by describing the image of F_1. The proof is long but the basic idea is that φ is exact near N, so $F_1(\alpha)$ is the derivative of a 2-form. The technical issue is that the 2-form is not guaranteed to lie in the correct weighted Sobolev space. This is resolved by specifying its behaviour near the singularities and then considering the cokernel of $d + d^*$.

Proposition 7.2. *In the notation of Proposition* 6.4 *and Definition* 6.5*, the map* F_1 *takes* $L^p_{k+1,\,\lambda}(\hat{U})$ *into* $\overline{d\big(L^p_{k+1,\,\lambda}(\Lambda^2 T^*\hat{N})\big)} \subseteq L^p_{k,\,\lambda}(\Lambda^3 T^*\hat{N})$.

Remark. The closure is a technicality forced upon us because a certain map is not guaranteed to be Fredholm and so may not have closed image.

Proof. Let $\alpha \in L^p_{k+1,\lambda}(\hat{U})$, recall the notation from the start of § 6 and let T be as in Proposition 6.19. As noted after that proposition, $[\varphi|_T] = 0$ in $H^3_{\mathrm{dR}}(T)$ and hence $\varphi|_T$ is exact. Thus, $\varphi|_T = d\psi$ for some $\psi \in C^\infty(\Lambda^2 T^*T)$. However, we want to select ψ in a particular way near the singularities. On $B(0;\eta) \subseteq \mathbb{R}^7$,

$$\chi_i^*(\varphi) = \varphi_0 + O(r_i).$$

If v is the dilation vector field on $\mathbb{R}^7$, given in coordinates $(x_1,\ldots,x_7)$ by

$$v = x_1\frac{\partial}{\partial x_1} + \cdots + x_7\frac{\partial}{\partial x_7},$$

then calculation shows that $d(v\cdot\varphi_0) = 3\varphi_0$. Thus, we can choose ψ to satisfy

$$\chi_i^*(\psi) = \frac{1}{3}(v\cdot\varphi_0) + O(r_i^2)$$

on V_i, then extend ψ smoothly to a form on T such that $d\psi = \varphi|_T$. Note that

$$(v \cdot \varphi_0)|_{\iota_i(C_i)} = v \cdot (\varphi_0|_{\iota_i(C_i)}) = 0$$

as $v \in T\big(\iota_i(C_i)\big)$. Hence $\chi_i^*(\psi) = O(r_i^2)$ on $\iota_i(C_i)$, for all i, and similar results hold for the derivatives of ψ. In the notation of Definition 6.5, define

$$H_1(\alpha) = f_\alpha^* \left(\psi|_{\hat{N}_\alpha}\right)$$

so that $F_1(\alpha) = d\big(H_1(\alpha)\big)$. Note that $\chi_i^*(\psi)|_{\iota_i(C_i)} = O(r_i^2)$ is dominated by $O(r_i^\lambda)$ terms as $r_i \to 0$ since $\lambda < 2$. Further, $H_1(\alpha)$ has the same growth as $\chi_i^*(\psi)|_{(\Phi_\alpha)_i((0,\epsilon)\times\Sigma_i)}$ as $r_i \to 0$, using the notation preceding Proposition 6.6. Continuing to use the aforementioned notation, we see that

$$\chi_i^*(\psi)|_{(\Phi_\alpha)_i((0,\epsilon)\times\Sigma_i)} = \chi_i^*(\psi)|_{((\Phi_\alpha)_i-\iota_i)((0,\epsilon)\times\Sigma_i)} + \chi_i^*(\psi)|_{\iota_i((0,\epsilon)\times\Sigma_i)}.$$

The first term on the right-hand side depends on $|(\Phi_\alpha)_i - \iota_i|$ and hence is $O(r_i^\lambda)$ as $r_i \to 0$. This dominates the second term by our observation above. Hence, $H_1(\alpha) \in L^p_{k,\lambda}$ because H_1 depends on α and $\nabla\alpha$. Notice that $H_1(\alpha)$ has one degree of differentiability less than expected.

We deduce that $F_1(\alpha) \in d\big(L^p_{k,\lambda}(\Lambda^2 T^*\hat{N})\big)$ and so lies in the image of $d + d^*\colon L^p_{k,\lambda}(\Lambda^2 T^*\hat{N} \oplus \Lambda^4 T^*\hat{N}) \to L^p_{k-1,\lambda-1}(\Lambda^3 T^*\hat{N})$, where we include a projection to 3-forms. Therefore, as the dual of a Sobolev space on $\hat{N}$ with weight μ has weight $-4-\mu$, $F_1(\alpha)$ is L^2-orthogonal to the kernel $\mathcal{K}^3$ of the adjoint map

$$d + d^*\colon L^q_{l+1,-3-\lambda}(\Lambda^3 T^*\hat{N}) \longrightarrow L^q_{l,-4-\lambda}(\Lambda^2 T^*\hat{N} \oplus \Lambda^4 T^*\hat{N}),$$

where $q > 1$ such that $1/p + 1/q = 1$ and $l \in \mathbb{N}$. (By elliptic regularity, we are free to choose any $l \in \mathbb{N}$ as $\mathcal{K}^3$ is independent of l.) We show below that

$$\overline{d\big(L^p_{k,\lambda}(\Lambda^2 T^*\hat{N})\big)} \oplus d^*\big(L^p_{k,\lambda}(\Lambda^4 T^*\hat{N})\big) \subseteq L^p_{k-1,\lambda-1}(\Lambda^3 T^*\hat{N})$$

is *characterized* as the subspace which is L^2-orthogonal to $\mathcal{K}^3$. Given this, as $\mathcal{K}^3$ is independent of k, the same holds with k replaced by $k+1$. Since $F_1(\alpha) \in L^p_{k,\lambda-1}$, we deduce that $F_1(\alpha)$ is the limit of a sequence $d\alpha_n + d^*\beta$ for some $(\alpha_n, \beta) \in L^p_{k+1,\lambda}(\Lambda^2 T^*\hat{N} \oplus \Lambda^4 T^*\hat{N})$. As $F_1(\alpha)$ is exact, the usual maximum principle argument applied to the harmonic 4-form β forces it to be zero and the result is proved.

Recall that $\lambda \notin \mathcal{D}$, so that the map (5.1), for $\mu = \lambda$, discussed in §5 is Fredholm. Therefore, its image, which is $d(L^p_{k,\lambda}(\Lambda^2_+ T^*\hat{N})) \oplus d^*(L^p_{k,\lambda}$

$(\Lambda^4 T^*\hat{N}))$ by the now familiar maximum principle argument, is closed, so each constituent part is closed as it is a direct sum. Consider

$$d + d^*\colon L^p_{k,\,\lambda}(\Lambda^{\mathrm{even}}T^*\hat{N}) \to L^p_{k-1,\,\lambda-1}(\Lambda^{\mathrm{odd}}T^*\hat{N}). \tag{7.1}$$

This elliptic map has image whose closure comprises precisely of those elements of $L^p_{k-1,\,\lambda-1}(\Lambda^{\mathrm{odd}}T^*\hat{N})$ which are L^2-orthogonal to the kernel $\mathcal{K}$ of

$$d + d^*\colon L^q_{l+1,\,-3-\lambda}(\Lambda^{\mathrm{odd}}T^*\hat{N}) \to L^q_{l,\,-4-\lambda}(\Lambda^{\mathrm{even}}T^*\hat{N}).$$

The space $\mathcal{K}$ can be written as the direct sum $\mathcal{K} = \mathcal{K}^1 \oplus \mathcal{K}^3 \oplus \mathcal{K}^m$, where

$$\mathcal{K}^j = \mathcal{K} \cap L^q_{l+1,\,-3-\lambda}(\Lambda^j T^*\hat{N})$$

for $j = 1$ and 3 and $\mathcal{K}^m$ is some transverse subspace. Recall that, in $L^p_{k-1,\,\lambda-1}$, the image of d^* on 4-forms is closed, and exact and coexact forms meet only at zero by the maximum principle. Thus, the projection of the image of (7.1) to 3-forms has closure

$$\overline{d\big(L^p_{k,\,\lambda}(\Lambda^2 T^*\hat{N})\big)} \oplus d^*\big(L^p_{k,\,\lambda}(\Lambda^4 T^*\hat{N})\big) = \{\alpha_3\colon \exists\, \alpha_1 \text{ such that } (\alpha_1, \alpha_3) \in \mathcal{K}^\perp\}.$$

Note that the projection $\pi_1(\mathcal{K}^m)$ of $\mathcal{K}^m$ onto the space of 1-forms must meet $\mathcal{K}^1$ in the zero form since, if $(\alpha_1, \alpha_3) \in \mathcal{K}^m$ and $\alpha_1 \in \mathcal{K}^1$, then $\alpha_3 \in \mathcal{K}^3$, which contradicts the direct sum decomposition of $\mathcal{K}$. Therefore, $\pi_1(\mathcal{K}^m)$ and $\mathcal{K}^1$ are transverse finite-dimensional subspaces of $L^q_{l+1,-3-\lambda}(\Lambda^1 T^*\hat{N})$. Hence, there exists a space $\mathcal{A}$ of smooth compactly supported 1-forms on $\hat{N}$ which is L^2-orthogonal to $\mathcal{K}^1$ and such that $\mathcal{A} \times \mathcal{K}^m \to \mathbb{R}$ given by $(\gamma, \xi) \mapsto (\gamma, 0) \cdot \xi$ is a dual pairing. If $\alpha_3 \in L^p_{k-1,\,\lambda-1}(\Lambda^3 T^*\hat{N})$ such that $\alpha_3 \in (\mathcal{K}^3)^\perp$, there exists a unique $\alpha_1 \in \mathcal{A}$ such that $(\alpha_1, 0) \cdot \xi = -(0, \alpha_3) \cdot \xi$ for all $\xi \in \mathcal{K}^m$, which implies that $(\alpha_1, \alpha_3) \in (\mathcal{K}^m)^\perp$. We conclude that

$$\begin{aligned}
(\mathcal{K}^3)^\perp &= \{\alpha_3 \in (\mathcal{K}^3)^\perp\colon \exists\, \alpha_1 \in (\mathcal{K}^1)^\perp \text{ such that } (\alpha_1, \alpha_3) \in (\mathcal{K}^m)^\perp\} \\
&= \{\alpha_3\colon \exists\, \alpha_1 \text{ such that } (\alpha_1, \alpha_3) \in \mathcal{K}^\perp\} \\
&= \overline{d\big(L^p_{k,\,\lambda}(\Lambda^2 T^*\hat{N})\big)} \oplus d^*\big(L^p_{k,\,\lambda}(\Lambda^4 T^*\hat{N})\big) \subseteq L^p_{k-1,\,\lambda-1}(\Lambda^3 T^*\hat{N})
\end{aligned}$$

as required. $\square$

We deduce from Propositions 6.6, 6.8, 6.10 and 7.2 that $\mathcal{M}_1(N,\lambda)$ is locally homeomorphic to the kernel of

$$\begin{aligned} G_1\colon L^p_{k+1,\lambda}(\hat{U}) \times L^p_{k+1,\lambda}(\Lambda^4 T^*\hat{N}) \\ \longrightarrow \overline{d\big(L^p_{k+1,\lambda}(\Lambda^2 T^*\hat{N})\big)} \oplus d^*\big(L^p_{k+1,\lambda}(\Lambda^4 T^*\hat{N})\big) \subseteq L^p_{k,\lambda-1}(\Lambda^3 T^*\hat{N}). \end{aligned}$$

By the implicit function theorem (Theorem 4.5), the regularity result Proposition 6.10 and Definition 7.1, if the target space of G_1 equals the image of

$$\begin{aligned} dG_1|_{(0,0)}\colon L^p_{k+1,\lambda}(\Lambda^2_+ T^*\hat{N} \oplus \Lambda^4 T^*\hat{N}) &\longrightarrow L^p_{k,\lambda-1}(\Lambda^3 T^*\hat{N}) \\ (\alpha,\beta) &\longmapsto d\alpha + d^*\beta, \end{aligned}$$

then $\mathcal{M}_1(N,\lambda)$ is a smooth manifold of dimension $\dim \mathcal{I}(N,\lambda)$. Therefore, our deformation theory will be obstructed if and only if the map

$$d\colon L^p_{k+1,\lambda}(\Lambda^2_+ T^*\hat{N}) \longrightarrow \overline{d\big(L^p_{k+1,\lambda}(\Lambda^2 T^*\hat{N})\big)}$$

is *not* surjective. This leads us to the next result and definition.

Proposition 7.3. *There exists a finite-dimensional subspace $\mathcal{O}_1(N,\lambda)$ of $L^p_{k,\lambda-1}(\Lambda^3 T^*\hat{N})$ such that*

$$\overline{d\big(L^p_{k+1,\lambda}(\Lambda^2 T^*\hat{N})\big)} = d\big(L^p_{k+1,\lambda}(\Lambda^2_+ T^*\hat{N})\big) \oplus \mathcal{O}_1(N,\lambda).$$

Proof. As noted in the proof of Proposition 7.2, the Fredholmness of (5.1) for $\mu = \lambda \notin \mathcal{D}$ implies that each part of $d(L^p_{k+1,\lambda}(\Lambda^2_+ T^*\hat{N})) \oplus d^*(L^p_{k+1,\lambda}(\Lambda^4 T^*\hat{N}))$ is closed. Moreover, the sum has finite codimension in $L^p_{k,\lambda-1}(\Lambda^3 T^*\hat{N})$. Since

$$\begin{aligned} d\big(L^p_{k+1,\lambda}(\Lambda^2_+ T^*\hat{N})\big) \oplus d^*\big(L^p_{k+1,\lambda}(\Lambda^4 T^*\hat{N})\big) \\ \subseteq \overline{d\big(L^p_{k+1,\lambda}(\Lambda^2 T^*\hat{N})\big)} \oplus d^*\big(L^p_{k+1,\lambda}(\Lambda^4 T^*\hat{N})\big), \end{aligned}$$

the latter has finite codimension in $L^p_{k,\lambda-1}(\Lambda^3 T^*\hat{N})$, and the former has finite codimension in the latter. Thus, $\mathcal{O}_1(N,\lambda)$ can be chosen as stated. $\square$

Definition 7.4. The *obstruction space* for Problem 1 is

$$\mathcal{O}_1(N,\lambda) \cong \frac{\overline{d\big(L^p_{k+1,\lambda}(\Lambda^2 T^*\hat{N})\big)}}{d\big(L^p_{k+1,\lambda}(\Lambda^2_+ T^*\hat{N})\big)}.$$

We now have all the ingredients necessary to apply the implicit function theorem. Use the notation introduced in § 6 and this section. Define

$$\begin{aligned}
U_1 &= L^p_{k+1,\lambda}(\hat{U}) \times L^p_{k+1,\lambda}(\Lambda^4 T^*\hat{N}),\\
X_1 &= L^p_{k+1,\lambda}(\Lambda^2_+ T^*\hat{N} \oplus \Lambda^4 T^*\hat{N}),\\
Y_1 &= \mathcal{O}_1(N,\lambda) \subseteq L^p_{k,\lambda-1}(\Lambda^3 T^*\hat{N}) \text{ and}\\
Z_1 &= \overline{d\big(L^p_{k+1,\lambda}(\Lambda^2 T^*\hat{N})\big)} \oplus d^*\big(L^p_{k+1,\lambda}(\Lambda^4 T^*\hat{N})\big) \subseteq L^p_{k,\lambda-1}(\Lambda^3 T^*\hat{N}).
\end{aligned}$$

Then X_1, Y_1 and Z_1 are Banach spaces: X_1 by definition, Y_1 because it is a finite-dimensional subspace of a Banach space and Z_1 because it is a closed subspace of a Banach space as noted in the proof of Proposition 7.2. Further, U_1 is an open neighbourhood of $(0,0)$ in X_1 because $L^p_{k+1,\lambda} \hookrightarrow C^1_1$ by Theorem 4.4 and $\hat{U}$ is an open subset of C^1_1 by Proposition 6.4. Thus, $W_1 = U_1 \times Y_1$ is an open neighbourhood of $\big((0,0),0\big)$ in $X_1 \times Y_1$. Define $\mathcal{G}_1\colon W_1 \to Z_1$ by:

$$\mathcal{G}_1\big((\alpha,\beta),\gamma\big) = G_1(\alpha,\beta) + \gamma.$$

Then $\mathcal{G}_1$ is well-defined by Propositions 7.2 and 7.3 and its derivative at $\big((0,0),0\big)$ acts surjectively from $X_1 \times Y_1$ to Z_1 as

$$d\mathcal{G}_1|_{((0,0),0)}\colon \big((\alpha,\beta),\gamma\big) \longmapsto d\alpha + d^*\beta + \gamma.$$

Using Definition 7.1 and the fact that $(d+d^*)(X_1) \cap Y_1 = \{0\}$, we see that

$$\begin{aligned}
\operatorname{Ker} d\mathcal{G}_1|_{((0,0),0)} &= \big\{\big((\alpha,\beta),\gamma\big) \in X_1 \times Y_1\colon d\alpha + d^*\beta + \gamma = 0\big\}\\
&\cong \{(\alpha,\beta) \in X_1\colon d\alpha + d^*\beta = 0\} = \mathcal{I}_1(N,\lambda).
\end{aligned}$$

The conclusion, by implementing the implicit function theorem (Theorem 4.5), is that $\operatorname{Ker}\mathcal{G}_1$ is a smooth manifold near zero locally diffeomorphic to an open neighbourhood $\hat{\mathcal{M}}_1(N,\lambda)$ of $(0,0)$ in $\mathcal{I}_1(N,\lambda)$. Formally, if we write $X_1 = \mathcal{I}_1(N,\lambda) \oplus A$ for some closed subspace A of X_1, there exist open sets $\hat{\mathcal{M}}_1(N,\lambda) \subseteq \mathcal{I}_1(N,\lambda)$, $V_A \subseteq A$, $V_Y \subseteq Y_1$, all containing zero, with $\hat{\mathcal{M}}_1(N,\lambda) \times V_A \subseteq U_1$, and smooth maps $\mathcal{V}_A\colon \hat{\mathcal{M}}_1(N,\lambda) \to V_A$ and $\mathcal{V}_Y\colon \hat{\mathcal{M}}_1(N,\lambda) \to V_Y$ such that

$$\operatorname{Ker}\mathcal{G}_1 \cap \Big(\big(\hat{\mathcal{M}}_1(N,\lambda) \times V_A\big) \times V_Y\Big) = \Big\{\Big(\big(x,\mathcal{V}_A(x)\big),\mathcal{V}_Y(x)\Big)\colon x \in \hat{\mathcal{M}}_1(N,\lambda)\Big\}.$$

Define a smooth map $\pi_1\colon \hat{\mathcal{M}}_1(N,\lambda) \to \mathcal{O}_1(N,\lambda)$ by $\pi_1(x) = \mathcal{V}_Y(x)$. Its kernel is locally homeomorphic to the kernel of G_1 and it can be considered as

a projection from a subset of the infinitesimal deformation space onto the obstruction space. Hence, the moduli space $\mathcal{M}_1(N,\lambda)$ near N is homeomorphic to the kernel of π_1 near 0. We write this result as a theorem.

Theorem 7.5. *Use the notation of Definitions* 6.1*,* 7.1 *and* 7.4*. There exist a smooth manifold* $\hat{\mathcal{M}}_1(N,\lambda)$*, which is an open neighbourhood of* $(0,0)$ *in* $\mathcal{I}_1(N,\lambda)$*, and a smooth map* $\pi_1\colon \hat{\mathcal{M}}_1(N,\lambda)\to\mathcal{O}_1(N,\lambda)$*, with* $\pi_1(0,0)=0$*, such that an open neighbourhood of* 0 *in* $\operatorname{Ker}\pi_1$ *is homeomorphic to an open neighbourhood of* N *in* $\mathcal{M}_1(N,\lambda)$*.*

We deduce that, if the obstruction space is zero, the moduli space for Problem 1 is a smooth manifold near N of dimension equal to that of the infinitesimal deformation space. We expect the obstruction space to be zero for generic choices of N and the G_2 structure on M.

7.2. Problem 2

Recall the notation introduced in Definitions 6.12, 6.14 and 6.16. We begin by defining the infinitesimal deformation space for this problem.

Definition 7.6. The *infinitesimal deformation space* for Problem 2 is

$$\begin{aligned}&\mathcal{I}_2(N,\lambda)\\&\quad=\{(\alpha,\beta,t)\in L^p_{k+1,\lambda}(\Lambda^2_+T^*\hat{N}\oplus\Lambda^4T^*\hat{N})\oplus T_0\hat{\mathcal{T}}\colon d\alpha+d^*\beta+L_2(t)=0\},\end{aligned}$$

which is finite-dimensional.

Using Proposition 6.18 and the maximum principle,

$$\begin{aligned}\mathcal{I}_2(N,\lambda)&=\{(\alpha,\beta,t)\in C^\infty_\lambda(\Lambda^2_+T^*\hat{N}\oplus\Lambda^4T^*\hat{N})\oplus T_0\hat{\mathcal{T}}\colon d\alpha\\&\qquad+d^*\beta+L_2(t)=0\}\\&\cong\{(\alpha,t)\in C^\infty_\lambda(\Lambda^2_+T^*\hat{N})\oplus T_0\hat{\mathcal{T}}\colon d\alpha+L_2(t)=0\},\end{aligned}$$

since $d\alpha+L_2(t)$ is exact by Definition 6.14.

Note. There is a subspace of $\mathcal{I}_2(N,\lambda)$ which is isomorphic to $\mathcal{I}_1(N,\lambda)$.

To start our consideration of obstructions, we have the easy generalization of Proposition 7.2.

Proposition 7.7. *The map* F_2 *given in Definition* 6.14 *takes* $L^p_{k+1,\lambda}(\hat{U})\times\hat{\mathcal{T}}$ *into* $\overline{d\big(L^p_{k+1,\lambda}(\Lambda^2T^*\hat{N})\big)}\subseteq L^p_{k,\lambda-1}(\Lambda^3T^*\hat{N})$*.*

Proof. Use the notation from Proposition 6.13 and its proof and from the proof of Proposition 7.2. Recall that we have an open set $T \supseteq \hat{T}$ in M containing N, which retracts onto N, and $\psi \in C^\infty(\Lambda^2 T^* T)$ such that $d\psi = \varphi|_T$.

Let $T(t) = \hat{T}(t) \cup \bigcup_{i=1}^s V_i(t)$. By making $\hat{T}(t)$ and $V_i(t)$ smaller if necessary, $T(t)$ is an open set in M, varying smoothly in t, which retracts onto N. We may choose $\psi(t)$ to vary smoothly with t such that

$$\chi_i(t)^*\big(\psi(t)\big) = \frac{1}{3}\,(v \cdot \varphi_0) + O(r_i^2)$$

on $V_i(t)$ and then extend smoothly to a form $\psi(t) \in C^\infty\big(\Lambda^2 T^* T(t)\big)$ such that $d\psi(t) = \varphi|_{T(t)}$. Notice that $\psi(0) = \psi$ and $T(0) = T$. Define

$$H_2(\alpha, t) = f_\alpha(t)^* \left(\psi(t)|_{\hat{N}_\alpha(t)}\right).$$

Then $d\big(H_2(\alpha, t)\big) = F_2(\alpha, t)$. Moreover, by the same reasoning that showed $H_1(\alpha) \in L^p_{k,\,\lambda}$ in the proof of Proposition 7.2, $H_2(\alpha, t)$ lies in $L^p_{k,\lambda}$. Therefore, $F_2(\alpha, t)$ lies in $d\big(L^p_{k,\lambda}(\Lambda^2 T^* \hat{N})\big)$. However, because $F_2(\alpha, t) \in L^p_{k,\lambda-1}$ $(\Lambda^3 T^* \hat{N})$, the proof of Proposition 7.2 implies the result. $\square$

We now define the obstruction space.

Definition 7.8. From Propositions 7.3 and 7.7, since L_2 is a linear map on a finite-dimensional vector space, there exists a finite-dimensional subspace $\mathcal{O}_2(N, \lambda)$ of $L^p_{k,\lambda-1}(\Lambda^3 T^* \hat{N})$, contained in $\mathcal{O}_1(N, \lambda)$, such that

$$\overline{d\big(L^p_{k+1,\,\lambda}(\Lambda^2 T^* \hat{N})\big)} = \Big(d\big(L^p_{k+1,\,\lambda}(\Lambda^2_+ T^* \hat{N})\big) + L_2(T_0\hat{\mathcal{T}})\Big) \oplus \mathcal{O}_2(N, \lambda).$$

We define $\mathcal{O}_2(N, \lambda)$ to be the *obstruction space* for Problem 2.

Following the scheme for Problem 1, we let

$$\begin{aligned}
U_2 &= L^p_{k+1,\lambda}(\hat{U}) \times L^p_{k+1,\lambda}(\Lambda^4 T^* \hat{N}) \times \hat{\mathcal{T}},\\
X_2 &= L^p_{k+1,\lambda}(\Lambda^2_+ T^* \hat{N} \oplus \Lambda^4 T^* \hat{N}) \oplus T_0\hat{\mathcal{T}},\\
Y_2 &= \mathcal{O}_2(N, \lambda) \subseteq L^p_{k,\lambda-1}(\Lambda^3 T^* \hat{N}) \text{ and}\\
Z_2 &= \overline{d\big(L^p_{k+1,\lambda}(\Lambda^2 T^* \hat{N})\big)} \oplus d^*\big(L^p_{k+1,\,\lambda}(\Lambda^4 T^* \hat{N})\big) \subseteq L^p_{k,\lambda-1}(\Lambda^3 T^* \hat{N}).
\end{aligned}$$

Recall that $\hat{\mathcal{T}} \subseteq \mathbb{R}^n \cong T_0\hat{\mathcal{T}}$ is open. As for Problem 1, X_2, Y_2 and Z_2 are Banach spaces, U_2 is an open neighbourhood of $(0, 0, 0)$ in X_2 and hence

$W_2 = U_2 \times Y_2$ is an open neighbourhood of $\big((0,0,0),0\big)$ in $X_2 \times Y_2$. Define $\mathcal{G}_2\colon W_2 \to Z_2$ by:

$$\mathcal{G}_2\big((\alpha,\beta,t),\gamma\big) = G_2(\alpha,\beta,t) + \gamma.$$

From Definition 7.8, $d\mathcal{G}_2|_{((0,0,0),0)}\colon X_2 \times Y_2 \to Z_2$ acts surjectively as

$$\big((\alpha,\beta,t),\gamma\big) \longmapsto d\alpha + d^*\beta + L_2(t) + \gamma.$$

Using the fact that the image of $dG_2|_{(0,0,0)}$ meets Y_2 at 0 only,

$$\begin{aligned}\operatorname{Ker} d\mathcal{G}_2|_{((0,0,0),0)} &= \big\{\big((\alpha,\beta,t),\gamma\big) \in X_2 \times Y_2\colon d\alpha + d^*\beta + L_2(t) + \gamma = 0\big\}\\ &\cong \{(\alpha,\beta,t) \in X_2\colon d\alpha + d^*\beta + L_2(t) = 0\} = \mathcal{I}_2(N,\lambda).\end{aligned}$$

As for Problem 1, Theorem 4.5 then gives us the following theorem.

Theorem 7.9. *Use the notation of Definitions* 6.11, 7.6 *and* 7.8. *There exist a smooth manifold* $\hat{\mathcal{M}}_2(N,\lambda)$, *which is an open neighbourhood of* $(0,0,0)$ *in* $\mathcal{I}_2(N,\lambda)$, *and a smooth map* $\pi_2 : \hat{\mathcal{M}}_2(N,\lambda) \to \mathcal{O}_2(N,\lambda)$, *with* $\pi_2(0,0,0) = 0$, *such that an open neighbourhood of zero in* $\operatorname{Ker}\pi_2$ *is homeomorphic to an open neighbourhood of* N *in* $\mathcal{M}_2(N,\lambda)$.

We deduce that, if $\mathcal{O}_2(N,\lambda) = \{0\}$, the moduli space for Problem 2 is a smooth manifold near N of dimension $\dim \mathcal{I}_2(N,\lambda) = \dim \mathcal{I}_1(N,\lambda) + \dim \hat{\mathcal{T}}$, which we expect to occur for generic choices of N and the torsion-free G_2 structure on M. In § 9, we see that if we choose a generic *closed* G_2 structure on M, within a certain family, we may drop the genericity assumption on N and still obtain a smooth moduli space.

7.3. Problem 3

We presume in this section that the reader is sufficiently familiar with the schemata we have used in § 7.1 and 7.2 to be able to generalize them to Problem 3. This allows us to present a tidier treatment of the problem.

Recall the notation of Definitions 6.20, 6.24 and 6.26.

Definition 7.10. The *infinitesimal deformation space* $\mathcal{I}_3(N,\lambda)$ for Problem 3 is

$$\begin{aligned}\mathcal{I}_3(N,\lambda) = \{(\alpha,\beta,t,f) \in L^p_{k+1,\lambda}(\Lambda^2_+ T^*\hat{N} \oplus \Lambda^4 T^*\hat{N}) \oplus T_0\hat{\mathcal{T}} \oplus T_0\hat{\mathcal{F}} \colon d\alpha \\ + d^*\beta + L_2(t) + L_3(f) = 0\}.\end{aligned}$$

By Proposition 6.28 and the maximum principle,

$$\begin{aligned}\mathcal{I}_3(N,\lambda) &= \{(\alpha,\beta,t,f) \in C^\infty_\lambda(\Lambda^2_+ T^*\hat{N} \oplus \Lambda^4 T^*\hat{N}) \oplus T_0\hat{\mathcal{T}} \oplus T_0\hat{\mathcal{F}} \colon d\alpha \\ &\quad + d^*\beta + L_2(t) + L_3(f) = 0\} \\ &\cong \{(\alpha,t,f) \in C^\infty_\lambda(\Lambda^2_+ T^*\hat{N}) \oplus T_0\hat{\mathcal{T}} \oplus T_0\hat{\mathcal{F}} \colon d\alpha \\ &\quad + L_2(t) + L_3(f) = 0\}.\end{aligned}$$

We then have the generalization of Propositions 7.2 and 7.7, which is proved in a similar manner.

Proposition 7.11. *In the notation of Definition* 6.24*,*

$$F_3\big(L^p_{k+1,\lambda}(\hat{U}) \times \hat{\mathcal{T}} \times \hat{\mathcal{F}}\big) \subseteq \overline{d\big(L^p_{k+1,\lambda}(\Lambda^2 T^*\hat{N})\big)} \subseteq L^p_{k,\,\lambda-1}(\Lambda^3 T^*\hat{N}).$$

This result leads us to define our final obstruction space.

Definition 7.12. Use the notation of Definitions 6.26 and 7.8. From Propositions 7.3 and 7.11, there exists a finite-dimensional subspace $\mathcal{O}_3(N,\lambda)$ of $L^p_{k,\lambda-1}(\Lambda^3 T^*\hat{N})$, contained in $\mathcal{O}_2(N,\lambda)$, such that

$$\begin{aligned}&\overline{d\big(L^p_{k+1,\lambda}(\Lambda^2 T^*\hat{N})\big)} \\ &\quad = \Big(d\big(L^p_{k+1,\lambda}(\Lambda^2_+ T^*\hat{N})\big) + L_2(T_0\hat{\mathcal{T}}) + L_3(T_0\hat{\mathcal{F}})\Big) \oplus \mathcal{O}_3(N,\lambda).\end{aligned}$$

We define $\mathcal{O}_3(N,\lambda)$ to be the *obstruction space* for Problem 3.

The use of the implicit function theorem (Theorem 4.5) in the derivation of Theorems 7.5 and 7.9 can be easily generalized to give the following.

Theorem 7.13. *Use the notation of Definitions* 6.21*,* 7.10 *and* 7.12*. There exist a smooth manifold* $\hat{\mathcal{M}}_3(N,\lambda)$*, which is an open neighbourhood of* $(0,0,0,0)$ *in* $\mathcal{I}_3(N,\lambda)$*, and a smooth map* $\pi_3\colon \hat{\mathcal{M}}_3(N,\lambda) \to \mathcal{O}_3(N,\lambda)$*, with* $\pi_3(0,0,0,0) = 0$*, such that an open neighbourhood of zero in* $\operatorname{Ker}\pi_3$ *is homeomorphic to an open neighbourhood of* $(N,0)$ *in* $\mathcal{M}_3(N,\lambda)$*.*

Hence, if $\mathcal{O}_3(N,\lambda)=\{0\}$, $\mathcal{M}_3(N,\lambda)$ is a smooth manifold near $(N,0)$ of dimension $\dim\mathcal{I}_3(N,\lambda)=\dim\mathcal{I}_2(N,\lambda)+\dim\hat{\mathcal{F}}$. Moreover, the projection map $\pi_{\hat{\mathcal{F}}}\colon \mathcal{M}_3(N,\lambda)\to\hat{\mathcal{F}}$ is smooth near $(N,0)$. We expect this to occur for generic choices of N and the torsion-free G_2 structure on M. We show in § 9 that if we work with *closed* G_2 structures, then we may drop our genericity assumptions on N and (φ,g) and get a smooth moduli space.

8. Dimension calculations

By Theorems 7.5, 7.9 and 7.13, the expected dimension of the moduli space for each problem is the difference between the dimension of the infinitesimal deformation space and the obstruction space. In this section, we first relate the expected dimension of the moduli space for Problem 1 to the index of (5.1) as discussed in § 5.2. We can then deduce an explicit lower bound for the dimension of the moduli space for each of our deformation problems.

Recall the notation introduced at the start of § 6 and § 7. In particular, we fix $\lambda\in(1,2)\setminus\mathcal{D}$, where $\mathcal{D}$ is given by Proposition 5.1, $p>4$ and $k\geq 2$, and let ρ be a radius function on $\hat{N}$ as in Definition 3.2.

Definition 8.1. Define

$$\mathcal{H}^m=\{\xi\in L^2(\Lambda^m T^*\hat{N})\colon d\xi=d^*\xi=0\}.$$

The Hodge star maps $\mathcal{H}^2$ into itself, so there is a splitting $\mathcal{H}^2=\mathcal{H}^2_+\oplus\mathcal{H}^2_-$, where $\mathcal{H}^2_\pm=\mathcal{H}^2\cap C^\infty(\Lambda^2_\pm T^*\hat{N})$.

Let $\mathcal{J}=\jmath\left(H^2_{\mathrm{cs}}(\hat{N})\right)$, where $\jmath\colon H^2_{\mathrm{cs}}(\hat{N})\to H^2_{\mathrm{dR}}(\hat{N})$ is the inclusion map. If $[\alpha],[\beta]\in\mathcal{J}$, there exist compactly supported closed 2-forms ξ and η such that $[\alpha]=\jmath([\xi])$ and $[\beta]=\jmath([\eta])$. We define a product on $\mathcal{J}\times\mathcal{J}$ by

$$[\alpha]\cup[\beta]=\int_{\hat{N}}\xi\wedge\eta. \tag{8.1}$$

Suppose that ξ' and η' are also compactly supported with $[\alpha]=\jmath([\xi'])$ and $[\beta]=\jmath([\eta'])$. Then there exist 1-forms χ and ζ such that $\xi-\xi'=d\chi$ and $\eta-\eta'=d\zeta$. Therefore,

$$\begin{aligned}\int_{\hat{N}}\xi'\wedge\eta'&=\int_{\hat{N}}(\xi-d\chi)\wedge(\eta-d\zeta)=\int_{\hat{N}}\xi\wedge\eta-d\chi\wedge\eta-\xi'\wedge d\zeta\\&=\int_{\hat{N}}\xi\wedge\eta-d(\chi\wedge\eta)-d(\xi'\wedge\zeta)=\int_{\hat{N}}\xi\wedge\eta,\end{aligned}$$

as both $\chi \wedge \eta$ and $\xi' \wedge \zeta$ have compact support. The product (8.1) on $\mathcal{J} \times \mathcal{J}$ is thus well-defined and is a symmetric topological product with a signature (a, b). By [14, Example 0.16], $\mathcal{H}^2 \cong \mathcal{J}$ and the isomorphism is given by $\xi \mapsto [\xi]$. Thus, $\dim \mathcal{H}^2_+ = a$ and hence is a topological number.

We use the notation of the next two definitions for the rest of this section.

Definition 8.2. For convenience, we introduce the notation

$$(d_+ + d^*)_\lambda = d + d^*\colon L^p_{k+1,\lambda}(\Lambda^2_+ T^*\hat{N} \oplus \Lambda^4 T^*\hat{N}) \longrightarrow L^p_{k,\lambda-1}(\Lambda^3 T^*\hat{N}).$$

By Definition 7.1, $\mathcal{I}_1(N, \lambda) = \operatorname{Ker}(d_+ + d^*)_\lambda$. Define the adjoint map by

$$\begin{aligned}(d^*_+ + d)_{-3-\lambda} = d^*_+ + d\colon L^q_{l+1,\,-3-\lambda}&(\Lambda^3 T^*\hat{N})\\ &\longrightarrow L^q_{l,\,-4-\lambda}(\Lambda^2_+ T^*\hat{N} \oplus \Lambda^4 T^*\hat{N}),\end{aligned}$$

where $q > 1$ such that $1/p + 1/q = 1$ and $l \geq k$. (Since we are only concerned with the kernel of this map, which is independent of $l \in \mathbb{N}$ by elliptic regularity, we are free to select l.) Then $\operatorname{Coker}(d_+ + d^*)_\lambda \cong \operatorname{Ker}(d^*_+ + d)_{-3-\lambda}$.

We now study the dimension of the kernel and cokernel of $(d_+ + d^*)_\mu$ to calculate its index. We begin with the kernel.

Proposition 8.3. *The kernel of $(d_+ + d^*)_{-2}$ is isomorphic to $\mathcal{H}^2_+$. Furthermore, if $\mu > -2$ is such that $(-2, \mu] \cap \mathcal{D} = \emptyset$, $\dim \operatorname{Ker}(d_+ + d^*)_\mu = \dim \mathcal{H}^2_+$.*

Proof. Using (4.1) and the maximum principle,

$$\begin{aligned}\mathcal{H}^2_+ &= \{\alpha \in L^2(\Lambda^2 T^*\hat{N}) \cap C^\infty(\Lambda^2_+ T^*\hat{N})\colon d\alpha = d^*\alpha = 0\}\\ &= \{\alpha \in L^2_{0,-2}(\Lambda^2_+ T^*\hat{N}) \cap C^\infty(\Lambda^2_+ T^*\hat{N})\colon d\alpha = 0\}\\ &\cong \{(\alpha, \beta) \in L^2_{0,-2}(\Lambda^2_+ T^*\hat{N} \oplus \Lambda^4 T^*\hat{N}) : \alpha \in C^\infty(\Lambda^2_+ T^*\hat{N}),\\ &\qquad d\alpha + d^*\beta = 0\}.\end{aligned}$$

This gives the first part of the proposition by elliptic regularity.

If $-2 \notin \mathcal{D}$, $[-2, \mu] \cap \mathcal{D} = \emptyset$ and thus, by Proposition 5.6, $\dim \operatorname{Ker}(d_+ + d^*)_\mu = \dim \operatorname{Ker}(d_+ + d^*)_{-2}$.

Suppose now that $-2 \in \mathcal{D}$, so that the kernel of $(d_+ + d^*)_\nu$ decreases, or its cokernel increases, as ν crosses -2 from below by Proposition 5.5. Recall the notation from the start of § 6. Suppose further that (α, β) corresponds to a self-dual 2-form and 4-form on $\hat{N}$ which are subtracted from the kernel. By

the work in [15, § 3 and § 4] this occurs if and only if (α, β) is asymptotic to an $O(r^{-2})$ form ξ on $(0, \epsilon) \times \Sigma_i \cong U_i \setminus \{z_i\}$, for some i, satisfying $(d + d^*)\xi = 0$. (The form ξ is determined by an element of $D(-2, i)$, using the notation of Proposition 5.2.) Therefore, (α, β) is of order $O(\rho^{-2})$ as $\rho \to 0$ and thus lies in L^2. We deduce that $(\alpha, \beta) \in \operatorname{Ker}(d_+ + d^*)_{-2}$, implying that the function $k(\nu) = \operatorname{Ker}(d_+ + d^*)_\nu$ is upper semi-continuous at -2 by Proposition 5.6. The second part of the proposition is thus proved. □

Proposition 8.4. *If $\mu < -1$ is such that $[\mu, -1) \cap \mathcal{D} = \emptyset$, the cokernel of $(d_+ + d^*)_\mu$ is isomorphic to $H^1_{\mathrm{dR}}(\hat{N})$.*

Proof. Recall the final part of the proof of Proposition 7.2 where we discussed the map $d + d^*$ acting on even forms, and its adjoint map which has kernel $\mathcal{K}$. Since $\mu < -1$, $-3 - \mu > \mu - 1$ so $\mathcal{K} \subseteq L^p_{k,\mu-1}(\Lambda^{\mathrm{odd}} T^* \hat{N})$ since $L^q_{l+1,-3-\mu} \hookrightarrow L^p_{k+1,\mu-1}$ by Theorem 4.4 (using $l \geq k$). Moreover, $L^q_{l+1,-3-\mu} \hookrightarrow L^2_{0,-2} = L^2$, which implies that $\mathcal{K}$ is graded and closed under the Hodge star.

Therefore,

$$L^p_{k,\mu-1}(\Lambda^{\mathrm{odd}} T^* \hat{N}) = \overline{(d + d^*)\big(L^p_{k+1,\mu}(\Lambda^{\mathrm{even}} T^* \hat{N})\big)} \oplus \mathcal{K}.$$

If $\gamma \in L^p_{k,\mu-1}(\Lambda^3 T^* \hat{N})$ then $(-*\gamma, \gamma) \in L^p_{k,\mu-1}(\Lambda^{\mathrm{odd}} T^* \hat{N})$ and hence there exist sequences (γ^m_n) in $L^p_{k+1,\mu}(\Lambda^m T^* \hat{N})$, for $m = 0, 2, 4$, and $\eta \in \mathcal{K}$ such that

$$(-*\gamma, \gamma) = \eta + \lim_{n\to\infty} (d + d^*)(\gamma^0_n, \gamma^2_n, \gamma^4_n).$$

By applying the Hodge star,

$$(*\gamma, \gamma) = *\eta + \lim_{n\to\infty} (d - d^*)(*\gamma^4_n, *\gamma^2_n, *\gamma^0_n).$$

Adding the above formulae and averaging gives:

$$\gamma = \tilde{\eta} + \lim_{n\to\infty} d\left(\frac{\gamma^2_n + *\gamma^2_n}{2}\right) + \lim_{n\to\infty} d^*\left(\frac{\gamma^4_n - *\gamma^0_n}{2}\right)$$

where $\tilde{\eta} \in \mathcal{K}^3$ by the properties of $\mathcal{K}$. We deduce that

$$\begin{aligned} d\big(L^p_{k+1,\mu}(\Lambda^2_+ T^* \hat{N})\big) + d^*\big(L^p_{k+1,\mu}(\Lambda^4 T^* \hat{N})\big) \\ = \overline{d\big(L^p_{k+1,\mu}(\Lambda^2 T^* \hat{N})\big) + d^*\big(L^p_{k+1,\mu}(\Lambda^4 T^* \hat{N})\big)} \end{aligned}$$

because $\mu \notin \mathcal{D}$ implies that the left-hand side is closed. In other words, the image of $(d_+ + d^*)_\mu$ is equal to the closure of the image of

$$(8.2) \quad (d + d^*)_\mu = d + d^*\colon L^p_{k+1,\,\mu}(\Lambda^2 T^*\hat{N} \oplus \Lambda^4 T^*\hat{N}) \to L^p_{k,\mu-1}(\Lambda^3 T^*\hat{N}).$$

Thus, $\operatorname{Coker}(d_+ + d^*)_\mu$ is isomorphic to the kernel of the adjoint map of (8.2):

$$(8.3) \quad (d^* + d)_{-3-\mu} = d^* + d\colon L^q_{l+1,\,-3-\mu}(\Lambda^3 T^*\hat{N}) \longrightarrow L^q_{l,-4-\mu}(\Lambda^2 T^*\hat{N} \oplus \Lambda^4 T^*\hat{N}).$$

Using (4.1) as in the proof of Proposition 8.3, the kernel of $(d^* + d)_{-3-(-1)} = (d^* + d)_{-2}$ is isomorphic to $\mathcal{H}^3$. By [14, Example 0.16], $\mathcal{H}^3 \cong H^1_{\mathrm{dR}}(\hat{N})$ and the isomorphism is given by $\gamma \mapsto [*\gamma]$. Since $[\mu, -1) \cap \mathcal{D} = \emptyset$, there are no changes in the cokernel in $[\mu, -1)$ by Proposition 5.6. Moreover, the dimension of the cokernel is lower semi-continuous in μ at -1; this fact can be demonstrated using similar methods to those employed in the proof of Proposition 8.3. $\square$

By Proposition 5.3, $(-2, -1] \cap \mathcal{D} = \emptyset$. Therefore, for any $\mu \in (-2, -1]$,

$$\dim \operatorname{Ker}(d_+ + d^*)_\mu = \dim \mathcal{H}^2_+ \quad \text{and} \quad \dim \operatorname{Coker}(d_+ + d^*)_\mu = b^1(\hat{N}),$$

using Propositions 8.3 and 8.4. Knowing the index of $(d_+ + d^*)_\mu$ for $\mu \in (-2, -1]$, we can calculate it for all growth rates using Theorem 5.5 and hence derive our first estimate of the expected dimension of the moduli space. For the rest of this section, we recall the quantity $d(\mu)$ for $\mu \in \mathcal{D}$, introduced in Definition 5.4, which appears in the statement of Theorem 5.5.

Proposition 8.5.

$$\operatorname{ind}(d_+ + d^*)_\lambda = \dim \mathcal{H}^2_+ - b^1(\hat{N}) - \sum_{\mu \in (-1,\lambda) \cap \mathcal{D}} d(\mu).$$

Since the obstruction space $\mathcal{O}_1(N, \lambda)$ is a subspace of $\operatorname{Coker}(d_+ + d^*)_\lambda$, by relating their dimensions we can improve our estimate.

Proposition 8.6. *In the notation of Definition* 7.4,

$$\dim \mathcal{O}_1(N, \lambda) \leq \dim \operatorname{Coker}(d_+ + d^*)_\lambda - b^1(\hat{N}).$$

Proof. From the proof of Proposition 7.2, the closure of the image of (8.2), for $\mu = \lambda$, is characterized as the subspace of $L^p_{k,\lambda-1}(\Lambda^3 T^*\hat{N})$ which is L^2-orthogonal to the kernel $\mathcal{L}$ of $(d^* + d)_{-3-\lambda}$ defined by (8.3). Furthermore, as noticed in the proof of Proposition 7.3, $\overline{\mathrm{Image}\,(d+d^*)_\lambda}$ has finite codimension in $L^p_{k,\lambda-1}(\Lambda^3 T^*\hat{N})$. Therefore, we may choose a finite-dimensional space $\mathcal{C}$ of smooth compactly supported 3-forms on $\hat{N}$ such that

$$L^p_{k,\lambda-1}(\Lambda^3 T^*\hat{N}) = \overline{\mathrm{Image}\,(d+d^*)_\lambda} \oplus \mathcal{C}$$

and so that the product $\mathcal{C} \times \mathcal{L} \to \mathbb{R}$ given by $(\gamma, \eta) \mapsto \langle \gamma, \eta \rangle_{L^2}$ is nondegenerate.

As $(d_+ + d^*)_\lambda$ is Fredholm, its image is the closed subspace of $L^p_{k,\lambda-1}$ $(\Lambda^3 T^*\hat{N})$ which is L^2-orthogonal to the kernel $\mathcal{L}'$ of $(d_+^* + d)_{-3-\lambda}$. Then $\mathcal{L}' \supseteq \mathcal{L}$ and $\mathcal{L}$ consists of closed and coclosed 3-forms, whereas $\mathcal{L}'$ consists of 3-forms η such that $d\eta = d_+^*\eta = 0$. Hence, we may choose a subspace $\mathcal{L}''$ of $\mathcal{L}'$, transverse to $\mathcal{L}$, comprising 3-forms which are *not* coclosed and such that $\mathcal{L}' = \mathcal{L} \oplus \mathcal{L}''$.

The next stage is to extend $\mathcal{C}$ to a space $\mathcal{C}' = \mathcal{C} \oplus \mathcal{C}''$, where $\mathcal{C}''$ consists of smooth *exact* compactly supported 3-forms on $\hat{N}$, such that

$$L^p_{k,\lambda-1}(\Lambda^3 T^*\hat{N}) = \mathrm{Image}\,(d_+ + d^*)_\lambda \oplus \mathcal{C}'$$

and the product $\mathcal{C}'' \times \mathcal{L}'' \to \mathbb{R}$ given by $(\gamma, \eta) \mapsto \langle \gamma, \eta \rangle_{L^2}$ is nondegenerate, which is possible as $\mathcal{L}''$ comprises forms which are not coclosed. By construction, $\mathcal{C}''$ is a valid choice for $\mathcal{O}_1(N, \lambda)$ by the proofs of Propositions 7.2 and 7.3. Therefore,

$$\dim \mathcal{O}_1(N,\lambda) = \dim \mathcal{C}' - \dim \mathcal{C} = \dim \mathrm{Coker}\,(d_+ + d^*)_\lambda - \dim \mathcal{L}.$$

If γ lies in the kernel of (8.3) for rate $\mu = -1$, then it also lies in the kernel for $\mu = \lambda$ by Theorem 4.4. Thus, the map from $\mathcal{L}$ to $H^1_{\mathrm{dR}}(\hat{N})$ given by $\gamma \mapsto [*\gamma]$ is surjective. This gives the result. □

We may now calculate a lower bound for the expected dimension of $\mathcal{M}_1(N, \lambda)$ using Propositions 8.5 and 8.6.

Proposition 8.7. *Using the notation of Definitions* 7.1 *and* 7.4,

$$\dim \mathcal{I}_1(N,\lambda) - \dim \mathcal{O}_1(N,\lambda) \geq \dim \mathcal{H}^2_+ - \sum_{\mu \in (-1,\lambda) \cap \mathcal{D}} d(\mu).$$

Recalling that the dimension of $\mathcal{T}$ given in Definition 6.12 is $21s$, we derive analogous results for our other problems.

Proposition 8.8. *In the notation of Definitions* 6.12*,* 6.20*,* 7.6*,* 7.8*,* 7.10 *and* 7.12*,*

$$\dim \mathcal{I}_2(N,\lambda) - \dim \mathcal{O}_2(N,\lambda) \geq \dim \mathcal{H}^2_+ + 21s - \dim \mathrm{H} - \sum_{\mu\in(-1,\lambda)\cap\mathcal{D}} d(\mu).$$

and

$$\dim \mathcal{I}_3(N,\lambda) - \dim \mathcal{O}_3(N,\lambda) \geq \dim \mathcal{H}^2_+ + 21s - \dim \mathrm{H} + \dim \hat{\mathcal{F}} - \sum_{\mu\in(-1,\lambda)\cap\mathcal{D}} d(\mu).$$

We note that Propositions 5.3, 8.4, 8.6 and 8.7 imply the following bound on $\dim \mathcal{O}_1(N,\lambda)$.

Proposition 8.9. *Using the notation of Definition* 7.4*,*

$$\dim \mathcal{O}_1(N,\lambda) \leq \sum_{\mu\in(-1,\lambda)\cap\mathcal{D}} d(\mu).$$

In Problem 2, we remove the obstructions corresponding to translations of the singularities and G_2 transformations of the tangent cones. These obstructions occur, respectively, at rates 0 and 1. Hence, $d(0) \geq 7s$, $d(1) \geq 14s - \dim \mathrm{H}$ and we have the following stronger bound on the dimension of $\mathcal{O}_2(N,\lambda)$.

Proposition 8.10. *In the notation of Definitions* 6.12 *and* 7.8*,*

$$\dim \mathcal{O}_2(N,\lambda) \leq -21s + \dim \mathrm{H} + \sum_{\mu\in(-1,\lambda)\cap\mathcal{D}} d(\mu).$$

9. φ-Closed 7-manifolds

For our deformation problems, we have assumed that N is a CS coassociative 4-fold in a G_2 manifold (M,φ,g); that is, φ satisfies $d\varphi = d^*\varphi = 0$. However, the results of McLean [17] we have used, which are based upon the linearization of the map we denoted F_1 in Definition 6.5, still hold if this condition on φ is relaxed to just $d\varphi = 0$. Thus, our deformation theory results hold if (M,φ,g) is a φ-closed 7-manifold in the sense of Definition 2.5.

Remark. The effect of $*\varphi$ not being closed on M means that coassociative 4-folds in M are no longer necessarily volume minimizing in their homology class.

The use of φ-closed 7-manifolds M is that closed G_2 structures occur in infinite-dimensional families, since the set of closed definite 3-forms on M, given in Definition 2.3, is open. Thus, if $m = \dim \mathcal{O}_1(N,\lambda)$, we show that we can always choose a family $\mathcal{F}$ of closed G_2 structures on M, parameterized by $\hat{\mathcal{F}} \subseteq \mathbb{R}^m$, such that $\mathcal{O}_3(N,\lambda) = \{0\}$. Hence, by Theorem 7.13, $\mathcal{M}_3(N,\lambda)$ is a smooth manifold and the projection $\pi_{\hat{\mathcal{F}}}\colon \mathcal{M}_3(N,\lambda) \to \hat{\mathcal{F}}$ is a smooth map near $(N,0)$.

Sard's theorem [12, p. 173] states that, if $h\colon X \to Y$ is a smooth map between finite-dimensional manifolds, the set of $y \in Y$ with some $x \in h^{-1}(y)$ such that $dh|_x\colon T_xX \to T_yY$ is *not* surjective is of measure zero in Y. Therefore, $h^{-1}(y)$ is a submanifold of X for almost all $y \in Y$.

Thus, by Sard's theorem, $\pi_{\hat{\mathcal{F}}}^{-1}(f)$ is a smooth manifold near (N,f) for almost all $f \in \hat{\mathcal{F}}$. As observed in Definition 6.21, $\pi_{\hat{\mathcal{F}}}^{-1}(f)$ corresponds to the moduli space of deformations for Problem 2 defined using the G_2 structure (φ^f, g^f). We deduce that, for any given N, a generic perturbation of the closed G_2 structure within $\mathcal{F}$ ensures that $\mathcal{M}_2(N,\lambda)$ is smooth near N.

We thus conclude with a generic smoothness result for the CS coassociative deformation theory in φ-closed 7-manifolds. This is similar to [7, Theorem 9.1] concerning CS special Lagrangian deformations in almost Calabi–Yau manifolds.

Theorem 9.1. *Let (M,φ,g) be a φ-closed 7-manifold and let N be a CS coassociative 4-fold in M at s points with rate $\lambda \in (1,2) \setminus \mathcal{D}$, where $\mathcal{D}$ is given in Proposition* 5.2. *Use the notation of Definitions* 5.4, 6.12, 7.4, 7.12 *and* 8.1. *Let $m = \dim \mathcal{O}_1(N,\lambda)$ and let $\hat{\mathcal{F}}$ be a small open ball about* 0 *in $\mathbb{R}^m$.*

There exists a smooth family $\mathcal{F} = \{(\varphi^f, g^f)\colon f \in \hat{\mathcal{F}}\}$ of closed G_2 structures on M such that $\mathcal{O}_3(N,\lambda) = \{0\}$. Hence, the moduli space of deformations for Problem 3 is a smooth manifold near $(N,0)$ with dimension

$$\dim \mathcal{M}_3(N,\lambda) \geq \dim \mathcal{H}^2_+ + 21s - \dim \mathrm{H} + \dim \mathcal{O}_1(N,\lambda) - \sum_{\mu \in (-1,\lambda)\cap\mathcal{D}} d(\mu).$$

Moreover, for generic $f \in \hat{\mathcal{F}}$, the moduli space of deformations in (M,φ^f,g^f) for Problem 2 is a smooth manifold near N with dimension

$$\dim \mathcal{M}_2(N,\lambda) \geq \dim \mathcal{H}^2_+ + 21s - \dim \mathrm{H} - \sum_{\mu \in (-1,\lambda)\cap\mathcal{D}} d(\mu).$$

Proof. Use the notation in the proof of Proposition 8.6. Recall that we have a subspace $\mathcal{L}''$ of $L^q_{l+1,\,-3-\lambda}(\Lambda^3 T^*\hat{N})$ consisting of forms η such that $d\eta = d^*_+\eta = 0$ but $d^*\eta \neq 0$. Moreover, $\mathcal{O}_1(N,\lambda)$ can be chosen to be a space of smooth compactly supported exact 3-forms γ such that if $\langle \gamma, \eta\rangle_{L^2} = 0$ for all $\eta \in \mathcal{L}'' \setminus \{0\}$, then $\gamma = 0$. Therefore, $\mathcal{L}'' \cong \big(\mathcal{O}_1(N,\lambda)\big)^*$ and hence has dimension m.

Let $\{\eta_1, \ldots, \eta_m\}$ be a basis for $\mathcal{L}''$ and choose a basis $\{d\upsilon_1, \ldots, d\upsilon_m\}$ for $\mathcal{O}_1(N,\lambda)$, where υ_j is a smooth compactly supported 2-form for all j, such that $\langle d\upsilon_i, \eta_j\rangle_{L^2} = \delta_{ij}$. This is possible because the L^2 product on $\mathcal{O}_1(N,\lambda) \times \mathcal{L}''$ is nondegenerate. For $f = (f_1, \ldots, f_m) \in \mathbb{R}^m$ define $\upsilon_f = \sum_{j=1}^m f_j \upsilon_j$.

Using the notation of Proposition 6.19, define (φ^f, g^f), for $f \in \hat{\mathcal{F}}$, to be a closed G_2 structure on M such that $\Xi([\varphi^f|_T]) = 0$ in $H^3_{\mathrm{cs}}(\hat{N})$ and $\varphi^f|_{\hat{N}} = d\upsilon_f$.

Recall from Definition 6.24 that we have a linear map $L_3\colon T_0\hat{\mathcal{F}} \cong \mathbb{R}^m \to L^p_{k,\,\lambda-1}(\Lambda^3 T^*\hat{N})$ arising from $dF_3|_{(0,0,0)}$. By construction, $L_3(f) = d\upsilon_f$ for $f \in \mathbb{R}^m$ and hence L_3 maps onto $\mathcal{O}_1(N,\lambda)$. Proposition 7.3 and Definition 7.12 imply that $\mathcal{O}_3(N,\lambda) = \{0\}$ as required.

The latter parts of the theorem now follow directly from the discussion preceding it and Proposition 8.8. □

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References

[1] R. Bartnik, *The mass of an asymptotically flat manifold*, Comm. Pure Appl. Math. **39**, (1986), 661–693.

[2] R.L. Bryant, *Some remarks on* G_2*-structures*, in Proceedings of Gökova Geometry — Topology Conference 2005, eds. S. Akbulut, T. Önder and R.J. Stern, International Press, Boston, 2006.

[3] R. Harvey and H.B. Lawson, *Calibrated geometries*, Acta Math. **148** (1982), 47–152.

[4] H. Federer, *Geometric measure theory*, Grundlehren Math. Wiss., **153**, Springer–Verlag, Berlin, 1969.

[5] D.D. Joyce, *Compact manifolds with special holonomy*, OUP, Oxford, 2000.

[6] D.D. Joyce, *Special Lagrangian submanifolds with isolated conical singularities. I. Regularity*, Ann. Global Ann. Geom. **25** (2004), 201–251.

[7] D.D. Joyce, *Special Lagrangian submanifolds with isolated conical singularities. II. Moduli spaces*, Ann. Global Ann. Geom. **25** (2004), 301–352.

[8] D.D. Joyce, *Special Lagrangian submanifolds with isolated conical singularities. III. Desingularization, the unobstructed case*, Ann. Global Ann. Geom. **26** (2004), 1–58.

[9] D.D. Joyce, *Special Lagrangian submanifolds with isolated conical singularities. IV. Desingularization, obstructions and families*, Ann. Global Ann. Geom. **26** (2004), 117–174.

[10] D.D. Joyce, *Special Lagrangian submanifolds with isolated conical singularities. V. Survey and applications*, J. Differential Geom. **63** (2003), 299–347.

[11] A. Kovalev, *Coassociative K3 fibrations of compact* G_2*-manifolds*, Preprint, 2005, http://www.arxiv.org/math.DG/0511150.

[12] S. Lang, *Differential manifolds*, Addison–Wesley, Reading, Massachusetts, 1972.

[13] S. Lang, *Real analysis*, 2nd edn, Addison–Wesley, Reading, Massachusetts, 1983.

[14] R.B. Lockhart, *Fredholm, Hodge and Liouville theorems on noncompact manifolds*, Trans. Amer. Math. Soc. **301** (1987), 1–35.

[15] R.B. Lockhart and R.C. McOwen, *Elliptic differential operators on noncompact manifolds*, Ann. Sci. Norm. Super. Pisa Cl. Sci. **12** (1985), 409–447.

[16] J. Lotay, *Deformation theory of asymptotically conical coassociative 4-folds*, Preprint, 2004, http://www.arxiv.org/math.DG/0411116.

[17] R.C. McLean, *Deformations of calibrated submanifolds*, Comm. Anal. Geom. **6** (1998), 705–747.

[18] S.P. Marshall, *Deformations of special Lagrangian submanifolds*, DPhil thesis, Oxford University, 2002.

[19] V.G. Maz'ya and B. Plamenevskij, *Elliptic boundary value problems*, Amer. Math. Soc. Transl. **123** (1984), 1–56.

[20] C.B. Morrey, *Multiple integrals in the calculus of variations*, Grundlehren Series, **130**, Springer–Verlag, Berlin, 1966.

[21] S. Salamon, *Riemannian geometry and holonomy groups*, Pitman Research Notes in Mathematics, **201**, Longman, Harlow, 1989.

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